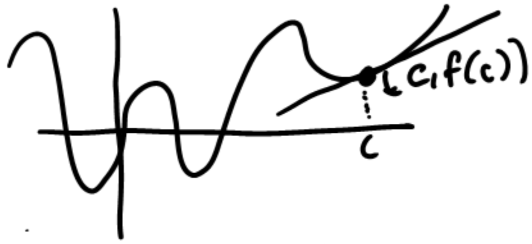
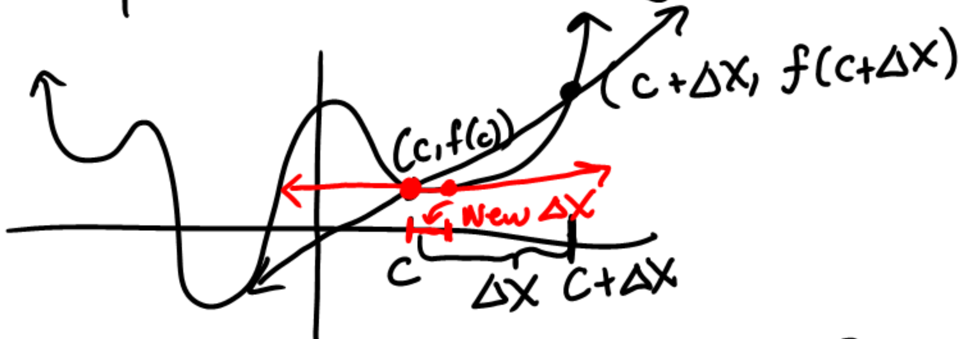
Lecture 5 The Derivative

We say a line is <u>tangent</u> to an object if it touches the object only once.



<u>The Tangent Problem</u> Given a graph f and a point c, determine the equation of the tangent line (tangent to f) at point



We can approximate a solution using a secant line. The slope of the secant line (<u>difference quotient</u>)

$$\text{Slope} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{f(c+\Delta x) - f(c)}{(c+\Delta x) - c} = \frac{f(c+\Delta x) - f(c)}{\Delta x} \approx \text{Slope of the tangent line}$$

<u>Notice</u>: We get a better approximation by moving closer to c (make Δx smaller)

<u>Def</u> The <u>derivative</u> of f at $x=c$ is the quantity

$$f'(c) \overset{\text{def}}{=} \lim_{\Delta x \to 0} \frac{f(c+\Delta x) - f(c)}{\Delta x}$$

If $f'(c)$ exists, we say f is <u>differentiable</u> at $x=c$

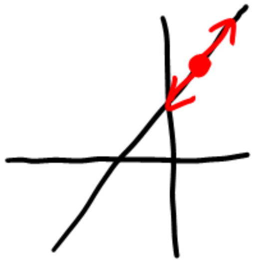
<u>NOTE:</u> ① Some textbooks use h instead of Δx

② If $f'(c)$ exists, the line with slope $f'(c)$ containing the point $(c, f(c))$

$$y - f(c) = f'(c)(x-c)$$

is the solution to the tangent problem.

Ex/ Find the equation of the tangent line of $f(x)=2x+3$ at $x=1$.



Sol/ Find Slope:

$$f'(1) = \lim_{\Delta x \to 0} \frac{f(\overbrace{1+\Delta x}) - f(1)}{\Delta x} = \lim_{\Delta x \to 0} \frac{2(1+\Delta x)+3-(2(1)+3)}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2+2\Delta x+3-5}{\Delta x} = \lim_{\Delta x \to 0} \frac{2\Delta x}{\Delta x} = \lim_{\Delta x \to 0} 2 = 2$$

Point: $(1, f(1)) = (1,5)$

Equation: $y - f(1) = f'(1)(x-1)$

$$y-5 = 3(x-1) \longrightarrow y = 3x+2$$

The Derivative As a Function

A point x $\longrightarrow$ $\boxed{f'(x) = \lim_{\Delta x \to 0} \frac{f(x+h)-f(x)}{h}}$ $\longrightarrow$ The slope of the tangent line at x

NOTATION:

Function	Evaluating at c
$f'(x)$	$f'(c)$
y'	$y'(c)$
$\frac{df}{dx}$	$\left.\frac{df}{dx}\right\vert_{x=c} = \left.\frac{df}{dx}\right]_{x=c}$
$\frac{dy}{dx}$	$\left.\frac{dy}{dx}\right\vert_{x=c}$

Ex/ If $f(x)=x^2$, find $f'(x)$

$$f'(x) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^2 - x^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{x^2+2x\Delta x+(\Delta x)^2-x^2}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{2x\Delta x+(\Delta x)^2}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x[2x+\Delta x]}{\Delta x} = \lim_{\Delta x \to 0}(2x+\Delta x) = 2x$$

Ex/ Find $\frac{d}{dx}(x^3)$ Google "Pascal's Triangle"

$$\begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 1 & & \\ & 1 & & 2 & & 1 & \\ 1 & & 3 & & 3 & & 1 \end{array}$$

$$\frac{d}{dx}(x^3) = \lim_{\Delta x \to 0} \frac{(x+\Delta x)^3 - x^3}{\Delta x} = \frac{x^3 + 3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3 - x^3}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \frac{3x^2\Delta x + 3x(\Delta x)^2 + (\Delta x)^3}{\Delta x} = \lim_{\Delta x \to 0} \frac{\Delta x\left[3x^2 + 3x\Delta x + (\Delta x)^2\right]}{\Delta x}$$

$$= \lim_{\Delta x \to 0} \left(3x^2 + 3x\Delta x + (\Delta x)^2\right) = 3x^2$$

$a^2 - b^2 = (a+b)(a-b)$

Ex/ If $y = \sqrt{x}$ $(x > 0)$, find y'

$$y' = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x} = \lim_{\Delta x \to 0} \frac{\sqrt{x+\Delta x} - \sqrt{x}}{\Delta x}\left(\frac{\sqrt{x+\Delta x} + \sqrt{x}}{\sqrt{x+\Delta x} + \sqrt{x}}\right)$$

$$= \lim_{\Delta x \to 0} \frac{x + \Delta x - x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})} = \lim_{\Delta x \to 0} \frac{\Delta x}{\Delta x(\sqrt{x+\Delta x} + \sqrt{x})}$$

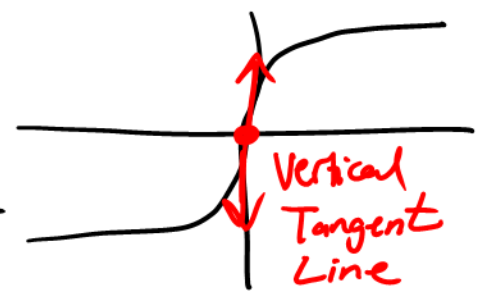
$$= \lim_{\Delta x \to 0} \frac{1}{\sqrt{x+\Delta x} + \sqrt{x}} \underset{\text{when } x>0}{=} \frac{1}{\sqrt{x} + \sqrt{x}} = \frac{1}{2\sqrt{x}}$$

<u>Question</u> When does differentiability fail?

<u>Answer 1</u> A vertical tangent line

Ex/ When is $y = \sqrt[3]{x}$ differentiable?

Take for granted $y' = \frac{1}{3}x^{-\frac{2}{3}} = \frac{1}{3\sqrt[3]{x^2}}$



<u>Answer 2</u> Discontinuity

<u>Theorem</u> ① If f is differentiable at a point c, f is continuous at c.

② If f is not continuous at c, it is not differentiable at c

Ex/ When is $\lim_{\Delta x \to 0} \frac{\frac{1}{x+\Delta x} - \frac{1}{x}}{\Delta x}$ well defined?

$$\lim_{\Delta x \to 0} \frac{x - x - \Delta x}{x(x+\Delta x)} \cdot \frac{1}{\Delta x} = \lim_{\Delta x \to 0} \frac{-\Delta x}{(\Delta x)x(x+\Delta x)} = -\frac{1}{x^2} = \frac{d}{dx}\left(\frac{1}{x}\right)$$

$\frac{1}{x}$ is discontinuous at 0, so

$$\frac{d}{dx}\left(\frac{1}{x}\right) = \begin{cases} \text{undefined} & x=0 \\ -\frac{1}{x^2} & x \neq 0 \end{cases}$$

<u>Answer 3</u> There is a "corner" or "kink" in the graph

Ex/ When is $y=|x|$ differentiable? We claim $|x|$ is not differentiable at 0

$$y'\Big|_{x=0} = \lim_{\Delta x \to 0} \frac{|0+\Delta x| - |0|}{\Delta x}$$

$$\lim_{\Delta x \to 0^+} \frac{|0+\Delta x| - |0|}{\Delta x} = \lim_{\Delta x \to 0^+} \frac{\Delta x}{\Delta x} = 1$$

$$\lim_{\Delta x \to 0^-} \frac{|0+\Delta x| - |0|}{\Delta x} = \lim_{\Delta x \to 0^-} \frac{-\Delta x}{\Delta x} = -1$$

$$|x| = \begin{cases} x & x \geq 0 \\ -x & x < 0 \end{cases}$$

$$y' = \begin{cases} 1 \text{ if } & x>0 \\ -1 \text{ if } & x<0 \\ \text{undefined if } & x=0 \end{cases} \overset{\text{def}}{=} \operatorname{sgn}(x)$$

$|x|$ is an example of a function that is continuous at 0, but not differentiable at 0.