Meromorphic Functions $\qquad (R \subseteq X \text{ rational})$

$X = Sp(A)$ has presheaf of meromorphic functions $M_X(R) :=$ total ring of fractions of $\mathcal{O}_X(R)$. The fact that restriction is actually well-behaved relies on flatness. Want to show M_X is sheaf.

Lemma: $f \in M_X(X)$, $R \subseteq X$ rational $\rightsquigarrow I(R) := \{a \in \mathcal{O}_X(R) : \overbrace{af \in \mathcal{O}_X(R)}^{(at \in n\mathcal{O}_X(R))}\}$. This is coherent ideal sheaf.

$\mathcal{O}_X(R) \otimes_{\mathcal{O}_X(X)} I(X)$

Pf: This must be $\widetilde{I(X)}$. We have $\widetilde{I(X)}(R) \to I(R)$, which we need to show is isom. Write $f = \frac{t}{n}$.

Consider the $\mathcal{O}(X)$-mod. $N = \frac{t\mathcal{O}(X) + n\mathcal{O}(X)}{n\mathcal{O}(X)}$. We have exact seq. of $\mathcal{O}_X(X) = A$-modules

$$(*) \qquad 0 \to I(X) \to \mathcal{O}(X) \xrightarrow{a \mapsto ta} N \to 0. \qquad \text{Now use flatness of } \mathcal{O}(X) \to \mathcal{O}(R).$$

□

Take tensor product w/ $\mathcal{O}(R)$ for $(*)$.

Thm: M_X is a sheaf.

Pf: Let $X = X_1 \cup \cdots \cup X_n$ be admissible cover. Enough to show $0 \to M_X(X) \to \prod_i M_X(X_i) \to \prod_{i,j} M_X(X_i \cap X_j)$ exact.

Suppose $f \in \ker(M_X(X) \to \prod_i M_X(X_i))$. Write $f = \frac{g}{n}$. Look at restrictions and use injectivity of localization in this case.

Sheafiness of $\mathcal{O}_X \Rightarrow f = 0$. Now we need to glue. $\{f_i \in M(X_i)\}$ w/ $f_i|_{X_i \cap X_j} = f_j|_{X_i \cap X_j}$. Find common denominators and glue. Use the ideal sheaves from before to clear denominators. The key is that $I(X)$ has an NZD. The only non-obvious thing is the alg. fact that if every elt. of $I(X)$ is a zerodivisor then there is some <u>global</u> zerodivisor.

□

Analytic Reductions

Affinoid $X = \mathrm{Sp}(A)$ has canonical reduction $\bar{X} = \mathrm{Spec}(\bar{A})$ for $\bar{A} = A^\circ / A^{\circ\circ}$. $\bar{A}$ is reduced finite type $\bar{k}$-alg.

Reduction map $\mathrm{red} : X \to \bar{X}$ (surj. on closed pts.), $x \in \mathrm{Sp}(A) \mapsto \ker(\bar{A} \to \bar{k}_x)$. This is functorial.

<u>Example</u>: Assume k alg. closed. (1) D closed unit disk $= \mathrm{Sp}\, k\langle z\rangle$. $\bar{D} = \mathrm{Spec}\, \bar{k}[z] = \mathbb{A}^1_{\bar{k}}$. $\mathrm{red}(z) = \bar{z} \in \bar{k}$.

Reduction as expected

(2) $T = \mathrm{Sp}\, k\langle z, z^{-1}\rangle = \mathrm{Sp}\, k\langle z,w\rangle / (zw-1) \Rightarrow \bar{T} = \mathrm{Spec}\, \bar{k}[z,w]/(zw-1)$.

(3) Fix pseudouniformizer $\pi \in k$ (so $0 < |\pi| < 1$). $X = \{ z \in k : |\pi| \le |z| \le 1 \}$ (annulus) $= \mathrm{Sp}\, k\langle z, \pi/z\rangle$. Reduction should be intermediate between previous two reductions. $X = \mathrm{Sp}\, k\langle z,w\rangle / (zw - \pi)$. $\bar{X} = \mathrm{Spec}\, \bar{k}[z,w]/(zw)$.

$\mathrm{red} : X \to \mathbb{A}^1_z \cup \mathbb{A}^1_w$. $|z| = 1 \Rightarrow \mathrm{red}(z) =$ pt. $\bar{z}$ on $\mathbb{A}^1_z$.

$|z| = |\pi| \Rightarrow \mathrm{red}(z) =$ pt. $\overline{(\frac{\pi}{z})}$ on $\mathbb{A}^1_w$.

$|\pi| < |z| < 1 \Rightarrow \mathrm{red}(z)$ gets sent to the crossing $w = 0 = z$.