

MA 16200: Plane Analytic Geometry and Calculus II

Lecture 15: Sequences and Series Intro

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Sections Covered: 10.1

$$9) \int_3^{\infty} \frac{8 \cos\left(\frac{\pi}{x}\right)}{x^2} dx$$

$$\text{Let } u = \frac{\pi}{x} \\ du = -\frac{\pi}{x^2} dx$$

$$= \frac{-8}{\pi} \int_3^{\infty} \cos\left(\frac{\pi}{x}\right) \frac{-\pi}{x^2} dx$$

$$\frac{2}{\sqrt{\frac{\pi}{3}}} \sqrt{3}$$

$$= -\frac{8}{\pi} \left[\sin\left(\frac{\pi}{x}\right) \right]_3^{\infty} = \lim_{t \rightarrow \infty} \left[-\frac{8}{\pi} \sin\left(\frac{\pi}{x}\right) \right]_3^t$$

$$= \lim_{t \rightarrow \infty} \left[-\frac{8}{\pi} \sin\left(\frac{\pi}{t}\right) + \frac{8}{\pi} \sin\left(\frac{\pi}{3}\right) \right] = \lim_{t \rightarrow \infty} \left[-\frac{8}{\pi} \sin t + \frac{8}{\pi} \cdot \frac{\sqrt{3}}{2} \right]$$

$$= \frac{4}{\pi} \sqrt{3}$$

$$10) \int_0^{\infty} 24 \tan^{-1}(3w) \left[\frac{1}{1+9w^2} \right] dw \quad \left| \begin{array}{l} u = \tan^{-1} 3w \\ du = \frac{1}{1+(3w)^2} \cdot 3 \, dx \end{array} \right.$$

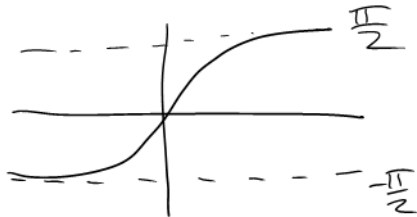
$$= \int_0^{\infty} 8 \underbrace{\tan^{-1}(3w)}_u \underbrace{\left[\frac{3}{1+(3w)^2} \right]}_{du} dx$$

I'm going to
Switch to
arc tan(x)

$$= \lim_{t \rightarrow \infty} \left[\frac{8}{2} \arctan^2 x \right]_0^t$$

$$= \lim_{t \rightarrow \infty} [4 \arctan^2(t)]$$

$$= 4 \left[\frac{\pi}{2} \right]^2 = 4 \frac{\pi^2}{4} = \pi^2$$



$$\int u = \frac{1}{2} u^2$$

$$\int \frac{e^x}{(e^x-1)(e^x+4)} dx \quad \begin{matrix} du = e^x \\ du = e^x dx \end{matrix}$$

multiply by
(u-1)(u+4)

$$= \int \frac{1}{(u-1)(u+4)} du \quad \left[\frac{1}{(u-1)(u+4)} = \frac{A}{u-1} + \frac{B}{u+4} \right]$$

$$= \int \left(\frac{1/5}{u-1} - \frac{1/5}{u+4} \right) du \quad \begin{aligned} 1 &= A(u+4) + B(u-1) \\ &= Au + 4A + Bu - B \end{aligned}$$

$$\begin{aligned} &= \frac{1}{5} \ln|u-1| - \frac{1}{5} \ln|u+4| + C \\ &= \frac{1}{5} \ln|e^x-1| - \frac{1}{5} \ln|e^x+4| + C \end{aligned} \quad \begin{aligned} 0u+1 &= (A+B)u + (4A-B) \\ \begin{cases} A+B=0 \rightarrow B=-A \\ 4A-B=1 \rightarrow 4A+A=1 \end{cases} &\rightarrow \begin{cases} B=-1/5 \\ A=1/5 \end{cases} \end{aligned}$$

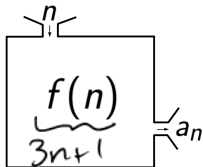
Enumerated Lists

Say we have an ordered list of numbers:

$$a_1, a_2, a_3, \dots, a_n, \dots$$

$$1, 4, 7, \dots, 3n+1, \dots$$

We can think of it as a function taking non-negative integers as inputs and real numbers as outputs.



Sequence Definition

Definition 1

A **sequence** is an ordered list of numbers of the form:

$$\{a_1, a_2, \dots, \underbrace{a_n}_{\leftarrow n\text{-th term}}, \dots\}$$

The subscript n is called the **index**. The number a_n is called the **n -th term** in the sequence.

Notation: Usually n starts at either 0 or 1 (but it is not required)

$$\begin{array}{ccccc} \{a_n\}_{n=1}^{\infty} & \{a_n\} & a_n; n \geq 1 & & \\ \uparrow & \uparrow & & & \\ \text{index} & \text{starting index} & & & \end{array}$$

Defining Sequences

One can define sequences:

- 1 **Explicitly** using a function $a_n = f(n)$.

$$\begin{aligned} \blacksquare \{ \sqrt{n-3} \}_{n=3}^{\infty} &= \{ 0, 1, \sqrt{2}, \sqrt{3}, \dots, \sqrt{n-3}, \dots \} \\ \blacksquare \left\{ \frac{1}{2}, -\frac{2}{3}, \frac{3}{4}, -\frac{4}{5}, \dots, (-1)^{n+1} \frac{n}{n+1}, \dots \right\} \end{aligned}$$

- 2 **Recursively** using a **recurrence relation** $a_{n+1} = f(a_n)$.

- The **Fibonacci Sequence**:

$$f_{n+1} = f_n + f_{n-1}; \quad f_1 = 1; \quad f_0 = 1$$

- Given a sequence $\{a_n\}$, define the **sequence of partial sums**

$$S_N = a_N + S_{N-1}; \quad S_1 = a_1 \quad S_N = a_1 + a_2 + \dots + a_N$$

- 3 “Abstractly” (when there is no obvious formula)

- Let $\{p_n\}$ be the sequence of prime numbers:

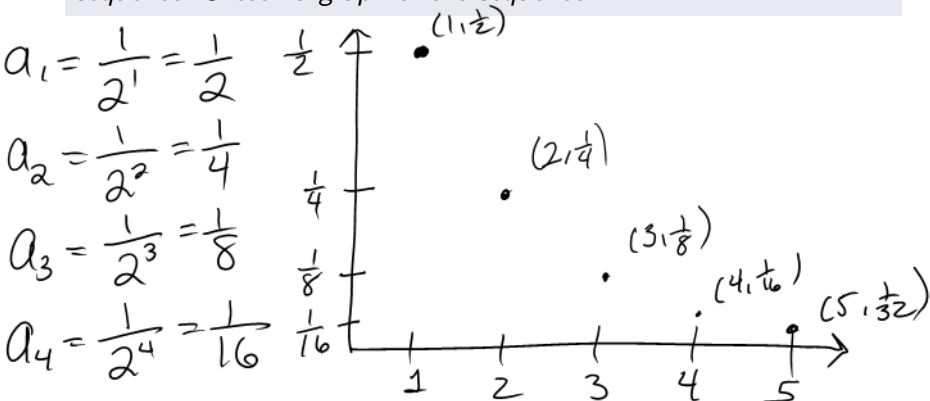
$$\{p_n\} = \{2, 3, 5, 7, 11, 13, \dots\}$$

Example (Explicit Formulas)

$$\{a_n\}_{n=1}^{\infty}$$

Problem 2

Use the explicit formula $\{\frac{1}{2^n}\}_{n=1}^{\infty}$ to write the first 4 terms of the sequence. Sketch a graph of the sequence.



Example (Explicit Formulas)

Alternate the signs
↙

Problem 3

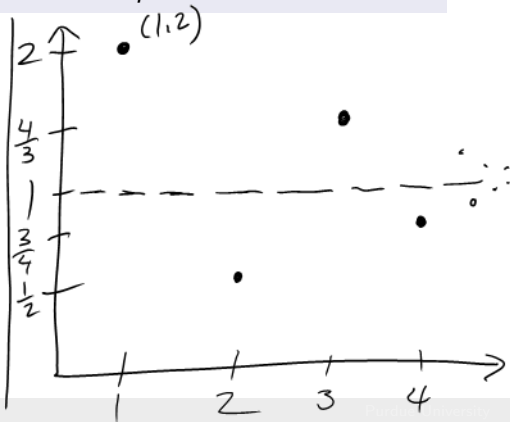
Use the explicit formula $\{1 + \frac{(-1)^{n+1}}{n}\}_{n=1}^{\infty}$ to write the first 4 terms of the sequence. Sketch a graph of the sequence.

$$a_1 = 1 + \frac{(-1)^{1+1}}{1} = 1 + 1 = 2$$

$$a_2 = 1 + \frac{(-1)^{2+1}}{2} = 1 + \frac{-1}{2} = \frac{1}{2}$$

$$a_3 = 1 + \frac{(-1)^{3+1}}{3} = 1 + \frac{1}{3} = \frac{4}{3}$$

$$a_4 = 1 + \frac{(-1)^{4+1}}{4} = 1 - \frac{1}{4} = \frac{3}{4}$$



Example (Recursion)

Problem 4

Use the recurrence relation to write the first 4 terms of the sequence.

$$\begin{cases} a_{n+1} = a_n + a_{n-1} \\ a_0 = 1 \\ a_1 = 1 \end{cases}$$

"Initial Values"
OR
"Base Case"
OR
"Seed Values"

$$\begin{aligned} a_2 &= a_1 + a_0 = 1 + 1 = 2 \\ a_3 &= a_2 + a_1 = 2 + 1 = 3 \\ a_4 &= a_3 + a_2 = 3 + 2 = 5 \\ a_5 &= a_4 + a_3 = 5 + 3 = 8 \end{aligned}$$

Example (Recursion)

Problem 5

Use the recurrence relation to write the first 4 terms of the sequence.

$$\begin{cases} a_{n+1} = 2a_n + 1 \\ a_1 = 1 \end{cases}$$

$$\begin{aligned} a_2 &= 2 \cdot a_1 + 1 = 2 \cdot 1 + 1 = 3 \\ a_3 &= 2 \cdot a_2 + 1 = 2 \cdot 3 + 1 = 7 \\ a_4 &= 2 \cdot a_3 + 1 = 2 \cdot 7 + 1 = 15 \\ a_5 &= 2 \cdot a_4 + 1 = 2 \cdot 15 + 1 = 31 \end{aligned}$$

$$a_n = 2^n - 1$$

Example (Finding Formulas)

a_n	-2	5	12	19
n	0	1	2	3

Problem 6

Consider the sequence $\{a_n\} = \{-2, 5, 12, 19, \dots\}$.

- Find 2 different formulas describing this sequence.
- Use either formula to find the next 2 terms in the sequence.

① Explicit Formula : $a_n = f(n) = 7n - 2 ; n \geq 0$
 $7(n-1) - 2 ; n \geq 1$

Recursive Formula:

$$a_{n+1} = 7(n+1) - 2 = 7n + 7 - 2 = \underbrace{(7n - 2)}_{a_n} + 7$$

$$= a_n + 7$$

$$\begin{cases} a_{n+1} = a_n + 7 \\ a_0 = -2 \end{cases}$$

$$\textcircled{2} \quad a_4 = a_3 + 7 = 19 + 7 = 26$$

$$a_5 = a_4 + 7 = 26 + 7 = 33$$

Example (Finding Formulas)

$n!$ is the fac. st.
① Domain integers
② $n! = n \cdot (n-1)!$

Problem 7

Consider the sequence $\{a_n\} = \{1, -2, 6, -24, 120, \dots\}$.

- 1 Find 2 different formulas describing this sequence.
- 2 Use either formula to find the next 2 terms in the sequence.

n	a_n	Explicit Formula
1	1	$a_n = (-1)^{n+1} \underbrace{n(n-1)(n-2)\dots 3 \cdot 2 \cdot 1}_{= (-1)^{n+1} \cdot n!}$
2	$-2 = (-1) 2 \cdot 1$	
3	$6 = 3 \cdot 2 \cdot 1$	
4	$-24 = (-1) \cdot 4 \cdot 3 \cdot 2 \cdot 1$	
5	$120 = 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1$	

Recursive:

$$a_{n+1} = (-1)^{(n+1)+1} (n+1)! = (-1)(-1)^{n+1} (n+1) \underbrace{n(n-1)(n-2)\dots 3\cdot 2\cdot 1}_{n!}$$

$$= (-1)(n+1) \boxed{(-1)^{n+1} n!}$$

$$= -(n+1) \cdot \widetilde{a_n} \quad a_n$$

Finally,

$$\begin{cases} a_{n+1} = -(n+1) \cdot a_n \\ a_1 = 1 \end{cases}$$

Convergence $a_n \rightarrow L$ means "a_n converges to L"

Definition 8

We say the sequence $\{a_n\}$ **converges** to a real number L (written $a_n \rightarrow L$) if

$$\lim_{n \rightarrow \infty} a_n = L$$

That is, the limit exists and equals L . Otherwise, we say the limit **diverges**.

Here is the formal definition, **you will not need to know this**.

Definition 9 (Formal Definition)

We say $a_n \rightarrow L$ if for any $\varepsilon > 0$ there is a positive integer N such that:

$$\text{If } n \geq N, \text{ then } |a_n - L| < \varepsilon$$

Example (Limits)

Problem 10

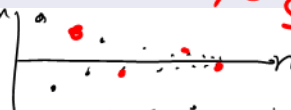
Make a conjecture about whether the following sequences converge or diverge. Explain why or why not.

① ■ $a_n = (-1)^n \frac{3}{n+5}; n \geq 0$ $a_n \rightarrow 0$

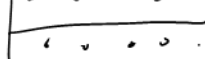
② ■ $a_n = \cos \pi n; n \geq 0$ a_n diverges (no limit point)

③ ■ $a_n = 2a_{n-1}; a_1 = 1$ a_n is unbounded, so $\lim a_n = \infty$.
So a_n diverges


① $\{a_n\} = \left\{ \frac{3}{5}, -\frac{3}{6}, \frac{3}{7}, -\frac{3}{8}, \dots \right\}$



② $\{\cos \pi n\}_{n=0}^{\infty} = \{1, -1, 1, -1, \dots\}$



③ $\{a_n\} = \{1, 2, 4, 8, \dots\}$

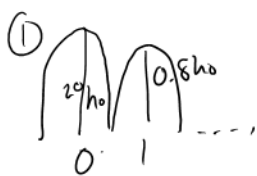


Example (Height of a Ball)

Problem 11

A basketball tossed straight up in the air reaches a high point and falls to the floor. Each time the ball bounces on the floor it rebounds to 0.8 of its previous height. Let $\{h_n\}$ be the high point after the n th bounce, with the initial height being $h_0 = 20$ ft.

- Find a recurrence relation and an explicit formula for the sequence $\{h_n\}$.
- What is the height of the peak after the 10th bounce? After the 20th bounce?
- Speculate the limit of the sequence $\{h_n\}$.



$$h_0 = 20$$

$$h_1 = 0.8 h_0$$

$$h_2 = 0.8 h_1 = 0.8(0.8 h_0) \\ = (0.8)^2 h_0$$

$$h_3 = 0.8 h_2 = 0.8(0.8)^2 h_0 \\ = (0.8)^3 h_0$$

Recursive

$$\begin{cases} h_n = 0.8 h_{n-1} \\ h_0 = 20 \end{cases}$$

Explicit

$$h_n = (0.8)^n h_0 = 20 \cdot (0.8)^n$$

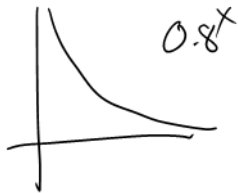
②

$$h_{10} = (0.8)^{10} \cdot 20 \approx 2.15 \text{ ft}$$

$$h_{20} = (0.8)^{20} \cdot 20 \approx 0.23 \text{ ft}$$

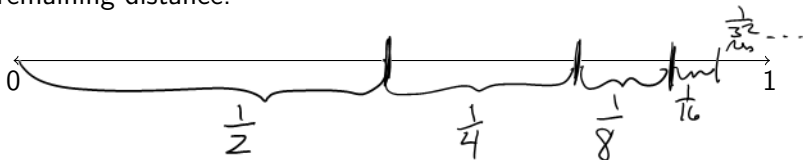
$$(3) \lim_{n \rightarrow \infty} h_n = \lim_{n \rightarrow \infty} 20 \cdot \underbrace{(0.8)^n}_{\substack{0 < 0.8 < 1}} = 0$$

$h_n \rightarrow 0$



Zeno's Paradox

Let's say we want to travel 1m. For each step, we go half the remaining distance.



What is the distance traveled after n steps? Will we ever reach 1m?

Distance traveled at n steps: $\frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \dots + \frac{1}{2^n} = \sum_{k=1}^n \frac{1}{2^k}$

We will at " $n = \infty$ "

We want to make sense of " $\sum_{k=1}^{\infty} \frac{1}{2^k} = 1$ "

Sequence of Partial Sums

For a given sequence $\{a_n\}$, we define the **sequence of partial sums** $\{S_N\}$ as:

$$S_1 = a_1$$

$$S_2 = a_1 + a_2$$

$$S_3 = a_1 + a_2 + a_3$$

$$\vdots$$

$$\boxed{S_N = a_1 + a_2 + \dots + a_N} = \underbrace{\sum_{n=1}^N a_n}_{\text{Explicit}} = \overbrace{a_N + S_{N-1}}^{\text{Recursive}}$$

That is, for the infinite sum $a_1 + a_2 + \dots + a_k + \dots$, S_N is the value when we “chop off” the first N terms and compute its sum.

Series Definition

Definition 12

Given a sequence $\{a_n\}_{n=1}^{\infty}$, the sum of its terms:

$$a_1 + a_2 + a_3 + \dots = \sum_{n=1}^{\infty} a_n$$

is called an **(infinite) series**. If the sequence of partial sums $\{S_N\}$ has a limit L , we say the series **converges** to L . We then write,

$$\sum_{n=1}^{\infty} a_n \stackrel{\text{def}}{=} \lim_{N \rightarrow \infty} \sum_{n=1}^N a_n = \lim_{N \rightarrow \infty} S_N = L$$

Otherwise, the series **diverges**.

$$0.9999999999\dots = 1?$$

$$0.\overline{b} = \frac{b}{9} \text{ for } 0 \leq b \leq 9$$

Problem 13

Make a conjecture about whether the sum:

$$\frac{9}{10} + \frac{9}{100} + \frac{9}{1000} + \dots = \sum_{n=1}^{\infty} 9 \cdot \left(\frac{1}{10}\right)^n$$

converges or diverges. If so, what is a plausible limit?

$$S_1 = 0.9$$

$$S_2 = 0.9 + 0.09 = 0.99$$

$$S_3 = 0.99 + 0.009 = 0.999$$

$$S_4 = 0.999 + 0.0009 = 0.9999$$

$$S_N = \sum_{n=1}^N 9 \left(\frac{1}{10}\right)^n = 0.\underbrace{999\dots 9}_{N \text{ times}}$$



$\lim_{N \rightarrow \infty} S_n$ should be 1

$$x = 0.555\dots$$

$$10x = 5.55\dots$$

$$\begin{array}{r} -x \quad -0.55\dots \\ \hline 9x = 5 \end{array}$$


$$x = \frac{5}{9}$$

Example

Telescoping Series

Problem 14

Make a conjecture about whether the sum:


$$\sum_{k=1}^{\infty} \frac{1}{k(k+1)} \xrightarrow{\text{Partial Fractions}} \sum_{k=1}^{\infty} \left[\frac{1}{k} - \frac{1}{k+1} \right]$$

converges or diverges. If so, what is a plausible limit?

$$\begin{aligned} S_1 &= 1 - \frac{1}{2} \\ S_2 &= 1 - \frac{1}{2} + \frac{1}{2} - \frac{1}{3} = 1 - \frac{1}{3} \\ S_3 &= 1 - \frac{1}{3} + \frac{1}{3} - \frac{1}{4} = 1 - \frac{1}{4} \\ &= 1 - \frac{1}{3+1} \end{aligned}$$

$$\begin{aligned} S_N &= 1 - \frac{1}{N+1} \\ \sum_{k=1}^{\infty} \frac{1}{k(k+1)} &= \lim_{N \rightarrow \infty} S_N = \lim_{N \rightarrow \infty} \left(1 - \frac{1}{N+1} \right) \\ &= 1 \end{aligned}$$

The Harmonic Series

Theorem 15

The **harmonic series**:

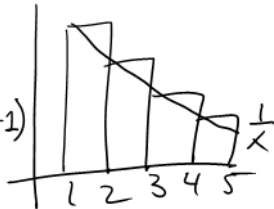
$$\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \cdots$$

diverges.

Why? For full details, see Section 10.4.

$$S_N = \sum_{n=1}^N \frac{1}{n} > \int_1^{N+1} \frac{1}{x} dx = \ln(N+1)$$

So S_N is unbounded (hence it diverges).



Example (Distance of a Ball)

Problem 16

Suppose a ball is thrown upward to a height of h_0 meters. Each time the ball bounces, it rebounds to a fraction $r = 0.5$ of its previous height. Let h_n be the height after the n th bounce and let S_n be the total distance the ball has traveled at the moment of the n th bounce.

- Find a formula for S_n and find a plausible value for the limit.



$$S_1 = 2h_0$$

$$\begin{aligned} S_2 &= S_1 + 2\left(\frac{h_0}{2}\right) = 2h_0 + \frac{1}{2}(2h_0) \\ &= 2h_0\left(1 + \frac{1}{2}\right) \end{aligned}$$

$$S_3 = S_2 + 2\left(\frac{h_1}{2}\right) = S_2 + 2\left(\frac{h_0}{4}\right)$$

$$S_3 = 2h_0 \left(1 + \frac{1}{2} + \frac{1}{4} \right)$$

$$S_N = 2h_0 \left(1 + \frac{1}{2} + \frac{1}{4} + \frac{1}{8} + \frac{1}{16} + \dots + \frac{1}{2^{N-1}} \right) = 2h_0 \sum_{n=1}^N \frac{1}{2^{n-1}}$$

Find a plausible limit for $2h_0 \sum_{n=1}^{\infty} \frac{1}{2^{n-1}}$



$$\sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = \lim_{N \rightarrow \infty} \sum_{n=1}^N \frac{1}{2^{n-1}} = 2$$

$$\text{Total Distance} : 2h_0 \sum_{n=1}^{\infty} \frac{1}{2^{n-1}} = 4h_0$$

Comparing to Functions

	Sequences/Series	Functions
Independent Variable	n	x
Dependent Variable	a_n	$f(x)$
Domain	Integers	Real Numbers
Accumulation	Sums	Integrals
Accumulation over finite interval	$\sum_{n=1}^N a_k$	$\int_1^N f(x) dx$
Accumulation over an infinite interval	$\sum_{n=1}^{\infty} a_n$	$\int_1^{\infty} f(x) dx$