Goal: Reverse the process of differentiation.

$\int kf(x)\ dx = kF(x) + C$	$\int (f(x) + g(x))\ dx = F(x) + G(x) + C$
$\int f(g(x))g'(x)\ dx = F(g(x)) + C$	$\int (f(x) - g(x))\ dx = F(x) - G(x) + C$
$\int x^n\ dx = \frac{1}{n+1}x^{n+1} + C$ (when $n \neq -1$)	$\int \frac{1}{x}\ dx = \ln\lvert x\rvert + C$
$\int e^x\ dx = e^x + C$	$\int e^{kx}\ dx = \frac{1}{k}e^{kx} + C$
$\int \sin x\ dx = -\cos x + C$	$\int \cos x\ dx = \sin x + C$
$\int \sec^2 x\ dx = \tan x + C$	$\int \sec x \tan x\ dx = \sec x + C$
$\int \csc x \cot x\ dx = -\csc x + C$	$\int \csc^2 x\ dx = -\cot x + C$
$\int \frac{1}{\sqrt{1-x^2}}\ dx = \sin^{-1} x + C$	$\int \frac{1}{1+x^2}\ dx = \tan^{-1} x + C$

Table 1: Table of Antiderivatives

Def A function F is an **antiderivative** of f on an interval I if $F'(x) = f(x)$ for all x in I.

Ex 1 (a) Is $F(x) = \frac{1}{2}x^2$ an antiderivative for x on $(-\infty, \infty)$

$F'(x) = [\frac{1}{2}x^2]' = \frac{1}{2} \cdot 2x = x$ ✓

(b) Is $F(x) = x^2$ an antiderivative for x

$F'(x) = 2x \neq x$ Not

(c) Is $F(x) = \frac{1}{2}x^2 + 1$ an antiderivative of x?

$F'(x) = [\frac{1}{2}x^2]' + 1' = x + 0 = x$ ✓

NOTE: Antiderivatives are not unique $\frac{1}{2}x^2 + C$ is an antiderivative for x

Q: Can we classify all antiderivatives of a fcn f on its domain

Theorem If $F' = 0$ on an interval, then F is constant on said interval

Why? Use Mean Value Theorem

Corollary Let F and G be two antiderivatives of a function f, then

$$F(x) = G(x) + C$$

for some constant C.

Why? $\frac{d}{dx}\big(F(x) - G(x)\big) = f(x) - f(x) = 0$

$\Rightarrow F(x) - G(x) = [\text{Constant}]$

Def The family of all antiderivatives of a fcn f is called the (indefinite) integral, denoted

$$\underbrace{\int}_{\text{Integral Sign}} \overbrace{f(x)}^{\text{Integrand}} \underbrace{dx}_{\text{Integration Variable}} \overset{\text{By Previous Corollary}}{=} \underbrace{F(x)}_{\text{Particular Antiderivative}} + \underbrace{C}_{\text{Constant of Integration}}$$

Ex 2/ Find all antiderivatives of $\sin x$

① Find a particular antiderivative

$$\frac{d}{dx}(\cos x) = -\sin x$$

$$-\frac{d}{dx}(\cos x) = \sin x$$

$$\frac{d}{dx}(\underbrace{-\cos x}) = \sin x$$

② Compute $\int \sin x \, dx$

$$\boxed{\int \sin x \, dx = -\cos x + C}$$

Ex3/ Compute $\int \frac{1}{x} \, dx$ on $(0, \infty)$

Recall $\frac{d}{dx}(\ln x) = \frac{1}{x}$ if $x > 0$

$$\int \frac{1}{x} \, dx = \ln x + C$$

In general, for $(-\infty, 0) \cup (0, \infty)$

$$\boxed{\int \frac{1}{x} \, dx = \ln(|x|) + C}$$

Ex 4/ Compute $\int x^2 \, dx$

1) Find particular antiderivative

$$\frac{d}{dx}(x^3) = 3x^2$$

$$\frac{1}{3}\left(\frac{d}{dx}(x^3)\right) = x^2$$

$$\frac{d}{dx}\left(\frac{1}{3}x^3\right) = x^2$$

Hence, $\int x^2 \, dx = \frac{1}{3}x^3 + C$

<u>Theorem (Reverse Power Rule)</u> When $n \neq -1$

$$\int x^n \, dx = \frac{x^{n+1}}{n+1} + C$$

Ex5/ $\int \sqrt{x}\,dx = \int x^{\frac{1}{2}}\,dx = \dfrac{x^{\frac{1}{2}+1}}{\frac{1}{2}+1} + C$

$= \dfrac{x^{\frac{3}{2}}}{\frac{3}{2}} + C = \boxed{\frac{2}{3}x^{\frac{3}{2}} + C}$

Verify it's correct:

$\dfrac{d}{dx}\left(\frac{2}{3}x^{\frac{3}{2}} + C\right) = \frac{2}{3}\cdot\frac{3}{2}x^{\frac{3}{2}-1} + 0$

$= x^{\frac{1}{2}} = \sqrt{x}$ ✓

<u>Theorem</u> Indefinite integrals are linear. I.e., if F and G are antiderivatives of f and g and k is a constant. Then,

① $\int (f(x) \pm g(x))\,dx = \int f(x)\,dx \pm \int g(x)\,dx$

$= F(x) \pm G(x) + C$

② $\int k f(x)\,dx = k\int f(x)\,dx = kF(x) + C$

Ex6/ Compute $\int \left(\dfrac{2x^5 - \sqrt{x}}{x}\right)dx$

$\int \left(\dfrac{2x^5 - \sqrt{x}}{x}\right)dx = \int\left(\dfrac{2x^5}{x} - \dfrac{\sqrt{x}}{x}\right)dx = \int (2x^4 - x^{-\frac{1}{2}})\,dx$

$x^{\frac{1}{2}}$

$= \int 2x^4\,dx - \int x^{-\frac{1}{2}}\,dx$

$$= 2\underbrace{\int x^4\,dx} - \int x^{-\frac{1}{2}}\,dx$$

$$= 2\cdot\frac{x^{4+1}}{4+1} - \frac{x^{-\frac{1}{2}+1}}{-\frac{1}{2}+1} + C$$

$$= \frac{2}{5}x^5 - \frac{x^{\frac{1}{2}}}{\frac{1}{2}} + C$$

$$= \frac{2}{5}x^5 - 2x^{\frac{1}{2}} + C$$

Remark $\int (x + x^2)\,dx = \int x\,dx + \int x^2\,dx$

$$= \frac{x^{1+1}}{1+1} + C_1 + \frac{x^{2+1}}{2+1} + C_2$$

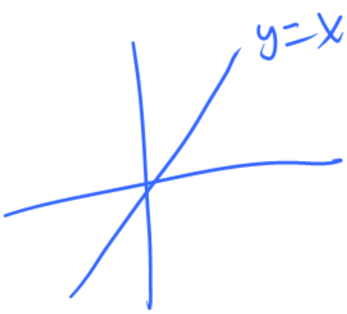
$$= \frac{1}{2}x^2 + \frac{1}{3}x^3 + (C_1 + C_2)$$

$$= \frac{1}{2}x^2 + \frac{1}{3}x^3 + C$$

Ex 7/ $\int \sec x(\sec x + \cos x)\,dx$

$$\int \sec x(\sec x + \cos x)\,dx = \int (\sec^2 x + \sec x\cos x)\,dx$$

$$= \int \sec^2 x\,dx + \int \underbrace{\sec x}_{=\frac{1}{\cos x}}\cos x\,dx$$



$$= \int \sec^2 x\,dx + \int 1\cdot x^0\,dx$$

$$= \tan x + x + C$$

Remark Include parantheses when the integrand has multiple terms

This won't be on the test, but you will see this in MA 16020

$$\int x + x^2\,dx \quad \times$$

$$\int (x + x^2)\,dx \quad \checkmark$$

Theorem (Reverse Chain Rule) Let F be an antiderivative of f

$$\int f(g(x))\,g'(x)\,dx = F(g(x)) + C$$

Ex 8/ $\int \cos(2x)\,dx$

$$\frac{d}{dx}(\sin(2x)) = 2 \cdot \cos(2x)$$

$$\frac{d}{dx}\left(\frac{1}{2}\sin(2x)\right) = \cos(2x)$$

$$\frac{2}{2}\int \cos(2x)\,dx = \frac{1}{2}\int 2\cos(2x)\,dx$$

$$= \frac{1}{2}\sin(2x) + C$$