Lecture 8 The Product Rule

<u>Goal</u>: Differentiate functions of the form $f(x)\,g(x)$

NOTE: $\frac{d}{dx}\left(f(x)\,g(x)\right) \overset{?}{\neq} \frac{d}{dx}(f(x))\,\frac{d}{dx}(g(x))$

$\frac{d}{dx}(x\cdot 1) = \frac{d}{dx}(x) = 1 \;\cancel{x}$

$\frac{d}{dx}(x)\cdot\frac{d}{dx}(1) = 1\cdot 0 = 0$

$$\frac{d}{dx}\left(f(x)\,g(x)\right) = \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\frac{f(x+\Delta x)g(x+\Delta x) - g(x+\Delta x)f(x) + g(x+\Delta x)f(x) - f(x)g(x)}{\Delta x}$$

$$= \lim_{\Delta x\to 0}\left[\underbrace{g(x+\Delta x)}\boxed{\frac{f(x+\Delta x)-f(x)}{\Delta x}} + \frac{g(x+\Delta x) - g(x)}{\Delta x}\,f(x)\right]$$

$$= g(x)\,f'(x) + f(x)\,g'(x)$$

<u>Theorem</u> Let f and g be differentiable. Then

$$\frac{d}{dx}\left(f(x)\;g(x)\right) = \frac{d}{dx}(f(x))\,g(x) + f(x)\,\frac{d}{dx}(g(x))$$

$$(\underbrace{f(x)}_{\text{"left"}}\underbrace{g(x)}_{\text{"right"}})' = \underbrace{f'(x)\,g(x) + f(x)\,g'(x)}_{\text{right d-left + left d-right}}$$

This is called the <u>product rule</u>.

EX/ Compute $\frac{d}{dx}(\underbrace{x}_{f}\,\underbrace{\sin x}_{g}) = \frac{d}{dx}(x)[\sin x] + x\cdot\frac{d}{dx}(\sin x)$

$= 1\cdot\sin x + x\cdot\cos x$

$= \sin x + x\cos x$

Ex/ Compute $(\underbrace{2\sin x}_{f}\ \underbrace{\cos x}_{g})' = [2\sin x]'\cos x + 2\sin x\,[\cos x]'$

$= (2\cos x)(\cos x) + 2\sin x\,(-\sin x)$

$= 2(\cos^2 x - \sin^2 x)$

Ex/ Compute $(\underbrace{2}_{f}\ \underbrace{\sin x\ \cos x}_{g})' = [2]'\sin x\cos x + 2[\sin x\cos x]'$

$= 2[\underbrace{\sin x}_{f}\ \underbrace{\cos x}_{g}]'$

$= 2[\cos^2 x + \sin x(-\sin x)]$

$(2\sin x\cos x)$ $[(2x)^5\cos x]$

Ex/ Compute h' if $h(x) = x^4(3x^3 + 4x + 1)$

$f(x) = x^4 \longrightarrow f'(x) = 4x^3$

$g(x) = 3x^3 + 4x + 1 \longrightarrow g'(x) = 9x^2 + 4$

$h'(x) = [f(x)g(x)]' = f'(x)\,g(x) + f(x)\,g'(x)$

$= (4x^3)(3x^3 + 4x + 1) + x^4(9x^2 + 4)$

Ex/ Compute h' if $h(x) = \underbrace{(x^{10} + x + 1)}_{f}\underbrace{(x^2 + 9)}_{g}$

$f(x) = x^{10} + x + 1 \longrightarrow f'(x) = 10x^9 + 1$

$g(x) = x^2 + 9 \longrightarrow g'(x) = 2x$

$h'(x) = f'(x)\,g(x) + f(x)\,g'(x)$

$= (10x^9 + 1)(x^2 + 9) + (x^{10} + x + 1)(2x)$

Ex/ Compute h' if $h(x) = 6e^x \sin x - 13e^x \cos x$

$$= \underbrace{e^x}_{f}\underbrace{(6\sin x - 13\cos x)}_{g}$$

$$h'(x) = [e^x]'(6\sin x - 13\cos x) + e^x[6\sin x - 13\cos x]'$$

$$= e^x(6\sin x - 13\cos x) + e^x[6\cos x - 13(-\sin x)]$$

$$= e^x[6\sin x - 13\cos x + 6\cos x + 13\sin x]$$

$$= e^x[19\sin x - 7\cos x]$$

Ex/ Compute h' if $h(x) = \underbrace{(x+1)}_{f}\underbrace{(2x+1)(3x+1)}_{g}$

$$h'(x) = [x+1]'(2x+1)(3x+1) + (x+1)[\underbrace{(2x+1)}_{f}\underbrace{(3x+1)}_{g}]'$$

$$= [x+1]'(2x+1)(3x+1) + (x+1)\left[[2x+1]'(3x+1) + (2x+1)[3x+1]'\right]$$

$$= 1\cdot(2x+1)(3x+1) + (x+1)[2(3x+1) + 3(2x+1)]$$

$$= (2x+1)(3x+1) + 2(x+1)(3x+1) + 3(x+1)(2x+1)$$

Ex/ Find the equation of the tangent line if

$$h(x) = \underbrace{(\sqrt{x}+4)}_{f}\underbrace{(x^{\frac{2}{3}} - 2x)}_{g} \quad \text{at} \quad x = 1$$

Slope $[h'(1)]$

$$h'(x) = [\sqrt{x}+4]'(x^{\frac{2}{3}} - 2x) + (\sqrt{x}+4)[x^{\frac{2}{3}} - 2x]'$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{d}{dx}(x^{\frac{1}{2}}) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}}$$

$$\frac{d}{dx}(x^{\frac{2}{3}}) = \frac{2}{3}x^{\frac{2}{3}-1} = \frac{2}{3}x^{-\frac{1}{3}} = \frac{2}{3\sqrt[3]{x}}$$

$$h'(x) = \left(\frac{1}{2\sqrt{x}}\right)\left(x^{\frac{2}{3}} - 2x\right) + (\sqrt{x}+4)\left(\frac{2}{3\sqrt[3]{x}} - 2\right)$$

$$h'(1) = \left(\frac{1}{2}\right)(1-2) + (1+4)\left(\frac{2}{3} - 2\right) = -\frac{67}{6}$$

Point:

$(1, h(1))$ $\quad h(x) = (\sqrt{x}+4)(x^{\frac{2}{3}} - 2)$

$$h(1) = (1+4)(1-2) = -5$$

$$(1, h(1)) = (1, -5)$$

Equation: $y - (-5) = -\frac{67}{6}(x-1)$

Ex/ When does $y = \overbrace{10x^5}^{f}\overbrace{e^x}^{g}$ have a horizontal tangent line?

$$y' = 50x^4e^x + 10x^5e^x = 10x^4e^x(5+x) \overset{set}{=} 0$$

Either $10x^4 = 0$ OR $e^x = 0$ OR $5+x=0$

$\boxed{x=0}$ $\qquad$ Impossible $\qquad$ $\boxed{x=-5}$