Section 13 Math 2202

Line Integrals, Gradient/Conservative Vector Fields and Green's Theorem

1. A conservative/gradient vector field $\mathbf{F}$ is one where $\mathbf{F} = \nabla f$ for some function $f(x, y)$.

   (a) Check that $\mathbf{F}(x, y) = xy^2\mathbf{i} + x^2y\mathbf{j}$ is a conservative vector field.

   (b) Find f such that $\nabla f = \mathbf{F}$.

   (c) Find the value of $\int_C \mathbf{F}\, d\mathbf{r}$ where C is the line between $(-1, 4)$ and $(3, 5)$. (Remember the Fundamental Theorem of Calculus for Line Integrals: if $\mathbf{F} = \nabla f$ and is a continuous vector field and C is smooth, then the integral $\int_C \mathbf{F}\, d\mathbf{r} = ...$)

(a) Let $P(x,y) = xy^2$, $Q(x,y) = x^2y$. We ~~must~~ need to check that $P_y = Q_x$. In our setting, both have value $2xy$. Since P, Q have continuous partial derivatives and $\vec{F}$ is defined on all of $\mathbb{R}^2$, $\vec{F}$ must be conservative.

(b) $f_x(x,y) = xy^2 \Rightarrow f(x,y) = \frac{1}{2}x^2y^2 + g(y)$

$\Rightarrow x^2y = f_y(x,y) = x^2y + g'(y) \Rightarrow g'(y) = 0 \Rightarrow g(y) = k$

$\Rightarrow f(x,y) = \frac{1}{2}x^2y^2 + k$

(c) The choice of k does not matter, so let's take $k = 0$ for simplicity.

Then, $\int_C \vec{F}\cdot d\vec{r} = f(3,5) - f(-1,4) = \frac{1}{2}3^2 5^2 - \frac{1}{2}(-1)^2 4^2 = \frac{1}{2}(225 - 16) = \frac{209}{2}$.

Important that we start at $(-1,4)$ and end at $(3,5)$. Going in the other direction would give us a minus sign.

   (d) (Extra) For some extra practice, parameterize the line between $(-1, 4)$ and $(3, 5)$ and compute the line integral without FTC.

Put $\vec{r}(t) = (1-t)\langle -1, 4\rangle + t\langle 3, 5\rangle = \langle t - 1 + 3t, 4 - 4t + 5t\rangle = \langle -1+4t, 4+t\rangle$. So, $\vec{r}\,'(t) = \langle 4, 1\rangle$

and $\int_C \vec{F}\cdot d\vec{r} = \int_0^1 \langle (-1+4t)(4+t)^2, (-1+4t)^2(4+t)\rangle \cdot \langle 4, 1\rangle\, dt$

$= \int_0^1 [4(-1+4t)(4+t)^2 + (-1+4t)^2(4+t)]\, dt = \frac{209}{2}$.

Note: Typo w/ minus sign

2. The following fields are conservative/gradient. Find f such that $\nabla f = \mathbf{F}$.

(a) $\mathbf{F}(x, y) = (3x^2 + 2y^2)\mathbf{i} + (4xy + 3)\mathbf{j}$

$$f_x(x,y) = 3x^2 + 2y^2 \Rightarrow f(x,y) = x^3 + 2xy^2 + g(y)$$

$$\Rightarrow 4xy + 3 = f_y(x,y) = 4xy + g'(y)$$

$$\Rightarrow g'(y) = 3 \Rightarrow g(y) = 3y + k$$

$$\Rightarrow f(x,y) = x^3 + 2xy^2 + 3y + k$$

(b) $\mathbf{F}(x, y) = (xy\cos(xy) + \sin(xy))\mathbf{i} + (x^2\cos(xy))\mathbf{j}$

$f_x(x,y) = xy\cos(xy) + \sin(xy) \Rightarrow f(x,y) = \cdots$ [This is a bit tricky, so start differently.]

$$f_y(x,y) = x^2\cos(xy) \Rightarrow f(x,y) = x\sin(xy) + g(x)$$

$$\Rightarrow \underset{\substack{\parallel \\ xy\cos(xy) + \sin(xy)}}{f_x(x,y)} = \sin(xy) + xy\cos(xy) + g'(x) \Rightarrow g'(x) = 0 \Rightarrow g(x) = k$$

$$\Rightarrow f(x,y) = x\sin(xy) + k$$

(c) $\mathbf{F}(x, y) = 2y^{3/2}\mathbf{i} + 3x\sqrt{y}\mathbf{j}$

Similarly to (b) we get $f(x,y) = 2xy^{3/2} + k$.

---

4. Compute the area of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (assume $a, b > 0$).

The ellipse E bounds a solid region D. A positive orientation parametrization of E which is a simple closed curve is given by $x(t) = a\cos(t)$, $y(t) = b\sin(t)$. Choosing suitable functions P, Q of x, y, we can arrange that $\int_E (P\,dx + Q\,dy) = \iint_D \underbrace{\left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y}\right)}_{=1} dA = \text{Area}(D)$. Take $P = -\frac{y}{2}$, $Q = \frac{x}{2}$.

In terms of t, $P\,dx + Q\,dy = -\frac{b\sin(t)}{2}(-a\sin(t)\,dt) + \frac{a\cos(t)}{2}(b\cos(t)\,dt) = \frac{ab}{2}\,dt$.

So, $\text{Area}(D) = \int_0^{2\pi} \frac{ab}{2}\,dt = \frac{ab}{2}(2\pi) = ab\pi$.