Assignment 3

due midnight (Eastern Time), Friday, Sept. 25, 2020

Prove all statements in the questions below. You must TeX your solutions and submit your writeup as a pdf at www.gradescope.com.
You can submit it any time until the deadline (try not to spend your Friday evening working on this!).

Question 1. (Folland 2.1.2) Suppose $f, g : X \to \overline{\mathbb{R}}$ are measurable.

1. fg is measurable (where $0 \cdot (\pm\infty) = 0$).
2. Fix $a \in \overline{\mathbb{R}}$ and define $h(x) = a$ if $f(x) = -g(x) - \pm\infty$ and $h(x) = f(x) + g(x)$ otherwise. Then h is measurable.

Question 2. (Folland 2.1.4) If $f : X \to \overline{\mathbb{R}}$ and $f^{-1}((r, \infty]) \in \mathcal{M}$ for each $r \in \mathbb{Q}$, then f is measurable.

Question 3. (Folland 2.1.6) The supremum of an uncountable family of measurable $\overline{\mathbb{R}}$-valued functions on X can fail to be measurable (unless the σ-algbra $\mathcal{M}$ is very special).

Question 4. (Folland 2.1.8) If $f : \mathbb{R} \to \mathbb{R}$ is monotone, then f is Borel measurable.

Question 5. (Folland 2.1.9) Let $f : [0, 1] \to [0, 1]$ be the Cantor function (§1.5), and let $f(x) = f(x) + x$.

1. g is a bijection from $[0, 1]$ to $[0, 2]$, and $h = g^{-1}$ is continuous from $[0, 2]$ to $[0, 1]$.
2. If C is the Cantor set, $m(g(C)) = 1$.
3. By Exercise 29 of Chapter 1, $g(C)$ contains a Lebesgue nonmeasurable set A. Let $B = g^{-1}(A)$. Then B is Lebesgue measurable but not Borel.
4. There exist a Lebesgue measurable function G and a continuous function G on $\mathbb{R}$ such that $F \circ G$ is not Lebesgue measurable.