MA 16200: Plane Analytic Geometry and Calculus II

Lecture 2: Dot and Cross Products

§13.3 The Dot Product

Definition

Definition 1 (Dot Product)

Let $\vec{u} = \langle u_1, u_2, \ldots, u_n \rangle$ and $\vec{v} = \langle v_1, v_2, \ldots, v_n \rangle$ be vectors. The **dot product** of $\vec{u}$ and $\vec{v}$ is:

$$\vec{u} \cdot \vec{v} \stackrel{\text{def}}{=} \sum_{i=1}^{n} u_i v_i = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n$$

In 2-Dimensions $(n = 2)$:

$$\vec{u} \cdot \vec{v} = \langle u_1, u_2 \rangle \cdot \langle v_1, v_2 \rangle = u_1 v_1 + u_2 v_2$$

In 3-Dimensions $(n = 3)$:

$$\vec{u} \cdot \vec{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Examples

Problem 2

Compute the following:

- $\langle 1, 2\rangle \cdot \langle 3, 4\rangle$
- $\langle 8, 2\rangle \cdot \langle 0, -2\rangle$
- $\langle 4, -1, 1\rangle \cdot \langle 1, 4, 0\rangle$

Extra Space

Properties Of Dot Products

Theorem 3

Let $\vec{u}$, $\vec{v}$, and $\vec{w}$ be vectors and c a scalar:

1. $\vec{0} \cdot \vec{u} = 0$
2. $\vec{u} \cdot \vec{u} = |\vec{u}|^2$
3. $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ *(Commutative Property)*
4. $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$ *(Associative Property)*
5. $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ *(Distributive Property)*

The dot product gives a rough idea of how "aligned" two vectors are

. . . can we be more precise than this?

Physics Definition of The Dot Product

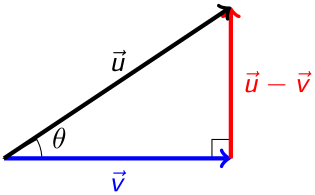
Theorem 4

If $\vec{u}$ and $\vec{v}$ are ***non-zero*** *vectors, then:*

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos\theta$$

where θ is the angle between $\vec{u}$ and $\vec{v}$

Why? By properties of the dot product:



$$\begin{aligned} |\vec{u} - \vec{v}|^2 &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) \\ &= \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} \\ &= |\vec{u}|^2 - 2\vec{u} \cdot \vec{v} + |\vec{v}|^2 \\ &= |\vec{u}|^2 - 2(\vec{u} \cdot \vec{v}) + |\vec{v}|^2 \end{aligned}$$

Extra Space

Now by the Law of Cosines:

$$
\begin{aligned}
|\vec{u} - \vec{v}|^2 &= |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}|\ |\vec{v}| \cos\theta \\
|\vec{u}|^2 - 2(\vec{u}\cdot\vec{v}) + |\vec{v}|^2 &= |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}|\ |\vec{v}| \cos\theta \\
-2(\vec{u}\cdot\vec{v}) &= -2|\vec{u}|\ |\vec{v}| \cos\theta \\
\vec{u}\cdot\vec{v} &= |\vec{u}|\ |\vec{v}| \cos\theta
\end{aligned}
$$

Angle Between Vectors

Definition 5

1. Solving for θ we get the **angle between two vectors** $\vec{u}$ and $\vec{v}$:

$$\theta = \cos^{-1}\left(\frac{\vec{u}\cdot\vec{v}}{|\vec{u}||\vec{v}|}\right)$$

If either $\vec{u}=\vec{0}$ or $\vec{v}=\vec{0}$, then θ is undefined. Note that $0 \le \theta \le \pi$.

2. If $\vec{u}\cdot\vec{v}=0$ (equivalently $\theta = \frac{\pi}{2}$), we say $\vec{u}$ and $\vec{v}$ are **orthogonal**.

In 2 and 3 dimensions, "orthogonal" and "perpendicular" mean the same thing.

Examples

Problem 6

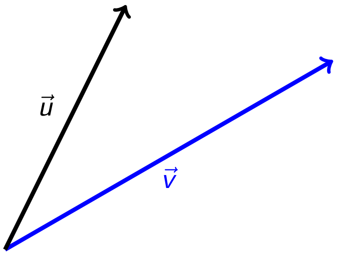
Use the dot product to show:

1. $2\vec{i} + 2\vec{j} - \vec{k}$ *and* $5\vec{i} - 4\vec{j} + 2\vec{k}$ *are orthogonal;*
2. $\langle 2, 1 \rangle$ *is parallel to* $\langle 10, 5 \rangle$.

Extra Space

Projecting Onto Other Vectors

Vectors can "cast a shadow" on another vector, what is the length and direction of that "shadow"?



Components and Projections

Definition 7

1. For a vector $\vec{u}$ and a non-zero vector $\vec{v}$, the **scalar component of $\vec{u}$ in the direction of $\vec{v}$ is**:

$$\mathrm{scal}_{\vec{v}}\vec{u} = |\vec{u}|\cos\theta = \frac{\vec{u}\cdot\vec{v}}{|\vec{v}|}$$

2. The **(orthogonal) projection of $\vec{u}$ onto $\vec{v}$ is**:

$$\mathrm{proj}_{\vec{v}}\vec{u} = [\mathrm{scal}_{\vec{v}}\vec{u}]\,\frac{\vec{v}}{|v|} = \left(\frac{\vec{u}\cdot\vec{v}}{|v|^2}\right)\vec{v}$$

Example

Problem 8

Find $\text{scal}_{\vec{v}}\vec{u}$ *and* $\text{proj}_{\vec{v}}\vec{u}$ *for* $\vec{u} = -4\vec{i} - 3\vec{j}$ *and* $\vec{v} = \vec{i} - \vec{j}$.

Orthogonal Decompositions

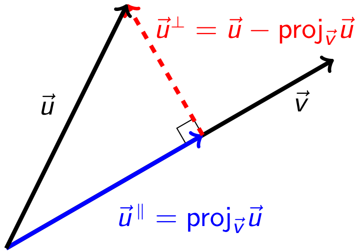
Theorem 9

For a vector $\vec{u}$ and a non-zero vector $\vec{v}$, $\vec{u}$ can be written as the sum of two vectors:

$$\vec{u} = \vec{u}^{\parallel} + \vec{u}^{\perp}$$

where $\vec{u}^{\parallel}$ is parallel to $\vec{v}$ and $\vec{u}^{\perp}$ is orthogonal to $\vec{v}$.

Why?



Application: Work

Problem 10

A wagon is pulled a distance of $100m$ along a horizontal path by a constant force of $50N$. The handle of the wagon is held at an angle of 30° above the horizontal.

1) Express the force vector $\vec{F} = 50\langle \cos\frac{\pi}{6}, \sin\frac{\pi}{6}\rangle = \langle 25\sqrt{3}, 25\rangle$ as the sum of two vectors: one parallel to the ground and one perpendicular to the ground.

Work Definition

Definition 11

Given a force vector $\vec{F}$ and displacement vector $\vec{D}$, the **work (W)** done by the force is defined as:

$$W \stackrel{\text{def}}{=} (Force)(Distance) = (|\vec{F}|\cos\theta)|\vec{D}| = \vec{F}\cdot\vec{D}$$

2) In the previous problem, how much work was done?

§13.4 The Cross Product

Review of Determinants

A **2-by-2 determinant** is defined by:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

A **3-by-3 determinant** is defined using 2-by-2 determinants:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Notice the negative sign next to the a_2.

Cross Product Definition

Definition 12

If $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, then the **cross product** of $\vec{u}$ and $\vec{v}$ is:

$$\vec{u} \times \vec{v} \overset{\text{def}}{=} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$$
$$= (u_2 v_3 - u_3 v_2)\vec{i} + (u_3 v_1 - u_1 v_3)\vec{j} + (u_1 v_2 - u_2 v_1)\vec{k}$$

Notice how $\vec{u} \times \vec{v}$ is a vector and is only defined in 3-dimensions.

Examples

Problem 13

Compute $\vec{u} \times \vec{v}$ for:

1. $\vec{u} = \langle 1, 2, 0 \rangle, \vec{v} = \langle 0, 3, 1 \rangle$
2. $\vec{u} = \langle 1, 1, 0 \rangle, \vec{v} = \langle -2, 3, 0 \rangle$

Extra Space

Properties of Cross Products

Theorem 14

*$\vec{u} \times \vec{v}$ is orthogonal to **both** $\vec{u}$ and $\vec{v}$.*

Theorem 15

Let $\vec{u}, \vec{v}$, and $\vec{w}$ be vectors and c a scalar:

1. $\vec{u} \times \vec{u} = \vec{0}$
2. $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ *(**anti**-commutativity)*
3. $(c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v}) = c(\vec{u} \times \vec{v})$
4. $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
5. $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
6. $\vec{u} \cdot (\vec{v} \times \vec{w}) = (u \times v) \cdot \vec{w}$
7. $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

Physics Definition of Cross Product

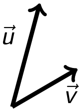
Theorem 16

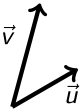
*Given two **non-zero** 3-dimensional vectors $\vec{u}$ and $\vec{v}$:*

$$|\vec{u} \times \vec{v}| = |\vec{u}|\ |\vec{v}| \sin\theta$$

where $0 \leq \theta \leq \pi$ is the angle between $\vec{u}$ and $\vec{v}$.

The direction of $\vec{u} \times \vec{v}$ is given by the **right-hand rule**: Line up the tails of $\vec{u}$ and $\vec{v}$. Curl your right hand from $\vec{u}$ to $\vec{v}$, then $\vec{u} \times \vec{v}$ points in the same direction as your thumb. If $\vec{u}$ or $\vec{v}$ is $\vec{0}$, the direction is undefined.





Why?

$$
\begin{aligned}
|\vec{u} \times \vec{v}|^2 =& (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2 \\
=& u_2^2 v_3^2 - 2 u_2 u_3 v_2 v_3 + u_3^2 v_2^2 + u_3^2 v_1^2 - 2 u_1 u_3 v_1 v_3 + u_1^2 v_3^2 \\
& \quad + u_1^2 v_2^2 - 2 u_1 u_2 v_1 v_2 + u_2^2 v_1^2 \\
=& \left(u_1^2 + u_2^2 + u_3^2\right)\left(v_1^2 + v_2^2 + v_3^2\right) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2 \\
=& |\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2 \\
=& |\vec{u}|^2 |\vec{v}|^2 - |\vec{u}|^2 |\vec{v}|^2 \cos^2 \theta \\
=& |\vec{u}|^2 |\vec{v}|^2 \left(1 - \cos^2 \theta\right) \\
=& |\vec{u}|^2 |\vec{v}|^2 \sin^2 \theta
\end{aligned}
$$

Since $\sin\theta \geq 0$ for $0 \leq \theta \leq \pi$ we have $\sqrt{\sin^2\theta} = \sin\theta$, so take the square root of both sides to get $|\vec{u} \times \vec{v}| = |\vec{u}|\ |\vec{v}| \sin\theta$

Test for Parallel Vectors

Corollary 17

1. *$\vec{u}$ and $\vec{v}$ are parallel if and only if $\vec{u} \times \vec{v} = \vec{0}$*
2. *Three points A, B, and C are collinear if and only if $\vec{AB} \times \vec{AC} = \vec{0}$*

Why?

Example: The points $A(-2, -4, 1)$, $B(1, 3, 7)$, and $C(4, 10, 13)$ are collinear.

Calculating Areas and Volumes

Theorem 18

1. *The area of the parallelogram determined by* $\vec{u}$ *and* $\vec{v}$ *is* $|\vec{u} \times \vec{v}|$;
2. *The area of the triangle determined by the non-collinear points* A, B, *and* C *is* $\frac{1}{2}|\vec{AB} \times \vec{AC}|$.

Why?

Examples

Problem 19

1. *Determine the area of the parallelogram with adjacent sides* $\vec{u} = -4\vec{i} + 3\vec{k}$ *and* $\vec{v} = \vec{i} + \vec{j} + 2\vec{k}$.
2. *Determine the area of the triangle with vertices* $(0, 1), (-1, 1)$, *and* $(1, -1)$

Extra Space

The Triple Scalar Product

Theorem 20

*The volume of the parallelepiped determined by vectors $\vec{u}$, $\vec{v}$, and $\vec{w}$ is the absolute value of the **triple scalar product**:*

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Why? This is a HW Problem.

Example

Problem 21

Are the vectors $\vec{u} = \langle 1, 4, -7 \rangle$, $\vec{v} = \langle 2, -1, 4 \rangle$, *and* $\vec{w} = \langle 0, -8, 18 \rangle$ *coplanar?*

Application: Torque

Definition 22

Applying a force $\vec{F}$ at a point P, the twisting effect (or **torque**) about a point O is a vector $\vec{\tau}$ with:

$$|\vec{\tau}| = |\vec{OP} \times \vec{F}|$$

The direction of $\vec{\tau}$ is governed by the right hand rule.

It is common to write $\vec{OP}$ as $\vec{r}$ (for "radius").

Torque Example

Problem 23

A bolt is tightened by applying a 40-N force to a 0.25-m wrench at a 75° angle. Find the magnitude and direction of the torque about the center of the bolt.

Application: Magnetism

Definition 24

A point charge q experiences a force $\vec{F}$ when it moves through a magnetic field $\vec{B}$ with velocity $\vec{v}$ governed by the equation:

$$\vec{F} = q(\vec{v} \times \vec{B})$$

Notice whether q is positive or negative influences the direction of the force.

Example

Problem 25

An electron $(q = -1.6 \times 10^{-19}\ C)$ *enters a magnetic field* $\vec{B} = \vec{i} + \vec{j}$ *with velocity* $\vec{v} = (2 \times 10^5\ m/s)\vec{k}$.

1. *Determine the magnitude and direction of the force.*
2. *Make a rough sketch of* $\vec{v}$, $\vec{B}$, *and* $\vec{F}$.

Extra Space