Local Godement Jacquet Theory

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Up until now we have been discussing Tate's thesis, which can be viewed as Godement-Jacquet theory for GL_1. We first handled the local case and then tackled the global case from an adèlic perspective, wrapping up with some results on compatibility. Our approach to Godement-Jacquet theory for general GL_n will be much the same. This is an important stepping stone to understanding the Braverman-Kazhdan-Ngô program. We will mostly follow the notational and linguistic conventions of [Luo], at times interpolating with [Wang].

1 Setup

Fix F a p-adic local field (i.e., a finite extension of $\mathbb{Q}_p$) with finite residue field of cardinality q and ϖ a uniformizer. Denote by M_n the affine algebraic F-group of $n \times n$ matrices. We equip each of $M_n(F), \mathrm{GL}_n(F)$ with the p-adic topology. As for Tate's thesis we need appropriate notions of Schwartz space and Fourier transform. It turns out that the obvious guesses work just fine.

Definition 1.1.

- *Define the* ***Schwartz space*** *to be* $\mathscr{S}(M_n) = \mathscr{S}(M_n(F)) := C_c^\infty(M_n(F))$*, the space of complex-valued, locally constant, compactly supported functions on* $M_n(F)$*.*
- *Let* $\psi = \psi_F : F \to \mathbb{T}$ *denote the unique additive character on* F *of conductor* $\mathcal{O}_F$*. This is given explicitly by* $\mathrm{tr}_{F/\mathbb{Q}_p} \circ \psi_0$*, where* ψ_0 *is the composition*

$$\mathbb{Q}_p \twoheadrightarrow \frac{\mathbb{Q}_p}{\mathbb{Z}_p} \xleftarrow{\sim} \frac{\mathbb{Z}[1/p]}{\mathbb{Z}} \hookrightarrow \frac{\mathbb{R}}{\mathbb{Z}} \xrightarrow{\sim} \mathbb{T}$$

  characterized by $\psi_0|_{\mathbb{Z}_p} = 1$ *and* $\psi_0(1/p^n) = \exp(2\pi i/p^n)$ *for every* $n \geq 1$*.*
- *Define the* ***Fourier transform*** *to be* $\widehat{\cdot} = \mathcal{F} = \mathcal{F}_\psi : \mathscr{S}(M_n) \to \mathscr{S}(M_n)$ *given by*

$$\widehat{f}(x) := \int_{M_n(F)} \psi(\mathrm{tr}(xy)) f(y)\, d^+y,$$

  where d^+y *is the unique additive Haar measure on* $M_n(F)$ *self-dual with respect to* ψ *in the sense that* $\mathcal{F}^2(f)(x) = f(-x)$ *for every* $f \in \mathscr{S}(M_n)$ *and* $x \in M_n(F)$*.*[1]

[1] A more general notion is that of a dual or Plancherel measure, which exists for any locally compact Hausdorff abelian topological group equipped with a Haar measure. Our Haar measure in this case is defined explicitly in terms of differents.

Fix (π, V) an irreducible admissible (Hermitian) representation of $\mathrm{GL}_n(F)$[2] and let $(\pi^\vee, V^\vee)$ denote the smooth contragredient.[3] This comes equipped with a pairing

$$\langle\cdot,\cdot\rangle : V \times V^\vee \to \mathbb{C}, \qquad (v, \lambda) \mapsto \lambda(v).$$

Let $\mathcal{C}(\pi)$ denote the space of matrix coefficients of π, which is by definition spanned by functions of the form

$$\varphi : \mathrm{GL}_n(F) \to \mathbb{C}, \qquad g \mapsto \langle \pi(g)v, \lambda\rangle$$

for fixed $v \in V$ and $\lambda \in V^\vee$.[4] To each such φ we may associate $\varphi^\vee$ via $\varphi^\vee(g) := \varphi(g^{-1})$, which defines a matrix coefficient of $\pi^\vee$ under the identification of (π, V) with its double smooth contragredient.

Given $f \in \mathscr{S}(M_n)$ and $\varphi \in \mathcal{C}(\pi)$ we have the **local zeta integral**

$$Z(s, f, \varphi) := \int_{\mathrm{GL}_n(F)} f(g)\varphi(g)|\det g|^{s+(n-1)/2}\, dg,$$

where dg is the unique Haar measure on $\mathrm{GL}_n(F)$ defined by $|\det g|^n \cdot dg = d^+g$ for d^+g inherited from $M_n(F)$.[5] For the sake of convenience we will also have reason to consider the shifted local zeta integral

$$\widetilde{Z}(s, f, \varphi) := \int_{\mathrm{GL}_n(F)} f(g)\varphi(g)|\det g|^{s}\, dg.$$

Here is the main result to which we will devote much of our attention today.[6]

Theorem 1.2.

(a) *$Z(s, f, \varphi)$ is absolutely convergent for $\mathrm{Re}(s) \gg 0$.*

(b) *$Z(s, f, \varphi)$ is a rational function in q^{-s}.*

(c) *The $\mathbb{C}[q^{\pm s}]$-submodule $I(s, \pi) \subseteq \mathbb{C}(q^{-s})$ spanned by $\{Z(s, f, \varphi) : f \in \mathscr{S}(M_n), \varphi \in \mathcal{C}(\pi)\}$ is a principal fractional ideal of $\mathbb{C}[q^{\pm s}]$, generated by **Euler factor** $L(s, \pi) := P(q^{-s})^{-1}$ for some $P(X) \in \mathbb{C}[X]$ with $P(0) = 1$.*[7]

(d) *There exists a unique **(local) γ-factor** $\gamma(s, \pi, \psi) \in \mathbb{C}(q^{-s})$ such that*

$$Z(1 - s, \widehat{f}, \varphi^\vee) = \gamma(s, \pi, \psi) Z(s, f, \varphi)$$

for every $f \in \mathscr{S}(M_n)$ and $\varphi \in \mathcal{C}(\pi)$.[8]

[2]How do we obtain such a representation? A theorem of Harish-Chandra guarantees that any irreducible smooth unitary representation of $G(F)$ is admissible for G a reductive algebraic group over F.

[3]Here, $\mathrm{GL}_n(F)$ acts on $\mathrm{Hom}(V, \mathbb{C})$ via $g \cdot \lambda := \lambda \circ \pi(g^{-1})$ and $V^\vee$ is the subspace of smooth linear functionals in $\mathrm{Hom}(V, \mathbb{C})$.

[4]Some sources choose not to take the span for defining matrix coefficients, a choice which ultimately does not matter. One reason to care about matrix coefficients is that, at least for unitary representations of compact td groups, they determine the representation in a sense that can be made precise. This leads to results like the Peter-Weyl theorem.

[5]Some authors take the exponent on $|\det g|$ to simply be s rather than $s + (n-1)/2$. Our convention ensures that the zeta functional equation is reminiscent to the one for Tate's thesis.

[6]One method of proof for this result which we will not touch on in these notes is to use the explicit classification of irreducible admissible representations of $\mathrm{GL}_n(F)$.

[7]The condition $P(0) = 1$ is achieved by scaling and ensures uniqueness of the Euler factor.

[8]The dependence of the local γ-factor on ψ comes from our choice of Fourier transform.

The uniqueness of $\gamma(s,\pi,\psi)$ is clear by (b) and the fact that it must have value $Z(1-s,\widehat{f},\varphi^\vee)/Z(s,f,\varphi)$ for **any** choice of $f \in \mathscr{S}(M_n)$ and $\varphi \in \mathcal{C}(\pi)$. Using the **local ϵ-factor**

$$\epsilon(s,\pi,\psi) := \gamma(s,\pi,\psi)\frac{L(s,\pi)}{L(1-s,\pi^\vee)},$$

the functional equation takes on the form

$$\frac{Z(1-s,\widehat{f},\varphi^\vee)}{L(1-s,\pi^\vee)} = \epsilon(s,\pi,\psi)\frac{Z(s,f,\varphi)}{L(s,\pi)}.$$

It follows that $\epsilon(s,\pi,\psi)$ is an element of $\mathbb{C}[q^{\pm s}]^\times$ hence a monomial in q^{-s}.

The set $\{Z(s,f,\varphi) : f \in \mathscr{S}(M_n), \varphi \in \mathcal{C}(\pi)\}$ need not be closed under addition. Assuming (b), however, we do have that this is set is closed under scaling by q^{ms} for $m \in \mathbb{Z}$. Indeed, given $Z(s,f,\varphi) \in I(\pi,s)$ we have

$$\begin{aligned} q^{ms}Z(s,f,\varphi) &= q^{ms}\int_{\mathrm{GL}_n(F)} f(g)\varphi(g)|\det g|^{s+(n-1)/2}\,dg \\ &= q^{-m(n-1)/2n}\int_{\mathrm{GL}_n(F)} f(q^{-m/n}g)\varphi(q^{-m/n}g)|\det g|^{s+(n-1)/2}\,dg \\ &= q^{-m(n-1)/2n}Z(s,f(q^{-m/n}\cdot),\varphi(q^{-m/n}\cdot)) \\ &\in I(s,\pi), \end{aligned}$$

where we have used the change of variables $g \mapsto q^{m/n}g$.[9] Since $\mathbb{C}[q^{\pm s}]$ is Noetherian, $I(s,\pi)$ is a fractional ideal of $\mathbb{C}[q^{\pm s}]$ if and only if it is finitely generated as a $\mathbb{C}[q^{\pm s}]$-module. Moreover, $I(s,\pi)$ is necessarily principal if it is fractional since $\mathbb{C}[q^{\pm s}]$ is a PID. The notation of the theorem then says that $I(s,\pi) = L(s,\pi)\mathbb{C}[q^{\pm s}]$.

We will now show that $I(s,\pi)$ is in fact finitely generated as a $\mathbb{C}[q^{\pm s}]$-module. The key to this is the following two observations.

(a) Pick some $h \in \mathrm{GL}_n(F)$ and define f_1, φ_1 via right translating f, φ by h – i.e., $f_1(g) := f(gh)$ and $\varphi_1(g) := \varphi(gh)$. Then, a simple change of variables gives

$$Z(s,f_1,\varphi_1) = |\det h|^{-s-(n-1)/2}Z(s,f,\varphi).$$

Writing $\det h = u\varpi^m$ for some $u \in \mathcal{O}_F^\times$ and $m \in \mathbb{Z}$ gives

$$|\det h|^{-s-(n-1)/2} = |u\varpi^m|^{-s-(n-1)/2} = (q^{-m})^{-s-(n-1)/2} = q^{m(n-1)/2}q^{ms} \in \mathbb{C}[q^{\pm s}].$$

(b) Suppose $\varphi = \langle \pi(\cdot)v, \lambda\rangle$ for some $v \in V$ and $\lambda \in V^\vee$. Using that π is smooth, choose $K_0 \leq \mathrm{GL}_n(F)$ compact open such that $v \in V^{K_0}$ and so the restriction of φ to K_0 has constant value φ_0. Then,

$$Z(s,\mathbb{1}_{K_0},\varphi) = \varphi_0\int_{\mathrm{GL}_n(F)} \mathbb{1}_{K_0}(g)\varphi(g)|\det g|^{s+(n-1)/2}\,dg$$

is a constant independent of s (which is nonzero if φ is nontrivial). Indeed, even though the integral in question is over K_0, we may translate and scale as in (a) to get that $K_0 = \mathrm{GL}_n(\mathcal{O}_F)$ and integrate with respect to the usual (normalized) Haar measure on this group. The determinant factor then goes away and we are left with some nonzero multiple of φ_0.

[9] By definition, the function $f(q^{-m/n}\cdot)$ applies f to the input scaled by $q^{-m/n}$. The same applies to $\varphi(q^{-m/n}\cdot)$.

One then uses that π is admissible (so V^{K_0} is finite dimensional for every $K_0 \leq \mathrm{GL}_n(F)$ compact open) and the fact that every element of $\mathscr{S}(M_n)$ is locally constant of compact support.

Note: The above comments seem a bit fishy. It seems like there is somehow "too much" information to account for. Maybe we need to use that something has finite index somewhere, or something like that...

Remark 1.3. *The above analysis tells us very little about what the Euler factor $L(s,\pi)$ actually looks like. This is no accident. In the case of Tate's thesis we already knew what our Euler factors should look like – namely,*

$$L(s,\eta) = \begin{cases} (1-q^{-s})^{-1}, & \eta \text{ is trivial,} \\ 1, & \text{otherwise,} \end{cases}$$

for $\eta : F^\times \to \mathbb{C}^\times$ unitary. Using this we were able to bootstrap our way up and prove that we had meromorphic continuation and a functional equation. For general GL_n we basically do things in the opposite direction since calculating Euler factors is a subtle matter. One indication of this is the fact that, given an elliptic curve $E/\mathbb{Q}$ with conductor N, the associated (Hasse-Weil) Euler factor at p is

$$L_p(s, E/\mathbb{Q}) = \begin{cases} (1-a_p(E)p^{-s}+p^{1-2s})^{-1}, & p \nmid N, \\ (1-a_p(E)p^{-s})^{-1}, & p \mid N, p^2 \nmid N, \\ 1, & p^2 \mid N, \end{cases}$$

where $a_p(E) := p+1-|E(\mathbb{F}_p)|$.

Corollary 1.4. *Let $\omega_\pi : F^\times \cong \mathcal{Z}(\mathrm{GL}_n(F)) \to \mathbb{C}^\times$ denote the central character of π, defined by $\pi(zg) = \omega_\pi(z)\pi(g)$ for every $z \in F^\times$ and $g \in \mathrm{GL}_n(F)$.*[10] *Then,*

$$\gamma(1-s,\pi^\vee,\psi)\gamma(s,\pi,\psi) = \omega_\pi(-1).$$

Proof. Let $f \in \mathscr{S}(M_n)$ and $\varphi \in \mathcal{C}(\pi)$. Twice applying the functional equation of the previous theorem gives

$$Z(s,\mathcal{F}^2(f),\varphi) = \gamma(1-s,\pi^\vee,\psi)Z(1-s,\widehat{f},\varphi^\vee) = \gamma(1-s,\pi^\vee,\psi)\gamma(s,\pi,\psi)Z(s,f,\varphi).$$

At the same time,

$$\begin{aligned} Z(s,\mathcal{F}^2(f),\varphi) &= \int_{\mathrm{GL}_n(F)} \mathcal{F}^2(f)(g)\varphi(g)|\det g|^{s+(n-1)/2}\,dg \\ &= \int_{\mathrm{GL}_n(F)} f(-g)\varphi(g)|\det g|^{s+(n-1)/2}\,dg \\ &= \int_{\mathrm{GL}_n(F)} f(g)\varphi(-g)|(-1)^n \det g|^{s+(n-1)/2}\,dg \\ &= \int_{\mathrm{GL}_n(F)} f(g)\omega_\pi(-1)\varphi(g)|\det g|^{s+(n-1)/2}\,dg \\ &= \omega_\pi(-1)Z(s,f,\varphi). \end{aligned}$$

[10] Note that ω_π need not be unitary. We can fix this by twisting by a power of $|\cdot|$ since then ω_π will be an extension of a character of $\mathcal{O}_F$ obtained by a choice of uniformizer and thus unitary. This corresponds to twisting π by a power of $|\det|$. The character ω_π exists since Schur's Lemma tells us that the restriction of π to the center factors through $\mathbb{C}^\times$.

The result follows. □

Our strategy for proving Theorem 1.2 has two major steps.

Step 1 Use Tate's thesis and the "niceness" of supercuspidal representations to prove the theorem in the supercuspidal case.

Step 2 Use parabolic induction to reduce to the case that π is supercuspidal.

These notes will address Step 1, leaving Step 2 to Héctor.

2 Step 1 – The Supercuspidal Case

We begin by recalling some things about supercuspidal representations. Let G denote a reductive algebraic group over F. Unless otherwise stated, (π, V) denotes a representation of $G(F)$.

Definition 2.1.

- *Let $\varphi \in \mathcal{C}(\pi)$ and $H \trianglelefteq G(F)$. We say φ is* ***compactly supported mod*** H *if the image of* $\mathrm{supp}(\varphi)$ *in $G(F)/H$ is compact. Equivalently, there exists $K \subset\subset G(F)$ such that* $\mathrm{supp}(\varphi)$ *is contained in HK.*
- *The representation π is* ***supercuspidal*** *(resp.,* ***quasicuspidal****) if it is admissible (resp., smooth) and each element of $\mathcal{C}(\pi)$ is compactly supported mod $\mathcal{Z}(G(F))$.*[11]
- *Let $P \leq G$ be parabolic with Levi subgroup M and unipotent radical N. Given (σ, W) a smooth representation of $M(F)$, the* ***induced representation*** $I(\sigma) = \mathrm{Ind}_P^G(\sigma) = \mathrm{Ind}_P^G(W)$ *whose elements are locally constant functions $f : G(F) \to W$ such that $f(mng) = \delta_P(m)^{1/2}\sigma(m)f(g)$ for every $m \in M(F), n \in N(F), g \in G(F)$, where δ_P is the modular quasicharacter of P.*[12] *There is also a* ***compactly induced representation*** $\mathrm{cInd}_P^G(\sigma)$ *that is defined in the same way with an extra compact support condition.*
- *Using the above setup, define $V(N) := \langle v - \pi(u)v : v \in V, u \in N(F)\rangle$. The* ***Jacquet module*** *or* ***coinvariant space*** *of N is $V_N := V/V(N)$. Alongside this we define $\pi_N := \pi_{M(F)} \otimes \delta_P^{1/2}$.*

From the above we obtain **parabolic induction functors**

$$\mathrm{cInd}_P^G, \mathrm{Ind}_P^G : \mathrm{Rep}_{\mathrm{sm}}(M(F)) \to \mathrm{Rep}_{\mathrm{sm}}(G(F))$$

and **Jacquet functors**

$$\cdot_N : \mathrm{Rep}_{\mathrm{sm}}(G(F)) \to \mathrm{Rep}_{\mathrm{sm}}(M(F)), \qquad (\pi, V) \mapsto (\pi_N, V_N).$$

Here, $\mathrm{Rep}_{\mathrm{sm}}(G(F))$ denotes the category of smooth complex representations of $G(F)$ with its symmetric monoidal tensor product structure.

Theorem 2.2 (Frobenius Reciprocity)**.** *The Jacquet functor $\cdot_N$ is left adjoint to Ind_P^G. More precisely, evaluation at the identity gives a natural $\mathbb{C}$-linear isomorphism*

$$\mathrm{Hom}_{G(F)}(V, \mathrm{Ind}_P^G(W)) \to \mathrm{Hom}_{M(F)}(V_N, W)$$

[11] If π is irreducible then one can show that π is supercuspidal if and only if it is quasicuspidal.

[12] Our normalization is such that $I(\sigma)$ is unitarizable if σ is unitarizable.

for every pair of smooth representations (π, V) of $G(F)$ and (σ, W) of $M(F)$.

Theorem 2.3 (Jacquet)**.** *The Jacquet functor preserves admissibility. Moreover, a smooth irreducible representation (π, V) of $G(F)$ is quasicuspidal if and only if $V_N = 0$ for every N the unipotent radical of a parabolic subgroup of G.*

Let's resume tackling the proof of Theorem 1.2 in the case that π is supercuspidal. By translating and scaling we may assume without loss of generality that $\mathrm{supp}(\varphi) \subseteq F^\times K$ for $K := \mathrm{GL}_n(\mathcal{O}_F)$. Choose suitable Haar measures da on $F^\times$ and dk on K such that $dg = dadk$ and $dk(K) = 1$.[13] Define $T(s, f, \varphi) : F \to \mathbb{C}$ via

$$T(s, f, \varphi)(a) := \int_K f(ak)\varphi(k)|\det k|^s \, dk = \int_K f(ak)\varphi(k) \, dk.$$

Then, $T(s, f, \varphi) \in \mathscr{S}(F)$ and

$$\widetilde{Z}(s, f, \varphi) = \int_{F^\times K} f(g)\varphi(g)|\det g|^s \, dg = \int_{F^\times} T(s, f, \varphi)\omega_\pi(a)|a|^{ns} \, da = Z(T(s, f, \varphi), \omega_\pi|\cdot|^{ns}),$$

with the latter a local zeta function of a character in the sense of Tate's thesis. Tate's thesis tells us that $Z(T(s, f, \varphi), \omega_\pi|\cdot|^{ns})$ converges absolutely for $\mathrm{Re}(s) \gg 0$ and so the same is true for $\widetilde{Z}(s, f, \varphi)$ hence $Z(s, f, \varphi)$.[14] Our goal now is to prove the existence of the desired local γ-factor $\gamma(s, \pi, \psi)$. The first step is to reinterpret our local zeta integrals in terms of operators. Given $s \in \mathbb{C}$ with $\mathrm{Re}(s) \gg 0$, we have

$$Z(s, \pi) : \mathscr{S}(M_n) \otimes V \otimes V^\vee \to \mathbb{C}, \qquad f \otimes v \otimes \lambda \mapsto Z(s, f, \langle \pi(\cdot)v, \lambda \rangle).$$

Equivalently, this may be viewed as a map $Z(s, \pi, \cdot) : \mathscr{S}(M_n) \to \mathrm{End}_{\mathbb{C}}(V)$ satisfying

$$\langle Z(s, \pi, f)v, \lambda \rangle = Z(s, f, \langle \pi(\cdot)v, \lambda \rangle)$$

for every $v \in V$ and $\lambda \in V^\vee$. From here there are two approaches.

(1) Work with all test functions in $\mathscr{S}(M_n)$, dealing with the "boundary" of $M_n(F)$ by sorting matrices by their rank. This is the approach taken by [Luo].

(2) Restrict attention to a suitably nice class of test functions in $\mathscr{S}(M_n)$. Show that such test functions satisfy the desired functional equation and then show that there is "enough" of these nice functions to get the functional equation for all of $\mathscr{S}(M_n)$. This is the approach taken by [Wang].

We will comment more on approach (2) in a little while. For now let's flesh out approach (1). Equip $\mathscr{S}(M_n)$ with the structure of a smooth $\mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$-module via

$$((g, h) \cdot f)(x) := f(g^{-1}xh).$$

For ease of notation we let $G := \mathrm{GL}_n(F) \times \mathrm{GL}_n(F)$. Consider now the $\mathbb{C}$-vector space $\mathscr{S}(M_n) \otimes V \otimes V^\vee$. We wish to equip this space with the structure of a smooth G-representation so that $Z(s, \pi)$ is a G-equivariant functional on $\mathscr{S}(M_n) \otimes V \otimes V^\vee$ – i.e., so that $Z(s, \pi) \in \mathrm{Hom}_G(\mathscr{S}(M_n) \otimes V \otimes V^\vee, \mathbb{C})$. To do this, suppose that the G-module structure on V is encoded by a (continuous) group

[13] Note that $|\det k| = 1$ for every $k \in K$ since by definition $K = \{g \in M_n(\mathcal{O}_F) : \det g \in \mathcal{O}_F^\times\}$.

[14] We can also just show the convergence directly, using the same argument as for Tate's thesis.

homomorphism $\rho : G \to \mathrm{Aut}_{\mathbb{C}}(V)$. This induces an action of G on $V^\vee$ via $(g,h)\cdot\lambda := \lambda\circ\rho(g^{-1},h^{-1})$ and hence a (diagonal) action of G on $\mathscr{S}(M_n)\otimes V\otimes V^\vee$. Fix now some $f\in\mathscr{S}(M_n)$, $v\in V$, and $\lambda\in V^\vee$. From the above we have

$$\begin{aligned} Z(s,\pi)((g,h)\cdot f\otimes v\otimes\lambda) &= Z(s,\pi)((g,h)\cdot f\otimes\rho(g,h)v\otimes\lambda\circ\rho(g^{-1},h^{-1})) \\ &= \int_{\mathrm{GL}_n(F)} f(g^{-1}xh)\langle\pi(x)(\rho(g,h)v),\lambda\circ\rho(g^{-1},h^{-1})\rangle\,|\det x|^{s+(n-1)/2}\,dx. \end{aligned}$$

TO DO: Finish defining the action ρ...

By the same token, the operator

$$Z^\vee(s,\pi):\mathscr{S}(M_n)\otimes V\otimes V^\vee\to\mathbb{C},\qquad f\otimes v\otimes\lambda\mapsto Z(1-s,\widehat{f},(\langle\pi(\cdot)v,\lambda\rangle)^\vee)$$

lies in the same Hom space for suitable s.

Theorem 2.4.

$$\dim\mathrm{Hom}_G(\mathscr{S}(M_n)\otimes V\otimes V^\vee,\mathbb{C})=1.$$

As an immediate corollary we get that $Z^\vee(s,\pi)=\gamma_s(\pi,\psi)Z(s,\pi)$ for some $\gamma_s(\pi,\psi)\in\mathbb{C}$. The function $s\mapsto\gamma_s(\pi,\psi)$ then defines $\gamma(s,\pi,\psi)$ as desired.[15]

Proof. The space $\mathscr{S}(M_n)$ admits a filtration

$$\{0\}=S_{n+1}\subsetneq S_n\subsetneq\cdots\subsetneq S_0=\mathscr{S}(M_n),$$

where S_k is defined to be the subspace of $\mathscr{S}(M_n)$ of functions supported on matrices of rank $\geq k$. Since the underlying action of G on $M_n(F)$ preserves rank, we have that each S_k inherits the structure of a smooth G-module. Each successive quotient S_k/S_{k+1} consists of functions in $\mathscr{S}(M_n)$ supported on matrices of rank exactly k and so there is a splitting

$$\mathscr{S}(M_n)\cong S_0/S_1\oplus S_1/S_2\oplus\cdots\oplus S_n/S_{n+1}.$$

Hence, we have an isomorphism

$$\mathrm{Hom}_G(\mathscr{S}(M_n)\otimes V\otimes V^\vee,\mathbb{C})\cong\bigoplus_{k=0}^{n}\mathrm{Hom}_G(S_k/S_{k+1}\otimes V\otimes V^\vee,\mathbb{C}).$$

We claim that

$$\dim\mathrm{Hom}_G(S_k/S_{k+1}\otimes V\otimes V^\vee,\mathbb{C})=\begin{cases}1, & k=n,\\ 0, & k\neq n,\end{cases}$$

noting that $S_n/S_{n+1}\leq C_c^\infty(\mathrm{GL}_n(F))$.[16] To begin, we have

$$C_c^\infty(\mathrm{GL}_n(F))\cong\mathrm{cind}_{\mathrm{GL}_n(F)}^G(\mathbb{C}),$$

where $\mathbb{C}$ denotes the trivial representation of $\mathrm{GL}_n(F)$ and $\mathrm{cind}_{\mathrm{GL}_n(F)}^G$ denotes the ordinary compact group theoretic induction associated to $GL_n(F)$ embedded diagonally in G. Explicitly, the RHS

[15] To be precise, we get the desired functional equation on a strip but then we can extend.

[16] Note that there is a difference between $C_c^\infty(\mathrm{GL}_n(F))$ and the "restriction" of $\mathscr{S}(M_n)=C_c^\infty(M_n(F))$ to $\mathrm{GL}_n(F)$ due to the difference in topology – i.e., the topology on $\mathrm{GL}_n(F)$ is **not** the subspace topology coming from $M_n(F)$.

consists of compactly supported locally constant functions $T : G \to \mathbb{C}$ such that $T(gg_1, gg_2) = T(g_1, g_2)$ for every $g, g_1, g_2 \in \mathrm{GL}_n(F)$. The isomorphism is given by sending f to the function that takes (g, h) to $f(g^{-1}h)$. It follows that

$$\begin{aligned}\mathrm{Hom}_G(C_c^\infty(\mathrm{GL}_n(F)) \otimes V \otimes V^\vee, \mathbb{C}) &\cong \mathrm{Hom}_G(\mathrm{cind}_{\mathrm{GL}_n(F)}^G(\mathbb{C}) \otimes V \otimes V^\vee, \mathbb{C}) \\ &\cong \mathrm{Hom}_G(V \otimes V^\vee, \mathrm{ind}_{\mathrm{GL}_n(F)}^G(\mathbb{C}) \\ &\cong \mathrm{Hom}_{\mathrm{GL}_n(F)}(V \otimes V^\vee, \mathbb{C}) \\ &\cong \mathrm{Hom}_{\mathrm{GL}_n(F)}(V, V) = \mathrm{End}_{\mathrm{GL}_n(F)}(V),\end{aligned}$$

where we have used ordinary group theoretic Frobenius reciprocity and ind denotes ordinary group theoretic induction. Since π is irreducible and admissible, the last space is 1-dimensional by Schur's Lemma. This settles the edge case $k = n$.

Let's now settle the edge case $k = 0$. In this case, S_0/S_1 is the trivial representation and so we are reduced to considering $\mathrm{Hom}_G(V \otimes V^\vee, \mathbb{C})$, which vanishes since (π, V) is a supercuspidal representation of $\mathrm{GL}_n(F)$ and so $\pi \otimes \pi^\vee$ (and any appropriate twist) is a supercuspidal representation of G.[17] We're now left with the case $0 < k < n$. Let P denote the standard parabolic subgroup of GL_n of type $(n-k, k)$. The key ingredient is the following isomorphism:

$$S_k/S_{k+1} \cong \mathrm{cInd}_{P \times P}^{\mathrm{GL}_n \times \mathrm{GL}_n}(C_c^\infty(\mathrm{GL}_k(F))).$$

We won't concern ourselves with proving this (and the indexing might be off).[18] From this we get

$$\mathrm{Hom}_G(S_k/S_{k+1} \otimes V \otimes V^\vee, \mathbb{C}) \cong \mathrm{Hom}_G(V \otimes V^\vee, \mathrm{Ind}_{P \times P}^{\mathrm{GL}_n \times \mathrm{GL}_n}(C_c^\infty(\mathrm{GL}_k(F))),$$

which vanishes as before by the fact that π is supercuspidal and the adjunction between parabolic induction and the Jacquet functor. $\square$

Remark 2.5. *In [Luo] we see further that $L(s, \pi) = 1$ for $n > 2$ as a result of the so-called matrix Paley-Wiener Theorem.*

Now, what can we say about approach (2)? [Wang] chooses to consider the space $\mathscr{S}_0(M_n)$ defined to be the collection of $f \in \mathscr{S}(M_n)$ such that $\mathrm{supp}(f) \subseteq \mathrm{GL}_n(F)$ and

$$\int_{N(F)} f(g_1 u g_2)\, du = 0$$

for every $g_1, g_2 \in \mathrm{GL}_n(F)$ and N a unipotent radical of some parabolic subgroup of GL_n. One of the nice things about this space is that it is stable under the action of the Fourier transform. $\mathscr{S}_0(M_n)$ is also "big enough" in the following sense.

Proposition 2.6.

(a) *Given $T \in \mathrm{End}_{\mathbb{C}}(V)$ and $s \in \mathbb{C}$, there exists $f \in \mathscr{S}_0(M_n)$ such that $Z(s, \pi, f) = T$.*

(b) *Fix $v \in V$ nonzero. Then, V is spanned by the collection of $u \in V$ such that*

there exists $f \in \mathscr{S}_0(M_n), c \neq 0, m \in \mathbb{Z}$ such that $Z(s, \pi, f)v = cq^{-ms}u$ for every $s \in \mathbb{C}$.

The idea is then to pair this proposition with a generalized version of Plancherel's Formula.

[17] The fact that S_0/S_1 is trivial is somewhat non-obvious but can be shown by considering minors.

[18] See pages 132-133 of *Automorphic Representations and L-Functions for the General Linear Group* by Goldfeld-Hundley. The key observation is that a rank k matrix is determined by an invertible $k \times k$ submatrix.

3 Spherical Representations

In general, local Euler factors associated to irreducible admissible representations of $\mathrm{GL}_n(F)$ can be hard to write down explicitly. One case where we can say something explicit is when (π, V) is *spherical* – i.e., $\dim V^K = 1$ for $K := \mathrm{GL}_n(\mathcal{O}_F)$. In this case, $\pi^\vee$ is also spherical and choosing $v_0 \in V^K$ and $v_0^\vee \in (V^\vee)^K$ such that $\langle v_0, v_0^\vee\rangle = 1$ allows us to define the so-called **zonal spherical function**

$$\Gamma : \mathrm{GL}_n(F) \to \mathbb{C}, \qquad g \mapsto \left\langle \pi(g)v_0, v_0^\vee \right\rangle.$$

Let $B = TU$ be the standard Levi decomposition of the standard Borel subgroup of GL_n. Choose $\chi_1, \ldots, \chi_n \in X(F^\times)$ unramified, so that then $\chi := \chi_1 \cdots \chi_n$ is a character of $T(F)$. Associate to this $\pi_\chi := \mathrm{Ind}_B^{\mathrm{GL}_n}(\chi)$. Letting V denote the $\mathbb{C}$-vector space of this representation, it turns out that $\dim V^K = 1$ and we may consider the spherical representation π_0 defined to be the irreducible component of π_χ containing V^K.

Theorem 3.1. *We have $\epsilon(s, \pi_0, \psi) = 1$ and $L(s, \pi_0) = L(s, \chi_1) \cdots L(s, \chi_n)$.*

Proof. The key to the proof is to choose $f \in \mathscr{S}(M_n)$ and $\varphi \in \mathcal{C}(\pi_0)$ which are particularly amenable to calculation. Choosing f is easy – anything bi-K-invariant will suffice for now but eventually we will want it to be $\mathbb{1}_{M_n(\mathcal{O}_F)}$. Choosing φ is a little more tricky. Consider the Iwasawa decomposition $\mathrm{GL}_n(F) = B(F)K$, which also comes with a Haar measure decomposition $dg = db dk$ with $dk(K) = 1$. A theorem proved independently by Borel-Matsumoto and Casselman tells us that there exists a unique vector $\phi \in \pi_\chi$ such that $\phi(bk) = \delta_B(b)^{1/2}\chi(b)$ for every $b \in B(F)$ and $k \in K$ (hence ϕ is identically 1 on K). We similarly get $\phi^\vee$ associated to $\pi_\chi^\vee \cong \mathrm{Ind}_B^{\mathrm{GL}_n}(\chi^{-1})$. To π_χ and hence π_0 we may associate the zonal spherical function $\Gamma_\chi \in \mathcal{C}(\pi_0)$ defined by

$$\Gamma_\chi(g) = \left\langle \pi_0(g)\phi, \phi^\vee \right\rangle = \int_K \phi(kg)\phi^\vee(k)\, dk = \int_K \phi(kg)\, dk.$$

We take this to be our choice of matrix coefficient. For $\mathrm{Re}(s) \gg 0$, we then have

$$\begin{aligned} Z(s, f, \Gamma_\chi) &= \int_{\mathrm{GL}_n(F)} f(g)\Gamma_\chi(g) |\det g|^{s+(n-1)/2}\, dg \\ &= \int_{\mathrm{GL}_n(F)} \int_K f(g)\phi(kg) |\det g|^{s+(n-1)/2}\, dg \\ &= \int_{\mathrm{GL}_n(F)} \int_K f(g)\phi(g) |\det g|^{s+(n-1)/2}\, dg, \end{aligned}$$

where the last equality comes from switching the order of integration, changing variables, and using the bi-K-invariance of f. Using the above Iwasawa decomposition this becomes

$$\int_{B(F)} f(b)\delta_B(b)^{1/2}\chi(b) |\det b|^{s+(n-1)/2}\, db.$$

Every element $b \in B(F)$ has the explicit form

$$b = \begin{pmatrix} a_1 & & u_{jk} \\ & \ddots & \\ 0 & & a_n \end{pmatrix}.$$

Choose Haar measures $d^\times a_i$ on $F^\times$ and du_{jk} on F such that $d^\times a_i(\mathcal{O}_F^\times) = 1$ and $du_{jk}(\mathcal{O}_F) = 1$. One explicitly computes that

$$db = \prod_{1\leq i\leq n} |a_i|^{-(n-i)} d^\times a_i \prod_{j,k} du_{jk}$$

and

$$\delta_B(b) = \prod_{1\leq i\leq n} |a_i|^{n+1-2i}.$$

Hence, our desired integral becomes

$$\int_{(F^\times)^n\times F^{n(n-1)/2}} f\begin{pmatrix} a_1 & & u_{jk} \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \prod_{1\leq i\leq n} \chi_i(a_i)|a_i|^s d^\times a_i \prod_{j,k} du_{j,k}.$$

Using the isomorphism $\mathscr{S}(M_n) \cong \mathscr{S}(F)^{\otimes n^2}$, we may identify the function

$$(a_1,\ldots,a_n) \mapsto \int_{F^{n(n-1)/2}} f\begin{pmatrix} a_1 & & u_{jk} \\ & \ddots & \\ 0 & & a_n \end{pmatrix} \prod_{j,k} du_{j,k}$$

with an element of $\mathscr{S}(F)^{\otimes n}$. Assuming without loss of generality that this gives a simple tensor $f_1 \otimes \cdots \otimes f_n$, we get

$$Z(s, f, \Gamma_\chi) = Z(s, f_1, \chi_1)\cdots Z(s, f_n, \chi_n).$$

Taking $f := \mathbb{1}_{M_n(\mathcal{O}_F)}$ gives the desired result on L-functions. The fact that $\epsilon(s, \pi_0, \psi) = 1$ follows from the fact that this choice of f is Fourier-stable. $\square$