Quasicoherent Sheaves

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1 Background

Let A be an animated (commutative) ring.[1] To this we may associate the ∞-category Mod_A of (left) A-modules, viewed in terms of modules over the underlying $\mathbb{E}_\infty$-algebra of A.[2] If A is discrete then Mod_A can be identified with the ∞-categorical derived category $\mathcal{D}(\mathsf{Mod}_A^\heartsuit)$ of the ordinary category $\mathsf{Mod}_A^\heartsuit$ of A-modules. The category Mod_A is nice in part because it is stable and presentable, hence complete and cocomplete. Mod_A is also symmetric monoidal, with structure encoded by ~~a~~ the tensor product $\otimes_A$.

We will primarily be interested in the full subcategory $\mathsf{Mod}_A^{\mathrm{cn}}$ of *connective* A-modules, which can equivalently be identified with the category of *animated* A-modules (which means it can roughly be obtained as some kind of free (co-)completion).

2 Basics of Quasicoherent Sheaves

Let $X \in \mathsf{PStk}$ be a derived prestack. Our goal in this section is to describe the category $\mathsf{QCoh}(X)$ of *quasicoherent sheaves* over X. In particular, we will compare different constructions. Let Arena denote the ∞-category of presentable ∞-categories whose morphisms are given by maps of functors admitting left adjoints – by the adjoint functor theorem these are the same as maps of functors commuting with small colimits.[3] The assignment $A \mapsto \mathsf{Mod}_A$ defines a functor $\mathsf{Mod} : \mathsf{SCR}^{\mathrm{op}} \to \mathsf{Arena}$ which we can then right Kan extend along the Yoneda embedding $\mathrm{Spec} : \mathsf{SCR}^{\mathrm{op}} \hookrightarrow \mathsf{PStk}$ to obtain QCoh. As a reminder, we have a homotopy coherent commutative diagram

$$\begin{tikzcd} & \mathsf{PStk} \arrow[dr, dotted, "\mathsf{QCoh}"] & \\ \mathsf{SCR}^{\mathrm{op}} \arrow[ur, "\mathsf{Spec}"] \arrow[rr, "\mathsf{Mod}"'] & & \mathsf{Arena} \end{tikzcd}$$

and QCoh is basically initial with this property. The way we compute this is to take

$$\mathsf{QCoh}(X) = \varprojlim_{\mathrm{Spec}\,A \to X} \mathsf{Mod}_A,$$

which in the case that $X = \mathrm{Spec}\,A$ yields Mod_A (via the universal property).[4] More explicitly,

[1] Yes, I am switching terminology once again. I will still use the notation SCR, though.

[2] In fact, we may work with the underlying $\mathbb{E}_1$-algebra of A in many situations, thereby having to keep track of less data.

[3] Our notation comes from Joyal, who instead uses the term "arena."

[4] In case it wasn't clear, this limit is taken in the ∞-category of arenas.

objects $\mathcal{F}$ of $\mathsf{QCoh}(X)$ are "quasicoherent" families of A-modules $\mathcal{F}_f$ for every $f : \operatorname{Spec} A \to X$, where we have "base change" equivalences satisfying a homotopy coherent cocycle condition. We see that $\mathsf{QCoh}(X)$ inherits many of the nice properties from its constituent categories – e.g., $\mathsf{QCoh}(X)$ is stable, presentable, and symmetric monoidal.[5] We have a canonical quasicoherent sheaf $\mathcal{O}_X \in \mathsf{QCoh}(X)$ which assigns A to every $\operatorname{Spec} A \to X$ and serves as the unit for the symmetric monoidal structure.

Given any morphism $f : X \to Y$ of derived prestacks, we get for free a symmetric monoidal colimit-preserving functor $f^* : \mathsf{QCoh}(Y) \to \mathsf{QCoh}(X)$ which admits a right adjoint f_*. One application of this is that, given any $\mathcal{F} \in \mathsf{QCoh}(Y)$, we may associate the anima of *global sections* $\Gamma(X, \mathcal{F}) := \operatorname{Hom}_{\mathsf{QCoh}(X)}(\mathcal{O}_X, f^*\mathcal{F})$. In a little while we will see that f_* admits a much more explicit description under a mild assumption on f.

Last time, we didn't get to properly discussing what it means to be a Zariski covering. First, let us say that a morphism $f : X \to Y$ of derived prestacks is **flat** if $f^* : \mathsf{QCoh}(Y) \to \mathsf{QCoh}(X)$ is exact. This agrees with our previous notion of flatness in the affine case. Namely, suppose $X = \operatorname{Spec} B$ and $Y = \operatorname{Spec} A$. Then, flatness amounts to $f_{\mathrm{cl}} : \operatorname{Spec} \pi_0 B \to \operatorname{Spec} \pi_0 A$ being flat and the canonical map $\pi_i A \otimes_{\pi_0 A} \pi_0 B \to \pi_i B$ being invertible for every i. For f to be a (Zariski) open immersion, all we demand is that f is flat and f_{cl} is an open immersion (which means that is the complement of a classical closed immersion). This is of course still just the affine case – the general case is handled via pullback. We get the notion of a **Zariski open covering** of X as a family $\{j_\alpha : U_\alpha \hookrightarrow X\}_{\alpha \in \Lambda}$ of open immersions such that $\coprod_{\alpha \in \Lambda} U_\alpha \to X$ is an effective epimorphism. At least when each U_α is affine, this is the same as $\mathsf{QCoh}(X) \to \prod_{\alpha \in \Lambda} \mathsf{QCoh}(U_\alpha)$ being conservative (i.e., reflecting isomorphisms). (More on this later...)

Last time, we commented on how (affine) derived schemes satisfy fpqc descent (hence Zariski descent).

We will today be concerned w/ various other descent statements.

Prop: Let $X \in \mathrm{DSch}$. Then, canon. functor $\overbrace{\mathsf{QCoh}(X) \to \varprojlim_{\mathcal{U} \hookrightarrow X} \mathsf{QCoh}(\mathcal{U})}^{(*)}$ is equiv., w/ limit taken over ~~open~~ immersions $\mathcal{U} \hookrightarrow X$ w/ $\mathcal{U}$ affine derived scheme.

Note that this is obvious in the affine case, via faithfully flat descent.

[5] All of this is purely formal.

Pf: QCoh is a right Kan ext. and so sends colimits of derived stacks to limits. Choose an affine Zariski cover $\{X_\alpha \hookrightarrow X\}_\alpha$ and let $\check{C}(X_\alpha/X)_\bullet$ be the Čech nerve of $\coprod_\alpha X_\alpha \to X$. By the observation we just made, the canon. functor $QCoh(X) \to \varprojlim_{n\in\Delta} QCoh(\check{C}(X_\alpha/X)_n)$ is an equiv. Given any open immersion $\mathcal{U} \hookrightarrow X$ w/ $\mathcal{U}$ affine, pulling back gives Zariski open cover $\{\mathcal{U}_\alpha \hookrightarrow \mathcal{U}\}_\alpha$ (so $\mathcal{U}_\alpha = X_\alpha \times_X \mathcal{U}$). Hence, $QCoh(\mathcal{U}) \xrightarrow{\sim} \varprojlim_{n\in\Delta} QCoh(\check{C}(\mathcal{U}_\alpha/\mathcal{U})_n)$. This sets us up for a reduction. The above immediately reduces us to showing that $QCoh(V) \to \varprojlim_{\mathcal{U}\hookrightarrow X} QCoh(V \times_X \mathcal{U})$ is equiv. for $V \hookrightarrow X$ any term of the Čech nerve above. By cofinality, this is the same as showing (*) w/ X replaced by any of the terms of the Čech nerve.

At this point, we are done if the pairwise intersections of the X_α are affine. These intersections are open subschemes of affine derived schemes hence are separated. These admits Zariski open covers w/ affine pairwise intersections and so we are good. $\square$

Thm: As a presheaf on the site of affine derived schemes, QCoh satisfies fpqc descent.

What's the idea here? First, we replace $QCoh(\cdot)$ by its connective counterpart $QCoh(\cdot)^{cn}$ (this is harmless since we can stabilize to go back). Let $\{f_\alpha : S_\alpha \to S\}_\alpha$ be fpqc covering family w/ induced $f: \tilde{S} \to S$ for $\tilde{S} := \coprod_\alpha S_\alpha$. We claim $QCoh(S)^{cn} \xrightarrow{\sim} \underbrace{Tot(QCoh(\check{C}(\tilde{S}/S)_\bullet)^{cn})}_{= \varprojlim_{n\in\Delta} QCoh(\check{C}(\tilde{S}/S)_n)^{cn}}$.

The key is to identify the RHS w/ the ∞-cat. of coalg.'s in $QCoh(\tilde{S})^{cn}$ over the comonad assoc. to the adjunction $f^*: QCoh(S)^{cn} \rightleftarrows QCoh(\tilde{S})^{cn} : f_*$ (c.f. [SAG, Lemma D.3.5.7]). We then apply the Barr-Beck-Lurie thm, which detects comonadicity for adjunctions. How does any of this work?

$SCR = \mathcal{P}_\Sigma(\mathrm{Poly})$, $\mathrm{Mod}^{cn} = \mathcal{P}_\Sigma(\mathrm{PolyMod})$

$$\begin{tikzcd} \mathrm{Mod}_R^{cn} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Mod}^{cn} \arrow[d] \\ \{R\} \arrow[r] & SCR \end{tikzcd}$$

<u>Q</u>: Does it follow that $\mathrm{Mod}_R^{cn} \simeq (\mathrm{Mod}_{\mathbb{Z}}^{cn})_{R/}$?

Can close the gap by stabilizing...

<u>Q</u>: How do we get Mod_R from this? [Morally, Mod_R and Mod_R^{cn} are "far" from each other.]

<u>Q</u>: Why do modules obtained in this way have good properties (stable, symm. monoidal, etc.)?

[SAG, 25.2.1.2]

↑

We get suitable sfp subcat.'s.

Barr-Beck gives nec. and suff. conditions for adjunction $\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathcal{D}$ to be monadic. What do we mean by monadicity? Given ∞-cat. $\mathcal{C}$, $\mathrm{Fun}(\mathcal{C},\mathcal{C})$ admits monoidal structure (composition). Given alg. object $T \in \mathrm{Alg}(\mathrm{Fun}(\mathcal{C},\mathcal{C}))$, we get ∞-cat. $\mathrm{LMod}_T(\mathcal{C})$ of left T-modules in $\mathcal{C}$. There is assoc. adjunction

$$\mathcal{C} \underset{G}{\overset{F}{\rightleftarrows}} \mathrm{LMod}_T(\mathcal{C}).$$

Need to explain gist of using Barr-Beck as descent tool. Before that, what does Wikipedia tell us?

$(\infty\text{-})$ $\mathcal{C}$ cat. $\rightsquigarrow$ $\mathrm{End}(\mathcal{C}) := \mathrm{Fun}(\mathcal{C},\mathcal{C})$. <u>Monad</u> on $\mathcal{C}$ is monoid in $\mathrm{End}(\mathcal{C})$. ← $\mathrm{Monad}(\mathcal{C})$ ~~Assume $(\mathcal{C}, \otimes, 1)$ is monoidal~~

What is the monoidal structure on $\mathrm{End}(\mathcal{C})$? We need suitable bifunctor on $\mathrm{End}(\mathcal{C})$ and identity object.

$\circ : \mathrm{End}(\mathcal{C}) \times \mathrm{End}(\mathcal{C}) \to \mathrm{End}(\mathcal{C})$ w/ identity object $\mathrm{id}_\mathcal{C}$. Object of $\mathrm{Monad}(\mathcal{C})$ is $T \in \mathrm{End}(\mathcal{C})$ w/ mult. $\mu : T \circ T \to T$ and unit $\eta : \mathrm{id}_\mathcal{C} \to T$.

$$\begin{tikzcd} (T\circ T)\circ T \arrow[r] \arrow[d, "\mu \otimes \mathrm{id}_T"'] & T\circ(T\circ T) \arrow[r, "\mathrm{id}_T \otimes \mu"] & T\circ T \arrow[d] \\ T\circ T \arrow[rr, "\mu"'] & \circlearrowright & T \end{tikzcd}$$

$$\begin{tikzcd} \mathrm{id}_\mathcal{C}\circ T \arrow[r, "\eta \otimes \mathrm{id}_T"] \arrow[dr] & T\circ T \arrow[d, "\mu"] & T \circ \mathrm{id}_\mathcal{C} \arrow[l, "\mathrm{id}_T \otimes \eta"'] \arrow[dl] \\ & T & \end{tikzcd}$$

$$\begin{tikzcd} T^3 \arrow[r, "T\mu"] \arrow[d, "\mu T"'] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd} \qquad \begin{tikzcd} T \arrow[r, "\eta T"] \arrow[d, "T\eta"'] \arrow[dr, equal] & T^2 \arrow[d, "\mu"] \\ T^2 \arrow[r, "\mu"'] & T \end{tikzcd}$$

$$\begin{tikzcd} T(T(T(X))) \arrow[r, "T(\mu_X)"] \arrow[d, "\mu_{T(X)}"'] & T(T(X)) \arrow[d, "\mu_X"] \\ T(T(X)) \arrow[r, "\mu_X"'] & T(X) \end{tikzcd} \qquad \begin{tikzcd} T(X) \arrow[r, "\eta_{T(X)}"] \arrow[d, "T(\eta_X)"'] \arrow[dr, equal] & T(T(X)) \arrow[d, "\mu_X"] \\ T(T(X)) \arrow[r, "\mu_X"'] & T(X) \end{tikzcd}$$

Let $f: R \to S$ be map of rings. What do we mean by assoc. descent datum?

Approach 1: Give me an S-mod. M w/ isom. $M \otimes_R S \xrightarrow{\sim} S \otimes_R M$ satisfying cocycle condition. This is basically identifying the two ways of making M into an $S \otimes_R S$-module, which amounts to agreement via both projections viewed geometrically.

Approach 2: Give me an S-mod. M w/ compatible $S \otimes_R S$-comod. structure. Here, $S \otimes_R S$ is a coring w/ structure map $\Delta: S \otimes_R S \to \cong S \otimes_R S \otimes_R S \otimes_R S$

Let $\phi: A \to B$ be map of rings. We have functor $\cdot \otimes_A B : Mod_A \to Mod_B$.

Q: Can I recover Mod_A from Mod_B?

This boils down to knowing whether $\cdot \otimes_A B$ is comonadic: Mod_A equivalent to cat. of $\phi^* \circ (\cdot \otimes_A B)$-comodules in Mod_B

Exercise: Cat. of comod.'s for this comonad is equiv. to cat. of $B \otimes_A B$-comod.'s where $B \otimes_A B$ is B-coalg. w/ comult. $B \otimes_A B \cong B \otimes_A A \otimes_A B \xrightarrow{id_B \otimes \phi \otimes id_B} B \otimes_A B \otimes_A B \cong (B \otimes_A B) \otimes_B (B \otimes_A B)$

Claim: $B \otimes_A B$-comod. structure on B-mod. N (so $\cdot \otimes_A B$ comonadic $\Rightarrow N \cong M \otimes_A B$ for unique $M \in Mod_A$) same as isom. $M \otimes_A B \cong M \otimes_B B \otimes_A B \xrightarrow{\sim} B \otimes_A B \otimes_B M \cong B \otimes_A M$ satisfying cocycle condition.

Key ingredient here is the adjunction $Hom_B(M, B \otimes_A M) \cong Hom_{B \otimes_A B}(M \otimes_A B, B \otimes_A M)$ ← Intuitively, this is endomorphisms up to a flip.

Translate this to the sheaf world to get a geometric statement.

Lemma [SAG, D.3.5.7.] : $f: A \to B$ morphism of $\mathbb{E}_\infty$-rings, $B^\bullet$ Čech nerve of f formed in ∞-cat. CAlg^{op}, $\mathcal{C}$ stable A-linear ∞-cat. TFAE:

(1) Base change functor $\mathcal{C} \to \mathrm{Mod}_B(\mathcal{C})$ is comonadic.

(2) Canonical map $\mathcal{C} \to \varprojlim \mathrm{Mod}_{B^\bullet}(\mathcal{C})$ is equiv. of ∞-cat.'s.

[2] = HA

[3] = SAG

Lecture 2

Descent for quasi-coherent sheaves

In this lecture we will continue our study of quasi-coherent sheaves by proving a descent theorem and looking at some of its basic consequences.

1. Fpqc descent.

1.1. We begin with the derived analogue of the "mother of all descent theorems", which is Grothendieck's faithfully flat descent:

Theorem 1.2. *The presheaf of ∞-categories on the site of affine derived schemes*

$$\mathrm{S} \mapsto \mathrm{Qcoh}(\mathrm{S})$$

satisfies fpqc descent.

We can replace $\mathrm{Qcoh}(-)$ by $\mathrm{Qcoh}(-)_{\geqslant 0}$: one recovers $\mathrm{Qcoh}(-)$ by stabilizing, which commutes with limits when the transition functors in the diagram are left-exact. Let $(f_\alpha : \mathrm{S}_\alpha \to \mathrm{S})_\alpha$ be an fpqc-covering family and let $f : \tilde{\mathrm{S}} \to \mathrm{S}$ where $\tilde{\mathrm{S}} = \coprod_\alpha \mathrm{S}_\alpha$. We want to show that the canonical functor

$$\mathrm{Qcoh}(\mathrm{S})_{\geqslant 0} \to \mathrm{Tot}(\mathrm{Qcoh}(\check{\mathrm{C}}(\tilde{\mathrm{S}}/\mathrm{S})_\bullet)_{\geqslant 0})$$

is an equivalence, where we have adopted the notation $\mathrm{Tot}(\mathrm{A}^\bullet) := \varprojlim_{n\in\Delta} \mathrm{A}^n$ for the totalization or limit of a cosimplicial diagram $\mathrm{A}^\bullet$. This totalization can be identified with the ∞-category of co-algebras in $\mathrm{Qcoh}(\tilde{\mathrm{S}})_{\geqslant 0}$ over the comonad associated to the adjunction $f^* : \mathrm{Qcoh}(\mathrm{S})_{\geqslant 0} \rightleftarrows \mathrm{Qcoh}(\tilde{\mathrm{S}})_{\geqslant 0} : f_*$ [3, Lem. D.3.5.7]. Thus it suffices to show that this adjunction is comonadic, for which we can apply the Barr–Beck–Lurie theorem to check two conditions:

Start here

(i) The functor f^* is conservative. (i.e., reflects isomorphisms)

(ii) The functor f_*f^* preserves limits of cosimplicial diagrams that admit a splitting after applying f^*.

The first holds by definition of faithfully flat morphism. The second is a more involved Bousfield–Kan type argument which we briefly sketch here (see [3, Prop. D.6.4.6] for details). Let $\mathcal{G}^\bullet$ be a cosimplicial diagram in $\mathrm{Qcoh}(\mathrm{S})_{\geqslant 0}$ which is f^*-split; the claim is that the canonical map $f_*f^*(\mathrm{Tot}(\mathcal{G}^\bullet)) \to \mathrm{Tot}(f_*f^*(\mathcal{G}^\bullet))$ is invertible. It suffices to show that it induces isomorphisms on homotopy groups

$$\pi_i f_*f^*(\mathrm{Tot}(\mathcal{G}^\bullet)) \to \pi_i \,\mathrm{Tot}(f_*f^*(\mathcal{G}^\bullet)) \tag{1.1}$$

This is why we passed to connective stuff.

for $i \geqslant 0$.

The fact that f is faithfully flat has the following consequences. First, the functor f_*f^* restricts to an exact functor between discrete objects, and $\pi_i f_*f^*(\mathcal{F}) = f_*f^*(\pi_i\mathcal{F})$ for each $\mathcal{F} \in \mathrm{Qcoh}(\mathrm{S})_{\geqslant 0}$ and $i \geqslant 0$. Second, a discrete object $\mathcal{F} \in \mathrm{Qcoh}(\mathrm{S})$ is zero iff $f_*f^*(\mathcal{F})$ is zero.

Skip this...

To compute the homotopy groups appearing in (1.1) we make use of the the Bousfield–Kan spectral sequence, in the form of the following lemma [2, Cor. 1.2.4.12]:

Lemma 1.3. *Let $\mathrm{E}^\bullet$ be a cosimplicial spectrum. Suppose that for each $i \geqslant 0$, the associated (unnormalized) cochain complex*

$$\pi_i(\mathrm{E}^0) \xrightarrow{\vartheta_i} \pi_i(\mathrm{E}^1) \to \pi_i(\mathrm{E}^2) \to \cdots$$

is an acyclic resolution of the kernel $\mathrm{K}_i = \mathrm{Ker}(\vartheta_i)$. Then for each $i \geqslant 0$, the map $\pi_i(\mathrm{Tot}(\mathrm{E}^\bullet)) \to \pi_i(\mathrm{E}_0)$ induces an isomorphism $\pi_i(\mathrm{Tot}(\mathrm{E}^\bullet)) \xrightarrow{\sim} \mathrm{K}_i$.

Symm. monoidal cat. $\mathcal{C} \rightsquigarrow$ cat. $CMon(\mathcal{C})$ of comm. monoids

Q: What about monads and comonads?

Comm. monoid $\mathcal{O}$ in $\mathcal{C} \rightsquigarrow$ notion of $\mathcal{O}$-module object in $\mathcal{C}$

Arenas = Presentable ∞ - Categories

Def: Cat. $\mathcal{C}$ is arena if $\exists$ small cat. $\mathcal{C}_0$ and $\underbrace{\text{accessible localization}}_{\gamma \text{ admits fully faithful accessible right adjoint}}$ $\gamma : \underset{\underset{\mathrm{Fun}(\mathcal{C}_0^{op}, \mathrm{Ani})}{\|}}{\mathcal{P}(\mathcal{C}_0)} \to \mathcal{C}$.

Morphisms in this cat. are required to commute w/ small colimits, Why is Arena nice?

- Arenas are complete and cocomplete.
- $\mathcal{C}, \mathcal{D}$ arenas $\Rightarrow \mathrm{Hom}_{\mathrm{Arena}}(\mathcal{C}, \mathcal{D}) = \mathrm{Fun}_!(\mathcal{C}, \mathcal{D})$ is an arena. ($\mathcal{C}, \mathcal{D} \in$ Arena ; might want to remove the "!")
- $u \in \mathrm{Fun}_!(\mathcal{C}, \mathcal{D})$ admits right adj. iff it commutes w/ small colimits.
- $u \in \mathrm{Fun}_!(\mathcal{C}, \mathcal{D})$ admits left adj. iff it commutes w/ small limits and is accessible.

] Adjoint Functor Thm

- $\mathcal{C} \in$ Arena $\Rightarrow$ functor $\mathcal{C}^{op} \to \mathrm{Ani}$ is representable iff it commutes w/ small limits.
- Local results...

[Hence, common to write Pr^L in place of Arena.]

Module Arenas

Arena admits canon. symm. monoidal structure. Let $\mathcal{C}_1, \mathcal{C}_2 \in$ Arena. There is canon. functor

$\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{C}_1 \otimes \mathcal{C}_2$ which commutes w/ small colimits in each argument and has univ. property :

$\forall \mathcal{D} \in$ Arena : canon. functor $\mathrm{Fun}_!(\mathcal{C}_1 \otimes \mathcal{C}_2, \mathcal{D}) \to \mathrm{Fun}(\mathcal{C}_1 \times \mathcal{C}_2, \mathcal{D})$ is fully faithful w/ ess. image

spanned by functors $\mathcal{C}_1 \times \mathcal{C}_2 \to \mathcal{D}$ commuting w/ small colimits in each argument.

We won't explain how to get this, though one could imagine cooking this up in several ways.