STANFORD UNIVERSITY MATHEMATICS CAMP (SUMAC) 2024

ADMISSION EXAM

For use by SUMaC 2024 applicants only. Not for distribution.

- Solve as many of the following problems as you can. Your work on these problems together with your grades in school, teacher recommendations, and answers to the questions on the application form are all used to evaluate your SUMaC application. Although SUMaC is very selective with a competitive applicant pool, correct answers on every problem are not required for admission.

- There is no time limit for this exam other than the application deadline.

- Please include clear, detailed explanations for all of your solutions; numerical answers or formulas with no explanation are not useful for evaluating your application.

- In the event you are unable to solve a problem completely, you are encouraged to write up any partial progress that you feel captures your ideas leading toward a solution.

- You will need to create a separate document with your solutions and explanations. This document may be typed or handwritten, as long as the final document you upload is legible for our review.

- None of these problems require a calculator or computer, and they are all designed so that they can be done without computational tools.

- You are expected to do your own work without the use of any outside source (books, teacher or parent help, internet search, etc). If you recognize one of the problems from another source, or if you receive any assistance, please indicate this in your write up.

- **Please do not share these problems or your solutions with anyone.**

1. A pair of distinct two-digit integers (n, d) is *misleading* if the first digit of n equals the second digit of d, and if you write the numbers as a fraction $\frac{n}{d}$ and "cancel" (i.e., cross out) the common digits, you get an equal fraction. For example, $(16, 64)$ is a misleading pair since

$$\frac{16}{64} = \frac{1}{4} = \frac{1\cancel{6}}{\cancel{6}4}.$$

   (a) Find all pairs of two-digit misleading integers. Explain how you know you have found all possible misleading two-digit pairs.

   (b) Are there any misleading pairs of three-digit integers? For a three-digit pair to be misleading, when written as a fraction, "cancelling" the first digit of n with either the second *or* third digit of d (where the digits being canceled are equal) results in a fraction equal in value to the original fraction. Furthermore, the two integers cannot both end in 0 (otherwise, the pair could be obtained directly from a pair of misleading two-digit numbers).

2. For an arbitrary set of numbers S, define $S + S$ to be the set of all sums $x + y$ where x and y are in S. Let A, B, C, D be sets of integers such that $(A \cup B) + (A \cup B) = (C \cup D) + (C \cup D)$. For example, if $A = B = C = D = \{1, 2\}$ then

$$(A \cup B) + (A \cup B) = (C \cup D) + (C \cup D) = \{1 + 1, 1 + 2, 2 + 1, 2 + 2\} = \{2, 3, 4\}.$$

Can you find four sets of integers A, B, C, D so that

$$(A \cup B) + (A \cup B) = (C \cup D) + (C \cup D) = (A \cup C) + (A \cup C) \neq (B \cup D) + (B \cup D)?$$

3. (a) On a 3×3 grid, draw a path that starts in the center of a square in the corner of the grid, and repeatedly moves from the center of one square to the center of an adjacent square, without visiting any squares twice, and such that every square in the grid is included. For example, the following is a path that meets these conditions:
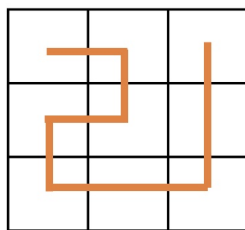

What are the possible ending positions for a path drawn according to these rules? What if you started from an edge square or the center square of the grid? Explain.

(b) Repeat the above problem for a 4×4 grid.

(c) Can you describe what happens in general for an $n \times n$ grid where n is a positive integer?

4. Let $P_1 = (x_1, y_1)$ and $P_2 = (x_2, y_2)$ be two points in the (x, y)−plane such that $y > 0$ and $x_1 < x_2$.
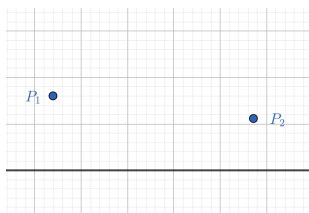

We call a circle C a *Yeppeun circle for P_1 and P_2* if C contains P_1 and P_2, C is tangent to the x−axis at a point $P_3 = (x_3, 0)$, and the x-coordinate of the point of tangency lies between the x-coordinates of P_1 and P_2; that is, $x_1 < x_3 < x_2$. For example, the circle C in the following diagram is a Yeppeun circle for the given P_1 and P_2
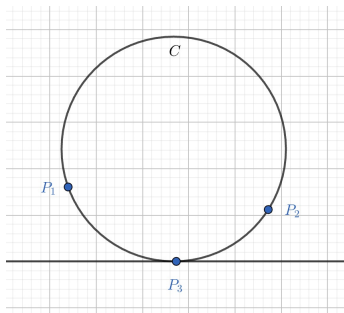
Given any two such points P_1 and P_2 above the x−axis, does a Yeppeun circle always exist? Explain.

5. Consider a disk D of radius 1. We say a collection of shapes *hides* D if we can place the shapes from that collection on top of the disk in such a way that each point of the disk is under some part of at least one of the shapes. For example, D can be hidden by a single disk of radius greater-than-or-equal-to 1. However, D cannot be hidden by a single disk of smaller radius than 1. As another example, D can be hidden by 7 disks of radius $3/4$ by placing one radius $3/4$ disk at the center and the remaining radius $3/4$ disks around the edge of D as follows:
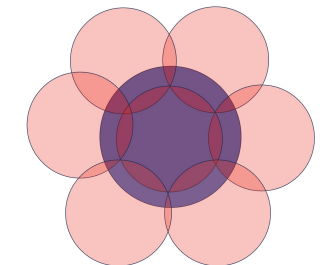
   (a) Find the smallest radius r so that two disks of radius r hide D (and any disk of radius less than r cannot hide it).

   (b) What is the smallest radius r you can find such that four disks of radius r can hide D?

   (c) What is the smallest radius r that *you* can find such that five disks of radius r can hide D? You don't need to prove this radius is optimal; you only need to show that you found a solution with your radius r. With the radius you found, can you hide D in such a way that the centers of the disks are not vertices of a regular pentagon?

6. Consider a game where you have five distinct coins in a row, and you are allowed to move them in pairs and insert them between other coins in the row (without changing the order of the pair). For example, here is an allowable move:

   As above you can slide the coins over horizontally to create space to insert the pair, or to remove extra space, as long as you do not change the order of the coins in the row.

   (a) Suppose the coins are placed in order of increasing value; is there a sequence of moves as described above that reverses the order (so they are in order of decreasing value)?

   (b) Again, suppose the coins are placed in order of increasing value; is there a sequence of moves that switches the first coin with the last, and where the middle three end up in their starting positions?

7. Students at the City High School are all required to join a club, and every club has at least two members. It is also the case that every two clubs have at least one common member. Show that students from the high school can be grouped into three groups, where each student is in exactly one group and every club is represented in at least two groups.