

Final Exam Review

This Friday 8/8, 1-3 PM, WALC B066

Volumes of Revolution:

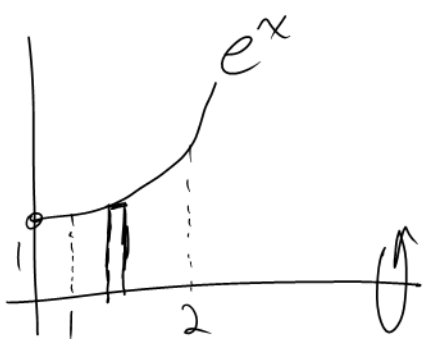
$$f(x) = e^x$$

Disk/Washer Method

$$\int \pi [f(x)]^2 dx$$

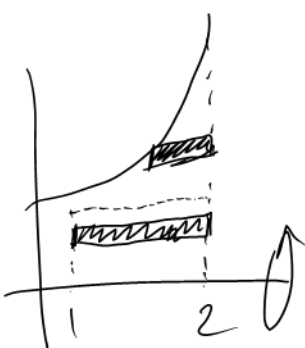
Shell Method

$$\int 2\pi x f(x) dx$$



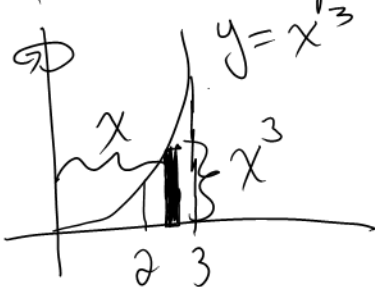
Disk Method:

$$\int_1^2 \pi [e^x]^2 dx = \frac{\pi}{2} \int_1^2 2e^{2x} dx = \frac{\pi}{2} [e^{2x}]_1^2 = \frac{\pi}{2} [e^4 - e^2]$$



$$\int_{y=1}^{e^2} 2\pi y [2 - 1] dy + \int_{e^2}^{e^4} 2\pi y [2 - \ln y] dy$$

Another Example



Formula: $\int 2\pi x f(x) dx$

$$\int_2^3 2\pi x [x^3] dx = \int_2^3 2\pi x^4 dx = 2\pi \cdot \frac{1}{5} [x^5]_2^3 = \frac{2\pi}{5} [3^5 - 2^5]$$

$$\int (\ln y) y dy = (\ln y) \left(\frac{1}{2} y^2 \right) - \int \frac{1}{2} y^2 \cdot \frac{1}{y} dy$$

$$u = \ln y \quad v = \frac{1}{2} y^2 \quad \left| \quad = (\ln y) \left(\frac{1}{2} y^2 \right) - \frac{1}{2} \int y dy \right.$$

$$du = \frac{1}{y} dy \quad dv = y dy \quad \left| \quad = (\ln y) \left(\frac{1}{2} y^2 \right) - \frac{1}{2} \cdot \frac{1}{2} y^2 + C$$

Arc Length:
 In Cartesian, $\int_a^b ds; ds = \sqrt{1 + [f'(x)]^2} dx$
 $= \int_a^b \sqrt{1 + [f'(x)]^2} dx; y = f(x); a \leq x \leq b$

In polar, $\int_a^b ds = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta; r = f(\theta); a \leq \theta \leq b$

Ex/ Compute the length of the curve for $y = x^2$
 from $0 \leq x \leq 1$.

$$y = x^2$$

$$\frac{dy}{dx} = 2x$$

$$L = \int ds = \int_0^1 \sqrt{1 + (2x)^2} dx = \int_0^1 \sqrt{1 + 4x^2} dx$$

$$\int \sqrt{1 + (2x)^2} dx = \int \sec \theta \left(\frac{1}{2} \sec^2 \theta \right) d\theta = \frac{1}{2} \int \underbrace{\sec \theta}_u \underbrace{\sec^2 \theta}_{dv} d\theta$$

$$\frac{2x}{1} = \tan \theta \quad \left| \quad dx = \frac{1}{2} \sec^2 \theta d\theta \right| \quad \sec \theta \tan \theta - \int \tan^2 \theta \sec \theta d\theta$$

$$x = \frac{1}{2} \tan \theta \quad \left| \quad \sec \theta = \sqrt{1 + (2x)^2} \right| = \sec \theta \tan \theta - \int (\sec^2 \theta - 1) \sec \theta d\theta$$

$$= \sec \theta \tan \theta - \int \sec^3 \theta d\theta + \int \sec \theta d\theta$$

$$2 \int \sec^3 \theta d\theta = \sec \theta \tan \theta + \int \sec \theta d\theta$$

$$\int \sec^3 \theta d\theta = \frac{1}{2} [\sec \theta \tan \theta + \ln |\sec \theta + \tan \theta|] + C_0$$



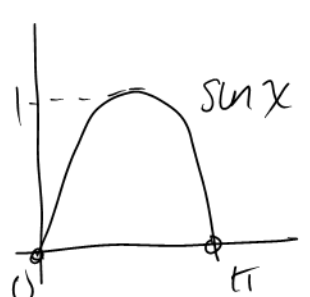
$$\int \sqrt{1+4x^2} dx = \frac{1}{2} [2x\sqrt{1+(2x)^2} + \ln(\sqrt{1+(2x)^2} + 2x)] + C$$

Finally,

$$\mathcal{L} = \int_0^1 \sqrt{1+4x^2} dx = \underset{x=1}{\text{This at}} = \sqrt{5} + \frac{1}{2} \ln(\sqrt{5} + 2)$$

Ex/ Determine the surface area of the region obtained when $f(x) = \sin x$ [$0 \leq x \leq \pi$] is revolved around the x -axis.

Surface Area: $\int_a^b 2\pi y ds = \int_a^b 2\pi y \sqrt{1 + \left(\frac{dy}{dx}\right)^2} dx$



$y = \sin x : \left(\frac{dy}{dx}\right)^2 = \cos^2 x$
 $\frac{dy}{dx} = \cos x$

$$\mathcal{L} = \int_0^\pi 2\pi [\sin x] \sqrt{1 + \cos^2 x} dx$$

Ex/ Find the length of the polar curve $r = \sqrt{1 + \sin 2\theta}$ from $0 \leq \theta \leq \frac{\pi}{2}$

Formula: $\mathcal{L} = \int_a^b ds = \int_a^b \sqrt{r^2 + \left(\frac{dr}{d\theta}\right)^2} d\theta$

$$r^2 = 1 + \sin 2\theta$$

$$2r \frac{dr}{d\theta} = 2 \cos 2\theta \Rightarrow \frac{dr}{d\theta} = \frac{\cos 2\theta}{r} \Rightarrow \left(\frac{dr}{d\theta}\right)^2 = \frac{\cos^2 2\theta}{r^2}$$

$$= \frac{\cos^2 2\theta}{1 + \sin 2\theta}$$

$$\mathcal{L} = \int_0^{\pi/2} \sqrt{(1 + \sin 2\theta) + \frac{\cos^2 2\theta}{1 + \sin 2\theta}} d\theta = \pi$$

Power Series, Polar Coordinates

§11.1 - §11.4 (Power Series)

$\sum_{n=0}^{\infty} C_n(x-a)^n$, There are 3 possibilities for convergence

① Only converges at a

② For all $x \in \mathbb{R}$

③ Converges only a certain distance away from a .

Ex/ Determine the radius and interval of convergence for

$$\sum_{n=1}^{\infty} \frac{(2x)^n}{\sqrt{n}}$$

Use Ratio Test

$$\lim_{n \rightarrow \infty} \left| \frac{(2x)^{n+1}}{\sqrt{n+1}} \cdot \frac{\sqrt{n}}{(2x)^n} \right| = \lim_{n \rightarrow \infty} \left| (2x)^{n+1-n} \cdot \frac{\sqrt{n}}{\sqrt{n+1}} \right| = \lim_{n \rightarrow \infty} |2x| \cdot \sqrt{\frac{n}{n+1}}$$

$$= |2x| \lim_{n \rightarrow \infty} \sqrt{\frac{n}{n+1}} = |2x| \cdot 1 = |2x| < 1 \Rightarrow |x| < \frac{1}{2}$$

Radius: $\frac{1}{2}$

Will converge on $(-\frac{1}{2}, \frac{1}{2})$, now check endpoints

When $x = -\frac{1}{2}$: $\sum_{n=1}^{\infty} \frac{(2 \cdot -\frac{1}{2})^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{(-1)^n}{\sqrt{n}}$ Check AST
 $\frac{1}{\sqrt{n}}$ positive and decreasing
 $\frac{1}{\sqrt{n}} \rightarrow 0$ as $n \rightarrow \infty$

When $x = \frac{1}{2}$: $\sum_{n=1}^{\infty} \frac{(2 \cdot \frac{1}{2})^n}{\sqrt{n}} = \sum_{n=1}^{\infty} \frac{1}{\sqrt{n}}$ divergent p-series
($p = \frac{1}{2}$)

Therefore, the interval is $[-\frac{1}{2}, \frac{1}{2})$

Term-by-Term Differentiation and Integration:

When x is within the radius of convergence

$$\frac{d}{dx} \sum_{n=0}^{\infty} C_n (x-a)^n = \sum_{n=0}^{\infty} C_n \frac{d}{dx} (x-a)^n$$

$$\int \sum_{n=0}^{\infty} C_n (x-a)^n dx = \sum_{n=0}^{\infty} C_n \int (x-a)^n dx$$

Quiz 9 #2 $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}; x \in (-\infty, \infty)$

① Determine a power series rep. for $x^2 e^x$

$$x^2 e^x = x^2 \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^2 \cdot x^n}{n!} = \sum_{n=0}^{\infty} \frac{x^{n+2}}{n!}$$

② Determine " " " " for e^{-x^2}

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} \Rightarrow e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-x^2)^n}{n!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!}$$

Suppose its interval of convergence is $(-\infty, \infty)$

③ Find an anti-derivative for e^{-x^2}

$$\begin{aligned} \int e^{-x^2} dx &= \int \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} dx = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \int x^{2n} dx \\ &= \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \cdot \frac{x^{2n+1}}{2n+1} + C \end{aligned}$$

④ Show that the error of approximating e^{-x^2} by the N^{th} term of the Taylor series goes to 0 as $N \rightarrow \infty$

$$e^{-x^2} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{n!} \quad \text{Error} = |R_N|$$

$$|R_N| \leq \left| \frac{(-1)^{N+1} x^{2(N+1)}}{(N+1)!} \right| \leq \frac{|x|^{2N+2}}{(N+1)!} \rightarrow 0$$

Taylor Series: For a function f , $\sum_{n=0}^{\infty} C_n (x-a)^n$ where

$$(1) C_n = \frac{f^{(n)}(a)}{n!}$$

$$(2) f(x) = \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!} (x-a)^n \text{ when } |R_N| \rightarrow 0$$

Ex The Taylor series for $\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$, Compute $f^{(2025)}(0)$

$$\text{When } |x| < 1, \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!} x^n = \frac{1}{1-x} = \sum_{n=0}^{\infty} x^n$$

$$\text{So, } \frac{f^{(n)}(0)}{n!} = 1 \text{ for all } n, \text{ when } n = 2025$$

$$\frac{f^{(2025)}(0)}{2025!} = 1 \Rightarrow f^{(2025)}(0) = 2025!$$

Another way Recognize $f^{(n)}(x) = \frac{n!}{(1-x)^{n+1}}$

Quiz 10 #1 Prove $\lim_{x \rightarrow 0} \frac{\sin x}{x} = 1$

$$\frac{1}{x} \sin x = \frac{1}{x} \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n+1}}{(2n+1)!} = \sum_{n=0}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!} = 1 + \sum_{n=1}^{\infty} \frac{(-1)^n x^{2n}}{(2n+1)!}$$

$1 + \left[\text{Something that goes to } 0 \text{ as } x \rightarrow 0 \right] \rightarrow 1 \text{ as } x \rightarrow 0$

Polar Coordinates

Conversion Formulas: Cartesian \rightarrow Polar
 $(x, y) \rightarrow (r, \theta)$
 $r^2 = x^2 + y^2$
 $\tan \theta = \frac{y}{x}$

Polar \rightarrow Cartesian
 $(r, \theta) \rightarrow (x, y)$
 $x = r \cos \theta$
 $y = r \sin \theta$

Ex/Convert the graph $y = x^2$ into Polar

$$r \sin \theta = (r \cos \theta)^2$$

$$r \sin \theta = r^2 \cos^2 \theta$$

Quiz 10 #3 Describe the curve $r = 6 \sin \theta$
 $r^2 = 6r \sin \theta$

$$x^2 + y^2 = 6y$$

$$x^2 + y^2 - 6y + 9 = 0 + 9$$

$$x^2 + (y-3)^2 = 9$$

Circle with
radius 3 and
center (0, 3)

Ex/ $\frac{1}{r} = 2 \ (r > 0)$

$$\frac{1}{\sqrt{x^2+y^2}} = 2 \Rightarrow \frac{1}{2} = \sqrt{x^2+y^2} \Rightarrow x^2+y^2 = \frac{1}{4} \quad [(x,y) \neq (0,0)]$$

Integration in Polar Coordinates $r = f(\theta) ; \alpha \leq \theta \leq \beta$

$$A = \int_{\alpha}^{\beta} \frac{1}{2} r^2 d\theta$$

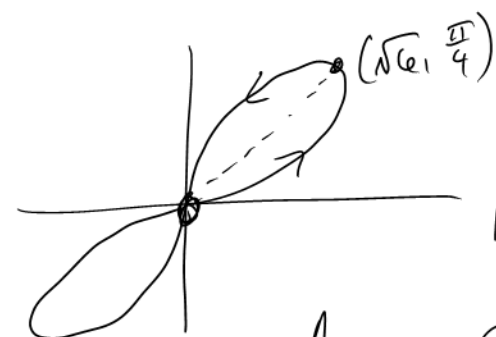
Ex/ Find the region bounded by the lemniscate

$r^2 = 6 \sin 2\theta$ Find Bounds: $0 = 6 \sin 2\theta$

$$\sin 2\theta = 0$$

$$2\theta = \pi n ; n \in \mathbb{Z}$$

$$\theta = \frac{\pi}{2} n$$



Bounds: $[0, \frac{\pi}{2}]$

$$\text{Area} = 2 \left[\text{Area of one half} \right] = 2 \int_0^{\pi/2} \frac{1}{2} (6 \sin 2\theta) d\theta$$

$$= \int_0^{\pi/2} 6 \sin 2\theta d\theta = 3 \int_0^{\pi/2} \underbrace{\sin 2\theta}_{u=2\theta} (2) d\theta$$

$$= 3 [-\cos 2\theta]_0^{\pi/2} = 3 [1 + 1] = 6$$

Complete Course Evaluations

Contact me via email if needed, pence11@purdue.edu