Goal: Differentiate functions of the form $F(x, y) = 0$. Use this to find the derivative of conic sections and inverse trigonometric functions.

Recall ① If y is a function of x,

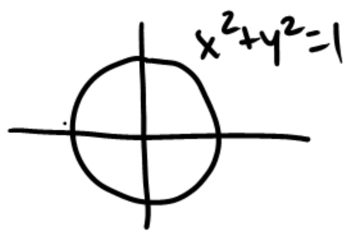
- $\frac{d}{dx}(y^2) = 2y\frac{dy}{dx}$ $\quad [y(x)]^2$
- $\frac{d}{dx}[\ln(y)] = \frac{1}{y} \cdot \frac{dy}{dx}$
- $\frac{d}{dx}(\sin(y)) = \cos(y) \cdot \frac{dy}{dx}$

② Taking the derivative does not change equality

$$\frac{d}{dx}(y-2x-1) = \frac{d}{dx}0$$
$$y' - 2 = 0$$
$$y' = 2$$

$$y = 2x+1$$
$$y' = 2$$

Q: How can we find the slope of a tangent line to the unit circle?



A: Recognize y is a function of x

$$\frac{d}{dx}(x^2+y^2) = \frac{d}{dx}(1)$$
$$2x + 2y\frac{dy}{dx} = 0$$
$$2y\frac{dy}{dx} = -2x$$
$$\frac{dy}{dx} = -\frac{2x}{2y} = -\frac{x}{y}$$

Q: Can we write $\frac{dy}{dx}$ solely as a function of x?

A: Sometimes!

$$x^2+y^2=1$$
$$y^2=1-x^2$$
$$y=\pm\sqrt{1-x^2}$$

$$\frac{dy}{dx}=\begin{cases} -\frac{x}{\sqrt{1-x^2}} & \text{if } y>0 \\ -\frac{x}{(-\sqrt{1-x^2})} & \text{if } y<0 \end{cases}$$

$x^2+y^2=1$ is an example of an <u>implicit</u> function of x. Finding $\frac{dy}{dx}$ is called <u>implicit differentiation.</u>

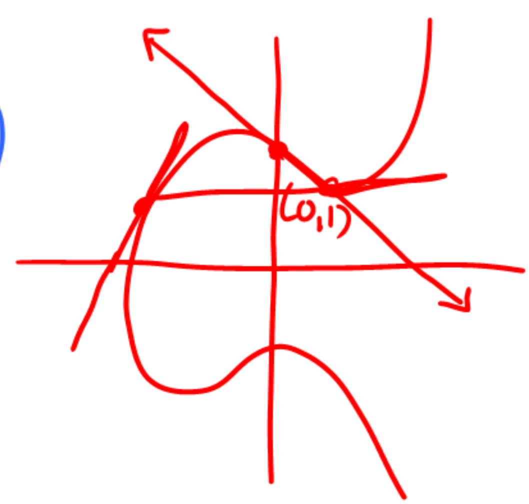
Ex/

<u>Explicit Functions</u>	<u>Implicit Functions</u>
$y=f(x)$	$F(x,y)=0$
$y=2x+1$	$x^2+y^2-1=0$
$y=\sin x$	$\sin(xy)=0$
$y=\ln(x^2)\cdot\cos x$	$e^{xy}+x^2=0$

Ex/ Find $\frac{dy}{dx}$ given the elliptic curve

$$y^2=x^3-x-1$$

at $(x,y)=(0,1)$



$$\frac{d}{dx}(y^2)=\frac{d}{dx}(x^3-x-1)$$
$$2y\frac{dy}{dx}=3x^2-1$$
$$\frac{dy}{dx}=\frac{3x^2-1}{2y}$$
$$\left.\frac{dy}{dx}\right]_{(0,1)}=\frac{3(0)^2-1}{2(1)}=\boxed{-\frac{1}{2}}$$

Ex2/ Find $\frac{dy}{dx}$ for $\sin(xy) = x$

$$\frac{d}{dx}\left(\sin(xy)\right) = \frac{d}{dx}(x)$$

$$(xy)' = (x)'\,y + x\,(y)'$$
$$= y + x\frac{dy}{dx}$$

$$\cos(xy)\cdot[xy]' = 1$$

$$\cos(xy)\left[y + x\frac{dy}{dx}\right] = 1$$

$$1 = \underbrace{y\cos(xy)} + x\cos(xy)\frac{dy}{dx}$$

$$1 - y\cos(xy) = \underbrace{x\cos(xy)}\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1 - y\cos(xy)}{x\cos(xy)}$$

Ex3/ Find $\frac{dy}{dx}$ given $xe^{y} = 9y$

$$\frac{d}{dx}\left(xe^{y}\right) = \frac{d}{dx}(9y)$$

$$e^{y} + xe^{y}\cdot\frac{dy}{dx} = 9\frac{dy}{dx}$$

$$e^{y} = 9\frac{dy}{dx} - xe^{y}\frac{dy}{dx}$$

$$e^{y} = (9 - xe^{y})\frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{e^{y}}{9 - xe^{y}}$$

Ex4/ Find $\frac{dy}{dx}$ given $\ln\left(\frac{x}{y}\right) = 6x$

$$\frac{d}{dx}\left(\ln(x) - \ln(y)\right) = \frac{d}{dx}\left(6x\right)$$

$$y\left(\frac{1}{x} - \frac{1}{y}\cdot\frac{dy}{dx}\right) = (6)\,y$$

$$\frac{y}{x} - \frac{dy}{dx} = 6y$$

$$-\frac{dy}{dx} = 6y - \frac{y}{x}$$

$$\frac{dy}{dx} = \frac{y}{x} - 6y$$

Ex/ Find $\frac{dy}{dx}$ given the algebraic curve

$$\frac{d}{dx}\left(y^3 + yx^2\right) = \frac{d}{dx}\left(x^3 + xy^2\right)$$

$$3y^2\frac{dy}{dx} + \frac{dy}{dx}x^2 + y(2x) = 3x^2 + y^2 + x(2y)\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} + x^2\frac{dy}{dx} + 2xy = 3x^2 + y^2 + 2xy\frac{dy}{dx}$$

$$3y^2\frac{dy}{dx} + x^2\frac{dy}{dx} - 2xy\frac{dy}{dx} = 3x^2 + y^2 - 2xy$$

$$\left(3y^2 + x^2 - 2xy\right)\frac{dy}{dx} = 3x^2 + y^2 - 2xy$$

$$\frac{dy}{dx} = \frac{3x^2 + y^2 - 2xy}{3y^2 + x^2 - 2xy}$$

Derivatives of Inverse Trig Functions

Ex 6/ Compute $\frac{dy}{dx}$ if $y = \sin^{-1}(x)$

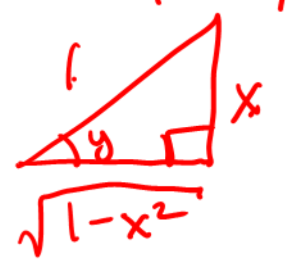
$$\sin(y) = \sin(\sin^{-1}x)$$

$$\frac{d}{dx}\left(\sin(y)\right) = \frac{d}{dx}(x)$$

$$\cos(y) \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\cos(y)} = \frac{1}{\sqrt{1-x^2}}$$

$\sin(y) = \frac{x}{1}$



Ex 7/ Repeat for $y = \tan^{-1}x$

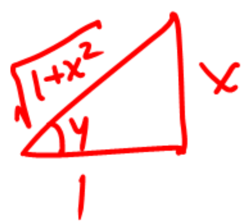
$$\frac{d}{dx}\left(\tan(y)\right) = \frac{d}{dx}(x)$$

$$\sec^2(y) \cdot \frac{dy}{dx} = 1$$

$$\frac{dy}{dx} = \frac{1}{\sec^2(y)}$$

$$\frac{dy}{dx} = \frac{1}{1+x^2}$$

$\tan(y) = \frac{x}{1}$



Other Examples

Recall $\frac{d}{dx}\left(\frac{1}{x}\right) = \frac{d}{dx}(x^{-1}) = -x^{-2}$

$= -\frac{1}{x^2}$

Ex/ $\frac{1}{x} + \frac{1}{y} = 1$

$$\frac{d}{dx}\left(\frac{1}{x} + \frac{1}{y}\right) = \frac{d}{dx}(1)$$

$$-\frac{1}{x^2} - \frac{1}{y^2} \cdot \frac{dy}{dx} = 0$$

$$-\frac{1}{y^2}\frac{dy}{dx} = \frac{1}{x^2}$$

$$\frac{dy}{dx} = -\frac{y^2}{x^2}$$

$$\frac{d}{dx}(\sqrt{x}) = \frac{1}{2\sqrt{x}}$$

Ex/ $$\frac{d}{dx}\left(\sqrt{x} + \sqrt{y}\right) = \frac{d}{dx}(4)$$

$$\frac{1}{2\sqrt{x}} + \frac{1}{2\sqrt{y}}\cdot\frac{dy}{dx} = 0$$

$$\frac{1}{2\sqrt{y}}\frac{dy}{dx} = -\frac{1}{2\sqrt{x}}$$

$$\frac{dy}{dx} = -\frac{2\sqrt{y}}{2\sqrt{x}} = -\sqrt{\frac{y}{x}}$$