Section 5 Math 2202

Parameterizations, Level Curves and Level Surfaces

Comments for Facilitator:

- Do Quiz (10 min). It includes both stuff on parameterizations and tangent lines to curves, as well as some derivative recall from their single variable days since I'm noticing some students are rusty and we're approaching derivatives.
- Go over quiz (10 min). Poll first on which one they most want to discuss (This we can use as a proxy for what students have most difficulty with.) NOTE: (a) You don't need to go over the Exploratory quiz question. They can refer to Stewart. (b) **Do not go over the derivatives except to put the answers up and say the relevant rules (product, chain, etc.) They can check solutions for details.**
- Problems (20 min): Two goals for today: 1. get them more comfortable with parameterizations and how to find them given curves. (They've done a lot in the other direction from a vector function to sketching the curve.) Emphasize (a) how we can often choose x or y (or z if we're in 3D) to be the parameter (b) For circle-type things, a polar coordinate viewpoint is useful.

  Do the ellipse first as a class - ask for suggestions - get the scaled sin, cos idea, which builds off Quiz #1. Then have them try the other two. The instinct may be to do some $\pm\sqrt{\ldots}$ stuff. Ask them if there is a way around that. Give about five minutes and then do together.

  After that they can try #3a (quick) and then make sure to get to #4.

1. **Parameterizing Ellipses, Parabolas and Hyperbolas** Find a parameterization for the following curves. Specify the domain of your parameter. In other words, what values must your parameter go through in order to trace out the entire curve.

   - The curve $x^2 + 4y^2 = 9$ in the xy-plane
   - The curve $x - z^2 = 4$ in the xz-plane
   - The curve $x^2 - y^2 = 1$ in the xy-plane
     Hint: Think about the trigonometric identity $\sec^2 t = 1 + \tan^2 t$.

   Comments for Facilitator: This is the one in which domain of t is especially surprising. Remind them that $\sec t = \frac{1}{\cos t}$ and $\tan t = \frac{\sin t}{\cos t}$ so these are defined for values of $t \neq -\pi/2$.
   Ask: Which part of the hyperbola is parameterized by t in $(-\pi/2, \pi/2)$? (the right half), etc.

2. **Parameterizing Level Curves**

(a) Parameterize the level curves of $f(x, y) = x^2 + 4y^2$ with $k = \pm 9, \pm 10$

> **Comments for Facilitator:** The idea is to realize we already did that in #1 for $k = 9$ and the others are similar (no need to do that). Also that $k < 0$ yields no level curves. Ask them what this function looks like - that this describes an elliptic paraboloid.

(b) Find a parameterization of the part of the curve $x^{2/3}y^{1/3} = 3$ for which $x, y \geq 0$. (Hint: can you rewrite this as a function of one variable?)

(This is a level curve of a Cobb-Douglas function. These are used in economics[1].)

3. Find a parameterization of the curve of intersection of the surfaces $x^2 + (y+2)^2 + (z-5)^2 = 4$ and $-3x + 4z = 20$.

   First ask yourself: what are these surfaces? what do you expect their intersection to look like?

> **Comments for Facilitator:** It's a circle of radius two lying in that plane centered at (0, -2, 5). .
> Solve the plane for z, plug into the sphere equation and you'll have an ellipse in x, y to parameterize, centered at (0, -2). Why an ellipse not a circle? The shadow the "slanted" circle casts in xy-plane is ellipse.
> *(They may recognize this as the curve from Practice Exam.*

4. **Level Surfaces, or Visualizing Functions Whose Graph is 4 Dimensional**

   To visualize something like $f(x, y, z) = x^2 + y^2 + z^2$, we would need 4 dimensions to plot this in. What we do instead is use level surfaces - we look at $f(x, y, z) = k$ for values of k and plot those in 3 space. This is analogous to using level curves to understand a function of 2 variables. Try it out for the following:

   (a) $f(x, y, z) = x^2 + y^2 + z^2$

   (b) $h(x, y, z) = x + 2y + z$

[1] A Cobb-Douglas function is of the form $P(L, K) = bL^aK^{1-a}$ and gives the total production (the monetary value of goods produced in a year) we get when the labor and capital invested is L and K. Labor, L, is measured in person-hours (the work done by an average worker in one hour) per year and K, the capital invested, is the value of machinery, equipment and buildings. Here a and b are positive constants (representing elasticities and a productivity factor). In this example x is L and y is K, and the constants are $b = 1$ and $a = 2/3$. What would $x^{2/3}y^{1/3} = 3$ mean in this context?