This is an optional assignment that will be worth 2 points of extra credit. You must show work to get credit.

NOTE: These are just review problems of Lectures 29-35; these are not necessarily representative of the problems of the exam. The exam can (and most likely will) have different problems.

Directions:

1. Complete each problem on the next page, make sure to show your work. Clearly mark the question number and final answer.
2. You have two options to turn in this assignment:
   (a) **In-person:** You can slip it under my office door located in MATH 615. Make sure your name is on it and that it is stapled together (if there are multiple pages).
   (b) **Email:** You may email your assignment to me at pence11@purdue.edu
      i. Scan your assignment so that it is one PDF (do not submit a bunch of images).
      ii. In the subject line, write "EXTRA CREDIT 4 [your name]".
3. The answers will be given...somewhere (either during the review session, or they will be posted to the GitHub; I haven't decided yet). Therefore, **no late submissions will be allowed**.

Topics for final: Everything (it is cumulative).

Topics After Exam 3:

- **Lecture 29-30:** Definite Integrals
- **Lecture 31-32:** The Fundamental Theorem of Calculus
- **Lecture 33:** Numerical Integration
- **Lecture 34:** Exponential Growth
- **Lecture 35:** Exponential Decay

Problem 1. The **error function** is defined as:

$$\text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_0^x e^{-t^2}\, dt$$

(a) Show that $\int_a^b e^{-t^2}\, dt = \frac{\sqrt{\pi}}{2}[\text{erf}(b) - \text{erf}(a)]$. [Hint: Combine the right-hand-side into 1 integral].

(b) Approximate the value of erf(2) by using 6 trapezoids. Round to 4 decimal places.

(a) Follow the hint:

$$\frac{\sqrt{\pi}}{2}[erf(b) - erf(a)] = \frac{\sqrt{\pi}}{2}\left[\frac{2}{\sqrt{\pi}}\int_0^b e^{-t^2}dt - \frac{2}{\sqrt{\pi}}\int_0^a e^{-t^2}dt\right]$$

$$= \int_0^b e^{-t^2}dt - \int_0^a e^{-t^2}dt = \int_0^b e^{-t^2}dt + \int_a^0 e^{-t^2}dt$$

$$= \int_a^b e^{-t^2}dt$$

(b) $erf(2) = \frac{2}{\sqrt{\pi}}\int_0^2 e^{-t^2}dt \approx \frac{2}{\sqrt{\pi}} T_6$. Find $\Delta x = \frac{2-0}{6} = \frac{1}{3}$

$$T_6 = \frac{1/3}{2}\left[f(0) + 2f(\tfrac{1}{3}) + 2f(\tfrac{2}{3}) + 2f(1) + 2f(\tfrac{4}{3}) + 2f(\tfrac{5}{3}) + f(2)\right]$$

$$= \frac{1}{6}\left[1 + 2e^{-\frac{1}{9}} + 2e^{-\frac{4}{9}} + 2e^{-1} + 2e^{-\frac{16}{9}} + 2e^{-\frac{25}{9}} + e^{-4}\right]$$

$$= \frac{1}{6}\left[1 + 1.7897 + 1.2824 + 0.7358 + 0.3380 + 0.1244 + 0.0183\right]$$

$$= \frac{1}{6}[5.2886] = 0.8814$$

So, $erf(2) \approx \frac{2}{\sqrt{\pi}} T_6 = \frac{2}{\sqrt{\pi}}(0.8814) = 0.9946$

For context, $erf(2) \approx 0.995322\ldots$

Problem 2. Compute the following integral:

$\frac{1-\cos x}{2}$

$$\int_{-\pi}^{\pi} \sin^2\left(\frac{x}{2}\right)\, dx$$

$\left[\text{Hint: } \sin^2\left(\frac{x}{2}\right) = \frac{\cancel{1-\sin(x)}}{2}\right]$

$$\int \sin^2\left(\frac{x}{2}\right) dx \overset{\text{Hint}}{=\!=} \int\left(\frac{1-\cos x}{2}\right) dx = \frac{1}{2}\int(1-\cos x)\, dx$$

$$= \frac{1}{2}\left[x + \sin x\right] + C$$

So,

$$\int_{-\pi}^{\pi} \sin^2\left(\frac{x}{2}\right) dx = \frac{1}{2}\left[x + \sin x\right]_{-\pi}^{\pi} = \frac{1}{2}\left[(\pi - 0) - (-\pi - 0)\right]$$

$$= \frac{1}{2}\left[2\pi - 0 + 0\right] = \frac{1}{2}\left[2\pi\right] = \pi$$

Problem 3. The marginal cost of manufacturing x yards of a certain fabric is $C'(x) = 3 - 0.01x + 0.000006x^2$ (in dollars per yard). Find the increase in cost if the production level is raised from 2000 yards to 4000 yards.

Need to find $C(4000) - C(2000)$

$$C(4000) - C(2000) \overset{\text{FTC / Net Change Thm}}{=\!=} \int_{2000}^{4000} C'(t)\,dt$$

$$\int_{2000}^{4000} C'(t)\,dt = \int_{2000}^{4000} \left[3 - 0.01x + 0.000006x^2\right] dx$$

$$= \left[3x - 0.01 \cdot \tfrac{1}{2}x^2 + 0.000006 \cdot \tfrac{1}{3}x^3\right]_{2000}^{4000}$$

$$= \left[3(4000) - \frac{0.01}{2}(4000)^2 + \frac{0.000006}{3}(4000)^3\right] - \left[3(2000) - \frac{0.01}{2}(2000)^2 + \frac{0.000006}{3}(2000)^3\right]$$

$$= 60000 - 2000 = \boxed{\$58000}$$

Problem 4. The half-life of Radium-226 ($^{226}_{88}$Ra) is 1590 years.

(a) If a sample of $^{226}_{88}$Ra has a mass of 100mg, what percentage of the sample is left after 1000 years?

(b) When will the mass be reduced to 30mg?

(a) Determine k:

$$\frac{1}{2}\cdot 100 = 100\, e^{1590k}$$

$$\frac{1}{2} = e^{1590k}$$

$$\ln\frac{1}{2} = 1590k$$

$$k = \frac{\ln\frac{1}{2}}{1590} = -\frac{\ln 2}{1590} \approx 0.000436$$

So, $P_{(t)} = 100\, e^{-0.000436t}$

Compute $P_{(1000)} = 100\, e^{-0.000436(1000)} = 100\, e^{-0.436}$

$$\approx 65\text{mg} \Longrightarrow \left[\frac{65}{100}\right]\cdot 100 = 65\% \text{ of the sample is left}$$

(b)

$$30 = 100\, e^{kt}$$

$$\frac{3}{10} = e^{kt}$$

$$\ln\frac{3}{10} = kt$$

$$t = -\frac{\ln(3/10)}{k}$$

We found $k = -\frac{\ln 2}{1590}$, so

$$t = -\frac{\ln(3/10)}{\left(-\frac{\ln 2}{1590}\right)} = \left(\frac{\ln(\frac{3}{10})}{\ln 2}\right)(1590) \approx 2761.78$$

It takes roughly $\boxed{2761.78}$ years to reduce the sample to 30 mg.

Problem 5. How long will be take for an investment to double if the interest rate is 6% compounded continuously?

$$2 = 1e^{0.06t}$$

$$\ln(2) = 0.06t$$

$$t = \frac{\ln(2)}{0.06} \approx 11.55 \text{ years}$$