Five main theories of rigid analytic geom.

(1) Tate's theory of rigid analytic varieties
(2) Huber's theory of adic spaces
(3) Berkovich's theory of Berkovich analytic spaces
(4) Raynaud's theory of bicat. geom. of formal schemes
(5) Fujiwara-Kato's theory

<u>More info</u>: (1) ~ Classical theory of var.'s over alg. closed fields, w/ same advantages / disadvantages

(2) ~ Theory of schemes

(3) Intermediate between theories of Huber and Tate w/ better top. behavior

(4) Makes precise idea that rigid analytic var.'s / $\mathbb{Q}_p$ are "generic fibers" of formal schemes over $\mathbb{Z}_p$.

(5) Generalizes Raynaud's theory by constructing analogue of so-called "Zariski-Riemann" space of formal scheme.

Fix k nonarch. complete (not nec. loc. compact) field w/ abs. value $|\cdot|$. Examples: $\mathbb{Q}_p$, $\widehat{\mathbb{Q}_p^{unr}}$, $\mathbb{C}_p$, $\mathbb{F}_p((t))$, $\mathbb{C}((t))$

Fix <u>pseudo-unif.</u> $\pi \in k$ - i.e., $0 < |\pi| < 1$.

<u>Prologue</u>

We want to work w/ sets like $\{x : |f(x)| \leq 1\}$ cut out by inequalities and analytic functions. [... as well as things "built up" from such functions ...]

<u>Remark</u>: We want $\mathbb{B}_{\mathbb{C}_p}$, closed unit disc in $\mathbb{C}_p$, to be conn. and qc. Conn. is forced if $\mathcal{O}_{\mathbb{B}_{\mathbb{C}_p}}(\mathbb{B}_{\mathbb{C}_p})$ is to capture something like conv. power series.

Classical theory of varieties proceeds as follows.

Step 1: Define affine space $\mathbb{A}^n_k$
Step 2: Define affine varieties as closed $V(I) \subseteq \mathbb{A}^n_k$
Step 3: Define general varieties by gluing affine varieties.

We mimic these steps.

<u>Step 1</u>

k being possibly non-alg. closed presents difficulties for naively defining polydiscs. Fortunately, $|\cdot|$ extends uniquely to $\bar{k}$ and we define $\mathbb{B}^n_k(\bar{k}) := \{(x_1, \ldots, x_n) \in \bar{k}^n : |x_i| \leq 1\}$. How do we "descend back down" to k? The first thought is that X FT k-scheme $\Rightarrow$ X (or at least closed pts.) recovered from $X(\bar{k}) / \mathrm{Aut}(\bar{k}/k)$. We'd like to say $\mathbb{B}^n_k := \mathbb{B}^n_k(\bar{k}) / \mathrm{Aut}(\bar{k}/k)$ but this is unwieldy. Inspiration comes from noting that if $X = \mathrm{Spec}\, A$ is FT k-scheme then $X(\bar{k}) / \mathrm{Aut}(\bar{k}/k) \leftrightarrow \mathrm{MaxSpec}\, A$. What should ring of analytic functions on closed unit polydisc $/k$ look like? Naive guess is

$$k\langle t_1, \ldots, t_n \rangle := \{ f(t_1, \ldots, t_n) \in k[\![t_1, \ldots, t_n]\!] : f(t_1, \ldots, t_n) \text{ conv. everywhere on } \mathbb{B}^n_k(\bar{k}) \}$$

$$= \{ \sum_{I \in \mathbb{N}^n} a_I t^I \in k[\![t_1, \ldots, t_n]\!] : \lim_{|I| \to \infty} |a_I| = 0 \}.$$