The Tate Curve (Silverman Vol. II, ch. V)

$$G_k(\tau) := \sum_{\substack{\lambda \in \mathbb{Z}+\mathbb{Z}\tau \\ \lambda \neq 0}} \frac{1}{\lambda^{2k}}$$

$\tau \in \mathcal{H} \rightsquigarrow \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. This is $\mathbb{C}$-pts. of $E: y^2 = 4x^3 - g_2(\tau)x - g_3(\tau)$.

$G_k(\tau)$ and $\wp(z,\tau)$ are unchanged by $\begin{matrix} z \mapsto z+1, \\ \tau \mapsto \tau+1. \end{matrix}$ We can use variables $q := e^{2\pi i \tau}$, $u := e^{2\pi i z}$.

$$s_k(q) := \sum_{n \geq 1} \frac{n^k q^k}{1-q^n} = \sum_{n \geq 1} \overbrace{\sigma_k(n)}^{\sum_{d \mid n} d^k} q^n.$$ We get an explicit uniformization

$$\begin{tikzcd} \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau) \arrow[r, "\sim"] \arrow[d, "\cong"', "z \mapsto u"] & E(\mathbb{C}) \\ \mathbb{C}^\times/q^{\mathbb{Z}} \arrow[ur, "u \mapsto (\wp(u,q),\, \wp'(u,q))"'] & \end{tikzcd}$$

We can in fact change variables to consider

$$E_q: \ y^2 + xy = x^3 + a_4(q)x + a_6(q).$$

Facts: (1) $a_4(q), a_6(q) \in \mathbb{Z}[[q]]$. $\underbrace{x(u,q),\, y(u,q)}$ also have integer coeffs., but not quite power series.

[Describe $\mathbb{C}^\times/q^{\mathbb{Z}} \to E_q(\mathbb{C})$.]

(2) Usual expansion for $j(E_q)$.

(3) $\Delta(E_q) = q \prod_{n \geq 1} (1-q^n)^{24}$.

[All of this works for $q \in$ unit disk in $\mathbb{C}$]

Let $L/\mathbb{Q}_p$ be fin. ext. $q \in L^\times$ w/ $|q|<1 \Rightarrow$ power series in (1) converge $\forall u \in L^\times$. We get Tate curve

$E_q: \ y^2 + xy = x^3 + a_4(q)x + a_6(q)$ over L. We also have $L^\times/q^{\mathbb{Z}} \xrightarrow{(*)} E_q(L)$, defined by x, y.

Thm (Tate): (1) $(*)$ is grp. isom. (2) E_q has split multiplicative reduction:

$$\begin{tikzcd} & \text{thing over } \mathcal{O}_L \arrow[dl, "\text{residue}"'] \arrow[dr, "\text{generic}"] & \\ \mathbb{G}_m & & E_q \end{tikzcd}$$

(3) E elliptic curve $/L$ w/ split mult. red. $\Rightarrow \exists!\ q \in L$ w/ $|q|<1$ s.t. $E \cong E_q$.

Remark: Every $\sigma \in \mathrm{Gal}(\bar{L}/L)$ is cont. so if $u \in \bar{L}^\times$ then $x(u,q)^\sigma = x(u^\sigma, q^\sigma) = x(u^\sigma, q)$. Same for y.

Hence, $\bar{L}^\times/q^{\mathbb{Z}} \xrightarrow{\sim} E_q(\bar{L})$ is $\mathrm{Gal}(\bar{L}/L)$-equivariant $\Rightarrow E_q[N](\bar{L}) \cong \langle \zeta_N, q^{1/N} \rangle \leq \bar{L}^\times$ as

$\mathrm{Gal}(\bar{L}/L)$-modules (so Tate module of Tate curve is something very explicit).

$(*)$ is an analytic thing, not algebraic because (convergent) power series!

Affinoids play the role of open affine subschemes but have compactness properties.

C.f. Ex. 2.1.1

Conn. affinoid := complement of finite union of open disks

Affinoid := finite union of conn. affinoids

Conn. components are unshaded bits

- closed disk w/ smaller open disk removed
- closed disk

<u>Ex 2.1.2</u>: $f(z) := \prod_{i=1}^{N} (z-a_i)^{n_i} \in k(z)$ non-constant rational function.

(1) $\{ a \in \mathbb{P} : |f(a)| \le c \}$ is affinoid.

(2) $f : \mathbb{P} \to \mathbb{P}$ pulls back affinoids to affinoids,

<u>Goal</u>: $F \subseteq \mathbb{P}$ affinoid subset $\Rightarrow$ we want to define ring of "holomorphic functions" $\mathcal{O}(F)$ which are Banach algebras

$F = F_1 \sqcup \cdots \sqcup F_n$ conn. component decomp. $\Rightarrow \mathcal{O}(F) = \mathcal{O}(F_1) \times \cdots \times \mathcal{O}(F_n)$ product of integral domains.

$\left(\underset{\tilde{k}}{\overset{\shortparallel}{k^\circ / k^{\circ\circ}}}\right)$

B Banach k-alg. $\rightsquigarrow \overline{B} := B^\circ / B^{\circ\circ}$ Banach $\tilde{k}$-alg.

<u>Ex</u>: $k\langle z \rangle := \{ f = \sum_{i \ge 0} a_i z^i \in k[\![z]\!] : |a_i| \to 0 \}$ w/ Gauss norm $\|f\| := \sup_{i \ge 0} \{ a_i \}$.

$k\langle z \rangle^\circ = k\langle z \rangle \cap k^\circ [\![z]\!]$, $k\langle z \rangle^{\circ\circ} = \ker(k\langle z \rangle^\circ \to \tilde{k}[z])$. ($\leftarrow$ polyn. ring) $\Rightarrow \overline{k\langle z \rangle} = \tilde{k}[z]$.

<u>Lemma</u> (Gauss): $f, g \in k\langle z \rangle \Rightarrow \|fg\| = \|f\| \cdot \|g\|$. (<u>Not</u> proved by Gauss, but uses technique of Gauss's Lemma).

(NB: Banach algebras only required to be submultiplicative in general)

<u>Prop</u>: $f \in k[\![z]\!]$ lies in $k\langle z \rangle$ iff it converges at every pt. of $\{ x \in \hat{k} : |x| \le 1 \}$.

<u>Prop</u>: $f \in k\langle z \rangle \Rightarrow \|f\| = \sup_{z \in D} \{ |f(z)| \}$ and this is actually a max.

<u>Pf</u>: Rescale to get $\|f\| = 1$ hence $f \in k^\circ [\![z]\!]$. $\bar{f} \in \tilde{k}[z]$ is nonzero. Easy to show $\sup_{z \in D} \{ |f(z)| \} \le 1$.

Now we need z s.t. $|f(z)| = 1$. $\tilde{k}$ infinite (achieved, e.g., if $\hat{k}$ is alg. closed) $\Rightarrow \exists\, \bar{z} \in \tilde{k}$ s.t. $\bar{f}(\bar{z}) \ne 0$. Choose lift $z \in D$ of $\bar{z}$.

$\overline{f(z)} = \bar{f}(\bar{z}) \ne 0 \Rightarrow f(z) \notin k^{\circ\circ} \Rightarrow |f(z)| = 1$. $\square$

k complete nonarch. valued field, always! This ring of integers k°, which is val. ring w/ maximal ideal $k^{\circ\circ}$.

We'll start w/ Ch. II, on the projective line. We will assume k is alg. closed, for simplicity.

$F \subseteq \mathbb{P}$ affinoid $\rightsquigarrow$ $Rat(F) \subseteq k(z)$ subring of rational functions w/ poles lying outside of F.

Given $f \in Rat(F)$, $\|f\| := \sup\{|f(a)| : a \in F\}$. $Rat(F)$ is normed k-alg.

$\mathcal{O}(F) :=$ completion of $Rat(F)$ w.r.t. $\|\cdot\|$.

Ex: Consider closed disk $D := \{z \in \mathbb{P} : |z| \le 1\}$. $|a| > 1$

$\Rightarrow \frac{1}{z-a} = \frac{-1/a}{1-(z/a)} = -\frac{1}{a}\left[1 + \frac{z}{a} + \frac{z^2}{a^2} + \cdots\right] \in k\langle z\rangle$. This shows $\mathcal{O}(D) = k\langle z\rangle$.

For a "disk centered at ∞", we get power series in $1/z$.

$F := \{z \in \mathbb{P} : |z| \ge r\}$

$\Rightarrow \mathcal{O}(F) \cong \{\sum_{i\ge 0} a_i \frac{1}{z^i} : \frac{1}{r^i}|a_i| \to 0\}$.

Thm (Divison Algorithm): Let $f \in k\langle z\rangle$ w/ $\|f\| = 1$ (hence $\tilde{f} \in \tilde{k}[z]$ is nonzero). Given $g \in k\langle z\rangle$, $\exists!$ $q \in k\langle z\rangle$ and $r \in k[z]$ s.t. $g = fq + r$ and $\deg(r) < \deg(\tilde{f})$. Moreover, $\|g\| = \max\{\|fq\|, \|r\|\} = \max\{\|q\|, \|r\|\}$.

We won't dwell on the proof, and in the future we will encounter much more elaborate versions.

Prop: Let $f \in k\langle z\rangle$.

(1) $\exists$ factorization $f(z) = u(z)p(z)$ w/ $u(z) \in k\langle z\rangle^\times$, $p(z) \in k[z]$ (hence $f(z)$ has fin. many zeros, if it is nonzero).

(2) TFAE: (i) $f \in k\langle z\rangle^\times$; (ii) $f = c(1+s)$ w/ $c \in k^\times$, $s \in k\langle z\rangle^{\circ\circ}$; (iii) f nowhere vanishing on D.

(iv) $\forall z \in D$: $|f(z)| = \|f\|$.

$F = \{z \in \mathbb{P} : |z| = 1\}$ is conn. affinoid (as complement of union of $|z| < 1$ and $|z| > 1$).

Prop: (1) $\mathcal{O}(F) = k\langle z, z^{-1}\rangle = \{f = \sum_{i\in\mathbb{Z}} a_i z^i : \lim_{i\to\infty} |a_i| = 0 = \lim_{i\to -\infty} |a_i|\}$, $\|f\| = \max\{|a_i|\}$.

(2) $\mathcal{O}(F)^\circ$ (resp. $\mathcal{O}(F)^{\circ\circ}$) same but w/ coeffs. in k° (resp. $k^{\circ\circ}$). $\widetilde{\mathcal{O}(F)} = \tilde{k}[z, z^{-1}]$.

(3) $f \in k\langle z, z^{-1}\rangle$ factors as $f = up$ w/ $u \in k\langle z, z^{-1}\rangle^{\times}$ and $p \in k[z]$.

(4) TFAE: Let $f \in k\langle z, z^{-1}\rangle$.

(i) f unit.

(ii) $f(z) = cz^n(1+s(z))$, $c \in k^{\times}$, $\underbrace{n \in \mathbb{Z}}$, $s \in k\langle z, z^{-1}\rangle^{\circ\circ}$ (This factor kind of interesting)

(iii) $\forall z \in F: |f(z)| = \|f\|$.

(iv) f nowhere vanishing on F.

Q: What is $\mathcal{O}(F)$ for general affinoid F?

Prop: $F = F_1 \sqcup \cdots \sqcup F_\ell$ w/ each conn. affinoid. Then, $\mathcal{O}(F) \cong \mathcal{O}(F_1) \times \cdots \times \mathcal{O}(F_\ell)$ w/ product norm. (not obvious!)

Pf: Suppose $E = E_1 \sqcup E_2$ w/ E_1 affinoid and E_2 conn. affinoid. After applying elt. of $PGL_2(k)$ can assume $\exists\ \pi \in k$ w/ $0 < |\pi| < 1$ s.t. $E_2 \subseteq \{|z| > |\pi^{-1}|\}$ and $E_1 \subseteq \{|z| < |\pi|\}$. [sort of separating into southern and northern hemispheres] Consider seq. of rational functions $f_n(z) := \dfrac{z^n}{1+z^n}$. On E_1: $f_n^{(z)} \to 0$ uniformly and on E_2: $f_n(z) \to 1$ uniformly. This gives indicator function which yields "orthogonal decomposition." $\square$

Lemma: Suppose $f \in \mathcal{O}(F)$ vanishes at $a \in F$. Then, $\exists\ g \in \mathcal{O}(F)$ s.t. $f(z) = (z-a)g(z)$. (key is that g must be "holomorphic")

Pf: Choose $f_n \in Rat(F)$ s.t. $f_n \to f$. Then, $f_n(a) \to f(a) = 0$. $\{f_n(z) - f_n(a)\}$ still Cauchy converging to f. So can assume $f_n(a) = 0$. So, $f_n(z) = (z-a)g_n(z)$ w/ $g_n \in Rat(F)$. We need g_n (uniformly) Cauchy.

First work on closed disk $D = \{z \in \mathbb{P} : |z-a| < r\}$ small enough s.t. $D \subseteq F$. So,

$\mathcal{O}(D) = \{g = \sum_{i \geq 0} c_i(z-a)^i : r^i|c_i| \to 0\}$. $\|(z-a)g\| = r\|g\|$ (comes from shifting the series) $\Rightarrow f_n|_D$ Cauchy $\Rightarrow g_n|_D$ Cauchy.

$\square$

Second work outside of D.

Thm: Let $F \subseteq \mathbb{P}$ be conn. affinoid w/ $\infty \notin F$. (There must be at least one pt. omitted, so just assume we omit ∞.)

(1) $f \in \mathcal{O}(F)$ factors as $f = up$ w/ $u \in \mathcal{O}(F)^\times$, $p \in k[x]$.

(2) $\mathcal{O}(F)$ is PID w/ maximal ideals of form $(z-a)\mathcal{O}(F)$ for some $a \in F$.

(3) Given $f \in \mathcal{O}(F)$, TFAE: (i) f is a unit; (ii) f is nowhere vanishing; (iii) $\inf\{|f(z)| : z \in F\} > 0$.

Pf: (Step I) Any nonzero $f \in \mathcal{O}(F)$ has fin. many zeros; (Step II) $f \in \mathcal{O}(F)$ nowhere vanishing $\Rightarrow \inf\{|f(z)| : z \in F\} > 0$.

One proves these by writing F as finite union of pieces each $PGL_2(k)$-equivalent to something of one of two shapes.

Each has an explicit division algorithm that can be deduced! We can use this (I+II) to easily show (iii) $\Rightarrow$ (i). $\square$

<u>Cauchy's Argument Principle</u>

$F \subseteq \mathbb{P}$ conn. affinoid $\Rightarrow \mathcal{O}(F)$ is integral domain. So, we get meromorphic functions on F by considering $\mathrm{Frac}(\mathcal{O}(F))$.

For general affinoid F, we simply look at elts. of the localization of $\mathcal{O}(F)$ at the 0-ideal.

$f \in k(z)$ rational function on $\mathbb{P} \Rightarrow \sum_{a \in \mathbb{P}} \mathrm{ord}_a(f) = 0$.

Def: let $\partial D := \{z \in \mathbb{P} : |z| = 1\}$ and suppose $f \in \mathcal{O}(\partial D) = k\langle z, z^{-1}\rangle$ is nowhere vanishing.

$\Rightarrow f(z) = cz^n(1+s)$, $n \in \mathbb{Z}$, $c \in k^\times$, $s \in k\langle z, z^{-1}\rangle^{oo}$. Define $\mathrm{ord}_{\partial D}(f) := n$.

f meromorphic on closed unit disk $|z| \leq 1$ w/ no zeros or poles on ∂D. Then,

Thm (Argument Principle):

$$\mathrm{ord}_{\partial D}(f) = \sum_{|a|<1} \mathrm{ord}_a(f).$$

Pf: (Case 1) f meromorphic on all of $\mathbb{P}$ (rational function). Then, $\sum_{a \in \mathbb{P}} \mathrm{ord}_a(f) = 0$. No zeros or poles at $\infty \Rightarrow$ can factor as product of $\frac{z-a}{z-b}$ (and some constant factor). More generally need $1/z$, etc. Check these by direct calculation.

(Case 2) For f holomorphic, choose uniformly approximating rational functions on closed unit disk.

(Case 3) For f meromorphic deal w/ the quotient. $\square$

Warning:

$\mathrm{ord}_{\partial D}(f)$ is not invariant under arbitrary automorphisms of ∂D.

The fix is we need some kind of "orientation." $\overline{\mathcal{O}(\partial D)} = \bar{k}[z,z^{-1}]$ w/ units given by powers of z times scalars. $\overline{\mathcal{O}(\partial D)}^\times / K^\times = \{\ldots, z^{-1}, 1, z, \ldots\}$.

Def: Orientation of ∂D is isom. $\mathbb{Z} \xrightarrow{\sim} \overline{\mathcal{O}(\partial D)}^\times / K^\times$.

Prop: Orientation-preserving automorphisms of ∂D preserve $\mathrm{ord}_{\partial D}$.

Remark: This principle is only relevant for curves. Maybe we can get extra mileage by thinking of this as having an arithmetic winding #.

Goal: Generalize the ~~···~~ argument principle.

Let $F \subseteq \mathbb{P}$ be conn. affinoid and write $F = \mathbb{P} - (D_1 \sqcup \cdots \sqcup D_r)$ w/ D_i open disk. Pick some $q \in F$ ("elect a mayor of F-ville"). For each D_i choose local coord. $t_i(z) = \dfrac{a_i z + b_i}{c_i z + d_i}$ s.t. (1) $t_i(q) = \infty$ and (2) $D_i \xrightarrow{\sim}$ open disk $|z|<1$ via $z \mapsto t_i(z)$. Define $\partial D_i := \{z \in \mathbb{P} : |t_i(z)| = 1\}$. This depends on q but not t_i! [Why do we need to do this?]

We get orientation $\mathbb{Z} \xrightarrow{\sim} \overline{\mathcal{O}(\partial D_i)}^\times / K^\times$, $1 \mapsto t_i$. Put "standard" orientation on ∂D. We have orientation-preserving isom. $\partial D_i \xrightarrow{\sim} \partial D$, $z \mapsto t_i(z)$. This gives us $\mathrm{ord}_{\partial D_i}$.

(reversed orientation from what we did before)

Thm: f meromorphic function on F w/ no zeros or poles on any ∂D_i. Then, $\sum_{a \in F} \mathrm{ord}_a(f) = -\sum_i \mathrm{ord}_{\partial D_i}(f|_{\partial D_i})$.

Residue Thm (classical): f meromorphic on open nbhd of closed unit disk in $\mathbb{C}$ w/ no zeros or poles on $|z|=1$

$$\Rightarrow \frac{1}{2\pi i} \int_{|z|=1} f(z)\, dz = \sum_{|a|<1} \mathrm{Res}_a(f).$$

Let $F \subseteq \mathbb{P}$ affinoid w/ $\infty \notin F$.

Def: Meromorphic differential form on F is (formal) $\omega = f(z)\,dz$ w/ f meromorphic.

Given $a \in F$ choose $t \in \mathcal{O}(F)$ w/ $\mathrm{ord}_a(t) = 1$. On small closed disk around a we have expansion $\omega = \left(\sum_{n \gg -\infty} c_n t^n\right) dt$.

Define $\mathrm{Res}_a(\omega) := c_{-1}$, which is independent of t. By algebraic geometry, $\omega = f(z)\,dz$ w/ $f \in k(z)$

$\Rightarrow \sum_{a \in \mathbb{P}} \mathrm{Res}_a(\omega) = 0$. Working on ∂D, suppose $\omega = f(z)\,dz$ w/ $f \in \mathcal{O}(\partial D) = k\langle z, z^{-1}\rangle$ a unit.

Expand $f(z) = \sum_{i=-\infty}^{\infty} a_i z^i$ and define $\mathrm{Res}_{\partial D}(\omega) := a_{-1}$.

Check (tedious): This is invariant under orientation-preserving automorphisms. ~~Miracle~~(Miracle: ∂D is "far" from the interior of D)

Residue Thm: ω meromorphic diff. form on closed unit disk $|z| \leq 1$ w/ no poles on ∂D. Then,

$$\mathrm{Res}_{\partial D}(\omega) = \sum_{|a|<1} \mathrm{Res}_a(\omega).$$

Pf: Write $\omega = f(z)\,dz$. Assume WLOG $f(z) = (z-a)^{-n}$ w/ $|a| < 1$. Using $z \mapsto z + a$ can assume $f(z) = z^{-n}$.

Now explicitly compute both sides. $\square$