

MA 16200: Plane Analytic Geometry and Calculus II

Lecture 2: Dot and Cross Products

§13.3 The Dot Product

Definition

Definition 1 (Dot Product)

Let $\vec{u} = \langle u_1, u_2, \dots, u_n \rangle$ and $\vec{v} = \langle v_1, v_2, \dots, v_n \rangle$ be vectors. The **dot product** of \vec{u} and \vec{v} is:

$$\vec{u} \cdot \vec{v} \stackrel{\text{def}}{=} \sum_{i=1}^n u_i v_i = u_1 v_1 + u_2 v_2 + \dots + u_n v_n$$

In 2-Dimensions ($n = 2$):

$$\vec{u} \cdot \vec{v} = \langle u_1, u_2 \rangle \cdot \langle v_1, v_2 \rangle = u_1 v_1 + u_2 v_2$$

In 3-Dimensions ($n = 3$):

$$\vec{u} \cdot \vec{v} = \langle u_1, u_2, u_3 \rangle \cdot \langle v_1, v_2, v_3 \rangle = u_1 v_1 + u_2 v_2 + u_3 v_3$$

Examples

Problem 2

Compute the following:

- $\langle 1, 2 \rangle \cdot \langle 3, 4 \rangle$
- $\langle 8, 2 \rangle \cdot \langle 0, -2 \rangle$
- $\langle 4, -1, 1 \rangle \cdot \langle 1, 4, 0 \rangle$

Extra Space

Properties Of Dot Products

Theorem 3

Let \vec{u} , \vec{v} , and \vec{w} be vectors and c a scalar:

1 $\vec{0} \cdot \vec{u} = 0$

2 $\vec{u} \cdot \vec{u} = |\vec{u}|^2$

3 $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$ (Commutative Property)

4 $c(\vec{u} \cdot \vec{v}) = (c\vec{u}) \cdot \vec{v} = \vec{u} \cdot (c\vec{v})$ (Associative Property)

5 $\vec{u} \cdot (\vec{v} + \vec{w}) = \vec{u} \cdot \vec{v} + \vec{u} \cdot \vec{w}$ (Distributive Property)

The dot product gives a rough idea of how “aligned” two vectors are

... can we be more precise than this?

Physics Definition of The Dot Product

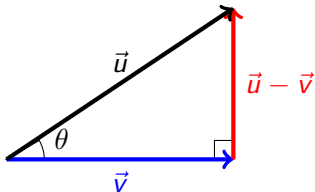
Theorem 4

If \vec{u} and \vec{v} are **non-zero** vectors, then:

$$\vec{u} \cdot \vec{v} = |\vec{u}||\vec{v}| \cos \theta$$

where θ is the angle between \vec{u} and \vec{v}

Why? By properties of the dot product:



$$\begin{aligned} |\vec{u} - \vec{v}|^2 &= (\vec{u} - \vec{v}) \cdot (\vec{u} - \vec{v}) \\ &= \vec{u} \cdot \vec{u} - \vec{u} \cdot \vec{v} - \vec{v} \cdot \vec{u} + \vec{v} \cdot \vec{v} \\ &= |\vec{u}|^2 - 2\vec{u} \cdot \vec{v} + |\vec{v}|^2 \\ &= |\vec{u}|^2 - 2(\vec{u} \cdot \vec{v}) + |\vec{v}|^2 \end{aligned}$$

Extra Space

Now by the Law of Cosines:

$$\begin{aligned} |\vec{u} - \vec{v}|^2 &= |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}| |\vec{v}| \cos \theta \\ |\vec{u}|^2 - 2(\vec{u} \cdot \vec{v}) + |\vec{v}|^2 &= |\vec{u}|^2 + |\vec{v}|^2 - 2|\vec{u}| |\vec{v}| \cos \theta \\ -2(\vec{u} \cdot \vec{v}) &= -2|\vec{u}| |\vec{v}| \cos \theta \\ \vec{u} \cdot \vec{v} &= |\vec{u}| |\vec{v}| \cos \theta \end{aligned}$$

Angle Between Vectors

Definition 5

- 1 Solving for θ we get the **angle between two vectors** \vec{u} and \vec{v} :

$$\theta = \cos^{-1} \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{u}| |\vec{v}|} \right)$$

If either $\vec{u} = \vec{0}$ or $\vec{v} = \vec{0}$, then θ is undefined. Note that $0 \leq \theta \leq \pi$.

- 2 If $\vec{u} \cdot \vec{v} = 0$ (equivalently $\theta = \frac{\pi}{2}$), we say \vec{u} and \vec{v} are **orthogonal**.

In 2 and 3 dimensions, “orthogonal” and “perpendicular” mean the same thing.

Examples

Problem 6

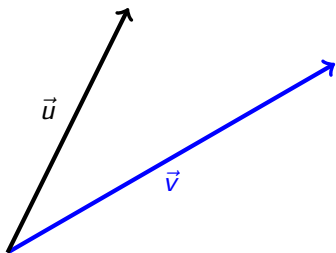
Use the dot product to show:

- 1 $2\vec{i} + 2\vec{j} - \vec{k}$ and $5\vec{i} - 4\vec{j} + 2\vec{k}$ are orthogonal;
- 2 $\langle 2, 1 \rangle$ is parallel to $\langle 10, 5 \rangle$.

Extra Space

Projecting Onto Other Vectors

Vectors can “cast a shadow” on another vector, what is the length and direction of that “shadow”?



Components and Projections

Definition 7

- 1 For a vector \vec{u} and a non-zero vector \vec{v} , the **scalar component of \vec{u} in the direction of \vec{v}** is:

$$\text{scal}_{\vec{v}} \vec{u} = |\vec{u}| \cos \theta = \frac{\vec{u} \cdot \vec{v}}{|\vec{v}|}$$

- 2 The **(orthogonal) projection of \vec{u} onto \vec{v}** is:

$$\text{proj}_{\vec{v}} \vec{u} = [\text{scal}_{\vec{v}} \vec{u}] \frac{\vec{v}}{|\vec{v}|} = \left(\frac{\vec{u} \cdot \vec{v}}{|\vec{v}|^2} \right) \vec{v}$$

Example

Problem 8

Find $\text{scal}_{\vec{v}}\vec{u}$ and $\text{proj}_{\vec{v}}\vec{u}$ for $\vec{u} = -4\vec{i} - 3\vec{j}$ and $\vec{v} = \vec{i} - \vec{j}$.

Orthogonal Decompositions

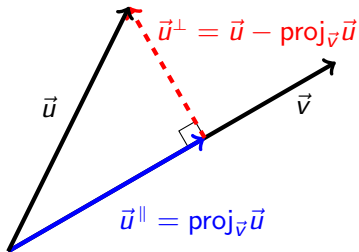
Theorem 9

For a vector \vec{u} and a non-zero vector \vec{v} , \vec{u} can be written as the sum of two vectors:

$$\vec{u} = \vec{u}^{\parallel} + \vec{u}^{\perp}$$

where \vec{u}^{\parallel} is parallel to \vec{v} and \vec{u}^{\perp} is orthogonal to \vec{v} .

Why?



Application: Work

Problem 10

A wagon is pulled a distance of 100m along a horizontal path by a constant force of 50N. The handle of the wagon is held at an angle of 30° above the horizontal.

1) Express the force vector $\vec{F} = 50\langle \cos \frac{\pi}{6}, \sin \frac{\pi}{6} \rangle = \langle 25\sqrt{3}, 25 \rangle$ as the sum of two vectors: one parallel to the ground and one perpendicular to the ground.

Work Definition

Definition 11

Given a force vector \vec{F} and displacement vector \vec{D} , the **work (W)** done by the force is defined as:

$$W \stackrel{\text{def}}{=} (\text{Force})(\text{Distance}) = (|\vec{F}| \cos \theta) |\vec{D}| = \vec{F} \cdot \vec{D}$$

2) In the previous problem, how much work was done?

§13.4 The Cross Product

Review of Determinants

A **2-by-2 determinant** is defined by:

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} = ad - bc$$

A **3-by-3 determinant** is defined using 2-by-2 determinants:

$$\begin{vmatrix} a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \\ c_1 & c_2 & c_3 \end{vmatrix} = a_1 \begin{vmatrix} b_2 & b_3 \\ c_2 & c_3 \end{vmatrix} - a_2 \begin{vmatrix} b_1 & b_3 \\ c_1 & c_3 \end{vmatrix} + a_3 \begin{vmatrix} b_1 & b_2 \\ c_1 & c_2 \end{vmatrix}$$

Notice the negative sign next to the a_2 .

Cross Product Definition

Definition 12

If $\vec{u} = \langle u_1, u_2, u_3 \rangle$ and $\vec{v} = \langle v_1, v_2, v_3 \rangle$, then the **cross product** of \vec{u} and \vec{v} is:

$$\begin{aligned}\vec{u} \times \vec{v} &\stackrel{\text{def}}{=} \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix} \\ &= (u_2 v_3 - u_3 v_2) \vec{i} + (u_3 v_1 - u_1 v_3) \vec{j} + (u_1 v_2 - u_2 v_1) \vec{k}\end{aligned}$$

Notice how $\vec{u} \times \vec{v}$ is a **vector** and is only defined in 3-dimensions.

Examples

Problem 13

Compute $\vec{u} \times \vec{v}$ for:

1 $\vec{u} = \langle 1, 2, 0 \rangle, \vec{v} = \langle 0, 3, 1 \rangle$

2 $\vec{u} = \langle 1, 1, 0 \rangle, \vec{v} = \langle -2, 3, 0 \rangle$

Extra Space

Properties of Cross Products

Theorem 14

$\vec{u} \times \vec{v}$ is orthogonal to **both** \vec{u} and \vec{v} .

Theorem 15

Let \vec{u} , \vec{v} , and \vec{w} be vectors and c a scalar:

- 1 $\vec{u} \times \vec{u} = \vec{0}$
- 2 $\vec{u} \times \vec{v} = -(\vec{v} \times \vec{u})$ (**anti-commutativity**)
- 3 $(c\vec{u}) \times \vec{v} = \vec{u} \times (c\vec{v}) = c(\vec{u} \times \vec{v})$
- 4 $\vec{u} \times (\vec{v} + \vec{w}) = \vec{u} \times \vec{v} + \vec{u} \times \vec{w}$
- 5 $(\vec{u} + \vec{v}) \times \vec{w} = \vec{u} \times \vec{w} + \vec{v} \times \vec{w}$
- 6 $\vec{u} \cdot (\vec{v} \times \vec{w}) = (\vec{u} \times \vec{v}) \cdot \vec{w}$
- 7 $\vec{u} \times (\vec{v} \times \vec{w}) = (\vec{u} \cdot \vec{w})\vec{v} - (\vec{u} \cdot \vec{v})\vec{w}$

Physics Definition of Cross Product

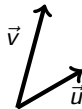
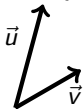
Theorem 16

Given two **non-zero** 3-dimensional vectors \vec{u} and \vec{v} :

$$|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta$$

where $0 \leq \theta \leq \pi$ is the angle between \vec{u} and \vec{v} .

The direction of $\vec{u} \times \vec{v}$ is given by the **right-hand rule**: Line up the tails of \vec{u} and \vec{v} . Curl your right hand from \vec{u} to \vec{v} , then $\vec{u} \times \vec{v}$ points in the same direction as your thumb. If \vec{u} or \vec{v} is $\vec{0}$, the direction is undefined.



Why?

$$\begin{aligned} |\vec{u} \times \vec{v}|^2 &= (u_2 v_3 - u_3 v_2)^2 + (u_3 v_1 - u_1 v_3)^2 + (u_1 v_2 - u_2 v_1)^2 \\ &= u_2^2 v_3^2 - 2u_2 u_3 v_2 v_3 + u_3^2 v_2^2 + u_3^2 v_1^2 - 2u_1 u_3 v_1 v_3 + u_1^2 v_3^2 \\ &\quad + u_1^2 v_2^2 - 2u_1 u_2 v_1 v_2 + u_2^2 v_1^2 \\ &= (u_1^2 + u_2^2 + u_3^2) (v_1^2 + v_2^2 + v_3^2) - (u_1 v_1 + u_2 v_2 + u_3 v_3)^2 \\ &= |\vec{u}|^2 |\vec{v}|^2 - (\vec{u} \cdot \vec{v})^2 \\ &= |\vec{u}|^2 |\vec{v}|^2 - |\vec{u}|^2 |\vec{v}|^2 \cos^2 \theta \\ &= |\vec{u}|^2 |\vec{v}|^2 (1 - \cos^2 \theta) \\ &= |\vec{u}|^2 |\vec{v}|^2 \sin^2 \theta \end{aligned}$$

Since $\sin \theta \geq 0$ for $0 \leq \theta \leq \pi$ we have $\sqrt{\sin^2 \theta} = \sin \theta$, so take the square root of both sides to get $|\vec{u} \times \vec{v}| = |\vec{u}| |\vec{v}| \sin \theta$

Test for Parallel Vectors

Corollary 17

- 1 \vec{u} and \vec{v} are parallel if and only if $\vec{u} \times \vec{v} = \vec{0}$
- 2 Three points A, B , and C are collinear if and only if $\vec{AB} \times \vec{AC} = \vec{0}$

Why?

Example: The points $A(-2, -4, 1)$, $B(1, 3, 7)$, and $C(4, 10, 13)$ are collinear.

Calculating Areas and Volumes

Theorem 18

- 1 The area of the parallelogram determined by \vec{u} and \vec{v} is $|\vec{u} \times \vec{v}|$;
- 2 The area of the triangle determined by the non-collinear points A , B , and C is $\frac{1}{2}|\vec{AB} \times \vec{AC}|$.

Why?

Examples

Problem 19

- 1 Determine the area of the parallelogram with adjacent sides $\vec{u} = -4\vec{i} + 3\vec{k}$ and $\vec{v} = \vec{i} + \vec{j} + 2\vec{k}$.
- 2 Determine the area of the triangle with vertices $(0, 1)$, $(-1, 1)$, and $(1, -1)$

Extra Space

The Triple Scalar Product

Theorem 20

The volume of the parallelepiped determined by vectors \vec{u} , \vec{v} , and \vec{w} is the *absolute value of the triple scalar product*:

$$\vec{u} \cdot (\vec{v} \times \vec{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix}$$

Why? This is a HW Problem.

Example

Problem 21

Are the vectors $\vec{u} = \langle 1, 4, -7 \rangle$, $\vec{v} = \langle 2, -1, 4 \rangle$, and $\vec{w} = \langle 0, -9, 18 \rangle$ coplanar?

Application: Torque

Definition 22

Applying a force \vec{F} at a point P , the twisting effect (or **torque**) about a point O is a vector $\vec{\tau}$ with:

$$|\vec{\tau}| = |\vec{OP} \times \vec{F}|$$

The direction of $\vec{\tau}$ is governed by the right hand rule.

It is common to write \vec{OP} as \vec{r} (for “radius”).

Torque Example

Problem 23

A bolt is tightened by applying a 40-N force to a 0.25-m wrench at a 75° angle. Find the magnitude and direction of the torque about the center of the bolt.

Application: Magnetism

Definition 24

A point charge q experiences a force \vec{F} when it moves through a magnetic field \vec{B} with velocity \vec{v} governed by the equation:

$$\vec{F} = q(\vec{v} \times \vec{B})$$

Notice whether q is positive or negative influences the direction of the force.

Example

Problem 25

An electron ($q = -1.6 \times 10^{-19} \text{ C}$) enters a magnetic field $\vec{B} = \vec{i} + \vec{j}$ with velocity $\vec{v} = (2 \times 10^5 \text{ m/s})\vec{k}$.

- 1 Determine the magnitude and direction of the force.*
- 2 Make a rough sketch of \vec{v} , \vec{B} , and \vec{F} .*

Extra Space