Thm (q-expansion principle): "Classical" notions of rationality and integrality for Siegel modular forms agree w/ notions thinking of sections of line bundles over the Siegel moduli space. (Can do half of this by looking at fields of definition for coeffs. of q-expansions).

Remark: q-expansions don't exist in general, so we really do need fine moduli space.

Hilbert Schemes

$S \in$ Sch Noe., $X \to S$ finite type, $\mathcal{E} \in \mathrm{Coh}(X)$. $\mathrm{Quot}_{\mathcal{E}/X/S} : \mathrm{Sch}_S \to \mathrm{Set}$ defined by

$\mathrm{Quot}_{\mathcal{E}/X/S}(T) := \{$ isom. classes of $(\mathcal{F}, q)$ w/ $\mathcal{F} \in \mathrm{Coh}(X_T)$ flat over $T$ w/ proper (over $T$) schematic support and $q : \mathcal{E}_T \twoheadrightarrow \mathcal{F} \}$. $(\mathcal{F}, q) \cong (\mathcal{F}', q')$ if $\mathcal{E}_T \overset{q}{\twoheadrightarrow} \mathcal{F}$, $\mathcal{E}_T \overset{q'}{\twoheadrightarrow} \mathcal{F}'$, $\exists \cong \mathcal{F} \to \mathcal{F}'$. The flatness condition means the stalk $\mathcal{F}_x$ is flat over $\mathcal{O}_{T, f(x)}$ $\forall x \in X_T$ ($f : X_T \to T$ arising from $X \to S$, $T \to S$). The schematic support $Y \subseteq X_T$ is smallest closed subscheme s.t. $\mathcal{F}$ is pushforward of coh. sheaf on $Y$. This is defined by the ideal sheaf $\mathrm{Ann}(\mathcal{F}) \subseteq \mathcal{O}_{X_T}$. [Look at the affine case.]

Remark: We could view the above in terms of subs rather than quotients, since we just need to look at kernels. [But conditions we want are more naturally imposed on the quotient.]

Case of much interest is $\mathcal{E} = \mathcal{O}_X$, giving $\mathrm{Hilb}_{X/S} := \mathrm{Quot}_{\mathcal{O}_X/X/S}$.

$\mathrm{Hilb}_{X/S}(T)$ is isom. classes of $q : \mathcal{O}_{X_T} \twoheadrightarrow \mathcal{F}$ w/ $\mathcal{F}$ flat over $T$ w/ proper schematic support.

$\leftrightarrow$ ideal sheaves $\mathcal{I} = \ker q \trianglelefteq \mathcal{O}_{X_T}$ s.t. $\mathcal{O}_{X_T}/\mathcal{I}$ is flat over $T$ w/ proper support.

$\leftrightarrow$ closed subschemes $Z \subseteq X_T$ flat and proper over $T$.

Thm (Grothendieck): $X \to S$ projective $\Rightarrow \mathrm{Quot}_{\mathcal{E}/X/S}$ representable by scheme loc. of finite type over $S$.

We can do better, however. We want to break $\mathrm{Quot}_{\mathcal{E}/X/S}$ into finite type parts.

Prop (Snapper's lemma): $X$ proper scheme / field $k$, $L \in \mathrm{Pic}(X)$, $F \in \mathrm{Coh}(X)$. $\exists$ polyn. $h \in \mathbb{Q}[x]$ (Hilbert polyn. of $F$ w.r.t. $L$) s.t. $\forall m \in \mathbb{Z}$: $h(m) = \sum_{i=0}^{\dim X} (-1)^i \dim_k H^i(X, F \otimes L^{\otimes m})$.

When $X = \mathbb{P}^n_k$, $L = \mathcal{O}_{\mathbb{P}^n_k}(1)$, $F = \pi_* \mathcal{O}_Z$ for $\pi : Z \hookrightarrow X$ closed emb. we have

$h(m) = \sum_{i \geq 0} (-1)^i \dim_k H^i(Z, \mathcal{O}_Z(m))$. This is Hilbert polyn. of $Z \subseteq \mathbb{P}^n$ (degree $= \dim Z$).

Suppose $X \to S$ proper w/ $S$ Noe., $L \in \mathrm{Pic}(X)$, $\mathcal{F} \in \mathrm{Coh}(X)$ flat$/S$. Given $s \in S$, taking the fiber gives Hilbert polyn. of $\mathcal{F}_s$ on $X_s$ w.r.t. $L_s$. This is locally constant.

$\mathcal{E} \in \mathrm{Coh}(X)$, $L \in \mathrm{Pic}(X)$, ~~...~~ $h(x) \in \mathbb{Q}[x] \rightsquigarrow \mathrm{Quot}^{h,L}_{\mathcal{E}/X/S} \subseteq \mathrm{Quot}_{\mathcal{E}/X/S}$ classifying (w/ $T$-pts.) $q: \mathcal{E}_T \to \mathcal{F}$ as in $\mathrm{Quot}_{\mathcal{E}/X/S}$ but s.t. $\mathcal{F}$ has Hilbert polyn. $h$ w.r.t. $L$.

Thm (Grothendieck): $L \in \mathrm{Pic}(X)$ very ample $\Rightarrow \mathrm{Quot}^{h,L}_{\mathcal{E}/X/S}$ is represented by projective $S$-scheme.

Remark: $X \to S$ proj. $\Rightarrow \mathrm{Quot}_{\mathcal{E}/X/S} = \coprod_{h \in \mathbb{Q}[x]} \mathrm{Quot}^{h,\mathcal{O}_X(1)}_{\mathcal{E}/X/S}$. So, Grothendieck actually proves the stronger result first.

Cor: $\mathrm{Hilb}^{h}_{\mathbb{P}^n/S} : \mathrm{Sch}_S \to \mathrm{Set}$ w/ $\mathrm{Hilb}^h_{\mathbb{P}^n/S}(T) = \{$ closed subschemes $Z \subseteq \mathbb{P}^n_T$ flat$/T$ w/ Hilbert polyn. $h\}$ is rep. by proj. scheme$/S$.

Remark: These constructions are very useful for constructing other moduli spaces.

There is universal object
$$\begin{array}{ccc} \mathcal{Z} & \hookrightarrow & \mathbb{P}^n \times \mathrm{Hilb}^h_{\mathbb{P}^n/\mathbb{Z}} \\ & \searrow & \downarrow \\ & & \mathrm{Hilb}^h_{\mathbb{P}^n/\mathbb{Z}} \end{array}$$
Given $k \geq 1$, let $\mathrm{Hilb}^{h,k}_{\mathbb{P}^n/\mathbb{Z}} := \mathcal{Z} \times_{\mathrm{Hilb}} \cdots \times_{\mathrm{Hilb}} \mathcal{Z}$.

[$\mathcal{Z}$ arises from pullback by universality...]

This parametrizes things w/ $k$ distinguished sections, $T \to \mathcal{Z}$ of the structure map $Z \hookrightarrow \mathbb{P}^n_T \to T$.

Mumford's idea for getting $\mathcal{H}_{g,d}$ as desired is to realize it as locally closed subscheme $\mathcal{H}_{g,d} \subseteq \mathrm{Hilb}^{h,1}_{\mathbb{P}^{6^g d-1}/\mathbb{Z}}$, where $h(x) = d(6x)^g$. [This comes from Mumford's Vanishing Thm.]

Remark: The marked pt. we want to be the origin/identity. So, we have to throw out the ones where this is not true (and show this is loc. closed condition).

Remark: For elliptic curves this says to look at $\mathbb{P}^5$! This is not what we get more classically (which is $\mathbb{P}^2$)! Katz-Mazur (or even Silverman) tell us that we just need to account for a $\mathbb{G}_m$-action (which corresponds to ambiguity in choosing a global nonvanishing alg. 1-form).