Assignment 1

due midnight (Eastern Time), Friday, Sept. 18, 2020

Prove all statements in the questions below. You must TeX your solutions and submit your writeup as a pdf at www.gradescope.com.
You can submit it any time until the deadline (try not to spend your Friday evening working on this!). (Use the course code listed in the syllabus to enroll in our class on gradescope.).

Question 1. (Folland 1.4.19) Let μ^* be an outer measure on X induced from a finite premeasure μ_0. If $E \subset X$, define the inner measure of E to be $\mu_*(E) = \mu_0(X) - \mu^*(E^c)$. Then E is μ^*-measurable iff $\mu^*(E) = \mu_*(E)$. (You may use the results Folland's exercise 1.4.18).

Question 2. (Folland 1.4.23) Let $\mathcal{A}$ be the collection of finite unions of sets of the form $(a, b] \cap \mathbb{Q}$ where $-\infty \le a < b \le \infty$.

a. $\mathcal{A}$ is an algebra on $\mathcal{Q}$. (Use Proposition 1.7.)

b. The σ-algebra generated by $\mathcal{A}$ is $\mathcal{P}(\mathbb{Q})$.

c. Define μ_0 on $\mathcal{A}$ by $\mu_0(\emptyset) = 0$ and $\mu_0(A) = \infty$ for $A \neq \emptyset$. Then μ_0 is a premeasure on $\mathcal{A}$, and there is more than one measure on $\mathcal{P}(\mathbb{Q})$ whose restriction go $\mathcal{A}$ is μ_0.

Question 3. (Folland 1.5.26) Prove Proposition 1.20. (Use Theorem 1.18.)

Question 4. (Folland 1.5.29) Let E be a Lebesgue measurable set.

a. If $E \subset N$ where N is the nonmeasurable set described in §1.1, then $m(E) = 0$.

b. If $m(E) > 0$, then E contains a nonmeasurable set. (It suffices to assume $E \subset [0, 1]$. In the notation of §1.1, $E = \bigcup_{r \in R} E \cap N_r$.)

Question 5. (Folland 1.5.30) If $E \in \mathcal{L}$ and $m(E) > 0$, for any $\alpha < 1$ there is an open interval I such that $m(E \cap I) > \alpha m(I)$.

Question 6. (Folland 1.5.31) If $E \in \mathcal{L}$ and $m(E) > 0$, the set

$$E - E := \{x - y : x, y \in E\}$$

contains an interval centered at 0.

(If I is as in Exercise 30 with $\alpha > \frac{3}{4}$, then $E - E$ contains $(-\frac{1}{2}m(I), \frac{1}{2}m(I))$.)