MA 16200: Plane Analytic Geometry and Calculus II

Lecture 9: Intro to Trigonometric Integrals

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Sections Covered: 8.3

Basics

- $\int \sin x = -\cos x + C$
- $\int \cos x = \sin x + C$
- $\int sec^2 x \ dx = \tan x + C$
- $\int \csc x \cot x \ dx = -\csc x + C$
- $\int \sec x \tan x \ dx = \sec x + C$
- $\int \csc^2 x \ dx = -\cot x + C$
- $\int \tan x = \ln|\sec x| + C$
- $\int \csc x = -\ln|\csc x + \cot x| + C$
- $\int \sec x = \ln|\sec x + \tan x| + C$
- $\int \cot x = \ln|\sin x| + C$

Basic Idea

We want to integrate functions like:

- $\int \sin^3 \, dx$
- $\int \sin^5 x \cos^3 x \, dx$
- $\int \tan^3 x \sec x \, dx$
- $\int \csc^5 x \cot^2 \, dx$

The main idea: Use trig identities to simplify the integrand until we can use u-substitution.

Pythagorean Identity

Problem 1

Compute $\int \sin^3 x \, dx$

Another Example

Problem 2

Compute $\int \cos^5 x \; dx$

Power Reduction Formulas

Problem 3

Compute $\int \sin^4 x \, dx$

When sine has an odd power

Problem 4

Compute $\int \sin^5 x \cos^2 x \, dx$

When cosine has an odd power

Problem 5

Compute $\int \sin^{-\frac{3}{2}} x \cos^3 x \; dx$

When they both have even powers

Problem 6

Compute $\int \sin^4 x \cos^2 x \, dx$

General Strategy

$\int \sin^m x \cos^n x \; dx$	**Strategy**
m odd and positive, n real	Split off $\sin x$, rewrite the resulting even power of $\sin x$ in terms of $\cos x$, then use $u = \cos x$.
n odd and positive, m real	Split off $\cos x$, rewrite the resulting even power of $\cos x$ in terms of $\sin x$, then use $u = \sin x$.
m and n are both even, non-negative integers	Use half-angle formulas to transform the integrand into a polynomial of $\cos 2x$, then apply the preceding strategies once again to powers of $\cos 2x$ greater than 1.

Reduction Formula (for sine) Derivation

Problem 7

Recursively compute $\int \sin^n x \, dx$

Reduction Formulas

Theorem 8

Assume n is a positive integer:

1. $\int \sin^n x \, dx = -\frac{\sin^{n-1} x \cos x}{n} + \frac{n-1}{n} \int \sin^{n-2} x \, dx$
2. $\int \cos^n x \, dx = \frac{\cos^{n-1} x \sin x}{n} + \frac{n-1}{n} \int \cos^{n-2} x \, dx$
3. $\int \tan^n x \, dx = \frac{\tan^{n-1} x}{n-1} - \int \tan^{n-2} x \, dx$, *provided* $n \neq 1$
4. $\int \sec^n x \, dx = \frac{\sec^{n-2} x \tan x}{n-1} + \frac{n-2}{n-1} \int \sec^{n-2} x \, dx$, *provided* $n \neq 1$

Powers of Tangent

Problem 9

Compute $\int \tan^3 x \, dx$

Factoring out $\sec^2 x$

Problem 10

Compute $\int \tan^6 x \sec^4 x \, dx$

Odd power of sec x

Problem 11

Compute $\int \tan^5 x \sec^7 x \, dx$

General Strategy

$\int \tan^m x \sec^n x \, dx$	**Strategy**
n even, m real	Split off $\sec^2 x$, rewrite the resulting even power of $\sec x$ in terms of $\tan x$, then use $u = \tan x$.
m odd and positive, n real	Split off $\sec x \tan x$, rewrite the resulting even power of $\tan x$ in terms of $\sec x$, then use $u = \sec x$.
m even and positive, n odd and positive	Rewrite the even power of $\tan x$ in terms of $\sec x$ to produce a polynomial of $\sec x$, then apply Reduction Formula 4 to each term.

Cotangent

Similar strategies can be used when the integrand is powers of cotangent and cosecant.

Problem 12

Compute $\int \cot^4 x \, dx$

Square Roots

Problem 13

Compute $\int \sqrt{1 - \sin x}\ dx$

Powers of Secant

Problem 14

Compute $\int \sec^3 x \, dx$

Product to Sum Formulas

Theorem 15

- $\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$
- $\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$
- $\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$

Using the Product to Sum Formulas

Problem 16

Compute $\int \sin 4x \cos 5x \; dx$