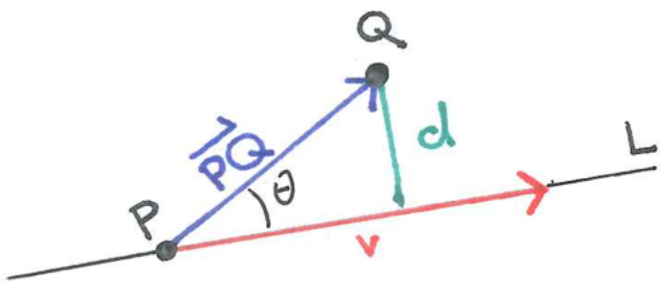
Section 3 Math 2202
Distances between Points, Lines and Planes

1. Find the distance from the point $Q = (2, -3, 1)$ to the line $L : x = 3 - t, y = 1 + 4t, z = 6$. (By 'distance', remember we mean the shortest distance between Q and any point on L.)

   Can you find the coordinates of the point on L closest to Q?

   Solution:



   As you can see in the figure above, the smallest distance between the point Q and the line L is given by $d = |\mathbf{PQ}| \sin \theta$. Since we do not explicitly know θ, it becomes necessary to use another method to make this computation. Luckily we have the projection

$$d = |\mathbf{PQ}| \sin \theta = \frac{|\mathbf{PQ} \times \mathbf{v}|}{|\mathbf{v}|}$$

   Computing the cross product for our given point Q and line L will give, numerically, the smallest distance between Q and L. Unfortunately, we still don't know much about **v**.

   In order to find the coordinates of the point on L closest to Q we can use the distance formula

$$\begin{aligned} d^2 &= (2 - 3 + t)^2 + (-3 - 1 - 4t)^2 + (1 - 6)^2 \\ &= (t - 1)^2 + (-4 - 4t)^2 + 25 \\ &= (t^2 - 2t + 1) + (16t^2 + 32t + 16) + 25 \\ &= 17t^2 + 30t + 42, \end{aligned}$$

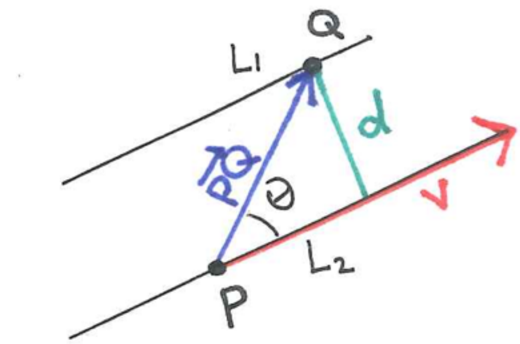
   and then solve for t to get t = -15/17. This tells us that d = 489/17 and the point on L closest to Q has coordinates (66/17, 111/17, 6).

2. For each case, how could you find the distance between two lines L_1 and L_2, using what we know about distance from a point to a line or to a plane?

   - L_1 and L_2 intersect
   - L_1 and L_2 are parallel
   - L_1 and L_2 are skew

   Solution: When L_1 and L_2 intersect, the smallest distance between the two lines is zero (which occurs at the point of intersection).

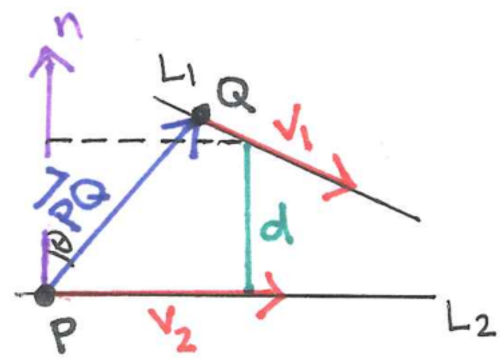
When L_1 and L_2 are parallel, we can use the methods in problem 1 to find the smallest distance between the two lines



so that

$$d = |\mathbf{PQ}| \sin\theta = \frac{|\mathbf{PQ} \times \mathbf{v}|}{|\mathbf{v}|}$$

When L_1 and L_2 are skew, this means that L_1 and L_2 lie in parallel planes.



Since parallel planes have the same normal unit vector, we can can compute it using the cross-product

$$\hat{\mathbf{n}} = \frac{\mathbf{v_1} \times \mathbf{v_2}}{|\mathbf{v_1} \times \mathbf{v_2}|}$$

Then projecting the vector $\mathbf{PQ}$ onto the normal vector, we get the minimal distance between L_1 and L_2

$$d = |\mathbf{PQ}||\cos\theta| = |\mathbf{PQ} \cdot \hat{\mathbf{n}}| = \frac{|\mathbf{PQ} \cdot (\mathbf{v_1} \times \mathbf{v_2})|}{|\mathbf{v_1} \times \mathbf{v_2}|}.$$

3. Find the distance between the given lines.

$$\begin{aligned} L_1 : \quad & x = 1 + 2t, \quad & y = 3t, \quad & z = 2 - t \\ L_2 : \quad & x = -1 + s, \quad & y = 4 + s, \quad & z = 1 + 3s. \end{aligned}$$

Solution: Here's how we know the lines are skew (they lie in parallel planes).

The direction vectors of the lines are $\mathbf{v}_1 = \langle 2, 3, -1 \rangle$ and $\mathbf{v}_2 = \langle 1, 1, 3 \rangle$, respectively. Since the direction vectors are not parallel, the lines are not parallel. Do the lines intersect? To see if this is the case, let us simultaneously solve the two equations: $1 + 2t = -1 + s$, $3t = 4 + s$

and $2 - t = 1 + 3s$. The y equations $3t = 4 + s$ gives us $s = 3t - 4$. Plugging this into the x equations, $1 + 2t = -1 + s$, we find $1 + 2t = -1 + (3t - 4)$. From this, we get $t = 6$ and $s = 14$. But now we check the z equations, $2 - t = 1 + 3s$, and see that when $t = 6$ an $s = 14$ this is equation fails: $-4 \neq 53$. Thus we have a contradiction, and lines do *not* intersect. That is, the lines are skew.

4. For each, determine the $x = k$ traces, the $y = k$ traces and the $z = k$ traces. Identify the quadric surface as a hyperboloid, cone, paraboloid, or ellipsoid, and give a rough sketch. If appropriate, describe any axes of symmetry.

   - $y = x^2 + 4z^2$.
   - $z = y^2 - x^2$

   Taking the $x = k$ trace here means setting x equal to the constant value k. For example, doing this with $y = x^2 + 4z^2$ gives $y = k^2 + 4z^2$, which is the equation of a parabola.