MA 16200: Plane Analytic Geometry and Calculus II

Lecture 15: Sequences and Series Intro

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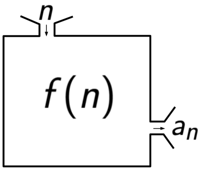
Sections Covered: 10.1

Enumerated Lists

Say we have an ordered list of numbers:

$$a_1, a_2, a_3, \ldots, a_n, \ldots$$

We can think of it as a function taking non-negative integers as inputs and real numbers as outputs.



Sequence Definition

Definition 1

A **sequence** is an ordered list of numbers of the form:

$$\{a_1, a_2, \ldots, a_n, \ldots\}$$

The subscript n is called the **index**. The number a_n is called the ***n*-th term** in the sequence.

Notation:

$$\{a_n\}_{n=1}^{\infty} \qquad \{a_n\} \qquad a_n;\ n \geq 1$$

Defining Sequences

One can define sequences:

1. **Explicitly** using a function $a_n = f(n)$.
   - $\{\sqrt{n-3}\}_{n=3}^{\infty}$
   - $\left\{\frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5} \dots, (-1)^{n+1}\frac{n}{n+1}, \dots\right\}$
2. **Recursively** using a **recurrence relation** $a_{n+1} = f(a_n)$.
   - The **Fibonacci Sequence:**

$$f_{n+1} = f_n + f_{n-1}; \quad f_1 = 1; \quad f_0 = 1$$

   - Given a sequence $\{a_n\}$, define the **sequence of partial sums**

$$S_N = a_N + S_{N-1}; \quad S_1 = a_1$$

3. "Abstractly" (when there is no obvious formula)
   - Let $\{p_n\}$ be the sequence of prime numbers:

$$\{p_n\} = \{2, 3, 5, 7, 11, 13, \dots\}$$

Example (Explicit Formulas)

Problem 2

Use the explicit formula $\left\{\frac{1}{2^n}\right\}_{n=1}^{\infty}$ *to write the first 4 terms of the sequence. Sketch a graph of the sequence.*

Example (Explicit Formulas)

Problem 3

Use the explicit formula $\{1 + \frac{(-1)^{n+1}}{n}\}_{n=1}^{\infty}$ *to write the first 4 terms of the sequence. Sketch a graph of the sequence.*

Example (Recursion)

Problem 4

Use the recurrence relation to write the first 4 terms of the sequence.

$$\begin{cases} a_{n+1} = a_n + a_{n-1} \\ a_0 = 1 \\ a_1 = 1 \end{cases}$$

Example (Recursion)

Problem 5

Use the recurrence relation to write the first 4 terms of the sequence.

$$\begin{cases} a_{n+1} = 2a_n + 1 \\ a_1 = 1 \end{cases}$$

Example (Finding Formulas)

Problem 6

Consider the sequence $\{a_n\} = \{-2, 5, 12, 19, \ldots\}$.

1. *Find 2 different formulas describing this sequence.*
2. *Use either formula to find the next 2 terms in the sequence.*

Example (Finding Formulas)

Problem 7

Consider the sequence $\{a_n\} = \{1, -2, 6, -24, 120, \ldots\}$.

1. *Find 2 different formulas describing this sequence.*
2. *Use either formula to find the next 2 terms in the sequence.*

Convergence

Definition 8

We say the sequence $\{a_n\}$ **converges** to a real number L (written $a_n \to L$) if

$$\lim_{n\to\infty} a_n = L$$

That is, the limit exists and equals L. Otherwise, we say the limit **diverges**.

Here is the formal definition, **you will not need to know this.**

Definition 9 (Formal Definition)

We say $a_n \to L$ if for any $\varepsilon > 0$ there is a positive integer N such that:

$$\text{If } n \geq N, \text{ then } |a_n - L| < \varepsilon$$

Example (Limits)

Problem 10

Make a conjecture about whether the following sequences converge or diverge. Explain why or why not.

- $a_n = (-1)^n \frac{3}{n+5}; \; n \geq 0$
- $a_n = \cos \pi n \; ; \; n \geq 0$
- $a_n = 2a_n; \; a_1 = 1$

Example (Height of a Ball)

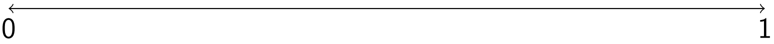
Problem 11

A basketball tossed straight up in the air reaches a high point and falls to the floor. Each time the ball bounces on the floor it rebounds to 0.8 of its previous height. Let h_n be the high point after the nth bounce, with the initial height being $h_0 = 20ft$.

- *Find a recurrence relation and an explicit formula for the sequence $\{h_n\}$.*
- *What is the height of the peak after the 10th bounce? After the 20th bounce?*
- *Speculate the limit of the sequence $\{h_n\}$.*

Zeno's Paradox

Let's say we want to travel 1m. For each step, we go half the remaining distance.



What is the distance traveled after n steps? Will we ever reach 1m?

Sequence of Partial Sums

For a given sequence $\{a_n\}$, we define the **sequence of partial sums** $\{S_N\}$ as:

$$\begin{aligned} S_1 &= a_1 \\ S_2 &= a_1 + a_2 \\ S_3 &= a_1 + a_2 + a_3 \\ &\vdots \\ S_N &= a_1 + a_2 + \ldots + a_N = \sum_{n=1}^{N} a_n = a_N + S_{N-1} \\ &\vdots \end{aligned}$$

That is, for the infinite sum $a_1 + a_2 + \ldots + a_k + \ldots$, S_N is the value when we "chop off" the first N terms and compute its sum.

Series Definition

Definition 12

Given a sequence $\{a_n\}_{n=1}^{\infty}$, the sum of its terms:

$$a_1 + a_2 + a_3 + \ldots = \sum_{n=1}^{\infty} a_n$$

is called an **(infinite) series**. If the sequence of partial sums $\{S_N\}$ has a limit L, we say the series **converges** to L. We then write,

$$\sum_{n=1}^{\infty} a_n \overset{def}{=} \lim_{N\to\infty} \sum_{n=1}^{N} a_n = \lim_{N\to\infty} S_n = L$$

Otherwise, the series **diverges**.

$0.9999999999\ldots = 1?$

Problem 13

Make a conjecture about whether the sum:

$$0.9 + 0.09 + 0.009 + \ldots$$

converges or diverges. If so, what is a plausible limit?

Example

Problem 14

Make a conjecture about whether the sum:

$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)}$$

converges or diverges. If so, what is a plausible limit?

The Harmonic Series

Theorem 15

*The **harmonic series**:*

$$\sum_{n=1}^{\infty} \frac{1}{n}$$

diverges.

Why? For full details, see Section 10.4.

$$S_N = \sum_{n=1}^{N} \frac{1}{n} > \int_1^N \frac{1}{x}\, dx = \ln N$$

So S_N is unbounded (hence it diverges).

Example (Distance of a Ball)

Problem 16

Suppose a ball is thrown upward to a height of h_0 meters. Each time the ball bounces, it rebounds to a fraction $r = 0.5$ of its previous height. Let h_n be the height after the nth bounce and let S_n be the total distance the ball has traveled at the moment of the nth bounce.

- *Find a formula for S_n and find a plausible value for the limit.*

Comparing to Functions

	Sequences/Series	**Functions**
Independent Variable	n	x
Dependent Variable	a_n	$f(x)$
Domain	Integers	Real Numbers
Accumulation	Sums	Integrals
Accumulation over finite interval	$\sum_{n=1}^{N} a_k$	$\int_1^N f(x)\,dx$
Accumulation over an infinite interval	$\sum_{n=1}^{\infty} a_n$	$\int_1^{\infty} f(x)\,dx$