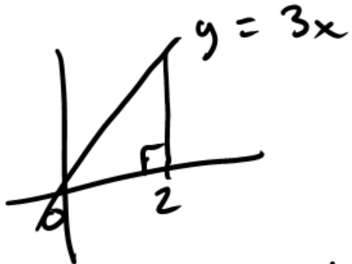
Lecture 31: The Fundamental Theorem of Calculus

GOAL: Compute integrals of more complicated functions.

Recall Our main tool for computing integrals so far has been geometry.



$$\int_0^2 3x\,dx = \frac{1}{2}(2)(6) = 6$$

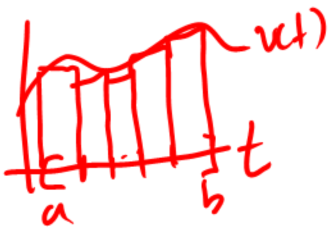
We want a way to compute these w/o relying on geometry. We call this the Fundamental Theorem of Calculus

Theorem (FTC) Let f be continuous on $[a,b]$. Then

$$\int_a^b f(x)\,dx = F(b) - F(a)$$

where F is any antiderivative of f.

Why? Think of this w.r.t. velocity and position



$$\int_{\text{START}}^{\text{END}} v(t)\,dx = \begin{array}{c}\text{CHANGE}\\ \text{IN}\\ \text{POSITION}\\ \text{(Displacement)}\end{array} = S(\text{END}) - S(\text{START})$$

Ex) $\displaystyle\int_0^2 3x\,dx = 3\int_0^2 x\,dx = 3\left[\frac{1}{2}2^2 - \frac{1}{2}0^2\right]$ $\quad \frac{x^{1+1}}{1+1} = \frac{1}{2}x^2$

$= 3[2-0] = 3\cdot 2 = 6$

Ex2/ $\int_1^3 e^x\,dx = e^3 - e^1 = e^3 - e$

e^x is an antiderivative

Remark 1 What happens when we pick another antiderivative?

$e^x + C$ $\quad \int_1^3 e^x\,dx = [e^3 + C] - [e^1 + C] = e^3 - e$

Constant Cancel. So we often choose the simplest antiderivative.

Remark 2 (Notation) We often write

$$\int_a^b f(x)\,dx = F(x)\Big|_a^b = F(x)\Big]_a^b = \Big[F(x)\Big]_a^b$$

$$= F(b) - F(a)$$

$F(x)\Big|_a^b$ is read "F(x) evaluated from a to b"

Ex3/ Find the area under the parabola $y = x^2$ from 0 to 1.

$$\int_0^1 x^2\,dx = \frac{x^{2+1}}{2+1}\Bigg|_0^1 = \frac{1}{3}x^3\Big|_0^1 = \left(\tfrac{1}{3}(1)^3\right) - \left(\tfrac{1}{3}(0)^3\right)$$

$$= \frac{1}{3} - 0 = \frac{1}{3}$$

Ex4/ $\int_3^6 \frac{dx}{x} = \int_3^6 \frac{1}{x}\,dx = \ln|x|\Big|_3^6 = \ln|6| - \ln|3|$

$$= \ln 6 - \ln 3 = \ln\frac{6}{3} = \ln 2$$

Ex5/ $\int_0^{\pi/2} \cos x \, dx = \sin x \Big|_0^{\frac{\pi}{2}} = \sin \frac{\pi}{2} - \sin 0$

$= 1 - 0 = 1$

Ex6/ $\int_5^{10} \frac{1}{x^2} \, dx = \int_5^{10} x^{-2} \, dx = \frac{x^{-2+1}}{-2+1} \Big|_5^{10}$

$= \frac{x^{-1}}{-1} \Big|_5^{10} = -\frac{1}{x} \Big|_5^{10}$

$= -\frac{1}{10} - \left(-\frac{1}{5}\right) = \frac{1}{5} - \frac{1}{10} = \frac{1}{10}$

More Complicated Examples

Ex7/ $\int_0^1 (3 + x\sqrt{x}) \, dx = \int_0^1 (3 + x^{\frac{3}{2}}) \, dx$

METHOD 1: Linearity

$\int_0^1 \underbrace{(3 + x^{\frac{3}{2}})} \, dx = \int_0^1 3 \, dx + \int_0^1 x^{\frac{3}{2}} \, dx$

$= 3 \int_0^1 1 \, dx + \int_0^1 x^{\frac{3}{2}} \, dx$

$\frac{x^{\frac{3}{2}+1}}{\frac{3}{2}+1} = \frac{2}{5} x^{\frac{5}{2}}$

$= 3 \cdot [x]_0^1 + \left[\frac{2}{5} x^{\frac{5}{2}}\right]_0^1$

$= 3(1-0) + \left(\frac{2}{5} - 0\right) = 3 + \frac{2}{5} = \boxed{\frac{17}{5}}$

METHOD 2:

$\int_0^1 (3 + x^{\frac{3}{2}}) \, dx = \left[3x + \frac{2}{5} x^{\frac{5}{2}}\right]_0^1 = \left(3 + \frac{2}{5}\right) - (0 + 0)$

$= \boxed{17/5}$

Ex 8/ Compute $\int_1^2 \left(\frac{4+t^2}{t^3}\right) dt$

$$\int_1^2 \left(\frac{4+t^2}{t^3}\right) dt = \int_1^2 \left(\frac{4}{t^3} + \frac{t^2}{t^3}\right) dt = \int_1^2 \left(\frac{4}{t^3} + \frac{1}{t}\right) dt$$

$\ln|t|$

$$\int \frac{4}{t^3}\,dt = 4\int t^{-3}\,dt = 4\cdot\frac{t^{-3+1}}{-3+1} = 4\,\frac{t^{-2}}{-2} = \frac{-2}{t^2}$$

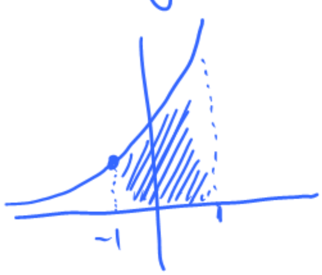
let $C=0$

$$\overset{FTC}{=} \left[\ln|t| - \frac{2}{t^2}\right]_1^2 = \left[\ln 2 - \frac{1}{2}\right] - \left[0 - 2\right]$$

$$= \ln 2 - \frac{1}{2} + 2 = \ln 2 + \frac{3}{2}$$

Often integration questions (especially in MA16020 & MA 261) are phrased like this

Ex9 Find the area of the region bounded by $y = e^{x+1}$, $y=0$, $x=-1$, and $x=1$.



Ie, compute $\int_{-1}^{1} e^{x+1}\,dx = \int_{-1}^{1} e^x \cdot e\,dx = e\int_{-1}^{1} e^x\,dx$

$$= e\left[e^x\right]_{-1}^{1} = e\left[e - \frac{1}{e}\right] = e^2 - 1$$

Ex10 $\int_0^{\pi/4} \sec x(\sec x + \cos x)\,dx = \int_0^{\pi/4} (\sec^2 x + 1)\,dx$

$$= \left[\tan x + x\right]_0^{\pi/4} = \left[\tan\frac{\pi}{4} + \frac{\pi}{4}\right] - \cancel{\left[\tan 0 + 0\right]}$$

$$= \boxed{1 + \pi/4}$$

Ex 11/ Find the area of the region bounded by

$y = x^2 - x - 6$, $y=0$, $x=-2$, and $x=3$

$$\int_{-2}^{3}(x^2-x-6)\,dx = \left[\tfrac{1}{3}x^3-\tfrac{1}{2}x^2-6x\right]_{-2}^{3}$$

$$= \left[9-\tfrac{9}{2}-18\right]-\left[-\tfrac{8}{3}-2+12\right]$$

$$= (9-18+2-12)+\left(-\tfrac{9}{2}+\tfrac{8}{3}\right)$$

$$= -19-\tfrac{11}{6} = -\tfrac{114}{6}-\tfrac{11}{6}$$

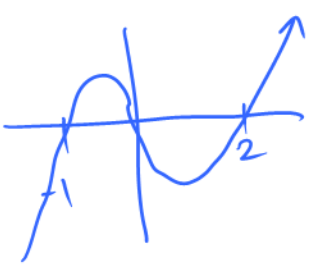
$$= -\tfrac{125}{6}.$$

$$\begin{array}{r} 19 \\ \times\ 6 \\ \hline 54 \\ 6\ \ \\ \hline 114 \end{array}$$

NOTE: $\int_a^b f(x)\,dx$ computes the <u>signed area</u>. To find area, we need $\int_a^b |f(x)|\,dx$

Ex12/ Consider the region bounded by $f(x) = 6x^3-6x^2-12x$,

$y=0$, $x=-1$, $x=2$



(a) Net Area

$$\int_{-1}^{2}(6x^3-6x^2-12x)\,dx = \left[\tfrac{3}{2}x^4-2x^3 \quad 6x^2\right]_{-1}^{2}$$

$$= [24-16-24]-\left[\tfrac{3}{2}+2-6\right] = -\tfrac{27}{2}$$

(b) Total Area

$$\int_{-1}^{2}|f(x)|\,dx = \int_{-1}^{0}(6x^3-6x^2-12x)\,dx+\int_{0}^{2}(-6x^3+6x^2+12x)\,dx = \tfrac{5}{2}+16$$

$$= \tfrac{37}{2}$$