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# The orbits of the satellites of Neptune

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**Abstract.** This article presents the results of a fit of numerically integrated Neptunian satellite orbits to Earth-based astrometric observations and early Voyager spacecraft observations. Ephemerides based on these orbits were used by the Voyager project as the final pre-encounter ephemerides. As a by-product of the orbit fits, estimates of the Neptune mass, the second zonal harmonic of Neptune, and the pole orientation of Neptune were also obtained.

**Key words:** astrometry – celestial mechanics – Nereid – satellites of Neptune – Triton

## 1. Introduction

On August 25, 1989 the Voyager 2 spacecraft flew through the Neptunian planetary system. The mission objectives included a number of scientific observations of the Neptunian satellites, Triton and Nereid. Success of the scientific measurements depended upon accurate navigation of the spacecraft and accurate pointing of the scientific instruments, both of which required precise knowledge of the locations of the planetary satellites, i.e., ephemerides. In order to provide the Voyager project with the necessary Neptunian satellite ephemerides, a model for the orbital motions of the satellites had to be developed and fit to observations.

Orbits were previously determined for Triton by Eichelberger (1926) and Harris (1984) and for Nereid by Rose (1974) and Veillet (1982, 1988). Both Eichelberger and Harris employed an inclined orbit precessing at a constant rate to represent Triton's motion. Eichelberger's orbit had a small eccentricity while Harris assumed a circular orbit. For Nereid's motion Rose used Keplerian elements, with independent semimajor axis and mean motion, while Veillet used the theory of Mignard (1981). Although these models are quite adequate for orbits fit to Earth-based data, they are not sufficiently complete nor precise enough to support Voyager navigation. For example, the Triton theories are kinematic in nature having no explicit dependence on dynamic parameters such as Neptune mass, and Mignard's Nereid theory is incomplete at the 100 km level needed for optical navigation. Consequently, numerical integration was adopted as the model for the orbits used to generate the Voyager satellite ephemerides. This approach proved highly successful during the Voyager Uranus encounter (Jacobson, 1986) and similar success was expected for the Neptune encounter.

This paper begins with a brief description of the mathematical model used in the integration and the integration method. It next discusses observation set and observation processing procedure. It closes with the results of the fit of the integration to the observations and an analysis of those results.

## 2. Model

The equations of motion used in the numerical integration, described in detail by Peters (1981), include the gravitational effects of an oblate primary, the mutual perturbations of the satellites, and the pertubations of external bodies, i.e., the planets of the solar system. The equations are formulated in Cartesian coordinates centered at the planetary system barycenter and referenced to the Earth mean equator and equinox of 1950.0 system.

The oblateness force depends upon the orientation of the pole of Neptune, a pole which is not fixed in inertial space but which precesses and nutates about the angular momentum vector of the Neptunian system. This polar motion is driven primarily by the torque due to the gravitational attraction of the Triton on the planet's equatorial bulge. When specifying the orientation of the pole, the nutation can for all practical purposes be neglected, but the precession cannot. Moreover, because of the latter's relatively short period of 600 years, Peters' representation of the pole right ascension and declination angles by linear functions of time is not valid over the time span of the observational data. Consequently, an alternative pole orientation model was adopted for use when numerically integrating the Neptunian satellite orbits. This model defines the pole as a unit vector precessing at constant rate about the system angular momentum vector. The model contains five parameters whose values must be determined: the right ascension and declination of the pole at some epoch, the right ascension and declination of the angular momentum vector, and the precession rate. The Appendix describes the model in more detail.

Since Nereid's effect on Triton is insignificant, it was assumed to be massless in order to eliminate computation of its attractive force and hence speed the integration. Triton's effect on Nereid, of course, was included.

To further simplify the integration process, the perturbations due to the planets were replaced by a pertubation due to a single fictitious body located at the solar barycenter and having a mass equal to the sum of the masses of the bodies interior to Neptune. This simplification, which has a negligible effect on the integration (causing changes of less than 200 m in Triton's orbit and less than 100 km in Nereid's orbit after an integration of 140 yr), was motivated by the observation that Neptune's orbital motion is nearly conic with respect to such a fictitious body. The masses and the Neptune solar system barycentric position were obtained from JPL planetary ephemeris DE 130 (Standish, 1987).

The integration included not only the satellite coordinates but also the partial derivatives of those coordinates with respect to a number of parameters to enable their values to be improved by the fit to the observations. The parameters were the epoch states and masses of both satellites, the mass of the Neptunian system (equivalent to the planet mass), the second zonal harmonic of Neptune's gravitational potential, the epoch values of the Neptune pole right ascension, declination, the pole precession rate, and the right ascension and declination of the precession axis.

The integration was carried out with a variable step size, variable order, Gauss-Jackson method with the step change and order selection algorithms due to Krogh (1973, 1974). An absolute truncation error limit of $10^{-11}\,\mathrm{km\,s^{-1}}$ imposed on the velocities controlled the integration step yielding an average step size of 8088 s. The maximum allowable order for any of the equations was 15, but the order at any step was automatically chosen to be compatible with the integration step size. Although the variational equations did not participate in the step size selection, their orders were varied as the step changed.

## 3. Observations

The astrometric observations of Triton, both photographic and visual, were obtained from the open literature (Alden, 1942, 1943; Crawford, 1928; Eichelberger, 1926; Greenwich Obs., 1902–1908, 1911; Harrington, 1984; Henry, 1884a, 1884b; Holden, 1876, 1877; Kostinsky, 1900, 1901; Lassell, 1851a, 1851b, 1853, 1865; Perrine, 1902; Perrotin, 1887; Newcomb, 1875; Pub. of USNO, 1911, 1929, 1953; Struve, 1894; Walker, 1978, 1988; Washington Obs., 1880–1882, 1885a, 1885b, 1887, 1889; Wirtz, 1905, 1910) as well as from private communications (Harrington, 1988; Taylor, 1988; Williams, 1988). The photographic observations consisted of 238 pairs of Neptune-relative differential right ascension and declination measurements, 571 position angle, and 572 separation distance measurements made during the period from February 1899 to October 1988. The visual observations included 2171 position angles and 2113 separation distances measured within the period of September 1847 to July 1947. The Earth-based data were supplemented with 56 imaging observations acquired by the Voyager spacecraft in the summer of 1988.

The photographic observation set included all of the observations used previously by Harris (1984); he did not use the visual observations. Newly processed observations were those published by Walker (1988), those privately communicated by Harrington, Taylor, and Williams, and those obtained by Voyager.

The observations of Nereid, all photographic, were made from May 1949 to July 1987, and included 64 pairs of Neptune-relative differential right ascension and declination measurements, 2 pairs of Triton-relative differential right ascension and declination measurements, and 7 pairs of absolute right ascension and declination measurements. These were all found in the open literature (Landgraf, 1988; Rose, 1974; Schaefer, 1988; van Biesbroeck, 1976; Veillet, 1982, 1988).

The astronometric observations were weighted by observer and opposition according to the RMS of observation residuals. Position angle weights were computed from

$$\sigma_p = \sigma_x / s$$

where $\sigma_x$ was the input weight and $s$ was the computed separation distance at the time of the observation. The Voyager observations were weighted at 0.5 pixel (about 1″).

## 4. Observation processing

### *4.1. Differential color refraction correction*

During his investigation of the orbit of Triton, Harris (1984) discovered that the post 1975 USNO observations of Triton relative to Neptune contained systematic errors. These errors were attributed to the effects of differences in astronomical refraction caused by the Triton aand Neptune color differences, effects which were magnified because of the low declination of the observations. Harris devised a differential refraction correction which removed most of the errors in the observations that he used.

For this paper a slightly different approach was taken to account for refraction. The total astronomical refraction was computed separately for Triton and Neptune and then explicitly differenced to obtain the differential refraction correction. The astronomical refraction was represented by a model developed by Garfinkel (1967) which gives the refraction as a function of five geophysical constants and five parameters. The former include the refractive index for the effective wavelength of the observations at standard temperature (273.15 K) and pressure (760 mm Hg), the radius of the Earth, the acceleration of gravity at sea level, the gas constant of air, and the height of the tropopause. The latter include the zenith distance, the temperature, pressure, and altitude at the observatory, and the assumed temperature gradient in the troposphere. The refractive index was determined from an expression given by Fukaya (1985), which defines the index in terms of effective wavelength, pressure of dry air, pressure of water vapor, and air temperature. In evaluating that expression, dry air was assumed (zero water vapor pressure). Table 1 provides the values for the various parameters used in the refraction computations.

Initially, the total refraction of each body was based on effective wavelengths provided by Harris. For observations acquired during the period of 1975 to 1984, this correction procedure worked well. However, for observations after 1984 the residuals exhibited a negative declination bias, suggesting that for some reason the corrections were no longer adequate. After further investigation, it was learned that the type of plates used in the

**Table 1.** Parameters in the refraction correction computation

| Parameter | Value |
|---|---|
| Radius of the Earth | 6378390 m |
| Acceleration of gravity | 9.80655 m/sec$^2$ |
| Gas constant of air | 287.053 m$^2$/sec$^2$/°C |
| Height of the tropopause | 11019 m |
| Temperature | 0.055 °C |
| Pressure | 574.445 mm Hg |
| Altitude | 2316 m |
| Temperature gradient | -0.0065 °C/m |

observations was changed in 1984, making it was necessary to determine new effective wavelengths.

Harris computed the effective wavelength for each body from its spectrum and the spectral response of the filter and plate combination used in making the observation. His values were:

Triton effective wavelength = 5684 Å

Neptune effective wavelength = 5597 Å

The Neptune spectrum was taken from Wamsteker (1973), and that of Triton was taken from Cruikshank (1979). The observations were made using a 3 mm GG 495 filter and plates with a IIaD emulsion (Walker, 1979), and the spectral response of this filter-plate combination was estimated from information obtained from Kodak and the Schott filter catalog.

All observations beginning in 1984 were made using a 3 mm GG 495 filter and plates with a IIIa-F emulsion. The spectral response for this filter-plate combination was obtained directly from R.L. Walker, and the effective wavelengths associated with it were found to be:

Triton effective wavelength = 6046 Å

Neptune effective wavelength = 5901 Å

### 4.2. *Estimation procedure*

For the observations of each Earth-based observer, Householder transformations (Lawson and Hanson, 1974) were used to pack the matrix of weighted observation partial derivatives and the weighted residual vector (actual minus computed observations) into an upper triangular square root information matrix and associated residual vector. This matrix and vector constitute the square root information array which is equivalent to the normal equations. Each column of the matrix and each element of the vector are associated with a particular satellite epoch state vector component or Neptunian system dynamical constant.

The Voyager data were processed with JPL's Optical Navigation and Orbit Determination Software to also obtain a square root information array. The data included not only the Triton imaging observations but also the radiometric spacecraft tracking data, and the information array contained information for the satellite related parameters, the Neptune planetary ephemeris parameters, and a number of spacecraft navigation parameters.

The square root information array for the complete data set was formed by combining the separate information arrays via Householder transformations. The solutions for the parameters (satellite states, dynamical constants, planetary ephemeris, and spacecraft navigation) were generated and analyzed by means of singular value decomposition techniques (Lawson and Hanson, 1974) applied to the composite square root information array.

When the observations were first processed, it was found that the postfit Earth-based data residuals, when grouped by observer and opposition, failed to exhibit zero mean statistics. Moreover, the means differed noticeably between observers. These results suggested the possibility of observer-dependent systematic errors in the observations. Analyses by Eichelberger and Newton (1926) had previously arrived at this same conclusion. To allow for systematic errors, the final processing included distinct biases for the observations of each observer. During the information array packing the observations were grouped into batches, where a batch contained all the observations for a particular opposition, and the biases were treated as batch sequential white noise parameters. At the start of processing for each batch, the portion of the information array corresponding to the biases was initialized with a priori uncertainties set to about twice the values of the residual means found during the preliminary fits. At the end of the processing for the batch, the accumulated information on the biases was saved for their later estimation. This procedure, discussed by Bierman (1978), is analogous to using distinct biases for each data batch.

The parameters in the precessing pole model were determined from a set of linear equality constraints which were imposed during the processing. The first two constraints were derived from linearizations of Eqs. (5) and (11) given in the Appendix. The remaining constraints were obtained by requiring Eqs. (16 and 17) of the Appendix to be satisfied in a least squares sense throughout the 140 yr data arc. A total of eight constraints were needed because the parameters to be determined included not only the five pole model parameters but also the inclination and node of Triton's mean orbit, and the magnitude of Triton's mean orbital angular momentum. The direct elimination technique described by Lawson and Hanson was used to introduce the constraints into the estimation procedure.

## 5. Processing results

The satellite epoch state vectors and system dynamical constants obtained from the fit to the observations are given in Tables 2 and 3. Eighteen digits are provided in the table entries in order to facilitate their use in future orbit integrations. The statistics appearing in Table 3 are actual statistics which represent the probable accuracy of the associated constant. Table 3 also contains the values of dynamical constants required in the integration but not estimated in the fit. Of these, the GM's of Triton and Nereid, the rotation period of Neptune, and the moment of inertia ratio of Neptune were treated as consider parameters (parameters not estimated but whose uncertainties affect the statistics of the estimated parameters) with the uncertainties presented in the table. The values of the unestimated parameters were those officially adopted by the Voyager project (Gerschultz, 1988).

For the Triton photographic observations, Figs. 1 and 2 give the right ascension and declination biases, which for the most part are less than 0.″01. Extremes for the USNO data include: a right ascension bias in 1977 of nearly 0.″05, a declination bias in 1981 of nearly 0.″08, and declination biases in 1987 and 1988 approaching −0.″025. Taylor's observations in 1988 exhibit a declination bias of about 0.″035. For the Triton position angle observations, both visual and photographic, the biases appear in

**Table 2.** Barycentric satellite state vectors at Julian ephemeris date 2447080.5 referred to the Earth mean equator and equinox of 1950.0

| Satellite | R(km) | V(km/sec) |
|---|---|---|
| Triton | 316189.885087448996 | 0.924045203790644080 |
| | 158244.787232475434 | -1.24562714299118278 |
| | 23036.0416651009713 | -4.10013261102776179 |
| Nereid | 4779370.16805217882 | -0.0207630977466919824 |
| | 4607135.52047063367 | 0.727501116888942569 |
| | 2673844.23838538887 | 0.373537695018904510 |

**Table 3.** Neptunian system dynamical constants

| Name | Value and Actual Uncertainty | Units |
|---|---|---|
| Neptune system GM | 6828017.86701104037 ± 11500.0 | ($km^3/sec^2$) |
| Triton GM‡ | 6185.0 ± 2850.0 | ($km^3/sec^2$) |
| Nereid GM‡ | 0.0 ± 1.0 | ($km^3/sec^2$) |
| External perturbing body GM | 132883680730.629637 | ($km^3/sec^2$) |
| Neptune $J_2$ | (3.70819375202521726 ± 0.180) × $10^{-3}$ | |
| Neptune radius | 25225 | (km) |
| Neptune pole right ascension† | 298.025418649317518 ± 1.2960 | (deg) |
| Neptune pole declination† | 40.6610939677269056 ± 1.1730 | (deg) |
| Precession axis right ascension | 298.298091476319170 ± 1.5383 | (deg) |
| Precession axis declination | 42.5111495984570507 ± 1.5805 | (deg) |
| Precession rate | 54.2996274463084697 ± 4.1342 | (deg/cent) |
| Neptune period of rotation‡ | 18.0 ± 1.0 | (hr) |
| Neptune moment of inertia ratio‡ | 0.26 ± 0.01 | |

† At Julian ephemeris date 2447763.5 (Voyager encounter date)
‡ Considered, not estimated

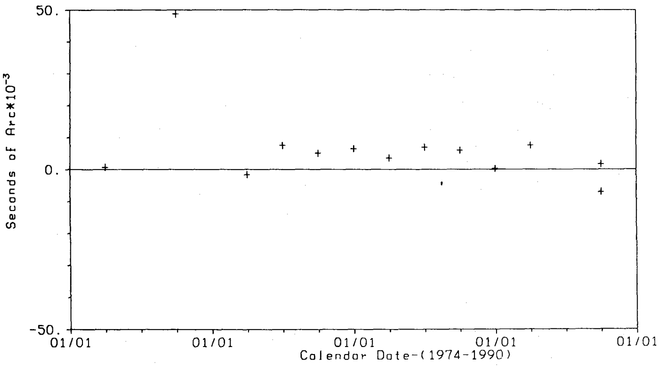


**Fig. 1.** Triton right ascension biases

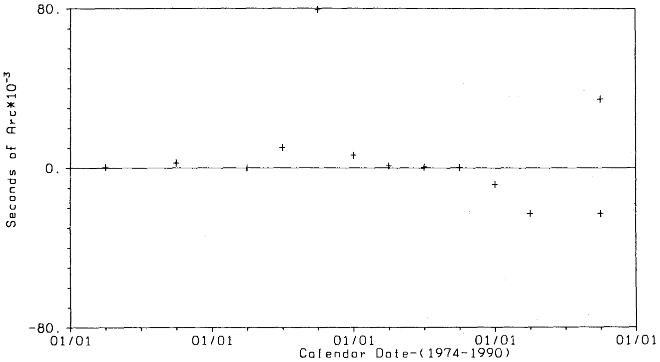


**Fig. 2.** Triton declination biases

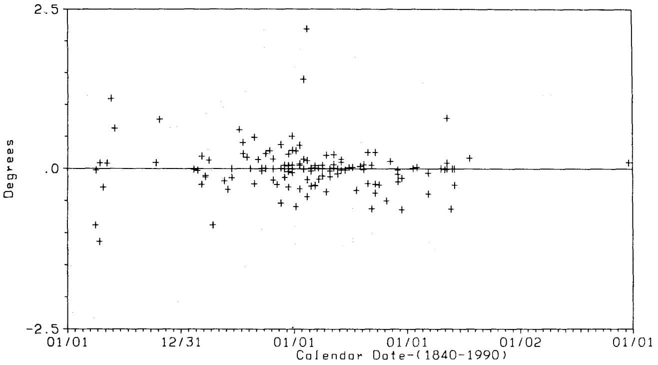


**Fig. 3.** Triton position angle biases

Fig. 3. Although several occur in the range from 1°.0 to 2°.5, the ensemble is nearly zero mean with a standard deviation of about 0°.4. The Triton separation distance biases, again for both visual and photographic observations, are shown in Fig. 4. The extremes of these biases approach 0″.6, and the ensemble mean and standard deviation are 0″.04 and 0″.13, respectively. From the figure and the value of mean it appears that the visual observers tended to over-estimate the Neptune-Triton separation. The biases in the Nereid right ascension and declination observations are given Figs. 5 and 6. Rather larger biases occur in the early observations, some exceeding 1″.0.

The sensitivity of the orbit estimate to the biases was tested by comparing orbits determined with and without estimating them. For Triton the largest difference, during the period of the Voyager encounter, was about 4 km in the in-orbit direction. For Nereid the largest difference, again in the in-orbit direction, was on the average 30 000 km with a peak of about 60 000 km at periapsis. On the basis of the comparison, it appears that the observer biases can be ignored for Triton, but that they should probably be included for Nereid.

The RMS of the postfit observation residuals for each observer at each opposition are given in Tables 4 and 5. The entries in the tables provide a concise summary of the number and type observations used and their accuracy. The observations by Landgraf are of Nereid relative to Triton and their statistics appear in both tables. For the Voyager observations the RMS has been converted from lines and pixels to seconds of arc for consistency with the other observations. To provide a comparable metric accuracy, the Voyager RMS should also be scaled by a factor of about 0.15 to account for the difference between the Voyager-Neptune range and the Earth-Neptune range. Figures 7 and 8 show time histories of the Triton residuals with the position angle and separation distance residuals converted into equivalent differential right ascension and declination residuals for display. The improvement in quality of the modern observations is readily apparent. Figures 9 and 10 show time histories of the Nereid residuals and clearly indicate the scarcity and rather poor quality of those observations.

## 6. Mean orbital elements

Tables 6 and 7 contain mean orbital elements for Triton and Nereid, respectively. These elements were obtained by fitting precessing ellipse orbit models to the integrated orbits over the

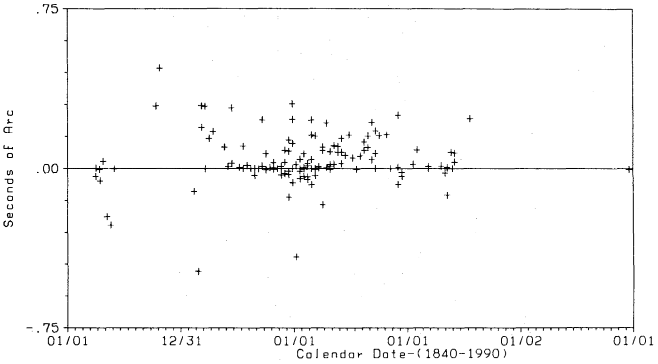


**Fig. 4.** Triton separation distance biases

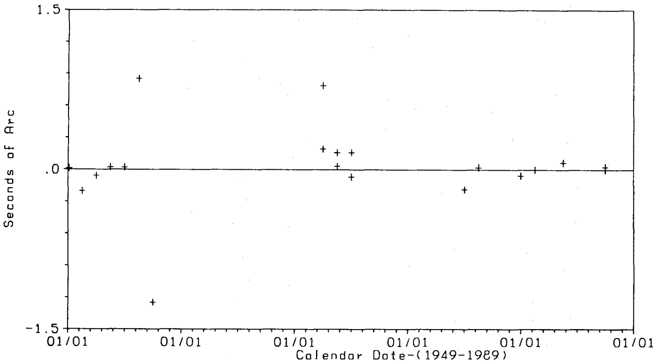


**Fig. 5.** Nereid right ascension biases

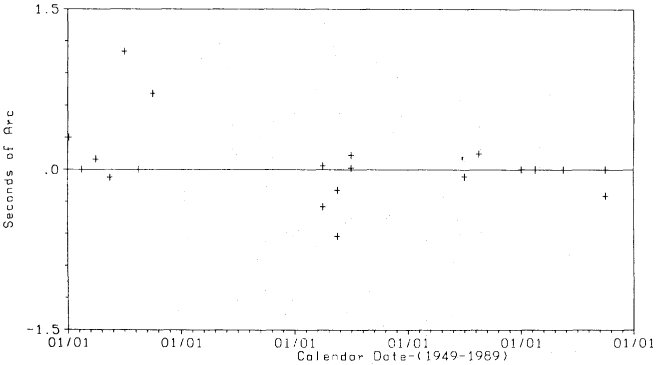


**Fig. 6.** Nereid declination biases

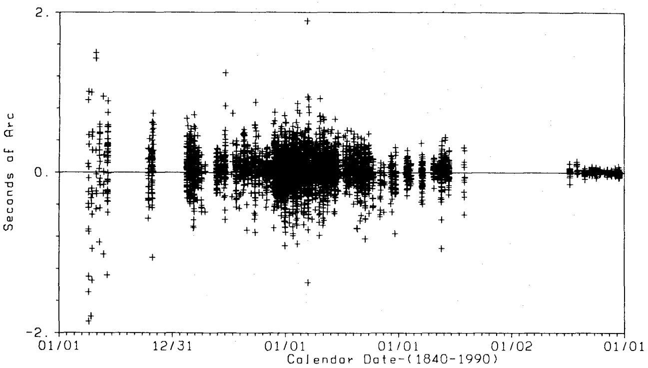


**Fig. 7.** Triton relative right ascension residuals

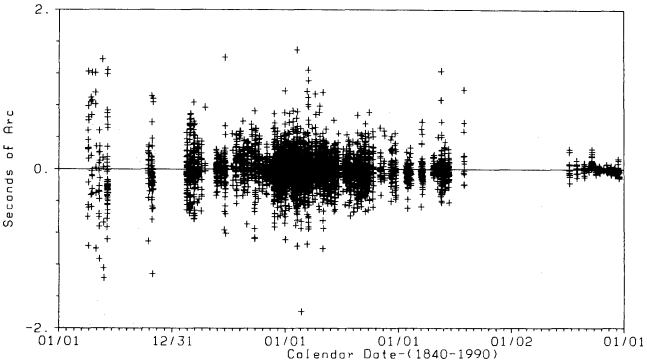


**Fig. 8.** Triton relative declination residuals

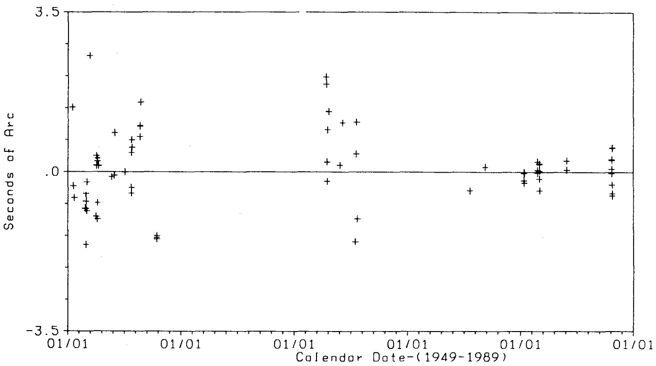


**Fig. 9.** Nereid relative right ascension residuals

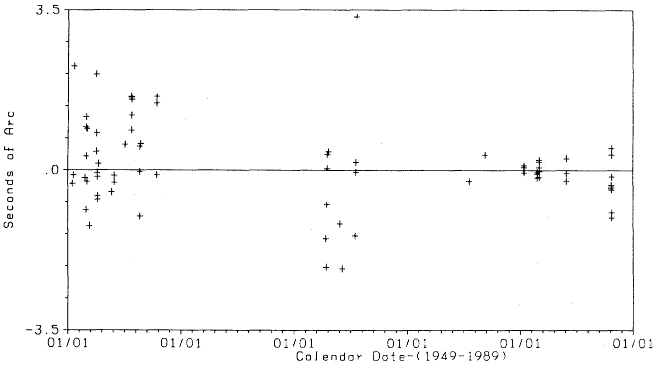


**Fig. 10.** Nereid relative declination residuals

140 year time span from 1847 to 1987. For Triton the ellipse was represented with retrograde equinoctial elements (Cefola, 1972) referred to the invariable plane, the plane normal to the system angular momentum vector (the orientation angles of which were also determined in the fit). The table contains these elements as well as classical elements derived from them. For Nereid the ellipse was represented with conventional posigrade equinoctial elements referred to the orbital plane of Neptune. Table 7 contains both the equinoctial elements and the clasical elements derived from them. It should be noted that the elements for Triton describe a planetocentric orbit whereas the elements for Nereid are for a barycentric orbit. All longitudes are measured from the node of the reference plane on the Earth mean equator of 1950.0.

**Table 4.** Triton observation residual statistics

| Opp. | Observer | No. | Type | RMS | No. | Type | RMS |
|---|---|---|---|---|---|---|---|
| 1847 | Bond | 7 | Δp | 2°77 | 7 | Δs | 0″988 |
| | Lassell | 9 | Δp | 2°04 | 7 | Δs | 0″439 |
| 1848 | Bond | 10 | Δp | 4°00 | 10 | Δs | 0″545 |
| | Lassell | 3 | Δp | 0°836 | 1 | Δs | 0″096 |
| 1849 | Lassell | 5 | Δp | 1°33 | 4 | Δs | 1″06 |
| 1850 | Lassell | 11 | Δp | 2°31 | 5 | Δs | 0″640 |
| 1851 | Lassell | 6 | Δp | 0°910 | 3 | Δs | 0″470 |
| 1852 | Lassell | 23 | Δp | 2°24 | 22 | Δs | 0″748 |
| 1863 | Lassell/Marth | 14 | Δp | 1°24 | 5 | Δs | 0″391 |
| 1864 | Lassell/Marth | 41 | Δp | 1°55 | 18 | Δs | 0″473 |
| 1873 | Newcomb | 24 | Δp | 1°58 | 43 | Δs | 0″219 |
| 1874 | Newcomb | 26 | Δp | 0°790 | 48 | Δs | 0″190 |
| 1875 | Hall | 21 | Δp | 1°45 | 20 | Δs | 0″215 |
| | Holden | 23 | Δp | 1°68 | 20 | Δs | 0″352 |
| 1876 | Hall | 6 | Δp | 0°880 | 6 | Δs | 0″274 |
| | Holden | 15 | Δp | 1°32 | 8 | Δs | 0″147 |
| 1877 | Holden | 8 | Δp | 1°00 | 9 | Δs | 0″344 |
| 1878 | Holden | 2 | Δp | 1°52 | 2 | Δs | 0″458 |
| 1881 | Hall | 31 | Δp | 0°890 | 29 | Δs | 0″177 |
| 1882 | Hall | 11 | Δp | 0°701 | 13 | Δs | 0″146 |
| 1883 | Hall | 23 | Δp | 1°45 | 20 | Δs | 0″133 |
| | Henry | 8 | Δp | 1°80 | 12 | Δs | 0″646 |
| 1885 | Struve, H. | 6 | Δp | 0°436 | 5 | Δs | 0″429 |
| 1886 | Perrotin | 13 | Δp | 0°687 | 13 | Δs | 0″302 |
| | Struve, H. | 15 | Δp | 0°929 | 15 | Δs | 0″200 |
| 1887 | Struve, H. | 25 | Δp | 0°953 | 22 | Δs | 0″224 |
| 1888 | Struve, H. | 64 | Δp | 0°881 | 51 | Δs | 0″238 |
| 1889 | Parrish | 7 | Δp | 1°78 | 5 | Δs | 0″398 |
| | Struve, H. | 20 | Δp | 0°710 | 20 | Δs | 0″140 |
| 1890 | Struve, H. | 30 | Δp | 0°535 | 23 | Δs | 0″253 |
| 1891 | Hall | 27 | Δp | 1°50 | 25 | Δs | 0″389 |
| | Struve, H. | 19 | Δp | 0°743 | 16 | Δs | 0″367 |
| 1892 | Barnard | 8 | Δp | 0°524 | 9 | Δs | 0″244 |
| | Struve, H. | 17 | Δp | 0°515 | 15 | Δs | 0″213 |
| 1893 | Barnard | 9 | Δp | 0°387 | 10 | Δs | 0″189 |
| 1894 | Barnard | 14 | Δp | 0°871 | 14 | Δs | 0″136 |
| | Brown | 13 | Δp | 0°536 | 13 | Δs | 0″221 |
| | Schaeberle | 10 | Δp | 0°531 | 10 | Δs | 0″181 |
| 1895 | Schaeberle | 15 | Δp | 0°526 | 12 | Δs | 0″100 |
| 1896 | Brown | 1 | Δp | 0°474 | 1 | Δs | 0″043 |
| | Drew | 45 | Δp | 1°59 | 40 | Δs | 0″365 |
| | Schaeberle | 7 | Δp | 0°407 | 7 | Δs | 0″159 |
| 1897 | Barnard | 53 | Δp | 0°995 | 53 | Δs | 0″225 |
| | Brown | 39 | Δp | 0°640 | 40 | Δs | 0″241 |
| | Schaeberle | 16 | Δp | 0°444 | 16 | Δs | 0″140 |
| 1898 | Aitken | 13 | Δp | 1°05 | 13 | Δs | 0″280 |
| | Barnard | 54 | Δp | 0°719 | 54 | Δs | 0″208 |
| | Drew | 12 | Δp | 0°938 | 12 | Δs | 0″259 |
| | Hussey | 12 | Δp | 0°751 | 12 | Δs | 0″267 |
| | Kostinsky | 7 | Δp | 0°463 | 7 | Δs | 0″250 |
| 1899 | Barnard | 72 | Δp | 0°777 | 71 | Δs | 0″184 |
| | Greenwich | 2 | Δp | 0°729 | 2 | Δs | 0″416 |
| | Hussey | 8 | Δp | 0°301 | 8 | Δs | 0″238 |
| | Kostinsky | 7 | Δp | 0°895 | 7 | Δs | 0″220 |
| | See | 62 | Δp | 1°18 | 64 | Δs | 0″289 |
| 1900 | Barnard | 48 | Δp | 0°839 | 47 | Δs | 0″232 |
| | Kostinsky | 5 | Δp | 1°14 | 5 | Δs | 0″260 |
| 1901 | Aitken | 9 | Δp | 0°460 | 13 | Δs | 0″187 |
| | Barnard | 47 | Δp | 0°822 | 47 | Δs | 0″236 |
| | Greenwich | 51 | Δp | 0°810 | 51 | Δs | 0″196 |
| | Perrine | 51 | Δp | 1°33 | 51 | Δs | 0″209 |
| 1902 | Barnard | 27 | Δp | 0°716 | 27 | Δs | 0″229 |
| | Dinwiddie | 22 | Δp | 1°30 | 22 | Δs | 0″330 |
| | Greenwich | 69 | Δp | 1°14 | 69 | Δs | 0″252 |
| | Wirtz | 10 | Δp | 2°20 | 10 | Δs | 0″475 |
| 1903 | Barnard | 22 | Δp | 1°02 | 22 | Δs | 0″276 |
| | Dinwiddie | 7 | Δp | 0°855 | 8 | Δs | 0″266 |
| | Greenwich | 54 | Δp | 0°644 | 54 | Δs | 0″211 |
| | Wirtz | 2 | Δp | 2°57 | 2 | Δs | 1″01 |
| 1904 | Barnard | 15 | Δp | 0°710 | 15 | Δs | 0″225 |
| | Greenwich | 57 | Δp | 0°922 | 57 | Δs | 0″235 |
| | Hammond | 11 | Δp | 0°815 | 10 | Δs | 0″246 |
| | Rice | 6 | Δp | 0°863 | 6 | Δs | 0″164 |
| | Wirtz | 1 | Δp | 0°806 | 1 | Δs | 0″061 |
| 1905 | Barnard | 11 | Δp | 1°01 | 10 | Δs | 0″309 |
| | Greenwich | 63 | Δp | 1°02 | 63 | Δs | 0″239 |
| | Hammond | 29 | Δp | 0°612 | 30 | Δs | 0″223 |
| | Wirtz | 19 | Δp | 3°34 | 19 | Δs | 0″628 |
| 1906 | Barnard | 17 | Δp | 0°493 | 16 | Δs | 0″147 |
| | Greenwich | 31 | Δp | 0°786 | 31 | Δs | 0″208 |
| 1907 | Barnard | 24 | Δp | 0°767 | 25 | Δs | 0″183 |
| | Hammond | 13 | Δp | 0°620 | 13 | Δs | 0″198 |
| | Greenwich | 26 | Δp | 1°24 | 26 | Δs | 0″225 |
| 1908 | Barnard | 26 | Δp | 0°765 | 25 | Δs | 0″218 |
| | Hall, Jr. | 27 | Δp | 0°787 | 23 | Δs | 0″356 |
| 1909 | Barnard | 25 | Δp | 1°00 | 25 | Δs | 0″317 |
| | Greenwich | 58 | Δp | 0°682 | 59 | Δs | 0″177 |
| | Hall, Jr. | 32 | Δp | 0°883 | 32 | Δs | 0″288 |
| | Wirtz | 1 | Δp | 0°269 | 1 | Δs | 0″075 |
| 1910 | Barnard | 25 | Δp | 0°996 | 23 | Δs | 0″244 |
| | Burton | 19 | Δp | 0°913 | 19 | Δs | 0″228 |
| | Hall, Jr. | 19 | Δp | 0°760 | 16 | Δs | 0″183 |
| 1911 | Barnard | 18 | Δp | 0°847 | 18 | Δs | 0″199 |
| | Burton | 44 | Δp | 0°597 | 44 | Δs | 0″152 |
| 1912 | Barnard | 31 | Δp | 0°815 | 31 | Δs | 0″274 |
| | Burton | 29 | Δp | 0°649 | 29 | Δs | 0″166 |
| | Hall, Jr. | 19 | Δp | 0°995 | 19 | Δs | 0″232 |
| 1913 | Barnard | 17 | Δp | 1°16 | 17 | Δs | 0″249 |
| 1914 | Barnard | 5 | Δp | 1°33 | 5 | Δs | 0″114 |
| 1915 | Barnard | 27 | Δp | 0°843 | 25 | Δs | 0″231 |
| 1916 | Barnard | 35 | Δp | 0°748 | 35 | Δs | 0″266 |
| 1917 | Barnard | 21 | Δp | 1°12 | 22 | Δs | 0″238 |
| 1918 | Barnard | 22 | Δp | 0°845 | 23 | Δs | 0″209 |
| | Hall, Jr. | 25 | Δp | 1°20 | 25 | Δs | 0″284 |
| 1919 | Barnard | 20 | Δp | 0°942 | 20 | Δs | 0″157 |
| | Hall, Jr. | 18 | Δp | 0°816 | 18 | Δs | 0″293 |
| 1920 | Barnard | 27 | Δp | 1°18 | 27 | Δs | 0″202 |
| | Hall, Jr. | 22 | Δp | 0°646 | 23 | Δs | 0″319 |
| 1921 | Barnard | 27 | Δp | 0°890 | 25 | Δs | 0″242 |
| | Burton | 3 | Δp | 0°805 | 3 | Δs | 0″130 |
| | Hall, Jr. | 3 | Δp | 0°572 | 3 | Δs | 0″432 |
| 1922 | Hall, Jr. | 12 | Δp | 1°00 | 12 | Δs | 0″262 |
| 1924 | Hall, Jr. | 11 | Δp | 0°842 | 9 | Δs | 0″296 |
| 1925 | Hall, Jr. | 4 | Δp | 0°532 | 5 | Δs | 0″374 |
| 1927 | Burton | 9 | Δp | 0°321 | 9 | Δs | 0″205 |
| | Hall, Jr. | 4 | Δp | 0°647 | 5 | Δs | 0″139 |
| | Crawford | 11 | Δp | 0°963 | 11 | Δs | 0″218 |
| 1928 | Burton | 11 | Δp | 0°963 | 11 | Δs | 0″222 |
| | Hall, Jr. | 3 | Δp | 0°805 | 3 | Δs | 0″596 |
| 1931 | Burton | 28 | Δp | 0°632 | 28 | Δs | 0″192 |
| 1932 | Bower | 12 | Δp | 0°611 | 12 | Δs | 0″119 |
| 1936 | Burton | 16 | Δp | 0°380 | 16 | Δs | 0″147 |
| | Lyons | 20 | Δp | 0°622 | 21 | Δs | 0″317 |

**Table 4** (continued)

| Opp. | Observer | No. | Type | RMS | No. | Type | RMS |
|---|---|---|---|---|---|---|---|
| 1939 | Alden | 30 | $\Delta$p | $0^s.356$ | 30 | $\Delta$s | $0''.076$ |
| 1940 | Alden | 21 | $\Delta$p | $0^s.303$ | 21 | $\Delta$s | $0''.094$ |
| 1941 | Burton | 2 | $\Delta$p | $0^s.264$ | 2 | $\Delta$s | $0''.150$ |
| | Lyons | 23 | $\Delta$p | $0^s.880$ | 24 | $\Delta$s | $0''.289$ |
| | Raynsford | 7 | $\Delta$p | $1^s.44$ | 7 | $\Delta$s | $0''.639$ |
| 1942 | Alden | 22 | $\Delta$p | $0^s.316$ | 22 | $\Delta$s | $0''.099$ |
| | Burton | 21 | $\Delta$p | $0^s.579$ | 21 | $\Delta$s | $0''.119$ |
| 1943 | Burton | 9 | $\Delta$p | $0^s.452$ | 9 | $\Delta$s | $0''.120$ |
| | unknown | 13 | $\Delta$p | $0^s.620$ | 13 | $\Delta$s | $0''.162$ |
| 1947 | Lyons | 6 | $\Delta$p | $0^s.765$ | 6 | $\Delta$s | $0''.493$ |
| 1975 | Walker | 18 | $\Delta\alpha\cos\delta$ | $0''.051$ | 18 | $\Delta\delta$ | $0''.098$ |
| 1977 | Walker | 10 | $\Delta\alpha\cos\delta$ | $0''.052$ | 10 | $\Delta\delta$ | $0''.061$ |
| 1979 | Harrington | 20 | $\Delta\alpha\cos\delta$ | $0''.030$ | 20 | $\Delta\delta$ | $0''.041$ |
| 1980 | Harrington | 16 | $\Delta\alpha\cos\delta$ | $0''.017$ | 16 | $\Delta\delta$ | $0''.016$ |
| 1981 | Harrington | 32 | $\Delta\alpha\cos\delta$ | $0''.024$ | 32 | $\Delta\delta$ | $0''.056$ |
| 1982 | Harrington | 27 | $\Delta\alpha\cos\delta$ | $0''.026$ | 27 | $\Delta\delta$ | $0''.026$ |
| 1983 | Harrington | 17 | $\Delta\alpha\cos\delta$ | $0''.017$ | 17 | $\Delta\delta$ | $0''.024$ |
| 1984 | Harrington | 4 | $\Delta\alpha\cos\delta$ | $0''.010$ | 4 | $\Delta\delta$ | $0''.011$ |
| 1985 | Harrington | 28 | $\Delta\alpha\cos\delta$ | $0''.020$ | 28 | $\Delta\delta$ | $0''.022$ |
| 1986 | Harrington | 24 | $\Delta\alpha\cos\delta$ | $0''.016$ | 24 | $\Delta\delta$ | $0''.024$ |
| 1987 | Landgraf | 5 | $\Delta\alpha\cos\delta$ | $0''.472$ | 5 | $\Delta\delta$ | $0''.679$ |
| | Walker | 28 | $\Delta\alpha\cos\delta$ | $0''.024$ | 28 | $\Delta\delta$ | $0''.023$ |
| 1988 | Walker | 14 | $\Delta\alpha\cos\delta$ | $0''.012$ | 14 | $\Delta\delta$ | $0''.016$ |
| | Taylor | 7 | $\Delta\alpha$ | $0''.046$ | 7 | $\Delta\delta$ | $0''.053$ |
| | Williams | 6 | $\Delta$p | $0^s.141$ | 6 | $\Delta$s | $0''.075$ |
| | Voyager | 56 | $\Delta$px | $0''.170$ | 56 | $\Delta$ln | $0''.170$ |

**Table 5.** Nereid observation residual statistics

| Opp. | Observer | No. | Type | RMS | No. | Type | RMS |
|---|---|---|---|---|---|---|---|
| 1949 | van Biesbroeck | 3 | $\Delta\alpha\cos\delta$ | $0''.895$ | 3 | $\Delta\delta$ | $1''.25$ |
| 1950 | van Biesbroeck | 8 | $\Delta\alpha\cos\delta$ | $1''.16$ | 8 | $\Delta\delta$ | $0''.828$ |
| 1951 | van Biesbroeck | 8 | $\Delta\alpha\cos\delta$ | $0''.579$ | 8 | $\Delta\delta$ | $0''.854$ |
| 1952 | van Biesbroeck | 3 | $\Delta\alpha\cos\delta$ | $0''.490$ | 3 | $\Delta\delta$ | $0''.265$ |
| 1953 | van Biesbroeck | 6 | $\Delta\alpha\cos\delta$ | $0''.457$ | 6 | $\Delta\delta$ | $0''.468$ |
| 1954 | van Biesbroeck | 4 | $\Delta\alpha\cos\delta$ | $0''.354$ | 4 | $\Delta\delta$ | $0''.638$ |
| 1955 | van Biesbroeck | 3 | $\Delta\alpha\cos\delta$ | $0''.192$ | 3 | $\Delta\delta$ | $0''.879$ |
| 1967 | van Biesbroeck | 5 | $\Delta\alpha\cos\delta$ | $0''.946$ | 5 | $\Delta\delta$ | $0''.980$ |
| | | 1 | $\alpha$ | $1''.24$ | 1 | $\delta$ | $0''.379$ |
| 1968 | van Biesbroeck | 1 | $\Delta\alpha\cos\delta$ | $0''.125$ | 1 | $\Delta\delta$ | $0''.391$ |
| | | 1 | $\alpha$ | $1''.01$ | 1 | $\delta$ | $2''.03$ |
| 1969 | van Biesbroeck | 3 | $\Delta\alpha\cos\delta$ | $1''.06$ | 3 | $\Delta\delta$ | $2''.12$ |
| | | 1 | $\alpha$ | $1''.02$ | 1 | $\delta$ | $0''.164$ |
| 1977 | Shelus | 1 | $\Delta\alpha\cos\delta$ | $0''.226$ | 1 | $\Delta\delta$ | $0''.178$ |
| 1978 | Mulholland | 1 | $\Delta\alpha\cos\delta$ | $0''.091$ | 1 | $\Delta\delta$ | $0''.184$ |
| 1981 | Veillet | 4 | $\Delta\alpha\cos\delta$ | $0''.117$ | 4 | $\Delta\delta$ | $0''.063$ |
| 1982 | Veillet | 8 | $\Delta\alpha\cos\delta$ | $0''.194$ | 8 | $\Delta\delta$ | $0''.135$ |
| 1984 | Veillet | 3 | $\Delta\alpha\cos\delta$ | $0''.110$ | 3 | $\Delta\delta$ | $0''.204$ |
| 1987 | Landgraf | 5 | $\Delta\alpha\cos\delta$ | $0''.472$ | 5 | $\Delta\delta$ | $0''.679$ |
| | Schaefer | 4 | $\alpha$ | $0''.191$ | 4 | $\delta$ | $0''.177$ |

## 7. Comparison with previous investigations

In order to provide some means of comparison with the previous orbit determinations, Tables 8 and 9 give the elements obtained by the different investigators for Triton and Nereid, respectively. The tables also include the derived Neptunian system mass, and Table 8 contains values for the second zonal harmonic of Neptune and the orientation angles of the pole of the invariable plane. Eichelberger's Triton elements have been converted from the mean of 1900 coordinate system to the mean of 1950 system, and his Neptune equator has been taken to be the invariable plane which is the plane actually determined from the motion of Triton. Both Veillet's (1988) mean and osculating elements are given for Nereid.

**Table 6.** Triton planetocentric mean elements at Julian ephemeris date 2433282.5 referred to the invariable plane

| Element | Value | Units |
|---|---|---|
| semi-major axis | 354611.773 | (km) |
| $h_r = e\sin(\omega-\Omega)$ | $-0.3429949528\times10^{-3}$ | |
| $k_r = e\cos(\omega-\Omega)$ | $-0.2250328532\times10^{-3}$ | |
| mean longitude $= M+\omega-\Omega$ | 49.85334766 | (deg) |
| $p_r = \cot(i/2)\sin\Omega$ | +0.0932102559 | |
| $q_r = \cot(i/2)\cos\Omega$ | −0.1738174468 | |
| longitude rate $=\dot M+\dot\omega-\dot\Omega$ | 61.25726751 | (deg/day) |
| apsidal rate $=\dot\omega-\dot\Omega$ | 0.5295275852 | (deg/yr) |
| nodal rate $=\dot\Omega$ | 0.5430763965 | (deg/yr) |
| invariable plane pole right ascension | 298.3065940 | (deg) |
| invariable plane pole declination | 42.51071244 | (deg) |
| eccentricity $= e$ | 0.0004102259410 | |
| longitude of periapsis $=\omega-\Omega$ | 236.7318362 | (deg) |
| inclination $= i$ | 157.6852321 | (deg) |
| longitude of ascending node $=\Omega$ | 151.7973992 | (deg) |

**Table 7.** Nereid barycentric mean elements at Julian ephemeris date 2433680.5 referred to the mean orbital plane of Neptune

| Element | Value | Units |
|---|---|---|
| semi-major axis | 5511233.255 | (km) |
| $h = e\sin(\omega+\Omega)$ | −0.7130006982 | |
| $k = e\cos(\omega+\Omega)$ | −0.2354680637 | |
| mean longitude $= M+\omega+\Omega$ | 251.14984688 | (deg) |
| $p = \tan(i/2)\sin\Omega$ | −0.04095843199 | |
| $q = \tan(i/2)\cos\Omega$ | +0.04240761392 | |
| longitude rate $=\dot M+\dot\omega+\dot\Omega$ | 0.9996465329 | (deg/day) |
| apsidal rate $=\dot\omega+\dot\Omega$ | 0.8696048083 | (deg/century) |
| nodal rate $=\dot\Omega$ | −3.650272562 | (deg/century) |
| Neptune orbit inclination | 22.313 | (deg) |
| Neptune orbit nodal longitude | 3.522 | (deg) |
| eccentricity $= e$ | 0.750876291 | |
| longitude of periapsis $=\omega+\Omega$ | 251.7242240 | (deg) |
| inclination $= i$ | 6.748231850 | (deg) |
| longitude of ascending node $=\Omega$ | 315.9958928 | (deg) |

**Table 8.** Triton element and constants comparison

| Element | Eichelberger | Harris | This Paper | Units |
|---|---|---|---|---|
| $a$ | 355247 | 354290 | 354612 | (km) |
| $e$ | 0.0049 | 0.0 | 0.0004 | |
| $i$ | 159.945 | 158.996 | 157.685 | (deg) |
| $u_0$ | 206.167 | 200.913 | 201.651 | (deg) |
| $\omega$ | 203.7 | 0.0 | 28.529 | (deg) |
| $\Omega$ | 158.098 | 151.401 | 151.797 | (deg) |
| $\dot u$ | 61.2589494 | 61.2588532 | 61.2587544 | (deg/day) |
| $\dot\omega$ | 1.94 | 0.0 | 1.07260 | (deg/yr) |
| $\dot\Omega$ | 0.61494 | 0.57806 | 0.54308 | (deg/yr) |
| $\alpha_{1950}$ | 295.571 | 297.813 | 298.306 | (deg) |
| $\delta_{1950}$ | 41.468 | 41.185 | 42.511 | (deg) |
| $\mu^{-1}$ | 19331 | 19490 | 19436 | |
| $J_2$ | — | 0.0037 | 0.0037 | |

Reference Date: Julian Ephemeris Date 2433282.5
Reference Plane: Invariable Plane

**Table 9.** Nereid element comparison

| Element | Rose | Veillet Mean | Veillet Osculating | This Paper | Units |
|---|---|---|---|---|---|
| $a$ | 5511380 | 5518665 | 5518665 | 5511233 | (km) |
| $e$ | 0.7482 | 0.7448 | 0.7486 | 0.7509 | |
| $i$ | 6.747 | 9.995 | 6.685 | 6.748 | (deg) |
| $M$ | 360.10 | 359.07 | 360.21 | 359.43 | (deg) |
| $\omega+\Omega$ | 251.55 | 252.22 | 251.58 | 251.72 | (deg) |
| $\Omega$ | 318.07 | 328.74 | 317.06 | 316.00 | (deg) |
| $n$ | 0.999721 | 0.999623 | 0.999623 | 0.999646 | (deg/day) |
| $\mu^{-1}$ | 19438 | 19366 | 19366 | 19436 | |

Reference Date: Julian Ephemeris Date 2433680.5
Reference Plane: Mean Orbital Plane of Neptune

The current determination of the Triton orbit is generally in agreement with the two previous ones. Eichelberger's orbit, which was based on only 34 yr of observations, differs more than that of Harris, which was based on 82 yr of observations and included high quality modern photographic data. The orbital periods of all three determinations agree within 2 s. The previous semimajor axes and masses bracket the current ones with Eichelberger's values being slightly larger and Harris' values slightly smaller. Eichelberger found a larger eccentricity, but his determination was very uncertain. Similarly, his argument of periapsis and its rate are quite different but are also uncertain due to the near circular nature of the orbit. Disagreement among the nodal precession rates and the invariable pole orientation angles can be attributed to the different observation time spans; Eichelberger's data arc covers 5% of the precession period, Harris' covers 12%, and the current arc covers 21%. The orbit orientation angles are close with most of the difference due to the differing invariable planes. In addition, for Eichelberger the propagation of the mean longitude and node from the 1900 epoch to the 1950 epoch introduces a nearly 4° difference due to rate differences. Harris' $J_2$ for the low mass Triton matches the currently determined value.

The Nereid orbit determined by Rose is in quite good agreement with the current orbit. This is somewhat surprising because Rose fit only van Biesbroeck's data which is significantly lower quality than recent observations. Except for semimajor axis, the orbit determined by Veillet also matches the current orbit. When comparing with Veillet's orbit his osculating elements should be used because the major periodic terms in Mignard's theory have periods of the order of thousands of years. Hence, the theory mean elements cannot be expected to agree with mean elements determined from a 140 yr integration.

## 8. Conclusions

This paper has presented the results of the effort to prepare the initial Neptunian satellite ephemerides to be used in support of the Voyager encounter. The ephemerides were based on a numerical integration of the satellite orbits fit to observations, and as a byproduct of the fit, new values for the Neptune system GM, the Neptune $J_2$, and the Neptune pole orientation angles were obtained.

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## Appendix

### *A.1. Precessing pole model*

The expression for a planet pole which is precessing about a fixed axis is:

$$\boldsymbol{h} = (\boldsymbol{h}_0 \cdot \boldsymbol{k})\boldsymbol{k} + \sin\psi(\boldsymbol{h}_0 \times \boldsymbol{k}) + \cos\psi[\boldsymbol{k} \times (\boldsymbol{h}_0 \times \boldsymbol{k})]\,, \tag{1}$$

where

$\boldsymbol{h}_0$ = the epoch pole vector,

$\boldsymbol{k}$ = the fixed axis about which the pole precesses,

$\psi$ = the precession angle.

The epoch pole is expressed in terms of the right ascension, $\alpha_0$, and declination, $\delta_0$, at epoch:

$$\boldsymbol{h}_0 = \begin{bmatrix} \cos\alpha_0\cos\delta_0 \\ \sin\alpha_0\cos\delta_0 \\ \sin\delta_0 \end{bmatrix}, \tag{2}$$

Analogously, the fixed axis is expressed in terms of right ascension, $\alpha_r$, and declination, $\delta_r$, as:

$$\boldsymbol{k} = \begin{bmatrix} \cos\alpha_r\cos\delta_r \\ \sin\alpha_r\cos\delta_r \\ \sin\delta_r \end{bmatrix}. \tag{3}$$

For precession at a constant rate, the precession angle is given by:

$$\psi = \dot{\psi}(t - t_0) \tag{4}$$

where

$\dot{\psi}$ = precession rate,

$t$ = time,

$t_0$ = epoch time.

### *A.2. Neptune pole parameter equations*

The Neptune pole, which is assumed to precess at a constant rate about the system angular momentum vector, is found from the expression for that vector. In determining the pole, the following assumptions are made:

1. The angular momentum vector of the Neptunian system is constant.
2. The system angular momentum can be represented by the sum of the angular momentum of Neptune and the angular momentum of Triton's mean orbit.
3. The contribution of Nereid to the system angular momentum is negligible.
4. Neptune is a rigid, axially symmetric body.
5. The nutation of Neptune can be ignored.

Under these assumptions the components of the Neptune angular momentum vector and the Triton mean orbital angular momentum vector which are normal to the system angular momentum vector must be equal in magnitude and opposite in direction. Hence, the vector magnitudes satisfy

$$\left[1-\left(1-\frac{J_2}{\gamma}\right)\left(\frac{\dot{\psi}}{\dot{W}}\right)\cos\epsilon\right]\sin\epsilon = \left(\frac{m_1}{m_0}\right)\left(1+\frac{m_1}{m_0}\right)\frac{H\sin I}{\gamma R^2 \dot{W}} \quad (5)$$

and the vector directions satisfy

$$\frac{\boldsymbol{k}\times\boldsymbol{h}}{\sin\epsilon} = -\frac{\boldsymbol{k}\times\boldsymbol{n}}{\sin I} \quad (6)$$

where

$\boldsymbol{h}$ = unit vector along the Neptune pole,
$\boldsymbol{k}$ = unit vector along the system angular momentum vector,
$\boldsymbol{n}$ = unit vector along the Triton mean barycentric orbital angular momentum vector,
$H$ = magnitude of the Triton mean barycentric orbital angular momentum vector,
$\epsilon$ = the angle between the Neptune pole and the system angular momentum vector,
$I$ = the angle between the Triton mean orbital angular momentum vector and the system angular momentum vector,
$\dot{W}$ = Neptune spin rate about its pole,
$\dot{\psi}$ = Neptune pole precession rate,
$m_0$ = mass of Neptune,
$m_1$ = mass of Triton,
$J_2$ = second zonal harmonic of the Neptune gravitational potential,
$R$ = equatorial radius of the Neptune,
$\gamma$ = the ratio of the axial moment of inertia to moment of inertia of a homogeneous sphere having Neptune's mass and a radius equal to the Neptune's equatorial radius.

Note that the angles $\epsilon$ and $I$ lie between 0° and 180° and that

$$\boldsymbol{k}\cdot\boldsymbol{h} = \cos\epsilon, \quad (7)$$

$$\boldsymbol{k}\cdot\boldsymbol{n} = \cos I. \quad (8)$$

In obtaining Eq. (5) the following relation between the polar moment of inertia, $C$, and $\gamma$ was used:

$$C = \gamma m_0 R^2 \quad (9)$$

as was the following relation between the polar and the equatorial moment of inertia, $A$:

$$A = C - J_2 m_0 R^2. \quad (10)$$

Since Eq. (6) is simply a requirement that two unit vectors be anti-parallel, it can be replaced by the equivalent scalar form:

$$(\boldsymbol{k}\times\boldsymbol{h})\cdot(\boldsymbol{k}\times\boldsymbol{n}) = -\sin\epsilon\sin I \quad (11)$$

Equations (5) and (11) together with the expressions from the previous section implicitly define the right ascension and declination of the Neptune pole.

The normal to Triton's mean orbit can be written as

$$\boldsymbol{n} = \cos I\boldsymbol{k} + \sin I(\sin\Omega\boldsymbol{p} - \cos\Omega\boldsymbol{q}) \quad (12)$$

where the vectors $\boldsymbol{p}$ and $\boldsymbol{q}$ are given by

$$\boldsymbol{p} = (\boldsymbol{e}_3\times\boldsymbol{k})\sec\delta_r, \quad (13)$$

$$\boldsymbol{q} = \boldsymbol{k}\times\boldsymbol{p}, \quad (14)$$

with

$\boldsymbol{e}_3$ = the unit vector (0,0,1),
$\Omega$ = the ascending node of the mean Triton orbit on the plane normal to the system angular momentum vector.

As a consequence of Eq. (6), Triton's mean orbit must precess at the same rate as the Neptune pole. Hence, its node is given by:

$$\Omega = \Omega_0 + \dot{\psi}(t-t_0). \quad (15)$$

The vector $\boldsymbol{k}$, the angles $I$ and $\Omega_0$, the magnitude of the orbital angular momentum $H$, and the precession rate $\dot{\psi}$ are found from the condition that the angle between Triton's mean orbital angular momentum vector and the system angular momentum vector is constant and from the condition that the former precesses at a constant rate about the latter. Those conditions are imposed by requiring the following equations

$$H\cos I = \boldsymbol{k}\cdot(\boldsymbol{r}\times\dot{\boldsymbol{r}}) \quad (16)$$

$$H\sin I\sin\Omega = \boldsymbol{p}\cdot(\boldsymbol{r}\times\dot{\boldsymbol{r}}) \quad (17)$$

where

$\boldsymbol{r}$ = barycentric position of Triton,
$\dot{\boldsymbol{r}}$ = barycentric velocity of Triton,

to hold in a least squares sense over the time span of the orbit data fit.

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