# MATH1101 – Spring 2022
# Intro Lab: Groupwork
# Thursday, January 20, 2022

**Group work guidelines:**

- Work in groups of 3-4 people on the problems below.
- You do not need to work on the problems in order. You may wish to start with problems that seem more familiar, or you may opt to work first on problems that seem harder and present more of a challenge. The choice is yours!
- **This week, the group work problems on this page and the next are not being collected this week.** In subsequent weeks, they will be and more directions will be given.

1. **Icebreaker** Write down the names and contact information (email) for the other people in your group. For future labs, you will need to know this information to compile your solutions and submit them on Gradescope. **Your answer for this problem should be the names and contact information for all group members.**

   What are three things ALL members of your group share in common[1]? What are three areas in which you are all unique or different?

2. **Setting the Stage**

   Discuss the following questions with your group. You may want to take notes; you will be asked to write up your ideas from this discussion in the homework.[2]

   - Why are you interested in taking this class? What do you hope to get out it?
   - How does a person learn something new? How do you learn something new?
   - What is the value of making mistakes in the learning process (for math and in general)? Give an example.

3. **Group Roles**

   Collaborating productively with others on math problems and communicating about math are two goals for the course. To support productive group work, we're asking you to use group roles for the first month or so, after which we'll adjust as needed. These roles help people learn to work together in a way that everyone contributes.

   Read the roles briefly. Then for each problem below, select a different role (or multiple roles) from the following. Roles should rotate throughout the group for each problem.

   - facilitator: Make sure that the task is clear to everyone, that the group is working together towards the agreed goals, and that everyone is participating and has their ideas heard.
   - recorder/reporter: Make sure important ideas and results are recorded. Record strategies and methods as well as solutions, and be ready to share the group's mathematical journey. Make sure that the team is ready to report at the end.
   - understanding coordinator: Make sure that calculations are checked and mathematical reasoning is justified. Make sure the group is making connections between ideas.
   - resource manager: Make sure everyone has a useful task to work on, and that they have the tools and information they need to complete the task. This role is the person who reaches out to the instructor if there is a group question.

   (Modified from the roles here: https://nrich.maths.org/7908.)

[1] Try to find something less obvious than being in this class together!!

[2] Problems modified from https://danaernst.com/setting-the-stage/

4. **Applications of Riemann Sums**

A new factory wants to maintain $CO_2$ emissions of no more than 20,000 lbs a week. In your role as environmental engineer, you are monitoring the factory (which operates 24 hours a day, seven days a week) to estimate the total emissions over a one-week period. Below is a table of data[3] where $r(t)$ is pounds of $CO_2$ per hour and $t$ is time measured in hours from 8 am on Monday morning.

| Day/Time | Mon 8 am | Tues 7 am | Wed 8 am | Thurs 9 am | Fri 8:30 am | Sun 8 am |
|---|---|---|---|---|---|---|
| $t$ | 0 | 23 | 48 | 73 | 96.5 | 144 |
| $r(t)$ | 131 | 135 | 120 | 150 | 127 | 130 |

(a) Use this data to estimate the total emissions during this week (from 8 am Monday to 8 am Monday). (There is more than one way.)

Include units with your computations, and explain your reasoning.

(b) Make a rough sketch of a *possible* graph of $r(t)$ as a function of $t$. How does your estimate relate to the graph? (Don't worry about it being perfectly to scale.)

(c) Can you tell if your estimate is an over- or underestimate for the total emissions? Are you making any assumptions? What other ways could you estimate that could be more precise or rely on different assumptions?

(d) Can you definitely conclude whether the factory is emitting over 20,000 lbs? Why or why not?

(e) Suppose you learn that the factory shut down briefly on Wed at noon until Thurs at midnight. Does this information affect your estimate and if so, how? Explain your reasoning.

(f) How does the method you used in this problem relate to the ideas from the first class about distance, speed and time?

**Solution:**

(a) Notice: We have a 7-day period, but only 6 data points. Also, the data points are not equally spaced - the first interval is 23 hours, the next is 25 hours, and so on.

What this means is we can use the process of a Riemann sum, but it won't be a traditional $R_n$ or $L_n$ with equal sized intervals.

The key idea is that rate of change times time interval = amount of change on that interval.

$$\text{rate of emission } \times \text{ hours of emission} = \text{ total emission}$$
$$\frac{\text{lbs CO2}}{\text{hour}} \times \text{ hours} = \text{ lbs CO2}$$

So we estimate the rate of emission and use that to estimate the total emission on each time interval.

We also have some choice to make about whether we use a left sum, an upper sum, a lower sum, or try to average the data on an interval.

For example, since $r(t)$ increases and decreases over the week, it makes sense to estimate using an average value of the endpoints for each measurement interval (except for the last interval where we only have a right endpoint, which we'll simply use).

In order to get an upper sum, we will also use the higher value given for each interval, and in order to get a lower sum, we will use the lower value given for each interval. (Note: the upper and lower may not actually be upper and lower estimates because we don't have perfect information, only data points. See next part of problem.)

Here are 4 possible ways.

[3] These numbers were created based off the assumption that the factory is using 100 kWh of electricity per hour and that the average electricity source emits 1.306 lbs of $CO_2$ per kWh, according to http://www.carbonfund.org/site/pages/carbon_calculators/category/Assumptions.

- For the average estimate, we have

$$\begin{aligned}\sum r(t)_{avg}\Delta t &= \left(\frac{131+135}{2}\right)\cdot 23+\left(\frac{135+120}{2}\right)\cdot 25+\left(\frac{120+150}{2}\right)\cdot 25\\ &\quad+\left(\frac{150+127}{2}\right)\cdot 23.5+\left(\frac{127+130}{2}\right)\cdot 47.5+130\cdot 24\\ &= 133\cdot 23+127.5\cdot 25+135\cdot 25+138.5\cdot 23.5+128.5\cdot 47.5+130\cdot 24\\ &= 22,100 \text{ lbs}\end{aligned}$$

- For each time interval, make an estimate using the upper of the values (it may be the left or right depending on the interval). This provides us our best guess at an upper estimate.

$$\begin{aligned}\sum r(t)_{upper}\Delta t &= 135\cdot 23+135\cdot 25+150\cdot 25+150\cdot 23.5+130\cdot 47.5+130\cdot 24\\ &= 23,050 \text{ lbs}\end{aligned}$$

- For each time interval, make an estimate using the lower of the values (it may be the left or right depending on the interval). This provides us our best guess at a lower estimate. If we look at the lower estimate:

$$\begin{aligned}\sum r(t)_{lower}\Delta t &= 131\cdot 23+120\cdot 25+120\cdot 25+127\cdot 23.5+127\cdot 47.5+130\cdot 24\\ &= 21,150 \text{ lbs}\end{aligned}$$

- Use the left-hand estimate. For each time interval, make an estimate using the left value on the interval. However, since $r(t)$ increases and decreases over the week it will be hard to know if this is giving a good lower or upper estimate.

$$\begin{aligned}\sum r(t)_{left}\Delta t &= 131\cdot 23+135\cdot 25+120\cdot 25+150\cdot 23.5+127\cdot 47.5+130\cdot 24\\ &= 22065.5 \text{ lbs}\end{aligned}$$

(b) Here is one possible graph.

(c) With only the data given and no other assumptions, we can't say whether the upper and lower estimates are truly over or underestimates.

Why? We don't know what the rate is between time points. For example, it's possible that emissions spiked or dipped at times when emissions were monitored.

However, if we make the assumption that the data points are connected by straight lines, then the lower sum will be an underestimate (since we always used the minimum rate on the interval) and the upper sum will be an overestimate (since we always used the maximum rate on the interval).

(d) Since even the lower estimate is greater than the 20,000 lbs per week, it is likely that the total number of emissions is greater than this. However, we can NOT conclude for sure that the emissions is over 20,000. For that we'd need more information about what is happening between the sample points.

Why? It's possible that emissions spiked at times when emissions were monitored and all of these are over-estimates.

(e) Since the factory shut down for 12 hours, it was not producing $CO_2$ during those hours and our previous estimates are too high.

In our average estimate, we subtract $135\cdot 12 = 1620$ lbs, as the 12-hour shutdown was during a time when the average emissions rate was $\left(\frac{120+150}{2}\right) = 135$ lbs/hr. Thus our average estimated emissions were $22,100 - 1620 = 20480$ lbs.

In our upper estimate, we subtract $150\cdot 12 = 1800$ lbs, giving us $23050 - 1800 = 21,250$ lbs total emissions.

In our lower estimate, we subtract $120\cdot 12 = 1440$ lbs, giving us $21150 - 1440 = 19,710$ lbs.

In our left estimate, we subtract $120\cdot 12 = 1440$ lbs, giving us $22065.5 - 1140 = 20925.5$ lbs.

With this extra information, it's possible that the actual emissions were lower than 20,000 lbs, but still not especially likely.

The factory should probably work on lowering their emissions in order to reach their goal of keeping emissions under 20,000 lbs.

(f) The key idea is that rate of change times time interval = amount of change on that interval.

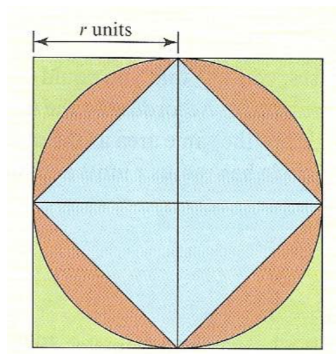
In class, we used this idea where the rate of change $r(t)$ was velocity (like miles per hour), and we found distance traveled. Here we used this idea where the rate of change $r(t)$ was lbs/hour, and we found total lbs emitted.

**5. Estimating $\pi$**

*Before you begin, select and record your roles for this problem.*

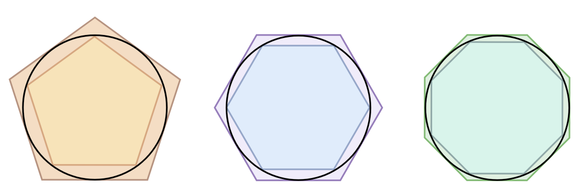
Estimating the area of a circle is a problem that has inspired humans across cultures.

- Pretend you don't know the value of $\pi$ at all, or the area of a circle formula. Use the picture below and geometry to get over- and underestimates for the area of a circle of radius $r$. (Don't use the area of a circle formula - you're just getting estimates.)
- Use your estimates in the case that $r = 1$ to find an estimate for the area of a circle of radius 1.
  We know the exact value of this area is the number $\pi$. How close are your estimates?
- How you might improve these estimates using similar area computations? Explain (but do not calculate). Include pictures with your explanation. How does this relate to the ideas from the first class?




**Solution:**

- The side length of the green square is $2r$ and so the area of the green square is $4r^2$. Thus we have that the area of the circle is at most $4r^2$.
  On the other hand, the diagonal of the blue square is $2r$ so the legs of each of the blue right triangles (that make up the blue square) have length $r$. Thus the sides of the blue square have length $r\sqrt{2}$ (via the Pythagorean Theorem or knowing the side-length ratios of 45-45-90 triangles). The area of the blue square is $(r\sqrt{2})^2 = 2r^2$, so the area of the circle is at least $2r^2$.
  Combining these estimates, we get that
  $$2r^2 \leq \text{ Area of circle radius } r \ \leq 4r^2.$$
- If $r = 1$, we get $2 <$ Area of circle radius 1 $< 4$, and so we get that $2 < \pi < 4$
  This is a pretty rough estimate, given that $\pi = 3.14.....$ But it is still a reasonable estimate and explains with a picture why $\pi$ should be somewhere near 3.
- One way to get a better estimate is to use a shape with more sides (a regular pentagon, hexagon, heptagon, octagon, etc.). By placing one of these shapes outside and inside the circle, we get over- and underestimates, respectively, for $\pi r^2$, and by dividing by $r^2$, we get over- and underestimates for $\pi$.



The areas of each of those regular polygons can be found by dividing the shape up into triangles, whose areas can be found more easily, and adding the areas of the triangles.