MA 16200: Plane Analytic Geometry and Calculus II

Lecture 26: Manipulaitng Power Series, Term-by-Term Differentiation/Integration

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Sections Covered: 11.2 (Part II)

The Radius of Convergence

Theorem 1

For a given power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ there are only 3 possibilities:

1. *There is a positive number R such that the series converges if $|x-a| < R$ and diverges if $|x-a| > R$.*
   - *This R is called the* ***Radius of Convergence****.*
2. *The series converges for all x.*
   - *By convention, $R = \infty$*
3. *The series converges only when $x = a$.*
   - *By convention, $R = 0$*

The radius of convergence is usually found by the Ratio or Root Test.

The Interval of Convergence

Definition 2

The **interval of convergence** for a power series $\sum_{n=0}^{\infty} c_n(x-a)^n$ is the set of all x where the series converges.

In the previous theorem,

1. In Case 1, the interval of convergence is 1 of 4 possibilities:

   $$(a-R, a+R) \quad [a-R, a+R) \quad (a-R, a+R] \quad [a-R, a+R]$$

2. In Case 2, the interval is $\mathbb{R} = (-\infty, \infty)$.
3. In Case 3, the interval is $\{a\}$.

Geometric Series Revisited

Theorem 3

The power series centered at 0:

$$\sum_{n=0}^{\infty} x^n = 1 + x + x^2 + \ldots = \frac{1}{1-x}$$

when $|x| < 1$.

Here the radius of convergence is 1 and the interval of convergence is $(-1, 1)$.

Example

Problem 4

Find a power series representation of the function $f(x) = \frac{1}{1+2x}$. *Determine the radius and interval of convergence.*

Example

Problem 5

Find a power series representation of the function $f(x) = \frac{1}{4-x}$.
Determine the radius and interval of convergence.

Example

Problem 6

Find a power series representation of the function $f(x) = \frac{x^2}{1-x}$*.*
Determine the radius and interval of convergence.

Example

Problem 7

Find a power series representation of the function $f(x) = \frac{1}{1+x^2}$. *Determine the radius and interval of convergence.*

Example

Problem 8

Determine the interval of convergence of:

$$\sum_{n=0}^{\infty} x^n + \sum_{n=0}^{\infty} \left(x - \frac{1}{2}\right)^n$$

What function does this power series converge to (on the interval of convergence)?

Combining Power Series Theorem

Theorem 9

Suppose $\sum c_n x^n \to f(x)$ on an interval I_1 and $\sum d_n x^n \to g(x)$ on an interval I_2. This also applies when the center isn't 0 (it is just less obvious why).

1. ***Sums and Differences:*** *The power series* $\sum (c_n \pm d_n) x^n \to f(x) + g(x)$ *on* $I_1 \cap I_2$.
2. ***Multiplication by x^m:*** *Suppose m is a positive integer such that $n + m \geq 0$. Then* $x^m \sum c_n x^n = \sum c_n x^{m+n} \to x^m f(x)$ *on* I_1 *(when $x \neq 0$). When $x = 0$, the series converges to* $\lim_{x \to 0} x^m f(x)$.
3. ***Composition:*** *If $h(x) = bx^m$, where m is a positive integer and b a non-zero real number, then $\sum c_n (h(x))^n \to f(h(x))$ on the set of all x such that $h(x)$ is in I_1.*

Multiplying and Dividing Power Series (Non-Examinable)

Let $f(x) = \sum_{n=0}^{\infty} a_n x^n$ and $g(x) = \sum_{n=0}^{\infty} b_n x^n$:

- Define the product $f(x)g(x)$ as:

$$f(x)g(x) \stackrel{def}{=} \sum_{n=0}^{\infty} c_n x^n \text{ where } c_n = \sum_{i=0}^{n} a_i b_{n-i}$$

- If $b_0 \neq 0$, define the quotient $\frac{f(x)}{g(x)}$ as the power series $h(x) = \sum_{n=0}^{\infty} c_n x^n$ such that $f(x) = h(x)g(x)$. The coefficients c_n are found recursively:

$$\begin{cases} c_n = \frac{1}{b_0}\left[a_n - \sum_{i=1}^{n} b_i c_{n-i}\right] \\ c_0 = \frac{a_0}{b_0} \end{cases}$$

The derivative and integral of a power series

Theorem 10 (Term-by-term differentiation/integration)

Suppose a power series $\sum c_n(x-a)^n \to f(x)$ when $|x-a| < R$:

1. *Then f is differentiable (hence is continuous) and:*

$$f'(x) = \frac{d}{dx}\sum_{n=0}^{\infty} c_n(x-a)^n = \sum_{n=0}^{\infty} c_n \frac{d}{dx}(x-a)^n = \sum_{n=1}^{\infty} nc_n(x-a)^{n-1}$$

2. *f can be integrated and:*

$$\int f(x)\,dx = \int \sum_{n=0}^{\infty} c_n(x-a)^n\,dx = \sum_{n=0}^{\infty} c_n \int (x-a)^n\,dx = C + \sum_{n=0}^{\infty} \frac{c}{n+1}(x-a)^{n+1}$$

where C is an arbitrary constant.

3. *f, f', and $\int f\,dx$ have the same center and radius of convergence.*

Example

Problem 11

Find a power series representation of $f(x) = \frac{1}{(1-x)^2}$. *Determine the radius and interval of convergence.*

Example

Problem 12

Find a power series representation of $f(x) = \ln(1 - x)$. Determine the radius and interval of convergence.

Example

Problem 13

Find a power series representation of $f(x) = \ln(1 + x)$. Determine the radius and interval of convergence.

Example

Problem 14

Find a power series representation of $f(x) = \ln \sqrt{1 - x^2}$.
Determine the radius and interval of convergence.

Example

Problem 15

Find a power series representation of $f(x) = \tan^{-1} x$. *Determine the radius and interval of convergence.*

Example

Problem 16

Find a power series representation of $f(x) = \ln \frac{1+x}{1-x}$. *Determine the radius and interval of convergence.*

Example

Problem 17

We will see in the next section that:

$$\cos x = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}; \quad x \in (-\infty, \infty)$$

Use this to find a power series representation for $\sin x$*. Determine the radius and interval of convergence.*

Application (Differential Equations)

Problem 18

Show that the series $f(x) = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ *is a solution to the Initial Value Problem (IVP):*

$$\begin{cases} f'(x) = f(x) \\ f(0) = 1 \end{cases}$$