Handling Constraints in Kalman Filter Framework

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Definitions:

Constraints: are the boundaries in which the optimization variables stay meaningful.

For example: we don't expect the price of an item to be negative.

Types of Constraints:

Source: Figure from Systems of Inequalities :: Mathspace.com

1. Linear.
   1.1. Equality: $Dx_k = d$
   1.2. Inequality: $Dx_k \leq d$
2. Non-Linear.
   2.1. Equality: $g(x_k) = d$
   2.2. Inequality: $g(x_k) \leq d$

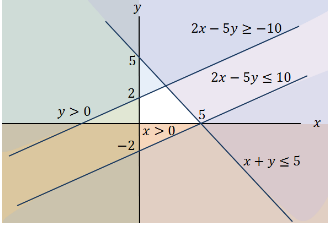


Figure: 2D Linear Inequality Constraints Illustration.

Constraints in Forecast Models

- Constraints are commonly neglected in forecasts models:

1. Some cases are easy to deal with: i.e. if forecast $f(x) \leq Min.Limit$, then $f(x) = Min.Limit$

2. It's not an easy task to modify the prediction model to incorporate the constraints.

When the Constraints are needed in Forecast models?

- Control the forecast trend.
- Forecast with Uncertainty.

Methods to Apply Constraints in Kalman Filter

Two methods will be presented in this seminar:

1. Space Transformation
2. PDF Truncation Method.

Other methods exist in the literature:
- Survey of KF constraining methods is available in the work of D. Simon

Applying Constraints through Space Transformation

- The original (constrained) space is remapped to (unconstrained) space by using transformation function.
- An example for transformation function: the step function.

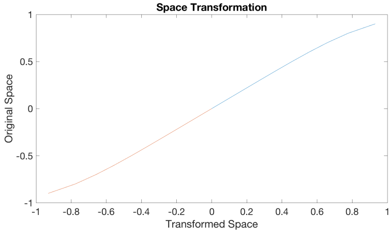
- Advantages:
  1. Easy to implement, minimal changes to the Kalman Filter code

- Disadvantages:
  1. Not effective on the speed and acceleration terms in the state. (Doesn't control the forecast trend).

Applying Constraints through Space Transformation

- Transformation Function: .

```
% Original to transformed-space
x=(gammaincinv(abs(y),1/n))^(1/n)
```



- Transform back to the original space:

```
% Transformed-space to original
y=gammainc(x.^n,1/n);
```

Pdf Truncation Method

- Advantages:
  1. Method for linear equality & inequality constraints.
  2. The Pdf of the state estimate computed by KF is truncated at the constraints edges.
  3. The mean of the truncated Pdf is the new constrained state estimate.

- Disadvantages:
  1. If the pdf of the state is too small, the constraint system may crash. i.e. applying the constraint after the Kalman Smoother.

Pdf Truncation Method

Source: Constrained Kalman Filtering via Density Function Truncation for Turbofan Engine Health Estimation

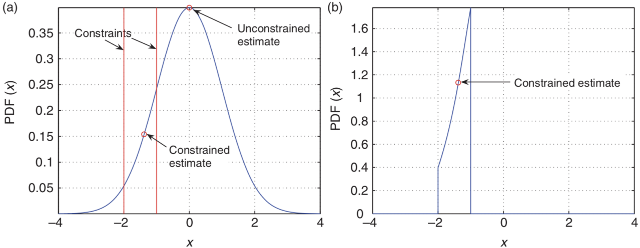


Figure: The unconstrained estimate violates the constraints. The constrained estimate is the centroid of the truncated PDF: (a) Unconstrained PDF; (b) contrained PDF [Simon D. et al 2010].

Pdf Truncation Method - Equations

- Consider the constraint: $a_i(k) \leq \phi_i(k)x_i(k) \leq b_i(k)$
- We are looking for:

$$\begin{bmatrix} v_i \\ \dot{v}_i \\ \ddot{v}_i \end{bmatrix} \rightarrow \begin{bmatrix} TS_i \\ 0 \\ 0 \end{bmatrix}$$

- Consequently, the covariance should be

$$\begin{bmatrix} \sigma_{11}^2 & \sigma_{12}^2 & \sigma_{13}^2 \\ \sigma_{21}^2 & \sigma_{22}^2 & \sigma_{23}^2 \\ \sigma_{31}^2 & \sigma_{32}^2 & \sigma_{33}^2 \end{bmatrix} \rightarrow \begin{bmatrix} \sigma_{trans}^2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- Moreover: $a_i(k) \rightarrow c_{trans}$ & $b_i(k) \rightarrow d_{trans}$

Pdf Truncation Method - Equations

- The Jordan canonical decomposition of the covariance is:

$$TWT^T = \Sigma_i(k)$$

- If the mean forecast violate the constraints, we wish to shift mean by δ to bring it within the constraints:

$$\mu_{trans} = RW^{-1/2}T^T(x_i(k) - \delta)$$

- The Gram-Schmidt orthogonalization:

$$RW^{1/2}T^T\phi_i(k) = [(\phi_i^T(k)\Sigma_i(k)\phi_i(k))^{1/2} \;\; 0 \; ... \; 0]$$

Pdf Truncation Method - Equations

- The PDF of the truncated distribution: $PDF(\eta) = \alpha exp(-\eta^2/2)$
- The parameter α is a normalization term:

$$\alpha = \frac{\sqrt{2}}{\sqrt{\pi}[erf(d_i(k)/\sqrt{2}) - erf(c_i(k)/\sqrt{2})]}$$

$$- \ \mu = \int_{c_i(k)}^{d_i(k)} \alpha \eta exp(-\eta^2/2)dx$$

$$- \ \sigma^2 = \int_{c_i(k)}^{d_i(k)} \alpha(\eta - \mu)^2 exp(-\eta^2/2)dx$$

$$- \ x_{constrained}(k) = TW^{1/2}R^T[\mu_{trans} \ 0...0 \ ] + x_i(k)$$

$$- \ \Sigma_{constrained}(k) = TW^{1/2}R^T diag(\sigma^2_{trans}, 1, 1, ..., 1)RW^{1/2}T^T$$

Quick Recap

Previously presented the following points:

1. The MDMDET Visual Inspections database of bridges.
2. Main Objective: Forecast structural elements condition over time.
3. Data Preprocessing (i.e. converting the 4 condition metric into 1 metric).
4. Performing Forecasts:
   - 4.a. Gaussian Process
   - 4.b. Kalman Filter
5. Tweaking Kalman Filter.

Kalman Filter (KF) Framework & Parameters

KF main components:

1. Known Components.
   1.a. Observations.
2. Unknown Components:
   2.a. Initial state, speed & acceleration.
   2.b. Initial uncertainties of state, speed & acceleration.
   2.c. Uncertainties σ_w (Model) & σ_v (Observations).

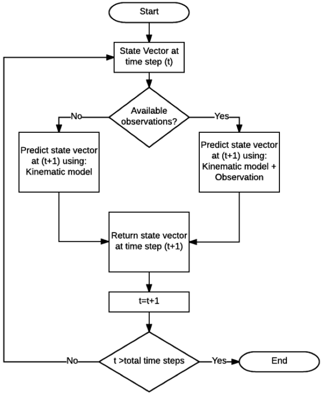


Figure: Kalman Filter (KF) Framework

Estimating σ_w (Model) & σ_v (Observations):

1. Max. Logliklihood.
2. Proposed a nonlinear model:
   2.1. $\sigma_v = n_v(-(M-62.5)^2+1406.25)+1$
   2.2. $\sigma_w =$
   $n_w(-(M-62.5)^2+1406.25)+10^{-6}$

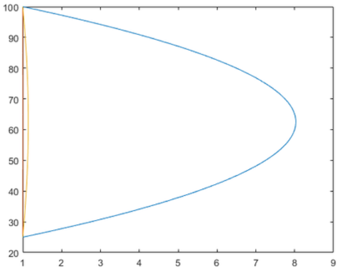


Figure: (σ_v or σ_w) vs. M% blue: Max., orange: Min.

Objective

- Constraint the state vector in KF within a feasible range.
- Prevent unrealistic state trends.
- Incorporate adaptive uncertainty in the framework.

Problem Constraints

- The condition feasible range is known [25 - 100].
- The condition without interventions is always decreasing. Thus, speed and acceleration should not be positive. ▸ Online Illustration

No Constraints

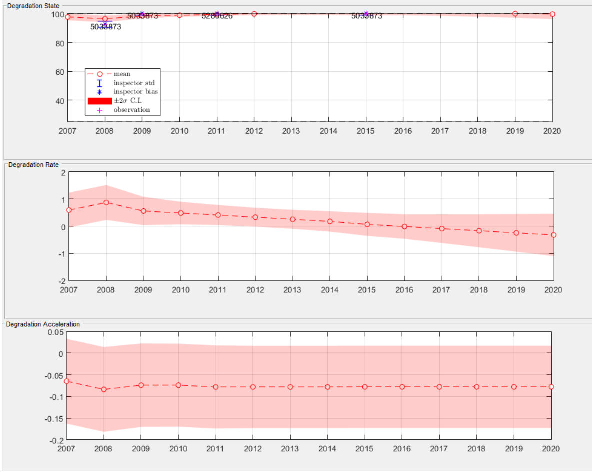


Figure: No Constraints with 5 years forecast

Space Transformation Only

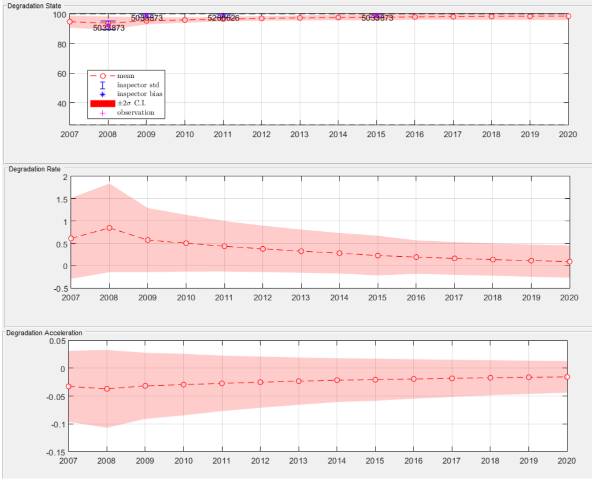


Figure: No Constraints with 5 years forecast

Space Transformation And Pdf Truncation

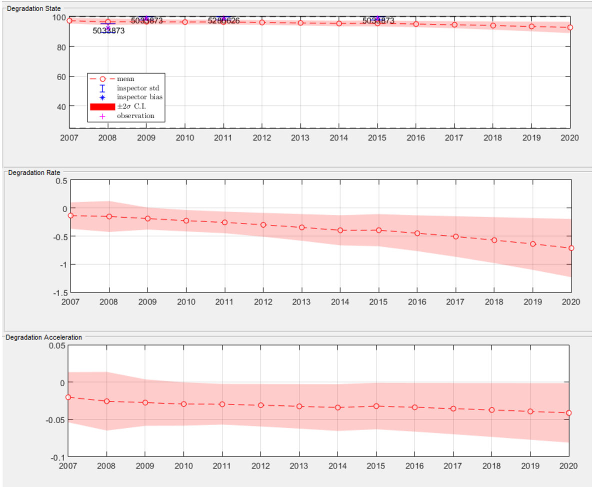


Figure: Full Constraints with 5 years forecast

Conclusions & Future Work

- Conclusions:
  1. Applying the constraints on forecasts shouldn't be neglected in some cases.
  2. Applying the constraints may improve forecasts, as it factors more knowledge in the prediction system.

- Future Work:
  1. The effectiveness of different optimization problems in KF framework will be assessed.