Section 9 Math 2202
Part I: Optimization
Part II: Integration Flashback

Comments for Facilitator:

- (20 min) Lagrange Problem: Have them start with this and give about 5 minutes until people have started to solve the system. Then go over the problem as a class, (solutions are included for you).

  The main goals are: practice with Lagrange multiplier method, understanding that absolute max and min are not guaranteed on a non-closed or non-bounded set and making sense of optimization and Lagrange multipliers with level curve pictures.

- (5 min) Open Questions

- (15) Integration Meaning and Riemann sum notation (Problem 2). Review together Riemann sum and build up to the definition of the integral using the problem. State definition of integral. Ask them how we generally compute integrals: Fundamental Theorem of Calculus, if we can find an antiderivative.

  Students often wonder after learning FTC why we bother with Riemann sums. One, it gives us the meaning behind the integral and what it is measuring and we'll see that as we try to extend it to multidimensions. Also, if we can't find an antiderivative (like for $y = e^{-x^2}$, a key function in probability) we can use sophisticated methods of Riemann sums to get very good estimates, called numerical integration.

- Do Quiz (10 min) This is meant to remind them of techniques of integration which many will need to review. The quiz points to some online resources and let them know you'll post quiz solutions and that there is no online quiz this week.

1. (Based on *Stewart 11.8 #6*) Consider the function $f(x, y) = e^{xy}$, and the constraint $x^3 + y^3 = 16$.

   (a) Use Lagrange multipliers to find the coordinates (x, y) of any points on the constraint where the function f could attain a maximum or minimum.

   Solution: We wish to find the extreme values of the function $f(x, y) = e^{xy}$ subject to the constraint $g(x, y) = x^3 + y^3 = 16$. Lagrange multipliers tells us that $\nabla f = \lambda \nabla g$ or $\langle ye^{xy}, xe^{xy}\rangle = \langle 3\lambda x^2, 3\lambda y^2\rangle$, and so we get the system

$$\begin{aligned} ye^{xy} &= 3\lambda x^2 \\ xe^{xy} &= 3\lambda y^2 \\ x^3 + y^3 &= 16. \end{aligned}$$

   Note that if either x or y is zero, then $x = y = 0$, which contradicts $x^3 + y^3 = 16$, so we can assume that $x \neq 0$ and $y \neq 0$. Then

$$\lambda = \frac{ye^{xy}}{3x^2} = \frac{xe^{xy}}{3y^2},$$

from which we get $x^3 = y^3$ and so $x = y$. Since $x^3 + y^3 = 16$, we get $2x^3 = 16$ or $x = y = 2$. Thus the point $(2, 2)$ is a point on the constraint where the function f could attain a maximum or minimum.

(b) For each point you found in part (a), is the point a maximum, a minimum, both or neither? Explain your answer carefully. What are the minimum and maximum values of f on the constraint? Please explain your answers carefully.

Solution: The maximum value is $f(2, 2) = e^4$.

We can see that there is no minimum value, since we can choose points satisfying the constraint $x^3 + y^3 = 16$ that make $f(x, y) = e^{xy}$ arbitrarily close to 0 (but never equal 0). More specifically, if x is given, then $y = \sqrt[3]{16 - x^3}$. If x grows positive without bound ($x \to +\infty$), then y grows *negative* without bound ($y \to -\infty$) and so e^{xy} approaches zero ($xy \to -\infty$ and so $e^{xy} \to 0$). So there is *no* minimum value of f.

(c) The extreme value theorem which we discussed in class (See 11.7 in Stewart) guarantees that under the right circumstances, we are guaranteed to find absolute minima and maxima for a function f on a certain constraint. Explain why parts (a) and (b) don't violate the extreme value theorem.

Solution: The extreme value theorem states that if a real-valued function f is continuous in a closed and bounded region, then f must attain its maximum and minimum value, each at least once.

But $x^3 + y^3 = 16$ is not a bounded region. The curve it describes cannot be contained in a circle. So it's possible f does not attain a max or min value.

In this case, we don't attain a min value since the constraint curve crosses level curves with ever-decreasing values (going toward 0) as $x \to \infty$ and $x \to -\infty$.

Note: the level curves are always positive valued, because $e^{xy} > 0$.

(d) Here we show the constraint $g(x, y) = 16$ and a number of level curves $f(x, y)$. These level curves are all powers of 2. In the first and third quadrant they are positive powers, increasing as we come out from the origin. In the second and fourth quadrants they are negative powers of 2, decreasing as we come out from the origin.

How does this figure match with what you determined above?

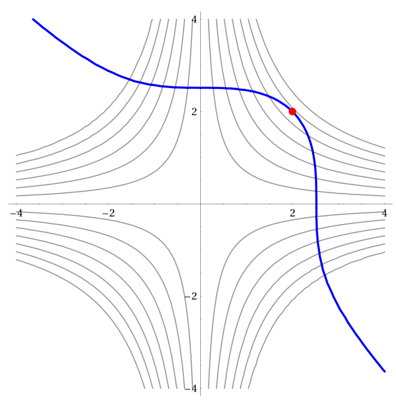


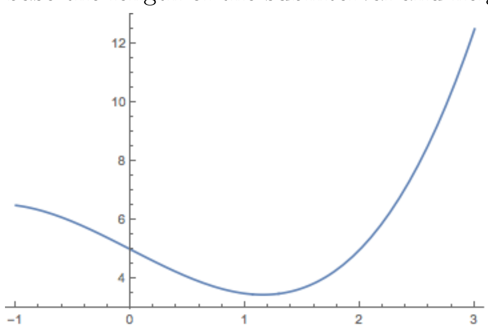
Figure 1: The constraint and level curves for Problem 1

2. Integration of $y = f(x)$ Review

Let $f(x) = \frac{x^3}{2} - 2x + 5$.

> **Comments for Facilitator:** Refresh the definition of the integral in single variable calculus, using notation like in the Stewart section 12.1 p. 830. The goal of this is to refresh the both the conceptual idea of integration and the notation used in Riemann sum.
>
> First review the notation for Riemann sums from 12.1 p. 830. Then do parts a and b. Then introduce the idea of area under the curve as a limit of Riemann sums and as an integral, also on p. 830. Then do parts c and d.

(a) Consider the interval $[-1, 3]$. Divide it into four equal parts and label the partition points x_0, x_1, x_2, x_3, x_4. (Note $x_0 = -1$ and $x_4 = 3$).
What is Δx, the length of each subinterval? For each subinterval, draw a rectangle with base the length of the subinterval and height so that the left endpoint touches the curve.



Write a Riemann sum (a sum of area of rectangles) to estimate the area under the curve and above the x-axis on the interval $[-1, 3]$ using 4 rectangles and the left endpoint rule.

(b) Write a Riemann sum (a sum of area of rectangles) that would estimate the area under the curve and above the x-axis on the interval $[-1, 3]$ using 20 rectangles with the left endpoint rule. What is Δx? Give a formula for x_i for $i = 0, ..., 20$.

(c) Write a limit of a Riemann sum that represents the exact area under the curve and above the x-axis on the interval $[-1, 3]$.

(d) Write an integral that measures the exact area under the curve $f(x)$ and above the x-axis from $x = -1$ to $x = 3$. Compute it.

(e) (Challenge) Can you write a Riemann sum that estimates the area under the curve and above the x-axis on the interval $[-1, 3]$ using n rectangles and the midpoint rule?