MA 16200: Plane Analytic Geometry and Calculus II

Lecture 4: Calculating Volumes & Solids of Revolution

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Sections Covered: 6.3

Derivation

General Slicing Method

Definition 1

Suppose a solid object extends from $x = a$ to $x = b$, and the cross-sectional area at a point x is given by a function $A(x)$ that can be integrated on $[a, b]$. Then the volume of the solid is:

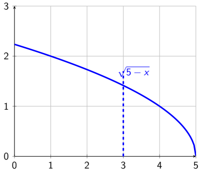
$$V = \int_a^b A(x)\ dx$$

Example

Problem 2

Consider a solid whose base is the region in the first quadrant bounded by the curve $y = \sqrt{5-x}$ and the line $x = 3$, and whose cross sections through the solid perpendicular to the x-axis are squares.

1. *Find an expression for the cross-sectional area $A(x)$ at a point $x \in [0,3]$.*
2. *Find the volume of the solid.*



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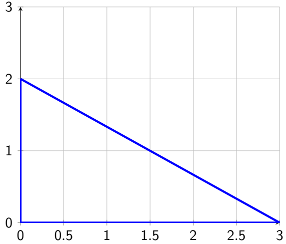
Examples

Problem 3

Use the general slicing method to find the volume of the solid whose base is the triangle with vertices $(0,0), (3,0)$, and $(0,2)$, and whose cross sections perpendicular to the base and parallel to the y-axis are semicircles



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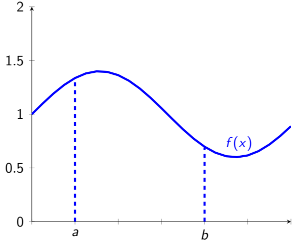
Solids of Revolution



Disk Method (about the x-axis)

Definition 4

Let $f \geq 0$ be continuous on $[a, b]$. If the region R bounded by the graph of f, the x-axis, and the lines $x = a$ and $x = b$ is revolved about the x-axis, the volume of the resulting solid of revolution is:

$$V = \int_a^b \pi [f(x)]^2 \, dx$$

Examples

Problem 5

Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region and a typical disk.

1. $y = e^x; x = 1; y = 0;$ *about the x-axis.*
2. $y = \frac{1}{x}; x = 1; x = 2; y = 0;$ *about the x-axis.*

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Washer Method (about the x-axis)

Definition 6

Let f and g be continuous function with $f(x) \geq g(x) \geq 0$ on $[a, b]$. Let R be the region bounded by $y = f(x)$, $y = g(x)$, $x = a$, and $x = b$. When R is revolved around the x-axis, the volume of the resulting solid of revolution is:

$$V = \int_a^b \pi[f(x)]^2\,dx - \int_a^b \pi[g(x)]^2\,dx = \int_a^b \pi([f(x)]^2 - [g(x)]^2)\,dx$$

Examples

Problem 7

Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region and a typical washer.

1. $y = \sec x; y = 1; x = -1; x = 1;$ *about the x-axis.*
2. $y = x^2; y^2 = x;$ *about the x-axis.*

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Rotating about the y-axis

Definition 8

Let p and q be continuous functions with $p(y) \geq q(y) \geq 0$ on $[c, d]$. Let R be the region bounded by $x = p(y), x = q(y)$, and the lines $y = c$ and $y = d$. When R is revolved about the y-axis, the volume of the resulting solid of revolution is given by

$$V = \int_c^d \pi[p(y)^2 - q(y)^2] \, dy$$

If $q(y) = 0$, the disk method results:

$$V = \int_c^d \pi[p(y)]^2 \, dy$$

Examples

Problem 9

Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region and a typical disk/washer.

1. $x = y - y^2; x = 0$; *about the y-axis.*
2. $y^2 = x; y = \frac{1}{2}x$; *about the y-axis.*

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Examples

Problem 10

Find the volume of the solid obtained by rotating the region bounded by the given curves about the specified line. Sketch the region and a typical disk/washer.

1. $y = x; y = \sqrt{x}$ *about* $y = 1$.
2. $y = \ln x; x = 0$; *on the interval* $0 \leq y \leq 1$; *about* $x = -1$.

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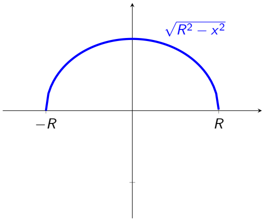
Volume of a Sphere

Problem 11

Show the volume of a sphere of radius R is:

$$V = \frac{4}{3}\pi R^3$$



Rotate $\sqrt{R^2 - x^2}$ about the x-axis, by the disk method:

$$V = \int_{-R}^{R} \pi(\sqrt{R^2 - x^2})^2 \, dx = 2\pi \int_0^R (R^2 - x^2) \, dx$$

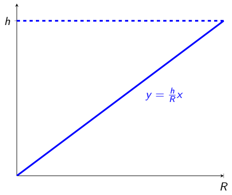
$$= 2\pi \left[R^2 x - \frac{1}{3}x^3\right]_0^R = 2\pi \left[\frac{2}{3}R^3\right] = \frac{4}{3}\pi R^3$$

Volume of a Cone

Problem 12

Show the volume of a cone of base radius R and height h is:

$$V = \frac{h}{3}\pi R^2$$



Rotate $x = \frac{R}{h}y$ about the y-axis, by the disk method:

$$V = \int_0^h \pi \left(\frac{R}{h}y\right)^2 \, dy = \pi \frac{R^2}{h^2}\left[\frac{1}{3}y^3\right]_0^h$$

$$= \pi \frac{R^2}{h^2}\left[\frac{1}{3}h^3\right] = \frac{h}{3}\pi R^2$$

Volume of a Wine Barrel

Problem 13

The sides of a wine barrel can be approximated by the parabola:

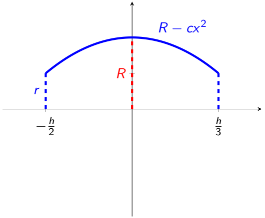
$$y = R - cx^2;\ -\frac{h}{2} \le x \le \frac{h}{2}$$

where R is the maximum radius, h is the height, and $c > 0$ a constant. Show that the volume is:

$$V = \frac{\pi h}{3}\left[2R^2 + r^2 - \frac{2}{5}(R-r)^2\right]$$

where r is the minimum radius.

Finding the Volume



Rotate $y = R - cx^2$ about the x-axis. By the Disk Method:

$$
\begin{aligned}
V &= \int_{-h/2}^{h/2} \pi(R - cx^2)^2\ dx = 2\pi \int_0^{h/2} (R - cx^2)^2\ dx \\
&= 2\pi \int_0^{h/2} (R^2 - 2Rcx^2 + c^2x^4)\ dx \\
&= 2\pi \left[R^2x - \frac{2Rc}{3}x^3 + \frac{c^2}{5}x^5 \right] \\
&= 2\pi \left[R^2(h/2) - \frac{2Rc}{3}(h/2)^3 + \frac{c^2}{5}(h/2)^5 \right] \\
&= \pi R^2 h - \frac{Rc\pi}{6}h^3 + \frac{c^2\pi}{80}h^5
\end{aligned}
$$

cont.

$$V = \pi h \left(R^2 - \frac{Rc}{6} h^2 + \frac{c^2}{80} h^4 \right) = \pi h \left(R^2 - \frac{2R}{3} \left(\frac{ch^2}{4} \right) + \frac{1}{5} \left(\frac{ch^2}{4} \right)^2 \right)$$

Let $d = \frac{ch^2}{4} = R - r$:

$$V = \pi h \left(R^2 - \frac{2R}{3} d + \frac{1}{5} d^2 \right) = \frac{\pi h}{3} \left(3R^2 - 2Rd + \frac{3}{5} d^2 \right)$$

$$= \frac{\pi h}{3} \left(2R^2 + R^2 + d^2 - \frac{2}{5} d^2 - 2Rd \right)$$

$$= \frac{\pi h}{3} \left(2R^2 - \frac{2}{5} d^2 + R^2 - 2Rd + d^2 \right) = \frac{\pi h}{3} \left(2R^2 - \frac{2}{5} d^2 + (R - d)^2 \right)$$

$$= \frac{\pi h}{3} \left(2R^2 + r^2 - \frac{2}{5} d^2 \right) = \frac{\pi h}{3} \left(2R^2 + r^2 - \frac{2}{5} (R - r)^2 \right)$$