Computing $\frac{d\sigma}{d\Omega}$ for our SIDM Candidate

Z. J. Hoelscher

December 27, 2025

Here we shall compute the S-matrix amplitude for a tree-level scattering process of the form $\mathrm{XX} \to \mathrm{XX}$, where X_μ is a dark photon, and scattering is mediated by a scalar, h. To avoid confusion with an index α, we are using X_μ (rather than α_μ) for the dark photon here. Consider a Lagrangian of the following form:

$$\mathcal{L} \supset -\frac{1}{4}\Psi_{\mu\nu}\Psi^{\mu\nu} + \frac{1}{2}m_{\mathrm{X}}^2 \mathrm{X}_\mu \mathrm{X}^\mu + \frac{1}{2}\partial_\mu h \partial^\mu h - \frac{1}{2}m_h^2 h^2 + gh\mathrm{X}_\mu \mathrm{X}^\mu, \tag{1}$$

where the Faraday tensor is defined as

$$\Psi_{\mu\nu} = \partial_\mu \mathrm{X}_\nu - \partial_\nu \mathrm{X}_\mu. \tag{2}$$

We have the following vertex factor, which could alternatively be written with the opposite sign, without altering $\mathcal{M}_t$, $\mathcal{M}_u$, or the final amplitude in our case. **Similarly, note that one could instead write the vertex factor with raised indices, and the polarization vectors with lowered indices, so long as everything is kept consistent.**

$$2ig\eta_{\mu\nu} \tag{3}$$

Incoming and outgoing vector bosons can be represented by the following:

$$\epsilon^\mu(p, \lambda) \tag{4}$$

and

$$\epsilon^{\mu *}(p, \lambda'). \tag{5}$$

The mediator has the following scalar propagator, where for brevity, we shall not write the $i\epsilon$ going forward:

$$\frac{i}{q^2 - m_h^2 + i\epsilon}. \tag{6}$$

The following expression is very useful here. Note that $\eta_{\mu\nu}$ is the Minkowski metric for flat spacetime, $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$.

$$\sum_{\lambda=1}^{3} \epsilon^\mu(p, \lambda)\epsilon^{\nu *}(p, \lambda) = -\eta^{\mu\nu} + \frac{p^\mu p^\nu}{m_{\mathrm{X}}^2} \tag{7}$$

This fills a role somewhat analogous to the following equation in quantum electrodynamics.

$$\sum_s u^s(p)\overline{u}^s(p) = \not{p} + m \tag{8}$$

We have the following tree-level Feynman diagrams. There is some flexibility on the labeling of the momenta, so we explicitly show our convention here. The same orientation and labeling is used for similar diagrams in Equation 4.119 in Peskin and Schroeder, where they are also considering $t+u$-channel. We have the t-channel process on the left, and the u-channel process on the right.

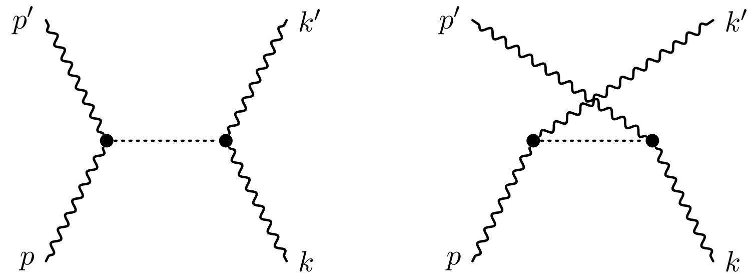

Figure 1: Tree-level XX $\to$ XX scattering via a scalar mediator h. The t-channel diagram is shown on the left and the u-channel diagram on the right. These are drawn so that time flows from the bottom to the top. The same orientation and momenta labeling is used in Equation 4.119 in Peskin and Schroeder.

We have the following matrix element, where we note that the t and u-channel diagrams have the same sign here (unlike fermions, which satisfy anti-commutation relations) because they are bosons, and the Pauli Exclusion Principle does not apply to them.

$$\begin{aligned} i\mathcal{M} = &\left[\epsilon^{\alpha*}(p', \lambda_1')\,(2ig\,\eta_{\alpha\beta})\,\epsilon^{\beta}(p,\lambda_1)\right]\left[\frac{i}{(p'-p)^2 - m_h^2}\right]\left[\epsilon^{\gamma*}(k', \lambda_2')\,(2ig\,\eta_{\gamma\delta})\,\epsilon^{\delta}(k,\lambda_2)\right] \\ &+ \left[\epsilon^{\alpha*}(p', \lambda_1')\,(2ig\,\eta_{\alpha\beta})\,\epsilon^{\beta}(k,\lambda_2)\right]\left[\frac{i}{(p'-k)^2 - m_h^2}\right]\left[\epsilon^{\gamma*}(k', \lambda_2')\,(2ig\,\eta_{\gamma\delta})\,\epsilon^{\delta}(p,\lambda_1)\right] \end{aligned} \tag{9}$$

Using $(AB)^\dagger = B^\dagger A^\dagger$, we obtain the following equation.

$$\begin{aligned} (i\mathcal{M})^* = &\left[\epsilon^{\gamma*}(k', \lambda_2')\,(2ig\,\eta_{\gamma\delta})\,\epsilon^{\delta}(k,\lambda_2)\right]^\dagger\left[\frac{-i}{(p'-p)^2 - m_h^2}\right]\left[\epsilon^{\alpha*}(p', \lambda_1')\,(2ig\,\eta_{\alpha\beta})\,\epsilon^{\beta}(p,\lambda_1)\right]^\dagger \\ &+ \left[\epsilon^{\gamma*}(k', \lambda_2')\,(2ig\,\eta_{\gamma\delta})\,\epsilon^{\delta}(p,\lambda_1)\right]^\dagger\left[\frac{-i}{(p'-k)^2 - m_h^2}\right]\left[\epsilon^{\alpha*}(p', \lambda_1')\,(2ig\,\eta_{\alpha\beta})\,\epsilon^{\beta}(k,\lambda_2)\right]^\dagger \end{aligned} \tag{10}$$

After simplifying, dropping λ, λ' for compactness, and noting $\eta^\dagger_{\alpha\beta} = \eta^{\mathrm{T}}_{\alpha\beta} = \eta_{\beta\alpha}$, we obtain the following equations. Note that $\left[\epsilon^{\gamma*}(k', \lambda_2')\,(2ig\,\eta_{\gamma\delta})\,\epsilon^{\delta}(k,\lambda_2)\right]^\dagger = \left[\epsilon^{\delta*}(k,\lambda_2)\,(-2ig\,\eta_{\delta\gamma})\,\epsilon^{\gamma}(k',\lambda_2')\right]$.

$$
\begin{aligned}
i\mathcal{M} = &\left[\epsilon^{\alpha *}(p')\,\epsilon_{\alpha}(p)\right]\left[\frac{-4ig^2}{(p'-p)^2-m_h^2}\right]\left[\epsilon^{\gamma *}(k')\,\epsilon_{\gamma}(k)\right] \\
&+\left[\epsilon^{\alpha *}(p')\,\epsilon_{\alpha}(k)\right]\left[\frac{-4ig^2}{(p'-k)^2-m_h^2}\right]\left[\epsilon^{\gamma *}(k')\,\epsilon_{\gamma}(p)\right]
\end{aligned} \tag{11}
$$

$$
\begin{aligned}
(i\mathcal{M})^* = &\left[\epsilon^{\delta *}(k)\,\epsilon_{\delta}(k')\right]\left[\frac{4ig^2}{(p'-p)^2-m_h^2}\right]\left[\epsilon^{\beta *}(p)\,\epsilon_{\beta}(p')\right] \\
&+\left[\epsilon^{\delta *}(p)\,\epsilon_{\delta}(k')\right]\left[\frac{4ig^2}{(p'-k)^2-m_h^2}\right]\left[\epsilon^{\beta *}(k)\,\epsilon_{\beta}(p')\right]
\end{aligned} \tag{12}
$$

We wish to compute what is often referred to as a spin-averaged, or unpolarized, cross section, where we average over initial polarizations, and sum over final polarizations. This ultimately leads to a factor of 1/9 via the expression shown below, as massive vector bosons have three polarizations. This is similar to how in quantum electrodynamics, a scattering process of the form $e^-e^- \to e^-e^-$ would receive a factor of 1/4, as electrons have two possible spins. Note that our cross section also receives a factor of 1/2 as a consequence of the fact that we have two identical bosons in the final state. See Peskin and Schroeder, Equation 4.100, for a similar factor of 1/2 applied to scattering in ϕ^4 theory. We go ahead and insert this factor of 1/2 here.

$$
|\mathcal{M}|^2_{\text{UNPOLARIZED}} = \frac{1}{2}\sum_{\lambda_1}\frac{1}{3}\sum_{\lambda_1'}\frac{1}{3}\sum_{\lambda_2}\sum_{\lambda_2'}|\mathcal{M}(\lambda_1\lambda_1' \to \lambda_2\lambda_2')|^2 \tag{13}
$$

For brevity, we split this into quantities labeled I, II, III and IV. We also use $|\mathcal{M}|^2 = (i\mathcal{M})(i\mathcal{M})^*$.

$$
|\mathcal{M}|^2_{\text{UNPOLARIZED}} = \text{I} + \text{II} + \text{III} + \text{IV} \tag{14}
$$

$$
\text{I} = \frac{8}{9}\left[\frac{g^2}{(p'-p)^2-m_h^2}\right]^2\left[\epsilon^{\alpha *}(p')\epsilon_{\alpha}(p)\right]\left[\epsilon^{\gamma *}(k')\epsilon_{\gamma}(k)\right]\left[\epsilon^{\delta *}(k)\epsilon_{\delta}(k')\right]\left[\epsilon^{\beta *}(p)\epsilon_{\beta}(p')\right] \tag{15}
$$

$$
\text{II} = \frac{8}{9}\left[\frac{g^2}{(p'-p)^2-m_h^2}\right]\left[\frac{g^2}{(p'-k)^2-m_h^2}\right]\left[\epsilon^{\alpha *}(p')\epsilon_{\alpha}(p)\right]\left[\epsilon^{\gamma *}(k')\epsilon_{\gamma}(k)\right]\left[\epsilon^{\delta *}(p)\epsilon_{\delta}(k')\right]\left[\epsilon^{\beta *}(k)\epsilon_{\beta}(p')\right] \tag{16}
$$

$$
\text{III} = \frac{8}{9}\left[\frac{g^2}{(p'-p)^2-m_h^2}\right]\left[\frac{g^2}{(p'-k)^2-m_h^2}\right]\left[\epsilon^{\alpha *}(p')\epsilon_{\alpha}(k)\right]\left[\epsilon^{\gamma *}(k')\epsilon_{\gamma}(p)\right]\left[\epsilon^{\delta *}(k)\epsilon_{\delta}(k')\right]\left[\epsilon^{\beta *}(p)\epsilon_{\beta}(p')\right] \tag{17}
$$

$$
\text{IV} = \frac{8}{9}\left[\frac{g^2}{(p'-k)^2-m_h^2}\right]^2\left[\epsilon^{\alpha *}(p')\epsilon_{\alpha}(k)\right]\left[\epsilon^{\gamma *}(k')\epsilon_{\gamma}(p)\right]\left[\epsilon^{\delta *}(p)\epsilon_{\delta}(k')\right]\left[\epsilon^{\beta *}(k)\epsilon_{\beta}(p')\right] \tag{18}
$$

We now simplify I, II, III, and IV separately.

1 Simplifying Term I:

We can rearrange the bracketed terms in the following quantity because they are c-numbers, and thus commute.

$$\mathrm{I} \propto \left[\epsilon^{\alpha *}(p')\epsilon_{\alpha}(p)\right]\left[\epsilon^{\gamma *}(k')\epsilon_{\gamma}(k)\right]\left[\epsilon^{\delta *}(k)\epsilon_{\delta}(k')\right]\left[\epsilon^{\beta *}(p)\epsilon_{\beta}(p')\right] \tag{19}$$

This yields:

$$\mathrm{I} \propto \Bigg(\left[\epsilon^{\beta *}(p)\epsilon_{\beta}(p')\right]\left[\epsilon^{\alpha *}(p')\epsilon_{\alpha}(p)\right]\Bigg)\Bigg(\left[\epsilon^{\gamma *}(k')\epsilon_{\gamma}(k)\right]\left[\epsilon^{\delta *}(k)\epsilon_{\delta}(k')\right]\Bigg) \tag{20}$$

We define the following.

$$\mathcal{A}_{\mathrm{I}} = \Bigg(\left[\epsilon^{\beta *}(p)\epsilon_{\beta}(p')\right]\left[\epsilon^{\alpha *}(p')\epsilon_{\alpha}(p)\right]\Bigg) \tag{21}$$

$$\mathcal{B}_{\mathrm{I}} = \Bigg(\left[\epsilon^{\gamma *}(k')\epsilon_{\gamma}(k)\right]\left[\epsilon^{\delta *}(k)\epsilon_{\delta}(k')\right]\Bigg) \tag{22}$$

$$\mathcal{A}_{\mathrm{I}} = \sum_{\lambda_1}\sum_{\lambda_1'} \epsilon_{\alpha}(p,\lambda_1)\epsilon^{\beta *}(p,\lambda_1)\epsilon_{\beta}(p',\lambda_1')\epsilon^{\alpha *}(p',\lambda_1') = \left(-\eta_{\alpha}{}^{\beta} + \frac{p_{\alpha}p^{\beta}}{m_{\mathrm{X}}^2}\right)\left(-\eta_{\beta}{}^{\alpha} + \frac{p_{\beta}'p'^{\alpha}}{m_{\mathrm{X}}^2}\right) \tag{23}$$

$$\mathcal{B}_{\mathrm{I}} = \sum_{\lambda_2}\sum_{\lambda_2'} \epsilon_{\delta}(k',\lambda_2')\epsilon^{\gamma *}(k',\lambda_2')\epsilon_{\gamma}(k,\lambda_2)\epsilon^{\delta *}(k,\lambda_2) = \left(-\eta_{\delta}{}^{\gamma} + \frac{k_{\delta}'k'^{\gamma}}{m_{\mathrm{X}}^2}\right)\left(-\eta_{\gamma}{}^{\delta} + \frac{k_{\gamma}k^{\delta}}{m_{\mathrm{X}}^2}\right) \tag{24}$$

$$\mathcal{A}_{\mathrm{I}} = 4 - \frac{\left(p'\cdot p'\right)}{m_{\mathrm{X}}^2} - \frac{\left(p\cdot p\right)}{m_{\mathrm{X}}^2} + \frac{\left(p\cdot p'\right)^2}{m_{\mathrm{X}}^4} = 2 + \frac{\left(p\cdot p'\right)^2}{m_{\mathrm{X}}^4} \tag{25}$$

$$\mathcal{B}_{\mathrm{I}} = 4 - \frac{\left(k\cdot k\right)}{m_{\mathrm{X}}^2} - \frac{\left(k'\cdot k'\right)}{m_{\mathrm{X}}^2} + \frac{\left(k'\cdot k\right)^2}{m_{\mathrm{X}}^4} = 2 + \frac{\left(k'\cdot k\right)^2}{m_{\mathrm{X}}^4} \tag{26}$$

After expanding $\mathcal{A}_{\mathrm{I}}\mathcal{B}_{\mathrm{I}}$, this yields:

$$\mathrm{I} = \frac{8}{9}\left[\frac{g^2}{(p'-p)^2 - m_h^2}\right]^2\left[4 + \frac{2\left(k'\cdot k\right)^2}{m_{\mathrm{X}}^4} + \frac{2\left(p\cdot p'\right)^2}{m_{\mathrm{X}}^4} + \frac{\left(p\cdot p'\right)^2\left(k'\cdot k\right)^2}{m_{\mathrm{X}}^8}\right] \tag{27}$$

We now move on to simplifying Terms II, III and IV.

2 Simplifying Term IV

We shall compute this before Terms II or III because the calculation resembles that which we just did for Term I.

$$\mathrm{IV} \propto \left[\epsilon^{\alpha *}(p')\epsilon_{\alpha}(k)\right]\left[\epsilon^{\gamma *}(k')\epsilon_{\gamma}(p)\right]\left[\epsilon^{\delta *}(p)\epsilon_{\delta}(k')\right]\left[\epsilon^{\beta *}(k)\epsilon_{\beta}(p')\right] \tag{28}$$

The bracketed terms commute, and can thus be rearranged.

$$\mathrm{IV} \propto \Bigg(\left[\epsilon^{\beta *}(k)\epsilon_{\beta}(p')\right]\left[\epsilon^{\alpha *}(p')\epsilon_{\alpha}(k)\right]\Bigg)\Bigg(\left[\epsilon^{\gamma *}(k')\epsilon_{\gamma}(p)\right]\left[\epsilon^{\delta *}(p)\epsilon_{\delta}(k')\right]\Bigg) \tag{29}$$

We now define the following.

$$\mathcal{A}_{\mathrm{IV}} = \Bigg(\left[\epsilon^{\beta *}(k)\epsilon_{\beta}(p')\right]\left[\epsilon^{\alpha *}(p')\epsilon_{\alpha}(k)\right]\Bigg) \tag{30}$$

$$\mathcal{B}_{\mathrm{IV}} = \Bigg(\left[\epsilon^{\gamma *}(k')\epsilon_{\gamma}(p)\right]\left[\epsilon^{\delta *}(p)\epsilon_{\delta}(k')\right]\Bigg) \tag{31}$$

Following a similar process to what we applied for Term I, this yields:

$$\mathcal{A}_{\mathrm{IV}} = \left(-\eta_{\alpha}{}^{\beta} + \frac{k_{\alpha}k^{\beta}}{m_{\mathrm{X}}^2}\right)\left(-\eta_{\beta}{}^{\alpha} + \frac{p'_{\beta}p'^{\alpha}}{m_{\mathrm{X}}^2}\right) = 2 + \frac{\left(k\cdot p'\right)^2}{m_{\mathrm{X}}^4} \tag{32}$$

$$\mathcal{B}_{\mathrm{IV}} = \left(-\eta_{\delta}{}^{\gamma} + \frac{k'_{\delta}k'^{\gamma}}{m_{\mathrm{X}}^2}\right)\left(-\eta_{\gamma}{}^{\delta} + \frac{p_{\gamma}p^{\delta}}{m_{\mathrm{X}}^2}\right) = 2 + \frac{\left(k'\cdot p\right)^2}{m_{\mathrm{X}}^4} \tag{33}$$

$$\mathrm{IV} = \frac{8}{9}\left[\frac{g^2}{(p'-k)^2 - m_h^2}\right]^2\left[4 + \frac{2\left(k\cdot p'\right)^2}{m_{\mathrm{X}}^4} + \frac{2\left(k'\cdot p\right)^2}{m_{\mathrm{X}}^4} + \frac{\left(k\cdot p'\right)^2\left(k'\cdot p\right)^2}{m_{\mathrm{X}}^8}\right] \tag{34}$$

We now move on to simplifying Term II and Term III.

3 Simplifying Term II

We have the following expression.

$$\mathrm{II} \propto \left[\epsilon^{\alpha *}(p')\epsilon_{\alpha}(p)\right]\left[\epsilon^{\gamma *}(k')\epsilon_{\gamma}(k)\right]\left[\epsilon^{\delta *}(p)\epsilon_{\delta}(k')\right]\left[\epsilon^{\beta *}(k)\epsilon_{\beta}(p')\right] \tag{35}$$

We can now rearrange the bracketed terms, as these commute.

$$\mathrm{II} \propto \left[\epsilon^{\beta *}(k)\epsilon_{\beta}(p')\right]\left[\epsilon^{\alpha *}(p')\epsilon_{\alpha}(p)\right]\left[\epsilon^{\delta *}(p)\epsilon_{\delta}(k')\right]\left[\epsilon^{\gamma *}(k')\epsilon_{\gamma}(k)\right] \tag{36}$$

We then have the following:

$$\mathrm{II} \propto \sum_{\lambda_1}\sum_{\lambda_1'}\sum_{\lambda_2}\sum_{\lambda_2'} \epsilon_{\gamma}(k,\lambda_2)\epsilon^{\beta *}(k,\lambda_2)\epsilon_{\beta}(p',\lambda_1')\epsilon^{\alpha *}(p',\lambda_1')\epsilon_{\alpha}(p,\lambda_1)\epsilon^{\delta *}(p,\lambda_1)\epsilon_{\delta}(k',\lambda_2')\epsilon^{\gamma *}(k',\lambda_2') \tag{37}$$

We then obtain

$$\mathrm{II} \propto \left(-\eta_{\gamma}{}^{\beta} + \frac{k_\gamma k^\beta}{m_{\mathrm{X}}^2}\right)\left(-\eta_{\beta}{}^{\alpha} + \frac{p'_\beta p'^\alpha}{m_{\mathrm{X}}^2}\right)\left(-\eta_{\alpha}{}^{\delta} + \frac{p_\alpha p^\delta}{m_{\mathrm{X}}^2}\right)\left(-\eta_{\delta}{}^{\gamma} + \frac{k'_\delta k'^\gamma}{m_{\mathrm{X}}^2}\right), \tag{38}$$

where we define

$$\mathcal{A}_{\mathrm{II}} = \left(-\eta_{\gamma}{}^{\beta} + \frac{k_\gamma k^\beta}{m_{\mathrm{X}}^2}\right)\left(-\eta_{\beta}{}^{\alpha} + \frac{p'_\beta p'^\alpha}{m_{\mathrm{X}}^2}\right) = \eta_{\gamma}{}^{\beta}\eta_{\beta}{}^{\alpha} - \frac{p'_\gamma p'^\alpha}{m_{\mathrm{X}}^2} - \frac{k_\gamma k^\alpha}{m_{\mathrm{X}}^2} + \frac{k_\gamma p'^\alpha\big(k\cdot p'\big)}{m_{\mathrm{X}}^4} \tag{39}$$

$$\mathcal{B}_{\mathrm{II}} = \left(-\eta_{\alpha}{}^{\delta} + \frac{p_\alpha p^\delta}{m_{\mathrm{X}}^2}\right)\left(-\eta_{\delta}{}^{\gamma} + \frac{k'_\delta k'^\gamma}{m_{\mathrm{X}}^2}\right) = \eta_{\alpha}{}^{\delta}\eta_{\delta}{}^{\gamma} - \frac{k'_\alpha k'^\gamma}{m_{\mathrm{X}}^2} - \frac{p_\alpha p^\gamma}{m_{\mathrm{X}}^2} + \frac{p_\alpha k'^\gamma\big(p\cdot k'\big)}{m_{\mathrm{X}}^4}. \tag{40}$$

Expanding $\mathcal{A}_{\mathrm{II}}\mathcal{B}_{\mathrm{II}}$ yields the following:

$$\begin{aligned}\mathrm{II} \propto &\left(4 - \frac{\big(k'\cdot k'\big)}{m_{\mathrm{X}}^2} - \frac{\big(p\cdot p\big)}{m_{\mathrm{X}}^2} + \frac{\big(p\cdot k'\big)^2}{m_{\mathrm{X}}^4}\right) \\ &+\left(-\frac{\big(p'\cdot p'\big)}{m_{\mathrm{X}}^2} + \frac{\big(p'\cdot k'\big)^2}{m_{\mathrm{X}}^4} + \frac{\big(p\cdot p'\big)^2}{m_{\mathrm{X}}^4} - \frac{\big(p'\cdot p\big)\big(k'\cdot p'\big)\big(p\cdot k'\big)}{m_{\mathrm{X}}^6}\right) \\ &+\left(-\frac{\big(k\cdot k\big)}{m_{\mathrm{X}}^2} + \frac{\big(k\cdot k'\big)^2}{m_{\mathrm{X}}^4} + \frac{\big(k\cdot p\big)^2}{m_{\mathrm{X}}^4} - \frac{\big(k\cdot p\big)\big(k\cdot k'\big)\big(p\cdot k'\big)}{m_{\mathrm{X}}^6}\right) \\ &+\left(\frac{\big(k\cdot p'\big)^2}{m_{\mathrm{X}}^4} - \frac{\big(k\cdot k'\big)\big(p'\cdot k'\big)\big(k\cdot p'\big)}{m_{\mathrm{X}}^6} - \frac{\big(k\cdot p\big)\big(p'\cdot p\big)\big(k\cdot p'\big)}{m_{\mathrm{X}}^6} + \frac{\big(k\cdot k'\big)\big(p'\cdot p\big)\big(k\cdot p'\big)\big(p\cdot k'\big)}{m_{\mathrm{X}}^8}\right)\end{aligned} \tag{41}$$

Simplifying:

$$\begin{aligned}\mathrm{II} = \frac{8}{9}&\left[\frac{g^2}{(p'-p)^2 - m_h^2}\right]\left[\frac{g^2}{(p'-k)^2 - m_h^2}\right]\left[\frac{\big(p\cdot k'\big)^2}{m_{\mathrm{X}}^4} + \frac{\big(p'\cdot k'\big)^2}{m_{\mathrm{X}}^4} + \frac{\big(p\cdot p'\big)^2}{m_{\mathrm{X}}^4}\right. \\ &-\frac{\big(p'\cdot p\big)\big(k'\cdot p'\big)\big(p\cdot k'\big)}{m_{\mathrm{X}}^6} + \frac{\big(k\cdot k'\big)^2}{m_{\mathrm{X}}^4} + \frac{\big(k\cdot p\big)^2}{m_{\mathrm{X}}^4} - \frac{\big(k\cdot p\big)\big(k\cdot k'\big)\big(p\cdot k'\big)}{m_{\mathrm{X}}^6} \\ &\left.+\frac{\big(k\cdot p'\big)^2}{m_{\mathrm{X}}^4} - \frac{\big(k\cdot k'\big)\big(p'\cdot k'\big)\big(k\cdot p'\big)}{m_{\mathrm{X}}^6} - \frac{\big(k\cdot p\big)\big(p'\cdot p\big)\big(k\cdot p'\big)}{m_{\mathrm{X}}^6} + \frac{\big(k\cdot k'\big)\big(p'\cdot p\big)\big(k\cdot p'\big)\big(p\cdot k'\big)}{m_{\mathrm{X}}^8}\right]\end{aligned} \tag{42}$$

We now move on to simplifying Term III.

4 Simplifying Term III

We shall see that this is equal to Term II, though rather than simply asserting this, we shall show the calculation. We have the following expression.

$$\mathrm{III} \propto \left[\epsilon^{\alpha *}(p')\epsilon_\alpha(k)\right]\left[\epsilon^{\gamma *}(k')\epsilon_\gamma(p)\right]\left[\epsilon^{\delta *}(k)\epsilon_\delta(k')\right]\left[\epsilon^{\beta *}(p)\epsilon_\beta(p')\right] \tag{43}$$

The bracketed terms commute, and can thus be rearranged, yielding the following.

$$\mathrm{III} \propto \left[\epsilon^{\beta *}(p)\epsilon_\beta(p')\right]\left[\epsilon^{\alpha *}(p')\epsilon_\alpha(k)\right]\left[\epsilon^{\delta *}(k)\epsilon_\delta(k')\right]\left[\epsilon^{\gamma *}(k')\epsilon_\gamma(p)\right] \tag{44}$$

This yields:

$$\mathrm{III} \propto \sum_{\lambda_1}\sum_{\lambda_1'}\sum_{\lambda_2}\sum_{\lambda_2'} \epsilon_\gamma(p,\lambda_1)\epsilon^{\beta *}(p,\lambda_1)\epsilon_\beta(p',\lambda_1')\epsilon^{\alpha *}(p',\lambda_1')\epsilon_\alpha(k,\lambda_2)\epsilon^{\delta *}(k,\lambda_2)\epsilon_\delta(k',\lambda_2')\epsilon^{\gamma *}(k',\lambda_2') \tag{45}$$

We then obtain

$$\mathrm{III} \propto \left(-\eta_\gamma{}^\beta + \frac{p_\gamma p^\beta}{m_{\mathrm{X}}^2}\right)\left(-\eta_\beta{}^\alpha + \frac{p'_\beta p'^\alpha}{m_{\mathrm{X}}^2}\right)\left(-\eta_\alpha{}^\delta + \frac{k_\alpha k^\delta}{m_{\mathrm{X}}^2}\right)\left(-\eta_\delta{}^\gamma + \frac{k'_\delta k'^\gamma}{m_{\mathrm{X}}^2}\right), \tag{46}$$

where we define

$$\mathcal{A}_{\mathrm{III}} = \left(-\eta_\gamma{}^\beta + \frac{p_\gamma p^\beta}{m_{\mathrm{X}}^2}\right)\left(-\eta_\beta{}^\alpha + \frac{p'_\beta p'^\alpha}{m_{\mathrm{X}}^2}\right) = \eta_\gamma{}^\beta\eta_\beta{}^\alpha - \frac{p'_\gamma p'^\alpha}{m_{\mathrm{X}}^2} - \frac{p_\gamma p^\alpha}{m_{\mathrm{X}}^2} + \frac{p_\gamma p'^\alpha\big(p\cdot p'\big)}{m_{\mathrm{X}}^4} \tag{47}$$

$$\mathcal{B}_{\mathrm{III}} = \left(-\eta_\alpha{}^\delta + \frac{k_\alpha k^\delta}{m_{\mathrm{X}}^2}\right)\left(-\eta_\delta{}^\gamma + \frac{k'_\delta k'^\gamma}{m_{\mathrm{X}}^2}\right) = \eta_\alpha{}^\delta\eta_\delta{}^\gamma - \frac{k'_\alpha k'^\gamma}{m_{\mathrm{X}}^2} - \frac{k_\alpha k^\gamma}{m_{\mathrm{X}}^2} + \frac{k_\alpha k'^\gamma\big(k\cdot k'\big)}{m_{\mathrm{X}}^4}. \tag{48}$$

Expanding $\mathcal{A}_{\mathrm{III}}\mathcal{B}_{\mathrm{III}}$ yields the following:

$$\mathrm{III} \propto \left(4 - \frac{\big(k'\cdot k'\big)}{m_{\mathrm{X}}^2} - \frac{\big(k\cdot k\big)}{m_{\mathrm{X}}^2} + \frac{\big(k\cdot k'\big)^2}{m_{\mathrm{X}}^4}\right) \tag{49}$$

$$+\left(-\frac{\big(p'\cdot p'\big)}{m_{\mathrm{X}}^2} + \frac{\big(p'\cdot k'\big)^2}{m_{\mathrm{X}}^4} + \frac{\big(p'\cdot k\big)^2}{m_{\mathrm{X}}^4} - \frac{\big(p'\cdot k'\big)\big(p'\cdot k\big)\big(k\cdot k'\big)}{m_{\mathrm{X}}^6}\right)$$

$$+\left(-\frac{\big(p\cdot p\big)}{m_{\mathrm{X}}^2} + \frac{\big(p\cdot k'\big)^2}{m_{\mathrm{X}}^4} + \frac{\big(p\cdot k\big)^2}{m_{\mathrm{X}}^4} - \frac{\big(p\cdot k'\big)\big(p\cdot k\big)\big(k\cdot k'\big)}{m_{\mathrm{X}}^6}\right)$$

$$+\left(\frac{\big(p\cdot p'\big)^2}{m_{\mathrm{X}}^4} - \frac{\big(p\cdot k'\big)\big(p'\cdot k'\big)\big(p\cdot p'\big)}{m_{\mathrm{X}}^6} - \frac{\big(p\cdot k\big)\big(p'\cdot k\big)\big(p\cdot p'\big)}{m_{\mathrm{X}}^6} + \frac{\big(p\cdot k'\big)\big(p'\cdot k\big)\big(p\cdot p'\big)\big(k\cdot k'\big)}{m_{\mathrm{X}}^8}\right)$$

Simplifying:

$$\text{III} = \frac{8}{9}\left[\frac{g^2}{(p'-p)^2 - m_h^2}\right]\left[\frac{g^2}{(p'-k)^2 - m_h^2}\right]\left[\frac{\left(k\cdot k'\right)^2}{m_X^4} + \frac{\left(p'\cdot k'\right)^2}{m_X^4} + \frac{\left(p'\cdot k\right)^2}{m_X^4}\right. \tag{50}$$

$$-\frac{\left(p'\cdot k\right)\left(k'\cdot p'\right)\left(k\cdot k'\right)}{m_X^6} + \frac{\left(p\cdot k'\right)^2}{m_X^4} + \frac{\left(p\cdot k\right)^2}{m_X^4} - \frac{\left(p\cdot k'\right)\left(p\cdot k\right)\left(k\cdot k'\right)}{m_X^6}$$

$$\left.+\frac{\left(p\cdot p'\right)^2}{m_X^4} - \frac{\left(p\cdot k'\right)\left(p'\cdot k'\right)\left(p\cdot p'\right)}{m_X^6} - \frac{\left(p\cdot k\right)\left(p'\cdot k\right)\left(p\cdot p'\right)}{m_X^6} + \frac{\left(p\cdot k'\right)\left(p'\cdot k\right)\left(p\cdot p'\right)\left(k\cdot k'\right)}{m_X^8}\right]$$

We see that the above expression is equivalent to that which we found for Term II.

5 Putting Terms I, II, III, and IV Together

$$|\mathcal{M}|^2_{\text{UNPOLARIZED}} = \text{I} + \text{II} + \text{III} + \text{IV} \tag{51}$$

$$= \frac{8}{9}\left[\frac{g^2}{(p'-p)^2 - m_h^2}\right]^2\left[4 + \frac{2\left(k'\cdot k\right)^2}{m_X^4} + \frac{2\left(p\cdot p'\right)^2}{m_X^4} + \frac{\left(p\cdot p'\right)^2\left(k'\cdot k\right)^2}{m_X^8}\right]$$

$$+\frac{16}{9}\left[\frac{g^2}{(p'-p)^2 - m_h^2}\right]\left[\frac{g^2}{(p'-k)^2 - m_h^2}\right]\left[\frac{\left(p\cdot k'\right)^2}{m_X^4} + \frac{\left(p'\cdot k'\right)^2}{m_X^4} + \frac{\left(p\cdot p'\right)^2}{m_X^4}\right.$$

$$-\frac{\left(p'\cdot p\right)\left(k'\cdot p'\right)\left(p\cdot k'\right)}{m_X^6} + \frac{\left(k\cdot k'\right)^2}{m_X^4} + \frac{\left(k\cdot p\right)^2}{m_X^4} - \frac{\left(k\cdot p\right)\left(k\cdot k'\right)\left(p\cdot k'\right)}{m_X^6}$$

$$\left.+\frac{\left(k\cdot p'\right)^2}{m_X^4} - \frac{\left(k\cdot k'\right)\left(p'\cdot k'\right)\left(k\cdot p'\right)}{m_X^6} - \frac{\left(k\cdot p\right)\left(p'\cdot p\right)\left(k\cdot p'\right)}{m_X^6} + \frac{\left(k\cdot k'\right)\left(p'\cdot p\right)\left(k\cdot p'\right)\left(p\cdot k'\right)}{m_X^8}\right]$$

$$+\frac{8}{9}\left[\frac{g^2}{(p'-k)^2 - m_h^2}\right]^2\left[4 + \frac{2\left(k\cdot p'\right)^2}{m_X^4} + \frac{2\left(k'\cdot p\right)^2}{m_X^4} + \frac{\left(k\cdot p'\right)^2\left(k'\cdot p\right)^2}{m_X^8}\right]$$

We now consider kinematics so we can write the above expression in terms of p, θ, and E_{CM}.

6 Kinematics

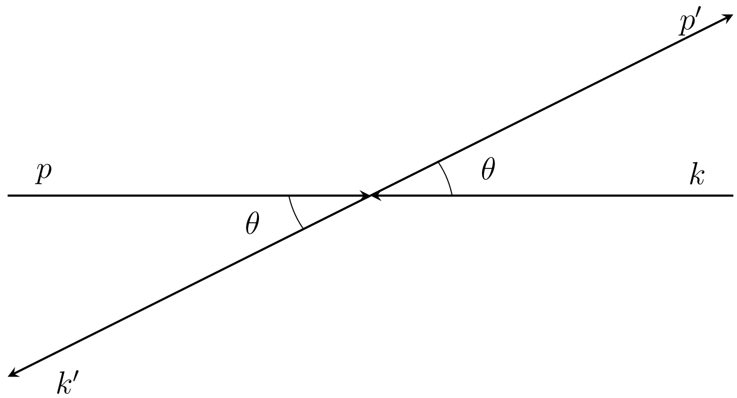

Figure 2: Here we show scattering kinematics.

Using $p = (\mathrm{E}, 0, 0, p)$, $k = (\mathrm{E}, 0, 0, -p)$, $p' = (\mathrm{E}, p\sin(\theta), 0, p\cos(\theta))$, $k' = (\mathrm{E}, -p\sin(\theta), 0, -p\cos(\theta))$, $\eta_{\mu\nu} = \mathrm{diag}(1, -1, -1, -1)$ and $p \cdot k = \eta_{\mu\nu} p^\mu k^\nu$, we obtain the following.

Mandelstam Variables:

$$s = (p + k)^2 = (p' + k')^2 = (2\mathrm{E})^2 = \mathrm{E}_{\mathrm{CM}}^2 \tag{52}$$

$$t = (p' - p)^2 = (k' - k)^2 = -2p^2(1 - \cos(\theta)) \tag{53}$$

$$u = (k' - p)^2 = (p' - k)^2 = -2p^2(1 + \cos(\theta)) \tag{54}$$

For future brevity, we define the following quantities:

$$C_1 = \tfrac{1}{4}\mathrm{E}_{\mathrm{cm}}^2 - \mathrm{p}^2\cos\theta \tag{55}$$

$$C_2 = \tfrac{1}{4}\mathrm{E}_{\mathrm{cm}}^2 + \mathrm{p}^2\cos\theta \tag{56}$$

$$C_3 = \tfrac{1}{4}\mathrm{E}_{\mathrm{cm}}^2 + \mathrm{p}^2 \tag{57}$$

We then obtain:

$$p \cdot p' = \mathrm{E}^2 - p^2\cos(\theta) = \frac{1}{4}\mathrm{E}_{\mathrm{CM}}^2 - p^2\cos(\theta) = \mathrm{C}_1 \tag{58}$$

$$k' \cdot k = \mathrm{E}^2 - p^2\cos(\theta) = \frac{1}{4}\mathrm{E}_{\mathrm{CM}}^2 - p^2\cos(\theta) = \mathrm{C}_1 \tag{59}$$

$$k \cdot p' = \mathrm{E}^2 + p^2\cos(\theta) = \frac{1}{4}\mathrm{E}_{\mathrm{CM}}^2 + p^2\cos(\theta) = \mathrm{C}_2 \tag{60}$$

$$k \cdot p = \mathrm{E}^2 + p^2 = \frac{1}{4}\mathrm{E}_{\mathrm{CM}}^2 + p^2 = \mathrm{C}_3 \tag{61}$$

$$p' \cdot k' = \mathrm{E}^2 + p^2 = \frac{1}{4}\mathrm{E}_{\mathrm{CM}}^2 + p^2 = \mathrm{C}_3 \tag{62}$$

$$p \cdot k' = \mathrm{E}^2 + p^2\cos(\theta) = \frac{1}{4}\mathrm{E}_{\mathrm{CM}}^2 + p^2\cos(\theta) = \mathrm{C}_2 \tag{63}$$

7 S-Matrix Amplitude

We can now write our result for $|\mathcal{M}|^2_{\text{UNPOLARIZED}}$ in the more compact form below, where we use Mathematica's FullSimplify function to condense the final result, after first writing $|\mathcal{M}|^2_{\text{UNPOLARIZED}}$ in terms of C_1, C_2, C_3, p, and $\cos(\theta)$.

$$\begin{aligned}|\mathcal{M}|^2_{\text{UNPOLARIZED}} = \frac{8g^4}{9m_{\text{X}}^8}\Bigg[&\frac{4m_{\text{X}}^4(C_1^2+C_2^2+C_3^2)+2C_1^2C_2^2-8C_1C_2C_3m_{\text{X}}^2}{(m_h^2+2p^2)^2-4p^4\cos^2\theta} \\ &+\frac{(C_1^2+2m_{\text{X}}^4)^2}{(m_h^2-2p^2\cos\theta+2p^2)^2}+\frac{(C_2^2+2m_{\text{X}}^4)^2}{(m_h^2+2p^2\cos\theta+2p^2)^2}\Bigg]\end{aligned} \tag{64}$$

Note again that we define C_1, C_2, and C_3 as the following.

$$\begin{aligned}C_1 &= \tfrac{1}{4}\mathrm{E}_{\mathrm{cm}}^2 - \mathrm{p}^2\cos\theta \\ C_2 &= \tfrac{1}{4}\mathrm{E}_{\mathrm{cm}}^2 + \mathrm{p}^2\cos\theta \\ C_3 &= \tfrac{1}{4}\mathrm{E}_{\mathrm{cm}}^2 + \mathrm{p}^2\end{aligned}$$

8 Differential Cross Section

This then yields the differential cross section, shown below.

$$\frac{d\sigma}{d\Omega} = \frac{|\mathcal{M}|^2_{\text{UNPOLARIZED}}}{64\pi^2{\mathrm{E}_{\mathrm{cm}}}^2} \tag{65}$$

$$d\Omega = d\phi\sin(\theta)d\theta \tag{66}$$