# Lecture 8   The Product Rule

Goal: Differentiate functions of the form $f(x)g(x)$

NOTE: $\frac{d}{dx}(f(x)g(x)) \neq \frac{d}{dx}(f(x))\frac{d}{dx}(g(x))$

$\frac{d}{dx}(x \cdot 1) = \frac{d}{dx}(x) = 1$

$\frac{d}{dx}(x) \cdot \frac{d}{dx}(1) = 1 \cdot 0 = 0$

$\frac{d}{dx}(f(x)g(x)) = \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \frac{f(x+\Delta x)g(x+\Delta x) - g(x+\Delta x)f(x) + g(x+\Delta x)f(x) - f(x)g(x)}{\Delta x}$

$= \lim_{\Delta x \to 0} \left[ g(x+\Delta x) \frac{f(x+\Delta x) - f(x)}{\Delta x} + f(x) \frac{g(x+\Delta x) - g(x)}{\Delta x} \right]$

$= g(x) f'(x) + f(x) g'(x)$

Theorem (Product Rule) Let $f$ and $g$ be differentiable

functions. Then,

$\frac{d}{dx}(f(x)g(x)) = \frac{d}{dx}(f(x)) \cdot g(x) + f(x) \cdot \frac{d}{dx}(g(x))$

$[f(x)g(x)]' = \underbrace{f'(x)}_{} \underbrace{g(x)}_{} + f(x) g'(x)$
left   right

"right d-left plus left d-right"

Ex/ Compute $\frac{d}{dx}(x \underbrace{\sin x}) = [x]' \sin x + x [\sin x]'$
                    $f$   $g$

$= 1 \cdot \sin x + x \cdot \cos x$

$= \sin x + x \cos x$

Ex/ Compute $\frac{d}{dx}[\underbrace{2\sin x}_{f} \underbrace{\cos x}_{g}] = [2\sin x]'\cos x + 2\sin x[\cos x]'$

$$= (2\cos x)\cos x + (2\sin x)(-\sin x)$$

$$= 2[\cos^2 x - \sin^2 x]$$

Ex/ Compute $h'$ if $h(x) = x^4(3x^3 + 4x + 1)$ ———— —

$f(x) = x^4 \longrightarrow f'(x) = 4x^3$

$g(x) = 3x^3 + 4x + 1 \longrightarrow g'(x) = 9x^2 + 4$

$h'(x) = [f(x) g(x)]' = f'(x) g(x) + f(x) g'(x)$

$$= [4x^3][3x^3 + 4x + 1] + [x^4][9x^2 + 4]$$

Ex/ Compute $h'$ if $h(x) = (x^{10} + x + 1)(x^2 + 9)$ ———— ~

$f(x) = x^{10} + x + 1 \longrightarrow f'(x) = 10x^9 + 1$

$g(x) = x^2 + 9 \longrightarrow g'(x) = 2x$ —— ——

$h'(x) = f'(x) g(x) + f(x) g'(x)$

$$= [\underbrace{10x^9 + 1}_{f'}][\underbrace{x^2 + 9}_{g}] + [\underbrace{x^{10} + x + 1}_{f}][\underbrace{2x}_{g'}]$$

Ex/ Compute $h'$ if $h(x) = \underbrace{(x+1)}_{f} \underbrace{(2x+1)(3x+1)}_{g}$ ✭

$h'(x) = [x + 1]'(2x+1)(3x+1) + (x+1)[\underbrace{(2x+1)}_{f} \underbrace{(3x+1)}_{g}]'$ ✭

$$= [x+1]'(2x+1)(3x+1) + (x+1)[[2x+1]'(3x+1) + (2x+1)[3x+1]']$$

$$= 1 \cdot (2x+1)(3x+1) + (x+1)[2(3x+1) + 3(2x+1)]$$
$$= (2x+1)(3x+1) + 2(x+1)(3x+1) + 3(2x+1)(x+1)$$

Ex/ Find the equation of the tangent line if
$$h(x) = \underbrace{(\sqrt{x} + 4)}_{f}\underbrace{(x^{\frac{2}{3}} - 2x)}_{g} \text{ at } x=1$$

Slope: $h'(1)$. $h'(x) = [\sqrt{x} + 4]'(x^{\frac{2}{3}} - 2x) + (\sqrt{x} + 4)[x^{\frac{2}{3}} - 2x]'$

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}(x^{\frac{2}{3}}) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$$

$$h'(x) = \left(\frac{1}{2\sqrt{x}}\right)(x^{\frac{2}{3}} - 2x) + (\sqrt{x} + 4)\left(\frac{2}{3\sqrt[3]{x}} - 2\right)$$

$$h'(1) = \left(\frac{1}{2}\right)(1-2) + (1+4)\left(\frac{2}{3} - 2\right) = \frac{-43}{6}$$

Point: $h(x) = (\sqrt{x} + 4)(x^{\frac{2}{3}} - 2x)$  
$(1, h(1))$    $h(1) = (1+4)(1-2) = -5$

Point: $(1, -5)$

<span style="color:red">Fractions, am I right?</span>

Equation: $y - (-5) = \frac{-43}{6}(x-1)$

Ex/ When does $y = \underbrace{10x^5}_{f}\underbrace{e^x}_{g}$ have a horizontal tangent line?

$$y' = 50x^4 e^x + 10x^5 e^x = 10x^4 e^x(5+x) \overset{\text{set}}{=} 0$$

Either $10x^4 = 0$   OR   $e^x = 0$   OR   $5+x = 0$

$\boxed{x = 0}$     Impossible     $\boxed{x = -5}$

Ex/ Compute $h'$ if $h(x) = 6e^x \sin x - 13 e^x \cos x$
$$= e^x (6 \sin x - 13 \cos x)$$
$\underbrace{e^x}_{f} \underbrace{(6 \sin x - 13 \cos x)}_{g}$

$$h'(x) = [e^x]'(6 \sin x - 13 \cos x) + e^x [6 \sin x - 13 \cos x]'$$

$$= e^x (6 \sin x - 13 \cos x) + e^x (6 \cos x + 13 \sin x)$$

$$= e^x (6 \sin x - 13 \cos x + 6 \cos x + 13 \sin x)$$

$$= e^x (19 \sin x - 7 \cos x)$$