AN ALGEBRAICITY CONJECTURE OF DRINFELD AND THE MODULI OF p-DIVISIBLE GROUPS

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ABSTRACT. We use the newly developed stacky prismatic technology of Drinfeld and Bhatt-Lurie to give a uniform, group-theoretic construction of smooth stacks $\mathrm{BT}_n^{G,\mu}$ attached to a smooth affine group scheme G over $\mathbb{Z}_p$ and 1-bounded cocharacter μ, verifying a recent conjecture of Drinfeld. This can be viewed as a refinement of results of Bültel-Pappas, who gave a related construction using (G,μ)-displays defined via rings of Witt vectors. We show that, when $G = \mathrm{GL}_h$ and μ is a minuscule cocharacter, these stacks are isomorphic to the stack of truncated p-divisible groups of height h and dimension d (the latter depending on μ). This gives a generalization of results of Anschütz-Le Bras, yielding a linear algebraic classification of p-divisible groups over very general p-adic bases, and verifying another conjecture of Drinfeld.

The proofs use derived algebraic geometry, combined with an animated variant of Lau's theory of higher frames and displays, and actually show representability of a wide range of stacks whose tangent complexes are 1-bounded in a suitable sense. In particular, we also prove algebraicity for the stack of perfect F-gauges of Hodge-Tate weights $0,1$ and level n.

CONTENTS

1. INTRODUCTION

The goal of this paper is to prove two recent conjectures of Drinfeld [21]. The first of these has to do with a Dieudonné theory for p-divisible groups over arbitrary p-adic formal schemes; that is, we aim to describe p-divisible groups, or more generally truncated p-divisible groups or Barsotti-Tate groups, in terms of linear algebraic data. For the purposes of this paper, this last phrase means a subcategory of vector bundles on a formal algebraic stack, though it has historically taken the form of a description in terms of modules equipped with a Frobenius semi-linear map along with certain additional structures.

The stacks we consider here arose in recent work of Bhatt-Lurie [8, 9, 6] and Drinfeld [23]. These authors have shown that one can associate with every p-adic formal scheme X a p-adic formal stack[1] X^{syn}, its *syntomification*, whose coherent cohomology computes the p-adic syntomic cohomology of X. If $X = \operatorname{Spf} R$ is affine, we will also denote this by R^{syn}.[2] Vector bundles of rank h on this stack and its mod-p^n fibers—which are examples of objects known as *F-gauges over R*—have a natural h-tuple of locally constant integer valued functions on $\operatorname{Spec} R$ associated with them: these are the *Hodge-Tate weights*.

Suppose that X is covered by p-adic formal affine schemes $\operatorname{Spf} R$ with $\Omega^1_{R/pR}$ finitely generated over R. For instance, X can be formally of finite type over $\mathbb{Z}_p$, or it can be of the form $\operatorname{Spf} R$ with R a *semiperfectoid* ring. We prove:

Theorem A. *Let $\mathcal{BT}_n(X)$ be the category of n-truncated Barsotti-Tate groups over X [35], and let $\mathrm{Vect}_{\{0,1\}}(X^{\mathrm{syn}})$ be the category of vector bundles on $X^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ with Hodge-Tate weights in $\{0,1\}$. Then there is a canonical equivalence of categories*

$$\mathcal{G}_n : \mathrm{Vect}_{\{0,1\}}(X^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\simeq} \mathcal{BT}_n(X)$$

compatible with Cartier duality.

Remark 1. Here are some (very incomplete) historical remarks, though see also [24, §7]. We will write $\mathcal{BT}(X)$ for the category of p-divisible groups over X:

- The first attempt at a complete description of this category for a particular X was probably by Dieudonné-Manin [53], where $X = \operatorname{Spec} \kappa$ with κ a perfect field of characteristic p.
- A uniform proof of a description of $\mathcal{BT}(\operatorname{Spec} \kappa)$ in terms of Dieudonné modules was given by Fontaine [28].
- A general construction of a *crystalline* Dieudonné functor was given in [4], and this was used by de Jong [18] to exhibit an equivalence between p-divisible groups and Dieudonné F-crystals over formally smooth formal schemes over $\mathbb{F}_p$ whose reduced scheme is of finite type over a field with finite p-basis.
- When $X = \operatorname{Spec} R$ with R a perfect $\mathbb{F}_p$-algebra, a form of Theorem A is due to Gabber and Lau [44]: One can show that $\mathrm{Vect}_{\{0,1\}}(X^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})$ is equivalent to a category of finite locally free $W_n(R)$-modules equipped with certain additional structures appearing in *loc. cit.*
- When X is quasisyntomic, Anschütz and Le Bras demonstrated in [1] an equivalence of categories between $\mathcal{BT}(X)$ and a certain category of *admissible* φ-modules over a sheaf of rings $\mathcal{O}^{\mathrm{pris}}$ obtained using prismatic cohomology. One can once again reformulate their result as proving Theorem A for such rings. See also the recent papers of Guo-Li [31] and Mondal [56], where a similar connection is made. Mondal actually proves a classification theorem for all finite locally free p-power torsion commutative group schemes over quasisyntomic X in terms of F-gauges.
- Therefore, the main content of the above theorem is its validity for formal schemes that are not quasisyntomic. This generality is important for instance to understand the deformation theory of p-divisible groups in terms of that of F-gauges.

Remark 2. As a prior footnote observed, X^{syn} is not in general a classical object, and, correspondingly, the category of vector bundles on X^{syn} is in general an *∞-category* that is not classical. However, the theorem shows that the subcategory spanned by the objects with Hodge-Tate weights in $\{0,1\}$ *is* classical.

Remark 3. The compatibility with Cartier duality takes the following shape: There is a canonical object $\mathcal{O}^{\mathrm{syn}}\{1\}$ in $\mathrm{Vect}_{\{0,1\}}(X^{\mathrm{syn}})$ of rank 1, the *Breuil-Kisin twist*, which we can tensor with any vector bundle $\mathcal{M}$ over $X^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ to obtain the twist $\mathcal{M}\{1\}$. If $\mathcal{M}$ has Hodge-Tate weights $0, 1$, then so does $\mathcal{M}^\vee\{1\}$, and we now have a canonical isomorphism of truncated Barsotti-Tate groups

$$\mathcal{G}_n(\mathcal{M}^\vee\{1\}) \xrightarrow{\simeq} \mathcal{G}_n(\mathcal{M})^*,$$

where the right hand side is the Cartier dual of $\mathcal{G}_n(\mathcal{M})$.

[1] This is actually a *derived* formal stack that is in general not a classical object. We will attempt to ignore this fact in this introduction.
[2] We have adopted this notation from the lecture notes of Bhatt [6].

Remark 4. One striking feature of the theorem to those familiar with Dieudonné theory hitherto is the natural direction of the functor realizing the equivalence. Usually, one associates linear algebraic objects with p-divisible groups and their truncations. Here, our functor $\mathcal{G}_n$ goes in the *other* direction and associates truncated Barsotti-Tate groups with objects that are more linear algebraic in nature. Its definition is in terms of syntomic cohomology: That is, for any map $f : \mathrm{Spf}\, C \to X$, we set

$$\mathcal{G}_n(\mathcal{M})(C) = \tau^{\leq 0} R\Gamma(C^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}, (f^{\mathrm{syn}})^*\mathcal{M}).$$

As such it is completely canonical and compatible with arbitrary base-change.

Remark 5. As noted above, in [56], Mondal extends the results of Anschütz-Le Bras and shows that the category of finite flat group schemes over quasisyntomic formal schemes is equivalent to the category of perfect F-gauges of Hodge-Tate weights $\{0, 1\}$ and Tor amplitude $[-1, 0]$. In forthcoming work [52], this will be generalized to the same context as that found in Theorem A.

1.1. **Method of proof.** Our proof is geometric in nature. Its starting point is the fundamental result of Grothendieck that the stack BT_n of n-truncated Barsotti-Tate groups is a smooth Artin stack [35]. We begin by showing the following analogue of Grothendieck's theorem:

Theorem B. *The assignment*[3]

$$X \mapsto \mathrm{Vect}_{\{0,1\}}(X^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})_{\simeq}$$

is represented by a smooth p-adic formal Artin stack over $\mathbb{Z}_p$.

Theorem A can be reduced to the assertion that this formal Artin stack—which we will denote for the purposes of this introduction by $\mathrm{Vect}^{\mathrm{syn}}_{\{0,1\},n}$—is canonically isomorphic to BT_n. To construct this isomorphism, we need another representability result.

Theorem C. *For any $\mathcal{M}$ in $\mathrm{Vect}_{\{0,1\}}(X^{\mathrm{syn}})$ the functor $\mathcal{G}(\mathcal{M})$ on formal schemes over X given for $f : \mathrm{Spf}\, C \to X$ by*

$$\mathcal{G}_n(\mathcal{M})(C) = \tau^{\leq 0} R\Gamma(C^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}, (f^{\mathrm{syn}})^*\mathcal{M})$$

is represented by a truncated Barsotti-Tate group scheme over X.

Theorems B and C together now give us a map of smooth p-adic formal Artin stacks

$$\mathcal{G}_n : \mathrm{Vect}^{\mathrm{syn}}_{\{0,1\},n} \to \mathrm{BT}_n.$$

To get a map in the other direction, by the smoothness of the stacks involved, and quasisyntomic descent, it suffices to define a canonical map $\mathcal{M} : \mathrm{BT}_n(X) \to \mathrm{Vect}^{\mathrm{syn}}_{\{0,1\},n}(X)$ when $X = \mathrm{Spf}\, R$ with R quasiregular semiperfectoid (qrsp). For this, we use the functor defined by Mondal [56], which is a reinterpretation of that of Anschütz-LeBras [1]. We could have also used the Dieudonné functor of Berthelot-Breen-Messing [4] for characteristic p qrsp inputs; see Remark 11.5.5.

With the functor $\mathcal{M}$ in hand, the verification that it is indeed an inverse proceeds via a direct and quite simple argument that comes down to the compatibility of the functor $\mathcal{G}_n$ with Cartier duality. This in turn relies on two things:

(1) A computation of Bhatt-Lurie showing that we have a canonical isomorphism $\mathcal{G}_n(\mathcal{O}^{\mathrm{syn}}_n\{1\}) \simeq \mu_{p^n}$;
(2) Results of de Jong [18], [17] on crystalline Dieudonné theory, though 'only' the case of complete DVRs in characteristic p is needed.

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[3] We write $\mathcal{C}_{\simeq}$ for the underlying groupoid of any category $\mathcal{C}$.

1.2. **Truncated (G,μ)-apertures.** The representability result in Theorem B is a special case of a more general result that proves another conjecture of Drinfeld from [21]. Here is the setup for this: We start with a smooth affine group scheme G over $\mathbb{Z}_p$ (not necessarily reductive!) and a cocharacter $\mu : \mathbb{G}_m \to G_{\mathcal{O}}$ defined over the ring of integers $\mathcal{O}$ of a finite unramified extension of $\mathbb{Q}_p$ that is **1-bounded** in the sense of Lau [42], so that the weights of the adjoint action of μ on the Lie algebra $\mathfrak{g}$ are bounded above by 1. For example, if G is reductive, then μ will simply be a minuscule cocharacter of $G_{\mathcal{O}}$. A standard example, for non-negative integers $d \leq h$ with $h > 0$, is $G = \mathrm{GL}_h$ with $\mu = \mu_d$ given by $z \mapsto \mathrm{diag}(\underbrace{z,\dots,z}_{d},1,\dots,1)$.

Drinfeld has given a definition for a stack $\mathrm{BT}_n^{G,\mu}$ associated with the pair (G,μ) that specializes to an open and closed substack of $\mathrm{Vect}^{\mathrm{syn}}_{\{0,1\},n}$ when $(G,\mu) = (\mathrm{GL}_h,\mu_d)$. He conjectured that this should be representable by a smooth 0-dimensional p-adic formal Artin stack over $\mathcal{O}$.

Here's a somewhat quick overview of the definition: We begin with a cartoon of how the syntomification is constructed. For any p-complete commutative ring R, the stack R^{syn} is obtained as follows. We have (derived) p-adic formal stacks $R^{\mathbb{\Delta}}, R^{\mathcal{N}}$: These are the *prismatization* of R and the *(Nygaard) filtered prismatization* of R, respectively. The second of these is a *filtered stack*: it lives naturally over $\mathbb{A}^1/\mathbb{G}_m$. The open locus lying over the point $\mathbb{G}_m/\mathbb{G}_m$ can be identified with $R^{\mathbb{\Delta}}$: this is the *de Rham* embedding of $R^{\mathbb{\Delta}}$ into $R^{\mathcal{N}}$. There is another open immersion of $R^{\mathbb{\Delta}}$ into $R^{\mathcal{N}}$, called the *Hodge-Tate* embedding, that is physically disjoint from the de Rham embedding. The syntomification is obtained by gluing these two copies of $R^{\mathbb{\Delta}}$ together.

Remark 6. When R is a perfect $\mathbb{F}_p$-algebra, we can identify $R^{\mathbb{\Delta}}$ with $\mathrm{Spf}\, W(R)$ and describe $R^{\mathcal{N}}$ via the Rees construction applied to the p-adic filtration on $W(R)$: this yields a stack isomorphic to $[\mathrm{Spf}\, W(R)[u,t]/(ut-p)/\mathbb{G}_m]$. Here, u has degree 1 and t has degree -1, and the de Rham and Hodge-Tate embeddings correspond respectively to the loci $\{t \neq 0\}$ and $\{u \neq 0\}$ (though the latter appears with a Frobenius twist). Objects over the mod-p fiber of the syntomification can be interpreted as giving two filtrations on objects over R—a decreasing Hodge filtration and an increasing conjugate filtration—along with an identification of their associated gradeds up to Frobenius twist. In other words, vector bundles over this stack are the F-zips of Moonen-Pink-Wedhorn-Ziegler [60].

Let us return to the question of defining $\mathrm{BT}_n^{G,\mu}$. Using the Breuil-Kisin twist $\mathcal{O}_n^{\mathrm{syn}}\{1\}$ and the cocharacter μ, we can produce a canonical G-torsor $\mathcal{P}_\mu$ over $\mathcal{O}^{\mathcal{N}}$.

We now define $\mathrm{BT}_n^{G,\mu}$ as the groupoid-valued functor on a certain full subcategory $\mathrm{CRing}^{f,p\text{-comp}}_{\heartsuit,\mathcal{O}/}$ of the category of p-complete commutative $\mathcal{O}$-algebras.[4] For $\mathcal{O}$-algebras R in this subcategory, $\mathrm{BT}_n^{G,\mu}(R)$ is the (∞)-groupoid of flat local G-torsors on $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ whose restriction to $R^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is isomorphic *flat locally on* $\mathrm{Spec}\, R$ to $\mathcal{P}_\mu$.

Remark 7. This local triviality condition should be viewed as an analogue of the possibly familiar Kottwitz signature condition appearing in the moduli description of Shimura varieties of PEL type: When $G = \mathrm{GL}_h$, the definition is essentially concerned with vector bundles on $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. Any such F-gauge gives rise to a filtered vector bundle over $\mathrm{Spec}(R \otimes \mathbb{Z}/p^n\mathbb{Z})$ equipped with a *Hodge filtration*. The triviality condition imposed here fixes the type of this filtration.

The next theorem proves [21, Conjecture C.3.1].[5]

Theorem D. *The formal prestack $\mathrm{BT}_n^{G,\mu}$ is represented by a zero-dimensional quasi-compact smooth Artin formal stack over $\mathcal{O}$ with affine diagonal. Moreover, the natural map $\mathrm{BT}_{n+1}^{G,\mu} \to \mathrm{BT}_n^{G,\mu}$ is smooth and surjective.*

As far as we are aware, the work of Bültel-Pappas [14] was the first to attempt to construct such stacks in generality. However, their construction—which involves working with a more direct generalization of the perfect case explained in Remark 6 using Witt vectors—has the expected properties only when restricted to what the

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[4]Explicitly, an $\mathcal{O}$-algebra R is in $\mathrm{CRing}^{f,p\text{-comp}}_{\mathcal{O}/,\heartsuit}$ if it is p-complete and is such that $\Omega^1_{\pi_0(R)/p\pi_0(R))/\mathbb{F}_p}$ is finitely generated over $\pi_0(R)$. This condition ensures that $\mathrm{Spf}\, R$ admits a p-quasisyntomic cover by the formal spectrum of a *semiperfectoid* algebra.

[5]Drinfeld takes the cocharacter μ to be a map $\mathbb{G}_m \to \mathrm{Aut}(G)$ defined over $\mathbb{Z}_p$ and gives a slightly different definition for $\mathrm{BT}_n^{G,\mu}$, so we are technically proving something very closely related to Drinfeld's conjecture. See Remark 9.1.4 for a discussion of this.

authors there call the 'adjoint nilpotent' locus. When considering the stack of p-divisible groups, this amounts to working only with the connected ones.

We should also make note of the work of K. Ito [37]: He defines the notion of a *prismatic G-display* using the prismatic site of Bhatt-Scholze. See the discussion in Section 7 of *loc. cit.* for the connection to the definitions here.

Remark 8. One should formulate and prove versions of the Theorem D 'with coefficients' (see for instance [37] or [55]), allowing smooth group schemes over the ring of integers of finite extensions of $\mathbb{Q}_p$. This will be considered in forthcoming work of the first author, Z. G.

There is also the very interesting question of allowing the cocharacter to be defined over a *ramified* extension of the field of coefficients, and, relatedly, to find the correct analogues of $\mathrm{BT}_n^{G,\mu}$ associated with parahoric group schemes, but this appears to require a genuinely new idea.

Let us now record some other results about $\mathrm{BT}_n^{G,\mu}$ that are of independent interest, and give some idea of the proof of Theorem D along the way.

Following Drinfeld, we first obtain a somewhat explicit description of the mod-p fiber $\mathrm{BT}_1^{G,\mu} \otimes \mathbb{F}_p$. To explain this, recall that we can associate with the pair (G, μ) the algebraic k-stack $\mathsf{Disp}_1^{G,\mu}$ of **F-zips with G-structure and type** μ; see [60]: It is a smooth zero-dimensional Artin stack over k with affine diagonal. We now have:

Theorem E. *There is a natural map $\mathrm{BT}_1^{G,\mu} \otimes \mathbb{F}_p \to \mathsf{Disp}_1^{G,\mu}$ that is a relatively representable by a smooth zero-dimensional Artin stack with relatively affine diagonal: in fact, it is a gerbe banded by a finite flat commutative p-group scheme of height one, the* Lau group scheme. *In particular, $\mathrm{BT}_1^{G,\mu} \otimes \mathbb{F}_p$ is a smooth zero-dimensional Artin stack over k with affine diagonal.*

Remark 9. When restricted to smooth inputs and μ is defined over $\mathbb{Z}_p$, this result is due to Drinfeld [21]. We verify here that his description continues to hold in general.

With Theorem E in hand, the rest of the proof of Theorem D comes down to a double bootstrapping argument. First, we inductively establish representability for $\mathrm{BT}_n^{G,\mu} \otimes \mathbb{F}_p$ for $n \geq 1$. For this, note that, given an object $\mathcal{P} \in \mathrm{BT}_n^{G,\mu}(R)$, we can twist the adjoint representation on $\mathfrak{g}$ by $\mathcal{P}$ to obtain a vector bundle $(\mathfrak{g})_{\mathcal{P}}$ over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. It is not difficult now to see that the fibers of $\mathrm{BT}_{n+1}^{G,\mu} \to \mathrm{BT}_n^{G,\mu}$ over $\mathcal{P}$ are controlled by the syntomic cohomology of this F-gauge. The main property that makes this F-gauge tractable is that it has Hodge-Tate weights bounded by 1: this is a direct consequence of the fact that μ is 1-bounded. The inductive argument therefore comes down to a special case of the following theorem, which is also an input into the proof of Theorem C:

Theorem F. *Suppose that $R \in \mathrm{CRing}^{f,p\text{-}comp}$ and suppose that $\mathcal{M}$ is an F-gauge over R corresponding to a perfect complex on $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ with Tor amplitude in $[-r, \infty)$ and Hodge-Tate weights bounded by 1. Then the assignment on p-complete R-algebras given by*

$$C \mapsto \tau^{\leq 0} R\Gamma(C^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{M}|_{C^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}})$$

is represented by a locally finitely presented p-adic formal derived algebraic r-stack over R.

The second bootstrapping argument involves a derived descent statement, encapsulated by:

Proposition 1. *The natural map*

$$\mathrm{BT}_n^{G,\mu}(R) \to \mathrm{Tot}\left(\mathrm{BT}_n^{G,\mu}(R \otimes^{\mathbb{L}} \mathbb{F}_p^{\otimes_{\mathbb{Z}}^{\mathbb{L}}(\bullet+1)})\right)$$

is an equivalence.

Note that, even to state this result, one needs to be working with *animated* commutative rings. We will do so systematically in the body of the paper.

To make full use of the proposition, we also need some finer control of the deformation theory of $\mathrm{BT}_n^{G,\mu}$. This involves an interesting (and in a sense elementary) technical tool: Weil restriction from $\mathbb{Z}/p^n\mathbb{Z}$ to $\mathbb{Z}_p$, an operation that is only fully sensible in the derived realm. This yields, for any p-adic formal Artin stack X, a new *derived* formal Artin stack $X^{(n)}$, whose values are characterized by

$$X^{(n)}(R) = X(R \otimes^{\mathbb{L}} \mathbb{Z}/p^n\mathbb{Z}).$$

Using this, for any animated divided power thickening $(R' \twoheadrightarrow R, \gamma)$ in $\mathrm{CRing}^{f,p\text{-comp}}_{\mathcal{O}/}$ we can write down a canonical commuting diagram

$$\begin{tikzcd} \mathrm{BT}^{G,\mu}_n(R') \arrow[r] \arrow[d] & BP^{-,(n)}_\mu(R') \arrow[d] \\ \mathrm{BT}^{G,\mu}_n(R) \arrow[r] & BP^{-,(n)}_\mu(R) \times_{BG^{(n)}(R)} BG^{(n)}(R') \end{tikzcd} \quad . \tag{1.2.0.1}$$

Here, $P^-_\mu \subset G_{\mathcal{O}}$ is the parabolic subgroup associated with the non-negative eigenspaces of the adjoint action of μ, and BH for any group scheme H denotes its classifying stack. The obstruction theory for $\mathrm{BT}^{G,\mu}_n$ is now captured by the following result:

Theorem G (Grothendieck-Messing theory)**.** *The above commuting square is* Cartesian *when the divided powers are* nilpotent*.*

This should be viewed as a truncated analogue of classical Grothendieck-Messing theory, which classifies liftings of p-divisible groups across classical nilpotent divided power thickenings in terms of lifts of the Hodge filtration on its crystalline realization. We first prove this when R' is an $\mathbb{F}_p$-algebra, and then lift it to general inputs using Proposition 1. It is now not hard to deduce the general case of Theorem D from its mod-p version (at least when $p > 2$) by applying Theorem G to the canonical nilpotent divided power thickening $R \to R/^{\mathbb{L}}p$. A very slightly more involved argument also works when $p = 2$.

1.3. **Higher animated frames.** A key technical device we use is a generalization to the animated realm of the notion of a *higher frame* introduced by Lau [42]. We call this an *animated higher frame* or simply *frame*. This is combined with an important structural result due to Bhatt-Lurie that—in the terminology introduced here—says that the syntomification of a semiperfectoid algebra can be realized from (and in fact determines) a canonical frame structure on its prismatization. We can now combine this with quasisyntomic descent in order to translate questions about stacks over syntomifications to assertions about objects living over frames.

The flexibility afforded by this translation turns out to be very useful, since the category of frames permits various constructions that are not visible on the level of the cohomological stacks. We exploit this flexibility[6] to prove Theorem G by using a unique lifting principle that is (by now) quite classical, is due essentially to Zink, and appears in some form or other already in various papers on related topics, including those of Lau [42], Bültel-Pappas [14], and also the recent work of Bartling [3] and Hedayatzadeh-Partofard [33].

We also similar techniques to prove Theorem E by reducing to the case of the Witt vector frame.

1.4. **Further remarks on the proofs.** All the results above are special cases of theorems about objects that we call 1-*bounded stacks*, whose precise definition is a bit technical and can be found in §4.8. Given such an object $\mathcal{X}$ over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$, we can define a functor on animated p-complete R-algebras given by:

$$\Gamma_{\mathrm{syn}}(\mathcal{X}) : C \mapsto \mathrm{Map}_{/R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}}(C^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}).$$

The condition of 1-boundedness is essentially the one that ensures that $\Gamma_{\mathrm{syn}}(\mathcal{X})$ is representable: The arguments sketched in §1.2 go through when applied to this prestack. Examples of 1-bounded stacks include:

- The 'stack' over $\mathcal{O}^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ parameterizing G-torsors that are isomorphic to $\mathcal{P}_\mu$ when restricted to $B\mathbb{G}_m \times \operatorname{Spec} \kappa$ for any algebraically closed field κ: this is of course relevant for Theorem D;

[6] In fact, we only use this device for semiperfect $\mathbb{F}_p$-algebras.

- The total spaces of vector bundles (and more generally perfect complexes) with Hodge-Tate weights bounded by 1: this is relevant for Theorem C.

The general representability result boils down to knowing the representability of certain Artin-Milne type cohomology groups, generalizing the fppf cohomology of finite flat group schemes of height one; see Section 7. For this, we use representability results of Bragg and Olsson [13].

This level of generality is responsible for some of the bulk of this paper, but it will be required for future applications, including, for instance, the construction of spaces of *isogenies* between objects in $\mathrm{BT}_\infty^{G,\mu}$ [45], leading to a general construction of Rapoport-Zink spaces as well as p-Hecke correspondences *without* the direct involvement of p-divisible groups. We also expect that it will help address some of the difficulty in constructing the correct analogues of $\mathrm{BT}_n^{G,\mu}$ when G is a parahoric, non-reductive group scheme; see Remark 8.

As a more immediate consequence, we are able to obtain an extension of Theorem B to perfect F-gauges.

Theorem H. *The prestack* $\mathrm{Perf}^{\mathrm{syn}}_{\{0,1\},n}$ *assigning to every p-complete ring R the ∞-groupoid of perfect complexes on $R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}$ with Hodge-Tate weights in $\{0,1\}$ is represented by a locally finitely presented p-adic derived Artin stack over $\mathbb{Z}_p$. Moreover, the prestack* $\mathrm{Perf}^{\mathrm{syn}}_{0,n}$ *classifying perfect complexes on $R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}$ with Hodge-Tate weights 0 is canonically isomorphic to the p-adic formal stack of perfect complexes of lisse $\mathbb{Z}/p^n\mathbb{Z}$-sheaves.*

1.5. **Application to Shimura varieties.** Theorem D also has a global application, which was the main motivation for one of us (K.M.) to pursue the work here. Suppose that (G,X) is a Shimura datum of abelian type with reflex field E. Suppose that G is unramified at p with reductive model $G_{\mathbb{Z}_p}$: this implies in particular that E is unramified over p. Fix a place $v\mid p$ of E, and choose an $\mathscr{O}_{E_v}$-rational representative $\mu^{-1}:\mathbb{G}_m\to G_{\mathscr{O}_{E_v}}$ for the (inverse of the) conjugacy class of Shimura cocharacters underlying X. Then, for any level subgroup $K\subset G(\mathbb{A}_f)$ with $K_p=G_{\mathbb{Z}_p}(\mathbb{Z}_p)$, we have the integral canonical model $\mathcal{S}_K$ over $\mathscr{O}_{E,(v)}$. Let $\mathcal{S}_K^{\mathfrak{F}}$ be its formal completion along the mod-v fiber. Combining the results here with those of Imai-Kato-Youcis in [36], one obtains the following theorem; when $p>2$, it is already contained in *loc. cit.*, and a proof without this condition will appear in (a revision of) [51].

Theorem I. *There exists a canonical formally étale map*

$$\varpi:\mathcal{S}_K^{\mathfrak{F}}\to\mathrm{BT}_\infty^{G^c_{\mathbb{Z}_p},\mu^{-1}}.$$

When (G,X) is of Siegel type, this agrees via the (polarized version of the) equivalence of Theorem A with the map carrying a polarized abelian variety to its corresponding polarized p-divisible group.

The polarized version alluded to here can be found in §11.6. The group G^c is the so-called *cuspidal* quotient of G, and μ is an $\mathscr{O}_{E_v}$-rational representative for the conjugacy class of the Shimura cocharacter associated with X. The map in the theorem is determined in a precise way by the canonical pro-étale $G^c(\mathbb{Z}_p)$-torsor over the generic fiber of the Shimura variety via a functor such as the one described in [6, §6.3] in the context of vector bundles over the syntomification.

The above theorem will be combined with Theorem F—and, in fact, with the general representability results in Section 8—in [51] to give a uniform construction of special cycle classes on the integral model $\mathcal{S}_K$.

1.6. **A note on the terminology.** Various categories of objects associated with the pair (G,μ) show up in this paper, and we have tried our best to find some coherent way for distinguishing between them. Here are some possibly helpful remarks for the reader:

- For objects appearing over p^n-torsion bases, we have used the adjective ***n*-truncated**: this is compatible via Theorem 11.1.4 with the corresponding terminology for Barsotti-Tate groups.
- For objects associated with the (higher animated) frames appearing in Section 5, we have used the term **(G,μ)-windows**: This harkens to Zink's original terminology in [65].
- Upon the advice of Drinfeld, we have reserved the term **(G,μ)-display** for objects associated with the Witt vector frame: this is compatible with the terminology in [14].
- Finally, for the fundamental objects living over the syntomification stacks, we have coined the term **(G,μ)-aperture**: The (admittedly vague) inspiration behind this choice is the aperture in a camera, which directs light onto the lens, a lens that can occasionally be a *prismatic* one.

- Given a particular frame, one can often produce (G, μ)-windows over that frame from (G, μ)-apertures over a quotient ring (see Remark 9.1.3): in this sense, frames can be viewed as a device for expanding apertures into windows.

1.7. **Structure of the paper.**

- We begin in Section 3 with some background on derived stacks. We also recall the notion of derived Weil restriction, and some facts about divided powers in the animated context.
- In Section 4, we recall the story of filtered animated rings and as well as of filtered derived stacks via C. Simpson's perspective of viewing such gadgets as objects over $\mathbb{A}^1/\mathbb{G}_m$. We give an account of our notion of a 1-bounded stack, give examples of such objects and prove some general facts about them.
- In Section 5, we present our generalization of Lau's theory of higher frames and displays from [42] in an animated context (though as mentioned above, we used the term 'window' instead of 'display'). We then use this to prove an abstract version of Grothendieck-Messing theory for 1-bounded stacks in § 5.8 using the unique lifting principle mentioned earlier, and we also prove an abstract version of the 'reduction to F-zips', Theorem E, in § 5.9.
- In Section 6, we review the stack-theoretic constructions of Drinfeld and Bhatt-Lurie from [6], [8], [9] and [23]. In particular, we recall in § 6.11 the *filtered affineness* of the various stacks when working with *semiperfectoid* rings, where the stacks of Drinfeld and Bhatt-Lurie are now obtained—via the Rees construction—from Nygaard filtered prismatic cohomology.
- Section 7 recalls a result of Bragg-Olsson on the representability of derived stacks that parameterize the fppf cohomology of certain 'perfect complexes' of finite flat group schemes of height one.
- We then prove our general representability theorems for stacks of sections associated with 1-bounded stacks: this takes up Section 8. We follow the strategy sketched above: Representability on the level of F-zips is first lifted to representability of the stack of sections over the mod-p syntomification of $\mathbb{F}_p$-algebras using filtered affineness for semiperfectoid inputs and the results of § 5.9. This is then bootstrapped to representability over the syntomification of $\mathbb{F}_p$-algebras, followed by a further bootstrapping up to arbitrary p-nilpotent algebras. We give some applications of our general representability results for stacks of F-gauges, and prove Theorems F and H.
- Section 9 is where we define the stacks $\mathrm{BT}_n^{G,\mu}$ and prove Theorems D, E and G as consequences of the general results of the previous section.
- In Section 10, we use deformation theory and a strategy introduced by Ito [37] to give explicit descriptions of the points of $\mathrm{BT}_n^{G,\mu}$ valued in certain regular complete local Noetherian rings, and show that the deformation rings defined by Faltings in [27, §7] are in fact providing explicit coordinates for the complete local rings of $\mathrm{BT}_\infty^{G,\mu} = \varprojlim_n \mathrm{BT}_n^{G,\mu}$.
- Finally, in Section 11, we gather our results together to prove Theorem A. The reader will also find some complements dealing (among other things) with polarizations and compatibility with the classical de Rham realization.
- The short appendix A collects some completeness results in the context of graded and filtered commutative rings that are used in Section 4.

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2. Notation and other conventions

(1) We adopt a resolutely ∞-categorical approach. This means that all operations, including (but not limited to) limits, colimits, tensor products, exterior powers etc. are always to be understood in a derived sense, unless otherwise stated.

(2) We will use Spc to denote the ∞-category of spaces, anima, or homotopy types: roughly speaking, this is the localization of the Quillen model category of simplicial sets with respect to homotopy equivalences.
(3) A map $X \to Y$ in Spc is **surjective** if the induced map $\pi_0(X) \to \pi_0(Y)$ is a surjective map of sets; we will denote surjective maps with $\twoheadrightarrow$.
(4) For any ∞-category $\mathcal{C}$ and an object c of $\mathcal{C}$, we will write $\mathcal{C}_{c/}$ (resp. $\mathcal{C}_{/c}$) for the comma ∞-categories of arrows $c \to d$ (resp. $d \to c$).
(5) We will in a few places make reference to the process of **animation**, as described say in [54, Appendix A]. This is a systematic way to get well-behaved ∞-categories and functors between them, starting from 'nice' classical categories $\mathcal{C}$ with a set $\mathcal{C}_0$ of compact, projective generators. The animation of such a category is the ∞-category $\mathcal{P}_\Sigma(\mathcal{C}_0)$ of presheaves of spaces on $\mathcal{C}_0$ that preserve finite products.
(6) We will denote by CRing the ∞-category of **animated commutative rings**, obtained via the process of animation from the usual category of commutative rings. Objects here can be viewed as being simplicial commutative rings up to homotopy equivalence.
(7) We will follow homological notation for CRing: For any $n \in \mathbb{Z}_{\geq 0}$, $\mathrm{CRing}_{\leq n}$ will be the subcategory of CRing spanned by those objects R with $\pi_k(R) = 0$ for $k > n$; that is, by the ***n*-truncated objects**. If $n = 0$, we will write $\mathrm{CRing}^\heartsuit$ instead of $\mathrm{CRing}_{\leq 0}$: its objects are the **discrete** or classical commutative rings, and the category can be identified with the usual category of commutative rings.
(8) Any animated commutative ring R admits a **Postnikov tower** $\{\tau_{\leq n}R\}_{n\in\mathbb{Z}_{\geq 0}}$ where $R \to \tau_{\leq n}R$ is the universal arrow from R into $\mathrm{CRing}_{\leq n}$ and the natural map $R \to \varprojlim_n \tau_{\leq n}R$ is an equivalence.
(9) We will also need the notion of a **stable ∞-category** from [47]: this is the ∞-category analogue of a triangulated category. The basic example is the ∞-category Mod_R, the derived ∞-category of R-modules. We will use *cohomological* conventions for these objects and so will write for instance $H^{-1}(M)$ instead of $\pi_1(M)$.
(10) An important feature of a stable ∞-category $\mathcal{C}$ is that it has an initial and final object 0, and, for any map $f : X \to Y$ in $\mathcal{C}$, we have the **homotopy cokernel** $\mathrm{hcoker}(f)$ defined as the pushout of $0 \to Y$ along f. We will sometimes abuse notation and write Y/X for this object.
(11) If $R \in \mathrm{CRing}^\heartsuit$ is a classical commutative ring, $M \in \mathrm{Mod}_R$ is a complex of R-modules, and $a_1, \ldots, a_m \in R$ form a regular sequence, we will write $M/^{\mathbb{L}}(a_1, \ldots, a_m)$ for the derived tensor product

$$M \otimes_R^{\mathbb{L}} R/(a_1, \ldots, a_m).$$

(12) In any stable ∞-category $\mathcal{C}$ and an object X in $\mathcal{C}$, we set $X[1] = \mathrm{hcoker}(X \to 0)$: this gives a shift functor $\mathcal{C} \to \mathcal{C}$ with inverse $X \mapsto X[-1]$, and we set $\mathrm{hker}(f : X \to Y) = \mathrm{hcoker}(f)[-1]$.
(13) Given an animated commutative ring R, we will write $\mathrm{Mod}_R^{\mathrm{cn}}$ for the sub ∞-category spanned by the connective objects (that is, objects with no cohomology in positive degrees), and $\mathrm{Perf}(R)$ for the sub ∞-category spanned by the perfect complexes.
(14) We have a truncation operator $\tau^{\leq 0} : \mathrm{Mod}_R \to \mathrm{Mod}_R^{\mathrm{cn}}$ defined as the right adjoint to the natural functor in the other direction. This leads to truncation operators $\tau^{\leq n}$ and cotruncation operators $\tau^{\geq n}$ for any $n \in \mathbb{Z}$ in the usual way.
(15) If $f : X \to Y$ is a map in $\mathrm{Mod}_R^{\mathrm{cn}}$, we set $\mathrm{hker}^{\mathrm{cn}}(f) = \tau^{\leq 0}\,\mathrm{hker}(f)$: this is the **connective (homotopy) kernel**.
(16) For any stable ∞-category $\mathcal{C}$, the mapping spaces $\mathrm{Map}_{\mathcal{C}}(X, Y)$ between any two objects have canonical lifts to the ∞-category of connective spectra. We will be interested in stable ∞-categories like Mod_R, which are $\mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}}$-modules, in the sense that the mapping spaces have canonical lifts to $\mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}}$. In this case, we can extend the mapping spaces $\mathrm{Map}_{\mathcal{C}}(X, Y)$ from $\mathrm{Mod}_{\mathbb{Z}}^{\mathrm{cn}}$ to objects $\underline{\mathrm{Map}}_{\mathcal{C}}(X, Y)$ in $\mathrm{Mod}_{\mathbb{Z}}$ by taking

$$\underline{\mathrm{Map}}_{\mathcal{C}}(X, Y) = \mathrm{colim}_{k\geq 0}\,\mathrm{Map}_{\mathcal{C}}(X, Y[k])[-k] \in \mathrm{Mod}_{\mathbb{Z}}.$$

When $\mathcal{C} = \mathrm{Mod}_R$ for an animated commutative ring R, this lifts to the **internal Hom** in Mod_R.
(17) We will write $\mathbf{\Delta}$ for the usual **simplex** category with objects the sets $\{0, 1, \ldots, n\}$ and morphisms given by the non-decreasing functions between them.

(18) A **cosimplicial object** $S^{(\bullet)}$ in an ∞-category $\mathcal{C}$ is a functor

$$\mathbf{\Delta} \mapsto \mathcal{C}$$
$$[n] \mapsto S^{(n)}.$$

If $\mathcal{C}$ admits limits, we will write $\mathrm{Tot} S^{(\bullet)}$ for the limit of the corresponding functor: this is the **totalization** of $S^{(\bullet)}$.

(19) Given any ∞-category $\mathcal{C}$ with finite coproducts, and any object S in $\mathcal{C}$ there is a canonical cosimplicial object $S^{(\bullet)}$ in $\mathcal{C}$, the **Čech conerve** with $S^{(n)} = \bigsqcup_{i\in[n]} S$.

(20) If X is a (derived) stack (resp. an object of Mod_R for some R), and $N \in \mathbb{Z}\backslash\{0\}$, we will write $X[N^{-1}]$ for the base change $\mathrm{Spec}\,\mathbb{Z}[N^{-1}] \times X \to \mathrm{Spec}\,\mathbb{Z}[N^{-1}]$ (resp. for the base change $\mathbb{Z}[N^{-1}] \otimes_{\mathbb{Z}} X$ in $\mathrm{Mod}_{\mathbb{Z}[N^{-1}]\otimes_{\mathbb{Z}} R}$). On the rare occasions when these notations collide, context will make the usage clear.

3. Stacks and other preliminaries

3.1. **Square-zero extensions and differential conditions.** Given a pair (R, M) with $R \in \mathrm{CRing}$ and $M \in \mathrm{Mod}^{\mathrm{cn}}_R$, we have a canonical object $R \oplus M \in \mathrm{CRing}_{R//R}$, the **trivial square-zero extension** of R by M: This is obtained by animating the construction on such pairs with R a polynomial algebra and M a finite free R-module to the usual square-zero extension $R \oplus M$.

If $R \in \mathrm{CRing}_{A/}$, we set

$$\mathrm{Der}_A(R, M) = \mathrm{Map}_{A//R}(R, R \oplus M).$$

This is the space of A-**derivations** of R **valued in** M. We always have the trivial A-derivation $d_{\mathrm{triv}} = (\mathrm{id}, 0)$.

A **square-zero extension** of R by M in $\mathrm{CRing}_{A/}$ is a surjective map $R' \twoheadrightarrow R$ in $\mathrm{CRing}_{A/}$ such that there exists an A-derivation $d : R \to R \oplus M[1]$ and an equivalence of A-algebras

$$R' \xrightarrow{\simeq} R \times_{d, R\oplus M[1], d_{\mathrm{triv}}} R.$$

We have the **cotangent complex** $\mathbb{L}_{R/A} \in \mathrm{Mod}^{\mathrm{cn}}_R$: this is obtained by animating the functor taking maps $S \to S'$ of polynomial rings over $\mathbb{Z}$ in finitely many variables to the module of differentials $\Omega^1_{S'/S}$, and is characterized by the property that, for any trivial square zero extension $R \oplus M \twoheadrightarrow R$, there is a canonical equivalence

$$\mathrm{Map}_{\mathrm{Mod}_R}(\mathbb{L}_{R/A}, M) \xrightarrow{\simeq} \mathrm{Der}_A(R, M).$$

Definition 3.1.1. An R-algebra $C \in \mathrm{CRing}_{R/}$ is **finitely presented** (over R) if the functor $S \mapsto \mathrm{Map}_{\mathrm{CRing}_{R/}}(C, S)$ respects filtered colimits. For any such finitely presented C, the cotangent complex $\mathbb{L}_{C/R} \in \mathrm{Mod}^{\mathrm{cn}}_C$ is perfect; see [47, (17.4.3.18)].

If in addition $\mathbb{L}_{C/R}$ is 1-connective, we say that C is **unramified** over R; if $\mathbb{L}_{C/R} \simeq 0$, we say that C is **étale** over R.

We say that a finitely presented $C \in \mathrm{CRing}_{R/}$ is **smooth** over R if $\mathbb{L}_{C/R} \in \mathrm{Mod}^{\mathrm{cn}}_C$ is locally free of finite rank. It is **quasi-smooth** if $\mathbb{L}_{C/R}$ is perfect with Tor amplitude $[-1, 0]$.

3.2. **Derived (pre)stacks.** Suppose that $\mathcal{C}$ is an ∞-category admitting all finite and sequential limits, totalizations of cosimplicial objects, and filtered colimits. A $\mathcal{C}$-**valued prestack over** $R \in \mathrm{CRing}$ is a functor

$$F : \mathrm{CRing}_{R/} \to \mathcal{C}.$$

If $\mathcal{C} = \mathrm{Spc}$, we will simply call F a **prestack over** R. Such objects organize into an ∞-category PStk_R.

We view such prestacks as presheaves on the ∞-category of derived affine schemes $\mathrm{Spec}\,R'$ over R (by definition opposite to $\mathrm{CRing}_{R/}$), and we can consider the subcategory of prestacks that are fpqc (resp. étale) sheaves—that is, presheaves satisfying descent along faithfully flat (resp. faithfully flat and étale) maps $\mathrm{Spec}\,R' \to \mathrm{Spec}\,R$.

Definition 3.2.1. Following Toën-Vezzosi [64], we will say that F is 0-**geometric** if we have $F \simeq \mathrm{Spec}\,R'$ for some $R' \in \mathrm{CRing}_{R/}$, and, inductively, that it is an n-**geometric derived Artin stack over** R for an integer $n \geq 1$ if it is an étale sheaf and admits a surjective cover $f : U = \sqcup_{i\in I} \mathrm{Spec}\,R'_i \to F$ of étale sheaves with $R'_i \in \mathrm{CRing}_{R/}$

satisfying the following condition: For every $S \in \mathrm{CRing}_{R/}$ and $x \in F(S)$, the base-change $U \times_{f,F,x} \operatorname{Spec} S \to \operatorname{Spec} S$ is represented by an $(n-1)$-geometric derived Artin stack over S.

Following Lurie [46, §5], we will say that F is a **derived Artin n-stack over** R if it is m-geometric for some m and is such that $F(R')$ is n-truncated for all discrete $R' \in \mathrm{CRing}_{R/,\heartsuit}$. A derived Artin 0-stack over R will be called a **derived algebraic space over** R.

A **derived Artin stack over** R is a prestack F that is a derived Artin n-stack for some $n \geq 0$. If $R = \mathbb{Z}$, then we will simply say 'derived Artin stack' instead.

A map of $X \to Y$ of prestacks over R is a **relative derived Artin stack** if, for every R-algebra C and every $y \in Y(C)$, the base-change $X_y \to \operatorname{Spec} C$ is a derived Artin stack over C.

Definition 3.2.2. A prestack F over R is **locally of finite presentation** or **locally finitely presented** if for every filtered system $\{C_i\}_{i\in I}$ in $\mathrm{CRing}_{R/}$ with colimit $C \in \mathrm{CRing}_{R/}$, the natural map

$$\operatorname{colim}_{i\in I} F(C_i) \to F(C)$$

is an equivalence.

Definition 3.2.3. A prestack F over R is **formally smooth** if for every square-zero extension $C' \twoheadrightarrow C$ in $\mathrm{CRing}_{R/}$, the map $F(C') \to F(C)$ is surjective.

A derived Artin stack over R is **smooth** if is locally finitely presented and formally smooth.

Definition 3.2.4. A prestack F over $A \in \mathrm{CRing}$ that is an fpqc sheaf is **classical** if it is equivalent as an fpqc sheaf to the left Kan extension to $\mathrm{CRing}_{A/}$ of its classical truncation $F_{\mathrm{cl}} : \mathrm{CRing}_{\pi_0(A)/} \to \mathrm{Spc}$: That is, it is a colimit of derived affine schemes $\operatorname{Spec} B$ with $B \in \mathrm{CRing}_{\pi_0(A)/}$ in the ∞-category of fpqc sheaves on $\mathrm{CRing}^{\mathrm{op}}_{A/}$. The functor $F \mapsto F_{\mathrm{cl}}$ is fully faithful when restricted to classical prestacks.

3.2.5. For any prestack $F \in \mathrm{PStk}_R$, we have an ∞-category $\mathrm{QCoh}(F)$ of **quasi-coherent sheaves on** F. The precise definition can be found in [49, §6.2.2]: roughly speaking, it is obtained by right Kan extension of the contravariant functor sending $S \in \mathrm{CRing}_{R/}$ to Mod_S. One can think of an object $\mathcal{M}$ in $\mathrm{QCoh}(F)$ as a way of assigning to every point $x \in F(S)$ an object $\mathcal{M}_x \in \mathrm{Mod}_S$ compatible with base-change. This ∞-category is particularly well-behaved when F is **quasi-geometric** [49, §9.1]: this means that F is an fpqc sheaf admitting a flat cover by an affine derived scheme with quasi-affine diagonal. Most of the prestacks we will encounter in this paper will be quasi-geometric or formal analogues of this notion; see Corollary 6.12.7.

Definition 3.2.6. We will say that $\mathcal{M}$ is **connective** if $\mathcal{M}_x$ belongs to $\mathrm{Mod}^{\mathrm{cn}}_S$ for each $x \in F(S)$ as above. We will say that it is **almost connective** if, for every $x \in F(S)$, there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\mathcal{M}_x[n]$ is connective. We will say that it is **perfect** if, for every $x \in F(S)$, $\mathcal{M}_x$ is perfect.

Write $\mathrm{QCoh}^{\mathrm{cn}}(F)$ (resp. $\mathrm{QCoh}^{\mathrm{acn}}(F)$, resp. $\mathrm{Perf}(F)$) for the ∞-category spanned by the connective (resp. almost connective, resp. perfect) objects in $\mathrm{QCoh}(F)$.

Definition 3.2.7. Following [49, §17.2.4], we will say that a morphism $f : F \to G$ in PStk_R **admits a cotangent complex** if there exists $\mathbb{L}_{F/G} \in \mathrm{QCoh}^{\mathrm{acn}}(X)$ such that, for every $C \in \mathrm{CRing}_{R/}$, every $M \in \mathrm{Mod}^{\mathrm{cn}}_R$, and every $x \in F(C)$, we have a canonical equivalence

$$\mathrm{Map}_{\mathrm{Mod}_C}(\mathbb{L}_{F/G,x}, M) \xrightarrow{\simeq} \mathrm{fib}_{(f(x)[M],x)}(F(C \oplus M) \to G(C \oplus M) \times_{G(C)} F(C)).$$

Here, $f(x)[M] \in G(C \oplus M)$ is the image of $f(x)$ along the natural section $G(C) \to G(C \oplus M)$.

If $F = \operatorname{Spec} C$ and $G = \operatorname{Spec} D$, then by Yoneda, any morphism $f : F \to G$ corresponds to an arrow $D \to C$ in $\mathrm{CRing}_{R/}$, and f admits a cotangent complex, namely $\mathbb{L}_{C/D}$.

Remark 3.2.8. Suppose that F is a locally finitely presented derived Artin stack over $R \in \mathrm{CRing}_{\heartsuit}$ such that the cotangent complex $\mathbb{L}_{F/R}$ is a perfect complex of *non-negative* Tor-amplitude. Then F is smooth and classical. By an argument via induction on n where F is an n-geometric derived Artin stack, this reduces to the fact that smooth R-algebras are flat over R and are hence classical; see [46, Prop. 3.4.9].

3.3. **Derived vector stacks.** We have the classical construction associating with every finite locally free R-module M the affine R-scheme $\mathbf{V}(M)$ with ring of functions $\mathrm{Sym}_R(M^\vee)$ the symmetric algebra of the R-dual $M^\vee$ of M. Its functor of points is given by $S \mapsto S \otimes_R M$.[7]

One can now consider, for any $R \in \mathrm{CRing}$ and any perfect complex $M \in \mathrm{Mod}_R$, the prestack

$$\mathrm{CRing}_{R/} \xrightarrow{S \mapsto S \otimes_R M} \mathrm{Spc}.$$

It is represented by a finitely presented derived Artin n-stack $\mathbf{V}(M)$ over R where n is such that M has Tor amplitude in $[-n, \infty)$: a somewhat droll way to see this is to reduce to the case where R is finitely presented over $\mathbb{Z}$, and then appeal to Artin-Lurie representability [46, Theorem 7.1.6].

When M has Tor amplitude in $[0, \infty)$, one can simply take the derived affine R-scheme associated with the animated symmetric algebra $\mathrm{Sym}_R(M^\vee)$ of the connective perfect complex $M^\vee$. The stacks associated with possibly non-coconnective perfect complexes can then be obtained as iterated classifying stacks of this derived $\mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}}$-valued scheme.

It is easy to see from the definition that $\mathbf{V}(M)$ has cotangent complex given by

$$\mathbb{L}_{\mathbf{V}(M)/R} \simeq \mathscr{O}_{\mathbf{V}(M)} \otimes_R M^\vee.$$

3.4. **p-adic formal stacks.** Let $\mathrm{CRing}^{p\text{-nilp}}$ be the subcategory of CRing spanned by those objects R such that p is nilpotent in $\pi_0(R)$.

Definition 3.4.1. A **p-adic formal prestack** over $R \in \mathrm{CRing}$ is simply a Spc-valued functur on $\mathrm{CRing}^{p\text{-nilp}}_{R/}$. We will usually extend such an object $\mathcal{Y}$ to (derived) p-complete R-algebras A by setting

$$\mathcal{Y}(A) \stackrel{\mathrm{defn}}{=} \varprojlim_n \mathcal{Y}(A/^{\mathbb{L}}p^n).$$

Definition 3.4.2. For any R-algebra S, the restriction of the affine scheme $\mathrm{Spec}\, S$ to $\mathrm{CRing}^{p\text{-nilp}}_{R/}$ yields a p-adic formal prestack, which, since p will be fixed in this paper, we will denote simply by $\mathrm{Spf}\, S$. This depends only on the p-completion of S.

Definition 3.4.3. A p-adic formal prestack is a **p-adic formal derived Artin stack** if, for each $n \geq 1$, its restriction to $\mathrm{CRing}_{(\mathbb{Z}/p^n\mathbb{Z})/}$ is represented by a derived Artin stack. Given such a formal derived Artin stack F, we will say that it is **foo**, if 'foo' is an attribute applicable to derived Artin stacks, and, if for each $n \geq 1$, the restriction of F to $\mathrm{CRing}_{(\mathbb{Z}/p^n\mathbb{Z})/}$ is an Artin stack that is foo.

3.4.4. Suppose that we have a surjective map $A \twoheadrightarrow \overline{A}$ in CRing with fiber J such that $\pi_0(\overline{A})_{\mathrm{red}}$ is an $\mathbb{F}_p$-algebra. Then we can consider the p-adic formal prestack $\mathrm{Spf}(A, J)$ given for each $C \in \mathrm{CRing}^{p\text{-nilp}}$ by

$$\mathrm{Spf}(A,J)(C) = \mathrm{Map}_{\mathrm{CRing}}(A, C) \times_{\mathrm{Map}_{\mathrm{CRing}}(\pi_0(A), \pi_0(C)_{\mathrm{red}})} \mathrm{Map}_{\mathrm{CRing}}(\pi_0(\overline{A})_{\mathrm{red}}, \pi_0(C)_{\mathrm{red}}).$$

In other words, we are looking at maps $A \to C$ such that J maps to a nilpotent ideal in $\pi_0(C)$. If J is clear from context, we will sometimes just write $\mathrm{Spf}(A)$ instead.

3.5. **Weil restrictions.** A very useful aspect of derived geometry is the ability to construct well-behaved Weil restrictions along certain non-flat maps. This will enable us to correctly identify the local models for our stacks from Theorem D.

3.5.1. Given $R \in \mathrm{CRing}$, for any prestack X over $R/^{\mathbb{L}}p^n$, we will define its **Weil restriction** $\mathrm{Res}_{(\mathbb{Z}/p^n\mathbb{Z})/\mathbb{Z}_p} X$ to be the p-adic formal prestack over R given by the composition

$$\mathrm{CRing}^{p\text{-nilp}}_{R/} \xrightarrow{C \mapsto C/^{\mathbb{L}}p^n} \mathrm{CRing}_{R/^{\mathbb{L}}p^n/} \xrightarrow{X} \mathrm{Spc}.$$

If Y is a p-adic formal prestack over R, we will set

$$Y^{(n)} = \mathrm{Res}_{(\mathbb{Z}/p^n\mathbb{Z})/\mathbb{Z}_p}(Y|_{\mathrm{CRing}_{R/^{\mathbb{L}}p^n/}}).$$

[7] Note that this is dual to Grothendieck's convention.

There is then a canonical map $\alpha^{(n)} : Y \to Y^{(n)}$.

3.5.2. There is a natural functor

$$\mathrm{QCoh}(Y \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\mathcal{F}\mapsto\mathcal{F}^{(n)}} \mathrm{QCoh}(Y^{(n)}).$$

With any $y \in Y^{(n)}(C)$ corresponding to $\overline{y} \in Y(C/^{\mathbb{L}}p^n)$ it associates the module $\mathcal{F}^{(n)}_y = \mathcal{F}_{\overline{y}}[-1]$.

The following lemma is easily verified:

Lemma 3.5.3. *For any p-adic formal prestack Z, write $i_n : Z \otimes \mathbb{Z}/p^n\mathbb{Z} \to Z$ for the associated canonical closed immersion. For any $\mathcal{F} \in \mathrm{QCoh}(Y \otimes_{\mathbb{Z}} \mathbb{Z}/p^n)$, we have a natural equivalence*

$$\alpha^{(n),*}\mathcal{F}^{(n)} \simeq i_{n,*}\mathcal{F}[-1] \in \mathrm{QCoh}(Y).$$

Lemma 3.5.4. *Suppose that we have a map of p-adic formal prestacks $Y \to Z$ such that*

$$Y \otimes \mathbb{Z}/p^n\mathbb{Z} \to Z \otimes \mathbb{Z}/p^n\mathbb{Z}$$

is a relative locally finitely presented (resp. smooth, resp. étale) derived Artin stack with cotangent complex $\mathcal{L} \overset{\mathrm{defn}}{=} \mathbb{L}_{Y\otimes\mathbb{Z}/p^n\mathbb{Z}/Z\otimes\mathbb{Z}/p^n\mathbb{Z}}$. Then $Y^{(n)} \to Z^{(n)}$ is once again a relative locally finitely presented (resp. smooth, resp. étale) p-adic formal derived Artin stack, and we have a canonical identification

$$\mathbb{L}_{Y^{(n)}/Z^{(n)}} \xrightarrow{\simeq} \mathcal{L}^{(n)}.$$

Proof. Given that this is a question of relative representability, we can assume that $Z = \mathrm{Spf}\, A$ is formally affine. Using the finite presentation condition, we can further reduce to the case where $A/^{\mathbb{L}}p$ is a finitely generated $\mathbb{F}_p$-algebra. Representability of $Y^{(n)}$ can now be verified by checking the criteria of Artin-Lurie representability [46, Theorem 7.1.6]. See [49, §19.1] for a much more general version of this in the context of spectral algebraic geometry.

Suppose that we have $y^{(m)} \in Y^{(n)}(C)$ corresponding to $\overline{y} \in Y(C/^{\mathbb{L}}p^n) = (Y \otimes \mathbb{Z}/p^n\mathbb{Z})(C/^{\mathbb{L}}p^n)$. Then we have:

$$\begin{aligned}
\mathrm{fib}_{y^{(m)}}(Y^{(n)}(C \oplus M) \to Y^{(n)}(C)) &= \mathrm{fib}_{\overline{y}}(Y(C/^{\mathbb{L}}p^n \oplus M/^{\mathbb{L}}p^n) \to Y(C/^{\mathbb{L}}p^n)) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_{C/^{\mathbb{L}}p^n}}(\mathcal{L}_{\overline{y}}, M/^{\mathbb{L}}p^n) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_{C/^{\mathbb{L}}p^n}}(\mathcal{L}_{\overline{y}}, \underline{\mathrm{Map}}_{\mathrm{Mod}_C}(C/^{\mathbb{L}}p^n, M[1])) \\
&\simeq \mathrm{Map}_{\mathrm{Mod}_C}(i_{n,*}\mathcal{L}_{\overline{y}}[-1], M).
\end{aligned}$$

This proves that the cotangent complex is as claimed.

Note that, if $Y \otimes \mathbb{Z}/p^n\mathbb{Z}$ is smooth over $Z \otimes \mathbb{Z}/p^n\mathbb{Z}$, so that $\mathcal{L}$ is a perfect complex with Tor-amplitude in $[0,\infty)$, then $(i_{n,*}\mathcal{L})^{(n)}$ is also perfect with Tor-amplitude in $[0,\infty)$, showing that $Y^{(n)}$ is a smooth Artin stack over $Z^{(n)}$. The same argument shows that $Y^{(n)}$ is étale when $Y \otimes \mathbb{Z}/p^n\mathbb{Z}$ is étale over $Z \otimes \mathbb{Z}/p^n\mathbb{Z}$. □

3.6. **Divided powers.** We will also need the notion of animated divided powers, which is an additional structure γ on surjective maps $R' \twoheadrightarrow R$ in CRing that 'animates' the classical notion.

3.6.1. This can be approached in two ways. First, we have the presentation from [54, §3.2], where one obtains an ∞-category AniPDPair via the process of animation: one takes the full subcategory $\mathcal{E}^0$ of the classical category PDPair of divided power thickenings $(R' \twoheadrightarrow R, \gamma)$ spanned by those thickenings of the form

$$(D_{(Y)}\mathbb{Z}[X,Y] \twoheadrightarrow \mathbb{Z}[X], \gamma)$$

where X, Y are finite sets of variables and $D_{(Y)}\mathbb{Z}[X,Y]$ is the divided power envelope of $\mathbb{Z}[X,Y] \xrightarrow{Y\mapsto 0} \mathbb{Z}[X]$ equipped with its tautological divided powers, and then takes $\mathrm{AniPDPair} = \mathcal{P}_{\Sigma}(\mathcal{E}^0)$ to be the ∞-category of finite product preserving presheaves on $\mathcal{E}^0$. The natural functor PDPair $\to$ AniPDPair obtained via the Yoneda map is fully faithful; see [54, Lemma 3.13].

3.6.2. There is a forgetful functor AniPDPair $\to$ AniPair that preserves all limits and finite colimits to the ∞-category AniPair of surjective maps $R' \twoheadrightarrow R$, and we can view a divided power structure γ on such a surjective map as being a lift along the forgetful functor. The forgetful functor admits a left adjoint, the **divided power envelope**, carrying $f : R' \twoheadrightarrow R$ to a surjection $D(f) \twoheadrightarrow R$ equipped with a divided power structure. For all this, see the discussion in [54, §3.2].

3.6.3. A second way to go about things is to first construct the animated divided power algebra $\Gamma_R(M)$ for any $R \in \mathrm{CRing}$ and any $M \in \mathrm{Mod}_R^{\mathrm{cn}}$: this is obtained by animating the usual divided power algebra on pairs (R, M) with R a polynomial algebra in finitely many variables and M is a finite free R-module defined for instance in [5, App. A]. This is in some sense a classical construction that goes back to the seminal work of Dold-Puppe [19].

By construction, $\Gamma_R(M)$ is an object in $\mathrm{CRing}_{R/}$, equipped with a map of R-modules $M \to \Gamma_R(M)$ that satisfies a certain universal property. To explain this, write $\mathbb{G}_a^\sharp$ for the affine scheme $\operatorname{Spec}\Gamma_{\mathbb{Z}}(\mathbb{Z})$: this is the divided power envelope of the origin in the additive group $\mathbb{G}_a$.

Lemma 3.6.4. *For any $M \in \mathrm{Mod}_R^{\mathrm{cn}}$ and for any other R-algebra C, we have a canonical equivalence*

$$\mathrm{Map}_{\mathrm{CRing}_{R/}}(\Gamma_R(M), C) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Mod}_R}(M, \mathbb{G}_a^\sharp(C)).$$

Proof. Both sides of the purported equivalence are evaluations on M of functors $\mathrm{Mod}_R^{\mathrm{cn}} \to \mathrm{Spc}^{\mathrm{op}}$ that preserve sifted colimits. Therefore, by [48, Prop. 5.5.8.15], it suffices to construct a canonical equivalence between these functors when evaluated on free R-modules of finite rank. That is, we want to construct isomorphisms

$$\mathrm{Map}_{\mathrm{CRing}_{R/}}(\Gamma_R(R^n), C) \xrightarrow{\simeq} \mathbb{G}_a^\sharp(C)^n;$$

or in other words isomorphisms

$$\Gamma_R(R^n) \xrightarrow{\simeq} \underbrace{\Gamma_R(R) \otimes_R \otimes \cdots \otimes_R \Gamma_R(R)}_{n}$$

in $\mathrm{CRing}_{R/}$. This is classical; see for instance [5, Prop. (A2)]. □

3.6.5. If $R' \twoheadrightarrow R$ is in AniPair with homotopy kernel I, then a divided power structure on it is also equivalent to giving a map of R'-algebras $\Gamma_{R'}(I) \to R'$ equipped with a homotopy equivalence between the induced map $I \to \Gamma_{R'}(I) \to R'$ with the tautological one.

A **divided power thickening** is a pair $(R' \twoheadrightarrow R, \gamma)$, where γ is a divided power structure on $R' \twoheadrightarrow R$.

3.7. **Nilpotent divided powers.** It will be useful to know what it means for a divided power thickening of animated rings to be **nilpotent** or **trivial**.

3.7.1. It's easiest to use the process of animation, this time applied to classical nilpotent divided power thickenings of a fixed order. That is, analogously to what we did in §3.6, we let $\mathcal{E}^{0,[n]}$ be the full subcategory of the classical category PDPair of divided power thickenings $(R' \twoheadrightarrow R, \gamma)$ spanned by those thickenings of the form

$$(D_{(Y)}\mathbb{Z}[X,Y]/D_{(Y)}^{\geq n}\mathbb{Z}[X,Y] \twoheadrightarrow \mathbb{Z}[X], \gamma)$$

where X, Y are finite sets of variables, and $D_{(Y)}^{\geq \bullet}\mathbb{Z}[X,Y]$ is the divided power filtration on $D_{(Y)}\mathbb{Z}[X,Y]$.

We now set $\mathrm{AniPDPair}^{[n]} = \mathcal{P}_\Sigma(\mathcal{E}^{0,[n]})$ and call this the ∞-category of **nilpotent animated divided power thickenings of order** $\leq n$. If $\mathrm{PDPair}^{[n]} \subset \mathrm{PDPair}$ is the full subcategory consisting of classical divided power thickenings that are nilpotent of order $\leq n$, then one checks as in [54, Lemma 3.13] that the Yoneda functor yields a fully faithful embedding $\mathrm{PDPair}^{[n]} \to \mathrm{AniPDPair}^{[n]}$.

Lemma 3.7.2. *The natural quotient functor $\mathcal{E}^0 \to \mathcal{E}^{0,[n]}$ induces a fully faithful functor $\mathrm{AniPDPair}^{[n]} \to \mathrm{AniPDPair}$.*

We now say that a divided power thickening $(R' \twoheadrightarrow R, \gamma)$ in AniPDPair is **nilpotent** if it is in the image of $\mathrm{AniPDPair}^{[n]}$ for some $n \geq 1$.

Remark 3.7.3. Suppose that $(R' \twoheadrightarrow R, \gamma)$ is a nilpotent divided power thickening of order $\leq n$. Then it admits a canonical triangle

$$\begin{tikzcd} (R' \twoheadrightarrow R, \gamma) \arrow[r] \arrow[dr] & (R' \twoheadrightarrow R_0, \gamma_0') \arrow[d] \\ & (R \twoheadrightarrow R_0, \gamma_0) \end{tikzcd}$$

in AniPDPair, where $(R' \twoheadrightarrow R_0, \gamma_0')$ is nilpotent of order $\leq (n-1)$ and $(R \twoheadrightarrow R_0, \gamma_0)$ is nilpotent of order ≤ 2. This follows by animating the triangle

$$\begin{tikzcd} (D_{(Y)}\mathbb{Z}[X,Y]/D^{\geq n}_{(Y)}\mathbb{Z}[X,Y] \twoheadrightarrow \mathbb{Z}[X], \gamma) \arrow[r] \arrow[dr] & (D_{(Y)}\mathbb{Z}[X,Y]/D^{\geq n}_{(Y)}\mathbb{Z}[X,Y] \twoheadrightarrow D_{(Y)}\mathbb{Z}[X,Y]/D^{\geq 2}_{(Y)}\mathbb{Z}[X,Y], \gamma_0') \arrow[d] \\ & (D_{(Y)}\mathbb{Z}[X,Y]/D^{\geq 2}_{(Y)}\mathbb{Z}[X,Y] \to \mathbb{Z}[X], \gamma_0). \end{tikzcd}$$

Remark 3.7.4. When $n = 2$, the natural map

$$\mathbb{Z}[X,Y]/(Y^2) \to D_{(Y)}\mathbb{Z}[X,Y]/D^{\geq 2}_{(Y)}\mathbb{Z}[X,Y]$$

is an isomorphism, and the divided powers on the right hand side are *trivial*, in the sense that all divided powers of order ≥ 2 are zero. In this case, one checks that the forgetful functor

$$\mathrm{AniPDPair}^{[2]} \to \mathrm{AniPair}$$

is fully faithful. We will call any object $R' \twoheadrightarrow R$ in its image a **trivial** divided power thickening. Note that any such object is a square-zero extension as in §3.1, since this is clearly true for all the generating objects. Conversely, it follows from [61, Lemma 1.4] that any square-zero extension is in this image.

4. Filtered abstractions

The purpose of this section is to introduce enough background about filtered animated rings and modules to discuss the key notion of a 1-*bounded stack*.

4.1. **Graded rings and modules.**

4.1.1. As usual, a graded ring or module can be viewed as a $\mathbb{G}_m$-equivariant object. Therefore, given $R \in \mathrm{CRing}$, we will define the ∞-category of **graded animated commutative R-algebras** to be the opposite to the ∞-category of relatively affine map of derived stacks $X \to B\mathbb{G}_m \times \mathrm{Spec}\, R$. Let $\mathcal{O}(1)$ be the inverse tautological bundle over $B\mathbb{G}_m$, and set $\mathcal{O}(i) = \mathcal{O}(1)^{\otimes i}$. Then, given a relatively affine map $X \to B\mathbb{G}_m \times \mathrm{Spec}\, R$, we will denote the coresponding graded animated ring symbolically by $B_\bullet = \oplus_i B_i$, where $B_i = R\Gamma(X, \mathcal{O}(i))$, so that $X = (\mathrm{Spec}\, B_\bullet)/\mathbb{G}_m$.

4.1.2. The ∞-category $\mathrm{GrMod}_{B_\bullet}$ of **graded $B_\bullet$-modules** is the category $\mathrm{QCoh}(X)$. Symbolically, if $\mathcal{F}$ is a quasicoherent sheaf over X, we can write the associated graded module in the form $M_\bullet = \oplus_i M_i$, where $M_i = R\Gamma(X, \mathcal{F} \otimes \mathcal{O}(i))$. Note that by construction this is a symmetric monoidal ∞-category, and we denote the associated graded tensor product of $M_\bullet$ and $N_\bullet$ in $\mathrm{GrMod}_{B_\bullet}$ by

$$M_\bullet \otimes_{B_\bullet} N_\bullet.$$

We will usually write $\mathrm{Map}_{B_\bullet}(M_\bullet, N_\bullet)$ for mapping spaces in this category.

Given any graded module $M_\bullet$ over $B_\bullet$ and an integer i, we obtain the **i-shifted module** $M_\bullet(i)$: If $M_\bullet$ is associated with a quasicoherent sheaf $\mathcal{F}$ over $(\mathrm{Spec}\, B_\bullet)/\mathbb{G}_m$, $M_\bullet(i)$ is associated with $\mathcal{F} \otimes \mathcal{O}(i)$ and satisfies $(M_\bullet(i))_m = M_{m+i}$.

4.1.3. Note that we can use this optic to speak of **graded perfect** $B_\bullet$**-modules** and **graded vector bundles** over $B_\bullet$: they will correpond to perfect complexes (resp. vector bundles) over X.

For any R-algebra C, the relatively affine map $B\mathbb{G}_m \times \operatorname{Spec} C \to B\mathbb{G}_m \times \operatorname{Spec} R$ corresponds to C with its *trivial* grading. In this case, we can speak simply of graded C-modules, etc.

4.2. **Filtered objects via the Rees construction.** We will make frequent use of the quotient stack $\mathbb{A}^1/\mathbb{G}_m$, where we view $\mathbb{G}_m$ as acting on the affine line via $(t,z) \mapsto tz^{-1}$. Explicitly, this stack parameterizes line bundles $\mathcal{L}$ equipped with a *co*section $t : \mathcal{L} \to \mathscr{O}$.

4.2.1. This stack gives a geometric method for dealing with filtered objects [58]. More precisely, for any $R \in \mathrm{CRing}$, there is a canonical equivalence

$$\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R) \xrightarrow{\simeq} \mathrm{FilMod}_R,$$

where the right hand side is the stable ∞-category of filtered objects in Mod_R: classically, if $R = \pi_0(R)$ is discrete, then its associated triangulated derived category is the usual filtered derived category.

Symbolically, under this equivalence, a filtered module $\mathrm{Fil}^\bullet M$ on the right is associated with the $\mathbb{G}_m$-equivariant $R[t]$-module

$$\mathrm{Rees}(\mathrm{Fil}^\bullet M) = \bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^i M \cdot t^{-i}.$$

Our convention is that t lives in graded degree 1. For the functor in the other direction, note that we have a canonical family of line bundles $\mathcal{O}(n) = \mathcal{O}(1)^{\otimes n}$ over $\mathbb{A}^1/\mathbb{G}_m$ indexed by integers $n \in \mathbb{Z}$: Here, $\mathcal{O}(1)$ is the inverse tautological line bundle $\mathcal{L}^{\otimes -1}$. Note that we have canonical maps $t : \mathcal{O}(i) \to \mathcal{O}(i+1)$. Given a quasi-coherent sheaf $\mathcal{F}$ over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$, we now obtain a filtered module $\mathrm{Fil}^\bullet M$ by setting $\mathrm{Fil}^i M = R\Gamma(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R, \mathcal{F} \otimes \mathcal{O}(-i))$ with the transition maps given by t.

Definition 4.2.2. Any R-module M, viewed as a quasi-coherent sheaf on $\operatorname{Spec} R$ pulls back to a quasi-coherent sheaf on $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$, and this yields a filtered R-module $\mathrm{Fil}^\bullet_{\mathrm{triv}} M$ with underlying R-module M. This filtration is just the **trivial filtration** with $\mathrm{Fil}^i_{\mathrm{triv}} M = M$ if $i \le 0$ and 0 otherwise.

Definition 4.2.3. A **filtered stack over** R is an R-stack X equipped with a map to $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$; we will view it as a filtration on the R-stack $X_{(t\neq 0)}$ with associated graded $X_{(t=0)} \to B\mathbb{G}_m$.

4.3. **Filtered animated commutative rings and filtered modules.**

4.3.1. The Rees equivalence also gives us a compact way of defining **filtered animated commutative** R**-algebras**. These correspond to relatively *affine* stacks over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$. Symbolically, given a filtered R-algebra $\mathrm{Fil}^\bullet S$, the $\mathbb{G}_m$-equivariant $R[t]$-module $\mathrm{Rees}(\mathrm{Fil}^\bullet S)$ has a canonical $\mathbb{G}_m$-equivariant structure of an animated commutative $R[t]$-algebra, and taking the quotient of the associated affine scheme over $\mathbb{A}^1 \times \operatorname{Spec} R$ yields the associated affine morphism

$$\mathcal{R}(\mathrm{Fil}^\bullet S) \to \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R.$$

We will call the source of this map the associated **Rees stack**. Note that the fiber of this stack over the open point $\mathbb{G}_m/\mathbb{G}_m$ is canonically isomorphic to $\operatorname{Spec} S$, and its fiber over $B\mathbb{G}_m$ is the relatively affine stack associated with the graded ring $\bigoplus_i \mathrm{gr}^{-i} S$.

Note that we can now give a precise meaning to the ∞-category of filtered commutative algebras over the filtered ring $\mathrm{Fil}^\bullet S$: it is opposite to the category of relatively affine stacks over $\mathcal{R}(\mathrm{Fil}^\bullet S)$.

Observe also that any animated commutative ring R admits a lift $\mathrm{Fil}^\bullet_{\mathrm{triv}} R$ to a *trivially* filtered animated commutative ring corresponding to $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$: We have $\mathrm{Fil}^i_{\mathrm{triv}} R = R$ if $i \le 0$ and $\mathrm{Fil}^i_{\mathrm{triv}} R = 0$ otherwise.

4.3.2. A **filtered module** over $\mathrm{Fil}^\bullet S$ is now just a quasi-coherent sheaf $\mathcal{F}$ over the associated Rees stack. Once again, concretely, one can write it in the form $\mathrm{Fil}^\bullet M$ where the S-modules $\mathrm{Fil}^i M$ are obtained as global sections of suitable twists of $\mathcal{F}$. Write $\mathrm{FilMod}_{\mathrm{Fil}^S}$ for the associated ∞-category. We will write mapping spaces in this category in the form $\mathrm{Map}_{\mathrm{Fil}^\bullet S}(\_,\_)$. If $\mathrm{Fil}^\bullet_{\mathrm{triv}} S$ is the *trivial* filtration, then we will also write $\mathrm{Map}_{\mathrm{FilMod}_S}(\_,\_)$ for this mapping space.

Note that this gives us a symmetrical monoidal ∞-category by definition, where the tensor product corresponds to that of quasicoherent sheaves on the Rees stack. We will denote the associated product between filtered $\mathrm{Fil}^\bullet S$-modules $\mathrm{Fil}^\bullet M$ and $\mathrm{Fil}^\bullet N$ by

$$\mathrm{Fil}^\bullet M \otimes_{\mathrm{Fil}^\bullet S} \mathrm{Fil}^\bullet N.$$

Using this optic, we can also systematically talk about **filtered perfect** complexes as well as **filtered vector bundles** over $\mathrm{Fil}^\bullet S$: these correspond to perfect complexes (resp. vector bundles) on the associated Rees stacks.

Pullback from $\mathcal{R}(\mathrm{Fil}^\bullet S)$ to the closed substack $\mathcal{R}(\mathrm{Fil}^\bullet S)_{(t=0)}$ yields a symmetric monoidal functor from $\mathrm{FilMod}_{\mathrm{Fil}^\bullet S}$ to $\mathrm{GrMod}_{\mathrm{gr}^\bullet S}$: this is just the functor taking a filtered module to its associated graded.

4.4. **Increasing filtrations.** There is a variant of the above that looks at objects over the stack $\mathbb{A}^1_+/\mathbb{G}_m$ classifying *sections* of line bundles $u : \mathscr{O} \to L$: this corresponds to the 'usual' action of $\mathbb{G}_m$ on $\mathbb{A}^1$. We will write $\mathbb{A}^1_+ \times \mathrm{Spec}\, R = \mathrm{Spec}\, R[u]$ where u has graded degree -1.

Quasi-coherent sheaves over this stack are now equivalent to *increasingly* filtered modules $\mathrm{Fil}_\bullet M$, and relatively affine schemes over it are now equivalent to increasingly filtered animated commutative rings $\mathrm{Fil}_\bullet S$. We will denote the corresponding Rees construction by $\mathcal{R}_+(\mathrm{Fil}_\bullet S)$. Symbolically, we have

$$\mathcal{R}_+(\mathrm{Fil}_\bullet S) = \mathrm{Spec}\left(\bigoplus_i \mathrm{Fil}_i S \cdot u^i\right)/\mathbb{G}_m.$$

Observe that any animated commutative ring R admits a lift $\mathrm{Fil}^{\mathrm{triv}}_\bullet R$ to a *trivially* increasingly filtered animated commutative ring corresponding to $\mathbb{A}^1_+/\mathbb{G}_m \times \mathrm{Spec}\, R$: We have $\mathrm{Fil}^{\mathrm{triv}}_i R = R$ if $i \geq 0$ and $\mathrm{Fil}^{\mathrm{triv}}_i R = 0$ otherwise.

4.5. **Filtered deformation theory.**

4.5.1. Every filtered animated commutative algebra $\mathrm{Fil}^\bullet S$ over a filtered animated commutative ring $\mathrm{Fil}^\bullet R$ admits a filtered cotangent complex $\mathbb{L}_{\mathrm{Fil}^\bullet S/\mathrm{Fil}^\bullet R}$: this is a filtered $\mathrm{Fil}^\bullet S$-module corresponding to the cotangent complex of the associated Rees stacks. This controls the filtered deformation theory as follows:

A map of filtered animated commutative rings $\mathrm{Fil}^\bullet S' \to \mathrm{Fil}^\bullet S$ is a **filtered square-zero extension** if the corresponding map of $\mathbb{G}_m$-equivarint affine schemes over $\mathbb{A}^1/\mathbb{G}_m$ is a square zero thickening. In this case the fiber of the map of filtered rings is a filtered $\mathrm{Fil}^\bullet S$-module $\mathrm{Fil}^\bullet M$.

Given a connective filtered module $\mathrm{Fil}^\bullet M$ over $\mathrm{Fil}^\bullet S$, we can consider the *trivial* square-zero extension $\mathrm{Fil}^\bullet S \oplus \mathrm{Fil}^\bullet M$. We then have a canonical equivalence:

$$\mathrm{Map}_{\mathrm{Fil}^\bullet R/}(\mathrm{Fil}^\bullet S, \mathrm{Fil}^\bullet S \oplus \mathrm{Fil}^\bullet M) \simeq \mathrm{Map}_{\mathrm{Fil}^\bullet S}(\mathbb{L}_{\mathrm{Fil}^\bullet S/\mathrm{Fil}^\bullet R}, \mathrm{Fil}^\bullet M).$$

Sections of either equivalent space will be called $\mathrm{Fil}^\bullet$ R**-derivations** from $\mathrm{Fil}^\bullet S$ to $\mathrm{Fil}^\bullet M$.

One way to obtain square-zero extensions with fiber $\mathrm{Fil}^\bullet M$ therefore is as the left vertical arrow of a Cartesian diagram of the form

$$\begin{tikzcd}
\mathrm{Fil}^\bullet S' \arrow[r] \arrow[d] & \mathrm{Fil}^\bullet S \arrow[d, "d_{\mathrm{triv}}"] \\
\mathrm{Fil}^\bullet S \arrow[r, "d"'] & \mathrm{Fil}^\bullet S \oplus \mathrm{Fil}^\bullet M[1]
\end{tikzcd}$$

where the right vertical arrow is the trivial map and the horizontal one on the bottom is a $\mathrm{Fil}^\bullet$ R-derivation.

4.5.2. Now suppose that $X \to \mathcal{R}(\mathrm{Fil}^\bullet S)$ is an fppf (or even smooth) sheaf admitting a relative cotangent complex $\mathbb{L}_X \overset{\text{defn}}{=} \mathbb{L}_{X/\mathcal{R}(\mathrm{Fil}^\bullet S)}$. For any filtered $\mathrm{Fil}^\bullet S$-algebra $\mathrm{Fil}^\bullet A$, set

$$X(\mathrm{Fil}^\bullet A) = \mathrm{Map}_{/\mathcal{R}(\mathrm{Fil}^\bullet S)}(\mathcal{R}(\mathrm{Fil}^\bullet A), X).$$

Then we obtain a Cartesian diagram

$$\begin{tikzcd} X(\mathrm{Fil}^\bullet S') \arrow[r] \arrow[d] & X(\mathrm{Fil}^\bullet S) \arrow[d, "d_{\mathrm{triv}}"] \\ X(\mathrm{Fil}^\bullet S) \arrow[r, "d"'] & X(\mathrm{Fil}^\bullet S \oplus \mathrm{Fil}^\bullet M[1]). \end{tikzcd}$$

Moreover, for any $x \in X(\mathrm{Fil}^\bullet S)$, pulling $\mathbb{L}_X$ along x yields a filtered module $\mathrm{Fil}^\bullet \mathbb{L}_{X,x}$ over $\mathrm{Fil}^\bullet S$, and we have a canonical equivalence:

$$\mathrm{fib}_x(X(\mathrm{Fil}^\bullet S \oplus \mathrm{Fil}^\bullet M[1]) \to X(\mathrm{Fil}^\bullet S)) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Fil}^\bullet S}(\mathrm{Fil}^\bullet \mathbb{L}_{X,x}, \mathrm{Fil}^\bullet M[1]).$$

4.6. **The attractor stack.** The terminology we will use here is borrowed (with a sign difference) from [20].

Definition 4.6.1. Suppose that we have a prestack $\mathcal{Y} \to B\mathbb{G}_m \times \operatorname{Spec} R$. The associated **fixed point locus** is the functor X^0 on R-algebras given by

$$Y^0(C) = \mathrm{Map}_{B\mathbb{G}_m \times \operatorname{Spec} R}(B\mathbb{G}_m \times \operatorname{Spec} C, \mathcal{Y}).$$

Definition 4.6.2. Suppose that we have a prestack $\mathcal{X} \to \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$; its associated **attractor stack** or simply **attractor** is the functor X^- on R-algebras given by:

$$X^-(C) = \mathrm{Map}_{/\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R}(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} C, \mathcal{X}).$$

We define its fixed point prestack X^0 to be that of the restriction $\mathcal{X}_{(t=0)}$ of $\mathcal{X}$ over the closed substack $B\mathbb{G}_m \times \operatorname{Spec} R$.

4.6.3. In other words, X^- (resp. X^0) is the Weil restriction of $\mathcal{X}$ (resp. $\mathcal{X}_{(t=0)}$) from $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$ (resp. $B\mathbb{G}_m \times \operatorname{Spec} R$) down to $\operatorname{Spec} R$. Note that the sequence of natural maps

$$B\mathbb{G}_m \times \operatorname{Spec} R \hookrightarrow \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R \to B\mathbb{G}_m \times \operatorname{Spec} R$$

yields maps

$$X^0 \leftarrow X^- \leftarrow X^0$$

whose composition is the identity.

4.6.4. If we have a prestack $\mathcal{X} \to \mathbb{A}^1_+/\mathbb{G}_m$ then we have the analogous notion of the **repeller** X^+ associated with $\mathcal{X}$, which also admits maps $X^0 \to X^+ \to X^0$ whose composition is the identity.

If $\mathcal{Y} \to B\mathbb{G}_m$ is a graded prestack, then we will define its attractor and repeller to be those associated with its pullback over $\mathbb{A}^1/\mathbb{G}_m$ and $\mathbb{A}^1_+/\mathbb{G}_m$, respectively.

Remark 4.6.5. If $\mathcal{X}$ is the pullback of an algebraic space over $B\mathbb{G}_m$, these notions are studied by Drinfeld in [20].

4.6.6. Suppose that $\mathcal{X}$ is locally finitely presented over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$ and that it admits a perfect relative cotangent complex. Note that, over X^-, we have a canonical filtered perfect complex $\mathrm{Fil}^\bullet \mathbb{L}^-_{\mathcal{X}}$: This associates with every $x \in X^-(C)$ the filtered module corresponding to the pullback of the cotangent complex $\mathbb{L}_{\mathcal{X}/(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R)}$ to $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} C$ along x.

Similarly, over X^0, we have a canonical graded perfect complex $\mathbb{L}^0_{\mathcal{X},\bullet}$: it is isomorphic to the associated graded of the restriction of $\mathrm{Fil}^\bullet \mathbb{L}^-_{\mathcal{X}}$ along $X^0 \to X^-$.

Lemma 4.6.7. *The prestack X^- admits a perfect cotangent complex over R; we have*

$$\mathbb{L}_{X^-/R} = \mathbb{L}^-_{\mathcal{X}}/\mathrm{Fil}^1 \mathbb{L}^-_{\mathcal{X}}.$$

Similarly, the prestack X^0 admits a perfect cotangent complex over X with

$$\mathbb{L}_{X^0/R} = \mathbb{L}^0_{\mathcal{X},0}.$$

Proof. From the discussion in §4.5, we see that, for $C \in \mathrm{CRing}_{R/}$, $M \in \mathrm{Mod}_C^{\mathrm{cn}}$, and $x \in X^-(C)$, we have

$$\mathrm{fib}_x(X^-(C \oplus M) \to X^-(C)) \simeq \mathrm{Map}_{\mathrm{FilMod}_C}(\mathrm{Fil}^\bullet \mathbb{L}^-_{\mathcal{X},x}, \mathrm{Fil}^\bullet_{\mathrm{triv}} M) \simeq \mathrm{Map}_{\mathrm{Mod}_C}(\mathbb{L}^-_{\mathcal{X},x}/\,\mathrm{Fil}^1 \mathbb{L}^-_{\mathcal{X},x}, M).$$

This proves the first part of the lemma. The proof of the second is entirely analogous. □

We will now give a general criterion for representability of X^-, X^+ and X^0 due to Halpern-Leistner and Preygel [32, Example 1.2.2].

Proposition 4.6.8. *Suppose that $\pi_0(R)$ is a G-ring and that $\mathcal{X} \to \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R$ is a locally finitely presented derived Artin 1-stack with quasi-affine (resp. affine) diagonal. Then X^-, X^0, X^+ are locally finitely presented derived Artin 1-stacks over R, and if $\mathcal{X}$ is* flat *over $\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R$, then X^-, X^0, X^+ all have quasi-affine (resp. affine) diagonal.*

Proof. We recall some key points of the proof, which uses Lurie's derived generalization of Artin's representability theorem [46, Theorem 7.1.6].

It is straightforward to see that X^-, X^0, X^+ are all étale sheaves that are locally finitely presented, nilcomplete and infinitesimally cohesive. We have already seen that they admit perfect cotangent complexes, and it is clear that their classical truncations are valued in 1-truncated spaces.

The main difficulty now is to show that they are *integrable* (condition (3) in *loc. cit.*). The authors of [32] appeal to a very general argument that applies to a wide class of quotient stacks, which are shown to be *cohomologically projective* and hence *formally proper*. We can translate this into rather concrete assertions in the particular cases we are dealing with here.

For X^0, one uses Proposition A.2.1.

For X^- (the argument for X^+ is identical), we need to know that the map

$$\mathrm{Map}_{/\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R}(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, C, \mathcal{X}) \to \varprojlim_m \mathrm{Map}_{/\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R}(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, C/\mathfrak{m}^m, \mathcal{X}). \tag{4.6.8.1}$$

is an isomorphism.

For this, one first finds from Proposition A.2.2 that, for any Noetherian $B \in \mathrm{CRing}_{\heartsuit,R/}$, we have

$$\mathrm{Map}_{/\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R}(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, B, \mathcal{X}) \xrightarrow{\simeq} \varprojlim_n \mathrm{Map}_{/\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R}((\mathbb{A}^1/\mathbb{G}_m)_{(t^n=0)} \times \mathrm{Spec}\, B, \mathcal{X}).$$

Via filtered deformation theory, the desired integrability for X^- now reduces to the already known assertion for X^0.[8]

It remains only to check the assertion about the diagonal, which is [32, Proposition 5.1.15]. □

4.7. **1-bounded fixed points.** Suppose that $\mathcal{Z} \to B\mathbb{G}_m \times \mathrm{Spec}\, R$ is a relative locally finitely presented Artin stack and let $Z^0 \to \mathrm{Spec}\, R$ be the corresponding fixed point locus. As observed in §4.6.6, over Z^0 we have the canonical graded perfect complex $\mathbb{L}^0_{\mathcal{Z},\bullet}$. Let $\mathbb{T}^0_{\mathcal{Z},\bullet}$ be its graded dual. The locus $Z^0_{\text{1-b}}$ of **1-bounded fixed points** is the *open* substack of Z^0 where we have $\mathbb{T}^0_{\mathcal{Z},i} \simeq 0$ for $i > 1$.

Example 4.7.1. Suppose that we have $\mathcal{M} \in \mathrm{Perf}(B\mathbb{G}_m \times \mathrm{Spec}\, R)$ corresponding to a graded R-module $M_\bullet$. Then we can take $\mathcal{Z} = \mathbf{V}(\mathcal{M}) \to B\mathbb{G}_m \times \mathrm{Spec}\, R$ to be the associated vector stack.

One checks that the corresponding fixed point locus Z^0 now is just the vector stack $\mathbf{V}(M_0) \to \mathrm{Spec}\, R$, while the graded perfect complex $\mathbb{T}^0_{\mathcal{Z},\bullet}$ corresponds simply to the restriction of $\mathcal{M}$ to $B\mathbb{G}_m \times \mathbf{V}(M_0)$. This implies that

$$Z^0_{\text{1-b}} = \mathbf{V}(M_0) \times_{\mathrm{Spec}\, R} (\mathrm{Spec}\, R)_{\text{1-b}},$$

where $(\mathrm{Spec}\, R)_{\text{1-b}} \subset \mathrm{Spec}\, R$ is the open locus M_i becomes nullhomotopic for $i > 1$.

Let us call $\mathcal{M}$ **1-bounded** if $M_i \simeq 0$ for $i > 1$ already over R. Then we see that, for such 1-bounded perfect complexes, we have

$$Z^0_{\text{1-b}} = Z^0 = \mathbf{V}(M_0).$$

[8]This argument is closely related to one appearing in [20].

Example 4.7.2. Consider the stack $\mathcal{P} : R \mapsto \mathrm{Perf}(R)^{\simeq}$ on CRing: this is represented by a locally finitely presented derived Artin stack over $\mathbb{Z}$; see [63, § 3].

Now take $\mathcal{Z} = \mathcal{P} \times B\mathbb{G}_m \to B\mathbb{G}_m$: the fixed point locus Z^0 associates with every $R \in \mathrm{CRing}$ the ∞-groupoid of graded perfect R-modules.

The tangent complex of $\mathcal{P}$ is $M_{\mathrm{taut}}^{\vee} \otimes M_{\mathrm{taut}}$, where $M_{\mathrm{taut}} \in \mathrm{Perf}(\mathcal{P})$ is the tautological perfect complex. From this, one finds that the graded perfect complex $\mathbb{T}^0_{\mathcal{Z},\bullet}$ is $M_{\mathrm{taut},\bullet}^{\vee} \otimes M_{\mathrm{taut},\bullet}$, where $M_{\mathrm{taut},\bullet}$ is the tautological graded perfect complex over Z^0.

Now, $Z^0_{\text{1-b}}$ is precisely the locus where $M_{\mathrm{taut},i} \otimes M_{\mathrm{taut},j} \simeq 0$ for all $i, j \in \mathbb{Z}$ with $j - i > 1$.

In particular, the locus where $M_{\mathrm{taut},i} \simeq 0$ for $i \neq 0, 1$ is an open substack of $Z^0_{\text{1-b}}$.

4.8. 1-bounded stacks.

Definition 4.8.1. Suppose that $A \in \mathrm{CRing}$ and $R \in \mathrm{CRing}_{A/}$. An **$R$-pointed graded prestack** over A is a prestack $\mathcal{Y} \to B\mathbb{G}_{m,A}$ equipped with a morphism $\iota : B\mathbb{G}_m \times \operatorname{Spec} R \to \mathcal{Y}$ of graded prestacks. In particular, for such a prestack, any relative derived Artin stack $\mathcal{Z} \to \mathcal{Y}$ has an associated fixed point locus $Z^0 \to \operatorname{Spec} R$ obtained from the base-change of $\mathcal{Z}$ over $B\mathbb{G}_m \times \operatorname{Spec} R$.

Usually R will be implicit, and we will simply call the pair $(\mathcal{Y}, \iota)$ a *pointed* graded prestack. If the 'point' ι is also clear from context, we will just refer to $\mathcal{Y}$ as a pointed graded prestack.

Definition 4.8.2. A **1-bounded stack** $\mathcal{X} = (\mathcal{X}^{\diamond}, X^0) \to (\mathcal{Y}, \iota)$ over $(\mathcal{Y}, \iota)$ (or simply $\mathcal{Y}$ if ι is clear from context) consists of the following data:

(1) A relative locally finitely presented derived Artin r-stack $\mathcal{X}^{\diamond} \to \mathcal{Y}$;
(2) An open immersion $X^0 \hookrightarrow X^{\diamond,0}_{\text{1-b}}$, which we will refer to as the **fixed point locus** of $\mathcal{X}$.

One can speak of maps between 1-bounded stacks over $\mathcal{Y}$ in the obvious way.

4.8.3. Suppose that $\mathcal{Y} \to \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} A$ is in fact a filtered prestack, and that ι lifts to a map of filtered stacks $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R \to \mathcal{Y}$ over A. Then we will associate with any 1-bounded stack $\mathcal{X} \to \mathcal{Y}$ its **attractor** $X^- \to \operatorname{Spec} R$ by setting $X^- \overset{\text{defn}}{=} X^{\diamond,-} \times_{X^{\diamond,0}} X^0$. Here $X^{\diamond,-}$ is the attractor of the base-change of $\mathcal{X}^{\diamond}$ over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$. Analogously, given a lift $\mathbb{A}^1_+/\mathbb{G}_m \times \operatorname{Spec} R \to \mathcal{Y}$, we can define an associated repeller X^+.

4.8.4. If $(\mathcal{Z}, \eta) \to (\mathcal{Y}, \iota)$ is a map of pointed graded prestacks with $\eta : B\mathbb{G}_m \times \operatorname{Spec} C \to \mathcal{Z}$ and $\mathcal{X} = (\mathcal{X}^{\diamond}, X^0)$ is a 1-bounded stack over $\mathcal{Y}$, we set

$$\mathrm{Map}_{/(\mathcal{Y},\iota)}((\mathcal{Z}, \iota), \mathcal{X}) = \mathrm{Map}_{/\mathcal{Y}}(\mathcal{Z}, \mathcal{X}^{\diamond}) \times_{X^{\diamond,0}(C)} X^0(C).$$

Here, the map

$$\mathrm{Map}_{/\mathcal{Y}}(\mathcal{Z}, \mathcal{X}^{\diamond}) \to X^{\diamond,0}(C) = \mathrm{Map}_{/\mathcal{Y}}(B\mathbb{G}_m \times \operatorname{Spec} C, \mathcal{X}^{\diamond})$$

is obtained via restriction along η.

If η and ι are clear from context, we will simply write $\mathrm{Map}_{/\mathcal{Y}}(\mathcal{Z}, \mathcal{X})$ for this space.

Here are our main examples:

Example 4.8.5. Example 4.7.1 shows that, if $\mathcal{M} \in \mathrm{Perf}(\mathcal{Y})$ is a perfect complex whose restriction over $B\mathbb{G}_m \times \operatorname{Spec} R$ is 1-bounded, then the stack $\mathcal{X}^{\diamond} \overset{\text{defn}}{=} \mathbf{V}(\mathcal{M}) \to \mathcal{Y}$ underlies a 1-bounded stack over $\mathcal{Y}$ with $X^0 = X^{\diamond,0}$.

Example 4.8.6. Example 4.7.2 shows that, when $\mathcal{Y} = B\mathbb{G}_m$ (viewed as a pointed graded stack in the tautological sense), then we obtain a 1-bounded stack $\mathcal{P}_{\{0,1\}} \to B\mathbb{G}_m$ with $\mathcal{P}^{\diamond}_{\{0,1\}} = \mathcal{P} \times B\mathbb{G}_m$, and $P^0_{\{0,1\}} \subset P^{\diamond,0}_{\{0,1\}}$ is the open substack parameterizing graded perfect complex $M_\bullet$ with $M_i \simeq 0$ for $i \neq 0, 1$.

For any pointed graded stack $(\mathcal{Z}, \eta) \to B\mathbb{G}_m$, the space

$$\mathrm{Perf}_{\{0,1\}}((\mathcal{Z}, \eta)) \overset{\text{defn}}{=} \mathrm{Map}_{/B\mathbb{G}_m}(\mathcal{Z}, \mathcal{P}_{\{0,1\}})$$

is the ∞-groupoid of perfect complexes on $\mathcal{Z}$ whose restriction along η is 1-bounded. We will refer to such objects simply as **1-bounded perfect complexes** over $(\mathcal{Z}, \eta)$.

Once again, if η is clear from context, we will denote this space simply by $\mathrm{Perf}_{\{0,1\}}(\mathcal{Z})$, and refer to its sections as 1-bounded perfect complexes over Z.

If we take $\mathbb{A}^1/\mathbb{G}_m \to B\mathbb{G}_m$ to be the canonical map, then the associated attractor is the stack

$$R \mapsto \mathrm{Perf}_{\{0,1\}}(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R)$$

of filtered perfect complexes $\mathrm{Fil}^\bullet M$ with $\mathrm{gr}^i M \simeq 0$ for $i \neq 0, -1$.

Similarly, the repeller associated with $\mathbb{A}^1_+/\mathbb{G}_m \to B\mathbb{G}_m$ is the stack of ascendingly filtered perfect complexes $\mathrm{Fil}_\bullet M$ with $\mathrm{gr}_i M \simeq 0$ for $i \neq 0, 1$.

All these stacks are representable: the only difficulty is integrability, but this is easily checked.

Example 4.8.7. We have an 'open substack' $\mathcal{V}_{\{0,1\}}$ of $\mathcal{P}_{\{0,1\}}$ by restricting to the open locus $\mathcal{V}^\diamond_{\{0,1\}} \subset \mathcal{P}^\diamond_{\{0,1\}}$, where the tautological perfect complex is in fact a vector bundle.

For any pair of non-negative integers $d \leq h$, we can further refine this to the 1-bounded stack $\mathcal{V}^{h,d}_{\{0,1\}} = (\mathcal{V}^\diamond_{\{0,1\}}, V^{0,h,d}_{\{0,1\}})$, where $V^{0,h,d}_{\{0,1\}}$ is the open and closed substack of the fixed point locus parameterizing graded vector bundles $M_\bullet$ such that $M_i \simeq 0$ for $i \neq 0, 1$, and such that M_1 is a vector bundle of rank d and M_0 is a vector bundle of rank $h - d$.

The attractor $V^{-,h,d}_{\{0,1\}}$ is the stack of filtered vector bundles $\mathrm{Fil}^\bullet V$ where: V has rank h; $\mathrm{gr}^i V \simeq 0$ for $i \neq 0, -1$; and $\mathrm{gr}^{-1} V$ has rank d.

4.9. **Cocharacters of group schemes and twisted group stacks.** This following discussion is essentially from [21, §2.3]. Suppose that G is a smooth affine group scheme over a classical commutative ring R and let $\mu : \mathbb{G}_{m,R'} \to G_{R'}$ be a cocharacter defined over some $R' \in \mathrm{CRing}_{R/}$, inducing a $\mathbb{G}_{m,R'}$-action on $G_{R'}$ via the adjoint action.

Remark 4.9.1. In [21], Drinfeld considers a seemingly more general situation where $\mathbb{G}_{m,R}$ acts on G via a map $\mathbb{G}_{m,R'} \to \mathrm{Aut}(G)_{R'}$. For instance, one can consider the case of $G = \mathbb{G}_a$ equipped with the natural action of $\mathbb{G}_m$.

This case can be subsumed—perhaps a bit unnaturally—into the setup here by viewing such a map as a cocharacter of the R'-group scheme $G \rtimes_\mu \mathbb{G}_m$, where the semi-direct product is defined by the action of $\mathbb{G}_{m,R'}$ on $G_{R'}$ via μ

See also Remark 9.1.4 below.

4.9.2. The fpqc quotient of $G_{R'}$ by the action of μ yields a group stack $G\{\mu\}$ over $B\mathbb{G}_{m,R'}$. We have subgroups $U^\pm_\mu \subset P^\pm_\mu \subset G_{R'}$ with $P^\pm_\mu/U^\pm_\mu \simeq M_\mu$ independent of sign: Namely, P^-_μ (resp. P^+_μ) is the attractor (resp. repeller) of the basechange of $G\{\mu\}$ over $\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R'$, and M_μ is the fixed point locus of $G\{\mu\}$.

Explicitly, given an R'-algebra S, we have

$$P^-_\mu(S) = \mathrm{Map}_{B\mathbb{G}_{m,R}}\left(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, S, G\{\mu\}\right) ;\ P^+_\mu(S) = \mathrm{Map}_{B\mathbb{G}_{m,R}}\left(\mathbb{A}^1_+/\mathbb{G}_m \times \mathrm{Spec}\, S, G\{\mu\}\right) ;$$
$$M_\mu(S) = \mathrm{Map}_{B\mathbb{G}_{m,R}}\left(B\mathbb{G}_m \times \mathrm{Spec}\, S, G\{\mu\}\right).$$

Restriction to the open point $\mathbb{G}_m/\mathbb{G}_m \subset \mathbb{A}^1/\mathbb{G}_m$ now gives a closed immersion of group schemes $P^\pm_\mu \hookrightarrow G$. The section $M_\mu \to P^\pm_\mu$ exhibits it as the centralizer in $P^\pm_\mu$ (and in G) of μ.

The subgroup $U^\pm_\mu \subset P^\pm_\mu$ is the kernel of the map to M_μ.

4.9.3. In terms of Lie algebras, the action of μ gives us a grading of the base-change of $\mathfrak{g} \overset{\mathrm{defn}}{=} \mathrm{Lie}\, G$ over R':

$$\mathfrak{g}_{R'} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i,$$

where $\mathbb{G}_m$ acts on $\mathfrak{g}_i$ via $z \mapsto z^{-i}$. We now have:[9]

$$\mathrm{Lie}\, P^\pm_\mu = \bigoplus_{\pm i\geq 0} \mathfrak{g}_i \ ;\ \mathrm{Lie}\, U^\pm_\mu = \bigoplus_{\pm i > 0} \mathfrak{g}_i \ ;\ \mathrm{Lie}\, M_\mu = \mathfrak{g}_0.$$

[9]We are following the sign conventions from [42]

When G is reductive, then what we have defined here are the parabolic and unipotent subgroups of G associated with μ.

4.9.4. Note that the cocharacter $\mu : \mathbb{G}_{m,R'} \to G_{R'}$ yields a map

$$B\mu : B\mathbb{G}_{m,R'} \to BG_{R'}$$

yielding a canonical G-torsor $\mathcal{P}_\mu \to B\mathbb{G}_{m,R'}$: The automorphisms of this torsor are represented by the group stack $G\{\mu\} \to B\mathbb{G}_{m,R'}$, and twisting by $\mathcal{P}_\mu$ yields an isomorphism

$$BG \times B\mathbb{G}_{m,R'} \xrightarrow{\simeq} BG\{\mu\}$$

of $B\mathbb{G}_{m,R'}$-stacks carrying $\mathcal{P}_\mu$ to the trivial $G\{\mu\}$-torsor.

In particular, we can canonically view every $G\{\mu\}$-torsor over a $B\mathbb{G}_{m,R'}$-stack as a G-torsor, and *vice versa*.

4.9.5. Consider the pointed graded stack $(B\mathbb{G}_{m,R}, \iota_{R'})$, where $\iota_{R'} : B\mathbb{G}_{m,R'} \to B\mathbb{G}_{m,R}$ is the structure morphism. If we take the stack $\mathcal{Z} = BG \times B\mathbb{G}_m \to B\mathbb{G}_{m,R}$, the associated fixed point locus Z^0 over R' parameterizes, for any $C \in \mathrm{CRing}_{R'/}$, G-torsors over $B\mathbb{G}_m \times \operatorname{Spec} C$.

Note that by the discussion in (4.9.4), for any R'-algebra C, we can also view $Z^0(C)$ as the ∞-groupoid of $G\{\mu\}$-torsors over $B\mathbb{G}_m \times \operatorname{Spec} C$. Unwinding definitions, one finds that, for any $C \in \mathrm{CRing}_{R'/}$, $Z^0(C)$ is the ∞-groupoid of the following equivalent kinds of objects:

- $G\{\mu\}$-torsors over $B\mathbb{G}_m \times \operatorname{Spec} C$;
- $G \rtimes_\mu \mathbb{G}_m$-equivariant schemes $\mathcal{P} \to \operatorname{Spec} C$ such that the underlying G action presents $\mathcal{P}$ as a G-torsor over C.

In the next lemma, given $C \in \mathrm{CRing}_{R'/}$ and a G-torsor $\mathcal{Q} \to B\mathbb{G}_m \times \operatorname{Spec} C$, we will write $\mathcal{Q}^\mu$ for the corresponding $G\{\mu\}$-torsor.

Lemma 4.9.6. *There is a canonical open and closed immersion $BM_\mu \to Z^0$ mapping isomorphically onto the locus of G-torsors $\mathcal{Q} \to B\mathbb{G}_m \times \operatorname{Spec} C$ satisfying the following equivalent conditions when $\operatorname{Spec} C$ is connected:*

(1a) There exists an étale cover $C \to C'$ such that the restriction of $\mathcal{Q}$ over $B\mathbb{G}_m \times \operatorname{Spec} C'$ is isomorphic to $\mathcal{P}_\mu \otimes_{\mathcal{O}} C'$;
(1b) There exists an étale cover $C \to C'$ such that the restriction of $\mathcal{Q}^\mu$ over $B\mathbb{G}_m \times \operatorname{Spec} C'$ is trivial;
(2a) For every geometric point $C \to \kappa$ of $\operatorname{Spec} C$, the G-torsor $x^*\mathcal{Q}$ over $B\mathbb{G}_m \times \operatorname{Spec} \kappa$ is isomorphic to $\mathcal{P}_\mu \otimes_{\mathcal{O}} \kappa$;
(2b) For every geometric point $C \to \kappa$ of $\operatorname{Spec} C$, the $G\{\mu\}$-torsor $x^*\mathcal{Q}^\mu$ over $B\mathbb{G}_m \times \operatorname{Spec} \kappa$ is trivial;
(3a) For some *geometric point $C \to \kappa$ of $\operatorname{Spec} C$, the G-torsor $x^*\mathcal{Q}$ over $B\mathbb{G}_m \times \operatorname{Spec} \kappa$ is isomorphic to $\mathcal{P}_\mu \otimes_{\mathcal{O}} \kappa$;*
(3b) For some *geometric point $C \to \kappa$ of $\operatorname{Spec} C$, the $G\{\mu\}$-torsor $x^*\mathcal{Q}^\mu$ over $B\mathbb{G}_m \times \operatorname{Spec} \kappa$ is trivial.*

Proof. The (a),(b) counterparts in each numbered pair are equivalent, so we will replace $\mathcal{Q}$ by $\mathcal{Q}^\mu$ and prove the (b) sides of each pair.

The map $BM_\mu \to Z^0$ associates with each M_μ-torsor $\mathcal{P}^0$ the G-torsor obtained via pushforward along the map $M_\mu \to G_{\mathcal{O}}$: Such a G-torsor is equipped with a canonical extension to an action of $G \rtimes_\mu \mathbb{G}_m$.

Let us now show that this yields an isomorphism of $BM_\mu(C)$ with the space of $G\{\mu\}$-torsors over $B\mathbb{G}_m \times \operatorname{Spec} C$ satisfying any of the three given conditions (1b), (2b) and (3b). For condition (1b), it is easy: Giving such an object over $B\mathbb{G}_m \times \operatorname{Spec} C$ is the same as giving an étale torsor over C for the fixed point group scheme of $G\{\mu\}$, which is of course M_μ.

To finish, it is enough to see that BM_μ is an open and closed substack of Z^0. The quickest way to see this is to observe, as Drinfeld does in [21, §C.2.3] that we have discrete invariants on Z^0 given by G-conjugacy classes of cocharacters $\mathbb{G}_m \to G \rtimes_\mu \mathbb{G}_m$ lifting the identity map of $\mathbb{G}_m$. Now, BM_μ is the open and closed substack of Z^0 associated with the trivial such lift. □

Remark 4.9.7. Let Z^- be the attractor (of the base change over $\mathbb{A}^1/\mathbb{G}_m$ of) $\mathcal{Z}$. Then we find that $Z^-(C) \times_{Z^0(C)} BM_\mu(C)$ is spanned by $G\{\mu\}$-torsors $\mathcal{Q}^\mu$ over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} C$ satisfying the following condition: There exists an étale cover $C \to C'$ such that the restriction of $\mathcal{Q}^\mu$ over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} C'$ is trivial. Indeed, this amounts to checking that a $G\{\mu\}$-torsor $\mathcal{Q}^\mu$ over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} C$ with trivial restriction over $B\mathbb{G}_m \times \operatorname{Spec} C$ is itself trivial. This is because $\mathcal{Q}^\mu$ is smooth over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} C$, and we have

$$\operatorname{Map}_{\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} C}(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} C, \mathcal{Q}^\mu) \xrightarrow{\simeq} \varprojlim_n \operatorname{Map}_{\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} C}((\mathbb{A}^1/\mathbb{G}_m)_{(t^m=0)} \times \operatorname{Spec} C, \mathcal{Q}^\mu).$$

See the proof of Proposition 8.6.4.

In particular, $Z^- \times_{Z^0} BM_\mu$ is isomorphic to the stack of étale torsors for the attractor group scheme associated with $G\{\mu\}$, which is of course P^-_μ. In other words, we have $Z^- \times_{Z^0} BM_\mu \simeq BP^-_\mu$.

4.10. **1-bounded cocharacters.** Here, we will present a key example of a 1-bounded stack relevant to Theorem D. The notation will be as above.

Definition 4.10.1. Following [42], we will say that μ is **1-bounded** if, under the adjoint action of μ, we have $\mathfrak{g}_i = 0$ for $i > 1$. In this case, we will set $\mathfrak{g}^+_\mu = \mathfrak{g}_1$.

If G is reductive, then this condition is equivalent to asking that μ be *minuscule.*

Lemma 4.10.2. *If μ is 1-bounded, the exponential map induces an equivalence:*

$$\mathbf{V}(\mathfrak{g}^+_\mu) \xrightarrow{\simeq} U^+_\mu.$$

In particular, for any $C \in \mathrm{CRing}_{R'/}$, we have an equivalence

$$\mathfrak{g}^+_\mu \otimes_R (C/^{\mathbb{L}}p^n) \xrightarrow[\simeq]{\exp} U^{+,(n)}_\mu(C).$$

Proof. See [42, Lemma 6.3.2]. □

4.10.3. We now isolate a particular open substack of $Z^0_{\text{1-b}}$ using the 1-bounded condition on μ. For this, we begin by noting that, for every representation W of G defined over R', the usual twisting process by the G-torsor $\mathcal{P}$ over $B\mathbb{G}_m \times \operatorname{Spec} C$ yields a canonical graded vector bundle $\mathcal{M}_\bullet(W)_{\mathcal{P}}$ over $\operatorname{Spec} C$.

In this way, we obtain a graded vector bundle $\mathcal{M}_\bullet(W)$ over Z^0. Now, the graded perfect complex $\mathbb{T}^0_{\mathcal{Z},\bullet}$ over Z^0 is simply $\mathcal{M}_\bullet(\mathfrak{g})[-1]$.

The 1-bounded locus $Z^0_{\text{1-b}}$ is the open and closed locus over which we have $\mathcal{M}_i(\mathfrak{g}) \simeq 0$ for $i > 1$.

Over BM_μ, for each $i \in \mathbb{Z}$, we have the vector bundles $\mathcal{M}^0(\mathfrak{g}_i)$ obtained by twisting the representation $\mathfrak{g}_i$ by the universal M_μ-torsor. The restriction of $\mathbb{T}^0_{\mathcal{Z},\bullet}$ to BM_μ is then seen to be isomorphic to the graded complex $\bigoplus_{i\in\mathbb{Z}} \mathcal{M}^0(\mathfrak{g}_i)(-i)[-1]$. In particular, the 1-boundedness of μ ensures exactly that this is a 1-bounded complex. That is, we have defined a map $BM_\mu \to Z^0_{\text{1-b}}$.

Definition 4.10.4. $\mathcal{B}(G,\mu)$ will be the 1-bounded stack over the pointed graded stack $(B\mathbb{G}_m \times \operatorname{Spec} R, \iota_{R'})$ given by the pair $(BG \times B\mathbb{G}_m, BM_\mu)$.

Remark 4.9.7 shows that the attractor $B(G,\mu)^-$ is simply BP^-_μ, and an analogous argument shows that the repeller is BP^+_μ.

Remark 4.10.5. It is easy to see from the definitions that $\mathcal{B}(G,\mu)$ depends only on the isomorphism class of the map $B\mu : B\mathbb{G}_{m,R'} \to BG_{R'}$. In particular, it depends on the cocharacter μ only up to conjugacy.

4.11. **Deformations of 1-bounded fixed points.** Suppose that $\mathcal{Y} = \operatorname{Spec}(B_\bullet)/\mathbb{G}_m$ for a non-positively graded animated commutative ring $B_\bullet$.

4.11.1. For any $m \geq 0$, we have the animated commutative graded 'quotient' $B_\bullet \to B_{\geq -m}$ with underlying graded B_0-module $\bigoplus_{i\geq -m} B_i(-i)$. To construct this, we need a different perspective on the ∞-category of non-positively graded animated commutative rings: They can also be obtained as the *animation* of the category of non-positively graded commutative rings, which admits a set of compact projective generators given by graded polynomial algebras in finitely many homogeneous variables in non-positive degrees. From this perspective, the quotient map $B_\bullet \to B_{\geq -m}$ is simply the animation of the usual construction for non-positively graded commutative rings. Note in particular, that $B_{\geq -1}$ is the *graded* trivial square-zero extension of B_0 by $B_{-1}(1)$.

4.11.2. Let $\mathcal{X} = (\mathcal{X}^{\diamond}, X^0) \to (\mathcal{Z}, \iota)$ be a 1-bounded stack over a pointed graded prestack with $\iota : B\mathbb{G}_m \times \operatorname{Spec} R \to \mathcal{Z}$. Suppose that we have a map of pointed prestacks $\mathcal{Y} \to \mathcal{Z}$.

Proposition 4.11.3. *Suppose that $\mathcal{X}^{\diamond} \to \mathcal{Z}$ has quasi-affine diagonal. Then the natural map*

$$\operatorname{Map}_{/\mathcal{Z}}(\mathcal{Y}, \mathcal{X}) \to \operatorname{Map}_{/\mathcal{Z}}((\operatorname{Spec} B_{\geq -1})/\mathbb{G}_m, \mathcal{X})$$

is an equivalence.

Proof. Consider the following general situation: Suppose that $C'_\bullet \to C_\bullet$ is a square-zero extension of animated non-positively graded commutative $B_\bullet$-algebras with fiber $I_\bullet$. By the graded analogue of the discussion in §4.5, one sees that the fiber of the map

$$\operatorname{Map}_{/\mathcal{Z}}((\operatorname{Spec} C'_\bullet)/\mathbb{G}_m, \mathcal{X}^{\diamond}) \to \operatorname{Map}_{/\mathcal{Z}}((\operatorname{Spec} C)/\mathbb{G}_m, \mathcal{X}^{\diamond})$$

over a section x has the following features:

- The obstruction to its being non-empty is given by a section of

$$\operatorname{Map}_{C_\bullet}(\mathbb{L}_{\mathcal{X}^{\diamond},x,\bullet}, I_\bullet[1])$$

  where $\mathbb{L}_{\mathcal{X}^{\diamond},x,\bullet}$ is the graded $C_\bullet$-module obtained via pulling the relative cotangent complex of $\mathcal{X}^{\diamond}$ over $\mathcal{Y}$ along x.
- If the obstruction is nullhomotopic, then the fiber is equivalent to

$$\operatorname{Map}_{C_\bullet}(\mathbb{L}_{\mathcal{X}^{\diamond},x,\bullet}, I_\bullet).$$

If $\mathbb{T}_{\mathcal{X}^{\diamond},x,\bullet}$ is the dual graded perfect module, then, up to shift, we see that we are looking at the spaces $\tau^{\leq i}(\mathbb{T}_{\mathcal{X}^{\diamond},x,\bullet} \otimes_{C_\bullet} I_\bullet)_0$ for $i = 0, 1$. Here, the subscript denotes the degree 0 component.

Let $x_0 \in \operatorname{Map}_{/\mathcal{Y}}((\operatorname{Spf} C_0)/\mathbb{G}_m, \mathcal{X}^{\diamond}) = X^{\diamond,0}(C_0)$ be the image of x. Then Lemma A.2.3 below gives us a canonical filtration $\operatorname{Fil}^\bullet_{\mathrm{wt}} \mathbb{T}_{\mathcal{X}^{\diamond},x,\bullet}$ with

$$\operatorname{gr}^i_{\mathrm{wt}} \mathbb{T}_{\mathcal{X}^{\diamond},x,\bullet} \simeq C_\bullet(-i) \otimes_{C_0} \mathbb{T}_{\mathcal{X}^{\diamond},x_0,i},$$

and we have

$$(\operatorname{gr}^i_{\mathrm{wt}} \mathbb{T}_{\mathcal{X}^{\diamond},x,\bullet} \otimes_{C_\bullet} I_\bullet)_0 \simeq (\mathbb{T}_{\mathcal{X}^{\diamond},x_0,i} \otimes_{C_0} I_\bullet(-i))_0 \simeq \mathbb{T}_{\mathcal{X}^{\diamond},x_0,i} \otimes_{C_0} I_{-i}$$

The right hand side here is is non-trivial only when $\mathbb{T}_{\mathcal{X}^{\diamond},x_0,i}$ is not nullhomotopic and when $i \leq 0$.

Now, if x_0 is in the image of $X^0(C_0)$, then it maps to a 1-bounded fixed point, and so $\mathbb{T}_{\mathcal{X}^{\diamond},x_0,i} \simeq 0$ for $i \geq 2$. From this, we deduce that the fiber over x depends only on the quotient $I_{-1}(1) \oplus I_0$ of $I_\bullet$. More precisely, the following square is Cartesian

$$\begin{tikzcd}
\operatorname{Map}_{/\mathcal{Z}}((\operatorname{Spec} C'_\bullet)/\mathbb{G}_m, \mathcal{X}) \arrow[r] \arrow[d] & \operatorname{Map}_{/\mathcal{Z}}((\operatorname{Spec} C_\bullet)/\mathbb{G}_m, \mathcal{X}) \arrow[d] \\
\operatorname{Map}_{/\mathcal{Z}}((\operatorname{Spec} C'_{\geq -1})/\mathbb{G}_m, \mathcal{X}) \arrow[r] & \operatorname{Map}_{/\mathcal{Z}}((\operatorname{Spec} C_{\geq -1})/\mathbb{G}_m, \mathcal{X})
\end{tikzcd}$$

Applying this with $C'_\bullet \to C_\bullet$ the map $\tau_{\leq (k+1)} B_\bullet \to \tau_{\leq k} B_\bullet$ for $k \geq 0$ and using the fact that $\mathcal{X}^{\diamond}$ is a nilcomplete smooth sheaf reduces to the situation where $B_\bullet$ is a *discrete* graded commutative ring, so that $\mathcal{Y}$ is now a classical stack.

We can now complete the proof by applying the same reasoning again to the square-zero thickenings $B_{\geq -m} \to B_{\geq -m+1}$ for $m \geq 2$ and using Proposition A.2.4. ☐

Definition 4.11.4. We will say that $\mathcal{X} \to (\mathcal{Z}, \iota)$ is **graded integrable** if, for all maps of pointed graded stacks $(\operatorname{Spec} B_\bullet)/\mathbb{G}_m \to \mathcal{Z}$ with $B_\bullet$ non-positively graded, the conclusion of Proposition 4.11.3 holds.

Remark 4.11.5. We see that $\mathcal{X} \to (\mathcal{Z}, \iota)$ is graded integrable whenever $\mathcal{X}^\diamond \to \mathcal{Z}$ has quasi-affine diagonal. However, this condition on the diagonal was used only in the application of Proposition A.2.4.

For instance, if $\mathcal{X}^\diamond = \mathcal{P} \times B\mathbb{G}_m \to \mathcal{Z} = B\mathbb{G}_m$, where $\mathcal{P}$ is as in Example 4.7.2, then, even though $\mathcal{P}$ does not have quasi-affine diagonal over $\operatorname{Spec}\mathbb{Z}$, we still know that the map

$$\operatorname{Map}((\operatorname{Spec} B_\bullet)/\mathbb{G}_m, \mathcal{P}) = \operatorname{Perf}((\operatorname{Spec} B_\bullet)/\mathbb{G}_m) \to \varprojlim_m \operatorname{Perf}((\operatorname{Spec} B_{\geq -m})/\mathbb{G}_m) = \varprojlim_m \operatorname{Map}((\operatorname{Spec} B_{\geq -m})/\mathbb{G}_m, \mathcal{P})$$

is an equivalence. In other words, the 1-bounded stack $\mathcal{P}_{\{0,1\}} \to B\mathbb{G}_m$ from Example 4.8.6 is graded integrable.

Similarly, the 1-bounded stack from Example 4.8.5 is also graded integrable.

4.12. **A useful cartesian square.**

4.12.1. Suppose that $\operatorname{Fil}^\bullet S$ is a non-negatively filtered animated commutative ring, and set $\overline{S} = \operatorname{gr}^0 S$. The map $S \to \overline{S}$ underlies an arrow $\operatorname{Fil}^\bullet S \to \operatorname{Fil}^\bullet_{\mathrm{triv}} \overline{S}$ of filtered animated commutative rings corresponding to a map of stacks

$$\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} \overline{S} \to \mathcal{R}(\operatorname{Fil}^\bullet S)$$

whose restriction over the open point of $\mathbb{A}^1/\mathbb{G}_m$ is the closed immersion $\operatorname{Spec} \overline{S} \to \operatorname{Spec} S$.

We will view $\mathcal{Y} \overset{\mathrm{defn}}{=} \mathcal{R}(\operatorname{Fil}^\bullet S)$ as a pointed graded stack via the composition

$$B\mathbb{G}_m \times \operatorname{Spec} \overline{S} \to \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} \overline{S} \to \mathcal{Y}.$$

4.12.2. Let $\mathcal{X} = (\mathcal{X}^\diamond, X^0) \to (\mathcal{Z}, \iota)$ be a 1-bounded stack over a pointed graded prestack with $\iota : B\mathbb{G}_m \times \operatorname{Spec} R \to \mathcal{Z}$, and let $X^- \to \operatorname{Spec} R$ be its associated attractor.

Suppose that we have a map of pointed prestacks $\mathcal{Y} \to \mathcal{Z}$. We now have a commutative diagram

$$\begin{tikzcd} \operatorname{Map}_{/\mathcal{Z}}(\mathcal{Y}, \mathcal{X}) \arrow[r] \arrow[d] & \operatorname{Map}_{/\mathcal{Z}}(\operatorname{Spec} S, \mathcal{X}^\diamond) \arrow[d] \\ X^-(S) = \operatorname{Map}_{/\mathcal{Z}}(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} \overline{S}, \mathcal{X}) \arrow[r] & \operatorname{Map}_{/\mathcal{Z}}(\operatorname{Spec} \overline{S}, \mathcal{X}^\diamond). \end{tikzcd} \tag{4.12.2.1}$$

Proposition 4.12.3. *Suppose that the kernel of the map*

$$\pi_0(S) \to \pi_0(\overline{S})$$

is nilpotent and that $\mathcal{X}^\diamond \to \mathcal{Z}$ has quasi-affine diagonal. Then (4.12.2.1) *is a Cartesian square.*

Proof. Let $\operatorname{Fil}^\bullet_{(S \twoheadrightarrow \overline{S})} S$ be the non-negatively filtered animated commutative ring with $\operatorname{Fil}^0_{(S \twoheadrightarrow \overline{S})} S \simeq S$ and

$$\operatorname{gr}^i_{(S \twoheadrightarrow \overline{S})} S \simeq \begin{cases} \overline{S} & \text{if } i = 0; \\ 0 & \text{otherwise.} \end{cases}$$

The associated Rees algebra sits in a Cartesian square of graded animated commutative rings

$$\begin{tikzcd} \bigoplus_i \operatorname{Fil}^i_{(S \twoheadrightarrow \overline{S})} S \cdot t^{-i} \arrow[r] \arrow[d] & S[t, t^{-1}] \arrow[d] \\ \overline{S}[t] \arrow[r] & \overline{S}[t, t^{-1}]. \end{tikzcd} \tag{4.12.3.1}$$

One can obtain this construction for instance by animating the obvious one for surjections of polynomial algebras over $\mathbb{Z}$.

For any non-negatively filtered animated commutative ring $\operatorname{Fil}^\bullet A$, view $\mathcal{R}(\operatorname{Fil}^\bullet A)$ as a pointed graded prestack with

$$\iota : B\mathbb{G}_m \times \operatorname{Spec} \operatorname{gr}^0 A \to \mathcal{R}(\operatorname{Fil}^\bullet A).$$

Also, set

$$\mathrm{Fil}^\bullet A' = \mathrm{Fil}^\bullet_{(A \twoheadrightarrow \mathrm{gr}^0(A))} A,$$

where the right hand side is defined as above. If $\mathcal{R}(\mathrm{Fil}^\bullet A)$ is a pointed graded stack over $(\mathcal{Z}, \iota)$, set

$$\mathcal{X}(\mathrm{Fil}^\bullet A) \overset{\mathrm{defn}}{=} \mathrm{Map}_{/\mathcal{Z}}(\mathcal{R}(\mathrm{Fil}^\bullet A), \mathcal{X})$$

The diagram (4.12.2.1) is Cartesian with $\mathcal{R}(\mathrm{Fil}^\bullet S)$ replaced by $\mathcal{R}(\mathrm{Fil}^\bullet S')$. This is because of the Cartesian nature of (4.12.3.1), and the *surjectivity* of the vertical maps; see (4) of [46, Theorem 7.5.1].

Note that there is a natural map $\mathcal{R}(\mathrm{Fil}^\bullet S') \to \mathcal{R}(\mathrm{Fil}^\bullet S)$. To complete the proof of the proposition it now suffices to show that the corresponding map

$$\mathcal{X}(\mathrm{Fil}^\bullet S) \to \mathcal{X}(\mathrm{Fil}^\bullet S')$$

is an equivalence. We will prove this using filtered deformation theory.

Suppose quite generally that $\mathrm{Fil}^\bullet B \to \mathrm{Fil}^\bullet A$ is a square-zero extension of non-negatively filtered with fiber $\mathrm{Fil}^\bullet K$. If $\mathcal{R}(\mathrm{Fil}^\bullet B)$ is a pointed graded stack over $(\mathcal{Z}, \iota)$, by the discussion in §4.5, we obtain a Cartesian square

$$\begin{tikzcd}
\mathcal{X}(\mathrm{Fil}^\bullet B) \arrow[r] \arrow[d] & \mathcal{X}(\mathrm{Fil}^\bullet A) \times_{X^0(\mathrm{gr}^0 A)} X^0(\mathrm{gr}^0 B) \arrow[d] \\
\mathcal{X}(\mathrm{Fil}^\bullet A) \times_{X^0(\mathrm{gr}^0 A)} X^0(\mathrm{gr}^0 B) \arrow[r] & \mathcal{X}(\mathrm{Fil}^\bullet A \oplus \mathrm{Fil}^\bullet K[1]) \times_{X^0(\mathrm{gr}^0 A)} X^0(\mathrm{gr}^0 B).
\end{tikzcd}$$

Moreover, if $\mathbb{L}_{\mathcal{X}^\diamond/\mathcal{Y}}$ is the relative cotangent complex, then for any $x \in \mathcal{X}(\mathrm{Fil}^\bullet A)$, we obtain a filtered $\mathrm{Fil}^\bullet A$-module $\mathrm{Fil}^\bullet \mathbb{L}_{\mathcal{X}^\diamond/\mathcal{Y},x}$, and we have a canonical equivalence:

$$\begin{aligned}
\mathrm{fib}_x(\mathcal{X}(\mathrm{Fil}^\bullet A \oplus \mathrm{Fil}^\bullet K[1]) \to \mathcal{X}(\mathrm{Fil}^\bullet A)) &\simeq \mathrm{Map}_{\mathrm{Fil}^\bullet A}(\mathrm{Fil}^\bullet \mathbb{L}_{\mathcal{X}^\diamond/\mathcal{Y},x}, \mathrm{Fil}^\bullet K[1]) \\
&\simeq \tau^{\leq 0} \mathrm{Fil}^0 \left(\mathrm{Fil}^\bullet \mathbb{T}_{\mathcal{X}^\diamond/\mathcal{Y},x} \otimes_{\mathrm{Fil}^\bullet A} \mathrm{Fil}^\bullet K[1]\right).
\end{aligned}$$

Now, the associated graded of the filtered tensor product on the right is $\mathrm{gr}^\bullet \mathbb{T}_{\mathcal{X}^\diamond/\mathcal{Y},x} \otimes_{\mathrm{gr}^\bullet A} \mathrm{gr}^\bullet K[1]$. By Lemma A.2.3, it admits a further filtration with associated gradeds isomorphic to

$$\mathrm{gr}^i \overline{\mathbb{T}}_{\mathcal{X}^\diamond/\mathcal{Y},x}(i) \otimes_{\mathrm{gr}^0 A} \mathrm{gr}^\bullet K[1],$$

where the corresponding degree j component is $\mathrm{gr}^i \overline{\mathbb{T}}_{\mathcal{X}/\mathcal{Y},x} \otimes_{\mathrm{gr}^0 A} \mathrm{gr}^{j-i} K[1]$. Our hypothesis of 1-boundedness now tells us that the first factor in this tensor product is zero for $i < -1$, while the second factor is zero for $j < i$. Using this, one finds that there is a fiber sequence

$$\mathrm{Fil}^0 \left(\mathrm{Fil}^\bullet \mathbb{T}_{\mathcal{X}^\diamond/\mathcal{Y},x} \otimes_{\mathrm{Fil}^\bullet A} \mathrm{Fil}^\bullet K[1]\right) \to \mathbb{T}_{\mathcal{X}^\diamond/\mathcal{Y},x} \otimes_A K[1] \to \mathrm{gr}^{-1} \overline{\mathbb{T}}_{\mathcal{X}^\diamond/\mathcal{Y},x} \otimes_{\mathrm{gr}^0 A} \mathrm{gr}^0 K[k+1].$$

In particular, this tells us that the first object in the sequence depends only on $\mathrm{Fil}^1 A \to A$. From this, one deduces that, if $\mathcal{X}(\mathrm{Fil}^\bullet A) \to \mathcal{X}(\mathrm{Fil}^\bullet A')$ is an isomorphism, then so is $\mathcal{X}(\mathrm{Fil}^\bullet B) \to \mathcal{X}(\mathrm{Fil}^\bullet B')$.

For every k, we have the truncated filtered animated commutative ring $\tau_{\leq k}(\mathrm{Fil}^\bullet S)$ obtained by taking the corresponding truncation for the associated Rees algebra: this is a square-zero extension of $\tau_{\leq (k-1)} \mathrm{Fil}^\bullet S$ by a filtered module $\pi_k(\mathrm{Fil}^\bullet S)[k]$. Via the deformation argument from the previous paragraph, induction on k, and using nilcompleteness, one therefore reduces to showing that the map

$$\mathcal{X}(\pi_0(\mathrm{Fil}^\bullet S)) \to \mathcal{X}(\pi_0(\mathrm{Fil}^\bullet S'))$$

is an isomorphism

So we can assume that S is discrete. For $m \geq 1$, let $\mathrm{Fil}^\bullet S_m$ be the induced filtration on the ring $S_m = S/(\mathrm{im}(\mathrm{Fil}^1 S))^m$. Since $\mathrm{Fil}^1 S$ is nilpotent by hypothesis, we also have $\mathrm{Fil}^\bullet S = \mathrm{Fil}^\bullet S_m$ for m sufficiently large. Therefore, using the deformation argument again reduces us to the case $S = S_1$, where the map $\mathrm{Fil}^1 S \to S$ is 0. Using the argument from the previous paragraph, we can assume that $\mathrm{Fil}^i S$ is discrete for all $i \geq 0$ and that the map $\mathrm{Fil}^1 S \to S$ is identically 0.

In the notation of (A.2.5), we now have $\mathrm{Fil}^\bullet_{(1)} S = \mathrm{Fil}^\bullet_{\mathrm{triv}} S = \mathrm{Fil}^\bullet_{(1)} S'$, and $\mathrm{Fil}^\bullet_{(m)} S' = \mathrm{Fil}^\bullet S'$ for $m \geq 2$. Therefore, the deformation theory argument above tells us that the proposition holds with $\mathrm{Fil}^\bullet S$ replaced with $\mathrm{Fil}^\bullet_{(m)} S$ for any $m \geq 1$. We now conclude using Proposition A.2.6. □

Remark 4.12.4. The above result can be viewed as a generalization of [42, Remark 6.3.3].

Definition 4.12.5. We will say that $\mathcal{X} \to (\mathcal{Z}, \iota)$ is **filtered integrable** if, for all maps of pointed graded stacks $\mathcal{R}(\mathrm{Fil}^\bullet S) \to \mathcal{Z}$ with $\pi_0(S) \to \pi_0(\overline{S})$ having nilpotent kernel the square (1.2.0.1) is Cartesian.

Remark 4.12.6. Proposition 4.12.3 tells us that $\mathcal{X}$ is filtered integrable whenever $\mathcal{X}^\diamond \to \mathcal{Z}$ has quasi-affine diagonal. However, this condition was used only in the application of Proposition A.2.6.

For instance, if $\mathcal{X} = \mathcal{P}_{\{0,1\}} \to B\mathbb{G}_m$ with $\mathcal{X}^\diamond = \mathcal{P} \times B\mathbb{G}_m$, then we still know that the map

$$\mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet S), \mathcal{P}^\circ) = \varprojlim_m \mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet_{(m)} S), \mathcal{P}^\circ)$$

is an equivalence. Therefore, $\mathcal{X}$ is filtered integrable.

Similarly, the 1-bounded stack from Example 4.8.5 is also filtered integrable.

5. Animated higher frames and windows

The purpose of this section is to give an account of the theory of [42] in an animated context. This will used in §9.2 to give a somewhat concrete description of the values of the stacks of interest in this paper on semiperfectoid inputs. We will also give an account of deformation theory in this context, which will later lead to an efficient proof of Theorem G.

5.1. Generalized Cartier divisors.

Definition 5.1.1. Recall that a **generalized Cartier divisor** for an animated commutative ring R is a surjective map $R \twoheadrightarrow \overline{R}$ whose homotopy kernel I is an invertible R-module. By abuse of notation we will refer to such an object via the cosection $s : I \to R$, which is the same as a map $s : \mathrm{Spec}\, R \to \mathbb{A}^1/\mathbb{G}_m$.

5.1.2. Any generalized Cartier divisor lifts R to a filtered animated commutative ring $\mathrm{Fil}^\bullet_I R$ where the filtration is the I-adic one given by

$$\mathrm{Fil}^k_I R = \begin{cases} I^{\otimes k} & \text{if } k \geq 0 \\ R & \text{if } k < 0, \end{cases}$$

and the transition maps are the identity for $k \leq 0$ and given by

$$I^{\otimes k} \simeq I \otimes_R I^{\otimes (k-1)} \xrightarrow{s \otimes 1} R \otimes_R I^{\otimes (k-1)} \simeq I^{\otimes (k-1)}$$

for $k > 0$. We will also have occasion to consider the **two-sided I-adic filtration** given by $\mathrm{Fil}^k_{I,\pm} R = I^{\otimes k}$ for all $k \in \mathbb{Z}$, which once again underlies a filtered animated commutative ring with

$$\mathcal{R}(\mathrm{Fil}^\bullet_{I,\pm} R) \simeq \mathrm{Spec}\, R.$$

To verify the assertions in the previous paragraph, using the classifying map $s : \mathrm{Spec}\, R \to \mathbb{A}^1/\mathbb{G}_m$, one reduces everything to the case where $R = \mathbb{Z}[x]$ with $I = x\mathbb{Z}[x]$, and here everything can be checked explicitly.

Remark 5.1.3. This also gives a concrete way of thinking of a point $\mathrm{Spec}\, R \to \mathcal{R}(\mathrm{Fil}^\bullet S)$ of the Rees stack corresponding to a filtered commutative ring $\mathrm{Fil}^\bullet S$: it is equivalent to giving a generalized Cartier divisor $I \to R$, along with a map of filtered animated commutative rings $\mathrm{Fil}^\bullet S \to \mathrm{Fil}^\bullet_{I,\pm} R$.

5.1.4. For any $M \in \mathrm{Mod}_R$, we will set $M[I^{-1}] = \mathrm{colim}_{k \geq 0} I^{-k} \otimes_R M$, where the transition maps are induced by s.

When we have an isomorphism $R \xrightarrow{\simeq} I$ of R-modules given by a section ξ of I, we will write $\mathrm{Fil}^\bullet_\xi R$ and $\mathrm{Fil}^\bullet_{\xi,\pm} R$ for these filtered rings.

For any R-module M, we will write $M/^{\mathbb{L}}(p, I)$ for $M/^{\mathbb{L}} p \otimes_R \overline{R}$.

If $R' \in \mathrm{CRing}_{R/}$ is an R-algebra, and $s' : I' = R' \otimes_R I \xrightarrow{1 \otimes s} R'$, then we will sometimes also denote the I'-adic filtrations on R' by $\mathrm{Fil}^\bullet_I R'$ and $\mathrm{Fil}^\bullet_{I,\pm} R'$.

5.2. **Formal Rees stacks.** Suppose that $I \to A$ is a generalized Cartier divisor with p-complete quotient $\overline{A}$, and suppose that A underlies a filtered animated commutative ring $\mathrm{Fil}^\bullet A$.

Definition 5.2.1. The **formal Rees stack** associated with this datum is the one associating with each $R \in \mathrm{CRing}^{p\text{-nilp}}$ the space of generalized Cartier divisors $J \to R$ along with maps $\mathrm{Fil}^\bullet A \to \mathrm{Fil}^\bullet_{J,\pm} R$ (see Remark 5.1.3) such that the underlying map $A \to R$ is in $\mathrm{Spf}(A, I)(R)$.

In the sequel, the Rees construction will only be appealed to in this formal context. Therefore, by abuse of notation, we will denote this formal stack once again by $\mathcal{R}(\mathrm{Fil}^\bullet A)$.

In particular, we have $\mathcal{R}(\mathrm{Fil}^\bullet_{I,\pm} A) \simeq \mathrm{Spf}(A, I)$.

5.3. **Witt vectors, δ-rings and prisms.**

5.3.1. We recall the notion of an animated δ-ring from [9, App. A]: First, one defines for any animated commutative ring R the level-2 Witt ring $W_2(R)$ with underlying space R^2 such that the projection onto the first coordinate is a map of animated commutative rings $W_2(R) \twoheadrightarrow R$. This amounts to the observation that the functor $C \mapsto W_2(C)$ on discrete commutative rings is represented by a smooth ring scheme, and so extends canonically to an animated ring scheme equipped with a map to $\mathbb{G}_a$.

Now, a **δ-structure** on an animated ring R is a section of the natural map $W_2(R) \to R$. Such a section is determined completely by its projection onto the second coordinate, which yields an operator $\delta : R \to R$ satisfying certain properties. When R is flat over $\mathbb{Z}_{(p)}$, giving such a δ is equivalent to giving a lift $\varphi : R \to R$ of the mod-p Frobenius satisfying $\varphi(r) = r^p + p\delta(r)$ for any $r \in R$. This interpretation (though perhaps not the explicit formula) is valid for all R if we treat Frobenius as an endomorphism of $R/^{\mathbb{L}}p$; see [9, Remark A.15].

An (animated) **δ-ring** is an animated ring R equipped with a δ-structure. We obtain an ∞-category CRing_δ of animated δ-rings in the usual fashion.

5.3.2. The forgetful functor from $\mathrm{CRing}_\delta \to \mathrm{CRing}$ admits both a left and right adjoint: The former is the free δ-ring functor and the latter is the Witt functor $A \mapsto W(A)$.

We will need a basic flatness result about the free δ-ring over $\mathbb{Z}_{(p)}$ in one variable: Let $\mathbb{Z}_{(p)}\{x\}$ be the free δ-ring obtained by hitting $\mathbb{Z}_{(p)}[x]$ with the left adjoint above, and let $\varphi : \mathbb{Z}_{(p)}\{x\} \to \mathbb{Z}_{(p)}\{x\}$ be the Frobenius lift. The next result follows from [11, Lemma 2.11]

Lemma 5.3.3. *The map $\mathbb{Z}_{(p)}\{x\} \xrightarrow{x \mapsto \varphi(x)} \mathbb{Z}_{(p)}\{x\}$ is faithfully flat.*

5.3.4. Following [9, Def. 2.4], we define a(n animated) **prism** to be an animated δ-ring A equipped with a generalized Cartier divisor $I \to A$ with quotient $\overline{A}$ such that the following conditions hold:

(1) A is (p, I)-complete.
(2) Given a perfect field k of characteristic p and a map $A \to W(k)$ of δ-rings, we have $W(k) \otimes_A \overline{A} \simeq k$.

If we want to emphasize the Cartier divisor, we will sometimes also denote the prism by $(A, s : I \to A)$.

A prism (A, I) is **transversal** if A is flat over $\mathbb{Z}_p$ and the map $I \to A$ is injective mod-p.

A prism (A, I) is **perfect** if the Frobenius lift $\varphi : A \to A$ is an isomorphism. In this case, $A/^{\mathbb{L}}p$ is a perfect $\mathbb{F}_p$-algebra and so is discrete; this implies that A is also discrete and p-torsion free. It is a result of Bhatt-Scholze [11, Theorem 3.10] that the assignment $(A, I) \mapsto \overline{A}$ is an equivalence of categories between perfect prisms and perfectoid rings. The inverse carries a perfectoid ring R to the perfect prism $(A_{\mathrm{inf}}(R), \ker\theta)$ where $A_{\mathrm{inf}}(R) = W(R^\flat)$ with $R^\flat$ the tilt of R and $\theta : A_{\mathrm{inf}}(R) \to R$ is the usual map.

5.3.5. Associated with any prism (A, I) is a canonical invertible module $A\{1\}$ over A constructed in [8, Proposition 2.5.1] and characterized by the following properties:

(1) ([8, (2.2.11)]) For any *transversal* prism (A, I), we have
$$A\{1\} = \varprojlim_k I_k/I_{k+1},$$
where $I_k = I\varphi_A^*(I) \cdots (\varphi_A^{k-1})^*(I)$, and the maps $I_k/I_{k+1} \to I_{k-1}/I_k$ are induced by dividing the natural maps by p.

(2) ([8, (2.5.3)]) For any map of prisms $(A,I)\to(B,J)$, there is a canonical isomorphism $B\otimes_A A\{1\}\xrightarrow{\simeq} B\{1\}$.

By [8, Remark 2.5.9], we have a canonical isomorphism $I\otimes_A\varphi^*(A\{1\})\xrightarrow{\simeq} A\{1\}$.

For any $M\in\mathrm{Mod}_A$ and $i\in\mathbb{Z}$, we will set $M\{i\}\overset{\text{defn}}{=}M\otimes_A A\{1\}^{\otimes i}$.

5.4. **Animated higher frames.**

Definition 5.4.1. An (animated higher) **frame** $\underline{A}$ is a tuple $(\mathrm{Fil}^\bullet A, I\xrightarrow{s}A,\Phi,A\{1\})$, where:

1. $\mathrm{Fil}^\bullet A$ is a non-negatively filtered (p,I)-complete animated commutative ring;
2. $s:I\to A$ is a generalized Cartier divisor;
3. $\Phi:\mathrm{Fil}^\bullet A\to\mathrm{Fil}^\bullet_I A$ is a map of filtered animated commutative rings such that the underlying endomorphism $\varphi:A\to A$ of animated commutative rings is a 'naïve' Frobenius lift in the sense that the induced endomorphism of $\pi_0(A)/p\pi_0(A)$ is Frobenius;
4. $A\{1\}$ is an invertible A-module equipped with an isomorphism
$$I\otimes_A\varphi^*A\{1\}\xrightarrow{\simeq}A\{1\}.$$

Frames organize into an ∞-category in the obvious way.

5.4.2. Given a frame $\underline{A}$, we will write R_A for the p-complete animated commutative ring $\mathrm{gr}^0 A$ and $\overline{A}$ for the quotient of $I\xrightarrow{s}A$.

Let $\mathrm{Fil}^\bullet_{I,\pm}A$ be the two-sided I-adic filtration on A; then we obtain a map
$$\Phi_\pm:\mathrm{Fil}^\bullet A\to\mathrm{Fil}^\bullet_{I,\pm}A,$$
which restricts to Φ in non-negative degrees, and which in filtered degree $-i$ (for $i\in\mathbb{Z}_{>0}$) is given by $s^{-i}\circ\varphi$.

Write $\varphi_i:\mathrm{Fil}^i A\to I^{\otimes i}$ for the filtered degree-i component of Φ.

For any $M\in\mathrm{Mod}_A$ and $i\in\mathbb{Z}$, set $M\{i\}=M\otimes_A A\{1\}^{\otimes i}$.

Definition 5.4.3. If $(I\xrightarrow{s}A)=(A\xrightarrow{p}A)$ with $A\{1\}\simeq A$ we will say that $\underline{A}$ is a **p-adic frame**.

Since $A\{1\}$ and $I\to A$ are superfluous here, we will denote a p-adic frame by a tuple $(A,\mathrm{Fil}^\bullet A,\Phi)$.

Remark 5.4.4. Our definition of an animated frame is inspired by the definition of Lau in [42, §2], but allows for more general objects. In fact, our notion of a p-adic frame is quite close to being the natural animated generalization of Lau's definition.

Indeed, suppose that we have a p-adic frame $\underline{A}$ such that A is a discrete $\mathbb{Z}_p$-algebra, and such that $\mathrm{Fil}^i A$ is also discrete for all $i\geq 0$. Then we obtain the graded Rees ring
$$S(\underline{A})\overset{\text{defn}}{=}\mathrm{Rees}(\mathrm{Fil}^\bullet A)=\bigoplus_{i=0}\mathrm{Fil}^i A\cdot t^{-i}$$
along with maps $\tau,\sigma:S(\underline{A})\to A$. Here, τ is given by
$$\tau:S(\underline{A})\to S(\underline{A})/(t-1)\simeq A$$
and σ as the composition
$$S(\underline{A})\xrightarrow{\mathrm{Rees}(\Phi_\pm)}\mathrm{Rees}(\mathrm{Fil}^\bullet_{p,\pm}A)\to\mathrm{Rees}(\mathrm{Fil}^\bullet_{p,\pm}A)/(t-p)\simeq A.$$
The triple $(S(\underline{A}),\sigma,\tau)$ is—after a sign change in the graded degrees—a (higher) frame as defined by Lau [42, Definition 2.0.1]

Remark 5.4.5. One can somewhat reverse this process: Suppose that (S,σ,τ) is a higher frame in the sense of Lau. Then one obtains an animated filtered commutative ring $\mathrm{Fil}^\bullet S_0$ with underlying ring S_0 and filtration given by $\mathrm{Fil}^i S_0=S_i\xrightarrow{\tau_i}S_0$. Note that this is in general *not* a filtered ring in the classical sense, since the S_0-modules $\mathrm{Fil}^i S_0$ are not necessarily ideals in S_0. This is the case for the truncated Witt frame from [42, Example 2.1.6].

As explained in [42, Remark 2.0.2], $\sigma_0:S_0\to S_0$ is a lift of the mod-p Frobenius on S/pS, and σ_i gives a map $\mathrm{Fil}^i S_0\to S_0$ such that $p\sigma_{i+1}=\sigma_i|_{\mathrm{Fil}^{i+1}S_0}$ for all $i\geq 0$. This means precisely that these maps organize into a map $\Phi:\mathrm{Fil}^\bullet S_0\to\mathrm{Fil}^\bullet_p S_0$ of filtered animated commutative rings.

If we now assume that each S_i is derived p-complete, then we have recovered our notion of a p-adic frame.

Remark 5.4.6. As noted in the previous remarks, our definition is a closely related to that of Lau if we restrict to p-adic frames. To motivate our more general definition, we need to look ahead to Lemma 6.11.4 and Theorem 6.11.5 below. These results show that the Nygaard filtered prismatization and syntomification of a semiperfectoid ring can be described in terms of an animated higher frame—as defined here—obtained from its Nygaard filtered prismatic cohomology. Thus, we will be able to apply the theory from this section to study objects living over the stacks that will appear in Section 6.

Definition 5.4.7. A frame $\underline{A}$ is **prismatic** if the following conditions hold:

- The pair (A, I) is a prism.
- The endomorphism $\varphi : A \to A$ is the one obtained from the underlying δ-ring structure on A.
- The invertible module $A\{1\}$ and the datum of the isomorphism $I \otimes_A \varphi^* A\{1\} \simeq A\{1\}$ are the canonical ones from (5.3.5).

Since the datum of $A\{1\}$ is superfluous here, we will denote a prismatic prism by the tuple $(A, I \to A, \mathrm{Fil}^\bullet A, \Phi)$. Note that any p-adic frame whose naïve Frobenius lift underlies a δ-ring structure is automatically prismatic.

Example 5.4.8 (The Witt frames)**.** Suppose that we have $R \in \mathrm{CRing}^{p\text{-comp}}$. If R is discrete, since $W(R)$ is a derived p-complete δ-ring, putting Remark 5.4.5 together with [42, Example 2.1.3] gives us the **Witt frame** $\underline{W(R)}$ associated with R. More generally, by applying this to the p-completed ring of functions of the Witt scheme W, we see that $W(R)$ underlies a p-adic frame $\underline{W(R)}$ for any p-complete animated commutative ring R. It is a p-adic frame and the filtration $\mathrm{Fil}^\bullet_{\mathrm{Lau}} W(R)$ is given by

$$\mathrm{Fil}^i_{\mathrm{Lau}} W(R) = \begin{cases} W(R) & \text{if } i \leq 0; \\ F_* W(R) & \text{if } i \geq 1, \end{cases}$$

with transition map $\mathrm{Fil}^1_{\mathrm{Lau}} W(R) = F_* W(R) \to W(R)$ given by the Verschiebung map, and that in higher degrees given by $F_* W(R) \xrightarrow{p} F_* W(R)$. The filtered Frobenius map is obtained in filtered degree 0 via the fact that we have $FV = p$, and is the identity in filtered degrees $i \geq 1$.

Note that this frame is prismatic.

Example 5.4.9 (The truncated Witt frames)**.** When R is an $\mathbb{F}_p$-algebra, in [42, Example 2.1.6], Lau also defines the *truncated* Witt frames with underlying ring $W_n(R)$ for $n \geq 1$. In particular, when $n = 1$, we obtain the **zip frame** from Example 2.1.7 of *loc. cit.*. Similarly to the previous example, Remark 5.4.5 now also gives us p-adic frames $\underline{W_n(R)}$; however, they will no longer be prismatic.

The Witt frames have some useful universal properties among p-adic frames.

Lemma 5.4.10. *Suppose that $\underline{A}$ is a p-adic frame.*

(1) If $\underline{A}$ is prismatic, then there is a canonical map of frames $\underline{A} \to \underline{W(R_A)}$.
(2) If R_A is an $\mathbb{F}_p$-algebra, then there is a canonical map of frames

$$\underline{A} \otimes \mathbb{F}_p \to \underline{W_1(R_A)}.$$

Proof. This uses the following description of the filtration on the Witt frame $\underline{W(R)}$: We have a Cartesian square of filtered animated commutative rings

$$\begin{tikzcd} \mathrm{Fil}^\bullet_{\mathrm{Lau}} W(R) \arrow[r] \arrow[d] & \mathrm{Fil}^\bullet_{\mathrm{triv}} R \arrow[d] \\ \mathrm{Fil}^\bullet_p F_* W(R) \arrow[r] & \mathrm{Fil}^\bullet_{\mathrm{triv}} F_* W(R)/^{\mathbb{L}} p, \end{tikzcd}$$

where the right vertical arrow is obtained from the map $R \to F_* W(R)/^{\mathbb{L}} p$ induced by the Frobenius lift $F : W(R) \to F_* W(R)$. There is a similar description of the truncated Witt frame $\underline{W_n(R)}$ with $F_* W(R)$ replaced by $F_* W_n(R)$.

If $\underline{A}$ is prismatic, then the δ-ring structure on A yields a map $\lambda_A : A \to W(R_A)$, and there is a canonical map $\mathrm{Fil}^\bullet A \to \mathrm{Fil}^\bullet_{\mathrm{Lau}} W(R_A)$ induced by

$$\mathrm{Fil}^\bullet A \xrightarrow{(\Phi,\mathrm{can})} \mathrm{Fil}^\bullet_p \varphi_* A \times_{\mathrm{Fil}^\bullet_{\mathrm{triv}} \varphi_* \overline{A}} \mathrm{Fil}^\bullet_{\mathrm{triv}} R_A \xrightarrow{(\mathrm{Fil}^\bullet_p \lambda_A,\mathrm{id})} \mathrm{Fil}^\bullet_p F_* W(R_A) \times_{\mathrm{Fil}^\bullet_{\mathrm{triv}} F_* W(R)/^{\mathbb{L}} p} \mathrm{Fil}^\bullet_{\mathrm{triv}} R_A.$$

This proves part (1), and the same argument also shows assertion (2). □

Example 5.4.11 (The A_{inf} frame)**.** Let (A, I) be a perfect prism with $R \overset{\mathrm{defn}}{=} A/I$ perfectoid. Then $\varphi^{-1}(I) \subset A$ is a principal ideal, and φ is naturally promoted to a map of filtered rings

$$\Phi : \mathrm{Fil}^\bullet_{\varphi^{-1}(I)} A \to \mathrm{Fil}^\bullet_I A.$$

The tuple $(A, I \to A, \mathrm{Fil}^\bullet_{\varphi^{-1}(I)} A, \Phi)$ now gives us a prismatic frame with $R_A \simeq R$. When $p = 0 \in R$, $A = A_{\mathrm{inf}}(R) = W(R)$ with $I = pW(R)$, and this gives a special case of Example 5.4.8.

Example 5.4.12 (Breuil-Kisin frames)**.** Take (A, I') to be a p-completely flat prism with $I' \subset A$ an ideal, and let $I = \varphi(I') \subset A$. Taking $\mathrm{Fil}^i A = \mathrm{Fil}^i_{I'} A$ and taking $A\{1\}$ to be trivial completes (A, φ) to a prismatic frame. When $A = W(k)[|u|]$ for a perfect field k with $\varphi(u) = u^p$, and $I' = (E(u))$ is generated by an Eisenstein ideal, this appears in the classical Breuil-Kisin theory.

Remark 5.4.13. For future reference, let us record some basic operations on frames:

1. (Base-change) Suppose that $\underline{A}$ is a frame and that $\varphi_A : A \to A$ is the underying naïve Frobenius lift. If $(A, \varphi_A) \to (B, \varphi_B)$ is a map of animated commutative p-complete rings equipped with naïve Frobenius lifts, then one sees that the tuple $(B \otimes_A \mathrm{Fil}^\bullet A, B \otimes_A I \to B, B \otimes_A A\{1\})$ underlies a canonical frame $B \otimes_A \underline{A}$.
2. (Reduction-mod-p^n) Applying this with $B = A/^{\mathbb{L}} p^n$ with its induced naïve Frobenius lift shows that we have a canonical frame $\underline{A/^{\mathbb{L}} p^n}$ obtained by reducing the original one mod p^n.
3. (Postnikov tower) For $k \geq 0$, there is a canonical frame structure $\underline{\tau_{\leq k} A}$ obtained by taking the k-truncations of all the defining data.

5.4.14. Let $\mathrm{Spf}\, A \overset{\mathrm{defn}}{=} \mathrm{Spf}(A, I)$ be the p-adic formal scheme obtained from A with its (p, I)-adic topology. If $\mathcal{R}(\mathrm{Fil}^\bullet A) \to \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spf}\, \mathbb{Z}_p$ is the associated formal Rees stack over $\mathrm{Spf}\, \mathbb{Z}_p$ as in §5.2, we obtain two maps

$$\tau, \sigma : \mathrm{Spf}\, A \hookrightarrow \mathcal{R}(\mathrm{Fil}^\bullet A)$$

as follows:

- τ is obtained by pulling back the open point

$$\mathbb{G}_m/\mathbb{G}_m \times \mathrm{Spf}\, \mathbb{Z}_p \hookrightarrow \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spf}\, \mathbb{Z}_p.$$

  It is in particular, an open immersion.
- σ is obtained as the composition

$$\mathrm{Spf}\, A \xrightarrow{\simeq} \mathcal{R}(\mathrm{Fil}^\bullet_{I,\pm} A) \xrightarrow{\mathcal{R}(\Phi_\pm)} \mathcal{R}(\mathrm{Fil}^\bullet A).$$

Remark 5.4.15. We also have a canonical map $\pi : \mathcal{R}(\mathrm{Fil}^\bullet A) \to \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spf}\, A$ arising from the map of filtered rings $\mathrm{Fil}^\bullet_{\mathrm{triv}} A \to \mathrm{Fil}^\bullet A$. From it, we obtain a map $\mathcal{R}(\mathrm{Fil}^\bullet A) \to \mathrm{Spf}\, A$ whose composition with τ is the identity, while its composition with σ is the endomorphism of $\mathrm{Spf}\, A$ obtained from the Frobenius lift φ.

Remark 5.4.16. The restriction of the map τ to $\mathrm{Spec}\, R_A$ extends along the open immersion

$$\mathbb{G}_m/\mathbb{G}_m \times \mathrm{Spec}\, R_A \hookrightarrow \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R_A$$

to a map $\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R_A \to \mathcal{R}(\mathrm{Fil}^\bullet A)$. On the level of filtered rings this corresponds to the map $\mathrm{Fil}^\bullet A \to \mathrm{Fil}^\bullet_{\mathrm{triv}} R_A$.

In particular, by restricting this last map to $B\mathbb{G}_m \times \mathrm{Spec}\, R_A$, we can view $\mathcal{R}(\mathrm{Fil}^\bullet A)$ as a pointed graded stack in the sense of Definition 4.8.1.

Before we state the next result, recall the following:

Definition 5.4.17. We will say that a map $B \to C$ in CRing is **Henselian** if $\pi_0(B) \to \pi_0(C)$ is surjective, and if $(\pi_0(B), \ker(\pi_0(B) \to \pi_0(C)))$ is a Henselian pair as defined for instance in [62, Tag 09XD].

The next result follows from [25, Thm. A.0.4].

Proposition 5.4.18. *Suppose that $B \to C$ is a Henselian map in $\mathrm{CRing}_{\mathbb{Z}_p/}$. Then, for any quasi-compact smooth algebraic stack X over $\mathbb{Z}_p$ with affine diagonal, the map $X(B) \to X(C)$ has non-empty fibers.*

Proposition 5.4.19. *Suppose that $\underline{A}$ is a (prismatic) frame. Then $A \twoheadrightarrow R_A$ is Henselian. Moreover, every p-completely étale map $R_A \to R_{A'}$ lifts uniquely to a (p, I)-completely étale map $A \to A'$[10], where A' underlies a (prismatic) frame $\underline{A}'$ uniquely determined by the fact that $\mathrm{Fil}^\bullet A' = \mathrm{Fil}^\bullet A \otimes_A A'$.*

Proof. This is an animated variant of [42, Lemma 4.2.3].

Let us check that $A \twoheadrightarrow R_A$ is Henselian. We follow the argument from [1, Lemma 4.1.28]. Since A is (p, I)-adically complete, it is enough to check that $\pi_0(A/^{\mathbb{L}}(p, I)) \to \pi_0(R_A/^{\mathbb{L}}(p, I))$ is Henselian, which is true since its kernel is locally nilpotent; indeed, our hypotheses imply that it is annihilated by the p-power Frobenius.

In fact, this argument also proves the assertion on lifting p-completely étale maps to (p, I)-completely étale ones; see [62, Tag 0ALI].

Similarly, we also have an endomorphism $\varphi' : A' \to A'$ extending $\varphi : A \to A$ and lifting the Frobenius endomorphism of $\pi_0(A')/p\pi_0(A')$. The corresponding filtered map $\Phi' : \mathrm{Fil}^\bullet A' \to \mathrm{Fil}^\bullet_I A'$ is now given by

$$\mathrm{Fil}^\bullet A' \simeq A' \otimes_A \mathrm{Fil}^\bullet A' \xrightarrow{\varphi' \otimes \Phi} A' \otimes_A \mathrm{Fil}^\bullet_I A' \simeq \mathrm{Fil}^\bullet_I A'.$$

If $\underline{A}$ is prismatic, we can interpret δ-ring structures on A' as sections $A' \to W_2(A')$, and the (p, I)-complete étaleness of A' over A guarantees that there exists a unique (up to homotopy) such section lifting the corresponding one for A. ∎

5.5. **(G, μ)-windows over higher frames.** Now, suppose that we are in the situation of §4.9, so that G is a smooth group scheme over $\mathbb{Z}_p$, $\mathcal{O}$ is the ring of integers in a finite unramified extension of $\mathbb{Z}_p$ and $\mu : \mathbb{G}_{m,\mathcal{O}} \to G_{\mathcal{O}}$ is a cocharacter.

5.5.1. Let k be the residue field of $\mathcal{O}$. If $\underline{A}$ is a frame such that R_A lifts to $\mathrm{CRing}_{k/}$, then A lifts canonically to $\mathrm{CRing}_{\mathcal{O}/}$: Lift the map $\mathcal{O} \to R_A/^{\mathbb{L}}(p, I)$ first to $\mathcal{O} \to A/^{\mathbb{L}}(p, I)$ using local nilpotence and the formal étaleness of $\mathcal{O}$, and then to A by (p, I)-completeness.

We will view $\mathcal{R}(\mathrm{Fil}^\bullet A)$ as living over the stack $B\mathbb{G}_m$ via the line bundle associated with the filtered module

$$\mathrm{Fil}^\bullet A\{1\} \overset{\mathrm{defn}}{=} \mathrm{Fil}^\bullet A(1) \otimes_A A\{1\}.$$

Note that the restriction of this line bundle along τ corresponds to the A-module $A\{1\}$, while that along σ corresponds to $I \otimes_A \varphi^* A\{1\} \simeq A\{1\}$. Therefore, if we take the structure map $\mathrm{Spf}\, A \to B\mathbb{G}_m$ classifying the line bundle associated with $A\{1\}$, both σ, τ can be viewed as maps of stacks over $B\mathbb{G}_m$.

Proposition 5.5.2. *The following are equivalent for an fpqc $G\{\mu\}$-torsor $\mathcal{Q}^\mu$ over $\mathcal{R}(\mathrm{Fil}^\bullet A) \otimes \mathbb{Z}/p^n\mathbb{Z}$.*

- *(1) $\mathcal{Q}^\mu$ is trivial étale-locally on $\mathrm{Spf}\, R_A$. That is, there exists a p-completely étale map $R_A \to R_{A'}$ such that the restriction of $\mathcal{Q}^\mu$ over $\mathcal{R}(\mathrm{Fil}^\bullet A') \otimes \mathbb{Z}/p^n\mathbb{Z}$ is trivial.*
- *(2) The restriction of $\mathcal{Q}^\mu$ to $\mathcal{R}(\mathrm{Fil}^\bullet A)_{(t=0)} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is trivial étale-locally on $\mathrm{Spf}\, R_A$.*
- *(3a) The restriction of $\mathcal{Q}^\mu$ over $\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R_A/^{\mathbb{L}}p^n$ is trivial étale-locally on $\mathrm{Spf}\, R_A$.*
- *(3b) For every geometric point $R_A \to \kappa$ of $\mathrm{Spf}\, R_A$, the restriction of $\mathcal{Q}^\mu$ over $\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, \kappa$ is trivial.*
- *(4a) The restriction of $\mathcal{Q}^\mu$ over $B\mathbb{G}_m \times \mathrm{Spec}\, R_A/^{\mathbb{L}}p^n$ is trivial étale-locally on $\mathrm{Spf}\, R_A$.*
- *(4b) For every geometric point $R_A \to \kappa$ of $\mathrm{Spf}\, R_A$, the restriction of $\mathcal{Q}^\mu$ over $B\mathbb{G}_m \times \mathrm{Spec}\, \kappa$ is trivial.*

If $\mathrm{Spf}\, R_A$ is connected, then this is also equivalent to: For some *geometric point $R_A \to \kappa$ of $\mathrm{Spf}\, R_A$, the restriction of $\mathcal{Q}^\mu$ over $B\mathbb{G}_m \times \mathrm{Spec}\, \kappa$ is trivial.*

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[10]By this, we mean that A' is (p, I)-complete, and $A/^{\mathbb{L}}(p, I) \to A'/^{\mathbb{L}}(p, I)$ is étale.

Proof. The equivalence of (3a), (3b), (4a) and (4b) follows from Lemma 4.9.6 and Remark 4.9.7, as does the equivalence of these statements with the last unnumbered assertion.

We can finish by showing (3a)⇒(1). Since the stacks involved are (p, I)-complete, it is enough to know that every section of the smooth relative scheme $\mathcal{Q}^\mu$ over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R_A/^{\mathbb{L}}(p, I)$ can be lifted to a section over $\mathcal{R}(\operatorname{Fil}^\bullet A/^{\mathbb{L}}(p, I))$.

Note that the kernel of the classical truncation of $A/^{\mathbb{L}}(p, I) \to R_A/^{\mathbb{L}}(p, I)$ is nilpotent, as observed in the proof of Proposition 5.4.19. If μ is 1-bounded, we can now conclude using Proposition 4.12.3.

In general, we can use the argument from the proof of the last proposition to prove the following claim: Given a non-negatively filtered animated commutative ring $\operatorname{Fil}^\bullet B$ where the kernel of $\pi_0(B) \to \pi_0(\operatorname{gr}^0 B)$ is nilpotent, and given a map $X \to \mathcal{R}(\operatorname{Fil}^\bullet B)$ fibered by smooth schemes, X admits a section over $\mathcal{R}(\operatorname{Fil}^\bullet B)$ if and only if it admits a section over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} B$. Indeed, using the invariance of this property under filtered nilpotent thickenings of $\operatorname{Fil}^\bullet B$, we reduce successively to the case where $\operatorname{Fil}^\bullet B$ is discrete, then to the case where $\operatorname{Fil}^1 B \to B$ is the zero map, and then using Proposition A.2.6 to the case where the Rees algebra of $\operatorname{Fil}^\bullet B$ is itself a nilpotent extension of $B[t]$. ☐

Definition 5.5.3. $\mathsf{Wind}^{G,\mu}_{n,\underline{A}}(R_A)$ is the ∞-groupoid of G-torsors $\mathcal{Q}$ over $\mathcal{R}(\operatorname{Fil}^\bullet A) \otimes \mathbb{Z}/p^n\mathbb{Z}$ equipped with an isomorphism of G-torsors $\sigma^*\mathcal{Q} \xrightarrow{\simeq} \tau^*\mathcal{Q}$, and satisfying the following equivalent conditions:

(1) The associated $G\{\mu\}$-torsor $\mathcal{Q}^\mu$ is trivial étale locally on $\operatorname{Spf} R_A$.
(2) For every geometric point $R_A \to \kappa$, the restriction of $\mathcal{Q}^\mu$ over $B\mathbb{G}_m \times \operatorname{Spec} \kappa$ is trivial.

We will refer to the objects of this ∞-groupoid as n**-truncated** (G, μ)**-windows** over $\underline{A}$.

We will have reason to consider the inverse limit

$$\mathsf{Wind}^{G,\mu}_{\infty,\underline{A}} = \varprojlim_n \mathsf{Wind}^{G,\mu}_{n,\underline{A}}.$$

Objects of this ∞-groupoid are (G, μ)**-windows** over $\underline{A}$.

Definition 5.5.4. We will call (G, μ)-windows over the Witt vector frame $\underline{W(R)}$ (G, μ)**-displays over** R, and write $\mathsf{Disp}^{G,\mu}_\infty(R)$ for the ∞-groupoid spanned by them.

If R is an $\mathbb{F}_p$-algebra, then, for $n \geq 1$, we will call (G, μ)-windows over the truncated Witt vector frame $\underline{W_n(R)}$ n**-truncated** (G, μ)**-displays**.[11]

For classical R, these definitions recover the definition from [14] (for $n = \infty$) and [42, §5].

When $n = 1$, one can show that, in the language of [60], $\mathsf{Disp}^{G,\mu}_1$ is the stack over k parameterizing G-zips of type μ.

Remark 5.5.5. Here is a slightly different perspective on the definition, closer to the treatment in [42, 14]. Suppose that we have a trivialization $A \xrightarrow{\simeq} A\{1\}$.

Let $\mathcal{Q}_0$ be the G-torsor over $\mathcal{R}(\operatorname{Fil}^\bullet A) \otimes \mathbb{Z}/p^n\mathbb{Z}$ obtained from the torsor $\mathcal{P}_\mu \to B\mathbb{G}_{m,\mathcal{O}}$: Its automorphisms are represented by the sheaf $L^+_{\underline{A}}G^{(n)}\{\mu\}$ given by

$$L^+_{\underline{A}}G^{(n)}\{\mu\}(R_{A'}) = \operatorname{Map}_{B\mathbb{G}_{m,\mathcal{O}}}(\mathcal{R}(\operatorname{Fil}^\bullet A') \otimes \mathbb{Z}/p^n\mathbb{Z}, G\{\mu\}).$$

Because of our chosen trivialization of $A\{1\}$, the restriction of $\mathcal{Q}_0$ along both τ and σ is also trivial, and so its automorphisms are represented on the p-completely étale site of R_A by the sheaf $L^+_{\underline{A}}G^{(n)}$ given by

$$L^+_{\underline{A}}G^{(n)}(R_{A'}) = G^{(n)}(A') = G(A'/^{\mathbb{L}}p^n).$$

Pullback along σ and τ gives two maps

$$\sigma, \tau : L^+_{\underline{A}}G^{(n)}\{\mu\} \to L^+_{\underline{A}}G^{(n)}$$

We now have

$$\mathsf{Wind}^{G,\mu}_{n,\underline{A}} = [L^+_{\underline{A}}G^{(n)} \ {}_{\sigma}/\!\!/_{\tau}\ L^+_{\underline{A}}G^{(n)}\{\mu\}],$$

[11]This is unfortunately inconsistent with our use of the adjective 'n-truncated' for windows, but it is compatible with the definition from [44]. For perfect R, the two notions of truncatedness agree.

where the right hand side indicates the quotient by the action given symbolically by

$$L^+_{\underline{A}}G^{(n)} \times L^+_{\underline{A}}G^{(n)}\{\mu\} \xrightarrow{(h,g)\mapsto\tau(h)^{-1}g\sigma(h)} L^+_{\underline{A}}G^{(n)}.$$

Remark 5.5.6. Under the same condition as in the previous remark, the ∞-groupoid of (G,μ)-windows over $\underline{A}$ can be viewed as the quotient

$$[L^+_{\underline{A}}G \;{}_{\sigma}\!/\!\!/_{\tau}\; L^+_{\underline{A}}G\{\mu\}]$$

with

$$L^+_{\underline{A}}G(R_{A'}) = G(A')\ ;\ L^+_{\underline{A}}G\{\mu\}(R_{A'}) = \mathrm{Map}_{B\mathbb{G}_{m,\mathcal{O}}}(\mathcal{R}(\mathrm{Fil}^\bullet A'), G\{\mu\}).$$

Note that, by Proposition 4.12.3, if μ is 1-bounded, we have

$$L^+_{\underline{A}}G\{\mu\}(R_{A'}) \simeq G(A') \times_{G(R_{A'})} P^-_\mu(R_{A'}).$$

Assume for the rest of this subsection that μ is 1-bounded.

Remark 5.5.7. Let $\mathrm{Syn}(\underline{A}))$ be the coequalizer of the two maps

$$\sigma,\tau : \mathrm{Spf}\, A \to \mathcal{R}(\mathrm{Fil}^\bullet A).$$

Unwinding definitions, $\mathsf{Wind}^{G,\mu}_{n,\underline{A}}(R_A)$ is simply the space

$$\mathrm{Map}_{/B\mathbb{G}_m}(\mathrm{Syn}(\underline{A}) \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{B}(G,\mu)),$$

where $\mathcal{B}(G,\mu) \to (B\mathbb{G}_m, \iota_{\mathcal{O}})$ is the 1-bounded stack from Definition 4.10.4 and we are viewing $\mathrm{Syn}(\underline{A})$ as a pointed graded stack via Remark 5.4.16.

Remark 5.5.8. If R is a discrete p-complete $\mathcal{O}$-algebra and $\underline{W(R)}$ is the Witt frame from Example 5.4.8, then the above description shows that our notion of a (G,μ)-display in terms of (G,μ)-windows over $\underline{W(R)}$, recovers the notion defined in [14].

Remark 5.5.9. Suppose that A is flat over $\mathbb{Z}_p$. Suppose also that we have $\mathrm{Fil}^1 A = (E)$ for some non-zero divisor $E \in A$, so that $R_A = A/(E)$, $I = (\varphi(E))$ and $A\{1\}$ is trivial. In particular, $\underline{A}$ is prismatic.

Let $H_\mu \to G_A$ be the *dilatation* of G_A along the subgroup $P^-_\mu \otimes R_A \subset G \otimes R_A$ (see for instance [12, §3.2]). This is a smooth affine group scheme over A characterized by the fact that, for any flat A-algebra B, we have $H_\mu(B) = G(B) \times_{G(B/(E))} P^-_\mu(B/(E))$. In particular, conjugation by $\mu(E)$ on the generic fiber of $G_{\mathcal{O}}$ restricts to a map $\mathrm{int}(\mu(E)) : H_\mu \to G_A$

Then we have $L^+_{\underline{A}}G\{\mu\}(R_{A'}) = H_\mu(A')$, τ is just the natural map $H_\mu(A') \to G(A')$, while σ is given by $\varphi \circ \mathrm{int}(\mu(E))$; see the argument in [42, Lemma 5.2.1].

Therefore, we see that $\mathsf{Wind}^{G,\mu}_{\infty,\underline{A}}$ is the étale sheafification of the functor

$$R_{A'} \mapsto [G(A') \;{}_{\sigma*}\!/\!\!/_{\tau^*}\; H_\mu(A')].$$

In other words, a section over A', amounts to giving an H_μ-torsor $\mathcal{P}$ over A for the p-completely étale topology, along with an isomorphism $\xi : \sigma^*\mathcal{P} \xrightarrow{\simeq} \tau^*\mathcal{P}$.

Remark 5.5.10. There is another situation in which the description of $\mathsf{Wind}^{G,\mu}_{\infty,\underline{A}}$ simplifies considerably: If A is flat over $\mathbb{Z}_p$, $\mathrm{Fil}^i A = 0$ for $i \geq 1$, $R_A = A$ and $I = (p)$. This is of course not a prismatic frame.

Then $L^+_{\underline{A}}G\{\mu\}(R_{A'}) = P^-_\mu(A')$, and $\tau^* : P^-_\mu(A') \to G(A')$ is the natural map, with σ is given by $\varphi \circ \mathrm{int}(\mu(p))$.

We find that an object of $\mathsf{Wind}^{G,\mu}_{\infty,\underline{A}}(A')$ is simply a P^-_μ-torsor $\mathcal{P}'$ over A' for the p-completely étale topology, along with an isomorphism $\sigma^*\mathcal{P}' \xrightarrow{\simeq} \tau^*\mathcal{P}'$

5.6. **$\underline{A}$-gauges and the case of GL_h.** Here, we explicate what the above definitions specialize to for $G = \mathrm{GL}_h$ for some $h \geq 1$. An **$\underline{A}$-gauge of level** n is a quasicoherent sheaf $\mathcal{M} \in \mathrm{QCoh}(\mathcal{R}(\mathrm{Fil}^\bullet A) \otimes \mathbb{Z}/p^n\mathbb{Z})$ equipped with an isomorphism $\xi : \sigma^*\mathcal{M} \xrightarrow{\simeq} \tau^*\mathcal{M}$ in $\mathrm{QCoh}(\mathrm{Spf}\, A/^{\mathbb{L}}p^n)$.

Remark 5.6.1. Explicitly, we can view $\mathcal{M}$ as a derived I-complete filtered module $\mathrm{Fil}^\bullet \mathsf{M}$ over $\mathrm{Fil}^\bullet A/^{\mathbb{L}}p^n$. Base-change along $\Phi_\pm : \mathrm{Fil}^\bullet A \to \mathrm{Fil}^\bullet_{I,\pm} A$ yields a filtered module $\Phi^*_\pm \mathrm{Fil}^\bullet \mathsf{M}$ over $\mathrm{Fil}^\bullet_{I,\pm} A/^{\mathbb{L}}p^n$ whose degree-0 filtered piece is an $A/^{\mathbb{L}}p^n$-module M_σ corresponding to $\sigma^*\mathcal{M}$. The isomorphism ξ now corresponds to an isomorphism $\mathsf{M}_\sigma \xrightarrow{\simeq} \mathsf{M}$ in $\mathrm{Mod}_{A/^{\mathbb{L}}p^n}$.

5.6.2. $\underline{A}$-gauges of level n organize into a symmetric monoidal stable ∞-category $\underline{A}{-}\mathsf{gauge}_n$, where the unit object is the structure sheaf $\mathscr{O}$ of $\mathcal{R}(\mathrm{Fil}^\bullet A) \otimes \mathbb{Z}/p^n\mathbb{Z}$ equipped with the canonical isomorphism ξ_0 between its pullbacks via σ^* and τ^*.

For any map $\underline{A} \to \underline{B}$ of frames, there is a natural base-change map

$$\underline{A}{-}\mathsf{gauge}_n \xrightarrow{(\mathcal{M},\xi)\mapsto \underline{B}\otimes_{\underline{A}}(\mathcal{M},\xi)} \underline{B}{-}\mathsf{gauge}_n.$$

The proof of the next result is as in [42, Example 5.3.5].

Proposition 5.6.3. *Suppose that $\mu : \mathbb{G}_m \to \mathrm{GL}_h$ is the cocharacter given by*

$$z \mapsto \mathrm{diag}(z^{m_1}, z^{m_2}, \ldots, z^{m_h})$$

for $m_1, m_2, \ldots, m_n \in \mathbb{Z}$. Then $\mathsf{Wind}^{\mathrm{GL}_h,\mu}_{n,\underline{A}}$ is equivalent to the ∞-groupoid of $\underline{A}$-gauges $\mathcal{M}$ of level n such that the underlying filtered module $\mathrm{Fil}^\bullet \mathsf{M}$ *satisfies the following condition: There exists an étale cover $R_A \to R_{A'}$ and an isomorphism*

$$\mathrm{Fil}^\bullet A' \otimes_{\mathrm{Fil}^\bullet A} \mathrm{Fil}^\bullet \mathsf{M} \xrightarrow{\simeq} \bigoplus_{i=1}^{h} \mathrm{Fil}^\bullet A'\{m_i\}/^{\mathbb{L}}p^n.$$

Here, we have set

$$\mathrm{Fil}^\bullet A'\{m_i\} \overset{\mathrm{defn}}{=} (\mathrm{Fil}^\bullet A')(m_i) \otimes_A A\{m_i\}.$$

5.7. **Relationship with the definitions of Anschütz-Le Bras.** The purpose of this subsection is to connect the definitions here with those of [1]. Similar discussions can be found in [31] and [56].

5.7.1. Suppose that $0 \leq d \leq h$ is such that $m_i = 1$ for $0 \leq i < d$ and $m_i = 0$ for $i \geq d$, so that μ is *minuscule*.

Suppose also that $\underline{A}$ is *classical* in the following sense: A is flat over $\mathbb{Z}_p$, $\mathrm{Fil}^\bullet A$ is a discrete filtered commutative ring filtered by A-submodules $\mathrm{Fil}^i A \subset A$, and also $I \subset A$ is a locally principal ideal.

Suppose that we have $\mathrm{Fil}^1 A = \varphi^{-1}(I) \subset A$; equivalently, we are assuming that the composition $A \xrightarrow{\varphi} A \to \overline{A}$ factors through an *injective* map $R_A \to \overline{A}$ allowing us to view $\overline{A}$ as an R_A-algebra.[12]

We now claim that $\mathsf{Wind}^{\mathrm{GL}_h,\mu}_{\infty,\underline{A}}(R_A)$ is equivalent to the (classical) category of pairs $(\mathsf{N}, \varphi_{\mathsf{N}})$ where:

(1) N is a finite locally free A-module of rank h;
(2) $\varphi_{\mathsf{N}} : \mathsf{N} \to \mathsf{N}$ is a φ-linear map such that the cokernel of the linearization $\varphi^*\mathsf{N} \to \mathsf{N}$ is killed by I;
(3) The image of the composition $\mathsf{N} \xrightarrow{\varphi_{\mathsf{N}}} \mathsf{N} \to \mathsf{N}/I\mathsf{N}$ is a locally free R_A-module F_{N} of rank d such that the induced map $\overline{A} \otimes_{R_A} F_{\mathsf{N}} \to \mathsf{N}/I\mathsf{N}$ is injective.

In harmony with [1, Definition 4.1.10], we will call such pairs **admissible $\underline{A}$-Dieudonné modules**.

5.7.2. To verify the above claim, first note that, by Proposition 5.6.3, $\mathsf{Wind}^{\mathrm{GL}_h,\mu}_{\infty,\underline{A}}(R_A)$ is equivalent to the category of pairs $(\mathrm{Fil}^\bullet \mathsf{M}, \xi)$ where:

(1) $\mathrm{Fil}^\bullet \mathsf{M}$ is a filtered $\mathrm{Fil}^\bullet A$-module such that for some étale cover $R_A \to R_{A'}$, we have an isomorphism

$$\mathrm{Fil}^\bullet A' \otimes_{\mathrm{Fil}^\bullet A} \mathrm{Fil}^\bullet \mathsf{M} \xrightarrow{\simeq} \mathrm{Fil}^\bullet A'\{1\}^{\oplus d} \oplus (\mathrm{Fil}^\bullet A')^{\oplus(h-d)}.$$

(2) $\Psi : \mathrm{Fil}^0 \Phi^*_\pm \mathrm{Fil}^\bullet \mathsf{M} \xrightarrow{\simeq} \mathsf{M}$ is an isomorphism of A-modules.

[12] This implies in particular that $\underline{A}$ is prismatic.

We claim that giving the filtered module $\mathrm{Fil}^\bullet \mathsf{M}$ is equivalent to giving a pair $(\mathsf{M}, \mathrm{Fil}^0 \mathsf{M})$ with M a finite locally free A-module of rank h and $\mathrm{Fil}^0 \mathsf{M} \subset \mathsf{M}$ a submodule such that $\mathsf{M}/\mathrm{Fil}^0 \mathsf{M}$ is locally free over R_A of rank d. Indeed, suppose we are given the latter data. Set $M = R_A \otimes_A \mathsf{M}$ and $P = \mathrm{coker}(\mathrm{Fil}^0 \mathsf{M} \to \mathsf{M})$, so that we have a quotient map $M \twoheadrightarrow P$ of locally free R_A-modules. Choose a splitting $M \simeq P \oplus Q$, and lift it to a splitting $\mathsf{M} \simeq \mathsf{P} \oplus \mathsf{Q}$ (possible by the Henselianness of $A \twoheadrightarrow R_A$; see [30]). Then the filtered module $\mathrm{Fil}^\bullet \mathsf{M}$ is given by

$$\mathrm{Fil}^\bullet \mathsf{M} = (\mathrm{Fil}^\bullet A(1) \otimes_A \mathsf{P})) \oplus (\mathrm{Fil}^\bullet A \otimes_A \mathsf{Q}). \tag{5.7.2.1}$$

It is not hard to see that this is independent of the choice of splitting and yields an inverse to the obvious forgetful functor in the other direction.

5.7.3. Set $\mathsf{M}_\sigma = \mathrm{Fil}^0 \Phi^*_{\pm} \mathrm{Fil}^\bullet \mathsf{M}$. Using the identity (5.7.2.1), we find that we have

$$\mathrm{Fil}^k(\Phi^*_{\pm} \mathrm{Fil}^\bullet \mathsf{M}) = (I^{\otimes(k+1)} \otimes_A \varphi^*\mathsf{P}) \oplus (I^{\otimes k} \otimes_A \varphi^*\mathsf{Q}),$$

and hence that

$$\mathsf{M}_\sigma = (I \otimes_A \varphi^*\mathsf{P}) \oplus \varphi^*\mathsf{Q} \subset \varphi^*\mathsf{P} \oplus \varphi^*\mathsf{Q} = \varphi^*\mathsf{M} \tag{5.7.3.1}$$

is the kernel of the map $\varphi^*\mathsf{M} \to \overline{A} \otimes_{R_A} P \simeq \varphi^*\mathsf{P}/I\varphi^*\mathsf{P}$. Note in particular that we have $I\varphi^*\mathsf{M} \subset \mathsf{M}_\sigma$.

Now, set $\mathsf{N} = \mathsf{M}\{-1\}$, so that we have

$$\varphi^*\mathsf{N} \simeq \varphi^*A\{-1\} \otimes_A \varphi^*\mathsf{M} \simeq A\{-1\} \otimes_A I\varphi^*\mathsf{M} \subset \mathsf{M}_\sigma\{-1\}.$$

Consider the composition

$$\varphi_{\mathsf{N}} : \mathsf{N} \xrightarrow{m \mapsto \varphi^* m} \varphi^*\mathsf{N} \to \mathsf{M}_\sigma\{-1\} \xrightarrow[\simeq]{\Psi\{-1\}} \mathsf{M}\{-1\} = \mathsf{N}.$$

We claim the pair $(\mathsf{N}, \varphi_{\mathsf{N}})$ is an admissible $\underline{A}$-Dieudonné module. Indeed, one checks using (5.7.3.1) that we have $\varphi_{\mathsf{N}}^{-1}(I\mathsf{N}) = (\mathrm{Fil}^0 \mathsf{M})\{-1\}$. Hence the composition of the projection onto $\mathsf{N}/I\mathsf{N}$ with φ_{N} has image $F_{\mathsf{N}} \simeq \left(\mathsf{M}/\mathrm{Fil}^0 \mathsf{M}\right)\{-1\}$. Similarly, one finds that the base-change along $R_A \to \overline{A}$ of F_{N} maps isomorphically onto a direct summand of $\mathsf{N}/I\mathsf{N}$.

5.7.4. Conversely, given such a pair, we set

$$\mathsf{M} = \mathsf{N}\{1\}\ ;\ \mathrm{Fil}^0 \mathsf{M} = \varphi_{\mathsf{N}}^{-1}(I\mathsf{N})\{1\}.$$

Then condition (2) for being an admissible $\underline{A}$-Dieudonné module tells us that $\mathsf{M}/\mathrm{Fil}^0 \mathsf{M}$ is locally free over R_A of rank d. Therefore, we can attach to this data a filtered module $\mathrm{Fil}^\bullet \mathsf{M}$ of type μ. To obtain the isomorphism Ψ, we first observe that we have

$$I\mathsf{M}_\sigma\{-1\} = (1 \otimes \varphi_{\mathsf{N}})^{-1}(I\mathsf{N}) \subset \varphi^*\mathsf{N}.$$

Indeed, since $I\mathsf{M}_\sigma\{-1\} = \ker(\varphi^*\mathsf{N} \to \overline{A} \otimes_{R_A} F_{\mathsf{N}})$, this is equivalent to condition (3), which asserts that $\overline{A} \otimes_{R_A} F_{\mathsf{N}}$ maps injectively into $\mathsf{N}/I\mathsf{N}$.

Now, tensoring this equality with $I^{\otimes -1}\{1\}$ gives us the desired isomorphism $\mathsf{M}_\sigma \simeq \mathsf{M}$.

5.8. **Abstract deformation theory for 1-bounded stacks.** In this subsection and the next, we will assume that all our frames are p-adic as in Definition 5.4.3. We will view $\mathcal{R}(\mathrm{Fil}^\bullet A)$ as a pointed graded stack using Remark 5.4.16.

5.8.1. Suppose that $\mathcal{X} = (\mathcal{X}^\diamond, X^0) \to \mathcal{R}(\mathrm{Fil}^\bullet A) \otimes \mathbb{Z}/p^n\mathbb{Z}$ is a 1-bounded and filtered integrable stack, equipped with an isomorphism $\xi : \sigma^*\mathcal{X}^\diamond \xrightarrow{\simeq} \tau^*\mathcal{X}^\diamond$ of stacks over $A/^{\mathbb{L}}p^n$. Then for any p-completely étale R_A-algebra $R_{A'}$ we will set

$$\Gamma_{\underline{A}}(\mathcal{X}, \xi)(R_{A'}) \overset{\mathrm{defn}}{=} \mathrm{eq}\left(\mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet A') \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}) \underset{\tau^*}{\overset{\xi \circ \sigma^*}{\rightrightarrows}} \mathcal{X}^\diamond(A'/^{\mathbb{L}}p^n)\right).$$

All mapping spaces here are over $\mathcal{R}(\mathrm{Fil}^\bullet A) \otimes \mathbb{Z}/p^n\mathbb{Z}$, and we are viewing $\mathrm{Spec}(A'/^{\mathbb{L}}p^n)$ as a scheme over it via the map τ.

Equivalently, if $\mathrm{Syn}(\underline{A})$ is as in Remark 5.5.7, then ξ gives a descent $\mathcal{X}_\xi \to \mathrm{Syn}(\underline{A})$ for $\mathcal{X}$, and we have

$$\Gamma_{\underline{A}}(\mathcal{X}, \xi)(R_{A'}) = \mathrm{Map}(\mathrm{Syn}(\underline{A}) \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}_\xi)$$

Remark 5.8.2. The prestack $\Gamma_{\underline{A}}(\mathcal{X},\xi)$ actually depends on much less structure than the entire apparatus of the frame. Indeed, by the filtered integrability of $\mathcal{X}$, we have

$$\begin{aligned}\mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet A')\otimes\mathbb{Z}/p^n\mathbb{Z},\mathcal{X}) &\simeq \mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet_{(A'\twoheadrightarrow R_{A'})} A')\otimes\mathbb{Z}/p^n\mathbb{Z},\mathcal{X})\\ &\simeq \mathrm{Map}(\mathbb{A}^1/\mathbb{G}_m\times\mathrm{Spec}\, R_{A'}/^{\mathbb{L}}p^n,\mathcal{X})\times_{\mathcal{X}^\diamond(R_{A'}/^{\mathbb{L}}p^n)}\mathcal{X}^\diamond(A'/^{\mathbb{L}}p^n).\end{aligned}$$

Similarly, if we view $\mathcal{R}(\mathrm{Fil}^\bullet_p\,\varphi_*A')$ as a stack over $\mathcal{R}(\mathrm{Fil}^\bullet A')$ via Φ, then we have

$$\begin{aligned}\mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet_p\,\varphi_*A')\otimes\mathbb{Z}/p^n\mathbb{Z},\mathcal{X}) &\simeq \mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet_{\varphi_*(A'\twoheadrightarrow\overline{A}')}\,\varphi_*A')\otimes\mathbb{Z}/p^n\mathbb{Z},\mathcal{X})\\ &\simeq \mathrm{Map}(\mathbb{A}^1/\mathbb{G}_m\times\mathrm{Spec}\,\overline{A}'/^{\mathbb{L}}p^n,\mathcal{X})\times_{(\sigma^*\mathcal{X}^\diamond)(\overline{A}'/^{\mathbb{L}}p^n)}(\sigma^*\mathcal{X}^\diamond)(A'/^{\mathbb{L}}p^n).\end{aligned}$$

The maps

$$\tau^*,\xi\circ\sigma^*:\mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet A')\otimes\mathbb{Z}/p^n\mathbb{Z},\mathcal{X})\to\mathcal{X}^\diamond(A'/^{\mathbb{L}}p^n)$$

admit simple descriptions: The first is isomorphic to the projection

$$\mathrm{Map}(\mathbb{A}^1/\mathbb{G}_m\times\mathrm{Spec}\, R_{A'}/^{\mathbb{L}}p^n,\mathcal{X})\times_{\mathcal{X}^\diamond(R_{A'}/^{\mathbb{L}}p^n)}\mathcal{X}^\diamond(A'/^{\mathbb{L}}p^n)\to\mathcal{X}^\diamond(A'/^{\mathbb{L}}p^n),$$

while the second is isomorphic to the composition

$$\begin{aligned}\mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet_{(A'\twoheadrightarrow R_{A'})} A')\otimes\mathbb{Z}/p^n\mathbb{Z},\mathcal{X}) &\to \mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet_{\varphi_*(A'\twoheadrightarrow\overline{A}')}\,\varphi_*A')\otimes\mathbb{Z}/p^n\mathbb{Z},\mathcal{X})\\ &\simeq \mathrm{Map}(\mathbb{A}^1/\mathbb{G}_m\times\mathrm{Spec}\,\overline{A}'/^{\mathbb{L}}p^n,\mathcal{X})\times_{(\sigma^*\mathcal{X}^\diamond)(\overline{A}'/^{\mathbb{L}}p^n)}(\sigma^*\mathcal{X}^\diamond)(A'/^{\mathbb{L}}p^n)\\ &\to (\sigma^*\mathcal{X}^\diamond)(A'/^{\mathbb{L}}p^n)\\ &\xrightarrow[\simeq]{\xi}\mathcal{X}^\diamond(A'/^{\mathbb{L}}p^n).\end{aligned}$$

Therefore, we only use the commuting square

$$\begin{tikzcd} \mathrm{Fil}^1 A \arrow[r, "\varphi_1"] \arrow[d] & A \arrow[d, "p"] \\ A \arrow[r, "\varphi"'] & A \end{tikzcd}$$

for the definition of $\Gamma_{\underline{A}}(\mathcal{X},\xi)$.

Definition 5.8.3. A **semiframe** $\underline{\underline{C}}$ is a tuple consisting of the following data:

(1) A surjective map $C\twoheadrightarrow R_C$ of p-complete animated commutative rings with fiber $\mathrm{Fil}^1 C$;
(2) A commuting diagram

$$\begin{tikzcd} \mathrm{Fil}^1 C \arrow[r, "\varphi_1"] \arrow[d] & C \arrow[d, "p"] \\ C \arrow[r, "\varphi"'] & C \end{tikzcd}$$

where $\varphi: C\to C$ is a naïve Frobenius lift.

Note in particular that every semiframe is equipped with a map $R_C\xrightarrow{\theta}\overline{C}\overset{\mathrm{defn}}{=}C/^{\mathbb{L}}p$ induced by the pair (φ,φ_1).

5.8.4. Every p-adic frame $\underline{A}$ has an underlying semiframe $\underline{\underline{A}}$. Note that, for any semiframe $\underline{\underline{C}}$, we have canonical maps

$$\tau, \sigma : \operatorname{Spf} C \to \mathcal{R}(\mathrm{Fil}^\bullet_{(C \twoheadrightarrow R_C)} C)$$

where τ is the open preimage of $\mathbb{G}_m/\mathbb{G}_m$, while σ is obtained from the composition

$$\operatorname{Spf} C \to \mathcal{R}(\mathrm{Fil}^\bullet_{\varphi_*(C \twoheadrightarrow \overline{C})} \varphi_* C) \to \mathcal{R}(\mathrm{Fil}^\bullet_{(C \twoheadrightarrow R_C)} C).$$

Here, we have set $\overline{C} = C/^{\mathbb{L}}p$, and the second map is induced by the pair (φ, φ_1). The notation is from the proof of Proposition 4.12.3.

Remark 5.8.2 now says that the prestack $\Gamma_{\underline{A}}(\mathcal{X}, \xi)$ only depends on the semiframe $\underline{\underline{A}}/^{\mathbb{L}}p^n$. Here, we are using that an analogue of Proposition 5.4.19 applies to semiframes as well.

In fact, given any semiframe $\underline{\underline{C}}$ such that C is a $\mathbb{Z}/p^n\mathbb{Z}$-algebra, equipped with a 1-bounded stack $\mathcal{X}$ over $\mathcal{R}(\mathrm{Fil}^\bullet_{(C \twoheadrightarrow R_C)} C)$ and an isomorphism $\xi : \sigma^*\mathcal{X}^\diamond \to \tau^*\mathcal{X}^\diamond$, we can define a prestack of sections $\Gamma_{\underline{\underline{C}}}(\mathcal{X}, \xi)$ by

$$\Gamma_{\underline{\underline{C}}}(\mathcal{X}, \xi)(R_{C'}) \overset{\mathrm{defn}}{=} \mathrm{eq}\Big(\mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet_{(C' \twoheadrightarrow R_{C'})} C', \mathcal{X}) \underset{\tau^*}{\overset{\xi \circ \sigma^*}{\rightrightarrows}} \mathcal{X}^\diamond(C')\Big).$$

This is compatible with the definition for 1-bounded stacks over frames, after taking their mod-p^n fibers.

Remark 5.8.5. Just as in Remark 5.4.13, we can take Postnikov truncations of semiframes. Also, if we have two maps of semiframes $\underline{\underline{A}} \to \underline{\underline{B}}$ and $\underline{\underline{A}} \to \underline{\underline{C}}$, we have the tensor product semiframe $\underline{\underline{B}} \otimes_{\underline{\underline{A}}} \underline{\underline{C}}$, whose underlying animated commutative ring is $B \otimes_A C$ sitting in a diagram

$$\begin{tikzcd}
\mathrm{Fil}^1(B \otimes_A C) \arrow[r] \arrow[d] & B \otimes_A C \arrow[r] \arrow[d, "\varphi_B \otimes \varphi_C"] & R_B \otimes_{R_A} R_C \arrow[d, "\theta_B \otimes \theta_C"] \\
B \otimes_A C \arrow[r, "p"] & B \otimes_A C \arrow[r] & \overline{B} \otimes_{\overline{A}} \overline{C}.
\end{tikzcd}$$

These observations will be used in the proof of Corollary 5.8.11 below.

5.8.6. We will now assume that we have a mod-p^n semiframe $\underline{\underline{A}}$ and a pair $(\mathcal{X}, \xi)$ over it. Suppose that we have a map of mod-p^n semiframes $q : \underline{\underline{B}} \to \underline{\underline{A}}$ such that the associated map $R_B \to R_A$ is a locally nilpotent thickening—that is, $\pi_0(R_B) \to \pi_0(R_A)$ is a surjection with locally nilpotent kernel. Suppose also that the pair $(\mathcal{X}, \xi)$ lifts over $\underline{\underline{B}}$, and denote this lift by the same symbol.

We now have a canonical equivalence of small étale sites $(R_B)_{\text{ét}} \xrightarrow{\simeq} (R_A)_{\text{ét}}$, and thus a natural base change functor

$$q_* : \Gamma_{\underline{\underline{B}}}(\mathcal{X}, \xi) \to \Gamma_{\underline{\underline{A}}}(\mathcal{X}, \xi)$$

of sheaves on $(R_B)_{\text{ét}}$.

Remark 5.8.7. Suppose that $B \twoheadrightarrow A$ is surjective, and set $K = \mathrm{hker}(B \twoheadrightarrow A)$. Suppose in addition that the map $B \to R_B$ factors through a map $\pi : A \to R_B$ lifting $A \to R_A$; equivalently, $K \to B$ factors through a map $\mathrm{Fil}^1 B \to B$. Note that this is trivially the case if either $B \xrightarrow{\simeq} A$ or $R_B \xrightarrow{\simeq} R_A$.

Let X^- be the attractor on R_B-algebras given by

$$R \mapsto \mathrm{Map}(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R, \mathcal{X}),$$

and let X be the stack $R \mapsto \mathcal{X}^\diamond(R)$.

Then we have a commuting diagram

$$\begin{tikzcd}
\Gamma_{\underline{\underline{B}}}(\mathcal{X},\xi)(R_B) \arrow[r] \arrow[d, "q_*"'] & X^-(R_B) \arrow[d] \\
\Gamma_{\underline{\underline{A}}}(\mathcal{X},\xi)(R_A) \arrow[r] & X^-(R_A) \times_{X(R_A)} X(R_B)
\end{tikzcd}.$$

where the top arrow arises from pullback along the map from Remark 5.4.16 (for $\underline{\underline{B}}$), while that on the bottom is from this map (for $\underline{\underline{A}}$) combined with pullback along π.

There is a trivial situation in which the square from Remark 5.8.7 is Cartesian.

Proposition 5.8.8. *Suppose that $q : B \xrightarrow{\simeq} A$, and the lift $A \xrightarrow{\simeq} B \to R_B$ is the obvious one. Then the square in Remark 5.8.7 obtained via the map $A \xrightarrow{\simeq} B \to R_B$ is Cartesian.*

Proof. For any p-completely étale map $R_B \to R_{B'}$ reducing to $R_A \to R_{A'}$, first note that our hypothesis says that $X(B') \xrightarrow{\simeq} X(A')$.

We now have

$$\mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet_{(B' \twoheadrightarrow R_{B'})} B'), \mathcal{X}) \simeq X^-(R_{B'}) \times_{X(R_{B'})} \mathcal{X}^\diamond(B');$$
$$\mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet_{(A' \twoheadrightarrow R_{A'})} A'), \mathcal{X}) \simeq X^-(R_{A'}) \times_{X(R_{A'})} \mathcal{X}^\diamond(B').$$

The proposition follows quite formally now from the two previous paragraphs. ☐

Remark 5.8.9. With the hypotheses of Remark 5.8.7, set $\widetilde{\mathrm{Fil}}^1 A = \mathrm{hker}(\pi : A \to R_B)$, which factors through $\mathrm{Fil}^1 A \to A$ via a map $v : \widetilde{\mathrm{Fil}}^1 A \to \mathrm{Fil}^1 A$. This gives us a canonical map $\dot{\varphi}_1 : K \to K$ such that we have a commuting diagram with exact rows:

$$\begin{tikzcd}
K \arrow[r] \arrow[d, "\dot{\varphi}_1"'] & \mathrm{Fil}^1 B \arrow[r] \arrow[d, "\mathrm{Fil}^1 \Phi"'] & \widetilde{\mathrm{Fil}}^1 A \arrow[d, "(\mathrm{Fil}^1 \Phi)\circ v"] \\
K \arrow[r] & B \arrow[r] & A.
\end{tikzcd}$$

Here is an important technical result, which is simply an animated elaboration of arguments of Lau [42] and Bültel-Pappas [14].

Proposition 5.8.10. *With the hypotheses and notation of Remark 5.8.9, suppose that the following additional conditions hold:*

(1) $\pi_0(\mathrm{Fil}^1 B) \twoheadrightarrow \pi_0(\mathrm{Fil}^1 A)$ *is surjective; equivalently,*

$$\mathrm{Fil}^1 K = \mathrm{hker}(\mathrm{Fil}^1 B \to \mathrm{Fil}^1 A)$$

is connective.

(2) The map $K \to K$ induced by $\dot{\varphi}_1$ is locally nilpotent; equivalently, the endomorphism induced on $K/^{\mathbb{L}}p$ is locally nilpotent.

Then the commuting square in Remark 5.8.7 obtained from condition (2) is Cartesian.

Corollary 5.8.11. *With the hypotheses as in Proposition 5.8.10, if $R_B \xrightarrow{\simeq} R_A$, then q_* is an equivalence.*

The proof of the proposition will be given below. We begin with a couple of remarks on the conditions involved in its statement.

Remark 5.8.12. By p-completeness, to see that $B \to A$ is surjective, it is enough to know that $\overline{B} \to \overline{A}$ is so. Moreover, in the situation of Remark 5.8.7, we have a commuting diagram with exact rows

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathrm{Fil}^1 B \arrow[r] \arrow[d] & \widetilde{\mathrm{Fil}}^1 A \arrow[d] \\ \mathrm{Fil}^1 K \arrow[r] \arrow[d] & \mathrm{Fil}^1 B \arrow[r] \arrow[d] & \mathrm{Fil}^1 A \arrow[d] \\ K \arrow[r] & B \arrow[r] & A \end{tikzcd}$$

where the composition of the vertical arrows on the left is isomorphic to the identity on K. In other words, we have a section $K \to \mathrm{Fil}^1 K$ splitting the fiber sequence

$$R_K[-1] \to \mathrm{Fil}^1 K \to K,$$

where $R_K = \mathrm{hker}(R_B \twoheadrightarrow R_A)$. Therefore, we have $\mathrm{Fil}^1 K \xrightarrow{\simeq} K \oplus R_K[-1]$. This shows in particular that condition (1) of Proposition 5.8.10 holds if and only if R_K is 1-connective.

5.8.13. We will assume from now on that the hypotheses of Proposition 5.8.10 hold. Suppose that we have a $\underline{\underline{B}}$-gauge $(\mathcal{M}, \xi)$: By this, we mean we have a quasicoherent sheaf $\mathcal{M}$ over $\mathcal{R}(\mathrm{Fil}^\bullet_{(B\twoheadrightarrow R_B)} B)$ corresponding to a filtered module $\mathrm{Fil}^\bullet \mathsf{M}$ over $\mathrm{Fil}^\bullet_{(B\twoheadrightarrow R_B)} B$, with underlying B-module M, and an isomorphism

$$\xi : \mathsf{M}_\sigma \xrightarrow{\simeq} \mathsf{M}$$

of B-modules. Here, M_σ is the B-module underlying the quasicoherent sheaf obtained from $\mathcal{M}$ via restriction along σ.

The associated $\underline{\underline{A}}$-gauge

$$(\mathcal{M}_A, \xi_A) \overset{\text{defn}}{=} \underline{\underline{A}} \otimes_{\underline{\underline{B}}} (\mathcal{M}, \xi)$$

corresponds to a filtered $\mathrm{Fil}^\bullet A$-module $\mathrm{Fil}^\bullet \mathsf{M}_A$. Set

$$R\Gamma_{\underline{\underline{B}}}(\mathcal{M}, \xi)(R_B) \overset{\text{defn}}{=} \mathrm{hker}(\mathrm{Fil}^0 \mathsf{M} \xrightarrow{\xi\circ\sigma^* - \tau^*} \mathsf{M}),$$

and similarly for the basechange over $\underline{\underline{A}}$.

Lemma 5.8.14. *Suppose that $R_B \xrightarrow{\simeq} R_A$ and that the restriction of $\mathcal{M}$ over $B\mathbb{G}_m \times \mathrm{Spec}(R_B/^{\mathbb{L}}p)$ is 1-bounded. Then the map*

$$q_* : R\Gamma_{\underline{\underline{B}}}(\mathcal{M}, \xi)(R_B) \to R\Gamma_{\underline{\underline{A}}}(\mathcal{M}_A, \xi_A)(R_A)$$

is an isomorphism.

Proof. By dévissage, we can assume that all the sheaves involved are killed by p. Since $\mathcal{M}$ is 1-bounded, if $\mathrm{Fil}^\bullet M$ is the filtered $R_B/^{\mathbb{L}}p$-module obtained from $\mathrm{Fil}^\bullet \mathsf{M}$, we have

$$\mathrm{Fil}^0 \mathsf{M} \simeq \mathrm{Fil}^0 M \times_M \mathsf{M}.$$

Equivalently, the natural map $\mathrm{gr}^{-1} \mathsf{M} \to \mathrm{gr}^{-1} M$ is an equivalence.

Set $\mathrm{Fil}^\bullet \mathsf{M}_K = \mathrm{hker}(\mathrm{Fil}^\bullet \mathsf{M} \to \mathrm{Fil}^\bullet \mathsf{M}_A)$. Then, since $R_B = R_A$, we find that $\mathrm{Fil}^0 \mathsf{M}_K \simeq \mathsf{M}_K \simeq K \otimes_B \mathsf{M}$, and that

$$\mathrm{hker}(q_*) \simeq \mathrm{hker}(\mathsf{M}_K \xrightarrow{\sigma^*_\xi - \mathrm{id}} \mathsf{M}_K).$$

Here, σ^*_ξ is the endomorphism of M_K obtained as the composition

$$\mathsf{M}_K \simeq \mathrm{Fil}^0 \mathsf{M}_K \to \mathrm{Fil}^0 \mathsf{M} \xrightarrow{\xi\circ\sigma^*} \mathsf{M},$$

We now claim that σ^*_ξ is given by the composition

$$\mathsf{M}_K \simeq K \otimes_B \mathsf{M} \to K \otimes_B \mathrm{gr}^{-1}\mathsf{M} \xrightarrow{\dot{\varphi}_1 \otimes \overline{\xi}_{-1}} K \otimes_B \mathsf{M} \simeq \mathsf{M}_K.$$

Assuming this, we will see that σ^*_ξ is locally nilpotent, which in turn will imply the lemma, since, for any locally nilpotent endomorphism ψ of $N \in \mathrm{Mod}_{\mathbb{Z}}$, the endomorphism $\psi - \mathrm{id}$ is a self-equivalence. To see the local nilpotence, it is enough to show that the composition

$$K \otimes_B \mathrm{gr}^{-1} M \xrightarrow{\dot{\varphi}_1 \otimes \overline{\xi}_{-1}} K \otimes_B \mathsf{M}_K \to K \otimes_B \mathrm{gr}^{-1} M$$

is locally nilpotent, which comes down to the assumed nilpotence of $\dot{\varphi}_1$ as an endomorphism of K.

Let us prove the claim about σ^*_ξ. Since the map σ^*_ξ for $\mathrm{Fil}^0\,\mathsf{M}$ factors through $\mathrm{gr}^0\,\mathsf{M}$ and the analogous map for M_A factors through $\mathrm{gr}^0\,\mathsf{M}_A$, we find that the restriction to M_K must factor through $\mathrm{gr}^0\,\mathsf{M}_K$. One can check (using Lemma A.2.3 for instance) that we have natural fiber sequences

$$\mathrm{gr}^1 B \otimes_{R_A} \mathrm{gr}^{-1} M \to \mathrm{gr}^0\,\mathsf{M} \to \mathrm{gr}^0 M\ ;\ \mathrm{gr}^1 A \otimes_{R_A} \mathrm{gr}^{-1} M \to \mathrm{gr}^0\,\mathsf{M}' \to \mathrm{gr}^0 M,$$

showing that we have

$$\mathrm{gr}^0\,\mathsf{M}_K \simeq \mathrm{gr}^1 K \otimes_{R_A} \mathrm{gr}^{-1} M.$$

From this, the desired claim is clear, since we have a canonical map $K \simeq \mathrm{Fil}^1 K \to \mathrm{gr}^1 K$. $\square$

We begin by proving the corollary.

Proof of Corollary 5.8.11. We want to show that the map $\Gamma_{\underline{\underline{B}}}(\mathcal{X},\xi) \to \Gamma_{\underline{\underline{A}}}(\mathcal{X},\xi)$ is an isomorphism of sheaves on $(R_B)_{\mathrm{\acute{e}t}}$.

Suppose that $\underline{\underline{D}} \to \underline{\underline{C}}$ is a map of semiframes over $\underline{\underline{B}}$ such that the underlying map $\mathrm{Fil}^\bullet_{(D \twoheadrightarrow R_D)} D \to \mathrm{Fil}^\bullet_{(C \twoheadrightarrow R_C)} C$ is a square-zero extension of filtered animated commutative rings: in this situation, we will simply say that $\underline{\underline{D}}$ is a square-zero extension of $\underline{\underline{C}}$.

If we set

$$(\mathrm{Fil}^1 J \to J) = \mathrm{hker}\big((\mathrm{Fil}^1 D \to D) \to (\mathrm{Fil}^1 C \to C)\big)$$

then this underlies a filtered $\mathrm{Fil}^\bullet_{(C \twoheadrightarrow R_C)} C$-module yielding a $\underline{\underline{C}}$-gauge $(\mathcal{J},\zeta)$.

The filtered deformation theory from §4.5 gives a canonical Cartesian diagram of étale sheaves

(5.8.14.1)
$$\begin{tikzcd}
\Gamma_{\underline{\underline{D}}}(\mathcal{X},\xi) \arrow[r] \arrow[d] & \Gamma_{\underline{\underline{C}}}(\mathcal{X},\xi) \arrow[d] \\
\Gamma_{\underline{\underline{C}}}(\mathcal{X},\xi) \arrow[r] & \Gamma_{\underline{\underline{C}}}(\mathcal{M}_D(\mathcal{X})[1])
\end{tikzcd}$$

Here, $\Gamma_{\underline{\underline{C}}}(\mathcal{M}_D(\mathcal{X})[1])$ is the sheaf over $\Gamma_{\underline{\underline{B'}}}(\mathcal{X},\xi)$ whose fiber over $x \in \Gamma_{\underline{\underline{C}}}(\mathcal{X},\xi)(R_{C'})$ is the space

$$\tau^{\leq 0} R\Gamma_{\underline{\underline{C'}}}(\mathcal{J} \otimes \mathcal{M}(\mathcal{X})_x[1], \zeta \otimes \xi_x[1]).$$

Here, $(\mathcal{M}(\mathcal{X})_x, \xi_x)$ is the 1-bounded $\underline{\underline{C}}'$-gauge obtained by pulling back the tangent complex of $(\mathcal{X},\xi)$ along x. The reader can find a few more details for this kind of argument in the proof of Proposition 8.8.1.

Now, for every $k \geq 1$, consider the coCartesian square of semiframes

$$\begin{tikzcd}
\tau_{\leq (k+1)}\underline{\underline{B}} \arrow[r] \arrow[d] & \tau_{\leq k}\underline{\underline{B}} \arrow[d] \\
\tau_{\leq (k+1)}\underline{\underline{B}} \otimes_{\underline{\underline{B}}} \underline{\underline{A}} \arrow[r] & \tau_{\leq k}\underline{\underline{B}} \otimes_{\underline{\underline{B}}} \underline{\underline{A}}.
\end{tikzcd}$$

Using the Cartesian square (5.8.14.1) for both horizontal maps, and combining this with Lemma 5.8.14, we see that it is enough to know that the map

$$\Gamma_{\pi_0(\underline{\underline{B}})}(\mathcal{X},\xi) \to \Gamma_{\pi_0(\underline{\underline{B}})\otimes_{\underline{\underline{B}}}\underline{\underline{A}}}(\mathcal{X},\xi)$$

is an equivalence. Therefore, we can assume that B and $\mathrm{Fil}^1 B$ are discrete. By applying the argument to the naïve p-adic filtration on B, we can also assume that B is an $\mathbb{F}_p$-algebra.

Similarly, we can also replace $\mathrm{Fil}^1 B$ by its image in B, and assume that R_B is also discrete. Now, working with the Postnikov filtration on $\underline{\underline{A}}$, we finally reduce to the case where A is also discrete.

At this point, $K \subset \mathrm{Fil}^1 B \subset B$ is an honest nilpotent ideal. So choose m such that $K^m = 0$, and apply the same deformation theoretic argument to a sequence of square-zero extensions of semiframes

$$\underline{\underline{B}} = \underline{\underline{B}}_m \to \underline{\underline{B}}_{m-1} \to \cdots \to \underline{\underline{B}}_1 = \underline{\underline{A}},$$

which is constructed as follows: For $j = 1,\dots,m$, $B_j = B/K^j$ and $\mathrm{Fil}^1 B_j = \mathrm{Fil}^1 B/K^j \subset B_j$ is the image of $\mathrm{Fil}^1 B$. The map $\varphi_1 : \mathrm{Fil}^1 B \to B$ is Frobenius semilinear: therefore, for $j \geq 2$, it satisfies

$$\varphi_1(K^j) = \varphi_1(K^{j-1}\cdot K) \subset \varphi(K^{j-1})\varphi_1(K) \subset K^{p(j-1)} \subset K^j.$$

This implies that it descends to a map $\mathrm{Fil}^1 B_j \to B_j$ yielding the desired semiframe $\underline{\underline{B}}_j$.

□

Proof of Proposition 5.8.10. By Remark 5.8.12, for condition (1) of the Proposition to hold, $\overline{K}$ must be 1-connective, and we must have $\mathrm{Fil}^1 K \xrightarrow{\simeq} K \oplus \overline{K}[-1]$. Now, set $\widetilde{\mathrm{Fil}}^1 A = \mathrm{hker}(A \twoheadrightarrow R_B)$. Then the fiber sequence

$$\widetilde{\mathrm{Fil}}^1 A \to \mathrm{Fil}^1 A \to \overline{K}$$

shows that $\widetilde{\mathrm{Fil}}^1 A$ is connective, and so $\widetilde{\mathrm{Fil}}^1 A \to A$ can be upgraded to the structure of a semiframe ${}_B\underline{\underline{A}}$ on A such that we have maps of frames

$$\underline{\underline{B}} \xrightarrow{q'} {}_B\underline{\underline{A}} \xrightarrow{\tilde{q}} \underline{\underline{A}}$$

whose composition is q.

Since $\tilde{q}$ is an equivalence of the underlying animated commutative rings, Proposition 5.8.8 shows we have

$$\Gamma_{{}_B\underline{\underline{A}}}(\mathcal{X},\xi)(R_A) \xrightarrow{\simeq} \Gamma_{\underline{\underline{A}}}(\mathcal{X},\xi)(R_A) \times_{X^-(R_A)\times_{X(R_A)}X(R_B)} X^-(R_B).$$

Therefore, by replacing q with q', we are reduced to showing that q_* is an equivalence when $R_B \xrightarrow{\simeq} R_A$, which is the content of the already known Corollary 5.8.11. □

5.9. **A pseudo-torsor structure associated with 1-bounded stacks.** Suppose that we have a p-adic frame as above, and suppose also that R_A is an $\mathbb{F}_p$-algebra. Let us take a 1-bounded stack $\mathcal{X} \to \mathcal{R}(\mathrm{Fil}^\bullet A)\otimes \mathbb{F}_p$ that is filtered *and* graded integrable, and is equipped with an isomorphism $\xi : \sigma^*\mathcal{X}^\diamond \to \tau^*\mathcal{X}^\diamond$.

We will now see that—under additional conditions on $\underline{A}$—the étale stack $\Gamma_{\underline{A}}(\mathcal{X},\xi)$ admits the structure of a pseudo-torsor over a simpler stack, $\Gamma_{F\mathrm{Zip}}(\mathcal{X})$. This will be applied in Section 8 to prove a general version of Theorem E. The proof is technical, but the basic idea involves the following steps, beginning with the observation from Lemma 5.4.10 that, for any p-adic frame $\underline{A}$ with R_A an $\mathbb{F}_p$-algebra, we have a canonical map of frames $\underline{A}\otimes\mathbb{F}_p \to \underline{W_1(R_A)}$ to the 1-truncated Witt frame:

(1) When $\underline{A}$ is the Witt frame $\underline{W(R)}$, $\mathrm{Syn}(\underline{W(R)})\otimes\mathbb{F}_p$ is a square-zero thickening of $\mathrm{Syn}(\underline{W_1(R)})$, and so we can use deformation theory to describe the stack $\Gamma_{\underline{W(R)}}(\mathcal{X},\xi)$ in terms of its counterpart over $\underline{W_1(R)}$, which we denote $\Gamma_{F\mathrm{Zip}}(\mathcal{X})$.
(2) In general, when $\underline{A}$ is prismatic, by Lemma 5.4.10, we have a map of frames $\underline{A} \to \underline{W(R_A)}$ lifting the mod-p map from above, and we can use it, along with the 1-boundedness of $\mathcal{X}$, to 'lift' this description from $\underline{W(R)}$ to $\underline{A}$. This step also uses deformation theory in the form of the graded and filtered integrability of $\mathcal{X}$.

5.9.1. The first assumption we will make is that the filtration $\mathrm{Fil}^\bullet A$ is *p-adic*, so that we have a map $\mathrm{Fil}^\bullet_p \mathbb{Z}_p \to \mathrm{Fil}^\bullet A$ of animated filtered commutative rings. More precisely, $\mathcal{R}(\mathrm{Fil}^\bullet A)$ is a p-adic formal stack over

$$\mathcal{R}(\mathrm{Fil}^\bullet_p \mathbb{Z}_p) \simeq \mathrm{Spf}\, \mathbb{Z}_p[u,t]/(ut-p)/\mathbb{G}_m,$$

where u has degree -1 and t has degree 1. We can interpret u as yielding, for every $i \in \mathbb{Z}$, a commuting diagram

$$\begin{tikzcd} \mathrm{Fil}^{i-1} A \cdot t^{-i+1} \arrow[r, "u"] \arrow[dr, "p"'] & \mathrm{Fil}^i A \cdot t^{-i} \arrow[d, "t"] \\ & \mathrm{Fil}^{i-1} A \cdot t^{-i+1}. \end{tikzcd}$$

Note that this is compatible with our hypothesis that $R_A = \mathrm{gr}^0 A$ is an $\mathbb{F}_p$-algebra.

5.9.2. The relatively affine map

$$\mathcal{R}(\mathrm{Fil}^\bullet A)_{(u=0)} \to \mathcal{R}(\mathrm{Fil}^\bullet_p \mathbb{Z}_p)_{(u=0)} \simeq \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, \mathbb{F}_p$$

corresponds to a lift $\mathrm{Fil}^\bullet_{\mathrm{Hdg}} \overline{A}$ of $\overline{A}$ to an animated filtered commutative ring: this is the **Hodge filtration** on $\overline{A}$. By construction, we have

$$\mathrm{Fil}^i_{\mathrm{Hdg}} \overline{A} = \mathrm{hcoker}(u : \mathrm{Fil}^{i-1} A \cdot t^{-i+1} \to \mathrm{Fil}^i A \cdot t^{-i}).$$

Example 5.9.3. Let us take the Witt frame from Example 5.4.8 associated with $R \in \mathrm{CRing}_{\mathbb{F}_p/}$. The filtration in this case is p-adic: The map u is an isomorphism in filtered degree $i \geq 1$, is the map $F : W(R) \to F_*W(R)$ in degree 0 and is multiplication by p in negative degrees. Thus we have

$$\mathrm{Fil}^i_{\mathrm{Hdg}} W(R)/^{\mathbb{L}}p \xrightarrow{\simeq} \begin{cases} \mathrm{hcoker}(F : W(R) \to F_*W(R)) & \text{if } i = 1 \\ W(R)/^{\mathbb{L}}p & \text{if } i \leq 0 \\ 0 & \text{otherwise.} \end{cases}$$

In particular, we find that $\mathrm{Fil}^\bullet_{\mathrm{Hdg}} W(R)/^{\mathbb{L}}p$ is a square-zero extension of $\mathrm{Fil}^\bullet_{\mathrm{triv}} R$, and we have

$$\mathrm{hker}(\mathrm{Fil}^\bullet_{\mathrm{Hdg}} W(R)/^{\mathbb{L}}p \xrightarrow{\simeq} \mathrm{Fil}^\bullet_{\mathrm{triv}} R) \simeq \mathrm{hcoker}(F : W(R) \to F_*W(R)) \otimes_R \mathrm{Fil}^\bullet_{\mathrm{triv}} R(1),$$

where $\mathrm{Fil}^\bullet_{\mathrm{triv}} R(1)$ is the free filtered module of rank 1 over R with associated graded supported in degree -1.[13]

Note also that the associated graded algebra $\mathrm{gr}^\bullet_{\mathrm{Hdg}} W(R)/^{\mathbb{L}}p$ is supported in degrees $-1, 0$, and is isomorphic as a graded R-algebra to

$$\mathrm{hcoker}(F : W(R) \to F_*W(R)) \oplus R$$

with R in degree 0.

5.9.4. Similarly, the relatively affine map

$$\mathcal{R}(\mathrm{Fil}^\bullet A)_{(t=0)} \to \mathcal{R}(\mathrm{Fil}^\bullet_p \mathbb{Z}_p)_{(t=0)} \simeq \mathbb{A}^1_+/\mathbb{G}_m \times \mathrm{Spec}\, \mathbb{F}_p$$

corresponds to an increasingly filtered animated commutative $\mathbb{F}_p$-algebra: The underlying animated commutative $\mathbb{F}_p$-algebra corresponds to the derived affine scheme $\mathcal{R}(\mathrm{Fil}^\bullet A)_{(t=0,u\neq 0)}$, and the degree $-i$ component of the associated Rees algebra is isomorphic to $\mathrm{gr}^i A \cdot u^i$.

The filtered Frobenius lift Φ yields a map

$$\mathrm{Spec}\, \overline{A} \simeq \mathcal{R}(\mathrm{Fil}^\bullet_p A)_{(p=0,u\neq 0)} \to \mathcal{R}(\mathrm{Fil}^\bullet_p A)_{(p=0)} \xrightarrow{\mathcal{R}(\Phi)} \mathcal{R}(\mathrm{Fil}^\bullet A)_{(p=0)}$$

that factors canonically through $\mathcal{R}(\mathrm{Fil}^\bullet_p A)_{(t=0,u\neq 0)}$. In applications, the resulting map $\mathrm{Spec}\, \overline{A} \to \mathcal{R}(\mathrm{Fil}^\bullet_p A)_{(t=0,u\neq 0)}$ will often be an isomorphism, and so we will slightly abuse notation by writing $\mathrm{Fil}_\bullet^{\mathrm{conj}} \overline{A}$ for the filtered animated commutative ring associated with $\mathcal{R}(\mathrm{Fil}^\bullet A)_{(t=0)}$, and call it the **conjugate filtration** on $\overline{A}$.

[13] Recall that according to our convention the i-th associated graded piece for a decreasing filtration is in graded degree $-i$.

Remark 5.9.5. Note that the composition

$$R_A = \mathrm{gr}^0 A \simeq \mathrm{Fil}^{\mathrm{conj}}_0 \overline{A} \to \overline{A} \to R_A$$

is the Frobenius endomorphism of R_A.

Remark 5.9.6. The restrictions of both $\mathcal{R}(\mathrm{Fil}^\bullet A)_{(t=0)}$ and $\mathcal{R}(\mathrm{Fil}^\bullet A)_{(u=0)}$ over $B\mathbb{G}_m \times \mathrm{Spec}\,\mathbb{F}_p$ are isomorphic to $\mathcal{R}(\mathrm{Fil}^\bullet A)_{(u=t=0)}$: this identification corresponds to an isomorphism of graded animated commutative rings

$$\mathrm{gr}^{\mathrm{conj}}_\bullet \overline{A} \xrightarrow{\simeq} \mathrm{gr}^\bullet_{\mathrm{Hdg}} \overline{A}.$$

Example 5.9.7. Let us return to the Witt frame from Example 5.9.3. Here, we have

$$\mathrm{Fil}^{\mathrm{conj}}_i W(R)/^{\mathbb{L}}p = \mathrm{gr}^i_{\mathrm{Lau}} W(R) \simeq \begin{cases} R & \text{if } i = 0 \\ F_*W(R)/^{\mathbb{L}}p & \text{if } i \geq 1 \\ 0 & \text{otherwise.} \end{cases}$$

The transition maps $\mathrm{Fil}^{\mathrm{conj}}_i W(R)/^{\mathbb{L}}p \to \mathrm{Fil}^{\mathrm{conj}}_{i+1} W(R)/^{\mathbb{L}}p$ are the identity when $i \geq 1$, while the map

$$R = \mathrm{Fil}^{\mathrm{conj}}_0 W(R)/^{\mathbb{L}}p \to \mathrm{Fil}^{\mathrm{conj}}_1 W(R)/^{\mathbb{L}}p = F_*W(R)/^{\mathbb{L}}p$$

is induced from the commuting diagram

$$\begin{tikzcd} W(R) \arrow[r, "F"] \arrow[d] & F_*W(R) \arrow[d] \\ R \arrow[r] & F_*W(R)/^{\mathbb{L}}p. \end{tikzcd}$$

The map $\mathrm{Fil}^{\mathrm{conj}}_i W(R)/^{\mathbb{L}}p \to W(R)/^{\mathbb{L}}p$ for $i \geq 1$ is simply the identity.

It's easy to see from this description that the associated graded for the conjugate filtration is isomorphic to that of the Hodge filtration.

5.9.8. We have canonical maps

$$\alpha_- : \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R_{A'} \to \mathcal{R}(\mathrm{Fil}^\bullet A')_{(u=0)}\ ;\ \alpha_+ : \mathbb{A}^1_+/\mathbb{G}_m \times \mathrm{Spec}\, R_{A'} \to \mathcal{R}(\mathrm{Fil}^\bullet A')_{(t=0)}$$

corresponding to the natural maps

$$\mathrm{Fil}^\bullet_{\mathrm{Hdg}} \overline{A}' \to \mathrm{Fil}^\bullet_{\mathrm{triv}} R_{A'}\ ;\ \mathrm{Fil}^{\mathrm{conj}}_\bullet \overline{A}' \to \mathrm{Fil}^{\mathrm{triv}}_\bullet R_{A'}$$

of animated filtered (resp. increasingly filtered) commutative rings.

In particular, α_- gives a map $\alpha_0 : B\mathbb{G}_m \times \mathrm{Spec}\, R_{A'} \to \mathcal{R}(\mathrm{Fil}^\bullet A')_{(u=t=0)}$, and we can use it to view $\mathcal{R}(\mathrm{Fil}^\bullet A')_{(u=0)}$, $\mathcal{R}(\mathrm{Fil}^\bullet A')_{(t=0)}$ and $\mathcal{R}(\mathrm{Fil}^\bullet A')_{(u=t=0)}$ as $R_{A'}$-pointed graded stacks.

Remark 5.9.9. The restriction of α_+ to $B\mathbb{G}_m \times \mathrm{Spec}\, R_{A'}$ gives another map from it to $\mathcal{R}(\mathrm{Fil}^\bullet A')_{(t=0)}$: By Remark 5.9.5, this differs from α_0 by a Frobenius twist. This is a manifestation of the fact that the maps α_- and α_+ in fact underlie a map of *frames* $\underline{A} \otimes \mathbb{F}_p \to \underline{W_1(R_A)}$ as observed in Lemma 5.4.10.

Definition 5.9.10. Define étale sheaves on R_A

$$\begin{aligned} \Gamma_{(t=0)}(\mathcal{X}) &: R_{A'} \mapsto \mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet A')_{(t=0)}, \mathcal{X}); \\ \Gamma_{(u=0)}(\mathcal{X}) &: R_{A'} \mapsto \mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet A')_{(u=0)}, \mathcal{X}); \\ \Gamma_{(u=t=0)}(\mathcal{X}) &: R_{A'} \mapsto \mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet A')_{(u=t=0)}, \mathcal{X}); \\ \Gamma_{\overline{A}}(\mathcal{X}) &: R_{A'} \mapsto \mathcal{X}(\overline{A}'). \end{aligned}$$

All spaces of maps are over $\mathcal{R}(\mathrm{Fil}^\bullet A) \otimes \mathbb{F}_p$, and we are viewing $\mathrm{Spec}\,\overline{A}'$ as a scheme over it via τ.

5.9.11. Let X^-, X^+, X^0 be the attractor, repeller and fixed point stacks of $\mathcal{X}$ obtained from α^-, α^+ and α_0, respectively. Then Remark 5.9.9 shows that we have maps $X^- \to X^0$ and $X^+ \to {}^{\varphi}X^0$, where

$${}^{\varphi}X^0(C) = X^0(\varphi_* C).$$

For each $x^- \in X^-(C)$, pulling back the tangent complex of $\mathcal{X}^\diamond$ along the associated map $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} C \to \mathcal{X}^\diamond$ yields a filtered perfect complex $\operatorname{Fil}^\bullet M_{x^-}$ over C; in this way, we get a filtered perfect complex $\operatorname{Fil}^\bullet_{\mathrm{Hdg}} M(\mathcal{X})$ over X^-. Note that the underlying perfect complex $M(\mathcal{X})$ is the pullback of the tangent complex of X over R_A.

Similarly, we have an increasingly filtered perfect complex $\operatorname{Fil}^{\mathrm{conj}}_\bullet M(\mathcal{X})$ over X^+, and a graded perfect complex $\operatorname{gr}^\bullet_{\mathrm{Hdg}} M(\mathcal{X})$ over X^0. The pullback of $\operatorname{gr}^\bullet_{\mathrm{Hdg}} M(\mathcal{X})$ over ${}^{\varphi}X^0$ is equipped with an isomorphism to $\operatorname{gr}^{\mathrm{conj}}_\bullet M(\mathcal{X})$, and its pullback over X^- is isomorphic to the associated graded for $\operatorname{Fil}^\bullet_{\mathrm{Hdg}} M(\mathcal{X})$, which explains the notation.

5.9.12. Let $Z^1_{\underline{A}}$ be the quasicoherent étale sheaf[14] over R_A given by

$$Z^1_{\underline{A}}(R_{A'}) = \operatorname{Fil}^{\mathrm{conj}}_1 \overline{A}' \times_{\overline{A}'} \operatorname{Fil}^1_{\mathrm{Hdg}} \overline{A}' \simeq \operatorname{hker}(\operatorname{Fil}^{\mathrm{conj}}_1 \overline{A}' \to \varphi_* R_{A'}).$$

The $R_{A'}$-module structure is given via the isomorphism $R_{A'} = \operatorname{gr}^0 A' \xrightarrow{\simeq} \operatorname{Fil}^{\mathrm{conj}}_0 \overline{A}'$.

Let $H^1_{\underline{A}}$ be the quasicoherent étale sheaf

$$H^1_{\underline{A}} : R_{A'} \mapsto \operatorname{gr}^{\mathrm{conj}}_1 \overline{A}'.$$

Note that we have two maps

$$q_1, q_2 : Z^1_{\underline{A}} \to H^1_{\underline{A}}.$$

The first of these is obtained via the map $\operatorname{Fil}^1_{\mathrm{conj}} \overline{A} \to \operatorname{gr}^{\mathrm{conj}}_1 \overline{A}$, and is $\mathcal{O}$-linear, while the second is obtained from the composition

$$\operatorname{Fil}^1_{\mathrm{Hdg}} \overline{A} \to \operatorname{gr}^1_{\mathrm{Hdg}} \overline{A} \xrightarrow{\simeq} \operatorname{gr}^{\mathrm{conj}}_1 \overline{A}.$$

This one is φ-semilinear.

Example 5.9.13. Let us return to the example of the Witt frame. Here, using Examples 5.9.3 and 5.9.7, we see that

$$\begin{aligned} H^1_{\underline{W(R)}}(R) &\simeq \operatorname{hcoker}(F : W(R) \to F_* W(R)); \\ Z^1_{\underline{W(R)}}(R) &\simeq \operatorname{hker}(F_* W(R)/^{\mathbb{L}} p = W(R)/^{\mathbb{L}} p \to R) \simeq \operatorname{hcoker}(F : W(R) \to F_* W(R)). \end{aligned}$$

Lemma 5.9.14. *Let $\varpi : X^0 \to \operatorname{Spec} R_A$ be the structure map.*

(1) There is a canonical equivalence

$$\Gamma_{\overline{A}}(\mathcal{X}) \times_X X^- \xrightarrow{\simeq} \Gamma_{(u=0)}(\mathcal{X}).$$

(2) There is a canonical Cartesian square of prestacks over R_A

$$\begin{tikzcd} \Gamma_{(t=0)}(\mathcal{X}) \arrow[r] \arrow[d] & X^0 \arrow[d, "0"] \\ X^+ \times_{{}^{\varphi}X^0} X^0 \arrow[r] & \tau^{\leq 0}(\operatorname{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X}) \otimes_{\mathcal{O}} \varpi^* Z^1_{\underline{A}}[1]) \end{tikzcd}$$

(3) There is a canonical map

$$\Gamma_{(u=t=0)}(\mathcal{X}) \to X^0$$

[14] Recall that all sheaves are with respect to the *small* étale site.

presenting the source as a trivial torsor under $\tau^{\leq 0}(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X})\otimes_{\mathcal{O}}\varpi^* H^1_{\underline{A}})$. In particular, there is a Cartesian square

$$\begin{tikzcd}
\Gamma_{(u=t=0)}(\mathcal{X}) \arrow[r] \arrow[d] & X^0 \arrow[d, "0"] \\
X^0 \arrow[r] & \tau^{\leq 0}(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X})\otimes_{\mathcal{O}}\varpi^* H^1_{\underline{A}}[1])
\end{tikzcd}$$

Proof. The first assertion is immediate from filtered integrability. The other two will use graded integrability. For instance, we have

$$\begin{aligned}
\Gamma_{(u=0)}(\mathcal{X})(R_{A'}) &\simeq \mathrm{Map}\left(\mathcal{R}_+(\mathrm{Fil}^{\mathrm{conj}}_\bullet \overline{A}'),\mathcal{X}\right)\\
&\simeq \mathrm{Map}\left(\mathrm{Spec}(\mathrm{Fil}^{\mathrm{conj}}_1\overline{A}'\cdot u\oplus \mathrm{Fil}^{\mathrm{conj}}_0\overline{A}')/\mathbb{G}_m,\mathcal{X}\right),
\end{aligned}$$

and

$$\begin{aligned}
X^+(R_{A'}) &\simeq \mathrm{Map}(\mathbb{A}^1_+/\mathbb{G}_m\times \mathrm{Spec}\, R_{A'},\mathcal{X})\\
&\simeq \mathrm{Map}\left(\mathrm{Spec}(\varphi_*(R_{A'}u\oplus R_{A'}))/\mathbb{G}_m,\mathcal{X}\right).
\end{aligned}$$

This description shows that $\Gamma_{(u=0)}(\mathcal{X})$ is a trivial torsor over X^0 under the sheaf $\tau^{\leq 0}(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X})\otimes_{\mathcal{O}}\varpi^*\,\mathrm{Fil}^{\mathrm{conj}}_1)$: this is a straightforward application of graded deformation theory. Here, $\mathrm{Fil}^{\mathrm{conj}}_1$ is the étale sheaf over R_A given by $R_{A'}\mapsto \mathrm{Fil}^{\mathrm{conj}}_1\overline{A}'$.

Similarly, X^+ is a trivial torsor over ${}^{\varphi}X^0$—and therefore $X^+\times_{{}^{\varphi}X^0} X^0$ is a trivial torsor over X^0—under the sheaf $\tau^{\leq 0}(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X})\otimes_{\mathcal{O}}\varpi^*\varphi_*\mathcal{O})$.

Assertion (2) follows from this and the definition of $Z^1_{\underline{A}}$. Assertion (3) is shown in similar fashion. □

5.9.15. We have closed immersions

$$\lambda_+:\mathcal{R}(\mathrm{Fil}^\bullet A)_{(u=t=0)}\to \mathcal{R}(\mathrm{Fil}^\bullet A)_{(t=0)}\ ;\ \lambda_-:\mathcal{R}(\mathrm{Fil}^\bullet A)_{(u=t=0)}\to \mathcal{R}(\mathrm{Fil}^\bullet A)_{(u=0)},$$

There is also an open immersion

$$j_-:\mathrm{Spec}\,\overline{A}\simeq \mathcal{R}(\mathrm{Fil}^\bullet A)_{(u=0,t\neq 0)}\hookrightarrow \mathcal{R}(\mathrm{Fil}^\bullet A)_{(u=0)},$$

through which the map $\tau:\mathrm{Spec}\,\overline{A}\to\mathcal{R}(\mathrm{Fil}^\bullet A)\otimes\mathbb{F}_p$ factors, and a map

$$j_+:\mathrm{Spec}\,\overline{A}\to \mathcal{R}(\mathrm{Fil}^\bullet A)_{(t=0,u\neq 0)}\hookrightarrow \mathcal{R}(\mathrm{Fil}^\bullet A)_{(t=0)}.$$

Via j_- and j_+ along with the datum of the isomorphism ξ gives us maps

$$j_-^*:\Gamma_{(u=0)}(\mathcal{X})\to\Gamma_{\overline{A}}(\mathcal{X})\ ;\ \xi\circ j_+^*:\Gamma_{(t=0)}(\mathcal{X})\to \Gamma_{\overline{A}}(\mathcal{X}).$$

Using the cohesiveness of $\mathcal{X}^\diamond$ and the fact that (u,t) is a regular sequence in $\mathbb{Z}_p[u,t]/(ut-p)$, one now finds:

Lemma 5.9.16. *We have an isomorphism of étale sheaves*

$$\Gamma_{\underline{A}}(\mathcal{X},\xi)\xrightarrow{\simeq}\mathrm{eq}\left(\Gamma_{(t=0)}(\mathcal{X})\times_{\Gamma_{\overline{A}}(\mathcal{X})}\Gamma_{(u=0)}(\mathcal{X}) \underset{\lambda_-^*\circ\mathrm{pr}_2}{\overset{\lambda_+^*\circ\mathrm{pr}_1}{\rightrightarrows}} \Gamma_{(u=t=0)}(\mathcal{X})\right).$$

5.9.17. We will now combine the presentation from Lemma 5.9.16 with the information from Lemma 5.9.14. For this, first note that we have two maps

$$\mathrm{pr}_1,\iota^*\circ\mathrm{pr}_3: X^0\times_{{}^{\varphi}X^0} X^+\times_X X^-\to X^0,$$

where $\iota^*:X^-\to X^0$ is the map induced from the closed immersion $B\mathbb{G}_m\times\mathrm{Spec}\,R_A\to\mathbb{A}^1/\mathbb{G}_m\times R_A$.

Set[15]

$$\Gamma_{F\mathrm{Zip}}(\mathcal{X}) = \mathrm{eq}\left(X^0 \times_{\varphi X^0} X^+ \times_X X^- \underset{\iota^*\circ \mathrm{pr}_3}{\overset{\mathrm{pr}_1}{\rightrightarrows}} X^0 \right)$$

5.9.18. We will need one more construction before we can state the main result of this subsection. The various filtered and graded complexes over X^-, X^+, X^0 considered in (5.9.11) can be pulled back to $\Gamma_{F\mathrm{Zip}}(\mathcal{X})$ along the natural maps out of it, and we will denote these pullbacks by the same symbol. Note tht we have isomorphisms

$$\varphi^* \operatorname{gr}^i_{\mathrm{Hdg}} M(\mathcal{X}) \xrightarrow{\simeq} \operatorname{gr}^{\mathrm{conj}}_i M(\mathcal{X})$$

over $\Gamma_{F\mathrm{Zip}}(\mathcal{X})$.

In particular, by 1-boundedness, this gives us maps

$$\varphi^* \operatorname{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X}) \xrightarrow{\simeq} \operatorname{Fil}^{\mathrm{conj}}_{-1} M(\mathcal{X}) \to M(\mathcal{X}) \to \operatorname{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X})$$

whose composition yields a φ-semilinear endomorphism

$$\psi : \varphi^* \operatorname{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X}) \to \operatorname{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X}).$$

Combining the maps $q_1, q_2 : Z^1_{\underline{A}} \to H^1_{\underline{A}}$ with ψ yields two further maps

$$q_{1,\psi}, q_{2,\psi} : \operatorname{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X}) \otimes_{\mathcal{O}} \omega^* Z^1_{\underline{A}} \to \operatorname{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X}) \otimes_{\mathcal{O}} \omega^* H^1_{\underline{A}}.$$

Here, $\omega : \Gamma_{F\mathrm{Zip}}(\mathcal{X}) \to \operatorname{Spec} R_A$ is the structure map of sheaves on $(R_A)_{\mathrm{\acute{e}t}}$. The map $q_{1,\psi}$ is simply given by $\mathrm{id} \otimes \omega^* q_1$, and is independent of ψ, while the map $q_{2,\psi}$ is given by $\psi \otimes \omega^* q_2$.

We now set for any $i \in \mathbb{Z}$

$$\Gamma_{\underline{A},\psi}(\operatorname{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X})[i]) \overset{\mathrm{defn}}{=} \tau^{\leq 0} \operatorname{hker}(q_{1,\psi}[i] - q_{2,\psi}[i]).$$

Proposition 5.9.19. *Suppose that $\underline{A}$ is prismatic. Then there is a canonical Cartesian diagram of étale sheaves*

$$\begin{tikzcd} \Gamma_{\underline{A}}(\mathcal{X}, \xi) \arrow[r] \arrow[d] & \Gamma_{F\mathrm{Zip}}(\mathcal{X}) \arrow[d, "0"] \\ \Gamma_{F\mathrm{Zip}}(\mathcal{X}) \arrow[r] & \Gamma_{\underline{A},\psi}(\operatorname{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X})[1]). \end{tikzcd}$$

5.9.20. The proof of the proposition will take some preparation. We begin by considering the linear case. Let us put ourselves in the situation of (5.8.13), so that we have a perfect $\underline{A}$-gauge $(\mathcal{M}, \xi)$ of level 1.

The restriction of $\mathcal{M}$ over $\mathcal{R}(\operatorname{Fil}^\bullet A)_{(u=0)}$ corresponds to a filtered module over $\operatorname{Fil}^\bullet_{\mathrm{Hdg}} \overline{A}$ that we denote by $\operatorname{Fil}^\bullet_{\mathrm{Hdg}} \mathsf{M}$. Similarly, the restriction over $\mathcal{R}(\operatorname{Fil}^\bullet A)_{(t=0)}$ corresponds to an increasingly filtered module over $\operatorname{Fil}^{\mathrm{conj}}_\bullet \underline{A}$ that we denote by $\operatorname{Fil}^{\mathrm{conj}}_\bullet \mathsf{M}$. The isomorphism ξ now tells us that the underlying $\overline{A}$-modules of both are identified with the same $\overline{A}$-module M. Furthermore, the identification with the common restriction over $\mathcal{R}(\operatorname{Fil}^\bullet A)_{(u=t=0)}$ tells us that we have an isomorphism

$$\eta_\bullet : \operatorname{gr}^\bullet_{\mathrm{Hdg}} \mathsf{M} \xrightarrow{\simeq} \operatorname{gr}^{\mathrm{conj}}_\bullet \mathsf{M}$$

compatible with the isomorphism of graded animated commutative rings $\operatorname{gr}^\bullet_{\mathrm{Hdg}} \overline{A} \xrightarrow{\simeq} \operatorname{gr}^{\mathrm{conj}}_\bullet \overline{A}$.

Now, in the notation used in Lemma 5.8.14, we have

$$R\Gamma_{\underline{A}}(\mathcal{M}, \xi)(R_A) \simeq \operatorname{hker}(\operatorname{Fil}^{\mathrm{conj}}_0 \mathsf{M} \times_{\mathsf{M}} \operatorname{Fil}^0_{\mathrm{Hdg}} \mathsf{M} \xrightarrow{\overline{\mathrm{pr}}_1 - \eta_0 \circ \overline{\mathrm{pr}}_2} \operatorname{gr}^{\mathrm{conj}}_0 \mathsf{M})$$

Here, the $\overline{(\cdot)}$ over the projection maps denotes that we are projecting and then mapping onto the corresponding associated graded component.

[15] The notation will be further explained in § (8.7): This is essentially the prestack of sections of $\mathcal{X}$ over $\underline{\mathit{Syn}}(\underline{W_1(R_A)})$, which we will later recognize as the stack $R_A^{F\mathrm{Zip}}$. Vector bundles over this stack are *F-zips* over R_A.

5.9.21. Restricting further along α_- gives us a filtered module $\mathrm{Fil}^\bullet_{\mathrm{Hdg}} M$ over R_A, and doing the same with α_+ gives us an increasingly filtered module $\mathrm{Fil}^{\mathrm{conj}}_\bullet M$ over R_A: the underlying R_A-modules for both are identified with the same module M.

Moreover, $\eta_\bullet$ induces an isomorphism[16]

$$\overline{\eta}_\bullet : \varphi^* \mathrm{gr}^\bullet_{\mathrm{Hdg}} M \xrightarrow{\simeq} \mathrm{gr}^{\mathrm{conj}}_\bullet M.$$

Write

$$\beta_\bullet : \mathrm{gr}^\bullet_{\mathrm{Hdg}} M \to \mathrm{gr}^{\mathrm{conj}}_\bullet M$$

for the underlying φ-semilinear map.

Set

$$R\Gamma_{F\mathrm{Zip}}(\mathcal{M},\xi)(R_A) \overset{\mathrm{defn}}{=} \mathrm{hker}\left(\mathrm{Fil}^{\mathrm{conj}}_0 M \times_M \mathrm{Fil}^0_{\mathrm{Hdg}} M \xrightarrow{\overline{\mathrm{pr}}_1 - \beta_0 \circ \overline{\mathrm{pr}}_2} \mathrm{gr}^{\mathrm{conj}}_0 M\right)$$

5.9.22. Suppose now that the restriction of $\mathcal{M}$ over $B\mathbb{G}_m \times \mathrm{Spec}\, R_A$ is 1-bounded; equivalently, we have

$$\mathrm{gr}^i_{\mathrm{Hdg}} M \simeq \mathrm{gr}^{\mathrm{conj}}_i M \simeq 0$$

for $i < -1$.

Just as in (5.9.18), the isomorphism $\overline{\eta}_{-1} : \varphi^* \mathrm{gr}^{-1}_{\mathrm{Hdg}} M \xrightarrow{\simeq} \mathrm{gr}^{\mathrm{conj}}_0 M$ can now be used to define a map

$$\psi : \varphi^* \mathrm{gr}^{-1}_{\mathrm{Hdg}} M \to \mathrm{gr}^{-1}_{\mathrm{Hdg}} M$$

and maps

$$q_{1,\psi}, q_{2,\psi} : \mathrm{gr}^{-1}_{\mathrm{Hdg}} M \otimes_{R_A} Z^1_{\underline{A}}(R_A) \to \mathrm{gr}^{-1}_{\mathrm{Hdg}} M \otimes_{R_A} H^1_{\underline{A}}(R_A)$$

Set

$$R\Gamma_{\underline{A},\psi}(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M)(R_A) \overset{\mathrm{defn}}{=} \mathrm{hker}\left(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M \otimes_{R_A} Z^1_{\underline{A}}(R_A) \xrightarrow{q_{1,\psi} - q_{2,\psi}} \mathrm{gr}^{-1}_{\mathrm{Hdg}} M \otimes_{R_A} H^1_{\underline{A}}(R_A)\right).$$

Lemma 5.9.23. *There is a canonical fiber sequence*

$$R\Gamma_{\underline{A},\psi}(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M)(R_A) \to R\Gamma_{\underline{A}}(\mathcal{M},\xi)(R_A) \to R\Gamma_{F\mathrm{Zip}}(\mathcal{M},\xi)(R_A).$$

Proof. The following facts can be seen exactly as in the proof of Lemma 5.9.14:

- We have
$$\mathrm{Fil}^0_{\mathrm{Hdg}} \mathsf{M} \simeq \mathsf{M} \times_M \mathrm{Fil}^0_{\mathrm{Hdg}} M.$$
- There is a canonical fiber sequence
$$\mathrm{gr}^{-1}_{\mathrm{Hdg}} M \otimes_{R_A} Z^1_{\underline{A}}(R_A) \to \mathrm{Fil}^{\mathrm{conj}}_0 \mathsf{M} \to \mathrm{Fil}^{\mathrm{conj}}_0 M \times_{\mathrm{gr}^{\mathrm{conj}}_0 M} \mathrm{gr}^0_{\mathrm{Hdg}} M,$$
where the map $\mathrm{Fil}^{\mathrm{conj}}_0 \mathsf{M} \to \mathrm{gr}^0_{\mathrm{Hdg}} M$ is obtained as the composition
$$\mathrm{Fil}^{\mathrm{conj}}_0 \mathsf{M} \to \mathrm{gr}^{\mathrm{conj}}_0 \mathsf{M} \xrightarrow{\simeq} \mathrm{gr}^0_{\mathrm{Hdg}} \mathsf{M} \to \mathrm{gr}^0_{\mathrm{Hdg}} M.$$
- There is a canonical fiber sequence
$$\mathrm{gr}^{-1}_{\mathrm{Hdg}} M \otimes_{R_A} H^1_{\underline{A}}(R_A) \to \mathrm{gr}^{\mathrm{conj}}_0 \mathsf{M} \to \mathrm{gr}^0_{\mathrm{Hdg}} M.$$

Furthermore, we have

$$R\Gamma_{F\mathrm{Zip}}(\mathcal{M},\xi)(R_A) \simeq \mathrm{hker}\left(\mathrm{gr}^0_{\mathrm{Hdg}} M \times_{\mathrm{gr}^{\mathrm{conj}}_0 M} \mathrm{Fil}^{\mathrm{conj}}_0 M \times_M \mathrm{Fil}^0_{\mathrm{Hdg}} M \xrightarrow{\mathrm{pr}_1 - \overline{\mathrm{pr}}_3} \mathrm{gr}^0_{\mathrm{Hdg}} M\right)$$

Combining this with the facts from the beginning of the proof shows that we have an isomorphism

$$\mathrm{hker}\left(R\Gamma_{\underline{A}}(\mathcal{M},\xi)(R_A) \to R\Gamma_{F\mathrm{Zip}}(\mathcal{M},\xi)(R_A)\right) \simeq \mathrm{hker}\left(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M \otimes_{R_A} Z^1_{\underline{A}}(R_A) \xrightarrow{r_1 - r_2} \mathrm{gr}^{-1}_{\mathrm{Hdg}} M \otimes_{R_A} H^1_{\underline{A}}(R_A)\right),$$

where r_1 and r_2 admit the following description:

[16]What we have here is a *perfect F-zip* over R_A in the terminology of § 8.2.

- r_1 sits in a commuting diagram where the rows are fiber sequences:

$$\begin{tikzcd}
\mathrm{gr}^{-1}_{\mathrm{Hdg}} M \otimes_{R_A} Z^1_{\underline{A}}(R_A) \arrow[r] \arrow[d, "r_1"'] & \mathrm{Fil}^{\mathrm{conj}}_0 \mathsf{M} \arrow[r] \arrow[d] & \mathrm{gr}^0_{\mathrm{Hdg}} M \times_{\mathrm{gr}^{\mathrm{conj}}_0 M} \mathrm{Fil}^{\mathrm{conj}}_0 M \arrow[d, "\mathrm{pr}_1"] \\
\mathrm{gr}^{-1}_{\mathrm{Hdg}} M \otimes_{R_A} H^1_{\underline{A}}(R_A) \arrow[r] & \mathrm{gr}^{\mathrm{conj}}_0 \mathsf{M} \arrow[r] & \mathrm{gr}^0_{\mathrm{Hdg}} M
\end{tikzcd}$$

- r_2 sits in a commuting diagram where the rows are fiber sequences:

$$\begin{tikzcd}
\mathrm{gr}^{-1}_{\mathrm{Hdg}} M \otimes_{R_A} Z^1_{\underline{A}}(R_A) \arrow[r] \arrow[d, "r_2"'] & \mathrm{Fil}^{\mathrm{conj}}_0 \mathsf{M} \arrow[r] \arrow[d, "\eta_0 \circ \mathrm{pr}_2"] & \mathrm{gr}^0_{\mathrm{Hdg}} M \times_{\mathrm{gr}^{\mathrm{conj}}_0 M} \mathrm{Fil}^{\mathrm{conj}}_0 M \times_M \mathrm{Fil}^0_{\mathrm{Hdg}} M \arrow[d, "\mathrm{pr}_3"] \\
\mathrm{gr}^{-1}_{\mathrm{Hdg}} M \otimes_{R_A} H^1_{\underline{A}}(R_A) \arrow[r] & \mathrm{gr}^{\mathrm{conj}}_0 \mathsf{M} \arrow[r] & \mathrm{gr}^0_{\mathrm{Hdg}} M
\end{tikzcd}$$

One sees immediately that r_1 can be identified canonically with $q_{1,\psi} = \mathrm{id} \otimes q_1$. To finish the proof of the lemma, we need to check that r_2 can be canonically identified with $q_{2,\psi}$.

For this, simply note that the map $\mathrm{gr}^{-1}_{\mathrm{Hdg}} M \otimes_{R_A} Z^1_{\underline{A}}(R_A) \to \mathrm{Fil}^{\mathrm{conj}}_0 \mathsf{M}$ arises via the equivalences

$$\mathrm{Fil}^{\mathrm{conj}}_{-1} \mathsf{M} \xrightarrow{\simeq} \mathrm{gr}^{-1}_{\mathrm{Hdg}} \mathsf{M} \xrightarrow{\simeq} \mathrm{gr}^{-1}_{\mathrm{Hdg}} M$$

and the natural R_A-linear map $\mathrm{Fil}^{\mathrm{conj}}_{-1} \mathsf{M} \to \mathrm{Fil}^{\mathrm{conj}}_0 \mathsf{M}$. □

Proof of Proposition 5.9.19. It is enough to show that we have the following commuting cubes where the front and back faces are Cartesian:

$$\begin{tikzcd}
\Gamma_{(t=0)}(\mathcal{X}) \arrow[rr] \arrow[dr, "\lambda_+^*"] \arrow[dd] & & X^0 \arrow[dr, equals] \arrow[dd, "0" near start] & \\
& \Gamma_{(u=t=0)}(\mathcal{X}) \arrow[rr] \arrow[dd] & & X^0 \arrow[dd, "0"] \\
X^0 \times_{\varphi X^0} X^+ \arrow[rr] \arrow[dr, "\mathrm{pr}_1"'] & & \tau^{\leq 0}(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X}) \otimes_{\mathcal{O}} Z^1_{\underline{A}}[1]) \arrow[dr, "q_{1,\psi}"] & \\
& X^0 \arrow[rr] & & \tau^{\leq 0}(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X}) \otimes_{\mathcal{O}} H^1_{\underline{A}}[1])
\end{tikzcd}$$

$$
\begin{tikzcd}
\Gamma_{(t=0)}(\mathcal{X}) \times_{\Gamma_{\underline{A}}(\mathcal{X})} \Gamma_{(u=0)}(\mathcal{X}) \arrow[rr] \arrow[dr, "\lambda_-^* \circ \mathrm{pr}_2"'] \arrow[dd] & & X^- \arrow[dr] \arrow[dd, "0"] & \\
& \Gamma_{(u=t=0)}(\mathcal{X}) \arrow[rr] \arrow[dd] & & X^0 \arrow[dd, "0"] \\
X^0 \times_{\varphi X^0} X^+ \times_X X^- \arrow[rr] \arrow[dr, "\iota^* \circ \mathrm{pr}_3"'] & & \tau^{\leq 0}(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X}) \otimes_{\mathcal{O}} Z^1_{\underline{A}}[1]) \times_{X^0} X^- \arrow[dr, "q_{2,\psi}"] & \\
& X^0 \arrow[rr] & & \tau^{\leq 0}(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X}) \otimes_{\mathcal{O}} H^1_{\underline{A}}[1])
\end{tikzcd}
\tag{5.9.23.1}
$$

The existence of the first commuting cube and also the Cartesianness of the front and back faces of both cubes are immediate from Lemma 5.9.14. It remains to verify that the second cube commutes. We will do so using deformation theory to reduce to the case of the Witt frame.

Suppose that we have a map of frames $\underline{A} \to \underline{B}$ such that the underlying map of filtered rings $\mathrm{Fil}^\bullet A \to \mathrm{Fil}^\bullet B$ is a square-zero thickening. We have analogous cubes that we can write down for the restriction of $(\mathcal{X}, \xi)$ over $\mathcal{R}(\mathrm{Fil}^\bullet B) \otimes \mathbb{F}_p$. Arguing as in the proof of Corollary 5.8.11 and using Lemma 5.9.23, one finds that it is enough to know that these latter cubes commute.

Therefore, we can use the nilcompleteness of the various prestacks involved, along with the Postnikov filtration, to reduce to the case where A is a discrete p-complete δ-ring and where $\mathrm{Fil}^i A$ is also a discrete A-module for all $i \geq 0$. Now, Lemma 5.4.10 gives us a canonical map of frames $\underline{A} \to \underline{W(\pi_0(R_A))}$. In fact, one finds from its construction that this map is surjective on the underlying filtered animated commutative rings.

Set $R = \pi_0(R_A)$. As in Remark 5.4.13, classical base-change along $A \to W(R)$ (viewed as the derived base-change, followed by taking the 0-truncation), yields a non-negatively filtered animated commutative ring $\mathrm{Fil}^\bullet W(R)$ lifting $W(R)$, underlying a frame ${}_A\underline{W(R)}$, and admitting maps of frames

$$\underline{A} \to {}_A\underline{W(R)} \to \underline{W(R)}.$$

The kernel of the first map is nilpotent mod-p, and so it is enough to know that the cubes for the frame ${}_A\underline{W(R)}$ commute. Moreover, if $\mathrm{Fil}^\bullet K$ is the kernel of $\mathrm{Fil}^\bullet W(R) \to \mathrm{Fil}^\bullet_{\mathrm{Lau}} W(R)$, then one finds:

- The map $\mathrm{Fil}^1 K \to W(R)$ is zero.
- For all $m \geq 1$, we have
$$\sum_{i_1 + \cdots + i_p = m} \mathrm{Fil}^{i_1} K \cdots \mathrm{Fil}^{i_p} K \subset p \, \mathrm{Fil}^m K.$$

Therefore, the kernel of the second map of frames is also nilpotent mod-p, and so we are finally reduced to the case of the Witt frame $\underline{W(R)}$. Here, using Example 5.9.3, we see that we have a Cartesian square

$$
\begin{tikzcd}
\Gamma_{(u=0)}(\mathcal{X}) \arrow[r] \arrow[d] & X^- \arrow[d] \\
X^- \arrow[r] & \tau^{\leq 0}(M(\mathcal{X}) \otimes_{\mathcal{O}} \mathrm{Fil}^1_{\mathrm{Hdg}}[1]) \times_X X^-.
\end{tikzcd}
$$

One can now verify the outstanding claims by using this as an intermediary between the front and back faces of the cube in (5.9.23.1). $\square$

Here is a useful corollary to Proposition 5.9.19.

Corollary 5.9.24. *Suppose that the following conditions hold:*

- $\mathcal{X}^\diamond$ *is* smooth *over* $\mathcal{R}(\mathrm{Fil}^\bullet A)\otimes\mathbb{F}_p$*;*
- *Its relative tangent complex* $\mathbb{T}_{\mathcal{X}^\diamond}$ *is* 1*-connective;*
- R_A *is semiperfect.*

Then $\Gamma_{\underline{A}}(\mathcal{X},\xi)\to\Gamma_{F\mathrm{Zip}}(\mathcal{X})$ *is a torsor under* $\Gamma_{\underline{A},\psi}(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X}))$.

Proof. It is enough to know that, in assertion (2) of Lemma 5.9.14, we have the stronger assertion that

$$\Gamma_{(t=0)}(\mathcal{X})\to X^+\times_{\varphi X^0} X^0$$

is a torsor under $\tau^{\leq 0}(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X})\otimes_{\mathcal{O}}\varpi^* Z^1_{\underline{A}})$.

Looking at the proof of that assertion, we find that we need to know that the map

$$\mathrm{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X})\otimes_{\mathcal{O}}\varpi^*\,\mathrm{Fil}^{\mathrm{conj}}_1\to\mathrm{gr}^{-1}_{\mathrm{Hdg}} M(\mathcal{X})\otimes_{\mathcal{O}}\varpi^*\varphi_*\mathcal{O}$$

is surjective on connective covers. But in fact our hypothesis on the tangent complex shows that the source and target of this map are already connective. Therefore, it is enough to know that the map

$$R_{A'}\simeq\mathrm{Fil}^{\mathrm{conj}}_0 A'\to\varphi_* R_{A'}$$

is surjective for any étale map $R_A\to R_{A'}$. But this is guaranteed to us by the semiperfectness of R_A, and the fact that any étale algebra over a semiperfect ring is also semiperfect. □

6. The stacks of Drinfeld and Bhatt-Lurie

6.1. **Transmutation.** Suppose that we have a map $\pi: Z\to Y$ of p-adic formal prestacks such that Z is a **relative ring prestack** over Y: For us, this will mean that we have specified a lift of the associated functor

$$\mathrm{CRing}^{p\text{-nilp}}_{Y/}\xrightarrow{(C,y)\mapsto Z((C,y))}\mathrm{Spc}$$

to a presheaf valued in CRing, which we will denote by the same symbol.

Here, $\mathrm{CRing}^{p\text{-nilp}}_{/Y}$ is the ∞-category of pairs (C,y) with $C\in\mathrm{CRing}^{p\text{-nilp}}$ and $y\in Y(C)$, and $Z((C,y))$ is the fiber of $Z(C)$ over y. Then, for any $R\in\mathrm{CRing}^{p\text{-nilp}}$, its **transmutation with respect to** π is the p-adic formal prestack over Y given by

$$\begin{aligned}\mathfrak{Tr}_\pi(R):\mathrm{CRing}^{p\text{-nilp}}_{/Y}&\to\mathrm{Spc}\\ (C,y)&\mapsto\mathrm{Map}_{\mathrm{CRing}}(R,Z((C,y)))\end{aligned}$$

This gives us a limit preserving functor

$$\mathrm{CRing}^{p\text{-nilp},\mathrm{op}}\xrightarrow{\mathrm{Spec}\,R\mapsto\mathfrak{Tr}_\pi(R)}\mathrm{PStk}_{/Y}.$$

6.2. **Cartier-Witt divisors and prismatizations.** Here, we quickly recall the story of (derived) absolute prismatizations from [9, §8].

6.2.1. To begin, we have the p-adic formal prestack $\mathbb{Z}_p^{\triangle}$ (the notation is from [6]) of Cartier-Witt divisors, denoted WCart in [8]. For $R\in\mathrm{CRing}^{p\text{-nilp}}$, $\mathbb{Z}_p^{\triangle}(R)$ parameterizes surjective maps $\pi: W(R)\twoheadrightarrow\overline{W(R)}$ of animated rings such that two properties hold:

- $I=\mathrm{hker}(\pi)$ is a locally free $W(R)$-module of rank 1;
- The map $\pi_0(I)\simeq I\otimes_{W(R)} W(\pi_0(R))\to W(\pi_0(R))$ is a Cartier-Witt divisor in the sense of [8, §3.1.1].

The second condition means that, Zariski-locally on $\mathrm{Spec}\,R$, we have a $W(\pi_0(R))$-linear isomorphism $\pi_0(I)\simeq W(\pi_0(R))$ such that the composition $W(\pi_0(R))\simeq\pi_0(I)\to W(\pi_0(R))$ is given by multiplication by a **distinguished element** $d\in W_{\mathrm{dist}}(\pi_0(R))$, given in Witt coordinates by $(d_0,d_1,\dots)$ with $d_0\in\pi_0(R)$ nilpotent mod-p and with $d_1\in\pi_0(R)^\times$.

In this situation, we will call the map $I\to W(R)$ a **Cartier-Witt divisor** over R.

This description shows (see [9, Proposition 8.4]):

Proposition 6.2.2. *We have $\mathbb{Z}_p^{\Delta} \simeq W_{\mathrm{dist}}/W^\times$, where W_{dist} is represented by the formal spectrum of $\mathbb{Z}_p[x_0, x_1^{\pm 1}, x_2, \ldots]^\wedge_{(x_0,p)}$, and inherits the $W^\times$-action from the natural one on W, where W is represented by the formal spectrum of $\mathbb{Z}_p[x_0, x_1, \ldots]^\wedge_p$. In particular, $\mathbb{Z}_p^{\Delta}$ is* classical.

6.2.3. Over $\mathbb{Z}_p^{\Delta}$, we have the tautological relative ring prestack $\mathbb{G}_a^{\Delta}$ given by $(W(R) \xrightarrow{\pi} \overline{W(R)}) \mapsto \overline{W(R)}$. Transmutation with respect to this (see §6.1) now gives a functorial assignment $R \to R^{\Delta}$ from $\mathrm{CRing}^{p\text{-nilp}}$ to $\mathrm{PStk}_{/\mathbb{Z}_p^{\Delta}}$. Concretely, R^{Δ} associates to any $(W(C) \twoheadrightarrow \overline{W(C)}) \in \mathbb{Z}_p^{\Delta}(C)$ the space $\mathrm{Map}_{\mathrm{CRing}}(R, \overline{W(C)})$. We call R^{Δ} the **prismatization** of R.

Remark 6.2.4. We have a canonical equivalence $\mathrm{Spf}\,\mathbb{Z}_p \xrightarrow{\simeq} \mathbb{F}_p^{\Delta}$ induced by the Cartier-Witt divisor $W(\mathbb{F}_p) = \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p = W(\mathbb{F}_p)$.

Remark 6.2.5. There is a canonical 'Frobenius lift' $\varphi : \mathbb{Z}_p^{\Delta} \to \mathbb{Z}_p^{\Delta}$ arising from the map $F : W \to W$, which carries a Cartier-Witt divisor $I \to W(R)$ to $F^*I \to W(R)$

Remark 6.2.6. If (A, I) is an animated prism as in [9, §2], then there is a canonical map $\iota_{(A,I)} : \mathrm{Spf}\,A \to \mathbb{Z}_p^{\Delta}$ associating to each p-nilpotent A-algebra C, the Cartier-Witt divisor $I \otimes_A W(C) \to W(C)$. Here, $A \to W(C)$ is the canonical lift of $A \to C$ afforded by the δ-ring structure on A. If we have a map $R \to A/I$—that is, if (A, I) lifts to an object in the (animated) prismatic site of R—then $\iota_{(A,I)}$ admits a lift to a map to R^{Δ}.

6.3. **The Hodge-Tate locus.**

6.3.1. Let $\hat{\mathbb{A}}^1$ be the p-adic formal completion of $\mathbb{A}^1$, equipped with the inverse of the usual action of $\mathbb{G}_m$ (as in §4.2). Then, $\hat{\mathbb{A}}^1/\mathbb{G}_m$ parameterizes line bundles $\mathcal{L}$ equipped with a cosection $t : \mathcal{L} \to \mathscr{O}$ that is nilpotent mod-p, meaning that for some (hence any) local trivialization of $\mathcal{L}$, t is given by a section of $\mathscr{O}$ that is nilpotent mod-p.

Then the natural map $W_{\mathrm{dist}} \to \hat{\mathbb{A}}^1$ descends to a map $\mathbb{Z}_p^{\Delta} \to \hat{\mathbb{A}}^1/\mathbb{G}_m$, and the **Hodge-Tate locus** $\mathbb{Z}_p^{\mathrm{HT}}$ is defined to be the fiber product (derived or classical: both are the same in this case)

$$\mathbb{Z}_p^{\mathrm{HT}} = \mathbb{Z}_p^{\Delta} \times_{\hat{\mathbb{A}}^1/\mathbb{G}_m} B\mathbb{G}_m.$$

This is a closed substack of $\mathbb{Z}_p^{\Delta}$ with locally invertible ideal sheaf, which Bhatt-Lurie make a very detailed study of in [8, §3.4]. They show that $\mathbb{Z}_p^{\mathrm{HT}}$ has a somewhat concrete description.

To explain this, first note that we have a canonical map $\mathrm{Spf}\,\mathbb{Z}_p \to \mathbb{Z}_p^{\mathrm{HT}}$ corresponding to the Cartier-Witt divisor $W(\mathbb{Z}_p) \xrightarrow{V(1)} W(\mathbb{Z}_p)$.

Proposition 6.3.2. *The above map presents $\mathbb{Z}_p^{\mathrm{HT}}$ as the formal classifying stack $B\mathbb{G}_m^\sharp$ over* $\mathrm{Spf}\,\mathbb{Z}_p$*. In particular, it is a flat surjection. Moreover, there is a natural equivalence*

$$\mathrm{Spf}\,\mathbb{Z}_p \times_{\mathbb{Z}_p^{\Delta}} \mathbb{F}_p^{\Delta} \xrightarrow{\simeq} \mathbb{G}_m^\sharp \times \mathrm{Spec}\,\mathbb{F}_p.$$

Proof. The first assertion is [8, Theorem 3.4.13]; see also [23, Lemma 4.5.2].

For the second, note that the left hand side is a canonically trivial $\mathbb{G}_m^\sharp$-torsor over

$$\mathbb{F}_p^{\mathrm{HT}} \stackrel{\mathrm{defn}}{=} \mathbb{Z}_p^{\mathrm{HT}} \times_{\mathbb{Z}_p^{\Delta}} \mathbb{F}_p^{\Delta},$$

but this base is canonically identified with the closed subscheme $\mathrm{Spec}\,\mathbb{F}_p \subset \mathrm{Spf}\,\mathbb{Z}_p$ via the equivalence of Remark 6.2.4. ☐

6.4. **The Nygaard filtered prismatization.** The underived story of the Nygaard filtered prismatization is explained in Bhatt's notes [6, §5] following Drinfeld in [23, §5]. We will now explain how to make sense of these constructions in the context of animated rings. This explanation was helped greatly by conversations with Juan Esteban Rodríguez Camargo.

6.4.1. View W as a ring stack over $\mathrm{Spf}\,\mathbb{Z}_p$, associating with each p-nilpotent R the ring of Witt vectors $W(R)$. We will consider modules for the ring stack W in the ∞-category of fppf sheaves over R, and refer to them simply as *W-modules over R*. Here are some examples of W-modules:

- Any quasicoherent sheaf can be viewed as a $\mathbb{G}_a$-module, and is hence equipped with the structure of a W-module via the map $W \to \mathbb{G}_a$.
- Given an invertible R-module L, we can take the divided power envelope of the identity section within the vector group scheme $\mathbf{V}(L)$ to obtain the W-module $\mathbf{V}(L)^\sharp$. Here, the W-action is via the map $W \to \mathbb{G}_a$.
- Given any W-module M, we obtain a new W-module F_*M via restriction along the map $F : W \to W$. If $M = W$, then F_*W is in fact an animated W-algebra.
- Given any $W(R)$-module M, we obtain an associated W-module $I \otimes_{W(R)} W$ that assigns to every R-algebra S the $W(S)$-module $I \otimes_{W(R)} W(S)$.

We now have a diagram of W-modules over $\mathrm{Spf}\,\mathbb{Z}_p$ where all the rows and columns are fiber sequences of W-modules.

$$\begin{tikzcd}
 & F_*W \arrow[r, equal] \arrow[d, "V"] & F_*W \arrow[d, "p"] \\
\mathbb{G}_a^\sharp \arrow[r] \arrow[d, equal] & W \arrow[r, "F"] \arrow[d] & F_*W \arrow[d] \\
\mathbb{G}_a^\sharp \arrow[r] & \mathbb{G}_a \arrow[r] & \mathbb{G}_a^{\mathrm{dR}} \simeq F_*W/^{\mathbb{L}}p.
\end{tikzcd} \tag{6.4.1.1}$$

See [6, Corollary 2.6.8].

6.4.2. Consider the (derived) p-adic formal stack $\mathbb{Z}_p^{\Delta} \times \mathbb{A}^1/\mathbb{G}_m$: over any p-nilpotent R, this parameterizes pairs $(I \xrightarrow{d} W(R), L \to R)$, consisting of a Cartier-Witt divisor and a generalized Cartier divisor. We will view the map of W-modules

$$F_*M' \overset{\mathrm{defn}}{=} F_*(I \otimes_{W(R)} W) \xrightarrow{d} F_*W$$

as a generalized Cartier divisor for the W-algebra F_*W. In turn this induces a generalized Cartier divisor

$$F_*M' \otimes_{F_*W} \mathbb{G}_a^{\mathrm{dR}} \to \mathbb{G}_a^{\mathrm{dR}}$$

for the W-algebra $\mathbb{G}_a^{\mathrm{dR}}$. On the other hand, over this formal prestack, we have another generalized Cartier divisor $L \otimes_R \mathbb{G}_a^{\mathrm{dR}} \to \mathbb{G}_a^{\mathrm{dR}}$ for the same W-algebra.

Suppose that we are given a map

$$\begin{tikzcd}
F_*M' \otimes_{F_*W} \mathbb{G}_a^{\mathrm{dR}} \arrow[rr, "\alpha"] \arrow[dr] & & L \otimes_R \mathbb{G}_a^{\mathrm{dR}} \arrow[dl] \\
 & \mathbb{G}_a^{\mathrm{dR}} &
\end{tikzcd}$$

of generalized Cartier divisors. Then pulling back the fiber sequence

$$\mathbf{V}(L)^\sharp \to L \otimes_R \mathbb{G}_a \to L \otimes_R \mathbb{G}_a^{\mathrm{dR}}$$

of W-modules along the composition

$$F_*M' \to F_*M' \otimes_{F_*W} \mathbb{G}_a^{\mathrm{dR}} \xrightarrow{\alpha} L \otimes_R \mathbb{G}_a^{\mathrm{dR}}$$

now gives us a sequence of W-modules

$$\mathbf{V}(L)^\sharp \to M(\alpha) \to F_*M'.$$

Lemma 6.4.3. *There is a canonical map of W-modules $M(\alpha) \xrightarrow{d_\alpha} W$ whose cofiber has a canonical structure of a W-algebra.*

Proof. We can lift F_*W to a filtered W-algebra with filtration $\mathrm{Fil}^\bullet_{M'} F_*W$ obtained from the generalized Cartier divisor $F_*M' \to F_*W$. Similarly, we can lift $\mathbb{G}_a^{\mathrm{dR}}$ (resp. $\mathbb{G}_a$) to a filtered W-algebra with filtration $\mathrm{Fil}^\bullet_L \mathbb{G}_a^{\mathrm{dR}}$ (resp. $\mathrm{Fil}^\bullet_L \mathbb{G}_a$).

The map α can be viewed now as yielding a map of non-negatively *filtered* W-algebras

$$\mathrm{Fil}^\bullet_{M'} F_*W \to \mathrm{Fil}^\bullet_L \mathbb{G}_a^{\mathrm{dR}},$$

and we can upgrade W to a non-negatively filtered W-algebra $\mathrm{Fil}^\bullet_\alpha W$ such that we have a Cartesian square

$$\begin{tikzcd}
\mathrm{Fil}^\bullet_\alpha W \arrow[r] \arrow[d] & \mathrm{Fil}^\bullet_{M'} F_*W \arrow[d] \\
\mathrm{Fil}^\bullet_L \mathbb{G}_a \arrow[r] & \mathrm{Fil}^\bullet_L \mathbb{G}_a^{\mathrm{dR}}.
\end{tikzcd}$$

This has the feature that $\mathrm{Fil}^\bullet_\alpha W = W$ and $\mathrm{Fil}^1_\alpha W = M(\alpha)$. In particular, we obtain a canonical map $M(\alpha) \to W$, and we also see that its cofiber is the W-algebra $\mathrm{gr}^0_\alpha W$. ☐

Definition 6.4.4. Let $\mathbb{Z}_p^{\mathcal{N}}$ be the p-adic formal prestack whose values on $R \in \mathrm{CRing}^{p\text{-nilp}}$ are given by triples $(I \to W(R), L \to R, \alpha)$, where $(I \to W(R), L \to R)$ is a section of $\mathbb{Z}_p^{\Delta} \times \mathbb{A}^1/\mathbb{G}_m$, and α is a map of generalized Cartier divisors on ring stacks

$$\alpha : (F_*M' \otimes_{F_*W} \mathbb{G}_a^{\mathrm{dR}} \to \mathbb{G}_a^{\mathrm{dR}}) \to (L \otimes_R \mathbb{G}_a^{\mathrm{dR}} \to \mathbb{G}_a^{\mathrm{dR}}).$$

By abuse of notation we will refer to this tuple by the symbol $M \xrightarrow{d} W$, where $M \overset{\mathrm{defn}}{=} M(\alpha)$ and $d = d_\alpha$, and refer to it as a **filtered Cartier-Witt divisor** over R.[17] We will also write $W/_dM$ instead of $\mathrm{gr}^0_\alpha W$. We will also write $L_M \to R$ for the underlying generalized Cartier divisor and $M' \xrightarrow{d'} W$ for the map of W-modules $I \otimes_{W(R)} W \to W$ arising from the Cartier-Witt divisor $I \to W(R)$, so that we have a diagram of fiber sequences of W-modules

$$\begin{tikzcd}
\mathbf{V}(L_M)^\sharp \arrow[r] \arrow[d] & M \arrow[r] \arrow[d, "d"] & F_*M' \arrow[d, "d'"] \\
\mathbb{G}_a^\sharp \arrow[r] & W \arrow[r] & F_*W
\end{tikzcd}$$

Definition 6.4.5. Over $\mathbb{Z}_p^{\mathcal{N}}$, we have the p-adic formal ring stack $\mathbb{G}_a^{\mathcal{N}}$ assigning to every filtered Cartier-Witt divisor $W \xrightarrow{d} M$ over R, the $W(R)$-algebra $(W/_dM)(R)$.

For any $R \in \mathrm{CRing}^{p\text{-nilp}}$, we now define the **Nygaard filtered prismatization** $R^{\mathcal{N}}$ to be the transmutation of $\mathrm{Spec}\, R$ with respect to $\mathbb{G}_a^{\mathcal{N}}$. That is, it is the p-adic formal prestack over $\mathbb{Z}_p^{\mathcal{N}}$ that associates with each $C \in \mathrm{CRing}^{p\text{-nilp}}$ and $(M \xrightarrow{d} W) \in \mathbb{Z}_p^{\mathcal{N}}(C)$ the space $\mathrm{Map}_{\mathrm{CRing}}(R, (W/_dM)(C))$.

Proposition 6.4.6. *The p-adic formal prestacks $\mathbb{Z}_p^{\mathcal{N}}$ and $\mathbb{G}_a^{\mathcal{N}}$ are classical. Moreover, their classical truncations agree with the descriptions in [6, §5].*

Proof. By construction $\mathbb{Z}_p^{\mathcal{N}}$ lives over the classical formal prestack $\mathbb{Z}_p^{\Delta} \times \mathbb{A}^1/\mathbb{G}_m$. So it is enough to know that it is classical after flat base-change over the latter prestack.

Consider the pullback X of $\mathbb{Z}_p^{\mathcal{N}}$ along the flat cover

$$W_{\mathrm{dist}} \times \mathbb{A}^1 \to \mathbb{Z}_p^{\Delta} \times \mathbb{A}^1/\mathbb{G}_m.$$

[17] We will see in Proposition 6.4.6 below this agrees with the definitions from [6] for discrete inputs. The notation and terminology have been adapted from Bhatt's notes.

Then the fiber of X over a section $(d',t) \in W_{\mathrm{dist}}(R) \times \mathbb{A}^1(R)$ is the stack of commuting squares of maps of F_*W-modules

$$\begin{tikzcd} F_*W \arrow[r] \arrow[d, "d'"'] & \mathbb{G}_a^{\mathrm{dR}} \arrow[d, "t^{\mathrm{dR}}"] \\ F_*W \arrow[r] & \mathbb{G}_a^{\mathrm{dR}} \end{tikzcd}$$

where the bottom arrow is the natural one, and where t^{dR} is the map induced by t and the $\mathbb{G}_a$-algebra structure on $\mathbb{G}_a^{\mathrm{dR}}$. Another way of looking at X is as the stack over $W_{\mathrm{dist}} \times \mathbb{A}^1 \times \mathbb{G}_a^{\mathrm{dR}}$ sitting in a Cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d] & W_{\mathrm{dist}} \times \mathbb{A}^1 \times \mathbb{G}_a^{\mathrm{dR}} \arrow[d, "(d',t,\alpha)\mapsto(d'^{,\mathrm{dR}},t^{\mathrm{dR}}\circ\alpha)"] \\ \mathbb{G}_a^{\mathrm{dR}} \arrow[r, "\Delta"'] & \mathbb{G}_a^{\mathrm{dR}} \times \mathbb{G}_a^{\mathrm{dR}} \end{tikzcd}$$

Here, $d'^{,\mathrm{dR}}$ is the section of $\mathbb{G}_a^{\mathrm{dR}}$ associated with d'.

Pulling back along the flat cover

$$W_{\mathrm{dist}} \times \mathbb{A}^1 \times \mathbb{G}_a \to W_{\mathrm{dist}} \times \mathbb{A}^1 \times \mathbb{G}_a^{\mathrm{dR}}$$

gives us a flat cover $Y \to X$ sitting in a Cartesian diagram

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & W_{\mathrm{dist}} \times \mathbb{A}^1 \times \mathbb{G}_a \arrow[d, "(d',t,u)\mapsto(d'^{,\mathrm{dR}},t^{\mathrm{dR}}\circ u^{\mathrm{dR}})"] \\ \mathbb{G}_a^{\mathrm{dR}} \arrow[r, "\Delta"] & \mathbb{G}_a^{\mathrm{dR}} \times \mathbb{G}_a^{\mathrm{dR}} \end{tikzcd}$$

By invoking (6.4.1.1), we can write this as a composition of two Cartesian squares

$$\begin{tikzcd} Y \arrow[r] \arrow[d] & W_{\mathrm{dist}} \times \mathbb{A}^1 \times \mathbb{G}_a \arrow[d, "(d',t,u)\mapsto(d',t\circ u)"] \\ W \arrow[r, "d\mapsto(F(d),\pi(d))"'] \arrow[d] & W \times \mathbb{G}_a \arrow[d, "(d',a)\mapsto(d'^{,\mathrm{dR}},a^{\mathrm{dR}})"] \\ \mathbb{G}_a^{\mathrm{dR}} \arrow[r, "\Delta"] & \mathbb{G}_a^{\mathrm{dR}} \times \mathbb{G}_a^{\mathrm{dR}} \end{tikzcd}$$

After all is said and done, the classicality of $\mathbb{Z}_p^{\mathcal{N}}$ is now reduced to the classicality of Y, which comes down to the assertion that the maps of formal schemes

$$W \xrightarrow{(F,\pi)} W \times \mathbb{G}_a \; ; \; W_{\mathrm{dist}} \times \mathbb{A}^1 \times \mathbb{G}_a \xrightarrow{(\mathrm{id},m)} W \times \mathbb{G}_a$$

are p-completely Tor-independent. Here, $m : \mathbb{A}^1 \times \mathbb{G}_a \xrightarrow{(t,u)\mapsto tu} \mathbb{G}_a$ is the multiplication map.

But these are actually maps of *affine* formal schemes corresponding to maps of p-complete rings given by:

$$\alpha : \mathbb{Z}_p[T, x_0, x_1, \ldots]_p^\wedge \xrightarrow{T\mapsto x_0, x_i\mapsto F^*x_i} \mathbb{Z}_p[x_0, x_1, \ldots]_p^\wedge$$
$$\beta : \mathbb{Z}_p[T, x_0, x_1, \ldots]_p^\wedge \xrightarrow{T\mapsto ut, x_i\mapsto x_i} \mathbb{Z}_p[u, t, x_0, x_1^{\pm 1}, \ldots]_{(x_0,p)}^\wedge.$$

The first map is the composition of the map $\alpha' : T \mapsto x_0, x_i \mapsto x_i$ with the *flat* map F^*. Therefore, it is enough to check that α' and β are p-completely Tor-independent. This comes down to the concrete (and easy) assertion that the element

$$ut - x_0 \in \mathbb{F}_p[u, t, x_0, x_1^{\pm 1}, \dots]^{\wedge}_{(x_0)}$$

is a non-zero divisor.

Let us now look at $\mathbb{G}_a^{\mathcal{N}}$. It suffices to check that it is classical after basechange to Y. Here, the above description shows that we have a commuting diagram of fiber sequences of W-modules

$$\begin{tikzcd}
\mathbb{G}_a^\sharp \arrow[r] \arrow[d, "u^\sharp"'] & W \arrow[r] \arrow[d] & F_*W \arrow[d, equal] \\
\mathbb{G}_a^\sharp \arrow[r] \arrow[d, "t^\sharp"'] & M \arrow[r] \arrow[d] & F_*W \arrow[d, "d'"] \\
\mathbb{G}_a^\sharp \arrow[r] & W \arrow[r] & F_*W
\end{tikzcd} \tag{6.4.6.1}$$

where the composition of the middle vertical arrows is multiplication by a section d of W with $F(d) = d'$, and where $M \to W$ is the universal filtered Cartier-Witt divisor over Y. This description shows that the W-module scheme M over Y is classical: It is a colimit of the classical schemes W and $\mathbb{G}_a^\sharp$, and so the restriction of $\mathbb{G}_a^{\mathcal{N}}$ to Y is also classical.

The last assertion about the values on discrete inputs can now be deduced from [6, Lemma 5.2.8].

□

Remark 6.4.7. The proof shows that every filtered Cartier-Witt divisor can be obtained flat locally as one sitting in the diagram (6.4.6.1) for sections u, t of $\mathbb{G}_a$ and d of W with d restricting to multiplication by $(tu)^\sharp$ on $\mathbb{G}_a^\sharp$.

More generally, for any invertible module L equipped with a section $R \xrightarrow{u} L$ and a cosection $L \xrightarrow{t} R$, and a section d of W restricting to $(t \circ u)^\sharp$ on $\mathbb{G}_a^\sharp$, we obtain a similar diagram with the middle sequence now an extension of F_*W by $\mathbf{V}(L)^\sharp$. For future reference, we will denote a filtered Cartier-Witt divisor obtained in this way by $M(L, d, t, u) \to W$.

Remark 6.4.8. The prestack $\mathbb{F}_p^{\mathcal{N}}$ admits a quite explicit description explained in [6, Prop. 5.4.2]. Suppose that we are given a filtered Cartier-Witt divisor $(M \xrightarrow{d} W) \in \mathbb{Z}_p^{\mathcal{N}}(R)$. Then it is not difficult to see that the existence of a map $\mathbb{F}_p \to (W/_d M)(R)$ implies that $M \to W$ is isomorphic canonically to a filtered Cartier-Witt divisor of the form $M(L, p, t, u) \to W$. Therefore we have

$$\mathbb{F}_p^{\mathcal{N}} \simeq Z(ut - p)/\mathbb{G}_m,$$

where $Z(ut - p) \subset \mathbb{A}^1 \times \mathbb{A}^1_+ = \operatorname{Spf} \mathbb{Z}_p[t, u]$ is the closed formal $\mathbb{G}_m$-equivariant subscheme defined by the equation $ut - p$.

6.5. **The de Rham and Hodge-Tate embeddings and syntomification.** We will now define two canonical open immersions $j_{\mathrm{dR}}, j_{\mathrm{HT}} : \mathbb{Z}_p^{\Delta} \to \mathbb{Z}_p^{\mathcal{N}}$, called the **de Rham** and **Hodge-Tate** embeddings. These are described in [23, §5.3, 5.6], and in [6, §5.3]. Let $M' \to W$ be the map of W-modules over $\mathbb{Z}_p^{\Delta}$ associated with the universal Cartier-Witt divisor over $\mathbb{Z}_p^{\Delta}$.

Definition 6.5.1. The open immersion j_{dR} is simply the open locus $\mathbb{Z}_p^{\Delta} \times_{\mathbb{A}^1/\mathbb{G}_m} (\mathbb{G}_m/\mathbb{G}_m)$ parameterizing filtered Cartier-Witt divisors $(M \xrightarrow{d} W) \in \mathbb{Z}_p^{\Delta}(R)$, where the underlying map $L_M \xrightarrow{\simeq} R$ is an isomorphism. Over this locus, the classifying map $(F_*M' \to F_*W) \to (\mathbb{G}_a^{\mathrm{dR}} \xrightarrow{\mathrm{id}} \mathbb{G}_a^{\mathrm{dR}})$ yields the filtered Cartier-Witt divisor given by

$$\begin{tikzcd} \mathbb{G}_a^\sharp \arrow[r] \arrow[d, equal] & M \arrow[r] \arrow[d] & F_*M' \arrow[d, "d'"] \\ \mathbb{G}_a^\sharp \arrow[r] & W \arrow[r] & F_*W \end{tikzcd}$$

where the right square is Cartesian.

The pullback of j_{dR} to the stack Y from the proof of Proposition 6.4.6 is the open locus where $t \neq 0$.

Definition 6.5.2. The open immersion j_{HT} is given by the natural map

$$(F_*F^*M' \xrightarrow{F^*(d')} F_*W) \to (F_*F^*M' \otimes_{F_*W} \mathbb{G}_a^{\mathrm{dR}} \to \mathbb{G}_a^{\mathrm{dR}})$$

which classifies the filtered Cartier-Witt divisor given by

$$\begin{tikzcd} \mathbf{V}(L)^\sharp \arrow[r] \arrow[d] & M' \arrow[r] \arrow[d] & F_*F^*M' \arrow[d] \\ \mathbb{G}_a^\sharp \arrow[r] & W \arrow[r] & F_*W \end{tikzcd}$$

Here $L \to \mathbb{G}_a$ is the generalized Cartier divisor for $\mathbb{G}_a$ obtained via base-change from $M' \to W$. The filtered Cartier-Witt divisors obtained in this fashion will be called **invertible**.

The pullback of j_{HT} to the stack Y from the proof of Proposition 6.4.6 is the open locus where $u \neq 0$.

Definition 6.5.3. We now define the prestack $\mathbb{Z}_p^{\mathrm{syn}}$ to be the coequalizer of the immersions $j_{\mathrm{dR}}, j_{\mathrm{HT}}$. Practically, what this means is that we have

$$\mathrm{QCoh}(\mathbb{Z}_p^{\mathrm{syn}}) \xrightarrow{\simeq} \mathrm{eq}\left(\mathrm{QCoh}(\mathbb{Z}_p^{\mathcal{N}}) \underset{j^*_{\mathrm{HT}}}{\overset{j^*_{\mathrm{dR}}}{\rightrightarrows}} \mathrm{QCoh}(\mathbb{Z}_p^{\mathbb{\Delta}})\right)$$

Definition 6.5.4. The process of transmutation now yields for every $R \in \mathrm{CRing}^{p\text{-nilp}}$ open immersions $j_{\mathrm{dR}}, j_{\mathrm{HT}} : R^{\mathbb{\Delta}} \to R^{\mathcal{N}}$, and we now define the **syntomification** of R, R^{syn}, to be their coequalizer. By construction we have a canonical structure map $R^{\mathrm{syn}} \to \mathbb{Z}_p^{\mathrm{syn}}$.

6.6. **The Breuil-Kisin twist.** Here we will describe a canonical line bundle $\mathcal{O}^{\mathrm{syn}}\{1\}$ on $\mathbb{Z}_p^{\mathrm{syn}}$ called the **Breuil-Kisin twist**.

6.6.1. Specifying such a line bundle is equivalent to specifying a line bundle $\mathcal{O}^{\mathcal{N}}\{1\}$ on $\mathbb{Z}_p^{\mathcal{N}}$ equipped with an isomorphism

$$j^*_{\mathrm{dR}}\mathcal{O}^{\mathcal{N}}\{1\} \xrightarrow{\simeq} j^*_{\mathrm{HT}}\mathcal{O}^{\mathcal{N}}\{1\}$$

of line bundles on $\mathbb{Z}_p^{\mathbb{\Delta}}$.

We begin by considering the pullback of the tautological line bundle on $B\mathbb{G}_m$: this gives a line bundle $\mathcal{O}^{B\mathbb{G}_m}\{1\}$ over $\mathbb{Z}_p^{\mathcal{N}}$. Note that we have a canonical trivialization

$$j^*_{\mathrm{dR}}\mathcal{O}^{B\mathbb{G}_m}\{1\} \xrightarrow{\simeq} \mathscr{O}_{\mathbb{Z}_p^{\mathbb{\Delta}}}. \tag{6.6.1.1}$$

Via the Hodge-Tate embedding, we obtain a canonical equivalence

$$j^*_{\mathrm{HT}}\mathcal{O}^{B\mathbb{G}_m}\{1\} \xrightarrow{\simeq} \mathscr{I}_{\mathbb{Z}_p^{\mathrm{HT}}} \tag{6.6.1.2}$$

where the right hand side is the ideal sheaf of the Hodge-Tate divisor.

6.6.2. Next, we consider a canonical line bundle $\mathcal{O}^{\Delta}\{1\}$ on $\mathbb{Z}_p^{\Delta}$ that is characterized up to isomorphism by any of the following properties:

- ([8, (2.2.11)]) For any *transversal* prism (A, I) the pullback of $\mathcal{O}^{\Delta}\{1\}$ to $\mathrm{Spf}\, A$ under the map $\iota_{(A,I)}$ from Remark 6.2.6 is canonically isomorphic to $A\{1\}$.
- ([23, §4.8, 4.9]) We have the line bundle $\mathscr{O}(\mathbb{Z}_p^{\mathrm{HT}})$ on $\mathbb{Z}_p^{\Delta}$ corresponding to the Hodge-Tate divisor. The endomorphism $\varphi^* - \mathrm{id}$ of the Picard group $\mathrm{Pic}(\mathbb{Z}_p^{\Delta})$ is an equivalence, and we take $\mathcal{O}^{\Delta}\{1\}$ to be the preimage of $\mathscr{O}(\mathbb{Z}_p^{\mathrm{HT}})$.
- ([8, (9.1.6)]) Let $f : (\mathbb{P}^1_{\mathbb{Z}_p})^{\Delta} \to \mathbb{Z}_p^{\Delta}$ be the map of prismatizations arising from the structure morphism $\mathbb{P}^1_{\mathbb{Z}_p} \to \mathrm{Spf}\, \mathbb{Z}_p$. Then, we have $\mathcal{O}^{\Delta}\{-1\} \simeq R^2 f_* \mathscr{O}$.

6.6.3. There is a canonical isomorphism

$$\varphi^* \mathcal{O}^{\Delta}\{1\} \xrightarrow{\simeq} \mathscr{O}(\mathbb{Z}_p^{\mathrm{HT}}) \otimes \mathcal{O}^{\Delta}\{1\} \tag{6.6.3.1}$$

of line bundles over $\mathbb{Z}_p^{\Delta}$.

We now set

$$\mathcal{O}^{\mathcal{N}}\{1\} = \mathcal{O}^{B\mathbb{G}_m}\{1\} \otimes \pi^* \mathcal{O}^{\Delta}\{1\}.$$

Since $\pi \circ j_{\mathrm{dR}} = \mathrm{id}$ and $\pi \circ j_{HT} = \varphi$, combining (6.6.3.1), (6.6.1.1) and (6.6.1.2) shows that this line bundle does admit a canonical descent to a line bundle $\mathcal{O}^{\mathrm{syn}}\{1\}$ over $\mathbb{Z}_p^{\mathrm{syn}}$.

6.7. The canonical sections x_{dR} and $x_{\mathrm{dR}}^{\mathcal{N}}$.

6.7.1. For any $R \in \mathrm{CRing}^{p\text{-nilp}}$, we have canonical maps

$$x_{\mathrm{dR}} : \mathrm{Spec}\, R \to R^{\Delta}\ ;\ x_{\mathrm{dR}}^{\mathcal{N}} : \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R \to R^{\mathcal{N}}$$

with the following properties:

- The composition
$$\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R \xrightarrow{x_{\mathrm{dR}}^{\mathcal{N}}} R^{\mathcal{N}} \xrightarrow{t} \mathbb{A}^1/\mathbb{G}_m$$
is the canonical structure map.
- The restriction of $x_{dR}^{\mathcal{N}}$ to the open point $\mathrm{Spec}\, R$ is isomorphic to x_{dR}.
- We have a commuting square
$$\begin{tikzcd} \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R \arrow[r, "x_{dR}^{\mathcal{N}}"] \arrow[d] & R^{\mathcal{N}} \arrow[d, "\pi"] \\ \mathrm{Spec}\, R \arrow[r, "x_{dR}"'] & R^{\Delta} \end{tikzcd} .$$

6.7.2. The map x_{dR} corresponds to the Cartier-Witt divisor $(W(R) \xrightarrow{p} W(R))$ with the map $R \to W(R)/^{\mathbb{L}}p$ the one through which the Frobenius endomorphism of $W(R)/^{\mathbb{L}}p$ canonically factors. Alternatively, there are equivalences of abstract animated rings

$$W(R)/^{\mathbb{L}}p \xrightarrow{\simeq} F_* W(R)/^{\mathbb{L}}p \xrightarrow{\simeq} \mathbb{G}_a^{dR}(R),$$

and the desired map is the evaluation of the natural map $\mathbb{G}_a \to \mathbb{G}_a^{dR}$ on R.

6.7.3. The map $x_{dR}^{\mathcal{N}}$ associates with every cosection $t : L \to C$ of a line bundle L over an R-algebra C the filtered Cartier-Witt divisor associated with the zero map $\mathbb{G}_a^{\mathrm{dR}} \to L \otimes_R \mathbb{G}_a^{\mathrm{dR}}$. Explicitly, it is given by the map

$$M(t) = F_* W \oplus \mathbf{V}(L)^{\sharp} \xrightarrow{d=(V,t^{\sharp})} W,$$

where the quotient $W/_d M(t)$ is also a quotient of $W/VW \simeq \mathbb{G}_a$, giving us the map

$$R \to C = \mathbb{G}_a(C) \to (W/_d M(t))(C).$$

6.8. **Relationship with divided powers.** The next observation will be used later to formulate abstract Grothendieck-Messing type statements.

Lemma 6.8.1. *Suppose that we have a divided power thickening $R' \twoheadrightarrow R$ in $\mathrm{CRing}^{f,p\text{-}comp}$. Then the canonical point $x_{\mathrm{dR},R'}$ from §6.7 factors canonically through a point $\tilde{x}_{\mathrm{dR},R'} : \operatorname{Spec} R' \to R^{\Delta}$.*

Proof. The claim here is that the canonical map $R' \to \mathbb{G}_a^{\mathrm{dR}}(R')$ admits a factoring through R that depends only on the divided powers on the fiber J of $R' \twoheadrightarrow R$. Since the constructions and the conclusion are compatible with sifted colimits, we reduce to the case where $R' \twoheadrightarrow R$ is a classical divided power thickening of flat p-complete $\mathbb{Z}_p$-algebras.

Now, it suffices to construct a canonical lift of R'-modules $J \to \mathbb{G}_a^{\sharp}(R')$. This is obtained from the following sequence of maps:

$$\begin{aligned} J &\xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Mod}_{R'}}(R', J) \\ &\to \mathrm{Map}_{\mathrm{CRing}_{R'/}}(\Gamma_{R'}(R'), \Gamma_{R'}(J)) \\ &\to \mathrm{Map}_{\mathrm{CRing}_{R'/}}(\Gamma_{R'}(R'), R') \simeq \mathbb{G}_a^{\sharp}(R'), \end{aligned}$$

where in the last row we have used the map $\Gamma_{R'}(J) \to R'$ arising from the divided powers on J. □

6.9. **Prismatic cohomology.** We will need the relationship between the stacks defined above and relative (Nygaard filtered) prismatic cohomology as constructed in [11] and [8] for semiperfectoid rings.

6.9.1. Suppose that we have a (classical) bounded prism (A, I) with $\overline{A} = A/I$. Then, with any $R \in \mathrm{CRing}_{\overline{A}/}$ we can associate its **relative prismatic cohomology** $\Delta_{R/A}$.

This can be obtained—see [8, Construction 4.1.3]—as the left Kan extension of its restriction to polynomial $\overline{A}$-algebras R, where $\Delta_{R/A}$ is defined to be an inverse limit in the ∞-category of p-complete *derived rings* over A (see also [9, §7]):

$$\Delta_{R/A} \xrightarrow{\simeq} \varprojlim_{(B,v) \in (R/A)_{\Delta}} B,$$

where $(R/A)_{\Delta}$ is the prismatic site for R relative to (A, I).

6.9.2. The base change $\overline{\Delta}_{R/A} \overset{\mathrm{defn}}{=} \overline{A} \otimes^{\mathbb{L}}_A \Delta_{R/A}$ over $\mathbb{F}_p$ is the **relative Hodge-Tate cohomology**, which is equipped with a canonical exhaustive increasing filtration $\mathrm{Fil}^{\mathrm{conj}}_{\bullet} \overline{\Delta}_{R/\mathbb{Z}_p}$ supported in non-positive degrees, and a canonical equivalence

$$\mathrm{gr}_i^{\mathrm{conj}} \overline{\Delta}_{R/A} \xrightarrow{\simeq} \wedge^i \mathbb{L}_{R/\overline{A}}[-i]$$

for each $i \geq 0$; see [8, Remark 4.1.7]. This is characterized by the property that it respects sifted colimits in $\overline{A}$-algebras and is isomorphic to the Postnikov filtration on $\overline{\Delta}_{R/A}$ for polynomial algebras R over $\overline{A}$.

Definition 6.9.3. A p-complete animated commutative ring R is **semiperfectoid** if there exists a *perfectoid* ring R_0 and a surjective map $R_0 \twoheadrightarrow R$.

In [8, §4.4] one finds the construction of the **absolute prismatic cohomology** Δ_R of any p-complete animated commutative ring R; see in particular Construction 4.4.10 of *loc. cit.* We will need to know the following result due to Bhatt-Lurie and Holeman [34]:

Theorem 6.9.4. *Suppose that R_0 is a perfectoid ring with corresponding perfect prism $(A, I) = (A_{\inf}(R_0), \ker\theta)$. Then:*

(1) There are canonical isomorphisms of commutative rings

$$\Delta_{R_0} \xrightarrow{\simeq} \Delta_{R_0/R_0} \xrightarrow{\simeq} A.$$

(2) In fact, for any R_0-algebra R, there is an isomorphism

$$\Delta_R \xrightarrow{\simeq} \Delta_{R/R_0}.$$

(3) If $R_0 \twoheadrightarrow R$ is surjective, so that R is semiperfectoid, then $\Delta_R \simeq \Delta_{R/R_0}$ is a (p, I)-complete animated commutative ring, and there is a canonical isomorphism (independent of the choice of R_0) of p-adic formal stacks

$$\mathrm{Spf}(\Delta_R) \xrightarrow{\simeq} R^{\Delta}.$$

Proof. The second statement follows from [8, Proposition 4.4.12], while the second isomorphism in assertion (1) is from the fact that (A, I) is an initial prism with a map $R_0 \to \overline{A}$.

The last assertion is [9, Corollary 7.18]. The only additional remark to be made is that $(\Delta_R, \Delta_R \otimes_A I)$ is an initial prism equipped with a map $R \to \overline{\Delta}_R$, and so is independent of the choice of the perfectoid ring R_0: this follows for instance from [34, Theorem 3.3.7]. The map $\mathrm{Spf}(\Delta_R) \to R^{\Delta}$ is canonical and arises from the general construction in Remark 6.2.6. □

Remark 6.9.5. Suppose that R is a semiperfect $\mathbb{F}_p$-algebra, then we can take $R_0 = R^{\flat}$ to be its perfection. In this case, we obtain isomorphisms

$$\Delta_R \xrightarrow{\simeq} \Delta_{R/\mathbb{Z}_p} \xrightarrow[\simeq]{\gamma^{\mathrm{crys}}_{R/\mathbb{Z}_p}} \Delta^{\mathrm{crys}}_{R/\mathbb{Z}_p} \xrightarrow{\simeq} \Delta^{\mathrm{crys}}_{R/W(R^{\flat})} \simeq A_{\mathrm{crys}}(R). \tag{6.9.5.1}$$

Here, $\Delta^{\mathrm{crys}}_{R/W(R^{\flat})}$ (resp. $\Delta^{\mathrm{crys}}_{R/\mathbb{Z}_p}$) is the relative derived crystalline cohomology of R over $W(R^{\flat})$ (resp. over $\mathbb{Z}_p$) and $A_{\mathrm{crys}}(R) \twoheadrightarrow R$ is the p-completed animated divided power envelope of the map $W(R^{\flat}) \twoheadrightarrow R$. The second isomorphism is explained in [8, Remark 4.6.5], the third is a consequence of the fact that $W(R^{\flat})$ is formally étale over $\mathbb{Z}_p$, and the last isomorphism is an animated enhancement of a result of Illusie [54, Prop. 4.64]. The composition of these isomorphisms is φ-semilinear over $W(R^{\flat})$ via the isomorphism $\Delta_{R^{\flat}} \xrightarrow{\simeq} W(R^{\flat})$.

In fact, Remark 4.6.5 of [8] shows that we have canonical isomorphism

$$\gamma^{\mathrm{crys}}_{R/\mathbb{Z}_p} : \Delta_{R/\mathbb{Z}_p} \xrightarrow{\simeq} \Delta^{\mathrm{crys}}_{R/\mathbb{Z}_p}$$

for *any* $R \in \mathrm{CRing}_{\mathbb{F}_p/}$. In particular, for any such R we have canonical isomorphisms

$$\overline{\Delta}_R \xrightarrow{\simeq} \Delta_R/^{\mathbb{L}}p \xrightarrow{\simeq} \left(\Delta^{\mathrm{crys}}_{R/\mathbb{Z}_p}\right)/^{\mathbb{L}}p \xrightarrow{\simeq} \mathrm{dR}_{R/\mathbb{F}_p}, \tag{6.9.5.2}$$

where on the right hand side we now have the derived de Rham cohomology of R over $\mathbb{F}_p$.

6.10. **Filtered Cartier-Witt divisors associated with frames.** Our purpose here is to extend the construction from Remark 6.2.6 to the filtered setting. This will require a slightly technical condition.

6.10.1. Suppose that we have a prismatic frame $\underline{A}$. Base-change of the natural map $A \to R_A$ along the Frobenius lift $\varphi : A \to A$ yields a map of A-algebras

$$A \to R_A \otimes_A \varphi_* A \to R_A \otimes_A \varphi_*(A/^{\mathbb{L}}p). \tag{6.10.1.1}$$

Definition 6.10.2. A **syntomic structure** on the prismatic frame $\underline{A}$ is the data of a factoring diagram of A-algebras:

$$\begin{tikzcd} A \arrow[d] \arrow[dr, "(6.10.1.1)"] & \\ \overline{A} \arrow[r, "\overline{\zeta}"] & R_A \otimes_A \varphi_*(A/^{\mathbb{L}}p) \end{tikzcd}$$

where the left vertical arrow is the natural one.

Example 6.10.3. If $\underline{A}$ is a p-adic prismatic frame, then it is automatically endowed with a syntomic structure, since (6.10.1.1) will factor canonically through $\overline{A} = A/^{\mathbb{L}}p$. This applies in particular to the Witt frame $\underline{W(R)}$ associated with any p-complete R.

Example 6.10.4. Suppose that A is p-completely flat over $\mathbb{Z}_p$ and $I \subset A$ is an honest ideal, generated by an element $d \in A$. Assume also that $\mathrm{Fil}^i A$ is a filtration of A by submodules. Then having a syntomic structure is equivalent to knowing that the map $A \to R_A \otimes_A \varphi_*(A/(p))$ kills d. If in addition $\varphi : A \to A$ is flat—as for instance in the case of the Breuil-Kisin frame from Example 5.4.12—then this becomes the quite concrete condition $d \in \varphi(\mathrm{Fil}^1 A) + pA$. This of course holds in the Breuil-Kisin case.

Proposition 6.10.5. *Suppose that we are given a prismatic frame $\underline{A}$ equipped with a syntomic structure. Then there exists a canonical map*

$$\iota_{\underline{A}} : \mathcal{R}(\mathrm{Fil}^\bullet A) \to R_A^{\mathcal{N}}$$

of p-adic formal stacks over $\mathbb{A}^1/\mathbb{G}_m$ such that we have Cartesian squares

$$\begin{tikzcd}
\mathrm{Spf}(A) \arrow[r, "\tau"] \arrow[d, "\iota_{(A,I)}"'] & \mathcal{R}(\mathrm{Fil}^\bullet A) \arrow[d, "\iota_{\underline{A}}"] \\
R_A^{\Delta} \arrow[r, "j_{\mathrm{dR}}"'] & R_A^{\mathcal{N}}
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathrm{Spf}(A) \arrow[r, "\sigma"] \arrow[d, "\iota_{(A,I)}"'] & \mathcal{R}(\mathrm{Fil}^\bullet A) \arrow[d, "\iota_{\underline{A}}"] \\
R_A^{\Delta} \arrow[r, "j_{\mathrm{HT}}"'] & R_A^{\mathcal{N}}
\end{tikzcd}$$

Proof. Set $(M' \to W) = (I \otimes_A W \to W)$: this is the canonical invertible filtered Cartier-Witt divisor over Spf A.

The map $\overline{\zeta}$ yields a map of A-modules $\zeta : I \to \varphi^* \mathrm{Fil}^1 A/^{\mathbb{L}}p$ and a commuting diagram

$$\begin{tikzcd}
I \arrow[r, "\zeta"] \arrow[d] & \varphi^* \mathrm{Fil}^1 A/^{\mathbb{L}}p \arrow[d] \\
A \arrow[r] \arrow[d] & A/^{\mathbb{L}}p \arrow[d] \\
\overline{A} \arrow[r, "\overline{\zeta}"] & R_A \otimes_A \varphi_*(A/^{\mathbb{L}}p)
\end{tikzcd}$$

where the columns are fiber sequences.

We also obtain maps of W-modules over A:

$$F_*M' = F_*(I \otimes_A W) \xrightarrow{F_*(\zeta\otimes 1)} F_*(\varphi^* \mathrm{Fil}^1 A/^{\mathbb{L}}p \otimes_{A/^{\mathbb{L}}p} W/^{\mathbb{L}}p) \xrightarrow{\simeq} \mathrm{Fil}^1 A \otimes_A (F_*W/^{\mathbb{L}}p) \xrightarrow{\simeq} \mathrm{Fil}^1 A \otimes_A \mathbb{G}_a^{\mathrm{dR}}.$$

Write $u_\zeta : F_*M' \to \mathrm{Fil}^1 A \otimes_A \mathbb{G}_a^{\mathrm{dR}}$ for the composition of these maps. It underlies a diagram

$$\begin{tikzcd}
F_*M' \arrow[r, "u_\zeta"] \arrow[d] & \mathrm{Fil}^1 A \otimes_A \mathbb{G}_a^{\mathrm{dR}} \arrow[d] \\
F_*W \arrow[r] \arrow[d] & \mathbb{G}_a^{\mathrm{dR}} \arrow[d] \\
F_*(\overline{A} \otimes_A W) \arrow[r] & R_A \otimes_A \mathbb{G}_a^{\mathrm{dR}}
\end{tikzcd}$$

where the bottom map is induced from $\overline{\zeta}$ via the identification

$$R_A \otimes_A \mathbb{G}_a^{\mathrm{dR}} \simeq F_* \left((R_A \otimes_A \varphi_*(A/^{\mathbb{L}}p)) \otimes_{A/^{\mathbb{L}}p} (W/^{\mathbb{L}}p)\right).$$

In particular, if we consider the map $M_\zeta \to W$ of W-modules over A defined so that we have a Cartesian diagram of maps

$$\begin{tikzcd}
(M_\zeta \to W) \arrow[r] \arrow[d] & (\mathrm{Fil}^1 A \otimes_A \mathbb{G}_a \to \mathbb{G}_a) \arrow[d] \\
(F_* M' \to F_* W) \arrow[r] & (\mathrm{Fil}^1 A \otimes_A \mathbb{G}_a^{\mathrm{dR}} \to \mathbb{G}_a^{\mathrm{dR}}),
\end{tikzcd}$$

then the quotients form a Cartesian diagram of $R_A \otimes_A W$-*algebras*

$$\begin{tikzcd}
W/M_\zeta \arrow[r] \arrow[d] & R_A \otimes_A \mathbb{G}_a \arrow[d] \\
F_*(W/M') \arrow[r] & R_A \otimes_A \mathbb{G}_a^{\mathrm{dR}}.
\end{tikzcd}$$

Now, suppose that we have a C-valued point η of $\mathcal{R}(\mathrm{Fil}^\bullet A)$ associated with a generalized Cartier divisor $L \to C$ and a map $\mathrm{Fil}^\bullet A \to \mathrm{Fil}^\bullet_L C$ of filtered animated commutative rings carrying I to a nilpotent ideal of C. This yields an arrow between maps of $\mathbb{G}_a^{\mathrm{dR}}$-modules over C

$$(\mathrm{Fil}^1 A \otimes_A \mathbb{G}_a^{\mathrm{dR}} \to \mathbb{G}_a^{\mathrm{dR}}) \to (L \otimes_C \mathbb{G}_a^{\mathrm{dR}} \to \mathbb{G}_a^{\mathrm{dR}}).$$

Composing this with u_ζ gives a map of generalized Cartier divisors

$$\Upsilon_\zeta(\eta) : (F_* M' \to F_* W) \to (L \otimes_C \mathbb{G}_a^{\mathrm{dR}} \to \mathbb{G}_a^{\mathrm{dR}})$$

over C classifying a filtered Cartier-Witt divisor $M_\zeta(\eta) \to W$. The quotient $W/M_\zeta(\eta)$ receives a map from W/M_ζ and is hence an $R_A \otimes_A W$-algebra. This gives the desired map $\mathcal{R}(\mathrm{Fil}^\bullet A) \to R_A^{\mathcal{N}}$.

The existence of the claimed Cartesian squares for the de Rham and Hodge-Tate loci is an immediate check from the construction. ☐

Corollary 6.10.6. *Suppose that we have a map of frames $\gamma : \underline{A} \to \underline{B}$ with $\underline{A}$ prismatic. Let $\mathrm{Syn}(\underline{B})$ be as in Remark 5.5.7. Then there is a canonical map*

$$\iota_\gamma : \mathrm{Syn}(\underline{B}) \to R_A^{\mathrm{syn}}$$

of p-adic formal prestacks.

Proof. Proposition 6.10.5 gives a map $\mathrm{Syn}(\underline{A}) \to R_A^{\mathrm{syn}}$, which we can compose with the map $\mathrm{Syn}(\underline{B}) \to \mathrm{Syn}(\underline{A})$ obtained from γ. ☐

Remark 6.10.7. If B is a $\mathbb{Z}/p^n\mathbb{Z}$-algebra, then ι_γ will factor through $R_A^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$.

6.11. **Nygaard filtered prismatic cohomology.** We now review the story of the Nygaard filtration on prismatic cohomology

6.11.1. We begin with the story relative to a bounded prism (A, I). For what we need, there is no harm in restricting even to perfect prisms, and we will do so. Write $\varphi^* \Delta_{A/I}$ for the base-change $A \otimes_{\varphi, A} \Delta_{A/I}$

Given $R \in \mathrm{CRing}^{p\text{-comp}}_{\overline{A}/}$, it is show in [8, §5.1] that there is now a canonical lift $\mathrm{Fil}^\bullet_{\mathcal{N}} \varphi^* \Delta_{R/A}$ of $\Delta_{R/A}$ to FilMod_A characterized by the following properties:

(1) It respects sifted colimits in R.
(2) It satisfies p-quasisyntomic descent with respect to R.
(3) If R is a p-quasisyntomic over $\overline{A}$ such that the quotient ring R/pR is generated by the images of $\overline{A}$ and $(R/pR)^\flat$, then $\mathrm{Fil}^i_{\mathcal{N}} \Delta_{R/A}$ is discrete and we have

$$\mathrm{Fil}^i_{\mathcal{N}} \varphi^* \Delta_{R/A} = \{x \in \varphi^* \Delta_{R/A} : (1 \otimes \varphi)(x) \in I^i \Delta_{R/A}\}.$$

Remark 6.11.2. The map $1\otimes\varphi:\varphi^*\Delta_{R/A}\to\Delta_{R/A}$ can be canonically lifted to a filtered map

$$\mathrm{Fil}^\bullet_{\mathcal{N}}\,\varphi^*\Delta_{R/A}\to\mathrm{Fil}^\bullet_I\,\Delta_{R/A}.$$

inducing for every $i\in\mathbb{Z}$ an equivalence

$$\mathrm{gr}^i_{\mathcal{N}}\,\varphi^*\Delta_{R/A}\xrightarrow{\simeq}I^i/I^{i+1}\otimes_{\overline{A}}\mathrm{Fil}^i_{\mathrm{conj}}\,\overline{\Delta}_{R/A}\simeq\mathrm{Fil}^i_{\mathrm{conj}}\,\overline{\Delta}_{R/A}\{i\}.$$

See [8, Remark 5.1.2].

6.11.3. In [8, §5.5], we find the construction of an *absolute* Nygaard filtration $\mathrm{Fil}^\bullet_{\mathcal{N}}\,\Delta_R$ on absolute prismatic cohomology. If (A,I) is a perfect prism and R is an $R_0=\overline{A}$-algebra, then, by [8, Theorem 5.6.2], the isomorphism

$$\Delta_R\xrightarrow[\simeq]{\text{Theorem 6.9.4(2)}}\Delta_{R/R_0}\xrightarrow{\simeq}\varphi^*\Delta_{R/R_0}$$

lifts to an isomorphism of filtered objects

$$\mathrm{Fil}^\bullet_{\mathcal{N}}\,\Delta_R\xrightarrow{\simeq}\mathrm{Fil}^\bullet_{\mathcal{N}}\,\varphi^*\Delta_{R/R_0}. \tag{6.11.3.1}$$

In fact, if R is qrsp and (Δ_R,I_R) is the canonical initial prism equipped with a map $R\to\overline{\Delta}_R$, then we have

$$\mathrm{Fil}^i_{\mathcal{N}}\,\Delta_R=\{x\in\Delta_R:\ \varphi(x)\in\mathrm{Fil}^i_{I_R}\,\Delta_R\}.$$

More generally, if R is semiperfectoid and (Δ_R,I_R) is the associated initial prism, then we have a canonical filtered map

$$\Phi:\mathrm{Fil}^\bullet_{\mathcal{N}}\,\Delta_R\to\mathrm{Fil}^\bullet_{I_R}\,\Delta_R$$

lifting the Frobenius endomorphism of Δ_R, which agrees with the relative counterpart from Remark 6.11.2 via the isomorphism (6.11.3.1). This follows from the construction explained in [8, Notation 5.7.5].

Lemma 6.11.4. *Suppose that R is semiperfectoid. Then, in the notation of §5.4, the tuple* $(\mathrm{Fil}^\bullet_{\mathcal{N}}\,\Delta_R,I_R\to\Delta_R,\Phi)$ *underlies a* prismatic frame $\underline{\Delta}_R$ *with* $\mathrm{gr}^0_{\mathcal{N}}\,\Delta_R\simeq R$ *endowed with a canonical syntomic structure as in Definition 6.10.2.*

Proof. The first thing to be checked is that $\mathrm{Fil}^\bullet_{\mathcal{N}}\,\Delta_R$ is a filtered animated commutative ring and that the map Φ is a map of filtered animated commutative rings. For this, it suffices to work with semiperfectoid algebras over a fixed perfectoid ring R_0. When R is quasiregular, then the assertion is clear, and to know it in general, it is easiest to note that the construction is via right Kan extension from R_0-algebras R as in (3) of (6.11.1) followed by left Kan extension from polynomial R_0-algebras. This endows $\mathrm{Fil}^\bullet_{\mathcal{N}}\,\Delta_R$ with the structure of a filtered *derived* commutative ring in general, and also shows that Φ is a map of filtered derived commutative rings. This assertion specializes when R is semiperfectoid to the structure desired.

To finish, we need to exhibit the existence of a canonical factoring

$$\begin{tikzcd}
\Delta_R \arrow[d] \arrow[dr] & \\
\overline{\Delta}_R \arrow[r] & R\otimes_{\Delta_R}\varphi_*(\Delta_R/^{\mathbb{L}}p).
\end{tikzcd}$$

The quickest way to see this is to mod-p Tot descent for prismatic cohomology [8, Proposition 5.5.24] to reduce to the case where R is an $\mathbb{F}_p$-algebra, where it follows from Example 6.10.3. ☐

The following description of the Nygaard filtered prismatization of semiperfectoid rings due to Bhatt-Lurie will be important for us.

Theorem 6.11.5. *Suppose that R is semiperfectoid. Then there is a canonical equivalence*

$$\iota_{\underline{\Delta}_R}:\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\,\Delta_R)\xrightarrow{\simeq}R^{\mathcal{N}}$$

of p-adic formal stacks.

Proof. The existence of the map ι_{Δ_R} follows from Lemma 6.11.4 and Proposition 6.10.5. Combined with Theorem 6.9.4, this already shows that the map is an isomorphism when restricted to the Hodge-Tate and de Rham loci.

When R is quasiregular, Bhatt constructs an explicit inverse using a map $R_0 \to R$ with R_0 perfectoid [6, Theorem 5.5.10]. The proof works essentially verbatim in our more general setting as well. Note, however, that the uniqueness argument from Corollary 5.5.11 of *loc. cit.* is no longer valid, and so it is not clear that the inverse is canonical and independent of the choice of R_0.

□

6.12. **Descent.** The stacks we are concerned with here carry p-quasisyntomic covers to covers in the fpqc topology. This is well-known to experts, but we include proofs here, since it is important for what follows.

Definition 6.12.1. A map $R \to S$ of p-complete animated commutative rings is **p-quasisyntomic** (or simply **quasisyntomic**) if it is p-completely flat (that is, $S/^{\mathbb{L}}p$ is flat over $R/^{\mathbb{L}}p$), and if $\mathbb{L}_{S/R}$ has p-complete Tor amplitude $[-1, 0]$: that is, $\mathbb{L}_{S/R} \otimes \mathbb{F}_p$ has Tor amplitude $[-1, 0]$ over $S/^{\mathbb{L}}p$.

We will say that $R \to S$ is a **quasisyntomic cover** if it is quasisyntomic and $S/^{\mathbb{L}}p$ is faithfully flat over $R/^{\mathbb{L}}p$.

These properties are invariant under base-change via maps $R \to R'$ of p-complete animated commutative rings.

Example 6.12.2. The key example of a quasisyntomic map is

$$\mathbb{Z}_p[T]^\wedge \to \mathbb{Z}_p[T^{1/p^\infty}]^\wedge,$$

where the superscript $^\wedge$ denotes p-adic completion.

Proposition 6.12.3. *Suppose that we have $g : R \to S$ in $\mathrm{CRing}^{p\text{-}comp}$. Then the associated maps $g^{\mathcal{N}} : S^{\mathcal{N}} \to R^{\mathcal{N}}$ and $g^{\Delta} : S^{\Delta} \to R^{\Delta}$ are surjective in the p-completely flat topology under either of the following assumptions.*

1. *g is a p-completely étale cover;*
2. *S is the p-completion of $\mathbb{Z}_p[T^{1/p^\infty}] \otimes_{\mathbb{Z}_p[T]} R$ for some map $\mathbb{Z}_p[T] \to R$.*

Proof. In case (1), we will actually see that g^{Δ} and $g^{\mathcal{N}}$ are p-completely étale covers. For the former, this is shown in [9, Remark 3.9]. The main point is that, for any $C \in \mathrm{CRing}^{p\text{-nilp}}$ and a Cartier-Witt divisor $I \to W(C)$ with quotient $\overline{W(C)}$, the small p-completely étale site of $\overline{W(C)}$ is equivalent to that of $W(C)$, and hence to that of C. This shows that, given $R \to \overline{W(C)}$, we have

$$S \otimes_R \overline{W(C)} \simeq \overline{W(C')}$$

for a canonical étale cover $C \to C'$.

Let us now look at $g^{\mathcal{N}}$: given $(M \xrightarrow{d} W)$ in $\mathbb{Z}_p^{\mathcal{N}}(C)$, $W/_dM$ is an extension of $\overline{W}$ by $\mathrm{hcoker}(t^\sharp)$, where $t^\sharp : \mathbf{V}(L)^\sharp \to \mathbb{G}_a^\sharp$ is the map associated with a cosection of a line bundle over $\operatorname{Spec} C$. Therefore, the map

$$(W/_dM)(C) \to \overline{W(C)}$$

is surjective as long as C is $\mathbb{G}_a^\sharp$-acyclic. For such C, the same argument as in the previous paragraph shows that

$$S \otimes_R (W/_dM)(C) \simeq (W/_dM)(C')$$

for a canonical étale map $C \to C'$. In general, we can choose a faithfully flat map $C \to D$ such that D is $\mathbb{G}_a^\sharp$-acyclic, and so the conclusion from the previous sentence follows for C by fpqc descent for étale C-algebras.

In case (2), by the limit-preserving properties of the prismatization and Nygaard filtered prismatization functors, we are reduced to the situation where $R = \mathbb{Z}_p[T]^\wedge$ and $S = \mathbb{Z}_p[T^{1/p^\infty}]^\wedge$.

Here, we will use the following consequence of Lemma 5.3.3: For any $C \in \mathrm{CRing}^{p\text{-nilp}}$ and any map $\beta : \mathbb{Z}_p[T] \to W(C)$, we can find a faithfully flat map $C \to C'$, and an extension $\mathbb{Z}_p[T^{1/p^\infty}] \to W(C')$ of β. Now, given any diagram $W(C) \twoheadrightarrow \overline{W(C)} \leftarrow \mathbb{Z}_p[T]$ corresponding to a C-valued point of R^{Δ}, we can first lift the second map to a map $\mathbb{Z}_p[T] \to W(C)$, and then find $C \to C'$ as in the previous sentence, so that we have a lift to C'-valued point of S^{Δ}.

This completes the proof in the case of the prismatizations. For the Nygaard filtered prismatization, suppose that we have a point in $R^{\mathcal{N}}(C)$. By replacing C with a flat cover if necessary we can assume that it is of the form $M(C,d,t,u)\to W$ (see Remark 6.4.7). We then have a surjection $W/^{\mathbb{L}}d\to W/M$ with homotopy kernel $\mathrm{hcoker}(u^\sharp)$. This means that, over $\mathbb{G}_a^\sharp$-acyclic algebras, any map $\mathbb{Z}_p[T]\to(W/M)(C)$ can be lifted étale locally to a map $\mathbb{Z}_p[T]\to W(C)/^{\mathbb{L}}d$, and thence to $W(C)$, and we can run the argument used for the prismatization again. $\square$

6.12.4. We will consider the full ∞-subcategory

$$\mathrm{CRing}^f\to\mathrm{CRing}$$

of animated commutative rings satisfying the following condition: The module of differentials $\Omega^1_{(\pi_0(R)/p\pi_0(R))/\mathbb{F}_p}$ is finitely generated over $\pi_0(R)$. The following algebras lie in this subcategory:

(1) Any R such that $\pi_0(R)$ is a finitely generated $\mathbb{Z}/p^m\mathbb{Z}$-algebra for some $m\geq 1$
(2) Any complete local Noetherian $\mathbb{Z}_p$-algebra R with residue field κ admitting a finite p-basis (that is, $[\kappa:\kappa^p]<\infty$); in fact, a complete local Noetherian $\mathbb{F}_p$-algebra belongs to CRing^f if and only if its residue field has finite p-basis.
(3) Any *semiperfectoid* $\mathbb{Z}_p$-algebra R.
(4) Any $\mathbb{Z}[1/p]$-algebra R: This, however, will be irrelevant for what follows.

Corollary 6.12.5. *For any $R\in\mathrm{CRing}^{f,p\text{-}comp}$, there exists a quasisyntomic cover $R\to R_\infty$ with $R_\infty^{\otimes_R m}$ semiperfectoid for all m, and such that $R_\infty^{\Delta}\to R^{\Delta}$ and $R_\infty^{\mathcal{N}}\to R^{\mathcal{N}}$ are surjective in the flat topology.*

Proof. This is as in the proof of [9, Theorem 7.20]: One begins by choosing a map $\mathbb{Z}_p[T_1,\dots,T_r]^\wedge\to R$ such that $\Omega^1_{\pi_0(R/pR)/\mathbb{F}_p}$ is generated by the images of $dT_1,\dots,dT_r$, and sets

$$R_\infty=\mathbb{Z}_p[T_1^{1/p^\infty},\dots,T_r^{1/p^\infty}]^\wedge\otimes_{\mathbb{Z}_p[T_1,\dots,T_r]^\wedge}R.$$

$\square$

Lemma 6.12.6. *Suppose that $R\to S$ is a quasisyntomic cover of semiperfectoid rings. Then the map*

$$S^{\mathcal{N}}\otimes\mathbb{Z}/p^n\mathbb{Z}\to R^{\mathcal{N}}\otimes\mathbb{Z}/p^n\mathbb{Z}$$

is an I_R-completely faithfully flat map of derived formal Artin 1-stacks over $\mathbb{Z}_p$.

Proof. Given Theorem 6.11.5, the proof is identical to that of assertion (1) in the proof of [31, Proposition 2.29]. $\square$

Corollary 6.12.7. *For any $R\in\mathrm{CRing}^{f,p\text{-}nilp}$, R^{Δ} and $R^{\mathcal{N}}$ are* quasi-geometric *formal stacks in the sense of [49, §9.1.1]. More precisely, if $R\to R_\infty$ is as in Corollary 6.12.5, then for any $n\geq 1$:*

(1) The maps

$$R_\infty^{\Delta}\otimes\mathbb{Z}/p^n\mathbb{Z}\to R^{\Delta}\otimes\mathbb{Z}/p^n\mathbb{Z}\ ;\ R_\infty^{\mathcal{N}}\otimes\mathbb{Z}/p^n\mathbb{Z}\to R^{\mathcal{N}}\otimes\mathbb{Z}/p^n\mathbb{Z}$$

are surjections of fpqc sheaves;

(2) $R_\infty^{\Delta}\otimes\mathbb{Z}/p^n\mathbb{Z}$ is a derived formal affine scheme and $R_\infty^{\mathcal{N}}\otimes\mathbb{Z}/p^n\mathbb{Z}$ is a derived formal Artin 1-stack over $\mathbb{Z}_p$;

(3) The natural maps

$$(R_\infty\hat{\otimes}_R R_\infty)^{\mathcal{N}}\otimes\mathbb{Z}/p^n\mathbb{Z}\to R_\infty^{\mathcal{N}}\otimes\mathbb{Z}/p^n\mathbb{Z}$$

are I-completely faithfully flat, where $I=I_{R_\infty}$.

Proof. This follows from Corollary 6.12.5, Theorem 6.11.5 and Lemma 6.12.6. $\square$

Combining Corollary 6.12.5 with Lemma 6.12.6 also yields:

Corollary 6.12.8. *Suppose that $R\to S$ is a quasisyntomic cover in $\mathrm{CRing}^{f,p\text{-}comp}$. Then the map*

$$S^{\mathcal{N}}\otimes\mathbb{Z}/p^n\mathbb{Z}\to R^{\mathcal{N}}\otimes\mathbb{Z}/p^n\mathbb{Z}$$

is an fpqc cover.

6.13. **A nilpotence result.** We will need a certain nilpotence result for applications to the abstract Grothendieck-Messing theory formulated in §8.9. Suppose that $S' \twoheadrightarrow S$ is a square-zero thickening in $\mathrm{CRing}_{\mathbb{F}_p/}$; then the natural map

$$\mathbb{\Delta}_{S'} \xrightarrow{\simeq} \mathbb{\Delta}_{S'/\mathbb{F}_p} \to S'$$

factors canonically through $\mathbb{\Delta}_S$. This can be viewed as a special case of Lemma 6.8.1, where we equip the thickening with the trivial divided power structure.

Set

$$K_{S' \twoheadrightarrow S} \overset{\text{defn}}{=} \mathrm{hker}\,(\mathbb{\Delta}_{S'} \to \mathbb{\Delta}_S)$$

Just as in Remark 5.8.9, the maps

$$\mathrm{Fil}^1\,\Phi : \mathrm{Fil}^1_{\mathcal{N}}\,\mathbb{\Delta}_{S'} \to \mathbb{\Delta}_{S'}\ ;\ \mathrm{Fil}^1\,\Phi : \mathrm{Fil}^1_{\mathcal{N}}\,\mathbb{\Delta}_S \to \mathbb{\Delta}_S$$

now give rise to a map

$$\dot{\varphi}_1 : K_{S' \twoheadrightarrow S} \to K_{S' \twoheadrightarrow S}.$$

Lemma 6.13.1. *The map $\dot{\varphi}_1$ induces a locally nilpotent endomorphism of $K_{S' \twoheadrightarrow S}/^{\mathbb{L}}p$.*

Proof. All constructions involved here are compatible with sifted colimits, as is the conclusion. So we can reduce to the case where we have $S' = \mathbb{F}_p[X,Y]/(Y^2) \to \mathbb{F}_p[X,Y]/(Y) = S$ for some finite sets of variables X and Y. Since the construction $S \mapsto \mathrm{Fil}^\bullet_{\mathcal{N}}\,\mathbb{\Delta}_S$ respects finite colimits, we can further reduce to the consideration of the case of the thickening

$$\mathbb{F}_p[T]/(T^2) \twoheadrightarrow \mathbb{F}_p.$$

Here, we claim that there is a nullhomotopy $\dot{\varphi}_1^2 \simeq 0$. For this, using quasisyntomic descent along the map $\mathbb{F}_p[T] \to \mathbb{F}_p[T^{1/p^\infty}]$, we can reduce still further to the case of a thickening of the form

$$\mathbb{F}_p[T_1^{1/p^\infty}, \ldots, T_r^{1/p^\infty}]/(T_1^2, T_1 - T_i :\ i \geq 2) \twoheadrightarrow \mathbb{F}_p[T_1^{1/p^\infty}, \ldots, T_r^{1/p^\infty}]/(T_1, T_1 - T_i :\ i \geq 2),$$

where, using the compatibility with finite colimits once again, we are finally reduced to the consideration of

$$\mathbb{F}_p[T^{1/p^\infty}]/(T^2) = S' \twoheadrightarrow \mathbb{F}_p[T^{1/p^\infty}]/(T) = S$$

equipped with trivial divided powers. In this case, one now checks using Remark 6.9.5 that we have

$$\overline{\mathbb{\Delta}}_{C'} \simeq \mathbb{F}_p[T^{1/p^\infty}]\langle T^{2/p}\rangle \to \mathbb{F}_p[T^{1/p^\infty}]\langle T^{1/p}\rangle \simeq \mathbb{\Delta}_S/^{\mathbb{L}}p, \tag{6.13.1.1}$$

where we have set

$$\mathbb{F}_p[T^{1/p^\infty}]\langle T^{2/p}\rangle \overset{\text{defn}}{=} \mathbb{F}_p[T^{1/p^\infty}] \otimes_{T^{2/p} \hookleftarrow T, \mathbb{F}_p[T]} \mathbb{F}_p\langle T\rangle\ ;\ \mathbb{F}_p[T^{1/p^\infty}]\langle T^{1/p}\rangle \overset{\text{defn}}{=} \mathbb{F}_p[T^{1/p^\infty}] \otimes_{T^{1/p} \hookleftarrow T, \mathbb{F}_p[T]} \mathbb{F}_p\langle T\rangle.$$

Explicitly, we have coordinates

$$\mathbb{F}_p[T^{1/p^\infty}]\langle T^{2/p}\rangle \xrightarrow[\simeq]{\gamma_p^n(T^{2/p}) \mapsto U_n} \mathbb{F}_p[T^{1/p^\infty}][U_i : i \in \mathbb{Z}_{\geq 0}]/(U_0 - T^{2/p}, U_i^p)$$

$$\mathbb{F}_p[T^{1/p^\infty}]\langle T^{1/p}\rangle \xrightarrow[\simeq]{\gamma_p^n(T^{1/p}) \mapsto V_n} \mathbb{F}_p[T^{1/p^\infty}][V_i : i \in \mathbb{Z}_{\geq 0}]/(V_0 - T^{1/p}, V_i^p),$$

and the map (6.13.1.1) is given by the map of $\mathbb{F}_p[T^{1/p^\infty}]$-algebras

$$\mathbb{F}_p[T^{1/p^\infty}][U_i : i \in \mathbb{Z}_{\geq 0}]/(U_0 - T^{2/p}, U_i^p) \xrightarrow{U_i \mapsto 0} \mathbb{F}_p[T^{1/p^\infty}][V_i : i \in \mathbb{Z}_{\geq 0}]/(V_0 - T^{1/p}, V_i^p).$$

The lift $\overline{\mathbb{\Delta}}_S \to S'$ is now given by the composition

$$\mathbb{F}_p[T^{1/p^\infty}][V_i : i \in \mathbb{Z}_{\geq 0}]/(V_0 - T^{1/p}, V_i^p) \to \mathbb{F}_p[T^{1/p^\infty}][V_0]/(V_0 - T^{1/p}, V_0^2) \xrightarrow{\simeq} \mathbb{F}_p[T^{1/p^\infty}]/(T^{2/p}),$$

where the first map kills V_i for $i \geq 1$. The kernel of this map is

$$\widetilde{\mathrm{Fil}}^1\,\overline{\mathbb{\Delta}}_S = (V_i :\ i \geq 1)$$

The kernel of the map $\overline{\mathbb{\Delta}}_{C'} \to S'$ is

$$\mathrm{Fil}^1\,\overline{\mathbb{\Delta}}_S = (U_i :\ i \geq 0).$$

In this case, we have a fiber sequence

$$K_{S'\twoheadrightarrow S}/^{\mathbb{L}}p \to \mathrm{Fil}^1\,\overline{\Delta}_S \to \widetilde{\mathrm{Fil}}^1\overline{\Delta}_S,$$

and one can check that the endomorphism of $K_{S'\twoheadrightarrow S}/^{\mathbb{L}}p$ arising from $\dot{\varphi}_1$ is induced by the diagram

$$\begin{tikzcd}
\mathrm{Fil}^1\,\overline{\Delta}_S \arrow[r] \arrow[d, "U_0\mapsto(p-1)!U_1"', "U_i\mapsto 0: i\geq 1"] & \widetilde{\mathrm{Fil}}^1\overline{\Delta}_S \arrow[d, "V_i\mapsto 0:\ i\geq 1"] \\
\mathrm{Fil}^1\,\overline{\Delta}_S \arrow[r] & \widetilde{\mathrm{Fil}}^1\overline{\Delta}_S
\end{tikzcd}$$

and so we clearly have $\dot{\varphi}_1^2 = 0$.

□

7. A result of Bragg-Olsson

The goal of this section is to state Theorem 7.1.5, which is due to Bragg-Olsson, and is an important ingredient in the general representability theorem that will appear in the next section.

7.1. **Formulation of the result.**

7.1.1. For $R \in \mathrm{CRing}_{\mathbb{F}_p/}$, set

$$Z^1_{\Delta}(R) = \mathrm{Fil}^{\mathrm{conj}}_1\,\overline{\Delta}_R \times_{\overline{\Delta}_R} \mathrm{Fil}^1_{\mathrm{Hdg}}\,\overline{\Delta}_R \simeq \mathrm{hker}(\mathrm{Fil}^{\mathrm{conj}}_1(R/^{\mathbb{L}}p) \to R)$$

Then we have two maps

$$q_1, q_2 : Z^1_{\Delta}(R) \to \mathrm{gr}^{\mathrm{conj}}_1\,\overline{\Delta}_R \simeq \mathbb{L}_{R/\mathbb{F}_p}[-1],$$

where q_1 arises from the natural map $\mathrm{Fil}^{\mathrm{conj}}_1\,\overline{\Delta}_R \to \mathrm{gr}^{\mathrm{conj}}_1\,\overline{\Delta}_R$, and q_2 arises from the natural map $\mathrm{Fil}^1_{\mathrm{Hdg}}\,\overline{\Delta}_R \to \mathrm{gr}^1_{\mathrm{Hdg}}\,\overline{\Delta}_R$ composed with the Cartier isomorphism.

$Z^1_{\Delta}(R)$ inherits the structure of an object in Mod_R from $\mathrm{Fil}^{\mathrm{conj}}_1\,\overline{\Delta}_R$. For this structure, q_1 is R-linear, while q_2 is φ-semilinear, and so corresponds to an R-linear map $1\otimes q_2 : \varphi^*Z^1_{\Delta}(R) \to \mathrm{gr}^{\mathrm{conj}}_1\,\overline{\Delta}_R$.

7.1.2. Let $\mathrm{Mod}^{\varphi}_{R/}$ be the ∞-category of pairs (N,ψ), where $N \in \mathrm{Mod}_R$, and $\psi : \varphi^*N \to N$ is a map in Mod_R. For each such pair, we obtain two maps

$$q_{1,\psi}, q_{2,\psi} : N \otimes_R Z^1_{\Delta}(R) \to N \otimes_R \mathrm{gr}^{\mathrm{conj}}_1\,\overline{\Delta}_R.$$

The first is simply $q_{1,\psi} = \mathrm{id}\otimes q_1$, and is independent of ψ, while the second is the composition

$$N \otimes_R Z^1_{\Delta}(R) \to \varphi^*N \otimes_R \varphi^*Z^1_{\Delta}(R) \xrightarrow{\psi\otimes(1\otimes q_2)} N \otimes_R \mathrm{gr}^{\mathrm{conj}}_1\,\overline{\Delta}_R.$$

Remark 7.1.3. Suppose that R is a smooth $\mathbb{F}_p$-algebra. Then we have

$$\mathrm{Fil}^{\mathrm{conj}}_i\,\overline{\Delta}_R \simeq \tau^{\leq i}\Omega^{\bullet}_{R/\mathbb{F}_p},$$

and therefore $Z^1_{\Delta}(R) \simeq \Omega^{1,\mathrm{cl}}_{R/\mathbb{F}_p}[-1]$ is the shifted module of closed differential forms, while $\mathrm{gr}^{\mathrm{conj}}_1\,\overline{\Delta}_R \simeq H^1(\Omega^{\bullet}_{R/\mathbb{F}_p})[-1]$.

The map q_1 now is the natural surjection, while q_2 is the composition

$$\Omega^{1,\mathrm{cl}}_{R/\mathbb{F}_p}[-1] \to \Omega^1_{R/\mathbb{F}_p}[-1] \xrightarrow{\simeq} H^1(\Omega^{\bullet}_{R/\mathbb{F}_p})[-1],$$

where the first map is the inclusion, and the second is the Cartier isomorphism.

When N is a vector bundle over R, there is an associated finite flat commutative group scheme $G(N,\psi)$ of height one (see §7.2 below), and the complex $R\Gamma_{\varphi}(R,(N,\psi))$ is (up to shift) precisely that of Artin-Milne appearing in [2, Prop. 2.4], which computes the flat cohomology of $G(N,\psi)$.

7.1.4. We can now associate with each $(N, \psi) \in \mathrm{Mod}_R^{\varphi}$ the space

$$R\Gamma_{\varphi}(R, (N, \psi)) = \mathrm{hker}(N \otimes_R Z^1_{\mathbb{\Delta}}(R) \xrightarrow{q_{1,\psi} - q_{2,\psi}} N \otimes_R \mathrm{gr}_1^{\mathrm{conj}} \overline{\mathbb{\Delta}}_R).$$

This yields a prestack

$$\begin{aligned} \mathsf{S}_{(N,\psi)} : \mathrm{CRing}_{R/} &\to \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{F}_p} \\ C &\mapsto \tau^{\leq 0} R\Gamma_{\varphi}(C, (C \otimes_R N, \mathrm{id} \otimes \psi)) \end{aligned}$$

Theorem 7.1.5. *Suppose that N is perfect of Tor amplitude $[-m, n]$; then $\mathsf{S}_{(N,\psi)}$ is represented by a derived Artin m-stack over R. If $n \leq -1$, then $\mathsf{S}_{(N,\psi)}$ is a smooth over R.*

7.2. **Flat cohomology of height one group schemes.** We now present the proof of Bragg-Olsson.

7.2.1. Begin by considering the following special case: Suppose that R is discrete and that N is locally free of finite rank over R. Then the pair (N, ψ) gives rise to a finite flat commutative p-group scheme $G(N, \psi)$ over R of height 1, meaning that its Frobenius endomorphism is trivial; see [29, Exp. VIIA], [16, §2]. Explicitly, the Cartier dual $G^{\vee}(N, \psi)$ is given as the kernel of the map

$$\mathbf{V}(N^{\vee}) \xrightarrow{\psi^{\vee} - F} \mathbf{V}(\varphi^* N^{\vee}) \simeq R \otimes_{\varphi, R} \mathbf{V}(N^{\vee}),$$

where F is the relative Frobenius map for $\mathbf{V}(N^{\vee})$ with respect to R.

We now have:

Theorem 7.2.2. *For any $C \in \mathrm{CRing}_{R/}$, we have a canonical isomorphism*

$$R\Gamma_{\varphi}(C, (C \otimes_R N, \mathrm{id} \otimes \psi)) \xrightarrow{\simeq} R\Gamma_{\mathrm{fppf}}(\mathrm{Spec}\, C, G(N, \psi)).$$

Proof. By considering the universal situation for pairs (N, ψ), one reduces to the case where R is a smooth $\mathbb{F}_p$-algebra. The arrow in question is obtained via left Kan extension from an isomorphism of functors for smooth R-algebras C, where it is a classical theorem of Artin-Milne [2, Proposition 2.4]; see Remark 7.1.3.

That this map is an isomorphism is a theorem of Bragg-Olsson [13, Theorem 4.8], who attribute the proof to Bhatt-Lurie.

□

Corollary 7.2.3. *The stack $\mathsf{S}_{(N,\psi)}$ is canonically isomorphic to the group scheme $G(N, \psi)$, and the stack $\mathsf{S}_{(N[1],\psi[1])}$ is isomorphic to the classifying stack $BG(N, \psi)$. For any $r \geq 2$, $\mathsf{S}_{(N,\psi)[r]}$ is isomorphic to $B^2 G(N, \psi)[r-2]$.*

Proof. Most of this is clear from Theorem 7.2.2. For the last assertion, we only need to note that the left hand side of the isomorphism in the theorem is always (-2)-connective. Indeed, $\mathrm{gr}_1^{\mathrm{conj}} \overline{\mathbb{\Delta}}_C$ and $Z^1_{\mathbb{\Delta}}(C)$ are both (-1)-connective for any $C \in \mathrm{CRing}_{\mathbb{F}_p/}$. □

Proof of Theorem 7.1.5. Suppose that we have a cofiber sequence

$$(N', \psi') \to (N, \psi) \to (N'', \psi'')$$

of φ-modules over R. Then we obtain a Cartesian diagram of prestacks

$$\begin{tikzcd} \mathsf{S}_{(N,\psi)} \arrow[r] \arrow[d] & \mathsf{S}_{(N'',\psi'')} \arrow[d, "0"] \\ \mathsf{S}_{(N'',\psi)} \arrow[r] & \mathsf{S}_{(N',\psi')[1]}. \end{tikzcd}$$

Therefore, $\mathsf{S}_{(N,\psi)}$ is representable as soon as $\mathsf{S}_{(N'',\psi'')}$ and $\mathsf{S}_{(N',\psi')[1]}$ are representable.

By working Zariski locally on $\mathrm{Spec}\, R$, we can assume that N is represented by a bounded complex of locally free modules over R and that ψ lifts to a φ-semilinear map of this complex. In particular, by considering the stupid filtration for this complex, we reduce to showing that $\mathsf{S}_{(N,\psi)[r]}$ is representable when N is a locally free R-module and $r \in \mathbb{Z}$. This follows from Corollary 7.2.3 for $r \geq 0$. For the representability of $\mathsf{S}_{(N,\psi)[r]}$ with $r < 0$, note that we have a Cartesian square of prestacks

$$\begin{tikzcd} \mathsf{S}_{(N,\psi)[r]} \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d, "0"] \\ \operatorname{Spec} R \arrow[r, "0"'] & \mathsf{S}_{(N,\psi)[r+1]}. \end{tikzcd}$$

It remains to show that $\mathsf{S}_{(N,\psi)}$ is smooth if $n \leq -1$: But the above proof tells us that it is enough to know that $\mathsf{S}_{(N,\psi)[r]}$ is smooth when N is locally free and $r \geq 1$, and this is also clear from Corollary 7.2.3. □

8. Representability theorems for 1-bounded stacks

In this section, the technical heart of the paper, we prove a representability theorem under somewhat general hypotheses. We also record some applications to stacks obtained from F-gauges.

8.1. The F-zip stack.

For $R \in \mathrm{CRing}_{\mathbb{F}_p/}$, we will now define an fpqc stack $R^{F\mathrm{Zip}}$ over $\mathbb{F}_p$: Vector bundles over this stack will be equivalent to the ∞-category of (vector bundle) F-zips over R.

8.1.1. We begin with the 1-truncated Witt frame $\underline{W_1(R)}$ from Example 5.4.8, and set $R^{F\mathrm{Zip}}$ to be the stack $\mathrm{Syn}(\underline{W_1(R)})$ from Remark 5.5.7.

This can be understood much more explicitly as follows: We first glue $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$ with $\mathbb{A}^1_+/\mathbb{G}_m \times \operatorname{Spec} R$ along the open substack $\operatorname{Spec} R \simeq \mathbb{G}_m/\mathbb{G}_m \times \operatorname{Spec} R$. Denote the resulting stack by $Y \times \operatorname{Spec} R$.

Consider the two canonical closed immersions

$$\lambda_+ : B\mathbb{G}_m \times \operatorname{Spec} R \to \mathbb{A}^1_+/\mathbb{G}_m \times \operatorname{Spec} R \lambda_- : B\mathbb{G}_m \times \operatorname{Spec} R \qquad \to \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R \tag{8.1.1.1}$$

and let ${}^{\varphi}\lambda_-$ be the composition

$$B\mathbb{G}_m \times \operatorname{Spec} R \xrightarrow{\mathrm{id}\times\varphi} B\mathbb{G}_m \times \operatorname{Spec} R \xrightarrow{\lambda_-} \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R.$$

We now take $R^{F\mathrm{Zip}}$ to be the coequalizer of the two maps

$$B\mathbb{G}_m \times \operatorname{Spec} R \xrightarrow{\lambda_+} \mathbb{A}^1_+/\mathbb{G}_m \times \operatorname{Spec} R \hookrightarrow Y \times \operatorname{Spec} R\ ;\ B\mathbb{G}_m \times \operatorname{Spec} R \xrightarrow{{}^{\varphi}\lambda_-} \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R \hookrightarrow Y \times \operatorname{Spec} R.$$

8.2. F-zips.

We will call objects in $\mathrm{QCoh}(R^{F\mathrm{Zip}})$ **F-zips over** R.

8.2.1. Unpacking the definitions, one sees that giving an F-zip M is equivalent to specifying the following data:

- A decreasingly filtered module $\mathrm{Fil}^\bullet_{\mathrm{Hdg}} M^-$ over R obtained via pullback along $x^{\mathcal{N}}_{dR}$;
- An increasingly filtered module $\mathrm{Fil}^{\mathrm{conj}}_\bullet M^+$ over R obtained via pullback along $x^{\mathcal{N}}_{HT}$;
- An isomorphism $\eta : M^+ \xrightarrow{\simeq} M^-$ in Mod_R identifying both with a common R-module M;
- An isomorphism of graded R-modules

$$\alpha : \mathrm{gr}^{\mathrm{conj}}_\bullet M^+ \xrightarrow{\simeq} \varphi^* \mathrm{gr}^{-\bullet}_{\mathrm{Hdg}} M^-.$$

In the sequel, we will use the identification $M^+ \simeq M^- \simeq M$ to drop all superscripts.

Remark 8.2.2. When M is a vector bundle over $R^{F\mathrm{Zip}}$, we essentially recover the definition of Moonen-Pink-Wedhorn-Ziegler from [60]. The de Rham cohomology of any smooth projective scheme over R with degenerating Hodge-to-de Rham spectral sequence, when equipped with its decreasing Hodge filtration and its increasing conjugate filtration, yields an example of such an F-zip.

8.2.3. Given an F-zip M over R, we obtain a $\mathrm{Mod}_{\mathbb{F}_p}$-valued prestack over R:

$$R\Gamma_{F\mathrm{Zip}}(\mathsf{M}) : C \mapsto R\Gamma(C^{F\mathrm{Zip}}, \mathsf{M}|_{C^{F\mathrm{Zip}}}).$$

We can make this 'F-zip cohomology' quite explicit. Let $(\mathrm{Fil}^\bullet_{\mathrm{Hdg}} M, \mathrm{Fil}^{\mathrm{conj}}_\bullet M, \eta, \alpha)$ be the tuple corresponding to $\mathsf{M} \stackrel{\mathrm{defn}}{=} \eta^*\mathcal{M}$.

Then we obtain two maps

$$q_1, q_2 : \mathrm{Fil}^{\mathrm{conj}}_0 M \times_M \mathrm{Fil}^0_{\mathrm{Hdg}} M \to \mathrm{gr}^{\mathrm{conj}}_0 M,$$

where the first is via the map $\mathrm{Fil}^{\mathrm{conj}}_0 M \to \mathrm{gr}^{\mathrm{conj}}_0 M$, and so is R-linear, while the second is via

$$\mathrm{Fil}^0_{\mathrm{Hdg}} M \to \varphi^* \mathrm{Fil}^0_{\mathrm{Hdg}} M \to \varphi^* \mathrm{gr}^0_{\mathrm{Hdg}} M \xrightarrow{\simeq} \mathrm{gr}^{\mathrm{conj}}_0 M,$$

and so is φ-semilinear. We now have:

$$R\Gamma_{F\mathrm{Zip}}(\mathrm{Spec}\, R, \mathsf{M}) \simeq \mathrm{hker}\left(\mathrm{Fil}^{\mathrm{conj}}_0 M \times_M \mathrm{Fil}^0_{\mathrm{Hdg}} M \xrightarrow{q_1 - q_2} \mathrm{gr}^{\mathrm{conj}}_0 M\right). \tag{8.2.3.1}$$

8.3. The map to the mod-p syntomification.

8.3.1. There is a canonical map $\eta : R^{F\mathrm{Zip}} \to R^{\mathrm{syn}} \otimes \mathbb{F}_p$ obtained from Remark 6.10.7 and the map of frames $\underline{W(R)} \to \underline{W_1(R)}$. Alternatively, this can be obtained from Lemma 5.4.10, Theorem 6.11.5 and quasisyntomic descent.

Once again, we can give a much more explicit description of η. Consider first the map

$$\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R \xrightarrow{x^{\mathcal{N}}_{\mathrm{dR}}} R^{\mathcal{N}} \otimes \mathbb{F}_p \to R^{\mathrm{syn}} \otimes \mathbb{F}_p. \tag{8.3.1.1}$$

Next, we will construct another map

$$\mathbb{A}^1_+/\mathbb{G}_m \times \mathrm{Spec}\, R \xrightarrow{x^{\mathcal{N}}_{\mathrm{HT}}} R^{\mathcal{N}} \otimes \mathbb{F}_p \to R^{\mathrm{syn}} \otimes \mathbb{F}_p \tag{8.3.1.2}$$

whose restriction to the open substack $\mathbb{G}_m/\mathbb{G}_m \times \mathrm{Spec}\, R$ agrees with that of (8.3.1.1), and thus yields a map $\tilde{\eta} : Y \times \mathrm{Spec}\, R \to R^{\mathrm{syn}} \otimes \mathbb{F}_p$.

This is constructed as follows: Given a point $(L, u : C \to L)$ of $\mathbb{A}^1_+/\mathbb{G}_m \times \mathrm{Spec}\, R$ over $C \in \mathrm{CRing}_{R/}$, we have the Cartier-Witt divisor $W \xrightarrow{p} W$ along with the map of trivial generalized Cartier divisors $(\mathbb{G}_a^{\mathrm{dR}} \xrightarrow{0} \mathbb{G}_a^{\mathrm{dR}}) \to (\mathbb{G}_a^{\mathrm{dR}} \otimes_C L \xrightarrow{0} \mathbb{G}_a^{\mathrm{dR}})$ induced by u. In the notation of Remark 6.4.7, the associated filtered Cartier-Witt divisor is simply $M(L, p, 0, u) \to W$; the quotient $W/M(L, p, 0, u)$ is also a quotient of $\mathbb{G}_a$, and this yields the structure map from R. The restriction to the open point corresponds to the natural structure map $R \to C \to W(C)/^{\mathbb{L}}p$, as desired.

8.3.2. Consider now the two maps

$$B\mathbb{G}_m \times \mathrm{Spec}\, R \to \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R \xrightarrow{x^{\mathcal{N}}_{\mathrm{dR}}} R^{\mathcal{N}} \otimes \mathbb{F}_p;$$

$$B\mathbb{G}_m \times \mathrm{Spec}\, R \to \mathbb{A}^1_+/\mathbb{G}_m \times \mathrm{Spec}\, R \xrightarrow{x^{\mathcal{N}}_{\mathrm{HT}}} R^{\mathcal{N}} \otimes \mathbb{F}_p.$$

The composition with the map to $\mathbb{F}_p^{\mathcal{N}}$ is the same for both of these: It attaches to every line bundle L the filtered Cartier-Witt divisor $M(L, p, 0, 0) \to W$; however the structure map $R \to (W/M(L, p, 0, 0))(C)$ associated with the second composition differs from that associated with the first via pre-composition with $\varphi : R \to R$. This shows that the map $\tilde{\eta}$ descends to the desired map $\eta : R^{F\mathrm{Zip}} \to R^{\mathrm{syn}} \otimes \mathbb{F}_p$.

8.4. F-gauges and F-zips.

8.4.1. Suppose that $R \in \mathrm{CRing}^{f}_{(\mathbb{Z}/p^m\mathbb{Z})/}$. An **$F$-gauge over R of level n** is a quasi-coherent sheaf $\mathcal{M}$ over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. Via the map $x^{\mathcal{N}}_{\mathrm{dR}}$ of §6.7, we obtain a symmetric monoidal functor of symmetric monoidal ∞-categories

$$\mathrm{QCoh}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}) \to \mathrm{QCoh}(R^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{(x^{\mathcal{N}}_{\mathrm{dR}})^*} \mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}(R/^{\mathbb{L}}p^n)),$$

which can be viewed as a functor $\mathcal{M} \to \mathrm{Fil}^{\bullet}_{\mathrm{Hdg}} M_n$ from F-gauges of level n to filtered modules over $R/^{\mathbb{L}}p^n$.

The **Hodge-Tate weights** of an F-gauge $\mathcal{M}$ are the integers i such that $\mathrm{gr}^{-i}_{\mathrm{Hdg}} M_n \neq 0$. We will say that $\mathcal{M}$ is **1-bounded** if its Hodge-Tate weights are bounded above by 1.

As an immediate consequence of Theorem 6.11.5, we find:

Proposition 8.4.2. *Suppose that R is semiperfectoid. Then there is a canonical symmetric monoidal equivalence of stable ∞-categories*

$$\mathrm{QCoh}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\simeq} \mathbb{\Delta}_R\text{−}\mathsf{gauge}_n,$$

where the right hand side is the ∞-category of $\mathbb{\Delta}_R$-gauges of level n from §5.6.

From this we obtain:

Proposition 8.4.3. *Suppose that $R \to R_\infty$ is as in Corollary 6.12.5. Then we have an equivalence of symmetric monoidal stable ∞-categories*

$$\mathrm{QCoh}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\simeq} \mathrm{Tot}\left(\mathbb{\Delta}_{R_\infty^{\otimes(\bullet+1)}}\text{−}\mathsf{gauge}_n\right).$$

8.4.4. Associated with $\mathcal{M}$ is the $\mathrm{Mod}_{\mathbb{Z}/p^n\mathbb{Z}}$-valued prestack over R

$$R\Gamma_{\mathrm{syn}}(\mathcal{M}) : \mathrm{CRing}_{R/} \to \mathrm{Mod}_{\mathbb{Z}/p^n\mathbb{Z}}$$
$$C \mapsto R\Gamma_{\mathrm{syn}}(\mathrm{Spec}\, C, \mathcal{M}|_{C^{\mathrm{syn}}}) \overset{\mathrm{defn}}{=} R\Gamma(C^{\mathrm{syn}}, \mathcal{M}|_{C^{\mathrm{syn}}}).$$

We will set $\Gamma_{\mathrm{syn}}(\mathcal{M}) = \tau^{\leq 0} R\Gamma_{\mathrm{syn}}(\mathcal{M})$.

Suppose now that R is an $\mathbb{F}_p$-algebra. Then pullback along $\eta : R^{F\mathrm{Zip}} \to R^{\mathrm{syn}} \otimes \mathbb{F}_p$. gives a symmetric monoidal functor

$$\eta^* : \mathrm{QCoh}(R^{\mathrm{syn}} \otimes \mathbb{F}_p) \to \mathrm{QCoh}(R^{F\mathrm{Zip}}).$$

Note that there is a natural map

$$R\Gamma_{\mathrm{syn}}(\mathcal{M}) \to R\Gamma_{F\mathrm{Zip}}(\eta^*\mathcal{M})$$

for any F-gauge $\mathcal{M}$ over R of level 1.

8.5. **Prestacks of sections.** We will now use the terminology and results of §4.7.

8.5.1. Suppose that $R \in \mathrm{CRing}^{f}_{(\mathbb{Z}/p^m\mathbb{Z})/}$. Note that $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is a pointed graded stack equipped with a canonical map of graded stacks

$$B\mathbb{G}_m \times \mathrm{Spec}(R/^{\mathbb{L}}p^n) \to R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}.$$

arising via the map

$$x^{\mathcal{N}}_{\mathrm{dR},R} \otimes \mathbb{Z}/p^n\mathbb{Z} : \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}(R/^{\mathbb{L}}p^n) \to R^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}.$$

Suppose that we have a 1-bounded stack $\mathcal{X} = (\mathcal{X}^{\diamond}, X^0) \to R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. Define a prestack $\Gamma_{\mathrm{syn}}(\mathcal{X})$ that associates with each $C \in \mathrm{CRing}_{R/}$ the space

$$\Gamma_{\mathrm{syn}}(\mathcal{X})(C) = \mathrm{Map}_{R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}}(C^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}).$$

Unpacking definitions, one sees that we have

$$\Gamma_{\mathrm{syn}}(\mathcal{X})(C) = \mathrm{Map}_{R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}}(C^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}^{\diamond}) \times_{X^{\diamond,0}(C/^{\mathbb{L}}p^n)} X^0(C/^{\mathbb{L}}p^n),$$

where $X^{\diamond,0} \to \mathrm{Spec}\, R/^{\mathbb{L}}p^n$ is the fixed point locus of $\mathcal{X}^{\diamond}$.

8.5.2. This has a slightly more explicit description. First, define prestacks $\Gamma_{\mathcal{N}}(\mathcal{X})$ and $\Gamma_{\Delta}(\mathcal{X})$ over R by

$$\Gamma_{\mathcal{N}}(\mathcal{X})(C) = \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(C^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X});$$
$$\Gamma_{\Delta}(\mathcal{X})(C) = \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(C^{\Delta} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}^{\diamond}).$$

Restriction along the de Rham and Hodge-Tate immersions yields maps

$$j^*_{\mathrm{dR}}, j^*_{\mathrm{HT}} : \Gamma_{\mathcal{N}}(\mathcal{X}) \to \Gamma_{\Delta}(\mathcal{X}),$$

and we now have

$$\Gamma_{\mathrm{syn}}(\mathcal{X}) \xrightarrow{\simeq} \mathrm{eq}\left(\Gamma_{\mathcal{N}}(\mathcal{X}) \underset{j^*_{\mathrm{HT}}}{\overset{j^*_{\mathrm{dR}}}{\rightrightarrows}} \Gamma_{\Delta}(\mathcal{X})\right). \tag{8.5.2.1}$$

8.6. **Some auxiliary stacks.** Suppose that $\mathcal{X} = (\mathcal{X}^{\diamond}, X^0)$ is a 1-bounded stack over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$.

8.6.1. Let $X^{(n)}, X^{-,(n)}, X^{-,0}$ be the prestacks over R given by

$$X^{(n)}(C) = \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(\mathrm{Spec}\, C/^{\mathbb{L}}p^n, \mathcal{X});$$
$$X^{-,(n)}(C) = \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, C/^{\mathbb{L}}p^n, \mathcal{X});$$
$$X^{0,(n)}(C) = \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(B\mathbb{G}_m \times \mathrm{Spec}\, C/^{\mathbb{L}}p^n, \mathcal{X}).$$

The last two stacks are simply the Weil restriction from $R/^{\mathbb{L}}p^n$ to R of the attractor and fixed point locus of $(\mathcal{X}^{\diamond}, X^0)$, respectively.

8.6.2. If R is in addition an $\mathbb{F}_p$-algebra, then also define

$$X^{+,(n)}(C) = \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(\mathbb{A}^1_+/\mathbb{G}_m \times \mathrm{Spec}\, C/^{\mathbb{L}}p^n, \mathcal{X}).$$

This is the Weil restriction of the repeller associated with $\mathcal{X}$ and the map $\mathbb{A}^1_+/\mathbb{G}_m \times \mathrm{Spec}\, R/^{\mathbb{L}}p^n \to R^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}$.

8.6.3. Suppose that $n = m$, so that R is a $\mathbb{Z}/p^n\mathbb{Z}$-algebra. In this case, for any R-algebra C, we have a section $C/^{\mathbb{L}}p^n \to C$, and so we can also define prestacks X, X^-, X^0, X^+ over R (the last only when $n = m = 1$) by:

$$X(C) = \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(\mathrm{Spec}\, C, \mathcal{X}^{\diamond});$$
$$X^-(C) = \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, C, \mathcal{X});$$
$$X^0(C) = \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(B\mathbb{G}_m \times \mathrm{Spec}\, C, \mathcal{X});$$
$$X^+(C) = \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{F}_p}(\mathbb{A}^1_+/\mathbb{G}_m \times \mathrm{Spec}\, C, \mathcal{X}).$$

The last three stacks are simply the attractor, fixed point locus and repeller for $\mathcal{X}$ base-changed from $C/^{\mathbb{L}}p^n$ to C.

Note a subtlety about base-points: The base-point $B\mathbb{G}_m \times \mathrm{Spec}\, C \to \mathbb{A}^1_+/\mathbb{G}_m \times \mathrm{Spec}\, C$ is a lift of the Frobenius endomorphism of $B\mathbb{G}_m \times \mathrm{Spec}\, C$. In particular, we actually have

$$X^+(C) = \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{F}_p}(\mathbb{A}^1_+/\mathbb{G}_m \times \mathrm{Spec}\, C, \mathcal{X}^{\diamond}) \times_{{}^{\varphi}X^{\diamond,0}(C)} {}^{\varphi}X^0(C).$$

Here, ${}^{\varphi}X^0$ is given by ${}^{\varphi}X^0(C) = X^0(\varphi_*C)$, where φ_*C is the R-algebra C with structure map given by $R \xrightarrow{\varphi} R \to C$.

Proposition 8.6.4. *Suppose that $\mathcal{X} \to R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is a 1-bounded stack and suppose that $\pi_0(R)$ is a G-ring. Suppose also that $\mathcal{X}^{\diamond}$ has quasi-affine diagonal over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. Then $X^{(n)}$, $X^{-,(n)}$, $X^{0,(n)}$ and $X^{+,(n)}$ (if $m = 1$)—and, if $n = m$, the prestacks X, X^-, X^0 and X^+ (if $n = m = 1$)—are locally finitely presented derived Artin stacks over R.*

Proof. By Lemmas 3.5.4 and 4.6.8 it follows that $X^{\diamond,-,(n)}$, $X^{\diamond,+,(n)}$, $X^{\diamond,0,n}$ and $X^{0,(n)}$ are locally finitely presented derived Artin stacks. Since we have

$$X^{\pm,(n)} = X^{\diamond,\pm,(n)} \times_{X^{\diamond,0,(n)}} X^{0,(n)},$$

we see that these are also locally finitely presented derived Artin stacks.

The argument for $X^{(n)}$ is even simpler, since it is a mod-p^n Weil restriction of a derived Artin stack by definition.

The argument for the remaining four prestacks (when $n = m$) is the same, but doesn't involve Weil restrictions. $\square$

Remark 8.6.5. Suppose that $\mathcal{X}^\diamond$ is graded integrable (this is in particular true if it has quasi-affine diagonal by Proposition 4.11.3). When $n = m = 1$, for any R-algebra C, the map

$$X^+(C) \to \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{F}_p}((\mathbb{A}^1_+/\mathbb{G}_m)_{(u^2=0)} \times \mathrm{Spec}\, C, \mathcal{X}^\diamond) \times_{{}^\varphi X^{\diamond,0}(C)} {}^\varphi X^0(C)$$

is an equivalence. In particular, as soon as X^0 is integrable (and hence representable), X^+ is also integrable and hence representable. Note also that X^+ is quasicompact as soon as X^0 is so.

8.7. **Dévissage to the F-zip stack.** Suppose that $n = 1$, and that we have a 1-bounded stack $\mathcal{X} \to R^{\mathrm{syn}} \otimes \mathbb{F}_p$ with $R \in \mathrm{CRing}^f_{\mathbb{F}_p/}$.

8.7.1. For any $\mathbb{F}_p$-algebra C, the stack $C^{F\mathrm{Zip}}$ is a pointed graded stack over $B\mathbb{G}_m$ equipped with a canonical map $B\mathbb{G}_m \times \mathrm{Spec}\, C \to C^{F\mathrm{Zip}}$. Therefore, we can define the prestack $\Gamma_{F\mathrm{Zip}}(\mathcal{X})$ over R by

$$\Gamma_{F\mathrm{Zip}}(\mathcal{X})(C) = \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{F}_p}(C^{F\mathrm{Zip}}, \mathcal{X}).$$

This can be explicitly understood as follows: Since R is an $\mathbb{F}_p$-algebra, there are two maps ${}^\varphi\lambda_-^*, \lambda_+^* : X^- \times_X X^+ \to {}^\varphi X^0$ induced via pullback along the maps from (8.1.1.1). We now have

$$\Gamma_{F\mathrm{Zip}}(\mathcal{X}) \xrightarrow{\simeq} \mathrm{eq}\left(X^- \times_X X^+ \underset{\lambda_+^*\circ\mathrm{pr}_2}{\overset{{}^\varphi\lambda_-^*\circ\mathrm{pr}_1}{\rightrightarrows}} {}^\varphi X^0\right). \tag{8.7.1.1}$$

We will actually need a slightly different formulation of this expression, which follows formally:

$$\Gamma_{F\mathrm{Zip}}(\mathcal{X}) \xrightarrow{\simeq} \mathrm{eq}\left(X^- \times_X X^+ \times_{{}^\varphi X^0} X^0 \underset{\mathrm{pr}_3}{\overset{\lambda_-^*\circ\mathrm{pr}_1}{\rightrightarrows}} X^0\right). \tag{8.7.1.2}$$

Lemma 8.7.2. *Suppose that $\mathcal{X}^\diamond$ is a relative r-stack with one of the following properties:*
- *$\mathcal{X}^\diamond \to R^{\mathrm{syn}} \otimes \mathbb{F}_p$ has quasi-affine diagonal and $\pi_0(R)$ is a G-ring;*
- *The stacks X^-, X^+, X^0 are locally finitely presented derived Artin r-stacks over R.*

Then $\Gamma_{F\mathrm{Zip}}(\mathcal{X})$ is a locally finitely presented derived Artin r-stack over R.

Proof. Proposition 8.6.4 shows that the first condition implies the second. We now conclude using the presentation (8.7.1.1). $\square$

8.7.3. Let $\mathbb{T}_{\mathcal{X}}$ be the relative tangent complex for $\mathcal{X}^\diamond$ over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. For every section $x \in \Gamma_{F\mathrm{Zip}}(\mathcal{X})(C)$, the perfect F-zip $\mathsf{M}(\mathcal{X})_x = x^*\mathbb{T}_{\mathcal{X}} \in \mathrm{Perf}(C^{F\mathrm{Zip}})$ is 1-bounded. We will view this as giving us an F-zip $\mathsf{M}(\mathcal{X})$ over $\Gamma_{\mathrm{FZip}}(\mathcal{X})$. Note that, if $\mathcal{X}^\diamond$ is a relative r-stack, then this will be a perfect F-zip with Tor amplitude in $[-r, \infty)$.

We now obtain from this a φ-module $(N(\mathsf{M}(\mathcal{X})), \psi(\mathsf{M}(\mathcal{X})))$ over $\Gamma_{F\mathrm{Zip}}(\mathcal{X})$. Explicitly this means the following: Given a section x of $\mathcal{X}$ over $C^{F\mathrm{Zip}}$, we obtain an F-zip $x^*\mathbb{T}_{\mathcal{X}}$, which corresponds to a tuple $(\mathrm{Fil}^\bullet_{\mathrm{Hdg}} M_x, \mathrm{Fil}_\bullet^{\mathrm{conj}} M_x, \eta, \alpha)$. We now have

$$N(\mathsf{M}(\mathcal{X}))_x = \mathrm{gr}_1^{\mathrm{conj}} M_x,$$

and the linearization of $\psi(\mathsf{M}(X))_x$ is given by

$$\varphi^* \mathrm{gr}_1^{\mathrm{conj}} M_x \xrightarrow{\simeq} \mathrm{gr}_1^{\mathrm{conj}} M_x = \mathrm{Fil}_1^{\mathrm{conj}} M_x \to M_x = \mathrm{Fil}^{-1}_{\mathrm{Hdg}} M_x \to \mathrm{gr}_1^{\mathrm{conj}} M_x. \tag{8.7.3.1}$$

8.7.4. Using Theorem 7.1.5, we now obtain, for every $i \in \mathbb{Z}$, a relative derived Artin stack $\mathsf{S}_i(\mathcal{X}) \to \Gamma_{F\mathrm{Zip}}(\mathcal{X})$ associating with every x as above the stack

$$\mathsf{S}_i(\mathcal{X})_x \overset{\mathrm{defn}}{=} \mathsf{S}_{(N(\mathsf{M}(\mathcal{X}))_x, \psi(\mathsf{M}(\mathcal{X}))_x)[i]}.$$

Explicitly, for any $C' \in \mathrm{CRing}^f_{C/}$, we have

$$\mathsf{S}_i(\mathcal{X})_x(C') \simeq \tau^{\leq i}\, \mathrm{hker}\left(\mathrm{gr}_1^{\mathrm{conj}} M_x \otimes_C Z^1_{\mathbb{\Delta}}(C') \xrightarrow{q_{1,x}-q_{2,x}} \mathrm{gr}_1^{\mathrm{conj}} M_x \otimes_C \mathrm{gr}_1^{\mathrm{conj}} \overline{\mathbb{\Delta}}_{C'}\right)[i] \tag{8.7.4.1}$$

for certain maps $q_{1,x}, q_{2,x}$. If $\mathcal{X}^\diamond$ is a relative r-stack, then we have already noted above that $N(\mathsf{M}(\mathcal{X}))$ has Tor amplitude in $[-r,\infty)$, and so $\mathsf{S}_i(\mathcal{X})$ is a relative r-stack.

Theorem 8.7.5. *Suppose that $\mathcal{X}^\diamond \to R^{\mathrm{syn}} \otimes \mathbb{F}_p$ satisfies one of the following conditions:*

- *It is graded and filtered integrable*[18]*;*
- *It has quasi-affine diagonal.*

Then:

(1) There is a canonical Cartesian diagram of prestacks over R:

$$\begin{tikzcd} \Gamma_{\mathrm{syn}}(\mathcal{X}) \arrow[r] \arrow[d] & \Gamma_{F\mathrm{Zip}}(\mathcal{X}) \arrow[d, "0"] \\ \Gamma_{F\mathrm{Zip}}(\mathcal{X}) \arrow[r] & \mathsf{S}_1(\mathcal{X}), \end{tikzcd}$$

where the right vertical map is the zero section.

(2) Suppose that $\mathcal{X}^\diamond$ is smooth *over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$, and that its relative tangent complex is* 1-connective. *Then $\Gamma_{\mathrm{syn}}(\mathcal{X}) \to \Gamma_{F\mathrm{Zip}}(\mathcal{X})$ is a quasisyntomic torsor under $\mathsf{S}_0(\mathcal{X})$.*

(3) In particular, if $\mathcal{X}$ satisfies the hypotheses in Lemma 8.7.2, then $\Gamma_{\mathrm{syn}}(\mathcal{X})$ is represented by a locally finitely presented derived Artin r-stack over R.

Proof. The last assertion is immediate from Lemma 8.7.2.

For the first two, begin by noting that the condition of having quasi-affine diagonal implies the other one by Propositions 4.11.3 and 4.12.3. Therefore, we can assume that $\mathcal{X}$ is graded and filtered integrable.

By quasisyntomic descent, we can reduce to checking the existence of the diagram in (1) for semiperfect inputs. Here, the result follows from Theorem 6.11.5 and Proposition 5.9.19.

Assertion (2) follows analogously from Corollary 5.9.24. □

Corollary 8.7.6. *Suppose that $\mathcal{M}$ is a 1-bounded perfect F-gauge of level 1 with Tor amplitude $[-r,s]$ over R.*

(1) The prestack $\Gamma_{\mathrm{syn}}(\mathcal{M})$ over R is represented by a finitely presented derived Artin r-stack taking values in t-connective spaces, where $t = \min\{s-1, 0\}$.

(2) If $s \leq -1$, then $\Gamma_{\mathrm{syn}}(\mathcal{M})$ is a smooth, faithfully flat Artin stack over R.

(3) If $r \leq 0$, then $\Gamma_{\mathrm{syn}}(\mathcal{M})$ is a derived affine scheme over R.

Proof. The vector stack $\mathcal{X} = \mathbf{V}(\mathcal{M}) \to R^{\mathrm{syn}} \otimes \mathbb{F}_p$ is a finitely presented r-stack, and Example 4.8.5 shows that it can be extended to a 1-bounded stack—denoted by the same symbol—by bringing along the entire fixed point locus.

Moreover, the associated stacks $X^{-,(n)}$, $X^{+,(n)}$ and $X^{0,(n)}$ admit the following explicit description: Associated with $\mathcal{M}$ is an F-zip giving in particular the pair $(\mathrm{Fil}^\bullet_{\mathrm{Hdg}} M, \mathrm{Fil}^{\mathrm{conj}}_\bullet M)$; then $X^{-,(n)}$ (resp. $X^{+,(n)}$, $X^{0,(n)}$) is the mod-p^n Weil restriction of the vector stack associated with $\mathrm{Fil}^0_{\mathrm{Hdg}} M$ (resp. $\mathrm{Fil}^{\mathrm{conj}}_0 M$, $\mathrm{gr}^0_{\mathrm{Hdg}} M$). In particular, all three stacks are representable, and so, if M is the underlying F-zip, then (8.7.1.1) shows that $\Gamma_{F\mathrm{Zip}}(\mathsf{M})$ is representable by a finitely presented derived Artin r-stack.

By Theorem 8.7.5 and Remarks 4.11.5 and 4.12.6, we have a Cartesian diagram

$$\begin{tikzcd} \Gamma_{\mathrm{syn}}(\mathcal{M}_1) \arrow[r] \arrow[d] & \Gamma_{F\mathrm{Zip}}(\mathsf{M}) \arrow[d] \\ \Gamma_{F\mathrm{Zip}}(\mathsf{M}) \arrow[r] & \mathsf{S}_{(N,\psi)[1]} \end{tikzcd}$$

[18]See Definitions 4.11.4, 4.12.5

where (N, ψ) is a perfect φ-module over $\Gamma_{F\mathrm{Zip}}(\mathsf{M})$ with $N = \mathrm{gr}^{-1}_{\mathrm{Hdg}} M$ of Tor amplitude in $[-r, s]$.

Therefore, Theorem 7.1.5 shows that $\Gamma_{\mathrm{syn}}(\mathbf{V}(\mathcal{M})) = \Gamma_{\mathrm{syn}}(\mathcal{M})$ is represented by a finitely presented derived Artin r-stack.

We now verify the assertion on the t-connectivity of its values. It is clear from (8.2.3.1) that $\Gamma_{F\mathrm{Zip}}(\mathcal{M})$ takes values in t-connective spaces. Combining this with the Cartesian diagram above and the fact that Z^1_{Δ} takes (-1)-connective values, now completes the proof of (1).

For (2), first note that $\Gamma_{F\mathrm{Zip}}(\mathcal{M})$ is smooth over R: This is because we can rewrite the formula (8.2.3.1) as

$$\Gamma_{F\mathrm{Zip}}(\mathsf{M})(R) \simeq R\Gamma_{F\mathrm{Zip}}(\mathrm{Spec}\, R, \mathsf{M}) \simeq \mathrm{hcoker}(\mathrm{gr}^{\mathrm{conj}}_0 M[-1] \to \mathrm{Fil}^{\mathrm{conj}}_0 M \times_M \mathrm{Fil}^0_{\mathrm{Hdg}} M),$$

where both source and target on the right are values of smooth vector stacks, faithfully flat over R. The proof is now completed by the second assertion from Theorem 7.1.5.

We now come to (3): Under our hypotheses, $\mathsf{M}^\vee$ is a connective perfect F-zip, and so (8.2.3.1) shows that $\Gamma_{F\mathrm{Zip}}(\mathsf{M})$ is a derived affine scheme over R. The proof of Theorem 7.1.5 reduces the relative affineness of $\Gamma_{\mathrm{syn}}(\mathcal{M})$ over $\Gamma_{F\mathrm{Zip}}(\mathsf{M})$ to that of $\mathsf{S}_{(N,\psi)}$ when (N, ψ) is a φ-module over R with N locally free. Here, the desired conclusion follows from Corollary 7.2.3, which shows that $\mathsf{S}_{(N,\psi)}$ is a finite flat group scheme over R. □

8.8. **Bootstrapping from characteristic p: coefficients.** Continue to assume that R is an $\mathbb{F}_p$-algebra. Suppose now that we have a 1-bounded stack $\mathcal{X} \to R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. For $m \le n$, write $\mathcal{X}_m$ for its restriction over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^m\mathbb{Z}$.

Over $\Gamma_{\mathrm{syn}}(\mathcal{X}_1)$, we have a 1-bounded perfect F-gauge $\mathcal{M}_1(\mathcal{X})$ of level 1: this associates with every $x \in \Gamma_{\mathrm{syn}}(\mathcal{X}_1)(C)$ the pullback to $C^{\mathrm{syn}} \otimes \mathbb{F}_p$ of the tangent complex of $\mathcal{X}_1^\diamond$ over $R^{\mathrm{syn}} \otimes \mathbb{F}_p$. In turn, for any $i \in \mathbb{Z}$, this gives us via Corollary 8.7.6 a $\mathrm{Mod}^{\mathrm{cn}}_{\mathbb{F}_p}$-valued relative locally finitely presented derived Artin stack $\Gamma(\mathcal{M}_1(\mathcal{X})[i]) \to \Gamma_{\mathrm{syn}}(\mathcal{X}_1)$ associating with each x the derived Artin stack

$$\Gamma(\mathcal{M}_1(\mathcal{X})[i])_x = \Gamma_{\mathrm{syn}}(\mathcal{M}_1(\mathcal{X})_x[i])$$

over C.

Proposition 8.8.1. *There is a canonical Cartesian diagram of prestacks*

$$\begin{tikzcd}
\Gamma_{\mathrm{syn}}(\mathcal{X}_{m+1}) \arrow[r] \arrow[d] & \Gamma_{\mathrm{syn}}(\mathcal{X}_m) \arrow[d, "0"] \\
\Gamma_{\mathrm{syn}}(\mathcal{X}_m) \arrow[r] & \Gamma(\mathcal{M}_1(\mathcal{X})[1]) \times_{\Gamma_{\mathrm{syn}}(\mathcal{X}_1)} \Gamma_{\mathrm{syn}}(\mathcal{X}_m),
\end{tikzcd}$$

In particular, if $\mathcal{X}_1$ satisfies the hypotheses of Theorem 8.7.5, then $\Gamma_{\mathrm{syn}}(\mathcal{X})$ is represented by a locally finitely presented derived Artin r-stack over R.

Proof. The last assertion follows from Theorem 8.7.5 and Corollary 8.7.6.

The existence of the Cartesian diagram of prestacks is an application of deformation theory. Let $\mathcal{M}_1^{\mathcal{N}}(\mathcal{X})$ (resp. $\mathcal{M}_1^{\Delta}(\mathcal{X})$) be the perfect complex over the Nygaard filtered prismatization of $\Gamma_{\mathcal{N}}(\mathcal{X}_1)$ (resp. over the prismatization of $\Gamma_{\Delta}(\mathcal{X}_1)$) obtained similarly to $\mathcal{M}_1(\mathcal{X})$ by pulling back the relative tangent complex of $\mathcal{X}^\diamond$ along the tautological map.

From the first relative vector stack, we obtain a prestack over $\Gamma_{\mathcal{N}}(\mathcal{X}_1)$ given on pairs (C, x) with $x \in \Gamma_{\mathcal{N}}(\mathcal{X}_1)(C)$ by

$$\Gamma_{\mathcal{N}}(\mathcal{M}_1(\mathcal{X}[1])) : (C, x) \mapsto \tau^{\le 0} R\Gamma(C^{\mathcal{N}} \otimes \mathbb{F}_p, \mathcal{M}_1^{\mathcal{N}}(\mathcal{X})_x[1]).$$

Similarly, over $\Gamma_{\Delta}(\mathcal{X}_1)$, we obtain a prestack given on pairs (C, x) with $x \in \Gamma_{\Delta}(\mathcal{X}_1)(C)$ by

$$\Gamma_{\Delta}(\mathcal{M}_1(\mathcal{X}[1])) : (C, x) \mapsto \tau^{\le 0} R\Gamma(C^{\Delta} \otimes \mathbb{F}_p, \mathcal{M}_1^{\Delta}(\mathcal{X})_x[1]).$$

Suppose that C is semiperfect to fix ideas: we can reduce to considering only such inputs by quasi-syntomic descent.

Let us look at the fibers of the map

$$\Gamma_{\mathcal{N}}(\mathcal{X}_{m+1})(C) \to \Gamma_{\mathcal{N}}(\mathcal{X}_m)(C).$$

By Theorem 6.11.5, $C^{\mathcal{N}} \otimes \mathbb{Z}/p^r\mathbb{Z}$ is the Rees stack $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}} \mathbb{\Delta}_C) \otimes \mathbb{Z}/p^r\mathbb{Z}$. Then by the discussion in §4.5 we have a canonical Cartesian square

$$\begin{tikzcd}
\Gamma_{\mathcal{N}}(\mathcal{X}_{m+1})(C) \arrow[r] \arrow[d] & \Gamma_{\mathcal{N}}(\mathcal{X}_m)(C) \arrow[d] \\
\Gamma_{\mathcal{N}}(\mathcal{X}_m)(C) \arrow[r] & \mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}} \mathbb{\Delta}_C/^{\mathbb{L}}p^m \oplus \mathrm{Fil}^\bullet_{\mathcal{N}} \overline{\mathbb{\Delta}}_C[1]), \mathcal{X}^\diamond) \times_{X^{\diamond,0,(1)}(C)} X^{0,(1)}(C).
\end{tikzcd}$$

Moreover, the prestack over R given by

$$C \mapsto \mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}} \mathbb{\Delta}_C/^{\mathbb{L}}p^m \oplus \mathrm{Fil}^\bullet_{\mathcal{N}} \overline{\mathbb{\Delta}}_C[1]), \mathcal{X}^\diamond) \times_{X^{\diamond,0,(1)}(C)} X^{0,(1)}(C)$$

is canonically isomorphic over $\Gamma_{\mathcal{N}}(\mathcal{X}_m)$ to the pullback of the stack $\Gamma_{\mathcal{N}}(\mathcal{M}_1(\mathcal{X})[1])$, showing that we have a Cartesian diagram of prestacks over $\Gamma_{\mathcal{N}}(\mathcal{X}_m)$:

$$\begin{tikzcd}
\Gamma_{\mathcal{N}}(\mathcal{X}_{m+1}) \arrow[r] \arrow[d] & \Gamma_{\mathcal{N}}(\mathcal{X}_m) \arrow[d, "0"] \\
\Gamma_{\mathcal{N}}(\mathcal{X}_m) \arrow[r] & \Gamma_{\mathcal{N}}(\mathcal{M}_1(\mathcal{X})[1]) \times_{\Gamma_{\mathcal{N}}(\mathcal{X}_1)} \Gamma_{\mathcal{N}}(\mathcal{X}_m)
\end{tikzcd}$$

There exists an analogous Cartesian diagram with $\mathcal{N}$ replaced with $\mathbb{\Delta}$. Combining these two diagrams with (8.5.2.1) now proves the Proposition. $\square$

Remark 8.8.2. Note that the establishment of the Cartesian diagrams in the proposition above did not use the hypothesis that R is an $\mathbb{F}_p$-algebra: One simply has to replace the word 'semiperfect' by 'semiperfectoid' in the proof.

Suppose that $\mathcal{X}^\diamond$ is as in (2) of Theorem 8.7.5. Then we have the following more streamlined assertion, which can be deduced from Proposition 8.8.1 and Corollary 8.7.6.

Corollary 8.8.3. *With these hypotheses,* $\Gamma_{\mathrm{syn}}(\mathcal{X}_{m+1}) \to \Gamma_{\mathrm{syn}}(\mathcal{X}_m)$ *is a quasisyntomic torsor under the smooth Artin stack* $\Gamma(\mathcal{M}_1(\mathcal{X}))$*, and is in particular smooth and surjective.*

8.9. **Deformation theory.** We continue with the assumptions of the previous subsection.

8.9.1. Suppose that we have a divided power thickening $(C' \twoheadrightarrow C, \gamma)$ of R-algebras. Then we have a canonical commuting square

$$\begin{tikzcd}
\Gamma_{\mathrm{syn}}(\mathcal{X})(C') \arrow[r] \arrow[d] & X^{-,(n)}(C) \arrow[d] \\
\Gamma_{\mathrm{syn}}(\mathcal{X})(C) \arrow[r] & X^{-,(n)}(C) \times_{X^{(n)}(C)} X^{(n)}(C').
\end{tikzcd} \tag{8.9.1.1}$$

This is obtained as follows: The top arrow and the first coordinate of the bottom arrow are obtained from the canonical map $\Gamma_{\mathrm{syn}}(\mathcal{X}) \to X^{-,(n)}$ obtained via pullback along the mod-p^n reduction of the map

$$x^{\mathcal{N}}_{\mathrm{dR}} : \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, C \to C^{\mathcal{N}}$$

for every R-algebra C. The second coordinate of the bottom arrow is obtained via pullback along the (mod-p^n reduction of the) lift

$$\tilde{x}_{\mathrm{dR},C'} : \mathrm{Spec}\, C' \to C^{\mathbb{\Delta}}$$

from Lemma 6.8.1.

Theorem 8.9.2. *Suppose that $\mathcal{X}$ is* 1*-bounded and filtered integrable, and that $\Gamma_{\mathrm{syn}}(\mathcal{X})$ is representable*[19]*. Let $(C' \twoheadrightarrow C, \gamma)$ be a* nilpotent *divided power thickening. Then the commuting square* (8.9.1.1) *is* Cartesian.

Proof. Write

$$\alpha_{(C' \twoheadrightarrow C, \gamma)} : \Gamma_{\mathrm{syn}}(\mathcal{X})(C') \to \Gamma_{\mathrm{syn}}(\mathcal{X})(C) \times_{X^{-,(n)}(C) \times_{X^{(n)}(C)} X^{(n)}(C')} X^{-,(n)}(C')$$

for the natural map. We would like to show that it is an equivalence when the divided power thickening is nilpotent.

Suppose that we have a triangle

$$\begin{tikzcd} C'' \arrow[r, two heads] \arrow[dr, two heads] & C' \arrow[d, two heads] \\ & C \end{tikzcd}$$

of maps underlying a triangle of nilpotent divided power thickenings

$$\begin{tikzcd} (C'' \twoheadrightarrow C', \gamma') \arrow[r] \arrow[dr] & (C'' \twoheadrightarrow C, \gamma'') \arrow[d] \\ & (C' \twoheadrightarrow C, \gamma). \end{tikzcd}$$

Then one easily finds that, if $\alpha_{(C' \twoheadrightarrow C, \gamma)}$ and $\alpha_{(C'' \twoheadrightarrow C', \gamma')}$ are equivalences, then so is $\alpha_{(C'' \twoheadrightarrow C, \gamma'')}$.

By Remark 3.7.3, we can factor $C' \twoheadrightarrow C$ into a sequence of divided power thickenings

$$C' = C'_m \twoheadrightarrow C'_{m-1} \twoheadrightarrow \cdots \twoheadrightarrow C'_1 \twoheadrightarrow C'_0 = C,$$

where each intermediate thickening $(C'_j \to C'_{j-1}, \gamma_j)$ is trivial and square-zero. Therefore, repeatedly using the discussion above, we can reduce to the case where $C' \twoheadrightarrow C$ is a square-zero extension with the trivial divided power structure.

In this case, since $\Gamma_{\mathrm{syn}}(\mathcal{X})$ is representable, and hence infinitesimally cohesive, it suffices to verify the theorem in the situation where $C' = C \oplus M[1]$ for some $M \in \mathrm{Mod}_C^{\mathrm{cn}}$. In particular, we can assume that $\mathrm{hker}(C' \to C)$ is 1-connective.

By quasisyntomic descent, we can reduce further to the case where C' and C are semiperfect. In this case, we are in the situation of Proposition 5.8.10 with $\underline{B} = \mathbb{\Delta}_{C'}$ and $\underline{A} = \mathbb{\Delta}_C$. Therefore, by Remark 5.8.12, it only remains to check two things:

(i) The map $\mathbb{\Delta}_{C'} \to \mathbb{\Delta}_C$ is surjective: this follows from [8, Remark 4.1.20]

(ii) If we set
$$K = \mathrm{hker}(\mathbb{\Delta}_{C'} \to \mathbb{\Delta}_C),$$
then the map $\dot{\varphi}_1 : K/^{\mathbb{L}}p \to K/^{\mathbb{L}}p$ is locally nilpotent: this follows from Lemma 6.13.1.

$\square$

8.10. **Bootstrapping from characteristic p: the base.** We will now take R to be in $\mathrm{CRing}^{f,p\text{-comp}}$. For any p-adic formal prestack Z over R set $\mathsf{R}(Z) = Z^{(1)}$, so that $\mathsf{R}(Z)(C) = Z(C/^{\mathbb{L}}p)$ for any $C \in \mathrm{CRing}^{f,p\text{-comp}}_{R/}$. This gives an endomorphism of the ∞-category of p-adic formal prestacks over R, and so can be iterated: We have

$$\mathsf{R}^t(Z)(C) = Z(C \otimes \mathbb{F}_p^{\otimes_{\mathbb{Z}} t}).$$

The key for us is the following systematic dévissage from characteristic p:

[19]We will only need it to be infinitesimally cohesive.

Proposition 8.10.1. *Let $\mathcal{X} \to R^{\mathrm{syn}} \otimes \mathbb{Z}/p^m\mathbb{Z}$ be a* 1*-bounded stack that is filtered integrable. For any $C \in \mathrm{CRing}^{f,p\text{-}comp}_{R/}$, the canonical map*

$$\Gamma_{\mathrm{syn}}(\mathcal{X})(C) \to \mathrm{Tot}\left(\Gamma_{\mathrm{syn}}(\mathcal{X})(C \otimes_{\mathbb{Z}} \mathbb{F}_p^{\otimes_{\mathbb{Z}} \bullet+1})\right)$$

is an equivalence. That is, we have an equivalence of p-adic formal prestacks

$$\Gamma_{\mathrm{syn}}(\mathcal{X}) \xrightarrow{\simeq} \mathrm{Tot}\left(\mathsf{R}^{\bullet+1}(\Gamma_{\mathrm{syn}}(\mathcal{X}))\right).$$

Now, if $(C' \twoheadrightarrow C, \gamma)$ is a divided power thickening of R-algebras, we obtain the canonical commuting square (8.9.1.1).[20]

Corollary 8.10.2 (Grothendieck-Messing)**.** *Suppose that $\mathcal{X} \to R^{\mathrm{syn}} \otimes \mathbb{Z}/p^m\mathbb{Z}$ is* 1*-bounded and filtered integrable, and that $\Gamma_{\mathrm{syn}}(\mathcal{X}) \otimes \mathbb{F}_p$ is representable. Then, if $(C' \twoheadrightarrow C, \gamma)$ is a* nilpotent *divided power thickening, the commuting square* (8.9.1.1) *is* Cartesian*.*

Proof. Nilpotence of divided powers is preserved under arbitrary base change along maps $C' \to D'$. Therefore, for every $m \geq 1$, the map

$$C' \otimes \mathbb{F}_p^{\otimes_{\mathbb{Z}} \bullet+1} \twoheadrightarrow C \otimes \mathbb{F}_p^{\otimes_{\mathbb{Z}} \bullet+1}$$

of cosimplicial $R \otimes \mathbb{F}_p$-algebras canonically lifts to a cosimplicial diagram of nilpotent divided power thickenings of $R \otimes \mathbb{F}_p$-algebras. This gives us a cosimplicial diagram of commuting squares as in (8.9.1.1), which are all Cartesian by Theorem 8.9.2. We conclude by Proposition 8.10.1, which now shows that the commuting square the corollary is concerned with is a limit of Cartesian ones. $\square$

As an immediate consequence, we obtain:

Corollary 8.10.3. *With the hypotheses above, write $\varpi_{\mathcal{X}} : \Gamma_{\mathrm{syn}}(\mathcal{X}) \to X^{-,(n)}$ for the canonical map. Then $\Gamma_{\mathrm{syn}}(\mathcal{X})$ admits a perfect cotangent complex over $X^{-,(n)}$, and we have canonical isomorphisms*

$$\mathbb{L}_{\Gamma_{\mathrm{syn}}(\mathcal{X})/R} \simeq \varpi_X^* \mathbb{L}_{X^{-,(n)}/X^{(n)}};$$
$$\mathbb{L}_{\Gamma_{\mathrm{syn}}(\mathcal{X})/X^{-,(n)}} \simeq \varpi_X^* \left(\mathbb{L}_{X^{(n)}/R}|_{X^{-,(n)}}\right)[1].$$

Assuming Proposition 8.10.1, we can now show:

Theorem 8.10.4. *Suppose that $\mathcal{X}$ is a* 1*-bounded r-stack over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. Suppose that one of the following holds:*

(1) $\mathcal{X}^\diamond$ has quasi-affine diagonal and $\pi_0(R)$ is a G-ring;
(2) $\mathcal{X}$ is filtered and graded integrable, and the p-adic formal stacks X^- and X^0 over $R/^{\mathbb{L}}p^n$ are representable.

Then:

(1) $\Gamma_{\mathrm{syn}}(\mathcal{X})$ is represented by a p-adic formal locally finitely presented Artin r-stack over R.
(2) If $\mathcal{X}^\diamond$ is flat over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ with (quasi-)affine diagonal, then $\Gamma_{\mathrm{syn}}(\mathcal{X})$ has (quasi-)affine diagonal.
(3) If X^- and X^0 are quasi-compact, then $\Gamma_{\mathrm{syn}}(\mathcal{X})$ is quasi-compact.

Proof. We can assume that R is a $\mathbb{Z}/p^m\mathbb{Z}$-algebra for some $m \geq 1$. As usual, the first hypothesis implies the second, and so we will assume that (2) is valid. Moreover, by Remark 8.6.5, when $n = m = 1$, our hypotheses also imply that the repeller X^+ is also representable.

One can now prove assertion (1) using a rather general argument involving Artin-Lurie representability; see Remark 8.10.13 below. Here, we give a more direct proof using Grothendieck-Messing theory.

By Proposition 8.8.1, we know that, under our hypotheses, $\Gamma_{\mathrm{syn}}(\mathcal{X}) \otimes \mathbb{F}_p$ is represented by a locally finitely presented Artin r-stack over $R/^{\mathbb{L}}p$.

[20]Strictly speaking, we had imposed the condition that R be an $\mathbb{F}_p$-algebra when we introduced this square; however, this hypothesis was not used in its construction.

If $p > 2$, then applying Corollary 8.10.2 to the natural nilpotent divided power structure on $R \twoheadrightarrow R/^{\mathbb{L}}p$, we obtain a Cartesian square of prestacks

$$\begin{tikzcd} \Gamma_{\mathrm{syn}}(\mathcal{X}) \arrow[r] \arrow[d] & X^{-,(n)} \arrow[d] \\ \mathsf{R}(\Gamma_{\mathrm{syn}}(\mathcal{X})) \arrow[r] & X^{(n)} \times_{\mathsf{R}(X^{(n)})} \mathsf{R}(X^{-,(n)}) \end{tikzcd}.$$

Assertion (1) now follows, since all the prestacks involved except for the one in the top left corner are known to be locally finitely presented derived Artin p-adic formal stacks over R.

If $p = 2$, then we can do something similar, by first considering the nilpotent divided power thickening $R \twoheadrightarrow R/^{\mathbb{L}}4$ to reduce to showing that $\Gamma_{\mathrm{syn}}(\mathcal{X}) \otimes \mathbb{Z}/4\mathbb{Z}$ is a locally finitely presented Artin stack over $\mathbb{Z}/4\mathbb{Z}$, and then using the trivial divided powers on the square zero extension $R \twoheadrightarrow R \otimes_{\mathbb{Z}/4\mathbb{Z}} \mathbb{F}_2$ (for $R \in \mathrm{CRing}^f_{(\mathcal{O}/4)/}$) to reduce further to the known case of $n = 1$.

Let us proceed to assertions (2) and (3): It is enough to prove them for the stack $\Gamma_{\mathrm{syn}}(\mathcal{X}) \otimes \mathbb{F}_p$. First, note that, under the hypotheses of (2) (resp. of (3)), $\Gamma_{F\mathrm{Zip}}(\mathcal{X}_1)$ has (quasi-)affine diagonal (resp. is quasicompact): This follows from Proposition 4.6.8 and the presentation (8.7.1.1).

Now, we claim that the map $\Gamma_{F\mathrm{Zip}}(\mathcal{X}_1) \to \mathsf{S}_1(\mathcal{X}_1)$ is quasi-compact with affine diagonal. For this, it is enough to know that $\mathsf{S}_0(\mathcal{X}_1)$ is a relative derived affine scheme over $\Gamma_{F\mathrm{Zip}}(\mathcal{X}_1)$: This follows from Corollary 8.7.6 and the fact that, under the hypotheses of (2), the cotangent complex for $\mathcal{X}$ over $R^{\mathrm{syn}} \otimes \mathbb{F}_p$ is a (-1)-connective perfect complex.

Combined with Theorem 8.7.5, this shows that, under the hypotheses of (2) (resp. of (3)), $\Gamma_{\mathrm{syn}}(\mathcal{X}_1) \otimes \mathbb{F}_p$ has (quasi-)affine diagonal (resp. is quasicompact).

By Proposition 8.8.1, the assertions for $n \geq 2$ are now reduced to the following assertion: For any 1-bounded perfect F-gauge $\mathcal{F}$ over $R^{\mathrm{syn}} \otimes \mathbb{F}_p$, the stack $\Gamma_{\mathrm{syn}}(\mathcal{F}) \to \mathrm{Spec}\, R$ is quasi-compact, and has affine diagonal when $\mathcal{F}$ has Tor amplitude in $[1, \infty)$.

The quasi-compactness follows from Corollary 8.7.6. The diagonal map for the stack is a torsor under $\Gamma_{\mathrm{syn}}(\mathcal{F}[-1])$, so it suffices to now observe that—when $\mathcal{F}$ has Tor amplitude in $[1, \infty)$—$\Gamma_{\mathrm{syn}}(\mathcal{F}[-1])$ is affine by the same corollary. $\square$

8.10.5. We now proceed towards the proof of Proposition 8.10.1. Let us say that a map $f : Z \to Y$ of p-adic formal prestacks over R^{syn} **satisfies Tot descent for** $\mathcal{X}^{\diamond}$ if the natural map

$$\mathrm{Map}_{/R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}}(Y \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}^{\diamond}) \to \mathrm{Tot}\left(\mathrm{Map}_{/R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}}(Z^{\times_Y(\bullet+1)} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}^{\diamond})\right)$$

is an equivalence. The map **satisfies universal Tot descent for** $\mathcal{X}^{\diamond}$ if, for any $Y' \to Y$, the base-change $Z \times_Y Y' \to Y'$ also satisfies Tot descent for $\mathcal{X}^{\diamond}$

Remark 8.10.6. Any flat cover satisfies universal Tot descent for $\mathcal{X}^{\diamond}$.

Remark 8.10.7. A composition of maps satisfying (universal) Tot descent for $\mathcal{X}^{\diamond}$ also satisfies (universal) Tot descent for $\mathcal{X}^{\diamond}$.

Remark 8.10.8. Suppose that we have maps $Z \xrightarrow{f} Y \xrightarrow{g} V$ such that:

- $f \circ g$ satisfies Tot descent for $\mathcal{X}^{\diamond}$;
- f satisfies universal Tot descent for $\mathcal{X}^{\diamond}$

Then g also satisfies Tot descent for $\mathcal{X}^{\diamond}$. This follows because, from our assumption on f, we find that the map $f^{\times_V m} : Z^{\times_V m} \to Y^{\times_V m}$ also satisfies Tot descent for $\mathcal{X}^{\diamond}$ for all $m \geq 1$.

Remark 8.10.9. We have the following observation of Halpern-Leistner and Preygel: Suppose that we have $A \in \mathrm{CRing}$ equipped with a map $\mathbb{Z}[T_1, \ldots, T_r] \to A$ such that A is derived J-complete, where $J = (T_1, \ldots, T_r) \subset$

$\mathbb{Z}[T_1,\dots,T_r]$; set $\overline{A} = A/^{\mathbb{L}}(T_1,\dots,T_r)$. Suppose that we have a J-adic formal Artin stack $\mathcal{Y}$ over A. Then, for $R \in \mathrm{CRing}_{A/}$ derived J-complete, the map

$$\mathcal{Y}(R) \to \mathrm{Tot}\left(\mathcal{Y}(R \otimes_A \overline{A}^{\otimes^{\mathbb{L}}_A \bullet+1})\right)$$

is an equivalence. In fact, one only needs for $\mathcal{Y}$ to be nilcomplete and infinitesimally cohesive; see [32, Cor. 3.1.4]. So Proposition 8.10.1 is certainly implied by Theorem D. Here we will use the former to complete the proof of the latter.

Lemma 8.10.10. *The map $C^{\mathbb{\Delta}} \to C^{\mathbb{\Delta}} \times_{\mathbb{Z}_p^{\mathbb{\Delta}}} \mathbb{Z}_p^{\mathrm{HT}}$ satisfies Tot descent for $\mathcal{X}^{\diamond}$.*

Proof. Via quasisyntomic descent once again, we reduce to the case where C is semiperfectoid. Let $I \xrightarrow{t} \mathbb{\Delta}_C$ be the generalized Cartier divisor on $\mathbb{\Delta}_C$ underlying its structure of a prism, so that we have

$$C^{\mathbb{\Delta}} \simeq \mathrm{Spf}(\mathbb{\Delta}_C)\ ;\ C^{\mathbb{\Delta}} \times_{\mathbb{Z}_p^{\mathbb{\Delta}}} \mathbb{Z}_p^{\mathrm{HT}} \simeq \mathrm{Spf}(\mathbb{\Delta}_C)_{(t=0)}.$$

Now, $\mathbb{\Delta}_C$ here is equipped with its I-adic topology with respect to which it is derived complete. Therefore, the lemma follows from Remark 8.10.9. ☐

Lemma 8.10.11. *The map $\mathbb{Z}_p^{\mathrm{HT}} \to \mathbb{F}_p^{\mathrm{HT}}$ satisfies universal Tot descent for $\mathcal{X}^{\diamond}$*

Proof. We will use Proposition 6.3.2, which shows (via Remark 8.10.6) that the map $\mathbb{Z}_p^{\mathrm{HT}} \to \mathrm{Spf}\,\mathbb{Z}_p$ satisfies universal Tot descent for $\mathcal{X}$.

It is now enough to show (see Remark 8.10.8) that the map $\mathrm{Spf}\,\mathbb{Z}_p \to \mathrm{Spec}\,\mathbb{F}_p$ satisfies universal Tot descent for $\mathcal{X}^{\diamond}$. This follows from Remark 8.10.9 and the fact that we are dealing with p-adic formal stacks. ☐

Lemma 8.10.12. *Suppose that we have $C \in \mathrm{CRing}_{R/}^{f,p\text{-}nilp}$. Then we have a canonical Cartesian square*

$$\begin{tikzcd}
\mathrm{Map}(C^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}) \arrow[r] \arrow[d] & \mathrm{Map}(C^{\mathbb{\Delta}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}^{\diamond}) \arrow[d] \\
X^{-}(C/^{\mathbb{L}}p^n) \arrow[r] & X(C/^{\mathbb{L}}p^n).
\end{tikzcd}$$

Proof. Via quasisyntomic descent, we reduce to the case of C semiperfectoid, where this follows from Theorem 6.11.5 and filtered integrability. ☐

Proof of Proposition 8.10.1. The limit preserving property of the functors $\mathrm{Spec}\,C \mapsto C^{\mathbb{\Delta}}$ and $\mathrm{Spec}\,C \mapsto C^{\mathcal{N}}$ shows that we have

$$C^{\mathbb{\Delta}} \times_{\mathbb{Z}_p^{\mathbb{\Delta}}} \underbrace{\mathbb{F}_p^{\mathbb{\Delta}} \times_{\mathbb{Z}_p^{\mathbb{\Delta}}} \times \cdots \times_{\mathbb{Z}_p^{\mathbb{\Delta}}} \mathbb{F}_p^{\mathbb{\Delta}}}_{\bullet+1} \xrightarrow{\simeq} (C \otimes \mathbb{F}_p^{\otimes \bullet+1})^{\mathbb{\Delta}};$$

$$C^{\mathcal{N}} \times_{\mathbb{Z}_p^{\mathcal{N}}} \underbrace{\mathbb{F}_p^{\mathcal{N}} \times_{\mathbb{Z}_p^{\mathcal{N}}} \times \cdots \times_{\mathbb{Z}_p^{\mathcal{N}}} \mathbb{F}_p^{\mathcal{N}}}_{\bullet+1} \xrightarrow{\simeq} (C \otimes \mathbb{F}_p^{\otimes \bullet+1})^{\mathcal{N}}.$$

Lemmas 8.10.10 and 8.10.11 together show that the composition

$$C^{\mathbb{\Delta}} \to C^{\mathbb{\Delta}} \times_{\mathbb{Z}_p^{\mathbb{\Delta}}} \mathbb{F}_p^{\mathbb{\Delta}} \to C^{\mathbb{\Delta}} \times_{\mathbb{Z}_p^{\mathbb{\Delta}}} \mathbb{F}_p^{\mathrm{HT}}$$

satisfies Tot descent for $\mathcal{X}^{\diamond}$, while the second map satisfies universal Tot descent for $\mathcal{X}^{\diamond}$. Therefore, Remark 8.10.8 now shows that $C^{\mathbb{\Delta}} \to C^{\mathbb{\Delta}} \times_{\mathbb{Z}_p^{\mathbb{\Delta}}} \mathbb{F}_p^{\mathbb{\Delta}}$ satisfies Tot descent for $\mathcal{X}^{\diamond}$.

This, combined with the discussion in the first paragraph, shows that we have

$$\mathrm{Map}(C^{\mathbb{\Delta}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}^{\diamond}) \xrightarrow{\simeq} \mathrm{Tot}\,\mathrm{Map}\big((C \otimes \mathbb{F}_p^{\otimes \bullet+1})^{\mathbb{\Delta}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}^{\diamond}\big).$$

Now, Lemma 8.10.12 combined with Remark 8.10.9 tells us that we also have

$$\mathrm{Map}(C^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}) \xrightarrow{\simeq} \mathrm{Tot}\,\mathrm{Map}\big((C \otimes \mathbb{F}_p^{\otimes \bullet+1})^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}\big).$$

The proof is now concluded by contemplating the identity (8.5.2.1). □

Remark 8.10.13. One can also deduce Theorem 8.10.4 from Proposition 8.8.1 and the following general assertion: Suppose that $\mathfrak{Y}$ is a p-adic formal prestack over R with the following properties:

(1) $\mathfrak{Y}\otimes\mathbb{F}_p$ is represented by a locally finitely presented derived Artin stack over $R/^{\mathbb{L}}p$;
(2) $\mathfrak{Y}$ satisfies Tot descent with respect to p: For every $C\in\mathrm{CRing}^{p\text{-nilp}}_{R/}$, the natural map
$$\mathfrak{Y}(C)\to\mathrm{Tot}\left(\mathfrak{Y}(C\otimes\mathbb{F}_p^{\otimes\cdot+1})\right)$$
is an equivalence.

Then $\mathfrak{Y}$ is represented by a p-adic formal derived Artin stack over R.

This is shown using Artin-Lurie representability [46, Theorem 7.1.6]. All the criteria involving limits are easily checked using our hypotheses. The existence of a p-completely almost perfect cotangent complex for $\mathfrak{Y}$ follows from the existence of an almost perfect cotangent complex for $\mathfrak{Y}\otimes\mathbb{F}_p$ and Tot descent along p for p-completely almost perfect complexes. From this, one also deduces the local finite presentation, which completes the verification of all the criteria in *loc. cit.*

8.11. **Functoriality.** Suppose that we have a map $\mathcal{X}_1\to\mathcal{X}_2$ of 1-bounded stacks over $R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}$ with quasi-affine diagonal. Then we obtain a map of derived stacks $\Gamma_{\mathrm{syn}}(\mathcal{X}_1)\to\Gamma_{\mathrm{syn}}(\mathcal{X}_2)$ over R. We also have the corresponding map of Weil restricted stacks $X_1^{-,(n)}\to X_2^{-,(n)}$ and $X_1^{(n)}\to X_2^{(n)}$.

The following is immediate from Corollary 8.10.3:

Proposition 8.11.1. *Let $\varpi_1:\Gamma_{\mathrm{syn}}(\mathcal{X}_1)\to X_1^{-,(n)}$ be the canonical map. Then we have a natural isomorphism*
$$\mathbb{L}_{\Gamma_{\mathrm{syn}}(\mathcal{X}_1)/\Gamma_{\mathrm{syn}}(\mathcal{X}_2)}\xrightarrow{\simeq}\varpi_1^*\mathbb{L}_{X_1^{-,(n)}/(X_1^{(n)}\times_{X_2^{(n)}}X_2^{-,(n)})}.$$

8.12. **Sections of 1-bounded perfect F-gauges.** Suppose that $\mathcal{M}$ is a 1-bounded perfect F-gauge of level n over $R\in\mathrm{CRing}^{f,p\text{-comp}}$ and Hodge-Tate weights bounded by 1. Pulling $\mathcal{M}$ back along $x^{\mathcal{N}}_{\mathrm{dR}}$ yields an increasingly filtered perfect complex $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}M_n$ over $R/^{\mathbb{L}}p^n$. The next theorem is immediate from Theorem 8.10.4 and Corollary 8.7.6.

Theorem 8.12.1. *The prestack $\Gamma_{\mathrm{syn}}(\mathcal{M})$ is represented by a p-adic formal locally finitely presented derived Artin stack over R with cotangent complex $\mathscr{O}_{\Gamma_{\mathrm{syn}}(\mathcal{M})}\otimes_R(\mathrm{gr}^{-1}_{\mathrm{Hdg}}M_n)^\vee[1]$. Moreover, if $(C'\twoheadrightarrow C,\gamma)$ is a nilpotent divided power thickening in $\mathrm{CRing}^{f,p\text{-}comp}_{R/}$, then we have a Cartesian square*
$$\begin{tikzcd}
\Gamma_{\mathrm{syn}}(\mathcal{M})(C') \arrow[r] \arrow[d] & C'\otimes_R\mathrm{Fil}^0_{\mathrm{Hdg}}M_n \arrow[d] \\
\Gamma_{\mathrm{syn}}(\mathcal{M})(C) \arrow[r] & (C\otimes_R\mathrm{Fil}^0_{\mathrm{Hdg}}M_n)\times_{C\otimes_R M_n}(C'\otimes_R M_n).
\end{tikzcd}$$
Moreover:

(1) If $\mathcal{M}$ has Tor amplitude in $(-\infty,-1]$, then $\Gamma_{\mathrm{syn}}(\mathcal{M})$ is a smooth faithfully flat p-adic formal stack over R.
(2) If $\mathcal{M}$ has Tor amplitude in $[0,\infty)$, then $\Gamma_{\mathrm{syn}}(\mathcal{M})$ is a derived affine p-adic formal scheme over R.

Proof. Only the numbered asssertions require proof. To show them, we can assume that R is an $\mathbb{F}_p$-algebra. We already know from Corollary 8.7.6 that the statements are true if $\mathcal{M}$ has level 1 and we now use the usual dévissage by power of p (say in the form of Proposition 8.8.1) to see that they are true in general. □

8.13. **Stacks of perfect F-zips of Hodge-Tate weights** $0,1$. For every $n\geq 1$, let $\mathcal{X}\to\mathbb{Z}_p^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}$ be the 1-bounded stack obtained via base-change from $\mathcal{P}_{\{0,1\}}\to B\mathbb{G}_m$ as described in Example 4.8.6.

We will denote the associated formal prestack $\Gamma_{\mathrm{syn}}(\mathcal{X})\to\mathbb{Z}_p$ by $\mathrm{Perf}^{\mathrm{syn}}_{\{0,1\},n}$. Concretely, this associates with every $R\in\mathrm{CRing}^{f,p\text{-nilp}}$ the ∞-groupoid $\mathrm{Perf}_{\{0,1\}}(R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z})^{\simeq}$ of perfect F-gauges of level n over R with Hodge-Tate weights $0,1$.

Over this prestack we have a canonical filtered perfect complex $\mathrm{Fil}^\bullet_{\mathrm{Hdg}} M_{\mathrm{taut}}$ obtained by viewing, for each R, the universal perfect F-gauge of level n as a perfect complex over R^{syn}, and pulling back along $x^{\mathcal{N}}_{\mathrm{dR}}$.

Since $\mathbb{Z}/p^m\mathbb{Z}$ is a G-ring for all m, we obtain the next theorem from Theorem 8.10.4 and the discussion in Example 4.8.6.

Theorem 8.13.1. *The prestack* $\mathrm{Perf}^{\mathrm{syn}}_{\{0,1\},n}$ *is represented by a p-adic formal locally finitely presented derived Artin stack over $\mathbb{Z}_p$ with cotangent complex* $(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M_{\mathrm{taut}})^\vee \otimes \mathrm{Fil}^0_{\mathrm{Hdg}} M_{\mathrm{taut}}$. *Moreover, if $(C' \twoheadrightarrow C, \gamma)$ is a nilpotent divided power thickening of p-complete algebras in* CRing^f*, then we have a Cartesian square*

$$\begin{tikzcd}
\mathrm{Perf}^{\mathrm{syn}}_{\{0,1\},n}(C') \arrow[r] \arrow[d] & \mathrm{Perf}_{\{0,1\}}(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, C'/^{\mathbb{L}}p^n) \arrow[d] \\
\mathrm{Perf}^{\mathrm{syn}}_{\{0,1\},n}(C) \arrow[r] & \mathrm{Perf}_{\{0,1\}}(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, C/^{\mathbb{L}}p^n) \times_{\mathrm{Perf}(C/^{\mathbb{L}}p^n)} \mathrm{Perf}(C'/^{\mathbb{L}}p^n).
\end{tikzcd}$$

Proof. The only thing that needs still to be verified is the assertion about the cotangent complex. For this, note that we have

$$\begin{aligned}
X^{-,(n)} &: C \mapsto \mathrm{Perf}_{\{0,1\}}(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, C/^{\mathbb{L}}p^n) \\
X^{(n)} &: C \mapsto \mathrm{Perf}(C/^{\mathbb{L}}p^n).
\end{aligned}$$

The fiber of the map $X^{-,(n)} \to X^{(n)}$ over a perfect complex M over $C/^{\mathbb{L}}p^n$ classifies two step filtrations $\mathrm{Fil}^\bullet M$ on M with $\mathrm{gr}^i M$ perfect for all i, and $\mathrm{gr}^i M \simeq 0$ for $i \neq -1, 0$. Giving such a datum is equivalent to specifying the map $f : \mathrm{Fil}^0 M \to \mathrm{Fil}^{-1} M = M$ with $\mathrm{Fil}^0 M$ perfect over $C/^{\mathbb{L}}p^n$.

This shows that the tangent space of the map at M is canonically isomorphic to the space of maps $\mathrm{Fil}^0 M \to \mathrm{gr}^{-1} M$, which is of course the C-module $(\mathrm{Fil}^0 M)^\vee \otimes_C \mathrm{gr}^{-1} M$. Taking duals and using Corollary 8.10.3 now gives the desired cotangent complex. □

Corollary 8.13.2. *Given $R \in \mathrm{CRing}^{f,p\text{-}nilp}$, write $\mathrm{Perf}^{\mathrm{syn}}_{\{0\},n}(R)$ for the ∞-groupoid of perfect F-gauges with Hodge-Tate weights* 0. *Then there is a canonical equivalence*

$$\mathrm{Perf}^{\mathrm{syn}}_{\{0\},n}(R) \xrightarrow{\simeq} D^b_{\mathrm{lisse}}(\mathrm{Spec}\, R, \mathbb{Z}/p^n\mathbb{Z})$$

where the right hand side is the bounded derived category of lisse $\mathbb{Z}/p^n\mathbb{Z}$-sheaves over $\mathrm{Spec}\, R$.

Proof. We can view $\mathrm{Perf}^{\mathrm{syn}}_{\{0\},n}$ as the open substack of $\mathrm{Perf}^{\mathrm{syn}}_{\{0,1\},n}$ parameterizing objects $\mathcal{M}$ such that $\mathrm{gr}^{-1}_{\mathrm{Hdg}} M_{\mathrm{taut}} \simeq 0$. Moreover, the description of the cotangent complex shows that this substack is *étale* over $\mathrm{Spf}\, \mathbb{Z}_p$. In particular, it is determined completely by its restriction to perfect $\mathbb{F}_p$-algebras R.

For perfect R, the left hand side in the statement can be identified with the ∞-groupoid of perfect complexes M of $W_n(R)$-modules equipped with an isomorphism $\varphi^*\mathsf{M} \xrightarrow{\simeq} \mathsf{M}$. We can now conclude by a classical result of Katz, as formulated in [10, Proposition 3.6]. □

Remark 8.13.3. Explicitly, the equivalence is given on connective objects by sending $\mathcal{M}$ to $\Gamma_{\mathrm{syn}}(\mathcal{M})$, where the latter is seen to be an étale stack over R by Theorem 8.12.1.

9. The algebraicity conjecture of Drinfeld

We can finally introduce the main protagonist of this paper. Fix a smooth affine group scheme G over $\mathbb{Z}_p$ and a 1-bounded cocharacter $\mu : \mathbb{G}_{m,\mathcal{O}} \to G_{\mathcal{O}}$ defined over the ring of integers $\mathcal{O}$ in a finite unramified extension of $\mathbb{Q}_p$ with residue field k.

9.1. **Definitions.** There is a canonical map $\mathbb{Z}_p^{\mathrm{syn}} \to B\mathbb{G}_m$ classifying the Breuil-Kisin twist from §6.6. The restriction to $\mathcal{O}^{\mathcal{N}}$ lifts to a map $B\mathbb{G}_{m,\mathcal{O}}$, which does not descend to $\mathcal{O}^{\mathrm{syn}}$; however, we do have a map of pointed graded p-adic formal stacks

$$\mathcal{O}^{\mathrm{syn}} \to (B\mathbb{G}_m \times \mathrm{Spf}\,\mathbb{Z}_p, \iota_{\mathcal{O}}).$$

Therefore, we can pull the 1-bounded stack $\mathcal{B}(G,\mu)$ from Definition (4.10.4) back over $\mathcal{O}^{\mathrm{syn}}$.

Definition 9.1.1. For any $R \in \mathrm{CRing}^{f,p\text{-comp}}_{\mathcal{O}/}$, we set $\mathrm{BT}^{G,\mu}_n(R) = \Gamma_{\mathrm{syn}}(\mathcal{B}(G,\mu) \otimes \mathbb{Z}/p^n\mathbb{Z})(R)$. Set

$$\mathrm{BT}^{G,\mu}_\infty = \varprojlim_n \mathrm{BT}^{G,\mu}_n.$$

For any $R \in \mathrm{CRing}^{f,p\text{-comp}}_{\mathcal{O}/}$, an n**-truncated** (G,μ)**-aperture** over R is an object of the ∞-groupoid $\mathrm{BT}^{G,\mu}_n(R)$. A (G,μ)**-aperture** over R is an object of $\mathrm{BT}^{G,\mu}_\infty(R)$.

Remark 9.1.2. For $R \in \mathrm{CRing}^{f,p\text{-nilp}}_{\mathcal{O}/}$, $\mathrm{BT}^{G,\mu}_n(R)$ can be described as the ∞-groupoid of G-torsors $\mathcal{Q}$ over $R^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ such that it is equipped with an equivalence $j^*_{dR}\mathcal{Q} \xrightarrow{\simeq} j^*_{HT}\mathcal{Q}$ of G-torsors over $R^{\Delta} \otimes \mathbb{Z}/p^n\mathbb{Z}$, satisfying the following condition: If $\mathcal{Q}^\mu$ is the associated $G\{\mu\}$-torsor, then, for every geometric point $R \to \kappa$ of $\mathrm{Spf}\,R$, the restriction of $(x^{\mathcal{N}}_{dR})^*\mathcal{Q}^\mu$ over $B\mathbb{G}_m \times \mathrm{Spec}\,\kappa$ is trivial.

Remark 9.1.3. Suppose that $\underline{A}$ is a prismatic frame equipped with a syntomic structure. Then pullback along the map $\iota_{\underline{A}}$ from Proposition 6.10.5—or better along the map $\iota_{\mathrm{id}:\underline{A}\to\underline{A}}$ from Corollary 6.10.6—gives us a canonical arrow

$$\mathrm{BT}^{G,\mu}_n(R_A) \to \mathsf{Wind}^{G,\mu}_{n,\underline{A}}(R_A).$$

More generally, if $\underline{B} \to \underline{A} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is any map of frames, we obtain a functor from $\mathrm{BT}^{G,\mu}_n(R_A)$ to the ∞-groupoid of windows over $\underline{B}$. In particular, when $\underline{A} = \underline{W(R)}$ is the Witt vector frame associated with $R \in \mathrm{CRing}_{\mathbb{F}_p/}$, and $\underline{B} = \underline{W_n(R)}$ is the n-truncated Witt frame, we obtain a canonical map $\mathrm{BT}^{G,\mu}_n(R) \to \mathsf{Disp}^{G,\mu}_n(R)$, underlying a map of formal prestacks over k.

Remark 9.1.4. When μ is defined over $\mathbb{Z}_p$, Drinfeld gives a different definition of $\mathrm{BT}^{G,\mu}_n$ in [21], which isn't quite the same as the one we give here, though it is *isomorphic* to ours.

To begin, he views μ as a cocharacter of the *automorphism* group $\mathrm{Aut}(G)$ acting on G. Since μ is defined over $\mathbb{Z}_p$, $G\{\mu\}$ lives over $B\mathbb{G}_m$ and hence it makes sense to talk about $G\{\mu\}$-torsors over R^{syn} for any $R \in \mathrm{CRing}^{p\text{-nilp}}$. Drinfeld's definition of $\mathrm{BT}^{G,\mu}_n(R)$ is as the ∞-groupoid of $G\{\mu\}$-torsors whose restriction to $B\mathbb{G}_m \times \mathrm{Spec}\,\kappa$ is trivial for any map $R \to \kappa$ to an algebraically closed field κ.

We can include this in our framework, though the definition of the 1-bounded stack $\mathcal{B}(G,\mu)$ will then have to be adjusted slightly: We will take it to be given by the pair $(BG\{\mu\}, BM_\mu)$ living over the tautologically pointed graded stack $B\mathbb{G}_{m,\mathbb{Z}_p}$, where BM_μ is still the trivial 1-bounded locus described in Lemma 4.9.6. If we define $\mathrm{BT}^{G,\mu}_n$ in the same way with this adjusted definition, we now recover Drinfeld's definition and Theorem 9.3.2 and its proof below hold verbatim. In particular, we obtain a proof of [24, Conjecture C.3.1].

If we have a cocharacter μ of G, then the stacks obtained from our definition and from Drinfeld's, viewing μ as a cocharacter of $\mathrm{Aut}(G)$ instead, are canonically isomorphic, since we have an equivalence between $G\{\mu\}$-torsors and G-torsors preserving the corresponding 1-bounded loci.

However, there is the following phenomenon: When, for instance, G is a torus, since a cocharacter of G will always act trivially, Drinfeld's definition would yield a stack that is *independent* of the cocharacter μ, since $G\{\mu\}$ would always be $G \times B\mathbb{G}_m$. But, in p-adic Hodge theory, it is important to keep track of the cocharacter, since it knows about the Hodge-Tate weights of étale or Galois realizations. For this reason, we have chosen the definition and presentation used here.

More importantly, as pointed out to us by the authors of [36], Drinfeld's description does not generalize directly when μ is only defined over some (unramified) ring of integers of $\mathbb{Z}_p$. See §10.4 below for more discussion of the case of tori, and note the role played by the cocharacter in Proposition 10.4.3.

9.2. **The semiperfectoid case.**

Lemma 9.2.1. *Suppose that R is semiperfectoid and that $R \to S$ is étale. Then S is also semiperfectoid. Moreover, the canonical map of frames $\underline{\Delta}_R \to \underline{\Delta}_S$ is (p, I_R)-completely étale, and we have*

$$\Delta_S \otimes_{\Delta_R} \mathrm{Fil}^\bullet_{\mathcal{N}} \Delta_R \xrightarrow{\simeq} \mathrm{Fil}^\bullet_{\mathcal{N}} \Delta_S.$$

Proof. Choose a perfectoid ring R_0 and a surjection $R_0 \twoheadrightarrow R$, lifting to a map $A_{\mathrm{inf}}(R_0) = \Delta_{R_0} \to \Delta_R$.

By [62, Tag 04D1], and by the equivalence of the étale sites of R and $\pi_0(R)$, there exists a p-completely étale map $R_0 \to R_0'$ such that $S = R_0' \otimes_{R_0} R$. We therefore reduce the first statement to showing that $\Delta_{R_0} \to \Delta_{R_0'}$ is (p, I_{R_0})-completely étale. This follows from the fact that R_0' is also perfectoid,[21] and the associated map of tilts $R_0^\flat \to R_0^{',\flat}$ is also étale, which implies that the map

$$\Delta_{R_0} = W(R_0^\flat) \to W(R_0^{',\flat}) = \Delta_{R_0'}$$

is (p, I_{R_0})-completely étale.

For the second assertion, first note that by (the proof of) assertion (1) of Proposition 6.12.3, $S^{\mathcal{N}} \to R^{\mathcal{N}}$ is (p, I_R)-completely étale. The proof is now completed by combining Proposition 5.4.19 and Theorem 6.11.5. ☐

The next two results tell us that $\mathrm{BT}_n^{G,\mu}$ can be understood in terms of the (G, μ)-windows from §5.5

Proposition 9.2.2 (Quasisyntomic descent)**.** *If $R \to R_\infty$ is as in Corollary 6.12.5, then we have:*

$$\mathrm{BT}_n^{G,\mu}(R) \xrightarrow{\simeq} \mathrm{Tot}\left(\mathrm{BT}_n^{G,\mu}(R_\infty^{\otimes_R \bullet+1})\right)$$

Proof. This is essentially immediate from the definitions and Proposition 6.12.3. ☐

Lemma 9.2.3. *If R is semiperfectoid, as sheaves on the small étale site $R_{\text{ét}}$ of $\mathrm{Spf}(R)$, we have a canonical equivalence*

$$\mathrm{BT}_n^{G,\mu}|_{R_{\text{ét}}} \xrightarrow{\simeq} \mathsf{Wind}^{G,\mu}_{n,\underline{\Delta}_R}$$

Proof. This is immediate from Lemma 9.2.1, Theorem 6.11.5 and Remark 9.1.3. ☐

Remark 9.2.4. Combining Proposition 9.2.2 and Lemma 9.2.3 with Proposition 5.5.2, we see that $\mathrm{BT}_n^{G,\mu}(R)$ can also be described as the ∞-groupoid of G-torsors $\mathcal{Q}$ over $R^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ equipped with an equivalence $j_{dR}^*\mathcal{Q} \xrightarrow{\simeq} j_{HT}^*\mathcal{Q}$ of G-torsors over $R^{\Delta} \otimes \mathbb{Z}/p^n\mathbb{Z}$, and satisfying the following equivalent conditions on the associated $G\{\mu\}$-torsor $\mathcal{Q}^\mu$:

(1) For every geometric point $R \to \kappa$ of $\mathrm{Spf}\, R$, the restriction of $(x_{\mathrm{dR}}^{\mathcal{N}})^*\mathcal{Q}^\mu$ over $B\mathbb{G}_m \times \mathrm{Spec}\,\kappa$ is trivial;
(2) The restriction of $\mathcal{Q}^\mu$ over $R^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is trivial locally in the p-quasisyntomic topology on $\mathrm{Spf}\, R$;
(3) The restriction of $\mathcal{Q}^\mu$ over $R^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is trivial flat locally on $\mathrm{Spf}\, R$.

If $\mathrm{Spf}\, R$ is connected, then these conditions are also equivalent to: For *some* geometric point $R \to \kappa$, the restriction of $(x_{\mathrm{dR}}^{\mathcal{N}})^*\mathcal{Q}^\mu$ over $B\mathbb{G}_m \times \mathrm{Spec}\,\kappa$ is trivial.

9.3. **Representability.** If we view $\mathcal{X} = \mathcal{B}(G, \mu) \otimes \mathbb{Z}/p^n\mathbb{Z}$ as a 1-bounded stack over $\mathcal{O}^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$, then the associated attractor $X^{-,(n)} \to \mathrm{Spec}\,\mathcal{O}$ is $BP_\mu^{-,(n)}$, and the stack $X^{(n)}$ is the Weil restricted classifying stack $BG_\mu^{(n)}$.

9.3.1. If $\mathcal{X}_1 = \mathcal{B}(G, \mu) \otimes \mathbb{F}_p$, then $\Gamma_{F\mathrm{Zip}}(\mathcal{X}_1)$ is a quasicompact smooth 0-dimensional Artin stack over k with affine diagonal. Indeed, it is not difficult to see using Remark 4.9.7 and (8.7.1.2) that this is just the stack of **F-zips with G-structure of type** μ; that is, of tuples $\mathsf{F} = (\mathcal{F}, \mathcal{F}^+, \mathcal{F}^-, \eta^+, \eta^-, \alpha)$ where:

- $\mathcal{F}$ is a G-torsor over R;
- $\mathcal{F}^+$ is a $P^+_{\mu^\varphi}$-torsor over R;
- $\mathcal{F}^-$ is a P^-_μ-torsor over R;
- $\eta^+ : \mathcal{F}^+ \to \mathcal{F}$ is a $P^+_{\mu^\varphi}$-equivariant map;
- $\eta^- : \mathcal{F}^- \to \mathcal{F}$ is a P^-_μ-equivariant map;
- $\alpha : \mathcal{F}^+/U^+_{\mu^\varphi} \xrightarrow{\simeq} \varphi^*\left(\mathcal{F}^-/U^-_\mu\right)$ is an isomorphism of M_{μ^φ}-torsors.

[21] This is a *much* easier assertion than the almost purity theorem for perfectoid algebras over fields!

The conclusion now follows from [60, §3.3]; see also the discussion in [21, §3.2], where it is denoted Disp_1^G. In the notation of this paper, we actually see that we have $\Gamma_{F\mathrm{Zip}}(\mathcal{X}_1) \simeq \mathsf{Disp}_1^{G,\mu}$.

The next result proves Theorems D and G

Theorem 9.3.2. $\mathrm{BT}_n^{G,\mu}$ *is a quasicompact smooth* 0*-dimensional* p*-adic formal Artin stack over* $\mathcal{O}$ *with affine diagonal. For every nilpotent divided power thickening* $(C' \twoheadrightarrow C, \gamma)$ *of* p*-complete* $\mathcal{O}$*-algebras, we have a Cartesian square*

$$\begin{tikzcd} \mathrm{BT}_n^{G,\mu}(C') \arrow[r] \arrow[d] & BP_\mu^{-,(n)}(C') \arrow[d] \\ \mathrm{BT}_n^{G,\mu}(C) \arrow[r] & BP_\mu^{-,(n)}(C) \times_{BG^{(n)}(C)} BG^{(n)}(C'). \end{tikzcd}$$

Moreover, the transition maps $\mathrm{BT}_{n+1}^{G,\mu} \to \mathrm{BT}_n^{G,\mu}$ *are smooth and surjective.*

Proof. Theorem 8.7.5 and the discussion above show that $\mathrm{BT}_n^{G,\mu}$ is a quasi-compact finitely presented p-adic formal derived Artin stack over $\mathcal{O}$ with affine diagonal.

The existence of the stated Cartesian square for nilpotent divided power thickenings follows from Corollary 8.10.2. Since $BP_\mu^{-,(n)} \to BG^{(n)}$ is smooth, this also shows that $\mathrm{BT}_n^{G,\mu}$ is a smooth p-adic formal Artin stack over $\mathcal{O}$.

The smooth surjectivity of the transition maps can be checked mod-p and here it is a special case of Corollary 8.8.3. Note that we can actually say a bit more: $\mathrm{BT}_1^{G,\mu} \otimes \mathbb{F}_p$ is a torsor over $\mathsf{Disp}_1^{G,\mu}$ under a group stack $\mathsf{S}_0(\mathcal{X}_1)$ by (2) of Theorem 8.7.5. In fact, from Corollary 7.2.3, one sees that $\mathsf{S}_0(\mathcal{X}_1)$ is the classifying stack of a certain finite flat group scheme of height one. This is nothing but the *Lau group scheme* from [24]. In particular, one sees that $\mathrm{BT}_1^{G,\mu} \otimes \mathbb{F}_p$ is a gerbe over $\mathsf{Disp}_1^{G,\mu}$ banded by the Lau group scheme: this recovers the main result of *loc. cit.* for smooth inputs.

Over $\mathrm{BT}_1^{G,\mu} \otimes \mathbb{F}_p$, we now have a canonical F-gauge $\mathcal{M}_1(\mathfrak{g})$ obtained by twisting the adjoint representation of G by the tautological G-torsor over $(\mathrm{BT}_1^{G,\mu} \otimes \mathbb{F}_p)^{\mathrm{syn}}$. Then we find that $\mathrm{BT}_n^{G,\mu} \otimes \mathbb{F}_p \to \mathrm{BT}_{n-1}^{G,\mu} \otimes \mathbb{F}_p$ is a torsor under $\Gamma(\mathcal{M}_1(\mathfrak{g}[1]))$, which, by assertion (2) of Corollary 8.7.6 is a smooth, surjective Artin 1-stack over $\mathrm{BT}_1^{G,\mu} \otimes \mathbb{F}_p$. $\square$

Remark 9.3.3. Remark 4.10.5 shows that the stacks $\mathrm{BT}_n^{G,\mu}$ depend only on the conjugacy class of μ.

Remark 9.3.4. The smooth map $\mathrm{BT}_1^{G,\mu} \otimes \mathbb{F}_p \to \mathsf{Disp}_1^{G,\mu}$ appearing in the proof above is nothing but the canonical one from Remark 9.1.3. One can also consider the pullback maps $\mathrm{BT}_n^{G,\mu} \otimes \mathbb{F}_p \to \mathsf{Disp}_n^{G,\mu}$ for all $n \geq 1$ associating an n-truncated (G,μ)-display to every n-truncated (G,μ)-aperture in characteristic p. Following Lau [44] and [22], one would expect that this map is a gerbe for a finite flat group scheme killed by the n-th power of Frobenius.

Remark 9.3.5. If G is a connected group scheme, $\mathrm{BT}_n^{G,\mu}$ is a connected p-adic formal Artin stack over $\mathcal{O}$. To begin, $\mathsf{Disp}_1^{G,\mu}$ is connected by the quotient presentation from [60, Proposition 3.11]. Since $\mathrm{BT}_1^{G,\mu} \otimes \mathbb{F}_p$ is a gerbe over $\mathsf{Disp}_1^{G,\mu}$ banded by a finite flat group scheme, we find that it is also connected.

The description in the proof of Theorem 9.3.2 now shows that it is enough to know the following: If $\mathcal{F}$ is a vector bundle F-gauge over R of level 1 and Hodge-Tate weights $\{0,1\}$, then $\Gamma_{\mathrm{syn}}(\mathcal{F}[1])$ is a connected algebraic stack over R. Once again, it is a gerbe banded by a finite flat group scheme over $\Gamma_{F\mathrm{Zip}}(\mathsf{F}[1])$, where F is the associated vector bundle F-zip, and the desired connectedness can be checked directly for the latter stack.

In fact, the argument shows that in general we have a bijection

$$\pi_0(\mathrm{BT}_n^{G,\mu} \otimes \mathbb{F}_p) \xrightarrow{\simeq} \pi_0(\mathsf{Disp}_1^{G,\mu}),$$

where the target is a certain quotient of $\pi_0(G \otimes \mathbb{F}_p)$.

9.4. **The case of trivial** μ. When $\mu = 0$ is the *trivial* cocharacter $z \mapsto 1$, then we have $P_\mu^- = G$, and Theorem 9.3.2 tells us that $\mathrm{BT}_n^{G,0}$ is an étale p-adic formal stack over $\mathbb{Z}_p$ with affine étale diagonal.

Proposition 9.4.1. *Suppose that G is connected. Then there is a canonical equivalence*

$$\mathrm{BT}_n^{G,0} \xrightarrow{\simeq} B\underline{G(\mathbb{Z}/p^n\mathbb{Z})},$$

where the right hand side is the classifying stack of the locally constant group scheme $\underline{G(\mathbb{Z}/p^n\mathbb{Z})}$.

Proof. Suppose that we have $\mathcal{Q} \in \mathrm{BT}_n^{G,0}(R)$. Consider the assignment $\Gamma_{\mathrm{syn}}(\mathcal{Q})$ on $\mathrm{CRing}_{R/}^{p\text{-nilp}}$ given by:

$$C \mapsto \mathrm{Map}_{BG(C^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z})}(G|_{C^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}, \mathcal{Q}).$$

This is represented over R by

$$\operatorname{Spec} R \times_{G,\mathrm{BT}_n^{G,0},\mathcal{Q}} \operatorname{Spec} R,$$

which is the pullback along $(G, \mathcal{Q})$ of the diagonal of $\mathrm{BT}_n^{G,0}$ and is hence affine and étale over R.

We claim that this is an étale $G(\mathbb{Z}/p^n\mathbb{Z})$-torsor over R, and hence gives a canonical map

$$\mathrm{BT}_n^{G,0} \to B\underline{G(\mathbb{Z}/p^n\mathbb{Z})} \tag{9.4.1.1}$$

To see this, it suffices to know that it is faithfully flat and that the natural map $\underline{G(\mathbb{Z}/p^n\mathbb{Z})} \to \Gamma_{\mathrm{syn}}(G)$ of étale schemes is an equivalence.

Both assertions can be checked over algebraically closed fields κ over R. Here, using Remark 5.5.5 and Lemma 9.2.3, we see that we have

$$\mathrm{BT}_n^{G,0}(\kappa) \simeq [G(\mathbb{\Delta}_\kappa/p^n\mathbb{\Delta}_\kappa)\ {}_{\sigma}/\!\!/_{\mathrm{id}}\, G(\mathbb{\Delta}_\kappa/p^n\mathbb{\Delta}_\kappa)],$$

where $\sigma : G(\mathbb{\Delta}_\kappa/p^n\mathbb{\Delta}_\kappa) \to G(\mathbb{\Delta}_\kappa/p^n\mathbb{\Delta}_\kappa)$ is pullback under the Frobenius lift. Using the connectedness of G and Lang's theorem, one sees that the natural map of groupoids

$$[*/G(\mathbb{Z}/p^n\mathbb{Z})] \to [G(\mathbb{\Delta}_\kappa/p^n\mathbb{\Delta}_\kappa)\ {}_{\sigma}/\!\!/_{\mathrm{id}}\, G(\mathbb{\Delta}_\kappa/p^n\mathbb{\Delta}_\kappa)]$$

is an equivalence. This proves both assertions, and also shows that the canonical map (9.4.1.1) is an equivalence. ☐

10. Explicit descriptions of $\mathrm{BT}_n^{G,\mu}(R)$

In this section, we will see, following the deformation theoretic method of Ito [37], that the above theory yields explicit descriptions for $\mathrm{BT}_n^{G,\mu}(R)$ in certain cases as the groupoid of n-truncated (G,μ)-windows over some quite concrete frames. All objects in this subsection will be discrete, unless otherwise noted, so we are back on firm classical ground.

We will also find that the deformation rings constructed by Faltings in [27], and which play a key role in the construction of integral canonical models in [40] admit a clean interpretation as universal deformation rings for $\mathrm{BT}_\infty^{G,\mu}$.

10.1. **An explicit description over some classical rings.**

10.1.1. We will put ourselves in the following situation (compare with [43, §6]):

- $(S, I' = (E))$ will be an oriented prism, flat over $\mathbb{Z}_p$, with associated Frobenius lift $\varphi : S \to S$;
- $J \subset S$ will be a finitely generated ideal such that $\varphi(J) \subset J^2$;
- We will assume that that S is J-adically complete, and that E and p map to non-zero divisors in S/J.

For $m \geq 1$, set $S_m = S/J^m$. Then φ descends to an endomorphism of S_m.

We will associate with this data the following prismatic frames that are special cases of Example 5.4.12: We set $I = \varphi(I') \subset S$. For each $m \geq 1$, we define $\underline{S}_m$ to be the frame with underlying non-negatively filtered ring $\mathrm{Fil}^\bullet_{I'} S_m$ with filtered Frobenius $\mathrm{Fil}^\bullet_{I'} S_m \to \mathrm{Fil}^\bullet_I S_m$. Set $R_m \overset{\mathrm{defn}}{=} S/(J^m + I) = S_m/\mathrm{Fil}^1_{I'} S_m$ and $R = S/I$, so that we have $R = \varprojlim_m R_m$.

Repeating the above construction with S_m replaced with S gives a frame $\underline{S}$ with $R_S = R$. We then have maps of frames $\underline{S}_{m+1} \to \underline{S}_m$ for each $m \geq 1$, and also

$$\underline{S} \xrightarrow{\simeq} \varprojlim_m \underline{S}_m.$$

Proposition 10.1.2. *There is a canonical map*

$$\mathrm{BT}_n^{G,\mu}(R_m) \to \mathsf{Wind}^{G,\mu}_{\underline{S}_m,n}(R_m),$$

and the square

$$\begin{tikzcd} \mathrm{BT}_n^{G,\mu}(R_{m+1}) \arrow[r] \arrow[d] & \mathsf{Wind}^{G,\mu}_{\underline{S}_{m+1},n}(R_{m+1}) \arrow[d] \\ \mathrm{BT}_n^{G,\mu}(R_m) \arrow[r] & \mathsf{Wind}^{G,\mu}_{\underline{S}_m,n}(R_m) \end{tikzcd}$$

is Cartesian. In particular, we have

$$\mathrm{BT}_n^{G,\mu}(R) \simeq \mathrm{BT}_n^{G,\mu}(R_1) \times_{\mathsf{Wind}^{G,\mu}_{\underline{S}_1,n}(S_1)} \mathsf{Wind}^{G,\mu}_{\underline{S},n}(R)$$

Proof. We begin by explicating the categories of windows. By Remarks 4.9.7, Proposition 4.12.3 and Proposition 5.5.2, giving a $G^{(n)}\{\mu\}$-torsor $\mathcal{Q}_m$ over $\mathcal{R}(\mathrm{Fil}^\bullet_E S_m)$ that is trivial étale locally on $\mathrm{Spf}\, R_m$ is equivalent to giving an étale G-torsor $\mathcal{P}_m$ over S_m/p^nS_m along with a reduction of structure group to P^-_μ of the induced G-torsor over $R_m/^{\mathbb{L}}p^n$ (this is a derived quotient). The map τ^* forgets the reduction of structure group, while the map σ^* will be described in the next couple of paragraphs.

Let $L^+_{\underline{S}_m}G^{(n)}$ and $L^+_{\underline{S}_m}G^{(n)}\{\mu\}$ be the étale sheaves of groups from Remark 5.5.5: Unwinding the definitions, and using Proposition 4.12.3 one find that, given an étale R_m-algebra R'_m, we have $L^+_{\underline{S}_m}G^{(n)}(R'_m) = G(S'_m/p^nS'_m)$, while we have

$$\sigma^*(L^+_{\underline{S}_m}G^{(n)}\{\mu\}(R'_m)) = P^-_\mu(S'_m/p^mS'_m) \times U^+_\mu(\mathrm{Fil}^1_I S'_m/p^m\,\mathrm{Fil}^1_I S'_m) \subset G(S'_m/p^mS'_m).$$

Here, we S'_m is the unique étale S_m-algebra lifting R'_m, and we are viewing the left hand side as a subset of the right via the multiplication map; see also [42, Remark 6.3.3]. The map σ^* corresponds to a homomorphism

$$\sigma^*_m : L^+_{\underline{S}_m}G^{(n)}\{\mu\} \to L^+_{\underline{S}_m}G^{(n)},$$

which, on the product decomposition of the left hand side, has the following description: Its restriction to $P^-_\mu(S'_m/p^nS'_m)$ is given by

$$\varphi^* \circ \mathrm{int}(\mu(p)^{-1}) : P^-_\mu(S'_m/p^nS'_m) \to (\varphi^*P^-_\mu)(S'_m/p^nS'_m) \subset G(S'_m/p^nS'_m),$$

while its restriction to $U^+_\mu(\mathrm{Fil}^1_I S'_m/p^m\,\mathrm{Fil}^1_I S'_m)$ is given by:

$$\begin{aligned} U^+_\mu(\mathrm{Fil}^1_I S'_m/p^n\,\mathrm{Fil}^1_I S'_m) &\xleftarrow[\simeq]{\exp} \mathfrak{g}^+_\mu \otimes_{\mathcal{O}} (\mathrm{Fil}^1_I S'_m/p^n\,\mathrm{Fil}^1_I S'_m) \\ &\xrightarrow{\mathrm{id}\otimes\varphi(E)^{-1}\circ\varphi} \varphi^*\mathfrak{g}^+_\mu \otimes_{\mathcal{O}} S_m/p^nS_m \\ &\xrightarrow[\simeq]{\exp} (\varphi^*U^+_\mu)(S_m/p^nS_m) \subset G(S_m/p^nS_m). \end{aligned}$$

Now, $\mathcal{Q}_m$ corresponds to an $L^+_{\underline{S}_m}G^{(n)}\{\mu\}$-torsor over R_m, and $\sigma^*\mathcal{Q}_m$ to the $L^+_{\underline{S}}G^{(n)}$-torsor obtained via pushforward along σ^*_m as described above.

Suppose that we have an object $\mathsf{Wind}^{G,\mu}_{\underline{S}_{m+1},n}(R_{m+1})$ corresponding to a $G^{(n)}\{\mu\}$-torsor $\mathcal{Q}_{m+1}$-torsor over $\mathcal{R}(\mathrm{Fil}^\bullet_E S_{m+1})$ equipped with an isomorphism $\xi_{m+1} : \sigma^*\mathcal{Q}_{m+1} \xrightarrow{\simeq} \tau^*\mathcal{Q}_{m+1}$. Let $(\mathcal{Q}_m, \xi_m)$ be the corresponding display over $\underline{S}_m$.

The hypothesis that $\varphi(J) \subset J^2$ implies that, for each m, the Frobenius lift $S_{m+1} \to S_{m+1}$ factors canonically through a map $S_m \to S_{m+1}$, and also that the map σ^*_{m+1} factors through a map

$$\overline{\sigma}^*_{m+1} : L^+_{\underline{S}_m}G^{(n)}\{\mu\} \to L^+_{\underline{S}_{m+1}}G^{(n)}.$$

Moreover, there exists a canonical identification

$$\sigma^*\mathcal{Q}_{m+1} \xrightarrow{\simeq} \overline{\sigma}^*_m\mathcal{Q}_m.$$

This shows that giving the lift $(\mathcal{Q}_{m+1}, \xi_{m+1})$ of $(\mathcal{Q}_m, \xi_m)$ is equivalent to specifying the lift $\mathcal{Q}_{m+1}$, along with an isomorphism $\tau^*\mathcal{Q}_{m+1} \xrightarrow{\simeq} \overline{\sigma}_m^*\mathcal{Q}_m$. We can further formulate this as follows: Let $\overline{P}_{m+1}$ be the G-torsor over $R_m/^{\mathbb{L}}p^n$ obtained from $\varphi_m^*\tau^*\mathcal{Q}_m$. Then giving the lift $(\mathcal{Q}_{m+1}, \xi_{m+1})$ is equivalent to specifying a reduction of structure group for $\overline{P}_{m+1}$ to a P_μ^--torsor. More precisely, we have a Cartesian diagram

$$\begin{tikzcd} \mathsf{Wind}^{G,\mu}_{\underline{S}_{m+1},n}(R_{m+1}) \arrow[r] \arrow[d] & BP_\mu^{-,(n)}(R_m) \arrow[d] \\ \mathsf{Wind}^{G,\mu}_{\underline{S}_m,n}(R_m) \arrow[r] & BP^{-,(n)}(R_m) \times_{BG^{(n)}(R_m)} BG_\mu^{(n)}(R_{m+1}). \end{tikzcd} \tag{10.1.2.1}$$

To finish the proof, it now suffices to observe that we have a canonical map $\mathrm{BT}_n^{G,\mu}(R_m) \to \mathsf{Wind}^{G,\mu}_{\underline{S}_m,n}(R_m)$ obtained from Example 6.10.4 and Remark 9.1.3 via pullback along the map $\iota_{\underline{S}_m}$. Moreover, one checks that the restriction of $\iota_{\underline{S}_m}$ to $\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spf}\, R_m$ agrees with the map $x^{\mathcal{N}}_{\mathrm{dR}}$, and also that the lift

$$\mathrm{Spf}\, R_{m+1} \to \mathrm{Spf}\, S_{m+1} \xrightarrow{\mathrm{Spf}(\varphi_m)} \mathrm{Spf}\, S_m \xrightarrow{\iota_{(S_m,(E))}} R_m^{\mathbb{\Delta}}$$

is that of Lemma 6.8.1 if we equip $R_{m+1} \twoheadrightarrow R_m$ with trivial divided powers.

The last part follows by taking the limit over m on the equivalence

$$\mathrm{BT}_n^{G,\mu}(R_m) \xrightarrow{\simeq} \mathrm{BT}_n^{G,\mu}(R_1) \times_{\mathsf{Wind}^{G,\mu}_{\underline{S}_1,n}(S_1)} \mathsf{Wind}^{G,\mu}_{\underline{S}_m,n}(R_m).$$

That this limit yields the desired equivalence follows from [7, Corollary 1.5]. □

Example 10.1.3. Let κ be a perfect field in characteristic p, with associated ring of Witt vectors $W(\kappa)$. We set $S = W(\kappa)[|t_1, \ldots, t_r|]$ for some $n \geq 0$, and take φ to be the Frobenius lift with $\varphi(t_i) = t_i^p$. Take $J = (t_1, \ldots, t_n)$, so that we have $\varphi(J) \subset J^p$. Suppose that E satisfies $\varphi(E) \equiv E^p \pmod p$, and is such that $S/(E)$ is p-torsion free.

Here, $R_m = S/((E) + J^m)$ with $R_1 = S/((E) + J) = W(\kappa)/(p) = \kappa$, and $\underline{S}_1$ is isomorphic to the frame $\underline{\mathbb{\Delta}}_\kappa$. In particular, the map

$$\mathrm{BT}_n^{G,\mu}(\kappa) = \mathrm{BT}_n^{G,\mu}(R_1) \to \mathsf{Wind}^{G,\mu}_{\underline{S}_1,n}(\kappa)$$

is an equivalence, and so we conclude that we have

$$\mathrm{BT}_n^{G,\mu}(R) \xrightarrow{\simeq} \mathsf{Wind}^{G,\mu}_{\underline{S},n}(R).$$

Note that the argument actually shows that we have

$$\mathrm{BT}_n^{G,\mu}(R_m) \xrightarrow{\simeq} \mathsf{Wind}^{G,\mu}_{\underline{S}_m,n}(R_m)$$

for all $m \geq 1$.

Using Remark 5.5.9, we can give a quite explicit description of $\mathsf{Wind}^{G,\mu}_{\underline{S}_m,\infty}(R_m)$: If H_μ is the dilatation of G_S along $P_\mu^- \otimes R$, viewed as an étale sheaf over S, along with the natural map $\tau : H_\mu \to G_S$ as well as the map $\sigma = \varphi \circ \mathrm{int}(E)$, then an object in $\mathsf{Wind}^{G,\mu}_{\underline{S}_m,\infty}(R_m)$ is an H_μ-torsor over S_m along with an isomorphism of G-torsors $\sigma^*\mathcal{P} \xrightarrow{\simeq} \tau^*\mathcal{P}$.

The case where $n = 1$ and E is an Eisenstein polynomial is the context for the classical story of Breuil-Kisin modules. In this case, $R = S/(E)$ is a totally ramified ring of integers over $W(\kappa)$, and $R_m = R/(\pi^m)$ where $\pi \in R$ is a uniformizer with minimal polynomial E.

Example 10.1.4. We have a non-Noetherian analogue of the previous example by taking $\underline{S} = \underline{\mathbb{\Delta}}_R$ for a perfectoid ring R, $E = \xi$ a generator for $\ker(\theta : \mathbb{\Delta}_R \to R)$, and $J = ([\varpi_1], [\varpi_2], \ldots, [\varpi_m])$ to be an ideal generated by Teichmüller lifts of topologically nilpotent elements $\varpi_i \in R^\flat$ that form a regular sequence. Suppose in addition that $[\varpi_1], \ldots, [\varpi_m], \xi$ also forms a regular sequence in $W(R^\flat)$. Then we are in a special case of the situation aboe with $S_1 = \mathbb{\Delta}_R/J$, and $R_1 = R/\theta(J)$.

In this case, we already know that $\mathrm{BT}_n^{G,\mu}(R) \simeq \mathsf{Wind}_{\underline{S},n}^{G,\mu}$ by Lemma 9.2.3. But Proposition 10.1.2 tells us that we also have

$$\mathrm{BT}_n^{G,\mu}(R) \simeq \mathrm{BT}_n^{G,\mu}(R_1) \times_{\mathsf{Wind}_{n,\underline{S}_1}^{G,\mu}(R_1)} \mathsf{Wind}_{n,\underline{S}}^{G,\mu}(R).$$

Since $\mathrm{BT}_n^{G,\mu}(R) \to \mathrm{BT}_n^{G,\mu}(R_1)$ is an effective epimorphism, this tells us that we in fact have

$$\mathrm{BT}_n^{G,\mu}(R_1) \simeq \mathsf{Wind}_{n,\underline{S}_1}^{G,\mu}(R_1).$$

This gives a somewhat concrete description of the left hand side, and also recovers—via Theorem 11.1.4 below—a description of p-divisible groups over R_1 which was first observed by Ito [37, Theorem 6.3.6] in the following situation: $R = \mathscr{O}_C$ is the ring of integers in a perfectoid field C, $m = 1$ and $\varpi = \varpi_1 \in \mathscr{O}_{C^\flat}$ is a topologically nilpotent non-zero element.

Example 10.1.5. Let S and J be as in Example 10.1.3, but assume now that $E = p$, so that we have a *crystalline* prism $(S, (p))$. In this case, and we obtain an equivalence

$$\mathrm{BT}_n^{G,\mu}(\kappa[|t_1,\ldots,t_r|]) \xrightarrow{\simeq} \mathsf{Wind}_{\underline{S},n}^{G,\mu}(\kappa[|t_1,\ldots,t_r|]) \times_{\mathsf{Wind}_{\underline{S}_1,n}^{G,\mu}(\kappa)} \mathrm{BT}_n^{G,\mu}(\kappa).$$

But note that $\underline{S}_1$ is simply the frame $\underline{\mathbb{A}}_\kappa$, and so as in *loc. cit.* the map $\mathrm{BT}_n^{G,\mu}(\kappa) \to \mathsf{Wind}_{\underline{S}_1,n}^{G,\mu}(\kappa)$ is an equivalence. This gives us an equivalence:

$$\mathrm{BT}_n^{G,\mu}(\kappa[|t_1,\ldots,t_r|]) \xrightarrow{\simeq} \mathsf{Wind}_{\underline{S},n}^{G,\mu}(\kappa[|t_1,\ldots,t_r|]).$$

By Remark 5.5.9, we obtain a rather explicit description of the limiting groupoid $\mathsf{Wind}_{\underline{S},\infty}^{G,\mu}(\kappa[|t_1,\ldots,t_r|])$: Its objects are H_μ-torsors $\mathcal{P}$ over S equipped with an isomorphism $\sigma^*\mathcal{P} \xrightarrow{\simeq} \tau^*\mathcal{P}$ of G-torsors. Here, H_μ is the dilatation of $G_{\mathcal{O}}$ along $P_\mu^- \otimes k$ and $\sigma = \varphi \circ \mathrm{int}(\mu(p))$, while τ is the natural map as usual.

10.2. **Relationship with Faltings deformation rings.** Here we will find that the deformation rings of $\mathrm{BT}_\infty^{G,\mu}$ can be described explicitly using results from the beginning of this subsection, combined with a construction of Faltings [27, §7]. Compare with the main result of Ito in [37], where one finds a version of such a result. There, however, the Faltings deformation space is only shown to have good descriptions for particular inputs from $\mathrm{Art}_{W(\kappa)}$; but, in [36], the authors combine Ito's work with ours here to complete the proof in general.

10.2.1. Maintain the notation from the previous subsection, but assume now that $I = (p)$; for instance, this is the case in the situation of Example 10.1.5. We can then also define (non-prismatic) frames $\underline{\widetilde{S}}$ and $\underline{\widetilde{S}}_m$, where we take the underlying filtered commutative ring to be S (resp. S_m) with the *trivial* filtration. We then have maps of frames $\underline{\widetilde{S}}_{m+1} \to \underline{\widetilde{S}}_m$ for each $m \geq 1$, and also

$$\underline{\widetilde{S}} \xrightarrow{\simeq} \varprojlim_m \underline{\widetilde{S}}_m.$$

In this case, we have $\widetilde{S}_m/\operatorname{Fil}^1 \widetilde{S}_m = S_m$ and $\widetilde{S}/\operatorname{Fil}^1 \widetilde{S} = S$.

Let us now use Remark 5.5.10. It tells us that $\mathsf{Wind}_{\infty,\underline{\widetilde{S}}_m}^{G,\mu}(S_m)$ can be described quite explicitly: Giving an object here is equivalent to giving a P_μ^--torsor $\mathcal{P}'$ over S_m along with an isomorphism $\sigma^*\mathcal{P}' \xrightarrow{\simeq} \tau^*\mathcal{P}'$ of G-torsors. Here, $\tau : P_\mu^- \to G_{\mathcal{O}}$ is the natural map and $\sigma = \varphi \circ \mathrm{int}(\mu(p))$.

In other words, we have

$$\mathsf{Wind}_{\infty,\underline{\widetilde{S}}_m}^{G,\mu}(S_m) \xrightarrow{\simeq} \mathsf{Wind}_{\infty,\underline{S}_m}^{G,\mu}(R_m) \times_{BP_\mu^-(R_m)\times_{BG(R_m)}BG(S_m)} BP_\mu^-(S_m). \tag{10.2.1.1}$$

10.2.2. We now have a canonical map

$$\mathrm{BT}_\infty^{G,\mu}(S_m) \to \mathsf{Wind}_{\infty,\underline{S}_m}^{G,\mu}(R_m) \times_{BP_\mu^-(R_m)\times_{BG(R_m)}BG(S_m)} BP_\mu^-(S_m) \simeq \mathsf{Wind}_{\infty,\underline{\widetilde{S}}_m}^{G,\mu}(S_m)$$

where the first coordinate is obtained from the composition

$$\mathrm{BT}_\infty^{G,\mu}(S_m) \to \mathrm{BT}_\infty^{G,\mu}(R_m) \to \mathsf{Wind}_{\infty,\underline{S}_m}^{G,\mu}(R_m),$$

while the second is pullback along $x^{\mathcal{N}}_{\mathrm{dR},S_m}$. From Proposition 10.1.2 and Grothendieck-Messing theory for $\mathrm{BT}^{G,\mu}_\infty$, one finds that, for each $m\geq 1$, there is a Cartesian square:

(10.2.2.1)
$$\begin{tikzcd} \mathrm{BT}^{G,\mu}_\infty(S_{m+1}) \arrow[r] \arrow[d] & \mathsf{Wind}^{G,\mu}_{\underline{\widetilde{S}}_{m+1},\infty}(S_{m+1}) \arrow[d] \\ \mathrm{BT}^{G,\mu}_\infty(S_m) \arrow[r] & \mathsf{Wind}^{G,\mu}_{\underline{\widetilde{S}}_{m},\infty}(S_m) \end{tikzcd}$$

Remark 10.2.3. When $p>2$, one can use Grothendieck-Messing theory to show that in fact all the horizontal arrows in the above square are equivalences. If $p=2$, then this assertion fails already for $m=1$.

10.2.4. Let κ be a perfect field, and suppose that we have a point $x\in\mathrm{BT}^{G,\mu}_\infty(\kappa)$. We can then consider the deformation problem on the usual category $\mathrm{Art}_{W(\kappa)}$ of Artin local $W(\kappa)$-algebras with residue field κ:

$$\begin{aligned} \mathrm{Def}_x : \mathrm{Art}_{W(\kappa)} &\to \mathrm{Spc} \\ A &\mapsto \mathrm{fib}_x(\mathrm{BT}^{G,\mu}_\infty(A)\to\mathrm{BT}^{G,\mu}_\infty(\kappa)). \end{aligned}$$

Grothendieck-Messing theory now tells us that, if $A'\twoheadrightarrow A$ is a square-zero thickening in $\mathrm{Art}_{W(\kappa)}$, then we have a Cartesian square

$$\begin{tikzcd} \mathrm{BT}^{G,\mu}_\infty(A') \arrow[r] \arrow[d] & BP^-_\mu(A') \arrow[d] \\ \mathrm{BT}^{G,\mu}_\infty(A) \arrow[r] & BP^-_\mu(A)\times_{BG(A)} BG(A'). \end{tikzcd}$$

Using this, we find:

Lemma 10.2.5. *For each $A\in\mathrm{Art}_{W(\kappa)}$, $\mathrm{Def}_x(A)$ is equivalent to a set, and Def_x is prorepresented by $\mathrm{Spf}\,R_x$ with $R_x\simeq W(\kappa)[|t_1,\dots,t_d|]$ where $d=\dim G-\dim P^-_\mu$.*

10.2.6. We begin by reformulating Faltings's construction in the language of torsors. Choose a lift $x'\in\mathrm{BT}^{G,\mu}_\infty(W(\kappa))$, which in turn yields a display (or more precisely a compatible family of n-truncated displays) over the frame $\underline{\widetilde{S}}_1$ associated with the trivial filtration on $W(\kappa)$.

Explicitly, this means the following: Let H_μ be the dilatation of $G_{\mathcal{O}}$ along $P^-_\mu\otimes k$. Under the equivalence

$$\mathrm{BT}^{G,\mu}_\infty(\kappa)\xrightarrow{\simeq}\varprojlim_n \mathsf{Wind}^{G,\mu}_{n,{\mathbb{\Delta}}_\kappa}(\kappa),$$

and Remark 5.5.9, x corresponds to an H_μ-torsor $\mathcal{P}_x$ over $W(\kappa)$, equipped with an isomorphism $\sigma^*\mathcal{P}_x\xrightarrow{\simeq}\tau^*\mathcal{P}_x$ of G-torsors over $W(\kappa)$. The lift x' gives rise to an object of $\mathsf{Wind}^{G,\mu}_{\infty,\underline{\widetilde{S}}_1}(W(\kappa))$, which, by Remark 5.5.10, amounts to refining the H_μ-torsor $\mathcal{P}_x$ to a P^-_μ-torsor $\mathcal{P}_{x'}$ over $W(\kappa)$.

10.2.7. Set $G_x=\mathrm{Aut}(\tau^*\mathcal{P}_x)$ and $P^-_x=\mathrm{Aut}(\mathcal{P}_{x'})$, so that G_x is a pure inner form over $W(\kappa)$ of G, and $P^-_x\subset G_x$ is associated with a cocharacter $\mu_x:\mathbb{G}_m\to G_x$ that is conjugate to μ, via the process explained in §4.9.

For such a choice of cocharacter, we can look at the 'opposite' unipotent $U^+_x\subset G_x$: this is a commutative unipotent group scheme over $W(\kappa)$ (see Lemma 4.10.2). We now define R^{Fal}_x to be the complete local ring of U^+_x at the identity: this is abstractly isomorphic to $W(\kappa)[|t_1,\dots,t_d|]$ as a $W(\kappa)$-algebra.

We equip R^{Fal}_x with the Frobenius lift φ arising from the p-power map on U^+_x, and take $J_x\subset R^{\mathrm{Fal}}_x$ to be the augmentation ideal: note that we have $\varphi(J_x)\subset J^p_x$.

10.2.8. We can now apply the setup from the beginning of the subsection with $(S, I') = (R_x^{\mathrm{Fal}}, (p))$ and $J = J_x$, and we find that we have:

$$\mathrm{fib}_{x'}(\mathrm{BT}_\infty^{G,\mu}(R_x^{\mathrm{Fal}}) \to \mathrm{BT}_\infty^{G,\mu}(W(\kappa))) \xrightarrow{\simeq} \mathrm{fib}_{\mathcal{P}_{x'}}(\mathsf{Wind}^{G,\mu}_{\underline{\widetilde{S}},\infty}(R_x^{\mathrm{Fal}}) \to \mathsf{Wind}^{G,\mu}_{\underline{\widetilde{S}}_1,\infty}(W(\kappa))).$$

One way to get an object on the right is as follows: Let $j : W(\kappa) \to R_x^{\mathrm{Fal}}$ be the structure map: this actually underlies a map of frames $\underline{\widetilde{S}}_1 \to \underline{\widetilde{S}}$, and so we can pull $\mathcal{P}_{x'}$ back to get the 'constant' lift $\mathcal{P}_{x'}^{\mathrm{con}}$ over $\underline{\widetilde{S}}$. More precisely, this corresponds to the P_μ^--torsor $\mathcal{P}_{x'}^{\mathrm{con}}$ over R_x^{Fal}, along with an isomorphism of G-torsors $\xi_{x'}^{\mathrm{con}} : \sigma^*\mathcal{P}_{x'}^{\mathrm{con}} \xrightarrow{\simeq} \tau^*\mathcal{P}_{x'}^{\mathrm{con}}$. All of this data is obtained simply via pullback from the corresponding data over $W(\kappa)$.

In $U_x^+(R_x^{\mathrm{Fal}})$, we have the tautological element g_x. We now define a new display $\mathcal{P}_{x'}^{\mathrm{Fal}}$ by keeping the P_μ^--torsor $\mathcal{P}_{x'}^{\mathrm{con}}$, but replacing $\xi_{x'}^{\mathrm{con}}$ with the composition $\xi_{x'}^{\mathrm{Fal}} = g_x \circ \xi_{x'}^{\mathrm{con}}$.

As explained above, this yields an object $x^{\mathrm{Fal}} \in \mathrm{BT}_\infty^{G,\mu}(R_x^{\mathrm{Fal}})$ lifting $x' \in \mathrm{BT}_\infty^{G,\mu}(W(\kappa))$, and so corresponds to a unique map $R_x \to R_x^{\mathrm{Fal}}$.

Proposition 10.2.9. *The map $R_x \to R_x^{\mathrm{Fal}}$ is an isomorphism.*

Proof. Let $\widehat{U}_x$ (resp. $\widehat{U}_x^{\mathrm{Fal}}$) be the deformation functor on $\mathrm{Art}_{W(\kappa)}$ represented by R_x (resp. R_x^{Fal}). If $\kappa[\epsilon]$ is the ring of dual numbers, we obtain maps of tangent spaces

$$\widehat{U}_x^{\mathrm{Fal}}(\kappa[\epsilon]) \to \widehat{U}_x(\kappa[\epsilon]) \xrightarrow{\simeq} \mathrm{fib}_{(\mathcal{P}_{x'},\sigma^*\mathcal{P}_{x'})}(BP_\mu^-(\kappa[\epsilon]) \to BP_\mu^-(\kappa) \times_{BG(\kappa)} BG(\kappa[\epsilon])),$$

where the second arrow is the isomorphism from Grothendieck-Messing theory.

The source of this composition is simply $\epsilon\kappa[\epsilon] \otimes_{W(\kappa)} \mathrm{Lie}\, U_x^+$, and the proof of Proposition 10.1.2 shows that this composition takes a tangent vector ϵN to $\exp(-\epsilon N) \cdot \mathcal{P}_{x'}$. In particular, it is an isomorphism onto its image.

Since both complete local rings are normal of the same dimension, the proposition now follows from Nakayama's lemma. $\square$

10.3. **The case of central μ.** Suppose that μ is *central* in G. For instance, this is the case whenever G is a torus over $\mathbb{Z}_p$. In this case, we have $P_\mu^- = G$, and so Theorem 9.3.2 shows that $\mathrm{BT}_n^{G,\mu}$ is an étale p-adic formal stack over $\mathcal{O}$.

10.3.1. We will now show that $\mathrm{BT}_\infty^{G,\mu}(\mathcal{O})$ is non-empty. Objects here will be called **Lubin-Tate (G,μ)-apertures**. This terminology is partially justified by Proposition 11.7.4 below.

By Example 10.1.3, this is equivalent to writing down objects in $\mathsf{Wind}^{G,\mu}_{\underline{S},\infty}(\mathcal{O})$ where $S = W(k)[|u|]$ is equipped with the E-adic filtration $\mathrm{Fil}_E^\bullet S$ with $E = u - p$, and the Frobenius lift $\varphi : u \mapsto u^p$, which lifts to a filtered map

$$\mathrm{Fil}_E^\bullet S \to \mathrm{Fil}_{\varphi(E)}^\bullet S.$$

In the notation of that example, we have $H_\mu = G$, and so a choice of trivialization of the module $S\{1\}$ now further identifies this groupoid with the groupoid of G-torsors $\mathcal{Q}$ over S equipped with an isomorphism $\varphi^*\mathcal{Q} \xrightarrow{\simeq} \mathcal{Q}$. Clearly, the trivial G-torsor has such structure, and so gives us an object in $\mathrm{BT}_n^{G,\mu}(\mathcal{O})$; note that the object that it corresponds to is not canonical and depends on all the choices we made, including that of the frame $\underline{S}$ as well as the trivialization of the module $S\{1\}$.

Remark 10.3.2. Alternatively, we could have first constructed objects in $\mathrm{BT}_\infty^{G,\mu}(k)$ and then used the formal étaleness of $\mathrm{BT}_\infty^{G,\mu}$ to obtain Lubin-Tate (G,μ)-apertures.

Remark 10.3.3. It would be interesting to give a direct construction of these G-torsors over $\mathcal{O}^{\mathrm{syn}}$. When μ is defined over $\mathbb{Z}_p$, one can use the composition

$$\mathbb{Z}_p^{\mathrm{syn}} \to B\mathbb{G}_m \xrightarrow{B\mu} BG$$

where the first map classifies the Breuil-Kisin twist.

Note that this gives a *canonical* Lubin-Tate (G,μ)-aperture over $\mathbb{Z}_p$. This is related to the fact that over $\mathbb{Z}_p$ we have a canonical choice of a Lubin-Tate formal group given by μ_{p^∞}.

Proposition 10.3.4. *Suppose that G is connected. Then there is a* non*-canonical isomorphism*

$$\mathrm{BT}_n^{G,\mu} \xrightarrow{\simeq} B\underline{G(\mathbb{Z}/p^n\mathbb{Z})}$$

of p-adic formal stacks over $\mathrm{Spf}\,\mathcal{O}$*. More precisely,* $\mathrm{BT}_n^{G,\mu}$ *is a gerbe over* $\mathrm{Spf}\,\mathcal{O}$ *banded by* $G(\mathbb{Z}/p^n\mathbb{Z})$ *that is non-canonically trivial.*

Proof. The proof is the same as that of Proposition 10.3.4. Instead of the trivial G-torsor, one uses one of the Lubin-Tate (G,μ)-apertures constructed above. ☐

10.4. **The case of tori.** As mentioned above, one special case is when $G = T$ is a torus. Here is a reinterpretation of Proposition 10.3.4:

Proposition 10.4.1. $\mathrm{BT}_n^{T,\mu}$ *is a non-canonically trivial* $B\underline{T(\mathbb{Z}/p^n\mathbb{Z})}$*-torsor over* $\mathrm{Spf}\,\mathcal{O}$*.*

10.4.2. As is well-known, there is an initial instance of data (T,μ) with μ defined over $\mathcal{O}$. Take $T_0 = \mathrm{Res}_{\mathcal{O}/\mathbb{Z}_p}\mathbb{G}_m$ and $\mu_0 : \mathbb{G}_{m,\mathcal{O}} \to T_{0,\mathcal{O}}$ obtained as follows: We have $T_{0,\mathcal{O}} \simeq \prod_{i=0}^{h-1}\mathbb{G}_{m,\mathcal{O}}$, where $h = [\mathcal{O}[1/p] : \mathbb{Q}_p]$, and the isomorphism is obtained from the map of $\mathcal{O}$-algebras

$$\mathcal{O}\otimes_{\mathbb{Z}_p}\mathcal{O} \xrightarrow{\simeq} \prod_{i=0}^{h-1}\mathcal{O}$$
$$a\otimes b \mapsto (a\varphi^i(b))_{0\le i\le h-1}.$$

We now take μ_0 to be the inclusion in the first factor.

For any other $\mathbb{Z}_p$-torus T with cocharacter $\mu : \mathbb{G}_{m,\mathcal{O}} \to T_{\mathcal{O}}$, we now see that the composition

$$\mathbb{G}_{m,\mathcal{O}} \xrightarrow{\mu_0} T_0 = \mathrm{Res}_{\mathcal{O}/\mathbb{Z}_p}\mathbb{G}_m \xrightarrow{\mathrm{Res}_{\mathcal{O}/\mathbb{Z}_p}\mu} \mathrm{Res}_{\mathcal{O}/\mathbb{Z}_p}T_{\mathcal{O}} \xrightarrow{\mathrm{Nm}_{\mathcal{O}/\mathbb{Z}_p}} T$$

is equal to μ.

We can understand $\mathrm{BT}_\infty^{T_0,\mu_0}$ somewhat explicitly. The following will be used to reinterpret it in terms of Lubin-Tate $\mathcal{O}$-modules in Proposition 11.7.4.

Proposition 10.4.3. *Giving a (T_0,μ_0)-aperture over $R \in \mathrm{CRing}^{f,p\text{-}nilp}_{\mathcal{O}/}$ is equivalent to giving a line bundle $\mathcal{F}$ over $R^{\mathrm{syn}}\times\mathrm{Spf}\,\mathcal{O}$ with the following property: For any algebraically closed field κ over R, the restriction of $\mathcal{F}$ to*

$$B\mathbb{G}_m\times\mathrm{Spec}(\kappa\otimes_{\mathbb{Z}_p}\mathcal{O}) \simeq \prod_{i=0}^{h-1}B\mathbb{G}_m\times\mathrm{Spec}\,\kappa$$

corresponds to a graded projective module of rank 1 over $\prod_{i=0}^{h-1}\kappa$ that is in graded degree 1 for $i=0$ and in graded degree 0 for $i>0$.

Proof. This is simply a reinterpretation of the definition using the fact that we have $BT_0 \simeq \mathrm{Res}_{\mathcal{O}/\mathbb{G}_m}B\mathbb{G}_m$. Note that the action of $B\underline{T_0(\mathbb{Z}_p)}$ under this optic is just given by tensor product of line bundles, where we use Proposition 10.3.4 to view $B\underline{T_0(\mathbb{Z}_p)}$ as the stack of line bundles over $R^{\mathrm{syn}}\times\mathrm{Spf}\,\mathcal{O}$ whose restrictions to $\prod_{i=0}^{h-1}B\mathbb{G}_m\times\mathrm{Spec}\,\kappa$ have graded degree 0 in every coordinate. ☐

11. The classification of truncated Barsotti-Tate groups

11.1. The statement of the theorem.

11.1.1. Recall that an n**-truncated Barsotti-Tate group scheme** over a discrete ring $R \in \mathrm{CRing}^{p\text{-nilp}}_{\heartsuit}$ is a finite flat commutative group scheme G over R with the following properties:

(1) G is p^n-torsion;
(2) The sequence $G \xrightarrow{p^{n-1}} G \xrightarrow{p} G$ is exact in the middle;
(3) If $n=1$, over R/pR, we have $\ker F = \mathrm{im}\,V \subset G\otimes\mathbb{F}_p$, where $F : G\otimes\mathbb{F}_p \to (G\otimes\mathbb{F}_p)^{(p)}$ and $V : (G\otimes\mathbb{F}_p)^{(p)} \to G\otimes\mathbb{F}_p$ are the Frobenius and Verschiebung homomorphisms, respectively.

See for instance [35, §I].

These organize into a category $\mathcal{BT}_n(R)$, and we will write $\mathrm{BT}_n(R)$ for the underlying groupoid obtained by jettisoning the non-isomorphisms. For $1 \leq r \leq n$, sending G to $G[p^r]$ yields a functor $\mathcal{BT}_n(R) \to \mathcal{BT}_r(R)$.

An important role will be played by the following fundamental result of Grothendieck [35]:

Theorem 11.1.2. *The assignment $R \mapsto \mathrm{BT}_n(R)$ on $\mathrm{CRing}^{f,p\text{-}nilp}$ is represented by a finitely presented smooth 0-dimensional p-adic formal Artin stack with affine diagonal.*

11.1.3. There is a canonical involution

$$\mathcal{BT}_n(R) \xrightarrow{G \mapsto G^*} \mathcal{BT}_n(R)$$

induced by Cartier duality, with $G^* = \underline{\mathrm{Hom}}(G, \mu_{p^n})$.

Let $\mathrm{Vect}_{\{0,1\}}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})$ be the ∞-category of vector bundles on $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ with Hodge-Tate weights $\{0,1\}$. There is once again a canonical involution

$$\mathrm{Vect}_{\{0,1\}}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\mathcal{M} \mapsto \mathcal{M}^*} \mathrm{Vect}_{\{0,1\}}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}),$$

with $\mathcal{M}^* = \mathcal{M}^\vee\{1\}$ is the twist of the dual vector bundle by the Breuil-Kisin twist $\mathcal{O}_n^{\mathrm{syn}}\{1\}$. In analogy with the involution on $\mathcal{BT}_n(R)$, we will refer to $\mathcal{M}^*$ as the **Cartier dual** of $\mathcal{M}$.

We can now state the main result of this section.

Theorem 11.1.4. *Suppose that R belongs to CRing^f. Then there is a canonical equivalence of ∞-categories*

$$\mathcal{G}_n : \mathrm{Vect}_{\{0,1\}}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\simeq} \mathcal{BT}_n(R)$$

compatible with Cartier duality, so that for every $\mathcal{M} \in \mathrm{Vect}_{\{0,1\}}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})$, there is a canonical isomorphism

$$\mathcal{G}_n(\mathcal{M})^* \xrightarrow{\simeq} \mathcal{G}_n(\mathcal{M}^*).$$

Remark 11.1.5. The theorem implies in particular that $\mathrm{Vect}_{\{0,1\}}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})$ is a classical category. This fact is not evident from the definitions, since the derived stack $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is in general not classical.

Remark 11.1.6. When R is quasisyntomic, Theorem 11.1.4 simply recovers the main result of Anschütz and Le Bras from [1] classifying p-divisible groups over R in terms of admissible prismatic Dieudonné crystals over R (see [1, Def. 4.1.5]).

This can be seen as follows: First, when R is qrsp, Proposition 8.4.2 shows that $\mathrm{Vect}_{\{0,1\}}(R^{\mathrm{syn}})$ can be described equivalently in terms of 1-bounded vector bundle $\underline{\mathbb{A}}_R$-gauges. The discussion in §5.7 shows that such $\underline{\mathbb{A}}_R$-gauges are canonically equivalent to the category of admissible prismatic Dieudonné modules over R as defined in [1, Def. 4.1.10].

The general quasisyntomic case now follows by descent.

One mildly interesting point: The proof of the equivalence exhibited in §5.7 shows that admissible prismatic Dieudonné modules should naturally be identified with the negative Breuil-Kisin twists of vector bundle F-gauges of Hodge-Tate weights $0, 1$: these of course span the category of vector bundle F-gauges of Hodge-Tate weights $-1, 0$.

Remark 11.1.7. When R is a totally ramified ring of integers over $W(\kappa)$ for some perfect field κ, then, combining the theorem with Example 10.1.3, one can recover a classification of p-divisible groups due to Kisin (the case $p > 2$ for R) [39], W. Kim (the case $p = 2$ for R) [38], and Lau (the general case of both R and R_m) [43].

Remark 11.1.8. When $R = \kappa[|x_1, \ldots, x_n|]$, using Example 10.1.5 and an argument such as the one used in §5.7, one recovers de Jong's description [18] of p-divisible groups over power series rings over perfect fields k in terms of certain filtered φ-modules $\mathrm{Fil}^\bullet M$ over $W(\kappa)[|x_1, \ldots, x_m|]$ equipped with the Frobenius lift satisfying $x_i \mapsto x_i^p$ (see also [27, §7]). *A priori*, de Jong's description is in terms of F-crystals, and so also requires a topologically nilpotent integrable connection on M compatible with the φ-semilinear structure; however, as observed by Faltings [27, Theorem 10], with this choice of Frobenius, the integrable connection is actually uniquely determined by the rest of the data.

We should note however that this 'recovery' is a bit circular: We implicitly make use de Jong's results in the case $m = 1$ in our proof below. See Remark 11.4.4.

11.2. **Height and dimension.**

11.2.1. The **height** of an n-truncated Barsotti-Tate group G over R is the $\mathbb{Z}_{\geq 0}$-valued locally constant function on $\operatorname{Spec} R$ such that $G[p]$ has degree p^h over R. The **dimension** d is the $\mathbb{Z}_{\geq 0}$-valued locally constant function such that $\ker F \subset G[p] \otimes \mathbb{F}_p$ has degree p^d over R/pR.

These are locally constant invariants of G, and yield decompositions

$$\mathrm{BT}_n = \bigsqcup_{d \leq h} \mathrm{BT}_n^{h,d},$$

of formal Artin stacks, where h ranges over the non-negative integers, d over the non-negative integers bounded by h, and $\mathrm{BT}_n^{h,d}$ is the locus of n-truncated Barsotti-Tate groups G of height h and dimension d.

11.2.2. On the F-gauge side of things, we have ∞-subgroupoids

$$\mathrm{Vect}_{h,d}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}) \subset \mathrm{Vect}_{\{0,1\}}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})^{\simeq}$$

spanned by vector bundles $\mathcal{M}$ over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ of rank h and Hodge-Tate weights $0, 1$ such that the associated graded $R/^{\mathbb{L}}p^n$-module $\mathrm{gr}^{-1}_{\mathrm{Hdg}} M$ is locally free of rank d.

Let us note the following:

Lemma 11.2.3. *Cartier duality yields equivalences*

$$\mathrm{Vect}_{h,d}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\simeq} \mathrm{Vect}_{h,h-d}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})$$

11.2.4. For $0 \leq d \leq h$, let $\mu_d : \mathbb{G}_m \to \mathrm{GL}_h$ be the cocharacter given by the diagonal matrix

$$\mu_d(z) = \mathrm{diag}(\underbrace{z, z, \ldots, z}_{d}, \underbrace{1, \ldots, 1}_{h-d}).$$

Associated with this, we have the smooth Artin stacks $\mathrm{BT}_n^{\mathrm{GL}_h, \mu_d}$ over $\mathbb{Z}_p$.

Proposition 11.2.5. *For $R \in \mathrm{CRing}^{f,p\text{-}nilp}$, there is a canonical equivalence of groupoids*

$$\mathrm{BT}_n^{\mathrm{GL}_h, \mu_d}(R) \xrightarrow{\simeq} \mathrm{Vect}_{h,d}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}).$$

Proof. One can see this by combining Lemma 9.2.3 with Propositions 5.6.3 and 9.2.2. ☐

Remark 11.2.6. Via the above proposition and Lemma 11.2.3, we find that there is a Cartier duality equivalence

$$* : \mathrm{BT}_n^{\mathrm{GL}_h, \mu_d} \xrightarrow{\simeq} \mathrm{BT}_n^{\mathrm{GL}_h, \mu_{h-d}}.$$

We now have:

Theorem 11.2.7. *There is a canonical equivalence of smooth p-adic formal Artin stacks*

$$\mathcal{G} : \mathrm{BT}_n^{\mathrm{GL}_h, \mu_d} \xrightarrow{\simeq} \mathrm{BT}_n^{h,d}$$

such that the following diagram commutes up to canonical isomorphism:

$$\begin{tikzcd}
\mathrm{BT}_n^{\mathrm{GL}_h, \mu_d} \arrow[r, "\mathcal{G}", "\simeq"'] \arrow[d, "*"'] & \mathrm{BT}_n^{h,d} \arrow[d, "G \mapsto G^*"] \\
\mathrm{BT}_n^{\mathrm{GL}_h, \mu_{h-d}} \arrow[r, "\simeq", "\mathcal{G}"'] & \mathrm{BT}_n^{h,h-d}.
\end{tikzcd}$$

Assuming this theorem, we can easily deduce Theorem 11.1.4 via a standard argument.

Proof of Theorem 11.1.4. Theorem 11.2.7, combined with Proposition 11.2.5, gives us an isomorphism of ∞-groupoids

$$\mathrm{Vect}_{\{0,1\}}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})^{\simeq} \xrightarrow{\simeq} \mathrm{BT}_n(R)$$

compatible with Cartier duality. To get an equivalence of ∞-categories, one now uses a graph construction: For $\mathcal{M}_1, \mathcal{M}_2$ in $\mathrm{Vect}_{\{0,1\}}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})$, the space of maps $\mathcal{M}_1 \to \mathcal{M}_2$ is equivalent to the space of isomorphisms $\mathcal{M}_1 \oplus \mathcal{M}_2 \xrightarrow{\simeq} \mathcal{M}_1 \oplus \mathcal{M}_2$ that are 'upper triangular' and project onto the identity endomorphisms of $\mathcal{M}_1$ and $\mathcal{M}_2$. A similar description holds for $\mathcal{G}_1$ and $\mathcal{G}_2$ in $\mathcal{BT}_n(R)$. □

11.3. **From F-gauges to Barsotti-Tate groups.** Suppose that we have $R \in \mathrm{CRing}^{f,p\text{-nilp}}$ and $\mathcal{M}$ in $\mathrm{Vect}_{\{0,1\}}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})$.

11.3.1. Set $\mathcal{G}_n(\mathcal{M}) = \Gamma_{\mathrm{syn}}(\mathcal{M})$. Then by Theorem 8.12.1 $\mathcal{G}(\mathcal{M})$ is locally finitely presented and *quasi-smooth* over R with cotangent complex given by $\mathscr{O}_{\mathcal{G}(\mathcal{M})} \otimes_R \mathrm{gr}^{-1}_{\mathrm{Hdg}} M[1]$. More generally, for all $r \leq n$, set

$$\mathcal{M}_r \overset{\mathrm{defn}}{=} \mathcal{M}\big|_{R^{\mathrm{syn}} \otimes \mathbb{Z}/p^r\mathbb{Z}} \ ; \ \mathcal{G}_r(\mathcal{M}) = \Gamma_{\mathrm{syn}}(\mathcal{M}_r).$$

11.3.2. By tensoring $\mathcal{M}$ with the canonical short exact sequence

$$0 \to \mathbb{Z}/p^r\mathbb{Z} \xrightarrow{a \mapsto p^{n-r}a} \mathbb{Z}/p^n\mathbb{Z} \to \mathbb{Z}/p^{n-r}\mathbb{Z} \to 0,$$

we obtain a fiber sequence in $\mathrm{Perf}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})$:

$$\mathcal{M}_r \to \mathcal{M} \to \mathcal{M}_{n-r}.$$

Taking derived global sections yields a fiber sequence

$$R\Gamma_{\mathrm{syn}}(\mathcal{M}_r) \to R\Gamma_{\mathrm{syn}}(\mathcal{M}) \to R\Gamma_{\mathrm{syn}}(\mathcal{M}_{n-r}) \tag{11.3.2.1}$$

of $\mathrm{Mod}_{\mathbb{Z}/p^n\mathbb{Z}}$-valued quasisyntomic sheaves over R.

Theorem 11.3.3. *Suppose that R is discrete and that $\mathcal{M}$ is in $\mathrm{Vect}_{h,d}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})$, then $\mathcal{G}_n(\mathcal{M})$ is a relative truncated Barsotti-Tate group scheme over R of height h and dimension d. In particular, for each $n \geq 1$, we have a canonical map of p-adic formal Artin stacks*

$$\mathcal{G}_n : \mathrm{BT}_n^{\mathrm{GL}_h, \mu_d} \to \mathrm{BT}_n^{h,d}.$$

Moreover, if $r < n$, there is a canonical short exact sequence of truncated Barsotti-Tate group schemes

$$0 \to \mathcal{G}_r(\mathcal{M}) \to \mathcal{G}_n(\mathcal{M}) \to \mathcal{G}_{n-r}(\mathcal{M}) \to 0$$

obtained by taking the connective truncation of (11.3.2.1).

Proof. Let us begin by considering the case $n = 1$, and let us also suppose that R is an $\mathbb{F}_p$-algebra. Here, we can consider the F-zip M underlying $\mathcal{M}$, which is given explicitly by the following data:

(1) A locally free R-module M equipped with direct summands $\mathrm{Fil}^0_{\mathrm{Hdg}} M \subset M$ and $\mathrm{Fil}^{\mathrm{conj}}_1 M \subset M$ of codimension d and $h - d$, respectively;
(2) Isomorphisms

$$\xi_1 : \mathrm{Fil}^{\mathrm{conj}}_1 M \xrightarrow{\simeq} \varphi^*(M/\mathrm{Fil}^0_{\mathrm{Hdg}} M) \ ; \ \xi_0 : \mathrm{gr}^0_{\mathrm{conj}} M = M/\mathrm{Fil}^{\mathrm{conj}}_1 M \xrightarrow{\simeq} \varphi^* \mathrm{Fil}^0_{\mathrm{Hdg}} M.$$

Now, consider the functor

$$\mathsf{G}(\mathsf{M}) = \tau^{\leq 0} R\Gamma_{F\mathrm{Zip}}(\mathsf{M}) : \mathrm{CRing}_{R/} \to \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{F}_p}$$

Unwinding definitions, one finds

$$\mathsf{G}(\mathsf{M})(C) = \{m \in C \otimes_R \mathrm{Fil}^0_{\mathrm{Hdg}} M : \ \xi_0(\overline{m}) = \varphi^* m\},$$

where $M \xrightarrow{m \mapsto \overline{m}} \mathrm{gr}^{\mathrm{conj}}_0 M$ is the natural quotient map. Viewing $m \mapsto \xi_0(\overline{m})$ as a map $\overline{\xi}_0 : \mathrm{Fil}^0_{\mathrm{Hdg}} M \to \varphi^* \mathrm{Fil}^0_{\mathrm{Hdg}} M$, we see that we have

$$\mathsf{G}(\mathsf{M}) = \ker(\mathbf{V}(\mathrm{Fil}^0_{\mathrm{Hdg}} M) \xrightarrow{\overline{\xi}_0 - F} \mathbf{V}(\varphi^* \mathrm{Fil}^0_{\mathrm{Hdg}} M)),$$

so that $\mathsf{G}(\mathsf{M})$ is the Cartier dual to a height one finite flat group scheme over R of rank p^{h-d}.

By Theorem 8.12.1, with input from Corollary 7.2.3, we see that the natural map $\mathcal{G}_1(\mathcal{M}) \to \mathsf{G}(\mathsf{M})$ presents the source as a torsor over the target under the finite flat height 1 group scheme $G(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M, \psi_{\mathsf{M}})$, which has rank p^d. Therefore, we have a short exact sequence of finite flat group schemes

$$0 \to G(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M, \psi_{\mathsf{M}}) \to \mathcal{G}_1(\mathcal{M}) \to \mathsf{G}(\mathsf{M}) \to 0. \tag{11.3.3.1}$$

In particular, we see that $\mathcal{G}_1(\mathcal{M})$ is finite flat over R of rank p^h.

We can now consider the general case of $R \in \mathrm{CRing}^{f,p\text{-nilp}}_{\heartsuit}$ and $n \geq 1$. Taking connective truncations of (11.3.2.1) shows that we have an isomorphism

$$\mathcal{G}_r(\mathcal{M}) \xrightarrow{\simeq} \mathrm{hker}^{\mathrm{cn}}\left(\mathcal{G}_n(\mathcal{M}) \to \mathcal{G}_{n-r}(\mathcal{M})\right) \tag{11.3.3.2}$$

of $\mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}/p^n\mathbb{Z}}$-valued quasisyntomic sheaves over R.

If $R = \kappa$ is an algebraically closed field, then this isomorphism, combined with a simple induction on r shows that $\mathcal{G}_n(\mathcal{M})(\kappa)$ is a finite set. Therefore, applying the following lemma to $Y = \mathcal{G}_n(\mathcal{M})$ shows that $\mathcal{G}_n(\mathcal{M})$ is quasi-finite and flat over R

Lemma 11.3.4. *Suppose that Y is a finitely presented quasi-smooth derived algebraic space over $R \in \mathrm{CRing}$ of virtual codimension 0. Set $S = \mathrm{Spec}\, R$. Then the following are equivalent:*

(1) Y is flat over R;
(2) $Y \otimes_R \pi_0(R)$ is flat and classicaly lci over $\pi_0(R)$;
(3) For every $x \in S(\kappa)$ with κ algebraically closed, $x^*Y \to \mathrm{Spec}\, \kappa$ is flat and classically lci;
(4) For every $x \in S(\kappa)$ with κ algebraically closed, $\pi_0((x^*Y)(\kappa))$ is a finite set.

Proof. Since Y is quasi-smooth of virtual codimension 0, étale locally on Y we can present it as a derived complete intersection subscheme of some affine space $\mathbb{A}^n_R$ over R cut out as the derived zero locus of n polynomials $f_1, \ldots, f_n$. If R is discrete, then such a derived zero locus is flat over R if and only if the animated ring $R[x_1, \ldots, x_n]/^{\mathbb{L}}(f_1, \ldots, f_n)$ has no higher homotopy groups. This is precisely equivalent to this ring being a classical lci algebra over R. Since flatness can be tested after derived basechange over the classical truncation, we see that (1) and (2) are equivalent.

If $R = \kappa$ is an algebraically closed field, then $\pi_0((x^*Y)(\kappa))$ is finite precisely when the classical truncation Y_{cl} is 0-dimensional and of finite type over κ. We now note that $\kappa[x_1, \ldots, x_n]/(f_1, \ldots, f_n)$ is 0-dimensional precisely when $f_1, \ldots, f_n$ form a regular sequence. This shows the equivalence of (3) and (4).

To see the equivalence of (1) and (3), we now only have to make the additional observation that $M \in \mathrm{Mod}_R$ is flat over R if and only if its derived base-change over every algebraically closed field over R is flat. □

Now, by considering the isomorphism (11.3.3.2) twice, first as given, and then again with r replaced by $n - r$, we find that we have

$$\mathrm{im}\left(\mathcal{G}_n(\mathcal{M}) \xrightarrow{p^{n-r}} \mathcal{G}_n(\mathcal{M})\right) = \mathcal{G}_r(\mathcal{M}) = \ker\left(\mathcal{G}_n(\mathcal{M}) \xrightarrow{p^r} \mathcal{G}_n(\mathcal{M})\right).$$

In particular, for every r, $\mathcal{G}_n(\mathcal{M})$ is a $\mathcal{G}_r(\mathcal{M})$-torsor over $\mathcal{G}_{n-r}(\mathcal{M})$. An inductive argument now shows that $\mathcal{G}_n(\mathcal{M})$ is a finite flat group scheme over R that is also a flat $\mathbb{Z}/p^n\mathbb{Z}$-module. Note that this also proves the last assertion of the theorem.

It still remains to verify that, when $n = 1$, $\mathcal{G}_1(\mathcal{M})$ is truncated Barsotti-Tate of height h and dimension d. From what we have just seen, to check that $\mathcal{G}_1(\mathcal{M})$ is a truncated Barsotti-Tate group scheme, it suffices to observe that, étale locally on $\mathrm{Spec}\, R$, $\mathcal{M}$ can be lifted to $\mathrm{Vect}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^2\mathbb{Z})$ by Theorem D and Proposition 11.2.7.

The last thing to check is that $\mathcal{G}_1(\mathcal{M})$ has dimension d. Once again, we can assume that $n = 1$ and that R is an $\mathbb{F}_p$-algebra. Let $F : \mathcal{G}_1(\mathcal{M}) \to \mathcal{G}_1(\mathcal{M})^{(p)}$ be the Frobenius map and let $V : \mathcal{G}_1(\mathcal{M})^{(p)} \to \mathcal{G}_1(\mathcal{M})$ be the Verschiebung. Then, since the Verschiebung on $\mathsf{G}(\mathsf{M})$ is identically zero (its Cartier dual is the Frobenius homomorphism for a height one group scheme), we have

$$\mathrm{im}\, V \subset G(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M, \psi_{\mathsf{M}}) \subset \ker F.$$

Since the outer subgroups are equal, it follows that they must equal the one in the middle, which has rank d over R by construction. □

11.4. **Cartier duality.** Let $\mathcal{O}_n$ be the structure sheaf of $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. We will have use for the following result, which is due to Bhatt-Lurie:

Proposition 11.4.1. *We have canonical isomorphisms*

$$\mathcal{G}_n(\mathcal{O}_n) \xrightarrow{\simeq} \mathbb{Z}/p^n\mathbb{Z}\ ;\ \mathcal{G}_n(\mathcal{O}_n\{1\}) \xrightarrow{\simeq} \mu\!\!\mu_{p^n}$$

in $\mathrm{BT}_n(R)$.

Proof. Unwinding definitions, the first isomorphism follows from [8, Theorem 8.1.9], while the second follows from [8, Theorem 7.5.6]

We can also give alternate proofs using the methods of this paper. As we already know from what we have seen above, $\mathcal{G}_n(\mathcal{O}_n)$ (resp. $\mathcal{G}_n(\mathcal{O}_n\{1\})$) is an n-truncated Barsotti-Tate group schemes over R, height 1 and dimension 0 (resp. dimension 1).

We can assume that $R = \mathbb{Z}/p^m\mathbb{Z}$ for some $m \geq 1$. Note that $\mathcal{G}_n(\mathcal{O}_n)$ is étale over $\mathbb{Z}/p^m\mathbb{Z}$, so it suffices to give a map $\mathbb{Z}/p^n\mathbb{Z} \to \mathcal{G}_n(\mathcal{O}_n)$ that is an isomorphism over $\overline{\mathbb{F}}_p$. This is given by the structure map $\mathbb{Z}/p^n\mathbb{Z} \to \mathcal{O}_n$.

For the case of the Breuil-Kisin twist, note that $\mathcal{G}_n(\mathcal{O}_n\{1\})$ is of multiplicative type. Once again, it suffices to give a canonical map $\mu\!\!\mu_{p^n} \to \mathcal{G}(\mathcal{O}_n\{1\})$ that is an isomorphism over $\overline{\mathbb{F}}_p$. In fact, since the scheme parameterizing such maps is finite étale over $\mathbb{Z}/p^m\mathbb{Z}$, we can assume that $m = 1$. In this case, by quasisyntomic descent, we only have to construct a canonical map

$$\mu\!\!\mu_{p^n}(C) \to (\mathrm{Fil}^1 {\mathbb{\Delta}}_C/p^n)^{\varphi=p} = \mathcal{G}_n(\mathcal{O}_n\{1\})(C) \tag{11.4.1.1}$$

for qrsp $\mathbb{F}_p$-algebras C. This is obtained using the isomorphism ${\mathbb{\Delta}}_C \xrightarrow{\simeq} A_{\mathrm{crys}}(C)$ from (6.9.5.1), and assigning to each $\alpha \in \mu\!\!\mu_{p^n}(C)$ the image of the logarithm $\log([\tilde{\alpha}]^{p^n}) \in A_{\mathrm{crys}}(C)$, where $\tilde{\alpha} \in C^\flat$ is a lift of α and $[\tilde{\alpha}]$ is its Teichmüller lift; see [8, §7.1].

To finish, it is enough to know that the map (11.4.1.1) is *injective* for all qrsp $\mathbb{F}_p$-algebras C. In fact, it suffices to verify that, when $C = \mathbb{F}_p[x^{1/p^\infty}]/(x)$ and $n = 1$, the element $\log([1-x]) \in A_{\mathrm{crys}}(C)$ is not divisible by p. But one has (see displayed equation (37) in [8, p. 168]):

$$\log([1-x]) \equiv -\sum_{d=1}^{p} \frac{[x]^d}{d} \pmod{p}.$$

To see that the right hand side is non-zero, we only have to note that the element

$$x + \frac{x^2}{2} + \ldots + \frac{x^{p-1}}{p-1} + (p-1)! \cdot x^{[p]} \in \mathbb{F}_p\langle x\rangle$$

in the standard divided power $\mathbb{F}_p$-algebra is non-zero.

□

11.4.2. For every vector bundle $\mathcal{M}$ over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ with Hodge-Tate weights $0, 1$, we will now define a canonical map

$$\mathcal{G}_n(\mathcal{M}^*) \to \mathcal{G}_n(\mathcal{M})^*. \tag{11.4.2.1}$$

This is obtained as follows: For every $x : R \to C$, we have

$$\mathcal{G}_n(\mathcal{M}^*)(C) \simeq \mathrm{Map}_{\mathrm{QCoh}(C^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})}(x^*\mathcal{M}, \mathcal{O}_n\{1\}),$$

and evaluation on global sections now yields a map

$$\mathcal{G}_n(\mathcal{M}^*)(C) \to \mathrm{Hom}(\mathcal{G}_n(x^*\mathcal{M}), \mathcal{G}_n(\mathcal{O}_n\{1\})) \simeq \mathrm{Hom}(x^*\mathcal{G}_n(\mathcal{M}), \mu\!\!\mu_{p^n}) \simeq \mathcal{G}_n(\mathcal{M})^*(C).$$

Here, we have used Proposition 11.4.1 for the penultimate isomorphism.

Theorem 11.4.3. *The map* (11.4.2.1) *is an isomorphism.*

Proof. Let's begin with the following easy observation that is immediate from Proposition 11.4.1: The map is an isomorphism for $\mathcal{M} = \mathcal{O}_n^{h-d} \oplus \mathcal{O}_n\{1\}^d$.

Denote the map (11.4.2.1) by α_n, and view it as a map of n-truncated Barsotti-Tate group schemes over the smooth p-adic formal algebraic stack $\mathrm{BT}_n^{\mathrm{GL}_h,\mu_d}$ of the same height and dimension. Taking the limit over n gives us a map α_∞ of p-divisible groups over $\mathrm{BT}_\infty^{\mathrm{GL}_h,\mu_d}$. We will actually show that α_∞ is an isomorphism: this is enough since $\mathrm{BT}_\infty^{\mathrm{GL}_h,\mu_d} \to \mathrm{BT}_n^{\mathrm{GL}_h,\mu_d}$ is a limit of smooth surjective maps.

Let R be a universal deformation ring for $\mathrm{BT}_\infty^{\mathrm{GL}_h,\mu_d} \otimes \mathbb{F}_p$ at a geometric point valued in a field κ (see Lemma 10.2.5): this is isomorphic to $\kappa[|t_1, \ldots, t_{d(h-d)}|]$ and is in particular normal. It is enough to know that the restriction of α_∞ over $\operatorname{Spf} R$ is an isomorphism. By [18, Lemma 2.4.4], this restriction algebraizes to a map of p-divisible groups over $\operatorname{Spec} R$. The observation from the beginning of the proof—combined with the connectedness of $\mathrm{BT}_\infty^{\mathrm{GL}_h,\mu_d}$ (see Remark 9.3.5)—shows that this map is generically an isomorphism. Therefore, we find from [17, Corollary 1.2]—and the subsequent discussion—that it is an isomorphism on the nose. □

Remark 11.4.4. The proof above ultimately relies on the results of de Jong in [17] and hence also on his results from [18] in the form of a fully faithfulness result for the classical Dieudonné functor of Berthelot-Breen-Messing [4] for complete DVRs in characteristic p. This use can be circumvented by a more careful study of the map α_n.

11.5. **From Barsotti-Tate groups to F-gauges.**

11.5.1. Let R be p-complete, p-quasisyntomic and either p-torsion free or an $\mathbb{F}_p$-algebra. These conditions ensure that there exist quasisyntomic covers $R \to R'$ with R' qrsp, and also that $\mathbb{\Delta}_{R'}$ is p-completely flat for such covers. In particular, the derived stacks $R^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ and $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ are actually *classical*.

Let
$$\epsilon_n : (R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})_{\mathrm{fl}} \to R_{\mathrm{qsyn}}$$
be the map of (classical) topoi arising via the functor $C \mapsto C^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ on p-quasisyntomic R-algebras. Here, the left hand side (resp. the right hand side) is the topos of sheaves on the ind-fppf site over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ (resp. the small p-quasisyntomic site over $\operatorname{Spf} R$).

We can view $\mathcal{O}_n$ as a sheaf of rings on $(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})_{\mathrm{fl}}$, and $\mathcal{O}_n\{1\}$ as a quasicoherent sheaf with respect to $\mathcal{O}_n$.

11.5.2. For $G \in \mathrm{BT}_n(R)$, set
$$\mathcal{M}(G) = \underline{\mathrm{Hom}}_{(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})_{\mathrm{fl}}}(\epsilon_n^{-1} G^*, \mathcal{O}_n\{1\})$$
where on the right we are considering the internal Hom sheaf over $(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})_{\mathrm{fl}}$. Note that by construction $\mathcal{M}(G)$ is a module over $\mathcal{O}_n$.

To alleviate notation, we will now drop the subscript n and write $\mathcal{G}$ for the functor $\mathcal{G}_n$. The next result can be found in a certain form in Anschütz-Le Bras [1], though this formulation (and most of the proof, though we give a different approach to Cartier duality) is due to Mondal [56]:

Proposition 11.5.3. *(1) $\mathcal{M}(G)$ is a vector bundle over $\mathcal{O}_n$ and yields an F-gauge in $\mathrm{Vect}_{\{0,1\}}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})$.*

(2) The functors
$$\mathcal{M} : \mathrm{BT}_n(R) \to \mathrm{Vect}_{\{0,1\}}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z});$$
$$\mathcal{G} : \mathrm{Vect}_{\{0,1\}}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\textit{Theorem 11.3.3}} \mathrm{BT}_n(R)$$
form an adjoint pair.

(3) The unit $\mathrm{id} \to \mathcal{G} \circ \mathcal{M}$ is an isomorphism.

(4) There is a natural isomorphism $\mathcal{M}(G^*)^* \to \mathcal{M}(G)$.

Proof. Most of the proof that we present here can be found in [56, §3].

For claim (1), via quasisyntomic descent we reduce to the case where R is qrsp. Here, the result follows from [56, Props. 3.56, 3.80, 3.81].

For the second claim, given $G \in \mathcal{BT}_n(R)$ and $\mathcal{M} \in \mathrm{Vect}_{\{0,1\}}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})$, we find canonical isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_n}(\mathcal{M}, \mathcal{M}(G)) &\simeq \mathrm{Hom}_{\mathcal{O}_n}\left(\mathcal{M}, \underline{\mathrm{Hom}}_{(R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z})_{\mathrm{fl}}}(\epsilon_n^{-1}G^*, \mathcal{O}_n\{1\})\right) \\
&\simeq \mathrm{Hom}_{(R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z})_{\mathrm{fl}}}\left(\epsilon_n^{-1}G^*, \underline{\mathrm{Hom}}_{\mathcal{O}_n}(\mathcal{M}, \mathcal{O}_n\{1\})\right) \\
&\simeq \mathrm{Hom}_{R_{\mathrm{qsyn}}}\left(G^*, \epsilon_{n,*}\underline{\mathrm{Hom}}_{\mathcal{O}_n}(\mathcal{O}_n, \mathcal{M}^*)\right) \\
&\simeq \mathrm{Hom}_{R_{\mathrm{qsyn}}}\left(G^*, \mathcal{G}(\mathcal{M}^*)\right) \\
&\simeq \mathrm{Hom}_{\mathcal{BT}_n(R)}(G^*, \mathcal{G}(\mathcal{M})^*) \\
&\simeq \mathrm{Hom}_{\mathcal{BT}_n(R)}(\mathcal{G}(\mathcal{M}), G).
\end{aligned}$$

Here, in the penultimate isomorphism, we have used Theorem 11.4.3.

For claim (3), suppose that we are given $G \in \mathrm{BT}_n(R)$. We then find:

$$\begin{aligned}
\mathcal{G}(\mathcal{M}(G))(R) &\simeq \mathrm{Hom}_{\mathcal{O}_n}(\mathcal{O}_n, \mathcal{M}(G)) \\
&\simeq \mathrm{Hom}_{\mathcal{BT}_n(R)}(\mathcal{G}(\mathcal{O}_n), G) \\
&\simeq \mathrm{Hom}_{\mathcal{BT}_n(R)}(\mathbb{Z}/p^n\mathbb{Z}, G) \\
&\simeq G(R).
\end{aligned}$$

Here, in the penultimate isomorphism, we have used Proposition 11.4.1. Since this isomorphism is valid with R replaced by any p-quasisyntomic R-algebra, claim (3) has been verified.

Finally, let us consider claim (4): We have

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_n}(\mathcal{M}(G^*)^*, \mathcal{M}(G)) &\simeq \mathrm{Hom}_{\mathcal{BT}_n(R)}(\mathcal{G}(\mathcal{M}(G^*)^*), G) \\
&\simeq \mathrm{Hom}_{\mathcal{BT}_n(R)}(\mathcal{G}(\mathcal{M}(G^*))^*, G) \\
&\simeq \mathrm{Hom}_{\mathcal{BT}_n(R)}((G^*)^*, G) \\
&\simeq \mathrm{Hom}_{\mathcal{BT}_n(R)}(G, G).
\end{aligned}$$

Here, the first isomorphism uses claim (2), the second uses Theorem 11.4.3 and the third uses claim (3). The identity endomorphism of G corresponds via these isomorphisms to the canonical arrow involved in claim (4).

That this arrow is an isomorphism is a consequence of the next lemma, applied with $\mathcal{M}_1 = \mathcal{M}(G^*)^*$ and $\mathcal{M}_2 = \mathcal{M}(G)$:

Lemma 11.5.4. *Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are two perfect F-gauges of level n over R with Hodge-Tate weights $0, 1$. Set $\mathcal{N}^* = \mathcal{N}^\vee\{1\}$ for any perfect F-gauge $\mathcal{N}$: this underlies an anti-involution on the ∞-category of perfect F-gauges of Hodge-Tate weights $0, 1$. Suppose that $f : \mathcal{M}_1 \to \mathcal{M}_2$ is an arrow such that*

$$\Gamma_{\mathrm{syn}}(f) : \Gamma_{\mathrm{syn}}(\mathcal{M}_1) \to \Gamma_{\mathrm{syn}}(\mathcal{M}_2) \ ; \ \Gamma_{\mathrm{syn}}(f^*) : \Gamma_{\mathrm{syn}}(\mathcal{M}_2^*) \to \Gamma_{\mathrm{syn}}(\mathcal{M}_1^*)$$

are equivalences of derived stacks over R. Then f is an isomorphism.

Proof. Set $\mathcal{N} = \mathrm{hker}(f)$; then we see that $\Gamma_{\mathrm{syn}}(\mathcal{N})(C) = 0$ for all $C \in \mathrm{CRing}_{R/}$. Similarly, $\Gamma_{\mathrm{syn}}(\mathcal{N}^*[-1])(C) = 0$ for all $C \in \mathrm{CRing}_{R/}$. Theorem 8.12.1 now tells us that, if $\mathrm{Fil}^\bullet_{\mathrm{Hdg}} N$ and $\mathrm{Fil}^\bullet_{\mathrm{Hdg}} N^*$ are the filtered perfect complexes over R obtained from $\mathcal{N}$ and $\mathcal{N}^*$ via pullback along $x^{\mathcal{N}}_{\mathrm{dR}}$, then we have

$$\mathrm{gr}^{-1}_{\mathrm{Hdg}} N \simeq 0 \simeq \mathrm{gr}^{-1}_{\mathrm{Hdg}} N^* \simeq (\mathrm{Fil}^0_{\mathrm{Hdg}} N)^\vee.$$

Since $\mathcal{N}$ has Hodge-Tate weights $0, 1$, this shows that $\mathrm{gr}^i_{\mathrm{Hdg}} N \simeq 0$ for all i, and hence that $\mathrm{Fil}^\bullet_{\mathrm{Hdg}} N \simeq 0$. This implies that $\mathcal{N} \simeq 0$: To see this, we can assume that R is semiperfectoid, in which case it follows from the observation that the map

$$\pi_0(\mathbb{\Delta}_R/^{\mathbb{L}}(p, I)) \to \pi_0(R/^{\mathbb{L}}(p, I))$$

has nilpotent kernel; see for instance the end of the proof of Proposition 4.12.3. □

□

We are now ready to prove Theorem 11.2.7

Proof of Theorem 11.2.7. We first note that Proposition 11.5.3 gives us a left inverse $\mathcal{M} : \mathrm{BT}_n^{h,d} \to \mathrm{BT}_n^{\mathrm{GL}_h,\mu_d}$ to the map $\mathcal{G}$ from Theorem 11.3.3. Indeed, by Theorem 11.1.2, $\mathrm{BT}_n^{h,d}$ is smooth, and therefore is a left Kan extension of its restriction to p-completely smooth $\mathbb{Z}_p$-algebras. This means that, to obtain the map $\mathcal{M}$ and verify that it is a left inverse to $\mathcal{G}$, it suffices to do so on such inputs, where it follows from the proposition.

From the same proposition, we find, for all $\mathcal{M} \in \mathrm{BT}_n^{\mathrm{GL}_h,\mu_d}(R)$, a canonical map of F-gauges $\mathcal{M} \to \mathcal{M}(\mathcal{G}(\mathcal{M}))$. To finish the proof of the theorem, we have to verify that this map is an isomorphism.

For this, we can assume without loss of generality that R is p-quasisyntomic. Now, we begin by observing that we also have a corresponding canonical map of Cartier dual F-gauges

$$\mathcal{M}^* \to \mathcal{M}(\mathcal{G}(\mathcal{M}^*)) \simeq \mathcal{M}(\mathcal{G}(\mathcal{M})^*),$$

where the last isomorphism using Theorem 11.4.3. Taking Cartier duals again yields a map

$$\mathcal{M}(\mathcal{G}(\mathcal{M})^*)^* \to \mathcal{M}.$$

and the composition

$$\mathcal{M}(\mathcal{G}(\mathcal{M})^*)^* \to \mathcal{M} \to \mathcal{M}(\mathcal{G}(\mathcal{M}))$$

is the canonical isomorphism in claim (4) of Proposition 11.5.3 applied with $G = \mathcal{G}(\mathcal{M})$. Alternatively, instead of using Cartier duality in this form, one can argue directly using Lemma 11.5.4.

This shows that $\mathcal{M} \to \mathcal{M}(\mathcal{G}(\mathcal{M}))$ is an epimorphism, and since it is a map of vector bundles of the same rank over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$, we conclude that it is in fact an isomorphism. □

Remark 11.5.5. In the proof above, since we already have the map $\mathcal{G}$, to verify that it is an equivalence, it suffices to do so over the special fiber. In particular, logically speaking, we only need Proposition 11.5.3 in the case where R is an $\mathbb{F}_p$-algebra. In this case, the verification of claim (1) of that proposition only requires the computations from [4] of the *crystalline* cohomology of abelian schemes in characteristic p. Formulating the proof this way would make it completely independent of the results of [1] or [56], though it wouldn't shed much light on what the inverse functor is, away from characteristic p.

Alternatively, if one is happy to assume classical Dieudonné theory over perfect fields as explicated, say, in [28], one could also reduce to the case of power series rings over perfect fields, where we could use Grothendieck-Messing theory on both sides of the purported equivalence to further reduce to the known case of perfect fields. However, here one would have to compare our construction of finite flat group schemes here with that used by Fontaine, which involves Witt covectors.

Another strategy is to reduce once again to the case of power series rings and then invoke the results of de Jong [18].

Remark 11.5.6. Combined with Theorem G, we find that $\mathrm{BT}_n^{h,d}$ also enjoys a Grothendieck-Messing type deformation theory. This should be a special case of a very general lifting result of Faltings [26, Theorem 17], but we have not verified the requisite compatibility with the constructions of Faltings.

11.6. **A polarized variant.** In this subsection, R will always denote a derived p-complete discrete ring in $\mathrm{CRing}^f_\heartsuit$.

Definition 11.6.1. Suppose that G is a finite flat commutative p^n-torsion group scheme over R. Then, given a $\mathbb{Z}/p^n\mathbb{Z}$-local system L over $\operatorname{Spec} R$, we will write $G \otimes L$ for the finite flat group scheme obtained by tensor product of fppf sheaves of abelian groups: that this is indeed such an object is clear from étale descent.

Alternatively we can also view $G \otimes L$ as the internal Hom sheaf $\underline{\mathrm{Hom}}(L^\vee, G)$ in fppf sheaves of abelian groups. This perspective shows:

Observation 11.6.2. *We have a canonical isomorphism $(G \otimes L)^* \simeq G^* \otimes L^\vee$, where $*$ denotes the Cartier dual as usual, and $L^\vee$ is the dual local system.*

Definition 11.6.3. Fix an integer $g \geq 1$. A **principal quasi-polarization** on $G \in \mathrm{BT}_n^{2g,g}(R)$ is the provision of the following data:

(1) A rank 1 $\mathbb{Z}/p^n\mathbb{Z}$-local system L over $\operatorname{Spf} R$;

(2) An isomorphism $\lambda : G \xrightarrow{\simeq} G^* \otimes L$ such that the composition

$$G^* \xrightarrow{\simeq} (G \otimes L)^* \otimes L \xrightarrow{\lambda^* \otimes 1} G^* \otimes L$$

is equal to $-\lambda$.

We will call the tuple (G, L, λ) a **principally quasi-polarized n-truncated Barsotti-Tate group** (of height $2g$), and denote the groupoid of such tuples by $\mathrm{BT}_n^{2g,\mathrm{qpol}}(R)$.

A **principal polarization on** G is a tuple (G, L, λ) as above satisfying the following additional condition: Étale locally on $\operatorname{Spf} R$, there exists a lift $(\tilde{G}, \tilde{L}, \tilde{\lambda})$ to $\mathrm{BT}_{n+1}^{2g,\mathrm{qpol}}(R)$. We will call such tuples **principally polarized n-truncated Barsotti-Tate groups** and write $\mathrm{BT}_n^{2g,\mathrm{ppol}}(R)$ for the groupoid spanned by them.

Remark 11.6.4. In the literature—see for instance [59] or [15]—one finds a definition of a principal quasi-polarization of p-divisible groups that is essentially the notion above (in the limit over n), except that L is taken to be *trivial*.

At finite level, we find a definition for the case $n = 1$ in [57, §2.6], which uses Dieudonné theory, and applies at the level of geometric points: it corresponds to our notion of a principal polarization given above. This is however a pointwise condition, and its moduli-theoretic interpretation is a little unclear. We have 'solved' this issue above via the lifting-based condition, which is justified primarily by Proposition 11.6.6 below. See also Remark 11.6.7.

11.6.5. Let GSp_{2g} be the generalized symplectic group over $\mathbb{Z}_p$ associated with the 'standard' symplectic space over $\mathbb{Z}_p$ of rank $2g$, and let $\mu_g : \mathbb{G}_m \to \mathrm{GSp}_{2g}$ be the minuscule cocharacter splitting a Lagrangian subspace, and is such that the standard representation of GSp_{2g} yields a map of pairs $(\mathrm{GSp}_{2g}, \mu_g) \to (\mathrm{GL}_{2g}, \mu_g)$. Note that we have the similitude character $\nu : \mathrm{GSp}_{2g} \to \mathbb{G}_m = \mathrm{GL}_1$ satisfying $\nu \circ \mu_g = \mu_1$. This yields a map

$$(\mathrm{GSp}_{2g}, \mu_g) \to (\mathrm{GL}_g \times \mathbb{G}_m, \mu_g \times \mu_1).$$

By Proposition 10.4.3 applied to the case $\mathcal{O} = \mathbb{Z}_p$, we see that $\mathrm{BT}_n^{\mathbb{G}_m,\mu_1}$ is the stack of line bundle F-gauges of the form $\mathcal{L}_0\{1\}$, where $\mathcal{L}_0$ is a line bundle F-gauge with Hodge-Tate weight 0, associated with a $\mathbb{Z}/p^n\mathbb{Z}$-local system $L = \Gamma_{\mathrm{syn}}(\mathcal{L}_0)$ of rank 1 over $\operatorname{Spf} R$. We will identify the stack of such local systems with the classifying stack $B\underline{(\mathbb{Z}/p^n\mathbb{Z})^\times}$.

Proposition 11.6.6. *There exists a canonical commuting diagram of p-adic formal classical Artin stacks over $\mathbb{Z}_p$ where the horizontal arrows are equivalences:*

$$\begin{tikzcd}
\mathrm{BT}_n^{\mathrm{GSp}_{2g},\mu_g} \arrow[r, "\simeq"] \arrow[d] & \mathrm{BT}_n^{2g,\mathrm{ppol}} \arrow[d, "(G,L,\lambda)\mapsto(G,L)"] \\
\mathrm{BT}_n^{\mathrm{GL}_{2g},\mu_g} \times \mathrm{BT}_n^{\mathbb{G}_m,\mu_1} \arrow[r, "\simeq"'] & \mathrm{BT}_n^{2g,g} \times B\underline{(\mathbb{Z}/p^n\mathbb{Z})^\times}.
\end{tikzcd}$$

Proof. All the formal stacks involved are smooth over $\mathbb{Z}_p$[22], so it suffices to construct such a diagram on p-torsion free qrsp inputs R. Since $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is a classical stack, using standard arguments, we find that $\mathrm{BT}_n^{\mathrm{GSp}_{2g},\mu_g}(R)$ is the groupoid of tuples $(\mathcal{M}, \mathcal{L}, \psi)$, where:

- $\mathcal{M}$ is in $\mathrm{BT}_n^{\mathrm{GL}_{2g},\mu_g}(R)$;
- $\mathcal{L}$ is in $\mathrm{BT}_n^{\mathbb{G}_m,\mu_1}(R)$;
- $\psi : \wedge^2\mathcal{M} \to \mathcal{L}$ is a map inducing an isomorphism of F-gauges

$$f_\psi : \mathcal{M} \xrightarrow{\simeq} \mathcal{M}^\vee \otimes \mathcal{L}.$$

Here, the exterior power $\wedge^2\mathcal{M}$ is of vector bundles over the classical stack $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$, constructed as a quotient of $\mathcal{M} \otimes \mathcal{M}$ in the usual way.

[22] For $\mathrm{BT}_n^{2g,\mathrm{ppol}}$, this is a 'well-known' result, for which we could not find a specific reference, though one should certainly be able to extract it from the results of Faltings in [26, §7]. In any case, the argument shows an unconditional result for smooth inputs, and so at least yields a canonical map $\mathrm{BT}_n^{\mathrm{GSp}_{2g},\mu_g} \to \mathrm{BT}_n^{2g,\mathrm{ppol}}$, which is an equivalence as soon as we know the smoothness of the target.

Note that f_ψ has the property that the composition

$$\mathcal{M} \xrightarrow{\simeq} (\mathcal{M}^\vee \otimes \mathcal{L})^\vee \otimes \mathcal{L} \xrightarrow[\simeq]{f_\psi^\vee \otimes 1} \mathcal{M} \otimes \mathcal{L}$$

is equal to $-f_\psi$. When $p > 2$, this condition on f_ψ is enough to recover the symplectic form ψ.

In general, we can say the following: Suppose that $f : \mathcal{M} \to \mathcal{M}^\vee \otimes \mathcal{L}$ can be lifted to an isomorphism

$$\tilde{f} : \tilde{\mathcal{M}} \xrightarrow{\simeq} \tilde{\mathcal{M}}^\vee \otimes \tilde{\mathcal{L}}$$

of vector bundle F-gauges of level $n+1$ satisfying the anti-symmetry property above. Then $f = f_\psi$ for a unique form $\psi : \wedge^2\mathcal{M} \to \mathcal{L}$. This is because, for any flat $\mathbb{Z}/p^{n+1}\mathbb{Z}$-algebra A, any 2-torsion element $a \in A$ maps to 0 in $\mathbb{Z}/p^n\mathbb{Z}$ (this is valid for all primes p!). Conversely, if $f = f_\psi$ for some ψ, then by the smoothness of $\mathrm{BT}_{n+1}^{\mathrm{GSp}_{2g},\mu_g} \to \mathrm{BT}_n^{\mathrm{GSp}_{2g},\mu_g}$ we see that it admits an étale local lift of level $n+1$.

Now, $\mathcal{L} \simeq \mathcal{L}_0\{1\}$ for some $\mathcal{L}_0$ of Hodge-Tate weight 0 associated with a $\mathbb{Z}/p^n\mathbb{Z}$-local system L of rank 1. Therefore, we have

$$\mathcal{M}^\vee \otimes \mathcal{L} \simeq \mathcal{M}^* \otimes \mathcal{L}_0.$$

Moreover, we have a canonical isomorphism

$$\mathcal{G}(\mathcal{M}^* \otimes \mathcal{L}_0) \xrightarrow{\simeq} \underline{\mathrm{Hom}}(L^\vee, \mathcal{G}(\mathcal{M})^*),$$

which is clear from the full faithfulness part of Theorem 11.1.4.

Theorem 11.2.7 now tells us that there is an equivalence of groupoids between tuples $(\mathcal{M}, \mathcal{L}, f')$ and (G, L, λ'), where $(\mathcal{M}, \mathcal{L})$ and (G, L) are as before, and $f' : \mathcal{M} \xrightarrow{\simeq} \mathcal{M}^\vee \otimes \mathcal{L}$ and $\lambda' : G \to G^* \otimes L$ are isomorphisms. As discussed above, the first kind of tuple lifts (uniquely) to $\mathrm{BT}_n^{\mathrm{GSp}_{2g},\mu_g}(R)$ if and only if it can be lifted, étale locally on $\mathrm{Spf}\, R$, to a tuple $(\tilde{\mathcal{M}}, \tilde{\mathcal{L}}, \tilde{f})$ of level $n+1$. We now conclude from the definition of $\mathrm{BT}_n^{2g,\mathrm{ppol}}(R)$. ☐

Remark 11.6.7. Frank Calegari has pointed out to us a *different* notion of polarization due to Hopkins and Lurie [50, §3], which gives a clean moduli-theoretic definition even when $p = 2$. More precisely, for every finite flat group scheme G over some ring R, they produce (see Definition 3.2.5 of *loc. cit.*) a group scheme $\mathrm{Alt}_G^{(2)}$ of *alternating* 2*-forms* on G that is a subscheme of the scheme $\mathrm{Skew}_G^{(2)}$ of alternating bilinear maps $G \times G \to \mathbb{G}_m$, and is equal to the latter when 2 is invertible in R. We can then define a polarization on G as a section of $\mathrm{Alt}_G^{(2)}$ whose associated alternating pairing is non-degenerate.

Now, suppose that R is p-nilpotent and that G is an n-truncated Barsotti-Tate group associated with an F-gauge $\mathcal{M} \in \mathrm{BT}_n^{\mathrm{GL}_{2g},\mu_g}$; then $\mathcal{F} \overset{\mathrm{defn}}{=} (\wedge^2\mathcal{M})^\vee\{1\}$ has Hodge-Tate weights $\{-1, 0, 1\}$. Therefore, by Theorem 8.12.1, $\Gamma_{\mathrm{syn}}(\mathcal{F})$ is a derived affine scheme over R, and its classical truncation is a group scheme whose points parameterize maps of F-gauges $\wedge^2\mathcal{M} \to \mathcal{O}_n\{1\}$.

We now expect that this group scheme is isomorphic to $\mathrm{Alt}_G^{(2)}$. One can reduce to knowing the following: A non-degenerate alternating pairing $\beta \in \mathrm{Skew}_G^{(2)}(R)$ lies in $\mathrm{Alt}_G^{(2)}(R)$ if and only if, étale locally on $\mathrm{Spec}\, R$, there exists an $(n+1)$-truncated Barsotti-Tate group scheme $\tilde{G}$ and a non-degenerate pairing $\tilde{\beta} \in \mathrm{Skew}_{\tilde{G}}^{(2)}(R)$ lifting β.

Remark 11.6.8. Here is a further aside: Hopkins and Lurie also define, for any $d \geq 2$, the group scheme of *alternating d-forms* $\mathrm{Alt}_G^{(d)}$, and show in [50, Theorem 3.5.1] that, when G is an n-truncated Barsotti-Tate group of height h and dimension 1, then $\mathrm{Alt}_G^{(d)}$ is represented by an n-truncated Barsotti-Tate group of height $\binom{h}{d}$ and dimension $\binom{h-1}{d}$. This should correspond simply to the fact that, with the notation of the previous remark, $(\wedge^d\mathcal{M})^\vee\{1\}$ is a vector bundle F-gauge of level n and Hodge-Tate weights $\{0, 1\}$ with height $\binom{h}{d}$ and dimension $\binom{h-1}{d}$.

11.7. **The de Rham realization.**

11.7.1. Let $P^-_{h,d} \subset \mathrm{GL}_h$ be the parabolic subgroup associated the cocharacter μ_d, so that $BP^-_{h,d}$ parameterizes filtered vector bundles $\mathrm{Fil}^\bullet \mathcal{V}$ with

$$\operatorname{rank} \operatorname{gr}^i \mathcal{V} = \begin{cases} d & \text{if } i = -1; \\ h-d & \text{if } i = 0; \\ 0 & \text{otherwise.} \end{cases}$$

Consider the natural map

$$\mathrm{BT}_\infty^{\mathrm{GL}_h,\mu_d} \to BP^-_{h,d},$$

obtained by taking the limit over n from the diagram in Theorem 9.3.2.

This associates with every F-gauge $\mathcal{M}$ in $\mathrm{BT}_\infty^{\mathrm{GL}_h,\mu_d}(R)$ over a derived p-complete ring R a filtered locally free R-module $\mathrm{Fil}^\bullet_{\mathrm{Hdg}} M$.

11.7.2. On the other hand, suppose that R is discrete and in CRing^f. Given such $\mathcal{M}$, we obtain a p-divisible group $\mathcal{G} \overset{\mathrm{defn}}{=} \mathcal{G}(\mathcal{M})$ over R of height h and dimension d, and the crystalline Dieudonné theory of [4] gives us a short exact sequence

$$0 \to \omega_{\mathcal{G}^*} \to \mathbb{D}(\mathcal{G}^*)(R) \to \mathrm{Lie}(\mathcal{G}) \to 0 \tag{11.7.2.1}$$

of locally free R-modules, where $\omega_{\mathcal{G}^*}$ (resp. $\mathrm{Lie}(\mathcal{G})$) has rank $h-d$ (resp. d); see in particular Corollaire 3.3.5 of *loc. cit.*

Proposition 11.7.3. *There is a canonical isomorphism of R-modules $M \xrightarrow{\simeq} \mathbb{D}(\mathcal{G}^*)(R)$ inducing isomorphisms*

$$\mathrm{Fil}^0_{\mathrm{Hdg}} M \xrightarrow{\simeq} \omega_{\mathcal{G}^*} \ ; \ \operatorname{gr}^{-1}_{\mathrm{Hdg}} M \xrightarrow{\simeq} \mathrm{Lie}(\mathcal{G}).$$

Proof. We can view $\mathbb{D}(\mathcal{G}^*)(R)$ as being a filtered locally free R-module supported in degrees $-1, 0$. From this perspective, we see that we have written down two maps from $\mathrm{BT}_\infty^{\mathrm{GL}_h,\mu_d}$ to $BP^-_{h,d}$, and we would like to know that they are isomorphic.

By derived descent and the pro-smoothness of the source, it suffices to verify this on qrsp $\mathbb{F}_p$-algebras R. The proposition now essentially follows from the discussion in [1, §4.3]. The point is to compare the construction here with the crystalline Dieudonné theory in [4]. Here are the details.

Let $(R/\mathbb{Z}_p)_{\mathrm{crys,pr}}$ be the big crystalline site for $\operatorname{Spec} R$ equipped with the pr topology given by extracting p-th roots: the definition of this site is due to Lau [41] and is used in [1, §4.3]. There is a map of topoi

$$u^{\mathrm{crys}} : \mathrm{Shv}\left((R/\mathbb{Z}_p)_{\mathrm{crys,pr}}\right) \to \mathrm{Shv}(R_{\mathrm{pr}})$$

such that the classical Dieudonné functor of [4] can be reinterpreted as the inner Ext

$$\mathbb{D}(\mathcal{G}^*) \simeq \varprojlim_n \underline{\mathrm{Ext}}^1((u^{\mathrm{crys}})^{-1}\mathcal{G}^*[p^n], \mathcal{O}_{\mathrm{crys}}),$$

where $\mathcal{O}_{\mathrm{crys}}$ is the structure sheaf over $(R/\mathbb{Z}_p)_{\mathrm{crys,pr}}$. The Hodge filtration is given (up to degree shift) by

$$\mathrm{Fil}^0 \mathbb{D}(\mathcal{G}^*) \simeq \varprojlim_n \underline{\mathrm{Ext}}^1((u^{\mathrm{crys}})^{-1}\mathcal{G}^*[p^n], \mathcal{J}_{\mathrm{crys}}),$$

where $\mathcal{J}_{\mathrm{crys}}$ is the kernel of the natural surjection $\mathcal{O}_{\mathrm{crys}} \to (u^{\mathrm{crys}})^{-1}\mathbb{G}_a$. When evaluated on the trivial divided power thickening $R \xrightarrow{\mathrm{id}} R$, this yields the short exact sequence (11.7.2.1).

To compare this with our constructions, let us begin by invoking the explicit inverse constructed in Proposition 11.5.3. This shows that the restriction of $\mathcal{M}$ to $R^{\mathcal{N}}$ corresponds to a filtered $\mathrm{Fil}^\bullet_{\mathcal{N}} \mathbb{\Delta}_R$-module $\mathrm{Fil}^\bullet \mathsf{M}$ admitting the following description: We have

$$\mathrm{Fil}^i \mathsf{M} = \varprojlim_n \mathrm{Hom}(\mathcal{G}^*[p^n], \mathrm{Fil}^{i+1}_{\mathcal{N}} \mathbb{\Delta}_{\_}/^{\mathbb{L}} p^n) \simeq \varprojlim_n \mathrm{Ext}^1(\mathcal{G}^*[p^n], \mathrm{Fil}^{i+1}_{\mathcal{N}} \mathbb{\Delta}_{\_}).$$

Here, we are looking at Hom (or Ext) between complexes of pr sheaves on $\operatorname{Spec} R$, where $\mathrm{Fil}^i_{\mathcal{N}} \mathbb{\Delta}_{\_}$ is the sheafification—as a derived p-complete complex—of the assignment $R' \mapsto \mathrm{Fil}^i_{\mathcal{N}} \mathbb{\Delta}_{R'}$ on quasisyntomic qrsp R-algebras (which form a basis with respect to the pr topology). The final isomorphism holds because $\mathrm{Fil}^i_{\mathcal{N}} \mathbb{\Delta}_{\_}$ takes values in p-completely flat objects on qrsp inputs, while $\mathcal{G}^*[p^n]$ is of course killed by p^n. Moreover, since we have an object with Hodge-Tate weights $0, 1$, the filtration is determined by the first two steps, in that we have, for $i \geq 2$,

$$\mathrm{Fil}^i\,\mathsf{M} = (\mathrm{Fil}^i_{\mathcal{N}}\,\mathbb{\Delta}_R)\cdot\mathrm{Fil}^0\,\mathsf{M} + (\mathrm{Fil}^{i+1}_{\mathcal{N}}\,\mathbb{\Delta}_R)\cdot\mathsf{M}.$$

The filtered module $\mathrm{Fil}^\bullet_{\mathrm{Hdg}}\,M$ is now obtained via filtered base-change along the map $\mathrm{Fil}^\bullet_{\mathcal{N}}\,\mathbb{\Delta}_R \to \mathrm{Fil}^\bullet_{\mathrm{triv}}\,R$

To finish, therefore, we only need to know that we have canonical isomorphisms of complexes of pr sheaves

$$Ru^{\mathrm{crys}}_*\mathcal{O}_{\mathrm{crys}} \xrightarrow{\simeq} \mathbb{\Delta}_{\_}\,;\; Ru^{\mathrm{crys}}_*\mathcal{J}_{\mathrm{crys}} \xrightarrow{\simeq} \mathrm{Fil}^1_{\mathcal{N}}\,\mathbb{\Delta}_{\_}.$$

It suffices to compare their values on qrsp quasisyntomic R-algebras. Here, this is a consequence of Remark 6.9.5 and the discussion in § 6.11: For any qrsp R-algebra R', let $R'_0 = R^{',\flat}$ be the inverse perfection of R' with prism structure given by $I_{R'_0} \subset W(R'_0) = \mathbb{\Delta}_{R'_0}$. we have a canonical isomorphism $\mathbb{\Delta}_{R'} \simeq A_{\mathrm{crys}}(R')$ as well as isomorphisms

$$\mathrm{Fil}^i_{\mathcal{N}}\,\mathbb{\Delta}_{R'} \simeq \{x \in A_{\mathrm{crys}}(R') :\ \varphi(x) \in I^i_{R'_0}A_{\mathrm{crys}}(R')\} \subset A_{\mathrm{crys}}(R').$$

On the other hand, $\theta_{R'} : A_{\mathrm{crys}}(R') \to R'$ with its divided powers is also the (pro-)initial object in $(R'/\mathbb{Z}_p)_{\mathrm{crys,pr}}$, and $\mathrm{Fil}^1_{\mathcal{N}}\,\mathbb{\Delta}_{R'}$ is identified with $\ker\theta_{R'}$ (see for instance [8, Proposition 5.3.6]). $\square$

Theorem 11.2.7 can now be applied to give a description of p-divisible groups of unramified 'EL type' in terms of apertures for appropriate group-theoretical data. As an example, we can combine it with Proposition 10.4.3 to get:

Proposition 11.7.4. *Let $\mathcal{O}$ and (T_0, μ_0) be as in § 10.4. Then, for any discrete, derived p-complete $\mathcal{O}$-algebra R in CRing^f, $\mathrm{BT}^{T_0,\mu_0}_\infty(R)$ is canonically equivalent to the groupoid of Lubin-Tate formal $\mathcal{O}$-modules over R of height h and dimension 1.*

Proof. It is well-known that the stack of Lubin-Tate formal $\mathcal{O}$-modules of height h and dimension 1 is represented by a pro-finite étale stack over $\mathrm{Spf}\,\mathcal{O}$ that is isomorphic to the trivial gerbe banded by $T_0(\mathbb{Z}_p)$, the automorphism group of any such Lubin-Tate formal $\mathcal{O}$-module over $\mathrm{Spf}\,\mathcal{O}$.

Therefore, it suffices to establish the stated equivalence on p-completely étale $\mathcal{O}$-algebras R. Here, it is essentially immediate from Proposition 10.4.3, except that we have to show the following: Given $\mathcal{F}$ as in that proposition, the Lie algebra $\mathrm{Lie}(\mathcal{G}(\mathcal{F}))$ of $\mathcal{G}(\mathcal{F}) = \Gamma_{\mathrm{syn}}(\mathcal{F})$ has rank 1, and the resulting action of $\mathcal{O}\otimes_{\mathbb{Z}_p}\mathcal{O}$ on the Lie algebra factors through the structure map $\mathcal{O} \to R$. This is immediate from Proposition 11.7.3. $\square$

11.8. **The étale realization.**
For completeness, we will finish by reviewing a result of Mondal.

11.8.1. Let R be a quasisyntomic ring. Then Bhatt shows that there is a canonical functor [6, Construction 6.3.1]

$$T_{\mathrm{\acute{e}t}} : \mathrm{Perf}(R^{\mathrm{syn}}) \to D^b_{\mathrm{lisse}}(\mathrm{Spf}(R)^{\mathrm{ad}}_\eta, \mathbb{Z}_p)$$

where the right hand side is the bounded derived category of lisse $\mathbb{Z}_p$-sheaves on the adic generic fiber $\mathrm{Spf}(R)_\eta \overset{\mathrm{defn}}{=} \mathrm{Spa}(R[1/p], R)$.

In particular, given $\mathcal{M} \in \mathrm{Vect}^{\mathrm{syn}}_{\{0,1\},n}(R)$, viewed as a perfect complex over R^{syn}, we obtain an object $T_{\mathrm{\acute{e}t}}(\mathcal{M})$ on the right hand side. On the other hand, we can also consider the finite flat p^n-torsion group scheme $G \overset{\mathrm{defn}}{=} \mathcal{G}(\mathcal{M})$ over R, and take its adic generic fiber G^{ad}_η, which can also be viewed as a perfect complex of lisse $\mathbb{Z}_p$-sheaves. The next result follows from [56, Proposition 3.99].

Proposition 11.8.2. *There is a canonical isomorphism $T_{\mathrm{\acute{e}t}}(\mathcal{M}) \xrightarrow{\simeq} G^{\mathrm{ad}}_\eta$.*

Appendix A. Some completeness results

A.1. **Tannaka duality.**

Definition A.1.1. For $R \in \mathrm{CRing}$, $M \in \mathrm{Mod}_R^{\mathrm{cn}}$ is almost perfect if, for all $m \geq 0$, $\tau^{\geq m}M$ is a compact object in the ∞-subcategory $\mathrm{Mod}_R^{\mathrm{cn},\geq m}$ spanned by the m-truncated objects; more generally $M \in \mathrm{Mod}_R$ is almost perfect if, for some $n \geq 0$, $M[n]$ is connective and almost perfect; see for instance [32, Appendix A].

If X is a prestack over R, then $\mathcal{M} \in \mathrm{QCoh}(X)$ is **almost perfect** if, for all $C \in \mathrm{CRing}_{R/}$ and all $x \in X(C)$, $\mathcal{M}_x \in \mathrm{Mod}_C$ is almost perfect.

Write $\mathrm{APerf}(X)$ (resp. $\mathrm{APerf}^{\mathrm{cn}}(X)$) for the symmetric monoidal ∞-category of almost perfect (resp. connective almost perfect) objects in $\mathrm{QCoh}(X)$.

We will need the following result of Bhatt and Halpern-Leistner [7, Theorem 5.1, Lemma 3.13] (see also the discussion in Step 2 of the proof of [32, Proposition 5.1.13]).

Theorem A.1.2 (Tannakian reconstruction)**.** *Suppose that X is a* classical *Noetherian Artin n-stack with quasi-affine diagonal; then, for any classical prestack S, there is an equivalence of ∞-groupoids*

$$\mathrm{Map}(S, X) \xrightarrow[\simeq]{f \mapsto f^*} \mathrm{Fun}^c_{\otimes}(\mathrm{APerf}^{\mathrm{cn}}(X), \mathrm{APerf}^{\mathrm{cn}}(S)).$$

Here the right hand side is the space of symmetric monoidal functors that preserve finite colimits. In particular, if $S' \to S$ is a map of classical prestacks inducing an equivalence of symmetric monoidal ∞-categories

$$\mathrm{APerf}^{\mathrm{cn}}(S) \xrightarrow{\simeq} \mathrm{APerf}^{\mathrm{cn}}(S'),$$

then the natural map

$$\mathrm{Map}(S, X) \to \mathrm{Map}(S', X)$$

is an equivalence.

A.2. **Completeness.**

Proposition A.2.1. *Suppose that R is a complete Noetherian local ring with maximal ideal $\mathfrak{m}$. Then, for any relatively locally finitely presented derived Artin stack $\mathcal{X} \to B\mathbb{G}_m \times \operatorname{Spec} R$ with quasi-affine diagonal, the map*

$$\mathrm{Map}_{/B\mathbb{G}_m \times \operatorname{Spec} R}(B\mathbb{G}_m \times \operatorname{Spec} R, \mathcal{X}) \to \varprojlim_m \mathrm{Map}_{/B\mathbb{G}_m \times \operatorname{Spec} R}(B\mathbb{G}_m \times \operatorname{Spec} R/\mathfrak{m}^m, \mathcal{X})$$

is an equivalence.

Proof. We can replace $\mathcal{X}$ with its classical truncation $X \overset{\mathrm{defn}}{=} \mathcal{X}_{\mathrm{cl}}$. By Theorem A.1.2, we are now reduced to the easy observation that the functor

$$\mathrm{APerf}^{\mathrm{cn}}(B\mathbb{G}_m \times \operatorname{Spec} R) \to \varprojlim_m \mathrm{APerf}^{\mathrm{cn}}(B\mathbb{G}_m \times \operatorname{Spec} R/\mathfrak{m}^m)$$

is an equivalence. $\square$

Proposition A.2.2. *Suppose that we have $R \in$* CRing *and a relatively locally finitely presented derived Artin stack $\mathcal{X} \to \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$ with quasi-affine diagonal. Then for any Noetherian $B \in \mathrm{CRing}_{\heartsuit, R/}$ the natural map*

$$\mathrm{Map}_{/\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R}(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} B, \mathcal{X}) \to \varprojlim_n \mathrm{Map}_{/\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R}((\mathbb{A}^1/\mathbb{G}_m)_{(t^n=0)} \times \operatorname{Spec} B, \mathcal{X})$$

is an equivalence.

Proof. Once again, we can replace $\mathcal{X}$ with its classical truncation, and so it suffices to know that the map

$$\mathrm{APerf}^{\mathrm{cn}}(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} B) \to \varprojlim_n \mathrm{APerf}^{\mathrm{cn}}((\mathbb{A}^1/\mathbb{G}_m)_{(t^n=0)} \times \operatorname{Spec} B)$$

is an equivalence.

This boils down to the fact that filtered almost perfect complexes over B are *complete* for their filtration, since they are inverse limits of complexes $\mathrm{Fil}^\bullet M$ with $\mathrm{Fil}^i M \simeq 0$ for i sufficiently large. $\square$

Lemma A.2.3. *Let $M_\bullet$ be a graded almost perfect module over a non-positively graded animated commutative ring $B_\bullet$. Write*

$$\overline{M}_\bullet = B_0 \otimes_{B_\bullet} M_\bullet$$

for the graded base change of $M_\bullet$. Then $M_\bullet$ admits a canonical decreasing filtration $\mathrm{Fil}^\bullet_{\mathrm{wt}} M_\bullet$ in $\mathrm{GrMod}_{B_\bullet}$ with

$$\mathrm{gr}^i_{\mathrm{wt}} M_\bullet \simeq B_\bullet(-i) \otimes_{B_0} \overline{M}_i.$$

Proof. This is shown as in the proof of [31, Theorem 2.44]. Here is a sketch: If we construct the filtration on $\tau^{\geq m} M_\bullet$ for all m, the inverse limit will give the desired filtration on $M_\bullet$. Therefore, we can and will assume that $M_\bullet$ is finitely presented as an m-truncated object. This implies in particular that there exists $b \in \mathbb{Z}$ such that $M_k \simeq 0$ for $k > b$.

We now claim that $\overline{M}_k \simeq 0$ for $k > b$ and that the natural map $M_b \to \overline{M}_b$ is an equivalence. To see this, observe that we have a fiber sequence

$$B_{\leq -1} \otimes_{B_\bullet} M_\bullet \to M_\bullet \to \overline{M}_\bullet,$$

and note that the left hand side is a graded module supported in degrees $\leq b-1$. This last observation can be checked using the bar resolution as in [31, Lemma 2.45].

Next, we claim that there exists $a \leq b$ such that $\overline{M}_k \simeq 0$ for $k < a$.

The proof will now proceed by induction on $b-a$, the case $b = a$ having already been verified above.

We can use the equivalence $M_b \xrightarrow{\simeq} \overline{M}_b$ to obtain a canonical map

$$B_\bullet \otimes_{B_0} \overline{M}_b(-b) \xrightarrow{\simeq} B_\bullet \otimes_{B_0} M_b(-b) \to M_\bullet$$

whose cofiber $M'_\bullet$ is supported in degrees $< b$. Graded base-change to B_0 yields a cofiber sequence

$$\overline{M}_b(-b) \to \overline{M} \to \overline{M}',$$

which shows that $\overline{M}_i \xrightarrow{\simeq} \overline{M}'_i$ for $i \leq b-1$ and $\overline{M}'_i \simeq 0$ for $i > b-1$.

By our inductive hypothesis, $M'_\bullet$ admits a filtration $\mathrm{Fil}^\bullet_{\mathrm{wt}} M'_\bullet$ with

$$\mathrm{gr}^i_{\mathrm{wt}} M'_\bullet \simeq B_\bullet \otimes_{B_0} \overline{M}'_i(-i) \simeq B_\bullet \otimes_{B_0} \overline{M}_i(-i)$$

for $i \leq b-1$.

We now obtain our desired filtration on $M_\bullet$ by setting

$$\mathrm{Fil}^i_{\mathrm{wt}} M_\bullet = \begin{cases} 0 & \text{if } i > b \\ \mathrm{Fil}^i_{\mathrm{wt}} M'_\bullet \times_{M'_\bullet} M_\bullet & \text{if } i \leq b. \end{cases}$$

□

Proposition A.2.4. *Suppose that $B_\bullet$ is a non-positively graded discrete commutative ring and that $\mathcal{X} \to \mathcal{Y} \overset{\mathrm{defn}}{=} \mathrm{Spec}(B_\bullet)/\mathbb{G}_m$ is a relatively locally finitely presented derived Artin stack with quasi-affine diagonal. Then the natural map*

$$\mathrm{Map}_{/\mathcal{Y}}(\mathcal{Y}, \mathcal{X}) \to \varprojlim_m \mathrm{Map}_{/\mathcal{Y}}(\mathrm{Spec}(B_{\geq -m})/\mathbb{G}_m, \mathcal{X})$$

is an equivalence.

Proof. We can replace $\mathcal{X}$ with its classical truncation, and by Noetherian approximation, we can also assume that $\mathcal{X}$ is the base-change over $\mathcal{Y}$ along a map $\mathcal{Y} \to \mathcal{Y}'$ of a locally finitely presented map of Noetherian stacks $\mathcal{X}' \to \mathcal{Y}'$ with quasi-affine diagonal. Thus, we are reduced via Theorem A.1.2 to knowing that the functor

$$\mathrm{APerf}^{\mathrm{cn}}(\mathcal{Y}) \to \varprojlim_m \mathrm{APerf}^{\mathrm{cn}}(\mathcal{Y}_m)$$

is an equivalence, where $\mathcal{Y}_m = (\mathrm{Spec}\, B_{\geq -m})/\mathbb{G}_m$.

Let us first check full faithfulness: we need to know that, for $M_\bullet, N_\bullet$ on the left hand side, the natural maps

$$\begin{aligned}\mathrm{Map}_{B_\bullet}(M_\bullet, N_\bullet) &\to \varprojlim_m \mathrm{Map}_{B_{\geq -m}}(B_{\geq -m}\otimes_{B_\bullet} M_\bullet, B_{\geq -m}\otimes_{B_\bullet} N_\bullet)\\ &\xrightarrow{\simeq} \varprojlim_m \mathrm{Map}_{B_\bullet}(M_\bullet, B_{\geq -m}\otimes_{B_\bullet} N_\bullet)\\ &\xrightarrow{\simeq} \mathrm{Map}_{B_\bullet}(M_\bullet, \varprojlim_m B_{\geq -m}\otimes_{B_\bullet} N_\bullet)\end{aligned}$$

are isomorphisms. This comes down to knowing that the natural map

$$N_\bullet \to \varprojlim_m B_{\geq -m}\otimes_{B_\bullet} N_\bullet$$

is an isomorphism. Using Lemma A.2.3, we can reduce to the case where $N_\bullet \simeq B_\bullet \otimes_{B_0} N'$, where N' is an almost perfect connective complex of B_0-modules. This reduces us to knowing that

$$B_\bullet \otimes_{B_0} N' \to \varprojlim_m B_{\geq -m}\otimes_{B_0} N'$$

is an equivalence. Comparing degree-by-degree, we are reduced to the case where $N' \simeq B_0$, where the assertion is clear.

For essential surjectivity, suppose that we are given an object on the right yielding a compatible sequence $(M_\bullet^{(m)})_m$ of graded almost perfect connective $B_{\geq -m}$-modules $M_\bullet^{(m)}$. We need to check that $M_\bullet = \varprojlim_m M_\bullet^{(m)}$ is a graded almost perfect connective $B_\bullet$-module, and that its graded base-change to $B_{\geq -m}$ yields $M_\bullet^{(m)}$. Once again, using Lemma A.2.3, we can reduce to the case where the compatible sequence is of the form $(B_{\geq -m}\otimes_{B_0} M)_m$ for some almost perfect connective complex of B_0-modules M. Here, the claim is obvious. ☐

A.2.5. Suppose that $\mathrm{Fil}^\bullet S$ is a filtered animated commutative ring with the following properties:

- $\mathrm{Fil}^i S$ is discrete for all $i \in \mathbb{Z}$;
- The map $\mathrm{Fil}^1 S \to S$ is identically zero.

Consider the ideal $J \subset \mathrm{Rees}(\mathrm{Fil}^\bullet S)$ of the Rees algebra given by

$$J = \bigoplus_{i=1}^{\infty} \mathrm{Fil}^i S \cdot t^{-i} \subset \bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^i S\cdot t^{-i} = \mathrm{Rees}(\mathrm{Fil}^\bullet S).$$

This is a homogeneous ideal, and so, for each $m \geq 1$, we can consider the quotient $\mathrm{Rees}(\mathrm{Fil}^\bullet S)/J^m$: it corresponds to a filtered animated commutative ring structure $\mathrm{Fil}^\bullet_{(m)} S$ on S with

$$\mathrm{Fil}^i_{(m)} S = \mathrm{Fil}^i S / \left(\sum_{\substack{k_1+\ldots+k_m=i\\ k_i\geq 1}} \mathrm{im}(\mathrm{Fil}^{k_1} S \otimes_S \cdots \otimes_S \mathrm{Fil}^{k_m} S)\right).$$

Proposition A.2.6. *Suppose that we have a relatively locally finitely presented derived Artin stack $\mathcal{X} \to \mathcal{Z} \overset{\mathrm{defn}}{=} \mathcal{R}(\mathrm{Fil}^\bullet S)$ with quasi-affine diagonal. Then the natural map*

$$\mathrm{Map}_{/\mathcal{Z}}(\mathcal{Z}, \mathcal{X}) \to \varprojlim_m \mathrm{Map}_{/\mathcal{Z}}(\mathcal{R}(\mathrm{Fil}^\bullet_{(m)} S), \mathcal{X})$$

is an equivalence.

Proof. As above, we reduce to knowing that

$$\mathrm{APerf}^{\mathrm{cn}}(\mathcal{R}(\mathrm{Fil}^\bullet S)) \to \varprojlim_m \mathrm{APerf}^{\mathrm{cn}}(\mathcal{R}(\mathrm{Fil}^\bullet_{(m)} S))$$

is an equivalence, which in turn follows easily from the observation that the map $\mathrm{Fil}^i S \to \mathrm{Fil}^i_{(m)} S$ is an equivalence for all $i < m$. ☐

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