# Virtual fundamental classes for Stacks

We consider $\mathcal{V}$ admissible category of schemes\Stacks

$$H(X) \in \mathrm{Spt} \backslash \mathrm{grAb} \quad \forall X \in \mathcal{V}$$

## Wanted structures:

- $f: X \to Y$ „proper\projective"
  $f_*: H(X) \to H(Y)$ push-forward
- $f: X \to Y$ smooth
  $f^*: H(Y) \to H(X)$ pullback
- $f: X \to Y$ quasi-smooth
  $f^!: H(Y) \to H(X)$ Gysin-pullback
- $\times: H(X) \otimes H(Y) \to H(X \times Y)$ exterior-product

+ „functoriality" and „interactions"

For ex we can intersect:

$$H(X) \otimes H(X) \xrightarrow{\times} H(X \times X) \xrightarrow{\Delta^!} H(X)$$

## Expected properties:

- $\mathbb{A}^1$-Invariance:
  $p: V \to X$ some vector bundle then
  $$p^*: H(X) \to H(V) \quad \text{is iso.}$$
- localisation:
  $$i: Z \underset{\text{closed}}{\hookrightarrow} X \underset{\text{open}}{\hookleftarrow} U = (X - Z) : j$$
  then have (co)fiber sequence
  $$H(Z) \xrightarrow[i_*]{} H(X) \xrightarrow[j^!]{} H(U)$$

**Examples:**

- $CH^*(X)$ for smooth schemes over over a field (Chow groups)
- $G(X)$ for quasi-smooth derived schemes over a noetherian derived base (G-theory)
- $KH(X)$ for derived schemes over a qcqs base ($\mathbb{A}^1$-invariant K-theory)
- $G^{\acute{e}t}(X)_{\mathbb{Q}}$ for derived Artin-Stacks over a noetherian base (étale G-theory w/ "$\mathbb{Q}$-coefficients")
- $\mathcal{F} \in SH_{\acute{e}t}(S)$

$$\begin{array}{l} X \\ \downarrow f \\ S \end{array} \text{ derived Artin Stack (of finite type)} \mapsto H^{BM}(X/S, \mathcal{F}(r)) = \mathrm{Map}_{SH_{\acute{e}t}(S)}(\mathbb{1}_S(r), f_* f^! \mathcal{F})$$

**Question:** Can we find fundamental classes $[X] \in H(X)$ s.t. $f^![X] = [Y]$ for $f: Y \to X$ quasi-smooth.

$SH(S)$ $\quad L_{\mathbb{A}}(Sh^{Nis}(S)_\bullet)$ $\quad \mathcal{F}(X) \to \mathcal{F}(X \times \mathbb{A}^1)$

$(\mathcal{F}_0, \mathcal{F}_1, \mathcal{F}_3, \ldots)$ $\quad \Omega_{\mathbb{P}^1} \mathcal{F}_{i+1} \simeq \mathcal{F}_i$

$\mathrm{Hom}_\bullet(\mathbb{P}^1, \mathcal{F}_{i+1})$

## The case of G-theory

**Def:** $X$ derived Artin-Stack then
$$[X]^{vir}_G \in G(X)$$
is just the image of
$$\mathcal{O}_X \in \mathrm{Perf}(X)^{\simeq} \to K(X) \to G(X)$$

Now we have for $f: X \to Y$ quasi-smooth

$$\begin{tikzcd}
{[Y]^{vir}_G} \arrow[rr, mapsto] \arrow[dd, mapsto] & & {[X]^{vir}_G} \arrow[dd, mapsto] \\
& G(Y) \arrow[r, "f^*"] \arrow[d, "\wr"] & G(X) \arrow[d, "\wr"] \\
\sum_{i \geq 0} (-1)^i [H^i(\mathcal{O}_Y)] & G(Y_{cl}) \arrow[r, "f^!"] & G(X_{cl}) & \sum_{i \geq 0} (-1)^i [H^i(\mathcal{O}_X)]
\end{tikzcd}$$

Want to justify this:

## the specialization map

$f: X \to Y$ quasi-smooth then have
**deformation to the normal stack**

$$V_X(\mathbb{L}_{X/Y}[-1]) \to X \qquad \mathbb{L}_{X/Y}[-1]|_S \to \mathcal{O}_S$$

$S$

$$\begin{tikzcd}
 & X \arrow[r] \arrow[d, "\theta"'] & X \times \mathbb{A}^1 \arrow[d] & X \times \mathbb{G}_m \arrow[l] \arrow[d, "f \times \mathrm{id}"] & \\
 & U_{X/Y} \arrow[r, "\hat{i}"] \arrow[dl] \arrow[d, "v"'] & D_{X/Y} \arrow[d, "u"] & Y \times \mathbb{G}_m \arrow[l, "\hat{j}"'] \arrow[d, equal] & 2. \\
X \arrow[r, "f"] & Y \arrow[r, "i"'] & Y \times \mathbb{A}^1 & Y \times \mathbb{G}_m \arrow[l, "j"] & 1.
\end{tikzcd}$$

take localization sequence ass to 1.

$$G(Y \times \mathbb{G}_m)[-1] \xrightarrow{\partial} G(Y) \xrightarrow{i_*} G(Y \times \mathbb{A}^1) \xrightarrow{j^*} G(Y \times \mathbb{G}_m)$$

$$Y \times \mathbb{A}^1 \xrightarrow[pr]{} Y \xrightarrow[s_1]{} Y \times \mathbb{G}_m$$

$$G(Y \times \mathbb{A}^1)[-1] \xrightarrow{j^*} G(Y \times \mathbb{G}_m)[-1] \xrightarrow{\partial} G(Y)$$

$$\mathcal{F} \mapsto q^* \mathcal{F} = \mathcal{F}[t^{\pm 1}] \xrightarrow{\cdot t} \mathcal{F}[t^{\pm 1}]$$

(up to sign) we can describe

$$\partial : G(X \times \mathbb{G}_m)[-1] \longrightarrow G(X)$$

by $\gamma_b : G(X) \xrightarrow[q^*]{} G(X \times \mathbb{G}_m) \xrightarrow[\cap b]{} G(X \times \mathbb{G}_m)[-1]$

$$\mathcal{F} \longmapsto \cdot t : \mathcal{F} \longrightarrow \mathcal{F}$$

Where $q : \mathbb{G}_m \times X \to X$

$$b = t \cdot : \mathcal{O}_{\mathbb{G}_m} \xrightarrow{t} \mathcal{O}_{\mathbb{G}_m} \otimes \mathrm{Perf}(X)^{\simeq \mathbb{A}^1} \longrightarrow K(X)[-1]$$

Now take also the loc. seq. of (2')

$$\begin{tikzcd}
G(Y \times \mathbb{G}_m)[-1] \arrow[r, "\partial"] \arrow[d, equal] & G(Y) \arrow[r, "i_*"] \arrow[d, "v^*"] \arrow[l, bend left, "\gamma_b"] & G(Y \times \mathbb{A}^1) \arrow[r, "j^*"] \arrow[d, "u^*"] & G(Y \times \mathbb{G}_m) \arrow[d, equal] \\
G(Y \times \mathbb{G}_m)[-1] \arrow[r, "\partial"'] & G(N_{X/Y}) \arrow[r, "i_*"'] & G(D_{X/Y}) \arrow[r] & G(Y \times \mathbb{G}_m)
\end{tikzcd}$$

<u>Def:</u> $f : X \to Y$ quasi-smooth

$$sp_{X/Y} : G(Y) \xrightarrow{\gamma_b} G(Y \times \mathbb{G}_m)[-1] \xrightarrow{\partial} G(N_{X/Y})$$

the specialization map ass. to $f$

<u>Prop:</u> $f : X \to Y$ quasi-smooth morph. of noetherian derived Artin Stacks (w) (affine stabilizers then)

$$\begin{tikzcd}
G(Y) \arrow[rr, "f^*"] \arrow[d, equal] & & G(X) \arrow[d, equal] \\
G(Y) \arrow[r, "sp_{X/Y}"'] & G(N_{X/Y}) \arrow[r, "(\pi^*)^{-1}"'] & G(X)
\end{tikzcd}$$

**Proof:** have $sp_{X/Y} = v^* = \pi^* \circ f^*$

$+$ $(\pi^*)^{-1}$ exists.

$\square$

**Rem:** Can drop ( ) by replacing

$\underline{G(X)}$ w/ $\underline{G^{et}(X)_{\mathbb{Q}}}$

$$\underline{X} \longmapsto \underline{\mathcal{X}}$$

$$\underline{\mathrm{Mod}_{kGL}(SH(S))} \simeq \underline{DA^{et}(S, \mathbb{Q})}$$

$G$

$\mathbb{G}_m \hookrightarrow \mathbb{A}^1 \ni *$

$L_{\mathbb{A}^1} \underline{Sh^{Nis}(S, S_{pt})}$

$$\begin{array}{ccc} & & \downarrow \\ * = \mathbb{A}^1 & \hookrightarrow & \mathbb{P}^1 \end{array}$$

$\mathbb{G}_m \longrightarrow k \setminus \{-1\}$

$S\mathbb{G}_m$

$$\mathbb{P}^1 \longrightarrow \underline{k}$$

$F$ $\quad \underline{\mathrm{Hom}}(k, F) \longrightarrow \underline{\mathrm{Hom}}(\mathbb{P}^1, F)$