Assignment 7

due by class time (2pm Eastern Time), Monday, November 16, 2020

Prove all statements in the questions below. You must TeX your solutions and submit your writeup as a pdf at www.gradescope.com.
You can submit it any time until the deadline.

Question 1. (Folland 5.3.29) Let $\mathcal{Y} = L^1(\mu)$ where μ is counting measure on $\mathbb{N}$, and let $\mathcal{X} = \{f \in \mathcal{Y} : \sum_1^\infty n|f(n)| < \infty\}$, equipped with the L^1 norm.

1. $\mathcal{X}$ is a proper dense subspace of $\mathcal{Y}$; hence $\mathcal{X}$ is not complete.
2. Define $T : \mathcal{X} \to \mathcal{Y}$ by $Tf(n) = nf(n)$. Then T is closed but not bounded.
3. Let $S = T^{-1}$. Then $S : \mathcal{Y} \to \mathcal{X}$ is bounded and surjective but not open.

Question 2. (Folland 5.3.37) Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces. If $T : \mathcal{X} \to \mathcal{Y}$ is a linear map such that $f \circ T \in \mathcal{X}^*$ for every $f \in \mathcal{Y}^*$, then T is bounded.

Question 3. (Folland 5.3.38) Let $\mathcal{X}$ and $\mathcal{Y}$ be Banach spaces, and let $\{T_n\}$ be a sequence in $L(\mathcal{X}, \mathcal{Y})$ such that $\lim T_n x$ exists for every $x \in X$. Let $Tx = \lim T_n x$; then $T \in L(\mathcal{X}, \mathcal{Y})$.

Question 4. (Folland 5.5.54) For any nonempty set A, $\ell^2(A)$ is complete.

Question 5. (Folland 5.5.56) If E is a subset of a Hilbert space $\mathcal{H}$. $(E^\perp)^\perp$ is the smallest closed subspace of $\mathcal{H}$ containing E.