Put $G = GL_2(\mathbb{Q}_p)$ and define other things as expected, w/ B standard Borel and A standard maximal torus.

Let π be irred. smooth fin. dim. rep. of G. Then, G has dim 1 by Schur's lemma. This has the form

$\pi(g) = \chi(\det g)$ for χ a smooth char. of $\mathbb{Q}_p^\times$, which we already know has an unramified part and a unitary part arising from $\mathbb{Z}_p^\times$.

Let ω_1, ω_2 be (normalized) unitary char.'s of $\mathbb{Q}_p^\times$ and $s_1, s_2 \in \mathbb{C}$. We have assoc. char.'s χ_1, χ_2 w/

$\chi_i(x) = \omega_i(x)\,|x|^{s_i}$. So, $\chi = (\chi_1, \chi_2)$ extends to Borel char. via $\chi\left[\begin{pmatrix} a & \\ & b\end{pmatrix}\begin{pmatrix} 1 & x \\ & 1\end{pmatrix}\right] = \chi_1(a)\chi_2(b)$.

Normalized parabolic induction is $V(\chi_1,\chi_2) := \left\{ f : G \to \mathbb{C} \underset{\text{(locally constant)}}{\text{smooth}} \;\middle|\; f\left(\begin{pmatrix} a & \\ & b\end{pmatrix}\begin{pmatrix} 1 & x \\ & 1\end{pmatrix} g\right) = \chi_1(a)\chi_2(b)\underbrace{\left|\tfrac{a}{b}\right|^{1/2}}_{} f(g) \right\}$

[normalization by modular quasicharacter]

This is the principal series rep. of G induced from (χ_1, χ_2), w/ action given by right trans.

$[(g\cdot f)(x) := f(xg)]$

Lemma: $\underbrace{V(\chi_1,\chi_2)^\vee}_{\text{[smooth contragredient]}} \cong V(\chi_1^{-1}, \chi_2^{-1})$.

Thm: $V(\chi_1,\chi_2)$ is admissible always and irred. unless $\chi_1\chi_2^{-1} = |\cdot|^{\pm 1}$.

(a) $\chi_1\chi_2^{-1} = |\cdot| \Rightarrow V(\chi_1,\chi_2)$ contains irred. adm. subspace of codim 1

(b) $\chi_1\chi_2^{-1} = |\cdot|^{-1} \Rightarrow V(\chi_1,\chi_2)$ contains inv. 1-dim subspace w/ irred. quotient

both called special rep. (take quotient in second case)

$$\pi(\chi_1,\chi_2) := \begin{cases} V(\chi_1,\chi_2), & V(\chi_1,\chi_2) \text{ irred.} \\ \text{special rep. assoc. to } V(\chi_1,\chi_2), & \text{otherwise} \end{cases}$$

Special rep.'s are all of the form $\pi(\chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2})$ for χ char. of $\mathbb{Q}_p^\times$. $\chi \equiv 1$ gives Steinberg rep. St.

We then identify $\pi(\chi|\cdot|^{1/2}, \chi|\cdot|^{-1/2})$ w/ twisted Steinberg rep. $St \otimes \chi$. More generally, given (π, V) rep. of G and char. $\chi : \mathbb{Q}_p^\times \to \mathbb{C}^\times$, we have twist $(\pi\otimes\chi, V)$ w/ $(\pi\otimes\chi)(g)v = \chi(\det g)\,\pi(g)v$.

So, every special rep. is twist of Steinberg. Also, $\pi(\chi_1,\chi_2)\otimes\chi \sim \pi(\chi_1\chi, \chi_2\chi)$.

Thm: $\pi(\chi_1,\chi_2) \sim \pi(\chi_1', \chi_2')$ iff $(\chi_1,\chi_2) = (\chi_1',\chi_2')$ up to ordering.

[Either $(\chi_1,\chi_2) = (\chi_1',\chi_2')$ or $(\chi_1,\chi_2) = (\chi_2',\chi_1')$.]

Let (π, V) be infinite irred. adm. rep. of G.

- Jacquet mod. V_N is adm. rep. of A w/ dim ≤ 2
- (π, V) is irred. principal series $\Leftrightarrow \dim V_N = 2$.
- (π, V) is special rep. $\Leftrightarrow \dim V_N = 1$.

Ex 3.1.6: (π, V) 1-dim rep. of $G \Rightarrow$ Jacquet mod. has dim 0.

$K = GL_2(\mathbb{Z}_p)$. One way to build supercuspidal rep's is to take an irrep. of $GL_2(\mathbb{F}_p)$ and lift via $K \twoheadrightarrow GL_2(\mathbb{F}_p)$. This gives supercuspidal rep.'s of depth 0.

The following might only work for GL_2. We have matrix coeffs. $g \overset{\varphi}{\mapsto} \langle \pi(g)v, v' \rangle$ for $v, v' \in V$.

- φ has compact support, if π is supercuspidal.
- π supercuspidal $\xRightarrow{\text{(Harish-Chandra)}}$ π is supercusp form – i.e., $\int_N f(g_1 n g_2)\,dn = 0 \ \forall g_1, g_2 \in G$.

Thm: Let π be irred. adm. rep. of G. Then, π falls into one and only one of the following equiv. classes.

(i) irred. principal series $\pi(\chi_1,\chi_2)$, w/ only relation $\pi(\chi_1,\chi_2) \sim \pi(\chi_2,\chi_1)$;

(ii) special rep. $St \otimes \chi$, w/ $St\otimes\chi \sim St\otimes\chi' \Leftrightarrow \chi = \chi'$;

(iii) supercuspidal rep.;

(iv) 1-dim, w/ form $\chi \circ \det$.

This is an interesting classification. How do things change if we impose triviality of the central char. or unitarizability?

Define $PGL_2(\mathbb{Q}_p) := G/Z$. Then, rep's of $PGL_2(\mathbb{Q}_p) \leftrightarrow$ rep's of G w/ triv central char.

Ex: $\pi = \pi(\chi_1,\chi_2)$ (irred. principal series or special) $\Rightarrow \omega_\pi = \chi_1\chi_2$. π any rep. of $G \Rightarrow \omega_{\pi\otimes\chi} = \chi^2\omega_\pi$.

[Ex] Cor: Here's classification of irred. adm. rep.'s of $PGL_2(\mathbb{Q}_p)$.

(i) irred. principal series $\pi(\chi,\chi^{-1})$.

(ii) $St\otimes\chi$ w/ $\chi^2 = 1$. [quad. twist of Steinberg]

(iii) supercuspidal w/ $\omega_\pi = 1$.

(iv) 1-dim, w/ form $\chi\circ\det$ for $\chi^2=1$.

Lemma: $\chi \in X(\mathbb{Q}_p^\times)$ is unitary iff $\chi = \omega|\cdot|^{ir}$ for $r\in\mathbb{R}$ and ω of finite order.

Thm: We classify irred. adm. unitary rep's of G as follows.

[Conjecturally, these do not occur as local components of automorphic rep's.]

(i) (cont. series) irred. principal series $\pi(\chi_1,\chi_2)$ w/ χ_1,χ_2 unitary.

(ii) (complementary series) irred. principal series $\pi(\chi,\bar{\chi}^{-1})$ w/ $\chi = |\cdot|^\sigma$ for $0<\sigma<1$.

(iii) special rep. w/ unitary central char.

(iv) supercuspidal rep. w/ unitary central char.

Given $n \geq 0$, define $K(n) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in K \;:\; c \in p^n \mathbb{Z}_p \right\}$.

<u>Def</u>: Let (π, V) be infinite dim. irred. adm. rep. of G. <u>Conductor</u> $c(\pi) := \inf\{ n \geq 0 : V^{K(n)} \neq 0 \}$.

π is <u>unramified</u> if $c(\pi) = 0$ and <u>ramified</u> otherwise.

<u>Facts</u>:
- $c(\pi) < \infty$.
- $V^{K(c(\pi))}$ has dim 1. $\leftarrow$ Vectors in $V^{K(c(\pi))}$ are called <u>new forms</u>.

<u>Thm</u>: (i) $\pi(\chi_1, \chi_2)$ irred. principal series $\Rightarrow c(\pi(\chi_1,\chi_2)) = c(\chi_1) + c(\chi_2)$.

(ii) For special rep.'s, $c(\mathrm{St} \otimes \chi) = \begin{cases} 1, & \chi \text{ unram.} \\ 2c(\chi), & \chi \text{ ram.} \end{cases}$

[This has important consequences for study of modular forms.] $\downarrow$

(iii) π supercuspidal $\Rightarrow c(\pi) \geq 2$.

<u>Cor</u>: Let (π, V) be as in definition above. π is unramified iff it is unram. principal series (i.e., χ_1, χ_2 unram.).

Let's now discuss L- and ε-factors. Let π be (infinite dim) irred. adm. rep. of G. Choose Kirillov model K of G. Given $\phi \in K$, $Z(s, \phi) := \int_{\mathbb{Q}_p^\times} \phi(y) |y|^{s - 1/2} \, d^\times y$. Let's define $L(s, \pi)$.

- $\pi = \pi(\chi_1, \chi_2)$ irred. principal series $\Rightarrow L(s,\pi) := \left((1 - \alpha_1 p^{-s})(1 - \alpha_2 p^{-s}) \right)^{-1}$ for $\alpha_i := \begin{cases} \chi_i(p), & \chi_i \text{ unram.} \\ 0, & \text{otherwise.} \end{cases}$
- $\pi = \mathrm{St} \otimes \chi$ special rep. $\Rightarrow L(s, \pi) := (1 - \alpha p^{-s})^{-1}$ for $\alpha := \begin{cases} \chi(p)|p|^{1/2} = p^{-1/2}\chi(p), & \chi|\cdot|^{1/2} \text{ unram.} \\ 0, & \text{otherwise.} \end{cases}$
- π supercuspidal $\Rightarrow L(s,\pi) := 1$.

Let ψ be standard additive char. of $\mathbb{Q}_p$ and suppose ω_π ~~has~~ is trivial. Define $\varepsilon(s, \pi, \psi) := \varepsilon_0 p^{c(\pi)(1/2 - s)}$ for $\varepsilon_0 \in \{\pm 1\}$. More specifically,

$$\varepsilon_0 := \begin{cases} \chi(-1), & \pi = \pi(\chi, \chi^{-1}) \text{ irred. principal series}, \\ -1, & \pi = \mathrm{St}, \\ \chi(-1), & \pi = \mathrm{St} \otimes \chi \text{ for } \chi \text{ nontriv. quadratic.} \end{cases}$$