MA 16200: Plane Analytic Geometry and Calculus II

Lecture 17: Series

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Sections Covered: 10.3

Review

Recall an infinite series takes the form:

$$\sum_{n=1}^{\infty} a_n$$

The series converges when the sequence of partial sums $\{S_N\}$ converges. I.e., :

$$\lim_{N\to\infty} S_n \stackrel{def}{=} \lim_{N\to\infty} \sum_{n=1}^{N} a_n = L$$

for some real number L. In that case, the series is equal L. Otherwise, it diverges.

Geometric Series Definition

Definition 1

A **geometric sum** is a sum of the form:

$$S_N = \sum_{n=0}^{N-1} ar^n = \sum_{n=1}^{N} ar^{n-1} = a(1 + r + r^2 + r^3 + \ldots + r^{N-1})$$

where $a \neq 0$ and r a real number. The number r is called the **common ratio**.

We eventually want to talk about the **geometric series** $\sum_{n=0}^{\infty} ar^n$

Examples

- Examples
  - $0.99999 = \sum_{n=1}^{5} \frac{9}{10}\left(\frac{9}{10}\right)^{n-1}$
  - $\sum_{n=0}^{9} 3^n$
  - $\sum_{n=1}^{3} 2\left(-\frac{3}{4}\right)^{n-1}$
  - $\sum_{n=0}^{\infty} 2^{-2n}5^{n+1}$
- Non-Examples
  - $\sum_{k=1}^{\infty} \frac{3}{k^2+5k+4}$
  - $\sum_{i=1}^{12} i$
  - $\sum_{n=1}^{\infty} \frac{x^n}{n!}$

Partial Sum Formula

For $S_N = \sum_{n=0}^{N-1} ar^n$, can we find an explicit formula for S_N?

Value of a Geometric Series

When is $\sum_{n=0}^{\infty} ar^n < \infty$?

Geometric Series Formula

Theorem 2 (Convergence of a Geometric Series)

Let $a \neq 0$ and r be real numbers.
If $|r| < 1$, then

$$\sum_{n=0}^{\infty} ar^n = \frac{a}{1-r}$$

If $|r| \geq 1$,

$$\sum_{n=0}^{\infty} ar^n \text{ diverges}$$

Example

Problem 3

Compute $\sum_{n=0}^{\infty} 5\left(-\frac{2}{3}\right)^n$, *or show that it diverges.*

Example

Problem 4

Compute $\sum_{n=1}^{\infty} \left(\frac{\pi}{e}\right)^{n-1}$*, or show that it diverges.*

Example

Problem 5

Compute $\sum_{n=1}^{\infty} 2^{2n}3^{1-n}$*, or show that it diverges.*

Repeating Decimals

Problem 6

Convert the repeating decimal $1.\overline{2} = 1.222...$ *into a fraction.*

Repeating Decimals

Problem 7

Convert the repeating decimal $2.3\overline{17} = 2.3171717\ldots$ *into a fraction.*

Sneak Peek into Power Series

Problem 8

Let f be the following function of x:

$$f(x) = \sum_{n=0}^{\infty} (-1)^n x^n$$

Find the domain and range of f.

Another Example

Problem 9

Let f be the following function of x:

$$f(x) = \sum_{n=0}^{\infty} (2x-1)^n$$

Find the domain of f.

Telescoping Series

Definition 10

Telescoping Series take the form:

$$\sum_{n=1}^{\infty}[f(n) - f(n+1)]$$

for some function f

Examples:

- $\sum_{k=1}^{\infty}\left(\frac{1}{k} - \frac{1}{k+1}\right)$
- $\sum_{n=0}^{\infty}\left(e^{-n} - e^{-(n+1)}\right)$

Partial Sums of Telescoping Series

When does $\sum_{n=1}^{\infty}[f(n) - f(n+1)]$ converge?

Convergence of Telescoping Series

Theorem 11 (Convergence of Telescoping Series)

If $f(n) \to L$, *then:*

$$\sum_{n=1}^{\infty}[f(n) - f(n+1)] = f(1) - L$$

Otherwise,

$$\sum_{n=1}^{\infty}[f(n) - f(n+1)] \textit{ diverges}$$

Example

Problem 12

Compute $\sum_{n=1}^{\infty} \left[\cos\frac{1}{n} - \cos\frac{1}{n+1}\right]$*, or show that it diverges.*

Example

Problem 13

Compute $\sum_{n=3}^{\infty} \frac{1}{(n-2)(n-1)}$*, or show that it diverges.*

Example

Problem 14

Compute $\sum_{n=1}^{\infty} \ln\left(1+\frac{1}{n}\right)$, *or show that it diverges.*

Properties of Convergent Series

Theorem 15

Let $\sum a_n$ *and* $\sum b_n$ *both be convergent series, then*

- *For any number* c, $\sum ca_n = c\sum a_n$;
- $\sum(a_n \pm b_n) = \sum a_n \pm \sum b_n$

Theorem 16

If $\sum a_n$ *diverges,*

- *For* $c \neq 0$, $\sum ca_k$ *diverges.*
- *If* $\sum b_n$ *converges,* $\sum(a_n \pm b_n)$ *diverges.*

Remark

If $\sum a_n$ and $\sum b_n$ both diverge, nothing can be said about $\sum(a_n \pm b_n)$.

- $\sum a_n = \sum 1$; $\sum b_n = \sum(-1)$; $\sum(a_n + b_n) = 0$
- $\sum a_n = \sum 1$; $\sum b_n = \sum 1$; $\sum(a_n - b_n) = 0$
- $\sum a_n = \sum 1$; $\sum b_n = \sum 1$; $\sum(a_n + b_n)$ diverges
- $\sum a_n = \sum 1$; $\sum b_n = \sum(-1)$; $\sum(a_n - b_n)$ diverges

Example

Problem 17

Compute $\sum_{n=1}^{\infty}\left(\frac{3}{n(n+1)}+\frac{1}{2^n}\right)$

Tails

Theorem 18

If M is a positive integer, then: $\sum_{n=1}^{\infty} a_n$ and $\sum_{n=M}^{\infty} a_n$ either both converge or both diverge.

In general, when determining convergence, adding or removing finitely many terms does not change anything.

$$\sum_{n=1}^{\infty} a_n = \underbrace{\sum_{n=1}^{M} a_n}_{\text{First } M \textbf{ leading terms}} + \underbrace{\sum_{n=M+1}^{\infty} a_n}_{M\textbf{-tail}}$$

However, the *value* of the series does change if non-zero terms are added or removed.