Derived Stuff

Zachary Gardner

When discussing (co-)chain complexes, the symbol $+$ indicates bounded below, the symbol $-$ indicates bounded above, and the symbol b indicates bounded. The symbol > 0 indicates a complex whose entries vanish for indices ≤ 0, with a similar convention for related symbols. Fix an abelian category $\mathcal{A}$. $\mathsf{Ch}(\mathcal{A})$ denotes the category of (co)chain complexes on $\mathcal{A}$. $\mathcal{C}$ denotes an arbitrary category with the minimal amount of structure needed to make sense in context. Let $\mathcal{I}$ (resp., $\mathcal{P}$) denote the full subcategory of $\mathcal{A}$ consisting of injective (resp., projective) objects. We use $\sim$ for homotopy equivalence, $\cong$ for (especially canonical/natural) isomorphism, and $\simeq$ for other notions of (weak) equivalence.[1] Given $f : X \to Y$ a morphism in $\mathsf{Ch}(\mathcal{A})$, we let $H(f)$ denote the induced morphism on cohomology. Let $\iota : \mathcal{A} \hookrightarrow \mathsf{Ch}(\mathcal{A})$ denote the fully faithful embedding that sends $A \in \mathcal{A}$ to the complex concentrated in degree 0.

[A] = Aluffi; [W] = Weibel

Recall that a category $\mathcal{A}$ is abelian if

- $\mathcal{A}$ is preadditive – i.e., enriched over Ab;
- $\mathcal{A}$ has a zero object;
- $\mathcal{A}$ has binary (and thus finite) biproducts;
- $\mathcal{A}$ has all kernels and cokernels;
- all monomorphisms are normal – i.e., obtained as the kernel of something; and
- all epimorphisms are conormal – i.e., obtained as the cokernel of something.

The aim of these notes is to give an overview of derived categories. Our focus will be on constructing and describing the derived category $D(\mathcal{A})$. The motivation behind derived categories comes from wanting to invert qis's and thereby obtain a more refined theory than that of the homotopy category $K(\mathcal{A})$. Assuming $\mathcal{A}$ has enough injectives and letting $F : \mathcal{A} \to \mathcal{B}$ be a left exact functor between abelian categories, we have (right) derived functors $R^iF : \mathcal{A} \to \mathcal{B}$ obtained by injectively resolving objects in $\mathcal{A}$, applying F to the resulting complex, and then taking cohomology. Assuming $\mathcal{A}$ has enough injectives, we will be able to adapt this procedure to construct $D(\mathcal{A})$ in terms of (co-)chain complexes. $D(\mathcal{A})$ will come with a fully faithful embedding $\mathcal{A} \hookrightarrow D(\mathcal{A})$ and F as above will induce a functor $RF : D(\mathcal{A}) \to D(\mathcal{B})$ such that $H^i(RF(A)) = R^iF(A)$ for every $A \in \mathcal{A}$.

Assume $\mathcal{A}$ has enough injectives. Given $A \in \mathcal{A}$ and an injective resolution $A \to I^\bullet$, A and $I^\bullet$ are the same object in $D(\mathcal{A})$ and so we should have $RF(A) = F(I^\bullet)$.

[1] Some care should be taken when comparing categories. There is a precise notion of isomorphism of categories, but what we really want in the practice is the notion of equivalence of categories.

Lemma 0.1 (A, Lemma 5.1)**.** *Let $F : \mathsf{Ch}(\mathcal{A}) \to \mathcal{C}$ be an additive functor sending qis's to isomorphisms. Then, homotopic morphisms in $\mathsf{Ch}(\mathcal{A})$ induce the same morphism in $\mathcal{C}$ under F. Stated another way, there is a unique factorization*

$$\begin{tikzcd}
\mathsf{Ch}(\mathcal{A}) \arrow[r, "F"] \arrow[d] & \mathcal{C} \\
K(\mathcal{A}) \arrow[ur, dotted, "\exists\,!"'] &
\end{tikzcd}$$

The same result applies with any desired boundedness assumptions.

Lemma 0.2 (A, Lemma 5.3)**.** *$K(\mathcal{A})$ is an additive category.*

Moreover, $K(\mathcal{A})$ is a triangulated category but is not abelian.

Theorem 0.3 (A, Cor 5.10)**.** *Homotopy classes of qis's induce isomorphisms in $K^-(\mathcal{P})$ and $K^+(\mathcal{I})$.*

Since homotopic morphisms in $\mathsf{Ch}(\mathcal{A})$ induce the same morphism on cohomology, the notion of qis extends from $\mathsf{Ch}(\mathcal{A})$ to $K(\mathcal{A})$ and so the previous theorem says that qis's in $K^-(\mathcal{P})$ and $K^+(\mathcal{I})$ are "already inverted." The following results help us prove the previous theorem and are useful to know in their own right.

Definition 0.4. *$X \in \mathsf{Ch}(\mathcal{A})$ is **split-exact** if $\mathrm{id}_X \sim 0$.*

Lemma 0.5 (A, Lemma 5.11)**.**

(a) *Let $P \in \mathsf{Ch}^{\leq 0}(\mathcal{P})$, $L \in \mathsf{Ch}(\mathcal{A})$ such that $H^i(L) = 0$ for $i > 0$, and $f : P \to L$ such that $H(f) = 0$. Then, $f \sim 0$.*

(b) *Let $I \in \mathsf{Ch}^{\geq 0}(\mathcal{I})$, $L \in \mathsf{Ch}(\mathcal{A})$ such that $H^i(L) = 0$ for $i < 0$, and $f : L \to I$ such that $H(f) = 0$. Then, $f \sim 0$.*

Lemma 0.6 (A, Cor 5.12)**.**

(a) *Let $P \in \mathsf{Ch}^-(\mathcal{P})$ and $L \in \mathsf{Ch}(\mathcal{A})$ exact. Then, $\mathrm{Hom}_{K(\mathcal{A})}(P, L) = 0$.*

(b) *Let $I \in \mathsf{Ch}^+(\mathcal{I})$ and $L \in \mathsf{Ch}(\mathcal{A})$ exact. Then, $\mathrm{Hom}_{K(\mathcal{A})}(L, I) = 0$.*

[Compare and contrast this with Sam's notions of projectivity and injectivity for complexes.]

Lemma 0.7 (A, Cor 5.13)**.** *Let $P \in \mathsf{Ch}^-(\mathcal{P})$ and $I \in \mathsf{Ch}^+(\mathcal{I})$ exact. Then, $P \sim 0$ and $I \sim 0$.*

The following result says that, under the appropriate assumptions, qis's are "NZDs up homotopy."

Lemma 0.8 (A, Lemma 5.14)**.** *Let $\rho : L \to M$ be a qis in $\mathsf{Ch}(\mathcal{A})$.*

(a) *Let $P \in \mathsf{Ch}^-(\mathcal{P})$ and $f : P \to L$ such that $\rho \circ f \sim 0$. Then, $f \sim 0$.*

(b) *Let $I \in \mathsf{Ch}^+(\mathcal{I})$ and $g : M \to I$ such that $g \circ \rho \sim 0$. Then, $g \sim 0$.*

Proposition 0.9 (A, Prop 5.15)**.**

(a) *A qis to an element of* $\mathsf{Ch}^-(\mathcal{P})$ *has a right homotopy inverse.*

(b) *A qis from an element of* $\mathsf{Ch}^+(\mathcal{I})$ *has a left homotopy inverse.*

Lemma 0.10 (A, Lemma 6.3)**.** *Let* $A \in \mathcal{A}$ *and* $M \in \mathsf{Ch}(\mathcal{A})$ *a resolution of* A.

(a) *Let* $P \in \mathsf{Ch}^{\leq 0}(\mathcal{P})$ *and* $\varphi : H^0(P) \to H^0(M) \cong A$*. Then, there exists* $f : P \to M$ *unique up to homotopy equivalence such that* $H^0(f) = \varphi$.

(b) *Let* $I \in \mathsf{Ch}^{\geq 0}(\mathcal{I})$ *and* $\psi : A \cong H^0(M) \to H^0(I)$*. Then, there exists* $g : M \to I$ *unique up to homotopy equivalence such that* $H^0(g) = \psi$.

It follows that projective (resp., injective) resolutions of $A \in \mathcal{A}$ are initial (resp., final) in the category of homotopy classes of qis's with fixed target (resp., source) $\iota(A)$.

Corollary 0.11. *Any two projective (resp., injective) resolutions of* $A \in \mathcal{A}$ *are homotopy equivalent.*

Lemma 0.12 (A, Prop 6.5)**.** *Let* $A_0, A_1 \in \mathcal{A}$ *and* $\varphi \in \mathrm{Hom}_{\mathcal{A}}(A_0, A_1)$.

(a) *Let* $P_0 \to A_0$ *and* $P_1 \to A_1$ *be projective resolutions. Then,* φ *is induced by some* $f : P_0 \to P_1$ *unique up to homotopy equivalence.*

(b) *Let* $A_0 \to I_0$ *and* $A_1 \to I_1$ *be injective resolutions. Then,* φ *is induced by some* $g : I_0 \to I_1$ *unique up to homotopy equivalence.*

Corollary 0.13.

(a) *Suppose* $\mathcal{A}$ *has enough projectives. Then, projective resolution identifies* $\mathcal{A}$ *as a full subcategory of* $K^-(\mathcal{P})$.

(b) *Suppose* $\mathcal{A}$ *has enough injectives. Then, injective resolution identifies* $\mathcal{A}$ *as a full subcategory of* $K^+(\mathcal{I})$.

Theorem 0.14 (A, Thm 6.6)**.** *Suppose* $\mathcal{A}$ *has enough projectives and let* $L \in \mathsf{Ch}^-(\mathcal{A})$.

(a) *There exists* $P \in \mathsf{Ch}^-(\mathcal{P})$ *unique up to homotopy equivalence such that* P *is qis to* L.

(b) *Every morphism in* $\mathsf{Ch}^-(\mathcal{A})$ *lifts uniquely to a corresponding morphism of projective resolutions in* $K^-(\mathcal{P})$.

We somewhat abusively refer to P as a **projective resolution** of L, the abuse coming from the fact that we do not keep track of the qis $P \to L$. The previous lemma allows us to construct a projective resolution functor $\mathscr{P} : \mathsf{Ch}^-(\mathcal{A}) \to K^-(\mathcal{P})$. Such a functor is not unique, but it is almost unique in a way that the following result makes precise.

Theorem 0.15 (A, Remark 6.8)**.** *Let* $\mathscr{P}, \mathscr{P}' : \mathsf{Ch}^-(\mathcal{A}) \to K^-(\mathcal{P})$ *be projective resolution functors.*

Then, there exists a unique natural isomorphism $\mathscr{P} \Rightarrow \mathscr{P}'$.

The following says that $K^-(\mathcal{P})$ solves the universal problem for the (bounded above) derived category $D^-(\mathcal{A})$.

Theorem 0.16 (A, Thm 6.9)**.** *Let $\mathscr{P} : \mathsf{Ch}^-(\mathcal{A}) \to K^-(\mathcal{P})$ be a projective resolution functor. Then, $\mathscr{P}$ sends qis's to isomorphisms and, moreover, given any additive functor $F : \mathsf{Ch}^-(\mathcal{A}) \to \mathcal{C}$ sending qis's to isomorphisms, there exists a functor $\tilde{F} : K^-(\mathcal{P}) \to \mathcal{C}$ unique up to natural isomorphism such that the diagram*

$$\begin{tikzcd} \mathsf{Ch}^-(\mathcal{A}) \arrow[r, "F"] \arrow[d, "\mathscr{P}"] & \mathcal{C} \\ K^-(\mathcal{P}) \arrow[ur, dotted, "\exists!\, \tilde{F}"'] & \end{tikzcd}$$

commutes up to natural isomorphism.

Dualizing the last few results allows us to construct an injective resolution functor $\mathscr{I} : \mathsf{Ch}^+(\mathcal{A}) \to K^+(\mathcal{I})$ (under the assumption that $\mathcal{A}$ has enough injectives) and show that $K^+(\mathcal{I})$ solves the universal problem for the (bounded below) derived category $D^+(\mathcal{A})$. One would of course like to construct a more general derived category $D(\mathcal{A})$ and relate $K^-(\mathcal{P})$ and $K^+(\mathcal{I})$ when $\mathcal{A}$ has both enough projectives and enough injectives.