Lecture 8 The Product Rule

Goal: Differentiate functions of the form $f(x)g(x)$

NOTE: $\frac{d}{dx}(f(x)\,g(x)) \neq \frac{d}{dx}(f(x))\frac{d}{dx}(g(x))$

$\frac{d}{dx}(x\cdot 1) = \frac{d}{dx}(x) = 1$ #

$\frac{d}{dx}(x)\cdot\frac{d}{dx}(1) = 1\cdot 0 = 0$

$$\frac{d}{dx}(f(x)g(x)) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)\,g(x+\Delta x) - f(x)\,g(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)\,g(x+\Delta x) - g(x+\Delta x)f(x) + g(x+\Delta x)f(x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\left[g(x+\Delta x)\,\frac{f(x+\Delta x)-f(x)}{\Delta x} + f(x)\,\frac{g(x+\Delta x)-g(x)}{\Delta x}\right]$$

$$= g(x)\,f'(x) + f(x)\,g'(x)$$

Theorem (Product Rule) Let f and g be differentiable functions. Then,

$$\frac{d}{dx}(f(x)\,g(x)) = \frac{d}{dx}(f(x))\cdot g(x) + f(x)\cdot\frac{d}{dx}(g(x))$$

$$[\underbrace{f(x)}_{\text{left}}\,\underbrace{g(x)}_{\text{right}}]' = \underbrace{f'(x)\,g(x) + f(x)\,g'(x)}_{\text{"right d-left plus left d-right"}}$$

Ex/ Compute $\frac{d}{dx}(\underbrace{x}_{f}\,\underbrace{\sin x}_{g}) = [x]'\sin x + x\,[\sin x]'$

$$= 1\cdot\sin x + x\cdot\cos x$$

$$= \sin x + x\cos x$$

Ex/ Compute $\frac{d}{dx}[\underbrace{2\sin x}_{f}\,\underbrace{\cos x}_{g}] = [2\sin x]'\cos x + 2\sin x\,[\cos x]'$

$$\star = (2\cos x)\cos x + (2\sin x)(-\sin x)$$

$$= 2[\cos^2 x - \sin^2 x]$$

Ex/ Compute h' if $h(x) = x^4(3x^3+4x+1)$

$f(x) = x^4 \longrightarrow f'(x) = 4x^3$

$g(x) = 3x^3+4x+1 \longrightarrow g'(x) = 9x^2+4$

$$h'(x) = [f(x)\,g(x)]' = f'(x)\,g(x) + f(x)\,g'(x)$$

$$= [4x^3][3x^3+4x+1] + [x^4][9x^2+4]$$

Ex/ Compute h' if $h(x) = (x^{10}+x+1)(x^2+9)$

$f(x) = x^{10}+x+1 \longrightarrow f'(x) = 10x^9+1$

$g(x) = x^2+9 \longrightarrow g'(x) = 2x$

$$h'(x) = f'(x)\,g(x) + f(x)\,g'(x)$$

$$= \underbrace{[10x^9+1]}_{f'}\underbrace{[x^2+9]}_{g} + \underbrace{[x^{10}+x+1]}_{f}\underbrace{[2x]}_{g'}$$

Ex/ Compute h' if $h(x) = \underbrace{(x+1)}_{f}\underbrace{(2x+1)(3x+1)}_{g}$ $\star$

$$h'(x) = [x+1]'(2x+1)(3x+1) + (x+1)[\underbrace{(2x+1)}_{f}\underbrace{(3x+1)}_{g}]' \quad \star$$

$$= [x+1]'(2x+1)(3x+1) + (x+1)\Big[[2x+1]'(3x+1) + (2x+1)[3x+1]'\Big]$$

$= 1 \cdot (2x+1)(3x+1) + (x+1)[2(3x+1) + 3(2x+1)]$

$= (2x+1)(3x+1) + 2(x+1)(3x+1) + 3(2x+1)(x+1)$

Ex/ Find the equation of the tangent line if

$h(x) = \underbrace{(\sqrt{x}+4)}_{f}\underbrace{(x^{\frac{2}{3}} - 2x)}_{g}$ at $x=1$

Slope: $h'(1)$. $h'(x) = [\sqrt{x}+4]'(x^{\frac{2}{3}}-2x) + (\sqrt{x}+4)[x^{\frac{2}{3}}-2x]'$

$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$

$\frac{d}{dx}(x^{\frac{2}{3}}) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$

$h'(x) = \left(\frac{1}{2\sqrt{x}}\right)(x^{\frac{2}{3}}-2x) + (\sqrt{x}+4)\left(\frac{2}{3\sqrt[3]{x}} - 2\right)$

$h'(1) = \left(\frac{1}{2}\right)(1-2) + (1+4)\left(\frac{2}{3}-2\right) = -\frac{67}{6}$

Point: $(1, h(1))$

$h(x) = (\sqrt{x}+4)(x^{\frac{2}{3}}-2x)$

$h(1) = (1+4)(1-2) = -5$

Point: $(1, -5)$

Equation: $y - (-5) = -\frac{67}{6}(x-1)$

Ex/ When does $y = \overbrace{10x^5}^{f}\overbrace{e^x}^{g}$ have a horizontal tangent line?

$y' = 50x^4e^x + 10x^5e^x = 10x^4e^x(5+x) \stackrel{\text{set}}{=} 0$

Either $10x^4 = 0$ OR $e^x = 0$ OR $5+x=0$

$\boxed{x=0}$ Impossible $\boxed{x=-5}$

Ex/ Compute h' if $h(x) = 6e^x \sin x - 13 e^x \cos x$

$$= \underbrace{e^x}_{f} \underbrace{(6\sin x - 13\cos x)}_{g}$$

$$h'(x) = [e^x]'(6\sin x - 13\cos x) + e^x[6\sin x - 13\cos x]'$$

$$= e^x(6\sin x - 13\cos x) + e^x(6\cos x + 13\sin x)$$

$$= e^x(6\sin x - 13\cos x + 6\cos x + 13\sin x)$$

$$= e^x(19\sin x - 7\cos x)$$