Goal: Differentiate functions of the form $f(g(x))$. $(2x)^5$, $\sin(x^2)$, e^{-t}

Summary:

	$(f(g(x)))' = f'(g(x)) \cdot g'(x)$ $\frac{d}{dx}(f \circ g) = \frac{df}{dg} \cdot \frac{dg}{dx}$	
$[g(x)^p]' = p(g(x))^{p-1} \cdot g'(x)$	$(b^x)' = b^x \cdot \ln(b)$	$(\sin(\theta^\circ))' = \frac{\pi}{180}\cos(\theta^\circ)$

Theorem (Chain Rule) Let f and g be differentiable.

Format I: $\frac{d}{dx}(f \circ g) = \frac{df}{dg} \cdot \frac{dg}{dx}$

Format II: $[f(g(x))]' = f'(g(x)) \cdot g'(x)$

Easy (but incorrect) reasoning why

$$[f(g(x))]' = \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta x}\left(\frac{\Delta g}{\Delta g}\right)$$

"outside" "inside"

$$= \lim_{\Delta x \to 0} \frac{\Delta f}{\Delta g} \cdot \frac{\Delta g}{\Delta x} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

Ex/ Compute $(\sin(x^2))'$ $g(x) = x^2$, $f(g) = \sin(g)$

Format I: $\frac{d}{dx}(f \circ g) = \frac{df}{dg} \cdot \frac{dg}{dx} = \frac{d}{dg}[\sin g] \cdot \frac{d}{dx}(x^2)$

$$= \cos(g) \cdot 2x = \cos(x^2) \cdot 2x$$

Format II: $[f(g(x))]' = f'(g) \cdot g' = \cos(g) \cdot 2x$

$$= \cos(x^2) \cdot 2x$$

Ex2/ Compute $[\sin^2 x]' = [(\underbrace{\sin x}))^2]'$

$g(x) = \sin x$
$f(g) = g^2$

Format I: $\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$

$$= \frac{d}{d(\sin x)}[(\sin x)^2] \cdot \frac{d}{dx}(\sin x)$$

$$= 2 \sin x \cos x$$

Format II:

$$[f(g(x))]' = f'(g(x)) \cdot g'(x) = [g^2]' \cdot [\sin x]'$$

$$= 2g \cdot \cos x = 2 \sin x \cos x$$

Ex 3/ Find y' if $y = e^{\underbrace{\cos x}_{g}}$; $f(g) = e^g$

$$[e^{\cos x}]' = \underbrace{[e^{\cos x}]'}_{\text{Differentiating w.r.t "}\cos x\text{"}} \cdot \underbrace{[\cos x]'}_{\text{Differentiating w.r.t } x} = e^{\cos x}(-\sin x)$$

$$= -e^{\cos x}(\sin x)$$

Applications of The Chain Rule

Ex 4/ Differentiate $y = (x^3-1)^{100}$

$g(x) = x^3 - 1$
$f(g) = g^{100}$

$$y' = \underbrace{[g^{100}]'}_{\text{w.r.t. } g} \cdot \underbrace{[x^3-1]'}_{\text{w.r.t. } x} = 100g^{99}(3x^2)$$

$$= 100(x^3-1)^{99} \cdot (3x^2)$$

$[f(x)]^{100}$

$100[f(x)]^{99} \cdot f'(x)$

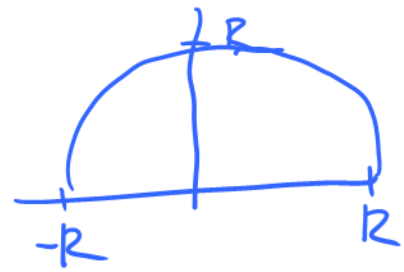
Theorem (Generalized Power Rule) For a differentiable function g and a real number p.

$$[g(x)]^p = p\,[g(x)]^{p-1} \cdot g'(x)$$

Remark: If $g(x)=x$, this is just the power rule.

Ex 5/ Let $R>0$. A semicircle of radius R is given by

$$y = \sqrt{R^2 - x^2}$$



Find y'

$$y' = \left[\sqrt{R^2 - x^2}\right]' = \left[(R^2 - x^2)^{\frac{1}{2}}\right]'$$

$f(g) = g^{\frac{1}{2}}$
$g(x) = R^2 - x^2$

$$= \frac{1}{2} g^{-\frac{1}{2}} \cdot (-2x) = \frac{1}{2\sqrt{g}} \cdot (-2x)$$

$$= \frac{-x}{\sqrt{g}} = -\frac{x}{\sqrt{R^2 - x^2}}$$

Ex 6/ What's $\frac{d}{dx}(2^x)$?

$$\frac{d}{dx}(2^x) = \frac{d}{dx}\left(e^{\ln(2^x)}\right) = \frac{d}{dx}\left(e^{x[\ln 2]}\right)$$

$g(x) = [\ln 2]x$
$f(g) = e^g$

$$= \frac{d}{d[(\ln 2)x]}\left(e^{x[\ln 2]}\right) \cdot [(\ln 2)x]' = e^{x[\ln 2]} \cdot (\ln 2)$$

$$= e^{\ln(2^x)} \cdot \ln(2) = 2^x \cdot \ln 2$$

In general, for any $b>0$

$$\boxed{\frac{d}{dx}(b^x) = b^x \cdot \ln(b)}$$

Ex 7/ $h(x) = \underbrace{\sin}_{f(g)=\sin(g)}(\underbrace{\cos(\tan x)}_{g})$

$$h'(x) = [\sin(g)]' \cdot [g']$$

$$= \cos(\cos(\tan x)) \cdot [\underbrace{\cos}_{f(g)=\cos(g)}(\underbrace{\tan x}_{g})]'$$

$$= \cos(\cos(\tan x)) \cdot [\cos g]' \cdot [\tan x]'$$

$$= \cos(\cos(\tan x)) \cdot [-\sin(\tan x)] \cdot \sec^2 x$$

Ex8/ Recall π radians $= 180°$ deg $\Longrightarrow \dfrac{\pi/180 \text{ radians}}{1 \text{ degree}}$

If an angle is measured in degrees

$$\sin(x°) = \sin\left(\underbrace{\frac{\pi}{180}x}_{\text{Radians}}\right)$$

What's $\frac{d}{dx}(\sin x)$ if x is measured in degrees?

$$\frac{d}{dx}[\sin(x°)] = \frac{d}{dx}\left[\underbrace{\sin}_{f(g)=\sin(g)}\left(\underbrace{\frac{\pi}{180}x}_{g}\right)\right] = [\sin(g)]' \cdot g'$$

$$= \cos(g) \cdot \frac{\pi}{180} = \frac{\pi}{180}\cos\left(\frac{\pi}{180}x\right) = \frac{\pi}{180}\cos(x°)$$

Ex 9 (Dampened Pendeleum) A pendeleum's position is modeled by its angle from its resting place

$$s(t) = e^{-t}\cos x$$

Find $v(t)$

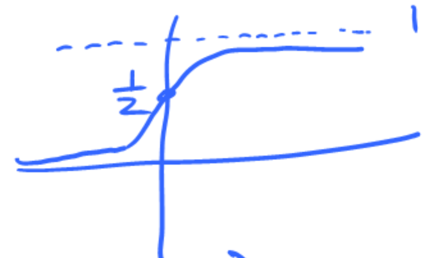
$$v(t) = s'(t) = [e^{-t}]'\cos x + e^{-t}[\cos x]'$$

$$= (-1)e^{-t}\cos x \quad + e^{-t}(-\sin x)$$

$$= -e^{-t}(\cos x + \sin x)$$

Ex 10 (Logistics Curve)

$$P(t) = \frac{1}{1+e^{-t}}$$



$$P'(t) = \left[(1+e^{-t})^{-1}\right]' = (-1)(1+e^{-t})^{-2}\cdot(-e^{-t})$$

$$= \frac{e^{-t}}{(1+e^{-t})^2}$$