MA 16200: Plane Analytic Geometry and Calculus II

Lecture 20: The Alternating Series Test

Zachariah Pence

Purdue University

Sections Covered: 10.6

Motivation

1. We have dealt with series where the terms are always positive (integral/comparison tests). But how can we deal with this series?
$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n}$$
2. Are there some series $\sum a_n$ where $\lim_{n\to\infty} a_n = 0$ implies the series converges?

Alternating Series Test

Theorem 1 (Alternating Series Test)

The alternating series $\sum(-1)^{n+1}a_n$ *converges if:*

1. *The terms* a_n *are non-increasing in magnitude (eventually):*

$$a_k \geq a_{k+1} > 0 \text{ for } k \text{ greater than some index } N$$

2. $\lim_{n\to\infty} a_n = 0$

Why? Use Monotone Convergence on S_{2N} and S_{2N+1} (see p.g. 689 of textbook)

The Alternating Harmonic Series

Theorem 2

*The **alternating harmonic series** converges. Moreover,*

$$\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = \ln 2$$

Why does it converge?

Example

Problem 3

Determine whether $\sum_{n=1}^{\infty} \frac{\cos \pi n}{n^{3/4}}$ *converges or diverges. State the test used.*

Example

Problem 4

Determine whether $\sum_{n=1}^{\infty} \frac{(-1)^n 3n}{4n-1}$ *converges or diverges. State the test used.*

Example

Problem 5

Determine whether $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{n^2}{n^3+1}$ *converges or diverges. State the test used.*

Example

Problem 6

Determine whether $\sum_{n=2}^{\infty}(-1)^n \frac{\ln n}{n}$ *converges or diverges. State the test used.*

Error Bound Derivation

For a series $\sum(-1)^{n-1}a_n$, what is a bound for $|R_N|$?

Error Bound Formula

Theorem 7 (Remainder in Alternating Series)

Let $\sum_{n=1}^{\infty}(-1)^{n+1}a_n$ be a convergent alternating series converging to S. Let $R_N = S - S_N = \sum_{n=N+1}^{\infty}(-1)^{n+1}a_n$ be the remainder in approximating S by the sum of the first N terms. Then:

$$|R_N| \leq a_{n+1}$$

In other words, the magnitude of the remainder is less than or equal to the magnitude of the first neglected term.

Approximating $\ln 2$

Problem 8

Recall $\sum_{n=1}^{\infty}(-1)^{n+1}\frac{1}{n} = \ln 2$*. How many terms of the series are required to approximate* $\ln 2$ *with an error less than* $\varepsilon = 10^{-6}$*?*

Approximating e^{-1}

Problem 9

Approximate $\frac{1}{e} = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!}$ *accurate to 3 decimal places.*

Approximating π

Problem 10

Leibniz's formula for π *(Proved in §11.2) states that:*

$$\pi = \sum_{n=0}^{\infty} (-1)^n \frac{4}{2n+1}$$

Bound the error for the approximation $\pi \approx \sum_{n=0}^{8} (-1)^n \frac{4}{2n+1}$

Definition

Definition 11

1. If $\sum |a_n|$ converges, we say $\sum a_n$ is **absolutely convergent**, or $\sum |a_n|$ **converges absolutely**.
2. If $\sum |a_n|$ diverges and $\sum a_n$ converges, we say $\sum a_n$ is **conditionally convergent**, or $\sum a_n$ **converges conditionally**.

The alternating harmonic series is conditionally convergent (Why?)

Example

Problem 12

Show $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n^2}$ *is absolutely convergent.*

Abs. Conv. Implies Convergence

Theorem 13 (Absolute Convergence Implies Convergence)

1. *If* $\sum |a_n|$ *converges, then* $\sum a_n$ *converges.*
2. *If* $\sum a_n$ *diverges, then* $\sum |a_n|$ *diverges.*

Why? $\sum(a_n + |a_n|) \leq 2\sum |a_n|$ converges by the comparison test. So,

$$\sum a_n = \sum(a_n + |a_n|) - \sum |a_n| < \infty$$

(2) is the contrapositive of (1).

Diagram

Example

Problem 14

Determine if $\sum_{n=1}^{\infty} \frac{\sin n}{n^2}$ *diverges, converges absolutely, or converges conditionally.*

Example

Problem 15

Determine if $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n}}$ *diverges, converges absolutely, or converges conditionally.*

Example

Problem 16

Determine if $\sum_{n=1}^{\infty} \frac{(-1)^{n+1} n}{n+1}$ *diverges, converges absolutely, or converges conditionally.*

Example

Problem 17

Determine if $\sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{\sqrt{n^3}}$ *diverges, converges absolutely, or converges conditionally.*

Example

Problem 18

Determine if $\sum_{n=1}^{\infty}(-1)^n \sin\frac{\pi}{n}$ *converges or diverges.*

Example

Problem 19

Determine if $\sum_{n=1}^{\infty}(-1)^n \frac{1}{n \ln n}$ *converges absolutely, conditionally, or diverges.*