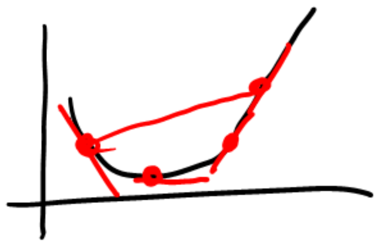
Goal: Use the 2nd derivative to determine properties of the original function.

Concave Upward [Convex]



f' is increasing.

$[f']' = f'' > 0$

Concave Downward

f' is decreasing,

$f'' < 0$

Theorem (Concavity Test) Let f be a fcn where f'' exists on an open interval I

① If $f'' > 0$ for all x in I, f is $\underbrace{\text{concave upward}}_{CU}$ on I

② If $f'' < 0$ for all x in I, f is $\underbrace{\text{concave downward}}_{CD}$ on I.

Ex) Discuss the shape of $f(x) = x^3 - 12x + 1$

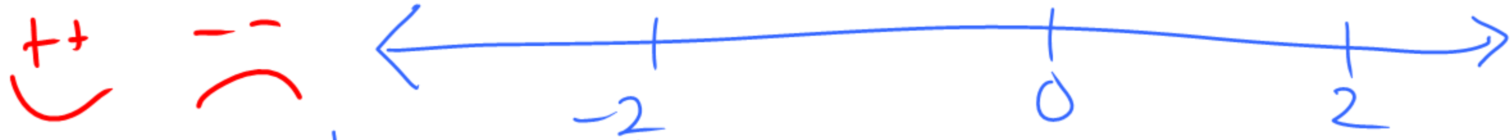
Step 1 Find when $f' = 0$ and $f'' = 0$

$$f'(x) = 3x^2 - 12 = 3(x-2)(x+2) \overset{\text{set}}{=} 0$$

$\Rightarrow x = -2, 2$

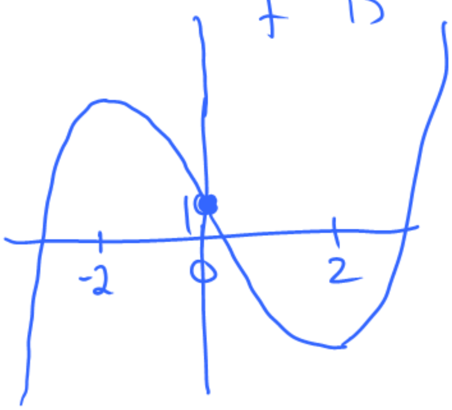
$$f''(x) = 6x \overset{\text{set}}{=} 0 \Rightarrow x = 0$$

Step 2 Make sign chart for both f' and f''



Test Points:	-100	-1	1	100
Sign of $3(x-2)(x+2)$	$+$	$-$	$-$	$+$
Sign of $6x$	$-$	$-$	$+$	$+$
Results of I/D Test	Inc	Dec	Dec	Inc
Results of C Test	CD	CD	CU	CU

Conclusion: f is inc and CD on $(-\infty,-2)$
f is dec and CD on $(-2,0)$
f is dec and CU on $(0,2)$
f is inc and CU on $(2,\infty)$

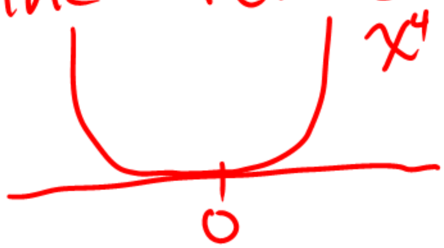


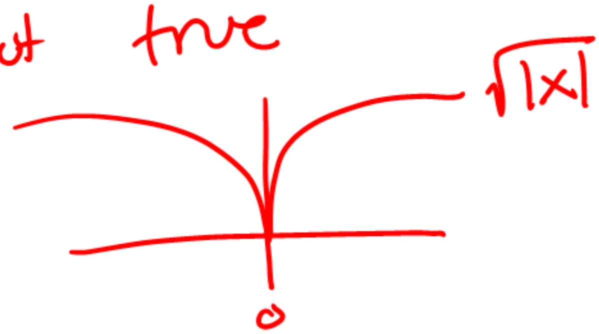
Def If f changes concavity at a point p, then p is an inflection point of f

Theorem If p is an inflection point, either $f''(p)=0$ or undefined

Why? Inflection points are crit. nums. of f'

NOTE The reverse is not true





Ex2/ Determine the location of the inflection points for $f(x) = 3x^4 - 4x^3 - 6x^2 + 12x + 1$

① Find when $f''(x) = 0$

$$f'(x) = 12x^3 - 12x^2 - 12x + 12$$

$$f''(x) = 36x^2 - 24x - 12 = 12(3x^2 - 2x - 1)$$

$$= 12(x-1)(3x+1) \underset{=}{\overset{\text{set}}{}} 0$$

Either $\cancel{12=0}$ OR $x - 1 = 0$ OR $3x + 1 = 0$

$\boxed{x = 1}$ $\quad$ $\boxed{x = -\frac{1}{3}}$

② Make sign chart for f''

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$-\frac{1}{3}$ $\quad$ 1

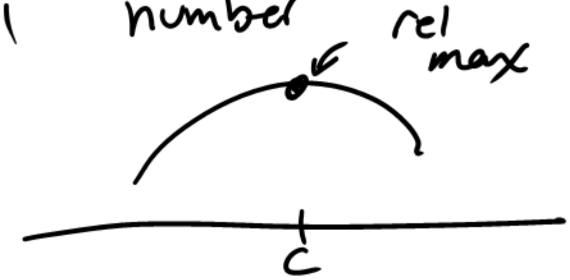
Test Points	-2	0	2
Sign of 12	+	+	+
Sign of $x-1$	−	−	+
Sign of $3x+1$	−	+	+
Sign of f''	+	−	+
Result of C Test	CU	CD	CU

Conclusion: Both $x = -\frac{1}{3}$ and $x = 1$ are inflection points.

To determine the 2D inflection point, find $\left(-\frac{1}{3}, f\left(-\frac{1}{3}\right)\right)$, $(1, f(1))$

The $2^{nd} \frac{d}{dx}$ can be used to detect rel. max/min

Let c be a critical number



$f''(c) > 0 \Rightarrow f$ is CU near c

$f''(c) < 0 \Rightarrow f$ is CD near c

<u>Theorem (2nd Derivative Test)</u> Let f'' be continuous near c and $f'(c) = 0$

① If $f''(c) > 0$, f has a rel. min at c.

② If $f''(c) < 0$, f has a rel. max at c.

③ If $f''(c) = 0$, then the <u>test is inconclusive.</u> (ie, pick another test)

Ex3/ Locate rel. extrema for $f(x) = x^3 - 3x^2$

<u>Step1</u> Find crit. nums.

$$f'(x) = 3x^2 - 6x = 3x(x-2) \overset{\text{Set}}{=} 0$$

Either $3x = 0$ OR $x - 2 = 0$

$\boxed{x=0}$ $\boxed{x=2}$

<u>Step2</u> Compute the sign of $f''(c)$ for all crit. nums. c.

$$f''(x) = 6x - 6 = 6(x-1)$$

$$f''(0) = 6(-1) = -6 < 0$$

$$f''(2) = 6(1) = 6 > 0$$

Step 3 State Conclusion

Conclusion: f has a rel. min. at $x=2$, while f has a rel. max at $x=0$

Ex 4 Repeat for $f(x)=e^{x}(x-7)$

① Find Crit. Nums.

$$f'(x)=e^{x}(x-7)+e^{x}\cdot 1=e^{x}(x-7+1)=\boxed{e^{x}(x-6)}$$

$$\overset{\text{set}}{=}0$$

$\Rightarrow x-6=0 \Rightarrow \boxed{x=6}$

② Compute the sign of $f''(6)$

$$f''(x)=e^{x}(x-6)+e^{x}=e^{x}(x-5)$$

$$f''(6)=e^{6}(6-5)=e^{6}>0$$

$++$

Conclusion: f has a rel. min at $x=6$

Optionally: The minimum value is $f(6)=-e^{6}$

Ex 5 Repeat for $f(x)=\dfrac{x}{x^2+4}$

$$f'(x)=\frac{x^2+4-x(2x)}{(x^2+4)^2}=\frac{4-x^2}{(x^2+4)^2}\overset{\text{set}}{=}0$$

$\Rightarrow 4-x^2=0 \Rightarrow (x+2)(x-2)=0 \Rightarrow \boxed{x=-2,2}$

② Compute the sign of $f''(-2)$ and $f''(2)$

$$f''(x) = \frac{(-2x)(x^2+4)^2 - (4-x^2)\,2(x^2+4)(2x)}{(x^2+4)^4}$$

$$= x\underbrace{\left[\frac{-2(x^2+4)^2}{(x^2+4)^4}\right]}_{\text{Always Negative}} - \overbrace{(4-x^2)}^{0 \text{ when } x=\pm 2}\frac{2(x^2+4)(2x)}{(x^2+4)^4}$$

$f''(-2) > 0 \Rightarrow f$ has a rel. min at $x=-2$

$f''(2) < 0 \Rightarrow f$ has a rel. max at $x=2$