Introduction

Our goal today is to talk about mixed Hodge structures as a prelude to discussing mixed Hodge modules. We will do this through the lens of concrete examples – hopefully you will walk away from this believing that mixed Hodge structures and their generalizations are interesting and worthwhile things to study.

Pure Hodge Structures

The notion of a (pure) Hodge structure comes, unsurprisingly, from Hodge theory.

X sm. proj. alg. var. Let $H := H^n(X^{an}, \mathbb{Q})$ for some $n \in \mathbb{Z}$. Its complexification may be identified w/ de Rham cohom. $H_{\mathbb{C}}$ has a certain family of subspaces $\{H^{p,q} : p+q=n,\ p,q \geq 0\}$ s.t. $H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}$ and $\overline{H^{p,q}} = H^{q,p}$.

Define $F^p(H_{\mathbb{C}}) := \bigoplus_{i \geq p} H^{i,n-i}$. This defines the decreasing <u>Hodge filtration</u> $F = F^\bullet$ on $H_{\mathbb{C}}$ w/

$H_{\mathbb{C}} = F^p \oplus \overline{F}^{n-p+1}\ \forall p$. Each $H^{p,q} = F^p \cap \overline{F}^q$. We axiomatize this construction.

<u>Def</u>: Let H be fin. dim. vec. space $/\mathbb{Q}$. Equip $H_{\mathbb{C}} := H \otimes_{\mathbb{Q}} \mathbb{C}$ w/ finite decreasing filtration $F = \{F^p(H_{\mathbb{C}})\}_{p \in \mathbb{Z}}$ by subspaces of $H_{\mathbb{C}}$ – i.e., $F^p(H_{\mathbb{C}}) \supseteq F^{p+1}(H_{\mathbb{C}})\ \forall p$, $F^p(H_{\mathbb{C}}) = 0\ \forall p \gg 0$, $F^{-p}(H_{\mathbb{C}}) = H_{\mathbb{C}}\ \forall p \gg 0$. This is the data of a pair (H, F). Given $n \in \mathbb{Z}$, a <u>Hodge structure of weight n</u> is a pair (H,F) s.t. $H_{\mathbb{C}} = F^p \oplus \overline{F}^{n-p+1}\ \forall p$. We call F the <u>Hodge filtration</u>. Defining $H^{p,q} := F^p \cap \overline{F}^q$ gives Hodge decomp. $H_{\mathbb{C}} = \bigoplus_{p+q=n} H^{p,q}$, $\overline{H^{p,q}} = H^{q,p}$.

A morphism is taken to preserve Hodge filtrations or, equivalently, Hodge Z-bigradings.

Remark:

- Like with so many things in math, Hodge structures serve to linearize a nonlinear problem.
- We often use the abbreviation HS for the term Hodge structure.
- This notion extends immediately to work for Z-modules as well.
- One could take the Hodge decomposition as the starting point and then define the filtration as in the classical Hodge theoretic setting above. However, the filtration definition is the more natural way to go about this in a way that can be made precise but I don't want to get into now.

One result that demonstrates the utility of Hodge structures is the following consequence of Hodge theory.

<u>Thm (Hodge Theory)</u> : $X \mapsto H^n(X; \mathbb{Q})$ is contravariant functor $\{$compact Kähler mflds$\} \to \{$pure Hodge structures of weight $n\}$.

Living among compact Kahler manifolds are smooth projective algebraic varieties. Can we extend this result to work for all complex algebraic varieties? Yes! This is where mixed Hodge structures enter the picture.

V vec. space / $\mathbb{Q}$ w/ $V_{\mathbb{C}} = \bigoplus_{p+q=n} V^{p,q}$ and $\overline{V^{p,q}} = V^{q,p}$. Morphism $\varphi : V \to V'$ is $\mathbb{Q}$-linear and respects decomp.'s of $V_{\mathbb{C}}, V'_{\mathbb{C}}$ – i.e., $\varphi_{\mathbb{C}}(V^{p,q}) \subseteq (V')^{p,q}$. $[p, q \geq 0]$

Filtered version: $V_{\mathbb{C}}$ carries dec. filtration $F^\bullet$ s.t. $V_{\mathbb{C}} = F^p \oplus \overline{F^{n-p+1}}$ w/ $\begin{cases} F^p V_{\mathbb{C}} = \bigoplus_{p' \geq p} V^{p', n-p'}, \\ V^{p,q} = F^p V_{\mathbb{C}} \cap \overline{F^q V_{\mathbb{C}}}. \end{cases}$

Morphism preserves filtrations.

<u>Thm (Hodge Theory)</u> : $X \mapsto H^n(X;\mathbb{Q})$ is contravariant functor $\{$compact kähler mflds$\} \to \{$pure Hodge structures of weight $n\}$.

Living among kähler mflds are smooth proj. alg. var.'s. Can we extend the above to work for all alg. var.'s / $\mathbb{C}$?

~~Thm (Deligne) : $X \mapsto H^n(X;\mathbb{Q})$ is contravariant functor $\{$ alg. var.'s $/\mathbb{C}\} \to \{$mixed Hodge structures$\}$~~

How does this work? Existence of Hodge filtration for smooth proper alg. var.'s / $\mathbb{C}$ follows from proj. case via Lefschetz-type argument + Chow [tell me more...]. For a quasi-proj. var., we can use compactifications and resolution of singularities to "resolve" by smooth proj. var.'s. Mixed Hodge structures essentially keep track of this data.

Hodge II : general smooth case ; Hodge III : singular case

<u>Intro to MHS</u>

<u>Goal</u> : Tackle the case $U = X \setminus Y$ for X smooth proj., Y smooth proj. of complex codim 1.

<u>Example</u> : (1) X curve of genus g, Y collection of $n+1$ pts., $U = X \setminus Y$.

i	rank $H^i(X)$	rank $H^i(U)$
0	1	1
1	$2g$	$2g+n$
2	1	0

For $H^1(U)$, n of the classes come from punctures and not X. What are their weights?

(2) $X = \mathbb{P}^n$, Y = hypersurface of deg d, $U = X \setminus Y$. What is cohom. of U? We have LES of the pair

$$\cdots \to H^i(X,U) \to H^i(X) \to H^i(U) \to H^{i+1}(X,U) \to \cdots$$ hence SES

$$0 \to \underbrace{H^i(X)/H^i(X,U)}_{\text{"Weight } i \text{ part"}} \to H^i(U) \to \underbrace{\ker(H^{i+1}(X,U) \to H^{i+1}(X))}_{\text{"Weight } i+1 \text{ part"}} \to 0.$$

We should have $H^i(X,U)$ equipped w/ pure Hodge structures s.t. $H^i(X,U) \to H^i(X)$ is morphism of pure HS's. $H^i(U)$ sub-object w/ pure weight i and quotient w/ pure weight $i+1$.

We want to make this precise.

Pf : (a) Let $L|K$ be fin. ext. w/ K finite. Then, L is finite as well. We have some isom. $K \cong \mathbb{F}_{p^n}$ and $L \cong \mathbb{F}_{p^m}$. ~~In fact, L is the splitting field of some irred.~~ We know $p^n \mid p^m$, so $m = nd$ for some $d \geq 1$:

$K \cong \text{spl}_{\mathbb{F}_p}(x^{p^n} - x)$, $L \cong \text{spl}_{\mathbb{F}_p}(x^{p^{nd}} - x)$

Def: MHS is fin. gen. $\mathbb{Q}$-vec. space w/ inc. $\mathbb{Q}$-lin. weight filtration $W^\bullet$ on V and dec. $\mathbb{C}$-lin. Hodge filtration $F^\bullet$ on $V_\mathbb{C}$ s.t.

$\forall k$: $gr_W^k(V) = W^k/W^{k-1}$ w/ Hodge filtration is HS of pure weight k. [Some authors write weight filtration w/ lower indices (subscripts).]

Example: From above we have $W^k(H^i(\mathcal{U})) = \begin{cases} 0, & k<i, \\ \mathrm{im}(H^i(X)\to H^i(\mathcal{U})), & k=i, \\ H^i(\mathcal{U}), & k>i, \end{cases}$ This gives

$$gr_W^k(H^i(\mathcal{U})) = \begin{cases} \mathrm{im}(H^i(X)\to H^i(\mathcal{U})) \cong H^i(X)/H^i(X,\mathcal{U}), & k=i, \\ H^i(\mathcal{U})/H^i(X) \cong \ker(H^{i+1}(X,\mathcal{U}) \to H^i(X)), & k=i+1, \\ 0, & \text{otherwise}. \end{cases}$$

We have not specified Hodge filtration on $H^i(\mathcal{U})$. This matters since cat. of MHS is not semisimple ($\exists$ non-split ext.'s) so pure HS's on assoc. graded is not in general enough data. In simple case this is not issue, however.

Consider $X = \mathbb{P}^1$, $Y = \{0,\infty\}$, $\mathcal{U} = X\setminus Y = \mathbb{G}_m$. We have exact seq. $0 \to H^1(\mathcal{U}) \to \ker(H^2(X,\mathcal{U}) \to H^2(X)) \to 0$.

This comes from SES $0 \to H^1(\mathbb{P}^1)/H^1(\mathbb{P}^1,\mathbb{G}_m) \to H^1(\mathbb{G}_m) \to \ker(H^2(\mathbb{P}^1, \mathbb{G}_m) \to H^2(\mathbb{P}^1)) \to 0$.

By $\mathbb{P}^1$, we mean $\mathbb{P}^1(\mathbb{C})^{an} = S^2$ the Riemann sphere, so $H^i(\mathbb{P}^1) = \begin{cases} \mathbb{Q}, & i=0,2, \\ 0, & \text{otherwise}. \end{cases}$

Similarly, $\mathbb{G}_m$ means $\mathbb{G}_m(\mathbb{C})^{an} = \mathbb{C}^\times \simeq S^1 \Rightarrow H^i(\mathbb{G}_m) = \begin{cases} \mathbb{Q}, & i=0,1, \\ 0, & \text{otherwise}. \end{cases}$ $H^1(\mathbb{G}_m)$ has dim 1 and weight 2, so is isom. as HS to $\mathbb{Q}(-1)$. In de Rham cohom. the nontriv. cohom. class α is rep. by $\frac{dz}{z}$ or by $-\frac{d\bar z}{\bar z}$. So, $\alpha \in F^1 \cap \overline{F^1}$ thinking of holo. and anti-holo. diff's.

Remark: Extending Hodge theory to non-proper setting requires weights different than k to appear for H^k. In general need Tate twists and mixed weights ~~in general~~, as seen by punctured curve example from before.

Gysin Map

[c.f. Thom class and Thom isom. thm.]

$H^k(X,\mathcal{U}) \to H^k(X)$ is map of HS's?

Why does $H^k(X,\mathcal{U})$ have nat. HS s.t. $H^k(X,\mathcal{U}) \to H^k(X)$ is map of HS's? Then,

We can retract $\mathcal{U}$ ($= X\setminus Y$) to complement V of tubular nbhd Y_ε of Y in X s.t. $H^i(X,\mathcal{U}) \cong H^i(X,V)$.

$H^i(X,V)$ = reduced cohom. of Thom complex $\tilde{T}_{N_X Y}$ of normal bundle $N_X Y$. Y has complex codim 1 $\Rightarrow N_X Y$ has rank 1 as complex vec. bundle / Y (so real rank 2) and we obtain Thom isom. $H^{k-2}(Y) \xrightarrow{\sim} \tilde{H}^k(\tilde{T}_{N_X Y}) \xrightarrow{\sim} H^k(X,V) \xrightarrow{\sim} H^k(X,\mathcal{U})$.

So, it seems we should put HS of $H^{k-2}(Y)$ on $H^k(X,\mathcal{U})$. Weights are off, so we fix this by tensoring w/ $\mathbb{Q}(-1)$.

So we obtain Gysin seq. (LES): $\cdots \to \underbrace{H^{k-2}(Y)\otimes\mathbb{Q}(-1) \to H^k(X)}_{\text{Need to show that this is map of HS's...}} \to H^k(\mathcal{U}) \to H^{k-1}(Y)\otimes\mathbb{Q}(-1) \to \cdots$

We do this using diff. forms.

Recall: X smooth proj. var. $/\mathbb{C}$, $Y \overset{i}{\hookrightarrow} X$ of codim 1 also smooth proj. var. $/\mathbb{C}$, $U = X \setminus Y$. Pass to analytifications.

X, Y compact $\Rightarrow$ Gysin map can be described using Poincaré duality. Its dual fits into comm. diag. ($n \equiv \dim X$)

(composite looks like...)

$$\begin{array}{ccccccc} H^{2n-k}(X) & \xrightarrow{\cap[X]} & H_k(X) & \longrightarrow & H^k(X)^\vee & \rightsquigarrow & \omega \mapsto (\eta \mapsto \int_X \omega \wedge \eta) \\ {\scriptstyle i^*}\downarrow & & & & \downarrow & & \\ H^{2n-k}(Y) & \xrightarrow[\cap[Y]]{} & H_{k-2}(Y) & \longrightarrow & H^{k-2}(Y)^\vee & \rightsquigarrow & \text{similar...} \end{array}$$

$$\begin{cases} f_\omega(\eta) = \int_X \omega \wedge \eta \\ f_{\omega_Y}(\alpha) = \int_Y \omega_Y \wedge \alpha \end{cases}$$

let $\omega \in H^{2n-k}(X)$ w/ image f_ω in $H^k(X)^\vee$. Let $\omega_Y := i^*\omega$. Then, f_ω maps to f_{ω_Y}. We seek $\psi: H^{k-2}(Y) \to H^k(X)$ (which should be the Gysin map). We want $f_{\omega_Y}(\alpha) = f_\omega(\psi(\alpha))$. So, we want $\psi(\alpha)$ form on X s.t.

$\int_Y \omega_Y \wedge \alpha = \int_X \omega \wedge \psi(\alpha)$. How do we do this?

Example: (1) Consider $X = \mathbb{P}^1$, $Y = \{0, \infty\}$. Only nonzero Gysin map is $H^0(Y) \to H^2(X)$, so $k = 2$ in the above. $\omega \in H^0(X)$ is constant w/ value c_0 on X, and ω_Y is restriction to $\{0,\infty\}$. Given $\alpha \in H^0(Y)$,

$\int_Y \omega_Y \wedge \alpha = (\alpha(\infty) + \alpha(0))\, c_0$. So, $\psi(\alpha) = \alpha(\infty) + \alpha(0)$. This case is "too special," ~~[Y is locally the locus of y.]~~ since everything is defined on the level of ordinary (C^∞) differential forms.

(2) Allow general X, Y, except require Y to be conn. Take $k = 2$. Locally on X we can construct unif. y for Y, unique up to mult. by non-vanishing hol. ϕ. We have $\frac{d(\phi y)}{\phi y} = d\log\phi + \frac{dy}{y}$. [Tells us how to change coords.] Using POU, we can construct form (only simple holo. thing) η s.t. for any local unif. y, η locally looks like $\frac{1}{2\pi i} \cdot \frac{dy}{y} + \theta$ for θ a smooth $(1,0)$-form on X.

Fix Hermitian metric on X and let $Y_\varepsilon \subseteq X$ be normal ε-ball for suff. small ε. ω closed 2-form on X, $\alpha \equiv c$ constant on Y (rep. of H^0). We want 2-form $\psi(c)$ on X s.t. $c\int_Y \omega_Y = \int_X \omega \wedge \psi(c)$.

$\int_X \omega \wedge d\eta = \lim_{\varepsilon \to 0} \int_{X \setminus Y_\varepsilon} \omega \wedge d\eta = \lim_{\varepsilon \to 0} \int_{\partial Y_\varepsilon} \omega \wedge \eta$ by Stokes. Local coords. x_i, z_i w/ y unif. for $Y \Rightarrow |x|$ (defined to be size of x in Hermitian metric on normal bundle) equal to nonzero multiple of $|y|$ up to first-order.

For ε small, $\int_{\partial Y_\varepsilon} \omega \wedge \eta = \int \left(\int_{|y| \sim \varepsilon} \frac{1}{2\pi i} \omega \wedge \frac{dy}{y} + \omega \wedge \theta \right)$. In the limit $\varepsilon \to 0$, we get $\int_Y \omega_Y$. Take $\psi(c)$ to be $c\,d\eta = d(\eta \wedge \tilde\alpha)$.

Similar argument shows $\psi(\alpha) = d(\eta \wedge \tilde\alpha)$ for every α k-form on Y and any ~~suitable~~ choice of ext. to X.

This is $d\eta \wedge \tilde\alpha + \eta \wedge d\tilde\alpha$, which lies in correct filtration. Unfortunately, this $\tilde\alpha$ is not C^∞ in general — if $d\tilde\alpha \neq 0$ $\Rightarrow \tilde\alpha$ has singularity along Y. So, we need to enlarge our space of forms.

We seek $K^\bullet \supseteq A^\bullet_X$ s.t. inclusion $A^\bullet_X \hookrightarrow K^\bullet$ is qis and the ψ map can be properly defined.

[de Rham complex for X, as denoted by complex geometer]

<u>Idea</u>: Even though de Rham isom. $H_k \to (H^k)^\vee$ may not be defined via int. for enlarged space of forms (including η), the composition map $H^{2n-k} \to (H^k)^\vee$ may be.

<u>Def</u>: <u>Log complex</u> $A^\bullet\langle\log Y\rangle$ is subcomplex of $A^\bullet_{\mathcal{U}}$ gen by $A^\bullet_X$ and η.

There is well-defined res. map $Res : A^\bullet\langle\log Y\rangle \to A^\bullet_Y[1]$. We define on a rep. $\omega_1 \wedge \eta + \omega_2$ (which need not be unique) via sending this to $\omega_1|_Y$. Define $K^\bullet := \ker Res$, which contains $A^\bullet_X$ and $\operatorname{im}\psi$.

<u>Thm</u>: $0 \to K^\bullet \to A^\bullet_X\langle\log Y\rangle \xrightarrow{Res} A^\bullet_Y[1] \to 0$ is exact.

(a) $K^\bullet$ computes the cohom. of X via $A^\bullet_X \hookrightarrow K^\bullet$.

(b) $A^\bullet_X\langle\log Y\rangle$ computes the cohom. of $\mathcal{U}$ via $A^\bullet_X\langle\log Y\rangle \hookrightarrow A^\bullet_{\mathcal{U}}$.

(c) $A^\bullet_Y[1]$ computes the cohom. of Y, shifted by 1. [This is obvious...]

(d) Induced LES is compatible w/ Gysin seq. More specifically, two can be identified using (a)-(c).

We're now in business to put MHS on $H^\bullet(\mathcal{U})$. Hodge filtration for X induced by filtration of de Rham complex $A^\bullet_X$ where $F^p A^\bullet_X$ is gen. by forms of type (p',q) for $p' \geq p$ ("at least p holomorphic diff's").

Filter $K^\bullet$, $A^\bullet_X\langle\log Y\rangle$ in similar way, and shift by 1 for $A^\bullet_Y$. qis $A^\bullet_X \hookrightarrow K^\bullet$ is strict (strong compatibility of filtrations) and so filtration on cohom. of X induced by filtration on $K^\bullet$ is Hodge filtration. Interpretation of (p,q) component of $H^\bullet(X)$ extends for $K^\bullet$. Filtration on $A^\bullet_X\langle\log Y\rangle \rightsquigarrow$ Hodge filtration on $H^\bullet(\mathcal{U})$.

<u>Thm</u>: Gysin map $\delta_k : H^{k-2}(Y) \otimes \mathbb{Q}(-1) \to H^k(X)$ is map of HS's of pure weight k. This induces HS's on $\ker\delta_k$ and $\operatorname{coker}\delta_k$ of weight k. Hodge filtrations on $\ker\delta_{k+1}$, $\operatorname{coker}\delta_k$ same as those induced by Hodge fil. on $H^k(\mathcal{U})$ via SES $0 \to \operatorname{coker}\delta_k \to H^k(\mathcal{U}) \to \ker\delta_{k+1} \to 0$ (arising from Gysin seq.).

<u>Cor</u>: $H^n(\mathcal{U})$ admits nat. MHS w/ weight filtration $W^k H^n(\mathcal{U}) = \begin{cases} 0, & k<n \\ \operatorname{im} H^n(X), & k=n \\ H^n(\mathcal{U}), & k>n \end{cases}$ and Hodge fil. $F^p H^n(\mathcal{U})$ given by classes rep. by $\geq p$ holomorphic logarithmic diff. forms s.t. $gr^k_W H^n(\mathcal{U}) = \begin{cases} 0, & \text{otherwise} \\ \operatorname{coker}\delta_n, & k=n \\ \ker\delta_{n+1}, & k=n+1 \end{cases}$.

<u>Thm (Deligne)</u>: $X \mapsto H^n(X;\mathbb{Q})$ is contravariant functor $\{$ alg. var.'s $/\mathbb{C}\} \to \{$ mixed Hodge structures $\}$.

Definition: Given an integer n we define the Tate twist Z(-n) to be the unique Z-HS of rank 1 and weight 2n. This is isomorphic to Z as a Z-module. Its remaining structure is described by taking the (p,q) piece of Z(-n) to be C if p=n=q and 0 otherwise. The Hodge filtration is described by taking F^p to be C if p \geq n and 0 otherwise. This has a Q-HS analogue denoted Q(-n).

The important function of Tate twists is that they allow us to adjust weights. As we will soon see, they arise naturally.

We've just shown that there is an isomorphism of Q-HS's between H^1(\G_m,Q) and Q(-1). In fact, thinking about this a bit more and using Poincare Duality gives a natural identification between H^1(G_m,Z) and Z(-1), with Z(-1) looking like (1/2\pi i)Z (the Residue Theorem should come to mind).

We can avoid mention of the Thom complex by cupping with the Thom class associated to the normal bundle of Y in X. Note that our vector bundle is complex hence canonically orientable. A choice of orientation amounts to a choice of square root of -1.

We cannot, however, expect a general smooth variety U to admit an embedding of this form. Instead, we must allow Y to be a union of smooth projective hypersurfaces with transverse intersections (a normal crossings divisor). There is a natural generalization of $A^*_X\langle\log Y\rangle$ to this case calculating the cohomology of U, however, we can no longer write this complex as an extension of two complexes associated to smooth projective varieties. Instead, it has a natural increasing filtration $W^*A^*_X\langle\log Y\rangle$ (in the case above $W^0 = K^*$, $W^1 = A^*_X\langle\log Y\rangle$ so the filtration is the same as the short exact sequence) such that

the graded components compute the cohomology of smooth projective varieties (given by intersections of the hypersurfaces in Y). Then, rather than a long exact sequence computing the cohomology of U out of smooth projective varieties, we have a spectral sequence (corresponding to the filtration W^*) with E_1 page the cohomology of smooth projective varieties that computes the cohomology of U. The spectral sequence expresses the E_∞ page, i.e. the graded components of $H^*(U)$ as sub-quotients of the E_1 page, and in fact due to compatibility of the differentials, this induces a Hodge structure on the graded components of $H^*(U)$ (via the Hodge structures on the E_1 page). Verifying that these Hodge structures on the graded components actually come from the filtration induced by our original filtration on $H^*(U)$ is analogous to verifying it in the case of the long exact sequence (in fact, the long exact sequence arises naturally from the spectral sequence associated to the two-term filtration above).

Thus, Section 1 of Hodge II is concerned largely with the exposition of the homological algebra necessary to track filtrations through spectral sequences. The most important result is 1.3.16 - The Two Filtrations Lemma, which generalizes Lemma 17 from above and is used to show that, just like with the long exact sequence, the (Hodge) filtration on $H^*(U)$ will agree on graded components with the Hodge filtration induced by the smooth projective varieties appearing in the weight spectral sequence (so that the "Hodge" filtration on $H^*(U)$ earns its name, i.e. it induces a mixed Hodge structure).

Section 2 of Hodge II is an exposition of the basic properties of mixed Hodge structures, viewed independent of their role in complex geometry. The most important result here is 2.3.5, which says that mixed Hodge structures form an abelian category. Note this is not at all obvious, as, for example, filtered vector spaces do not form an abelian category. The difficultly is that for a filtered morphism in general there is no reason for the cokernel with its quotient topology to have the same filtration as the image with its sub-object topology – indeed we have already seen that this is the case only when the morphism is strict. Thus, the fact that MHS is an abelian category is intimately tied to the fact that morphisms in MHS are automatically strict with respect to all filtrations, and we have already seen how this strictness/abelianness comes into play when we defined the Hodge structure in our example with the long exact sequence.

How does our example fit into a more sheaf theoretic perspective? Let X be an algebraic variety over k \subset C, i: Y \inj X a closed embedding, and U the complement of Y in X. Given F a constructible (complex of) sheave(s) on X, we have exact sequences

$$i_\star i^! F \to F \to Rj_\star j^\star F \xrightarrow{+1} \tag{2}$$

and

$$Rj_! j^\star F \to F \to i_\star i^\star F \xrightarrow{+1} \tag{3}$$

Plug F=Z_X into the first triangle. We get

$$i_\star i^! \mathbb{Z}_X \to \mathbb{Z}_X \to Rj_\star \mathbb{Z}_U \xrightarrow{+1}$$

If X is nonsingular of (algebraic) dimension $d = \dim X$, then $D_X = \mathbb{Z}_X[2d](d)$ so, in this case, $i^!\mathbb{Z}_X = D_Y[-2d](-d)$ and the triangle gives rise to an exact sequence of MHS:

$$\cdots H^{BM}_{2d-m}(Y;\mathbb{Z})(-d) \to H^m(X;\mathbb{Z}) \to H^m(U;\mathbb{Z}) \to H^{BM}_{2d-m-1}(Y;\mathbb{Z})(-d) \cdots$$

D_X here is the dualizing sheaf of X. Notice the Tate twists are here as always to make sure the weights work out.

An even further specialization is when $Y \subset X$ is a **nonsingular divisor**: in this case $i^!\mathbb{Z}_X = \mathbb{Z}_Y[-2](-1)$ and we get an exact sequence of MHS:

$$\cdots H^{m-2}(Y;\mathbb{Z})(-1) \to H^m(X;\mathbb{Z}) \to H^m(U;\mathbb{Z}) \to H^{m-1}(Y;\mathbb{Z})(-1) \cdots \quad (5)$$

Let S be a complex manifold. A *variation of Hodge structure of weight* k on S consists of the following data:

(1) a local system $\mathbb{V}_{\mathbb{Z}}$ of finitely generated Abelian groups on S;
(2) a finite decreasing filtration $\{\mathcal{F}^p\}$ of the holomorphic vector bundle $\mathcal{V} := \mathbb{V}_{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathcal{O}_S$ by holomorphic subbundles (the *Hodge filtration*).

These data must satisfy the following conditions:

(1) for each $s \in S$ the filtration $\{\mathcal{F}^p(s)\}$ of $\mathbb{V}(s) \simeq \mathbb{V}_{\mathbb{Z},s} \otimes_{\mathbb{Z}} \mathbb{C}$ defines on the finitely generated Abelian group $\mathbb{V}_{\mathbb{Z},s}$ a Hodge structure of weight k;
(2) the connection ∇: $\mathcal{V} \to \mathcal{V} \otimes_{\mathcal{O}_S} \Omega^1_S$ whose sheaf of horizontal sections is $\mathbb{V}_{\mathbb{C}}$ satisfies the *Griffiths' transversality condition* $\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega^1_S$.

EXAMPLES. (1) Let V be a Hodge structure of weight k and $s_0 \in S$ a base point. Suppose that one has a representation ρ: $\pi_1(S, s_0) \to \mathrm{Aut}(V)$. Then the local system $\mathbb{V}(\rho)$ associated to ρ underlies a locally constant variation of Hodge structure. In this case the Hodge bundles $\mathcal{F}^p$ are even locally constant, so that $\nabla(\mathcal{F}^p) \subset \mathcal{F}^p \otimes \Omega^1_S$. This property characterizes the local systems of Hodge structures among the variations of Hodge structure. In case ρ is the trivial representation, we denote the corresponding variation by V_S.

(2) Let f: $X \to S$ be a proper and smooth morphism of complex algebraic manifolds. We have seen that the cohomology groups $H^k(X_s)$ of the fibres X_s fit together into a local system. This local system, by the fundamental results of Griffiths underlies a variation of Hodge structure on S such that the Hodge structure at s is just the Hodge structure we have on $H^k(X_s)$. This case will be referred to as *the geometric case*.