MA 16200: Plane Analytic Geometry and Calculus II

Lecture 13: Partial Fractions (Irreducible Quadratic Cases)

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Sections Covered: 8.5

Non-Repeating Linear Terms

Problem 1

Compute $\int \frac{1}{x^2-4}\ dx$

Repeating Linear Terms

Problem 2

Compute $\int \frac{x-1}{x^3+x^2}\ dx$

General Strategy

For each factor of the form $(x - r)$ in the denominator, the term in the Partial Fraction Decomposition is:

$$\frac{A}{x - r}$$

For each factor of the form $(x - a)^n$ for $n > 1$,

$$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \ldots + \frac{A_n}{(x - a)^n}$$

Example

Problem 3

Compute $\int \frac{3x^2+1}{x(x^2+1)}\ dx$

General Strategy

For each **irreducible** factor of the form $(ax^2 + bx + c)$ in the denominator, the term in the Partial Fraction Decomposition is:

$$\frac{Ax + B}{ax^2 + bx + c}$$

For each **irreducible** factor of the form $(ax^2 + bx + c)^n$ for $n > 1$,

$$\frac{A_1x + B_1}{ax^2 + bx + c} + \frac{A_2x + B_2}{(ax^2 + bx + c)^2} + \ldots + \frac{A_n + B_n}{(ax^2 + bx + c)^n}$$

A Trivial Example

Problem 4

Compute $\int \frac{x+1}{x^2+1}\ dx$

Example

Problem 5

Compute $\int \frac{2x^2 - x + 4}{x^3 + 4x}\ dx$

Example

Problem 6

Compute $\int \frac{7x^2-13x+13}{(x-2)(x^2-2x+3)}\,dx$

Example with Long Division

Problem 7

Compute $\int \frac{4x^2-3x+2}{4x^2-4x+3}\, dx$

Combining everything

Problem 8

Set up the PFD for $\frac{x^3+x^2+1}{x(x-1)(x^2+x+1)(x^2+1)^3}$

Repeating Quadratics Example

Problem 9

Compute $\int \frac{1-x+2x^2-x^3}{x(x^2+1)^2}\, dx$

Rationalizing Integrands

Problem 10

Compute $\int \frac{\sqrt{x+4}}{x}\,dx$

Making Substitutions

Problem 11

Compute $\int \frac{\cos x}{\sin^2 x + \sin x}\ dx$

Another Example

Problem 12

Compute $\int \frac{1}{\sqrt{x}-\sqrt[3]{x}}\,dx$

Weierstrass Substitution (Non-Examinable)

Theorem 13 (Weierstrass)

Any rational function of $\sin x$ *and* $\cos x$ *can be converted to a rational function of* t *by making the substitution* $t = \tan \frac{x}{2}$.

Why? One can check that $\cos \frac{x}{2} = \frac{1}{\sqrt{1+t^2}}$ and $\sin \frac{x}{2} = \frac{t}{\sqrt{1+t^2}}$. So,

$$\cos x = \frac{1-t^2}{1+t^2}; \quad \sin x = \frac{2t}{1+t^2}; \quad dx = \frac{2}{1+t^2}\, dt$$

Example (Non-Examinable)

Problem 14

Compute $\int \frac{1}{3\sin x - 4\cos x}\, dx$

$$\int \frac{1}{3\sin x - 4\cos x}\, dx = \int \frac{1}{3\left(\frac{2t}{1+t^2}\right) - 4\left(\frac{1-t^2}{1+t^2}\right)} \cdot \frac{2}{1+t^2}\, dt$$

$$= \int \frac{1}{(t+2)(2t-1)}\, dt$$

$$= \int \left(\frac{-1/5}{t+2} + \frac{2/5}{2t-1} \right) dt$$

(cont.)

$$\begin{aligned}\int \frac{1}{3\sin x - 4\cos x}\,dx &= -\frac{1}{5}\ln|t+2| + \frac{2}{5}\ln|2t-1| + C_0 \\ &= \frac{1}{5}\ln\left|\frac{2t-1}{t+2}\right| + C_0 \\ &= \frac{1}{5}\ln\left|\frac{2\tan\frac{x}{2}-1}{\tan\frac{x}{2}+2}\right| + C\end{aligned}$$