Section 13 Math 2202

Line Integrals, Gradient/Conservative Vector Fields and Green's Theorem

Comments for Facilitator:

- Quiz and Going Over (20 min): 8 minutes for quiz, then go over. Make sure to emphasize how in general we can compute given a parameterization, then discuss how FTC for Line integrals makes this easier: it only depends on endpoints. That will be the focus of the next example.
- (10 min) Remind them that for a gradient/conservative vector field the line integral is the net change of the function from point A to point B. (Fundamental Theorem). They know the test for this. If we know a field is conservative, how do we find the function f? You can lead them one examples using "partial" integration as discussed p. 929-930, then let them try another. (The last two are for practice on their own outside of section.)
- (15 min) Green's theorem for finding an area of an ellipse.
- (5 min) Open Questions

1. A conservative/gradient vector field $\mathbf{F}$ is one where $\mathbf{F} = \nabla f$ for some function $f(x, y)$.

   (a) Check that $\mathbf{F}(x, y) = xy^2\mathbf{i} + x^2y\mathbf{j}$ is a conservative vector field.

   Solution:
   Here $P(x, y) = xy^2$, so $P_y = 2xy$.
   Here $Q(x, y) = x^2y$, so $Q_x = 2xy$.
   Thus $P_y = Q_x$. We know for $\mathbf{F}$ defined on all of $\mathbf{R}^2$ with P, Q having continuous partial derivatives, then $\mathbf{F}$ must be conservative if $P_y = Q_x$. (The converse is also true: if $\mathbf{F}$ is conservative, then $P_y = Q_x$.

   (b) Find f such that $\nabla f = \mathbf{F}$.

   Solution:
   If there is a function $f(x, y)$ such that:

$$\langle f_x(x, y), f_y(x, y)\rangle = \nabla f(x, y) = \mathbf{F}(x, y) = \langle xy^2, x^2y\rangle$$

   Then the components must be equal:

$$f_x(x, y) = xy^2 \tag{1}$$
$$f_y(x, y) = x^2y \tag{2}$$

   Since $f_x(x, y) = xy^2$, we know that:

$$f(x, y) = \frac{1}{2}x^2y^2 + g(y)$$

   for some function $g(y)$. (Here, we're taking "an antiderivative with respect to x" - you can check that $\frac{\partial}{\partial x}(\frac{1}{2}x^2y^2 + g(y)) = xy^2$, for any $g(y)$.)
   But we also know something about $f_y(x, y)$. Using equation (2):

$$\begin{aligned} x^2y &= f_y(x, y) \\ &= \frac{\partial}{\partial y}(\frac{1}{2}x^2y^2 + g(y)) \\ &= x^2y + g'(y) \end{aligned}$$

   So, $g'(y) = 0$, and by single-variable calculus $g(y) = K$ for some constant K. We've shown, if $\mathbf{F}(x, y) = \nabla f(x, y)$, then $f(x, y) = \frac{1}{2}x^2y^2 + K$. Let's check: fix some constant K and define $f(x, y) = \frac{1}{2}x^2y^2 + K$. Then:

$$\begin{aligned} \nabla f(x, y) &= \langle \frac{\partial}{\partial x}(\frac{1}{2}x^2y^2 + K), \frac{\partial}{\partial y}(\frac{1}{2}x^2y^2 + K)\rangle \\ &= \langle xy^2, x^2y\rangle \\ &= \mathbf{F}(x, y) \end{aligned}$$

(c) Find the value of $\int_C \mathbf{F}\, d\mathbf{r}$ where C is the line between $(-1,4)$ and $(3,5)$.
(Remember the Fundamental Theorem of Calculus for Line Integrals: if $\mathbf{F} = \nabla f$ and is a continuous vector field and C is smooth, then the integral $\int_C \mathbf{F}\, d\mathbf{r} = ...$)

Solution:

In particular, let's choose $K = 0$, so $f(x,y) = \frac{1}{2}x^2y^2$. This is one function f such that $\nabla f = \mathbf{F}$, and we can use this to compute:

$$\begin{aligned}\int_C \mathbf{F}\ d\mathbf{r} &= \int_C \nabla f\ d\mathbf{r}\\ &= f(3,5) - f(-1,4)\\ &= \frac{1}{2}3^2 \cdot 5^2 - \frac{1}{2}(-1)^2 \cdot 4^2\\ &= \frac{1}{2}(225 - 16)\\ &= \frac{209}{2}\end{aligned}$$

(d) (Extra) For some extra practice, parameterize the line between $(-1,4)$ and $(3,5)$ and compute the line integral without FTC.

2. The following fields are conservative/gradient. Find f such that $\nabla f = \mathbf{F}$.

(a) $\mathbf{F}(x,y) = (3x^2 - 2y^2)\mathbf{i} + (4xy + 3)\mathbf{j}$

(b) $\mathbf{F}(x,y) = (xy\cos(xy) + \sin(xy))\mathbf{i} + (x^2\cos(xy))\mathbf{j}$

(c) $\mathbf{F}(x,y) = 2y^{3/2}\mathbf{i} + 3x\sqrt{y}\mathbf{j}$

Solution:

(a) $\mathbf{F}(x,y) = (3x^2 + 2y^2)\mathbf{i} + (4xy + 3)\mathbf{j}$

Here's the idea: If $\mathbf{F}(x,y) = \nabla f(x,y)$,

$$f_x(x,y) = 3x^2 + 2y^2$$
$$f_y(x,y) = 4xy + 3$$

From the first equation,

$$f(x,y) = x^3 + 2y^2x + g(y)$$

For some function $g(y)$. Take the partial derivative of this with respect to y and compare with our second equation to get:

$$4xy + 3 = f_y(x,y) = 4yx + g'(y)$$

So, $g'(y) = 3$ and $g(y) = 3y + K$ for some constant K. Let's choose $K = 0$, so $f(x,y) = x^3 + 2y^2x + 3y$, and check:

$$\begin{aligned}\nabla f(x,y) &= \langle \frac{\partial}{\partial x}(x^3 + 2y^2x + 3y), \frac{\partial}{\partial y}(x^3 + 2y^2x + 3y)\rangle \\ &= \langle 3x^2 + 2y^2, 4yx + 3\rangle \\ &= \mathbf{F}(x,y)\end{aligned}$$

(b) Here's the idea: If $\mathbf{F}(x,y) = \nabla f(x,y)$,

$$f_x(x,y) = xy\cos(xy) + \sin(xy)$$
$$f_y(x,y) = x^2\cos(xy)$$

In this case, it's easier to work from the second equation. Taking antiderivative with respect to y, we get

$$f(x,y) = x\sin(xy) + g(x)$$

for some function $g(x)$. Take the partial derivative of this with respect to x (requires product rule!) and compare with our first equation to get:

$$xy\cos(xy) + \sin(xy) = f_x(x,y) = +g'(x) = \sin(xy) + xy\cos(xy) + g'(x)$$

where $g'(x)$ is derivative with respect to x.

So, $g'(y) = 0$ and thus $g(x) = K$ for some constant K. Let's choose $K = 0$, so $f(x,y) = x\sin(xy)$.

We check:

$$\begin{aligned}\nabla f(x,y) &= \langle \frac{\partial}{\partial x}(x\sin(xy)), \frac{\partial}{\partial y}(x\sin(xy))\rangle \\ &= \langle xy\cos(xy) + \sin(xy), x^2\cos(xy)\rangle \\ &= \mathbf{F}(x,y)\end{aligned}$$

(c) $\mathbf{F}(x,y) = 2y^{3/2}\mathbf{i} + 3x\sqrt{y}\mathbf{j}$

Repeat the same procedure! You'll find, if $\mathbf{F}(x,y) = \nabla f(x,y)$,

$$f(x,y) = 2y^{3/2}x + K$$

for some constant K. Let's choose $K = 0$. You can check that $\mathbf{F}(x,y) = \nabla f(x,y)$ for $f(x,y) = 2y^{3/2}x$.

3. Consider a piece-wise smooth, simple closed curve C, oriented counterclockwise, and let D be the region enclosed by C. Let $\mathbf{F}(x, y) = \langle -y, x \rangle$.

   Why does $\int_C \mathbf{F} \cdot d\mathbf{r}$ equal twice the area of D? (Hint: What does Green's Theorem say about the value of the line integral?)

   Solution: Green's Theorem applies since C satisfies the conditions and $\mathbf{F}$ is defined on all of $\mathbf{R}^2$ with continuous partial derivatives of its components, $P = -y$ and $Q = x$.

   $$\int_C \mathbf{F} \cdot d\mathbf{r} = \int\int_D Q_x - P_y \, dA = \int\int_A 1 - (-1) \, dA = \int\int_D 2 \, dA$$

   This is twice the integral $\int\int_D 1 \, dA$ which measures the area of D.

4. Consider the ellipse $\frac{x^2}{4} + \frac{y^2}{9} = 1$. Find the area of the region inside the ellipse by computing a line integral around the ellipse (the curve).

   Solution: We'll use the idea to compute the area of the region enclosed by the ellipse by computing the line integral.

   We need to parameterize $\frac{x^2}{4} + \frac{y^2}{9} = 1$. This ellipse can be parameterized as $\mathbf{r}(t) = \langle 2\cos t, 3\sin t \rangle$, where $t = 0$ to 2π.

   Then $\int_C \mathbf{F} \cdot d\mathbf{r} = \int_0^{2\pi} \mathbf{F}(\mathbf{r}(t)) \cdot \mathbf{r}'(t) \, dt = \int_0^{2\pi} \langle -3\sin t, 2\cos t \rangle \cdot \langle 2\sin t, -3\cos t \rangle \, dt = \int_0^{2\pi} 6\sin^2 t + 6\cos^2 t \, dt = \int_0^{2\pi} 6 \, dt = 6 \cdot 2\pi$.

   Thus the area of the ellipse is 6π square units.

   Note: we could do this more generally for $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ to deduce that the area of such an ellipse is $ab\pi$.