Final Exam

Wed. Dec. 16, 2020

This is a closed book exam; you may not consult any references or notes. Upload your completed exam to gradescope immediately after completing the exam.

Do any 5 of the questions below. All questions are worth the same number of points.

Question 1. Prove that $C([0,1])$ is not complete in the L^1 metric (with respect to Lebesgue measure).

Question 2. Let $f \in L^1([0,\infty))$ for Lebesgue measure. Suppose f is uniformly continuous. Prove that $f \in C_0([0,\infty))$.

Question 3. Prove that the closed unit ball in ℓ^2 is not compact.

Question 4. Let $C([0,1])$ be the space of continuous functions on $[0,1]$ with the uniform norm. Let P be the subspace of polynomials. Give an example of an unbounded linear functional $T : P \to \mathbb{R}$.

Question 5. Let $f : [0,1] \to \mathbb{R}$ be continuous. Prove that

$$\lim_{n\to\infty} \int_0^1 f(x^n)dx$$

exists and evaluate the limit. Does the limit exist if f is only assumed to be in L^1 (and not necessarily continuous)?

Question 6. Let E be a Banach space. Suppose that $e_1, \ldots, e_n \in E$ are linearly independent. Fix any n elements $t_1, \ldots, t_n$ in E. Prove that there is a bounded linear map $T : E \to E$ such that $T(e_i) = t_i$ for all i.

Question 7. Let X be a normed linear space with norm $||\cdot||$. Prove that the linear map $\iota : X \to X^{**}$ given by

$$\iota(x)(f) = f(x)$$

is an isometry.
(You may use without proof the fact that for each $x \in X$, there exists $f \in X^*$ such that $||f||_{op} = 1$ and $||x|| = f(x)$.)