Section 5 Math 2202

Parameterizations, Level Curves and Level Surfaces

1. **Parameterizing Ellipses, Parabolas and Hyperbolas** Find a parameterization for the following curves. Specify the domain of your parameter. In other words, what values must your parameter go through in order to trace out the entire curve.

   - The curve $x^2 + 4y^2 = 9$ in the xy-plane

This is the equation of an ellipse. If we divide through by 9 we get

$\left(\frac{x}{3}\right)^2 + \left(\frac{2y}{3}\right)^2 = 1$. This suggests that $\frac{1}{3}x(t) = \cos(t)$ and $\frac{2}{3}y(t) = \sin(t)$.

That is, $x(t) = 3\cos(t)$ and $y(t) = \frac{3}{2}\sin(t)$ w/ $0 \le t < 2\pi$.

   - The curve $x - z^2 = 4$ in the xz-plane

This one is easy since we can treat x like the dependent variable and z like the independent variable. We have $z(t) = t$ and $x(t) = 4 + t^2$ w/ $-\infty < t < \infty$.

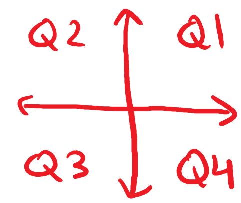
   - The curve $x^2 - y^2 = 1$ in the xy-plane
     Hint: Think about the trigonometric identity $\sec^2 t = 1 + \tan^2 t$.

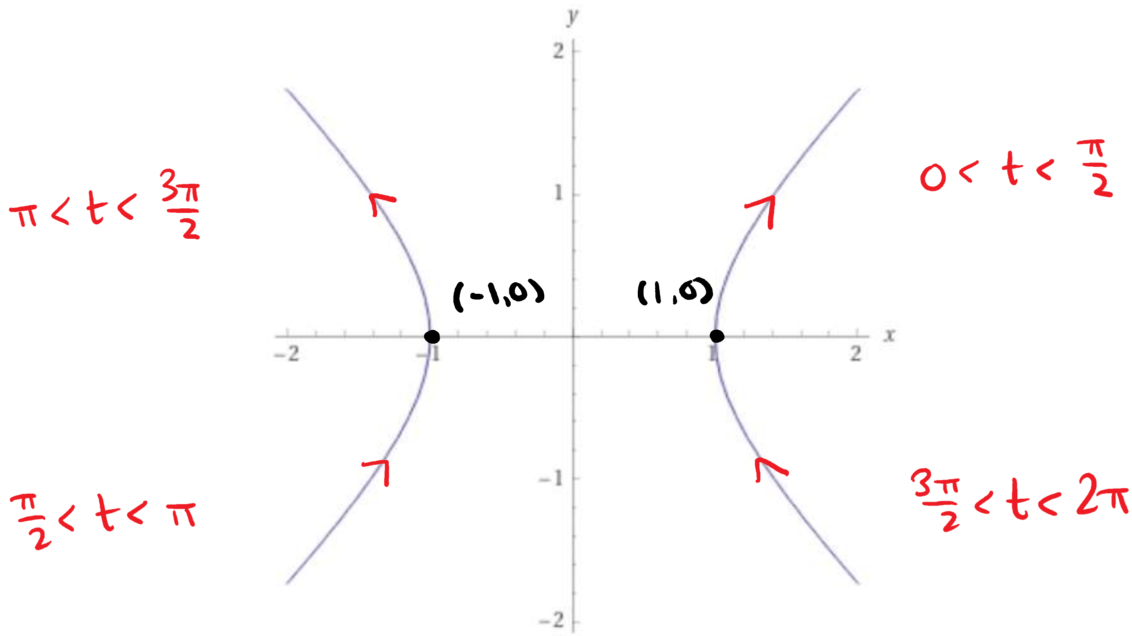
As per the hint we should take $x(t) = \sec(t)$ and $y(t) = \tan(t)$. What are the domains of these functions? Recall $\sec(t) = \frac{1}{\cos(t)}$ and $\tan(t) = \frac{\sin(t)}{\cos(t)}$. Both

are undefined when $\cos(t) = 0$, which happens when $t = \pm\frac{\pi}{2}, \pm\frac{3\pi}{2}, \ldots$

We only need t between 0 and 2π (since all functions here are periodic w/ period 2π), so we get domain $[0, \frac{\pi}{2}) \cup (\frac{\pi}{2}, \frac{3\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)$

Interval	Sign of sec(t)	Sign of tan(t)	Quadrant
$(0, \frac{\pi}{2})$	+	+	1
$(\frac{\pi}{2}, \pi)$	−	−	3
$(\pi, \frac{3\pi}{2})$	−	+	2
$(\frac{3\pi}{2}, 2\pi)$	+	−	4





Direction depends on whether we "start at" or "go to" an asymptote.

2. **Parameterizing Level Curves**

(a) Parameterize the level curves of $f(x, y) = x^2 + 4y^2$ with $k = \pm 9, \pm 10$

Note first of all that $f(x,y) = x^2 + 4y^2 = k$ has no solutions if $k < 0$.

For $k = 9$ we already did this: $x(t) = 3\cos(t)$, $y(t) = \frac{3}{2}\sin(t)$.

For $k = 10$ things are similar: $x(t) = \sqrt{10}\cos(t)$, $y(t) = \frac{\sqrt{10}}{2}\sin(t)$.

(b) Find a parameterization of the part of the curve $x^{2/3}y^{1/3} = 3$ for which $x, y \geq 0$. (Hint: can you rewrite this as a function of one variable?)
(This is a level curve of a Cobb-Douglas function. These are used in economics[1].)

It's reasonable to try $x(t) = 3^a t^b$ and $y(t) = 3^c t^d$. Then,

$3 = (3^a t^b)^{2/3} (3^c t^d)^{1/3} = 3^{2a/3 + c/3} t^{2b/3 + d/3}$. So, we want

$\frac{2a+c}{3} = 1$ and $\frac{2b+d}{3} = 0$. One way to achieve this is to take

$a = 1 = c$, $b = 1$, $d = -2$. Then, $x(t) = 3t$ and $y(t) = 3t^{-2}$.

3. Find a parameterization of the curve of intersection of the surfaces $x^2 + (y+2)^2 + (z-5)^2 = 4$ and $-3x + 4z = 20$.

   First ask yourself: what are these surfaces? what do you expect their intersection to look like?

$x^2 + (y+2)^2 + (z-5)^2 = 4$ is sphere of radius 2 w/ center $(0,-2,5)$.

$-3x + 4z = 20$ is plane that is "slanted" with respect to coordinate planes.

Intersection should be an ellipse. We have $z = \frac{20+3x}{4} = 5 + \frac{3}{4}x$ on intersection

and $4 = x^2 + (y+2)^2 + \left(\frac{3}{4}x\right)^2 = \frac{25}{16}x^2 + (y+2)^2$.

4. **Level Surfaces, or Visualizing Functions Whose Graph is 4 Dimensional**

   To visualize something like $f(x,y,z) = x^2 + y^2 + z^2$, we would need 4 dimensions to plot this in. What we do instead is use level surfaces - we look at $f(x,y,z) = k$ for values of k and plot those in 3 space. This is analogous to using level curves to understand a function of 2 variables. Try it out for the following:

   (a) $f(x,y,z) = x^2 + y^2 + z^2$
   (b) $h(x,y,z) = x + 2y + z$

(a) For $k > 0$, the level surfaces are concentric spheres centered at $(0,0,0)$.

(b) The level surfaces are a bunch of parallel planes.