Final Exam

Math 2202

Thursday, Dec 16, 2021

Name: ________________________________

Problem	Points	Score
1	16	
2	12	
3	12	
4	15	
5	14	
6	12	
7	20	
8	22	
9	15	
10	12	
Total	150	

You have 150 minutes for this exam. Do not spend an inordinate amount of time on any problem. **Notes and books allowed. A calculator is not needed, you may leave numerical answers in terms of things like $\sqrt{2}$ and $\cos^{-1}(\frac{4}{15})$.** Good luck!

Please sign below.

I affirm my awareness of the Standards of Academic Integrity of Boston College and the fact that cheating (presentation of others' work as my own or using outside aid) and collusion (assisting another student in academic dishonesty) are violations of these standards and undermine the educational process.

Signature ____ Date ______________

1. (16 points)

(a) Write an equation for the line L through point $(1, 2, 3)$ and parallel to $\vec{\mathbf{v}} = \langle 0, -2, 5\rangle$.

(b) Does the line T given by

$$x = 2 + s, y = 1 + s, z = 15 + 7s$$

intersect line L? If so, at what point? If not, why not?

(c) Write an equation for the plane P through the point $(1, 0, 2)$ and perpendicular to $\vec{\mathbf{w}} = \langle 1, 2, 3\rangle$.

(d) Consider the curve of intersection of $z = y^2$ and the plane P. Find a parameterization of this curve.

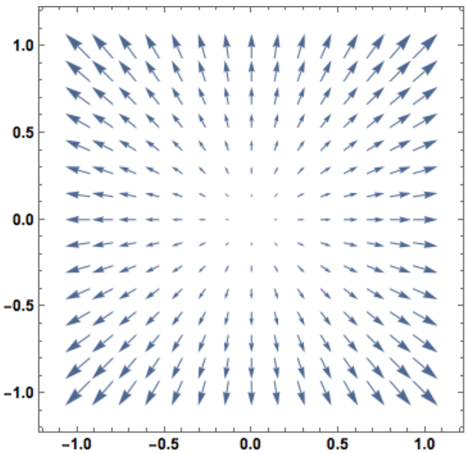
2. (12 points)

Each of the plots below depicts one of the vector fields labeled (I)-(VI). Match each plot with its vector field by writing the appropriate numeral in each box. NO EXPLANATION NEEDED.

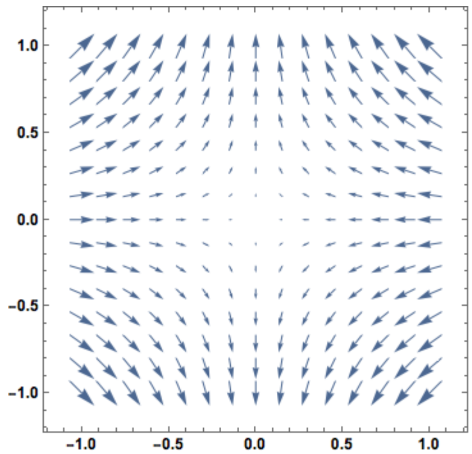
(I) $\overrightarrow{\mathbf{F}}(x,y) = \langle x, y \rangle$ (II) $\overrightarrow{\mathbf{F}}(x,y) = \langle y, -x \rangle$

(III) $\overrightarrow{\mathbf{F}}(x,y) = \left\langle x^2, y^2 \right\rangle$ (IV) $\overrightarrow{\mathbf{F}}(x,y) = \langle -x, y \rangle$

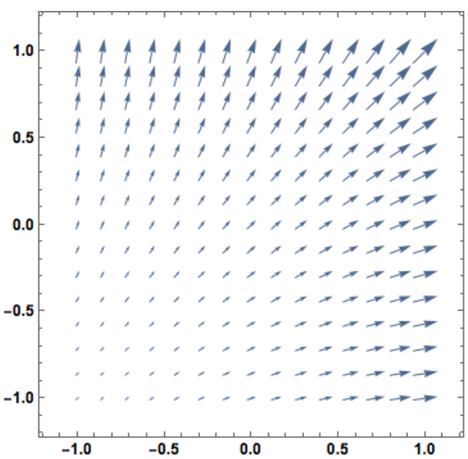
(V) $\overrightarrow{\mathbf{F}}(x,y) = \langle e^x, e^y \rangle$ (VI) $\overrightarrow{\mathbf{F}}(x,y) = \langle -y, 1 \rangle$



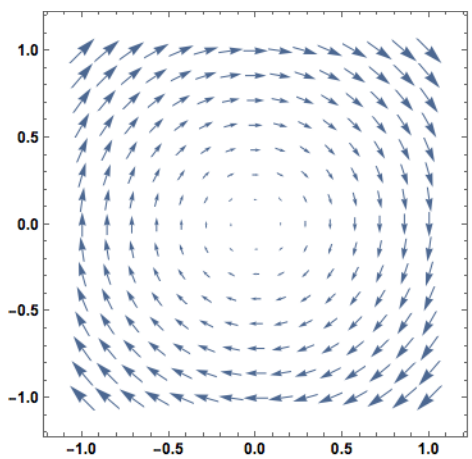
(i) The plot above best matches the vector field



(ii) The plot above best matches the vector field



(iii) The plot above best matches the vector field



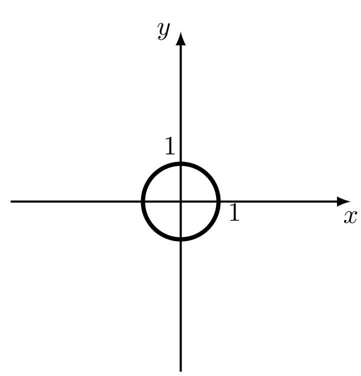
(iv) The plot above best matches vector field

3. (12 points)

Suppose that the temperature at the point (x, y) in plane $\mathbf{R}^2$ is given by $T(x, y) = 2x^2 + 4(y-1)^2$, measured in degrees Celsius.

(a) Using the method of Lagrange Multipliers, find the maximum temperature $T(x, y)$ on the circle $x^2 + y^2 = 1$. *Show all work for partial credit.*

(b) On the graph of $x^2 + y^2 = 1$ below, indicate the point(s) at which temperature T is maximum and sketch the level curve of $T(x, y)$ which corresponds to the maximum value of $T(x, y)$. *A rough sketch is fine, as long as proportions and orientation are reasonable.*



4. (18 points)

(a) Sketch (roughly) the curve C given by the parametrization

$$\mathbf{r}(t) = \langle t\cos(t), t\sin(t)\rangle$$

with $0 \leq t \leq 3\pi/4$. Be sure to label the coordinates of at least three points on the curve.

(b) Recall that the arc length of a curve C in $\mathbf{R}^2$ is the line integral along C of the function $f(x, y) = 1$. Write an integral for the arc length of the curve C. Do NOT evaluate the integral, but simplify enough so there are no vectors, or dot or cross products in the integrand.

YOU CAN DO THESE PARTS EVEN IF YOU DID NOT DO (a) and (b).

(c) Check that vector field

$$\mathbf{F}(x, y) = \langle \frac{y^2}{1 + x^2}, 2y \arctan(x) \rangle$$

is conservative (also known as *gradient*).

(d) Find a function f such that $F = \nabla f$.

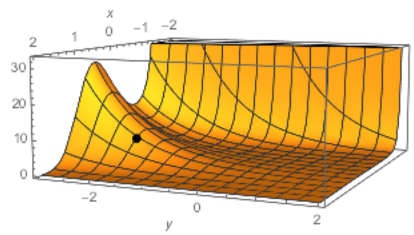
(e) We are still considering the curve C given by $\mathbf{r}(t) = \langle t\cos(t), t\sin(t) \rangle$ with $0 \leq t \leq 3\pi/4$. Calculate $\int_C \mathbf{F} \cdot \mathbf{dr}$ using the method of your choice.

5. (14 points)

Consider the function

$$f(x,y) = e^{-x^3+2x-y}.$$

Here is a graph of $f(x,y)$ on $[-2,2] \times [-3,2]$.



Assume $z = f(x,y)$ is the elevation above sea level of a region, where x, y and z are measured in meters.

(a) Compute the directional derivative of $f(x,y)$ at $(1,-2,e^3)$ in the direction $\vec{\mathbf{v}} = \langle 3,4\rangle$. Interpret the value in words, using appropriate units.

(b) At the point $(1,-2,e^3)$, in which direction(s) relative to the xy-plane is the elevation not changing?

(c) Consider a hiker on this surface, whose x and y coordinates at time t are given by $\vec{\mathbf{r}}(t) = \langle t^2, t-1\rangle$, where t is in minutes. How is the hiker's elevation along this path changing with time, at the instant $t = 1$? Give a precise numerical answer for the rate of change, including units.

6. (12 points) Let $f(x,y) = \sqrt{x^4 + y^4}$.

(a) Find the tangent plane to the surface $z = f(x,y)$ at the point $(1,0,1)$.

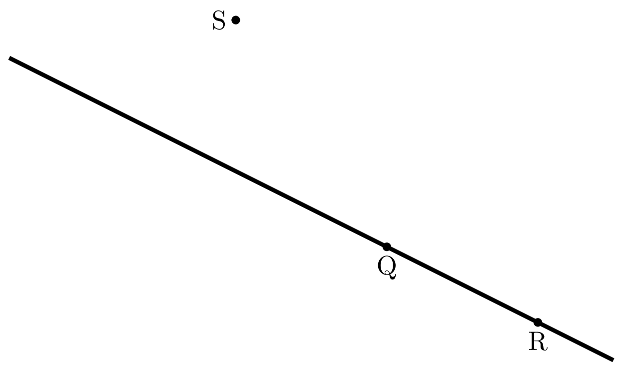
(b) Use linear approximation to estimate $\sqrt{(0.93)^4 + (0.02)^4}$.

(c) Consider the level curve in the xy-plane given by $f(x,y) = 1$. Find the tangent line to this curve at the point $(1,0)$.

7. **SHORT ANSWER** The following parts are NOT related to each other and NO explanation is needed.

(a) (7 points)

i. On the diagram below, sketch $\text{proj}_{\overrightarrow{\mathbf{QR}}} \overrightarrow{\mathbf{QS}}$.



ii. Which of the following is equal to the distance from point S to the line through Q and R? **Choose all possible.**

A.
$$\frac{\overrightarrow{\mathbf{QS}} \cdot \overrightarrow{\mathbf{QR}}}{|\overrightarrow{\mathbf{QR}}|}$$

B.
$$\frac{|\overrightarrow{\mathbf{QR}} \times \overrightarrow{\mathbf{QS}}|}{|\overrightarrow{\mathbf{QR}}|}$$

C.
$$|\,\text{proj}_{\overrightarrow{\mathbf{QR}}} \overrightarrow{\mathbf{QS}}|$$

D.
$$|\overrightarrow{\mathbf{QS}} - \text{proj}_{\overrightarrow{\mathbf{QR}}} \overrightarrow{\mathbf{QS}}|$$

E. None of the above, but the following expression will work: ______________________________

(b) (6 points) **True/False**

i. $\int_0^\pi \int_0^3 \int_0^{2\pi} \rho^2 \sin\phi \, d\theta \, d\rho \, d\phi$ is the volume of a sphere of radius 3.

True **False**

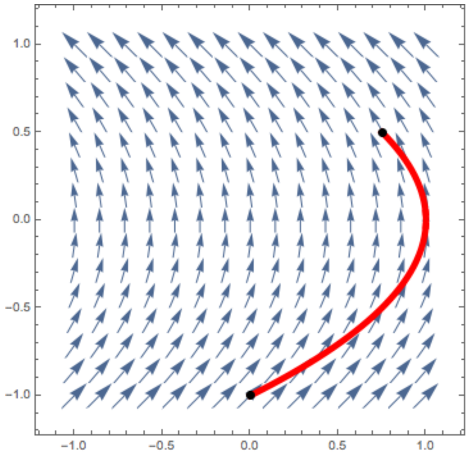
ii. Let C_1 and C_2 be two smooth curves in $\mathbf{R}^2$ which start at point $(1, 2)$ and end at point $(3, 4)$ and let $\overrightarrow{\mathbf{F}}$ be a vector field defined on all of $\mathbf{R}^2$. If $\int_{C_1} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{dr}} = \int_{C_2} \overrightarrow{\mathbf{F}} \cdot \overrightarrow{\mathbf{dr}}$, then $\overrightarrow{\mathbf{F}}$ is a conservative vector field.

True **False**

iii. If $\overrightarrow{\mathbf{v}} = \langle 0, 1, 2\rangle$ and $\overrightarrow{\mathbf{w}} = \langle 1, 2, 0\rangle$, then $\overrightarrow{\mathbf{v}} \times \overrightarrow{\mathbf{w}} = -4\,\overrightarrow{\mathbf{i}} + 2\,\overrightarrow{\mathbf{j}} - \overrightarrow{\mathbf{k}}$.

True **False**

(c) (3 points) Consider the vector field $\overrightarrow{\mathbf{F}}$ shown below and the curve C shown. **Choose one**. NO EXPLANATION NEEDED.



i. The work done by F along path C from $(0, -1)$ to $(0.75, 0.5)$ is positive.

ii. The work done by F along path C from $(0, -1)$ to $(0.75, 0.5)$ is negative.

iii. The work done by F along path C from $(0, -1)$ to $(0.75, 0.5)$ is zero.

iv. We cannot determine the sign of the work done by F along path C from $(0, -1)$ to $(0.75, 0.5)$.

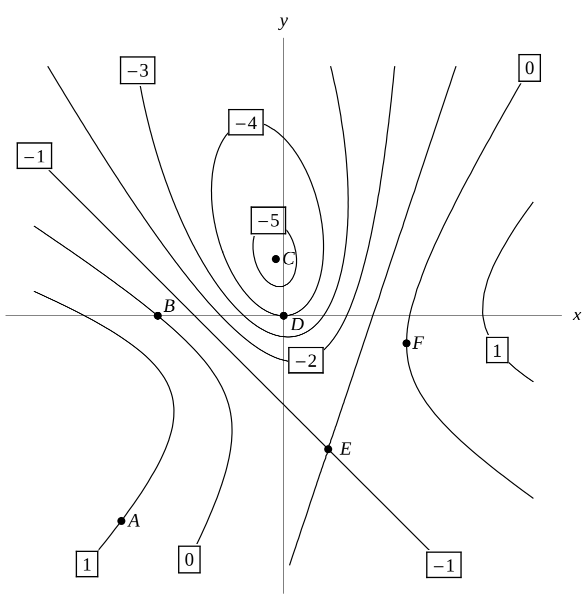
(d) (4 points) Below is a contour map of a differentiable function $f(x, y)$. Using the contour map of $f(x, y)$ shown below,

i. Determine which of the following labeled points has $f_y = 0$. CIRCLE ALL.

A **B** **D** **F**

ii. What can you say about the value $D_{\overrightarrow{\mathbf{u}}} f$ at point B if $\overrightarrow{\mathbf{u}} = \langle 1/\sqrt{2}, 1/\sqrt{2} \rangle$? CIRCLE ONE.

the value is positive **the value is negative** **the value is zero**

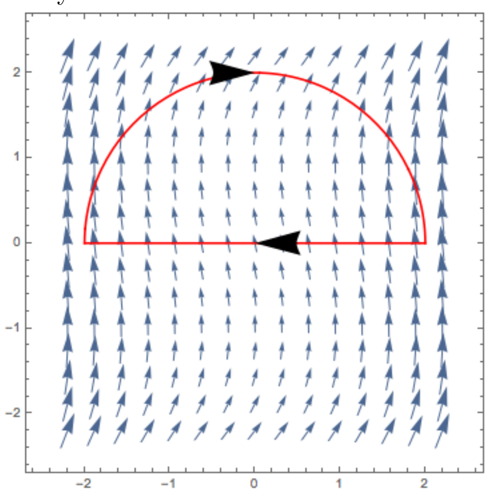


8. (22 points) A vector field $\overrightarrow{\mathbf{F}}(x,y)$ in the plane is given by $\overrightarrow{\mathbf{F}}(x,y) = \langle y^2 - 1, x^2 + 5\rangle$.

(a) Consider the line C_1 from $(2,0)$ to $(1,3)$. Compute $\int_{C_1} \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{r}}$, showing all your work.

(b) Consider the closed curve C, oriented as shown. The curve consists of a semi-circle of radius 2 and a line segment from $(2,0)$ to $(-2,0)$.

Compute $\int_C \overrightarrow{\mathbf{F}} \cdot d\overrightarrow{\mathbf{r}}$. Show all your work, and refer explicitly to any theorems or results you use, if any.



YOU CAN DO THESE PARTS EVEN IF YOU DID NOT DO (a) and (b).

We are still considering

$$\overrightarrow{\mathbf{F}}(x, y) = \langle y^2 - 1, x^2 + 5 \rangle.$$

(c) Find all the critical points of $g(x, y) = |\overrightarrow{\mathbf{F}}(x, y)|^2$ and classify them.

(d) Consider the length of the vectors in the vector field, $|\overrightarrow{\mathbf{F}}(x, y)|$, within the region R described by the closed curve C (shown on the previous page). NOTE: $|\overrightarrow{\mathbf{F}}(x, y)|$ has its largest value(s) exactly where $g(x, y) = |\overrightarrow{\mathbf{F}}(x, y)|^2$ has its largest value(s).

Given what you know about $g(x, y)$ from part (c), which of the following is true about $|\overrightarrow{\mathbf{F}}(x, y)|$? **Choose one.**

i. There is no largest value of $|\overrightarrow{\mathbf{F}}(x, y)|$ in this region.

ii. The largest value of $|\overrightarrow{\mathbf{F}}(x, y)|$ occurs on the boundary of the region.

iii. The largest value of $|\overrightarrow{\mathbf{F}}(x, y)|$ occurs inside the region.

iv. There is a largest value of $|\overrightarrow{\mathbf{F}}(x, y)|$ in this region, but we cannot determine whether it is on the boundary or inside the region.

If you did NOT do part (c), then explain what you can conclude about the largest value of F on the region.

9. (15 points) Let E be the solid in bounded by the parabolic cylinder $y = x^2$ and the planes $z = 0, z = 4$ and $y = 9$.

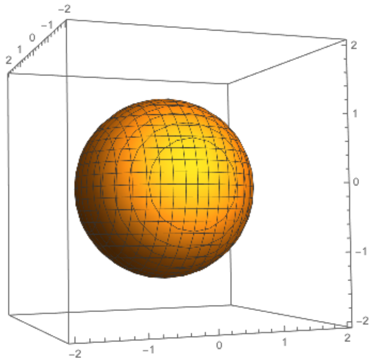
   (a) Use a triple integral to calculate the volume of the solid E.

   (b) Assuming that the solid has mass density $f(x, y, z) = y$ grams per cubic unit, write an integral that represents the total mass of the solid. Do NOT compute.

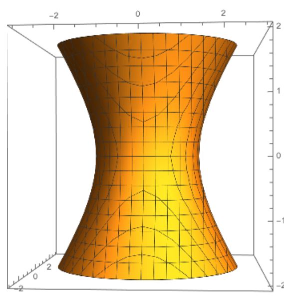
   (c) Suppose that we want to cut the solid into two parts along a plane $y = y_0$. What value of y_0 should we choose so that the two parts of the solid have the same volume?

10. (12 points)

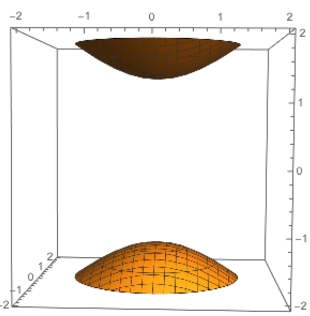
(a) Consider the quadric surface $z^2 = x^2 + y^2 + 2$. Which of the following is a graph of this surface? NO EXPLANATION NEEDED.



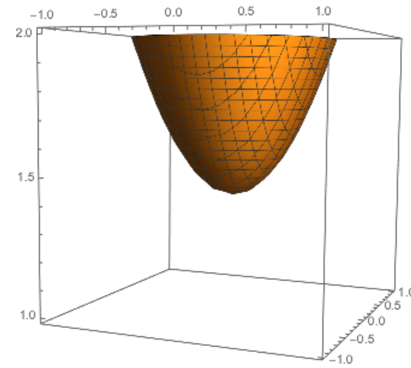
i.



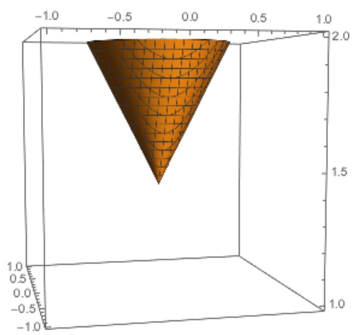
ii.



iii.



iv.



v.

(b) Find a normal vector to the surface $z = y^3 - 2xy - 1$ at the point $(-1, 1, 2)$.

(c) Consider the intersection of these two surfaces:

$$z = y^3 - 2xy - 1$$

$$z^2 = x^2 + y^2 + 2$$

Find the tangent line to the curve of intersection at point $(-1, 1, 2)$. (Hint: you can do this without parameterizing the curve.)