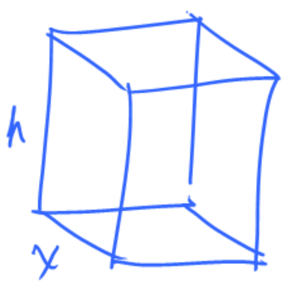
Goal: Solve optimization problems involving volume, surface area, and distances.

Ex 1/ We want to construct a box with a square base and no top. If the volume is $500\ in^3$, then what is the minimum amount of material required?



Obj: Minimize $A_{(x,h)} = x^2 + 4xh$

Given: $500 = x^2 h$; $x, h > 0$

$$h = \frac{500}{x^2}$$

$$A_{(x)} = x^2 + 4x\left(\frac{500}{x^2}\right) = x^2 + \frac{2000}{x}$$

$$\frac{dA}{dx} = 2x - \frac{2000}{x^2} \overset{\text{set}}{=} 0$$

$$2x = \frac{2000}{x^2}$$

$$x = \frac{1000}{x^2}$$

$$\sqrt[3]{x^3} = \sqrt[3]{1000}$$

$$x = 10$$

Verify it is a minimum: number line: 0, 1, 10; < 0 on $(0,10)$, > 0 on $(10,\infty)$

Local Min (By 1st $\frac{d}{dx}$ test) + Only 1 crit. value $\Rightarrow$ Abs Min

$x = 10$

$h = \frac{500}{10^2} = 5$

$$A_{(10,5)} = 10^2 + 4(10)(5) = 300\ in^2$$

Conclusion: We only need 300cm^2 worth of material to construct such a box.

Ex2 (Ideal Soup Can)

A company is designing a cylindrical soup can to hold 250π (≈ 785.4) cm^3 of liquid. What should the dimensions be to minimize costs?

Obj: Minimize $A_{(r,h)} = 2\pi r^2 + 2\pi r h$

Given: $250\pi = \pi r^2 h$

$$h = \frac{250\pi}{\pi r^2} = \frac{250}{r^2}$$

$$A_{(r)} = 2\pi r^2 + 2\pi r\left(\frac{250}{r^2}\right) = 2\pi r^2 + \frac{500\pi}{r}$$

$$\frac{dA}{dr} = 4\pi r - \frac{500\pi}{r^2} \underset{=}{\text{set}} 0$$

$$4\pi r = \frac{500\pi}{r^2}$$

$$r = \frac{125}{r^2} \rightarrow r^3 = 125$$

$$r = 5$$

Verify it's a minimum: $\underset{<0}{\longleftarrow} 5 \underset{>0}{\longrightarrow}$ Indeed it is

$$h = \frac{250}{r^2} \Rightarrow h\Big]_{r=5} = \frac{250}{5^2} = \frac{250}{25} = 10$$

Conclusion: The can needs to have a radius of 5 cm and the height needs to be 10 cm to minimize costs.

Ex3 (Ideal Soup Can II)

Due to the manufacturing process, each can has to use 294π (≈ 923.63) cm^2 worth of aluminium. Find the maximum volume.

Obj: Maximize $V(r,h) = \pi r^2 h$

Given: $294\pi = 2\pi r^2 + 2\pi r h$

$$2\pi r h = 294\pi - 2\pi r^2$$

$$h = \frac{294\pi - 2\pi r^2}{2\pi r}$$

$$h = \frac{147}{r} - r$$

$$\boxed{V(r) = \pi r^2\left(\frac{147}{r} - r\right)} = 147\pi r - \pi r^3$$

$$\frac{dV}{dr} = 147\pi - 3\pi r^2 \underset{=}{\text{set}} 0$$

$$3\pi r^2 = 147\pi$$

$$r^2 = 49 \longrightarrow r = 7$$

$$V(7) = \pi \cdot 7^2\left(\frac{147}{7} - 7\right) = \pi \cdot 7^2 (21 - 7) = \pi \cdot 7^2 \cdot 14$$

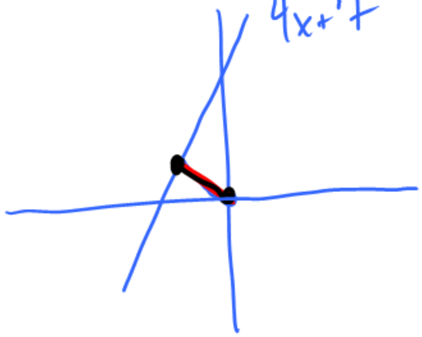
$$= 2\pi \cdot 7^3 = 686\pi$$

$$\begin{array}{r} 49 \\ \times\ 7 \\ \hline 63 \\ 280 \\ \hline 343 \\ \times\ 2 \\ \hline 686 \end{array}$$

Conclusion: The maximum volume is 686π (≈ 2155.13) cm^3.

Optimizing Distances

Ex 4/ What is the smallest distance from the line $y = 4x + 7$ and the origin?



Recall distance formula: The distance between the points (x, y) and (x_0, y_0) is

$$d = \sqrt{(x - \underbrace{x_0}_{=0})^2 + (y - \underbrace{y_0}_{y_0 = 0})^2}$$

To find the location where the optimal occurs, we can look at the loc. of the optimal squared distance

$$(\max)^2 \geq \left(\begin{matrix}\text{All other} \\ \text{distance}\end{matrix}\right)^2$$

$$\sqrt{(\max)^2} \geq \sqrt{(\text{The rest})^2} \text{ ; b/c distance are postive.}$$

$$\max \geq (\text{The rest})$$

Back to ex: Obj: Minimize $S(x,y) = d^2(x,y) = x^2 + y^2$

Given: $y = 4x + 7$

$$S(x) = x^2 + (4x+7)^2 = x^2 + 16x^2 + 56x + 49$$

$$S(x) = 17x^2 + 56x + 49$$

$$\frac{dS}{dx} = 34x + 56 \underset{=}{\text{set}} 0$$

$$x = -\frac{56}{34}$$

$$x = -\frac{28}{17}$$

Verify it's a minimum:

$$\frac{d^2S}{dx^2} = 34 > 0$$

local Min by 2^{nd} $\frac{d}{dx}$ test $\xRightarrow{\text{only 1 crit. value}}$ Abs. Min.

(y-coordinate) of the location of the minimum (squared) distance

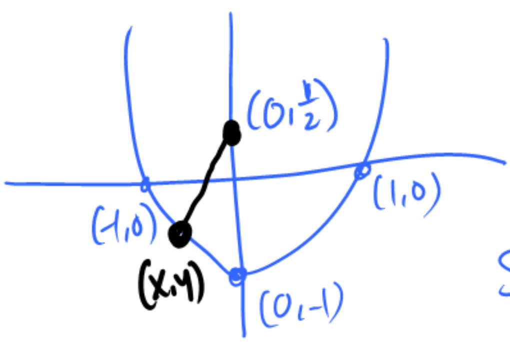
$$y = 4\left(\frac{-28}{17}\right) + 7 = \frac{7}{17}$$

Minimum Distance: Distance between the origin and $\left(\frac{-28}{17}, \frac{7}{17}\right)$

$$d = \sqrt{\left(\frac{-28}{17}\right)^2 + \left(\frac{7}{17}\right)^2} = \frac{7}{\sqrt{17}}$$

Should of been 1/2, not -1/2

Ex 5 Where is the distance between $(0, \frac{1}{2})$ and the parabola $y = x^2 - 1$ at a minimum?



Obj: Minimize $S(x,y) = d^2(x,y) = x^2 + \left(y - \frac{1}{2}\right)^2$

Given: $y = x^2 - 1$

$$S(x) = x^2 + \left(x^2 - 1 - \frac{1}{2}\right)^2 = x^2 + \left(x^2 - \frac{3}{2}\right)^2$$
$$= x^2 + x^4 - 3x^2 + \frac{9}{4}$$
$$= x^4 - 2x^2 + \frac{9}{4}$$

$$\frac{dS}{dx} = 4x^3 - 4x \overset{\text{set}}{=} 0$$
$$4x(x^2 - 1) = 0$$
$$4x(x-1)(x+1) = 0$$

$x = \pm 1, 0$

See which ones are local mins

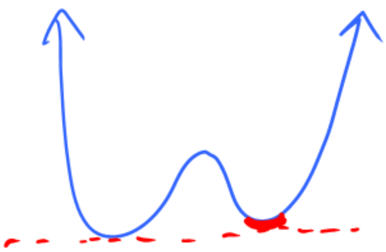
$$\frac{d^2S}{dx^2} = 12x^2 - 4$$

$$\left.\frac{d^2S}{dx^2}\right]_{x=-1} = 8 > 0 \qquad \left.\frac{d^2S}{dx^2}\right]_{x=1} = 8 > 0$$

$$\left.\frac{d^2S}{dx^2}\right]_{x=0} = -4 < 0$$

By the 2^{nd} $\frac{d}{dx}$ test, $x=\pm 1$ are the locs. of local mins. But does a global min exist?

Looking at end behavior $S \to +\infty$ as $x \to \pm\infty$



$\Leftarrow$ A global min exists, but not a global max

NOTE: Since $S(1) = S(-1)$, the global min occurs at both places.

Ie, the minimum distance occurs at $(\pm 1, 0)$.