Quiz 3 Math 2202

Guidelines

- This quiz is for you to test yourself on what we've been studying recently.
- Full credit is given for taking the quiz, if you arrive on time to start. No credit otherwise.
- You have 10 minutes. As a section, we will go over the quiz (or part of it). Solutions will be posted online as well.

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1. Write the equation for the following planes in linear equation form. Are any of these planes are parallel? Are any perpendicular to each other?

   (a) xy-plane
   (b) zy-plane
   (c) $y = 5$
   (d) $y = \frac{3}{5}x - 2z + 1$

   Solution:

   (a) $z = 0$
   (b) $x = 0$
   (c) $y - 5 = 0$
   (d) $\frac{3}{5}x - y - 2z + 1 = 0$

2. Find the equation of a plane parallel to the plane $y = \frac{3}{5}x - 2z + 1$ and containing the point $(0, 3, 4)$. Is the vector $\langle 0, 3, 4\rangle$ parallel to this plane?

   Solution:

   Let P be the plane parallel to the plane $y = \frac{3}{5}x - 2z + 1$ and containing the point $(0, 3, 4)$.

   The plane $y = \frac{3}{5}x - 2z + 1$ can be written in linear equation form as $\frac{3}{5}x - y - 2z + 1 = 0$, so it has normal vector $\mathbf{n} = \langle \frac{3}{5}, -1, -2\rangle$. Since parallel planes have parallel normal vectors, $\mathbf{n}$ is also a normal vector to P.

   Since P contains the point $(0, 3, 4)$ and has normal vector $\mathbf{n} = \langle \frac{3}{5}, -1, -2\rangle$,

   $$P = \{(x, y, z) | \frac{3}{5}(x-0) + (-1)(y-3) + (-2)(z-4) = 0\} = \{(x, y, z) | \frac{3}{5}x - (y-3) - 2(z-4) = 0\}$$

   The vector $\mathbf{v} = \langle 0, 3, 4\rangle$ is parallel to P if $\mathbf{v}$ is perpendicular to the normal vector $\mathbf{n}$. And,

   $$\begin{aligned} \mathbf{v} \cdot \mathbf{n} &= \langle 0, 3, 4\rangle \cdot \langle \frac{3}{5}, -1, -2\rangle \\ &= 0 \cdot \frac{3}{5} + 3 \cdot (-1) + 4 \cdot (-2) \\ &= -11 \end{aligned}$$

   Since $\mathbf{v} \cdot \mathbf{n} = -11 \neq 0$, $\mathbf{v}$ and $\mathbf{n}$ are not perpendicular, and so $\mathbf{v} = \langle 0, 3, 4\rangle$ is not parallel to the plane P.

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3. Let $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ be non-zero vectors. Which of the following is a meaningful quantity? If so, is it a scalar or a vector?

   (a) $|\mathbf{a} \times \mathbf{b}|$

   (b) $|\mathbf{a}| \times |\mathbf{b}|$

   (c) $|\mathbf{a} \cdot \mathbf{b}|$

   (d) $|\frac{\mathbf{a} \cdot \mathbf{b}}{|\mathbf{b}|}|$

   (e) $\frac{|\mathbf{a} \times \mathbf{b}|}{|\mathbf{b}|}$

   (f) $(\mathbf{a} \cdot \mathbf{b}) \times \mathbf{c}$

   (g) $(\mathbf{a} \times \mathbf{b}) \cdot \mathbf{c}$

   For those that are meaningful quantities, what do they measure? Sketch a picture of vectors $\mathbf{a}$, $\mathbf{b}$ and $\mathbf{c}$ to illustrate, when appropriate.

   Solution:

   (a) $|\mathbf{a} \times \mathbf{b}|$ is a scalar, equal to the area of the parallelogram determined by $\mathbf{a}$ and $\mathbf{b}$.

   (b) $|\mathbf{a}| \times |\mathbf{b}|$ is not a meaningful quanitity, because we can only take cross products of two vectors, and $|\mathbf{a}|, |\mathbf{b}|$ are real numbers.

   (c) $|\mathbf{a} \cdot \mathbf{b}|$ is a scalar. Since $\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos\theta$, $0 \leq |\mathbf{a} \cdot \mathbf{b}| \leq |\mathbf{a}||\mathbf{b}|$. Also, $|\mathbf{a} \cdot \mathbf{b}| = 0$ exactly when $\mathbf{a}$ and $\mathbf{b}$ are orthogonal, and $|\mathbf{a} \cdot \mathbf{b}| = |\mathbf{a}||\mathbf{b}|$ exactly when $\mathbf{a}$ and $\mathbf{b}$ are parallel.

(d) $|\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|}|$ is a scalar, equal to the length of the projection vector $\mathrm{proj}_{\mathbf{b}}\mathbf{a}$:

$$|\mathrm{proj}_{\mathbf{b}}\mathbf{a}| = \left| \left(\frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} \right) \frac{\mathbf{b}}{|\mathbf{b}|} \right| = \left| \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} \right|$$

You can also see that like this:

$$||\mathbf{a}|\cos\theta| = \frac{|\mathbf{a}||\mathbf{b}|\cos\theta|}{|\mathbf{b}|} = \left| \frac{\mathbf{a}\cdot\mathbf{b}}{|\mathbf{b}|} \right|$$

(e) $\frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|}$ is a scalar, equal to the length of the orthogonal projection of $\mathbf{a}$ onto $\mathbf{b}$:

$$||\mathbf{a}|\sin\theta| = \frac{|(|\mathbf{a}||\mathbf{b}|\sin\theta)|}{|\mathbf{b}|} = \frac{|\mathbf{a}\times\mathbf{b}|}{|\mathbf{b}|}$$

(f) $(\mathbf{a}\cdot\mathbf{b})\times\mathbf{c}$ is not a meaningful quantity, because we can only take cross products of two vectors, and $(\mathbf{a}\cdot\mathbf{b})$ is a scalar.

(g) $(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ is a scalar, and $|(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}| = \pm(\mathbf{a}\times\mathbf{b})\cdot\mathbf{c}$ is the volume of the parallelepiped determined by the vectors $\mathbf{a}, \mathbf{b}$, and $\mathbf{c}$:

Think about it... Find the distance from the point $Q = (2, -3, 1)$ to the line $L : x = 3 - t, y = 1 + 4t, z = 6$. (By 'distance', remember we mean the shortest distance between Q and any point on L.) Can you find the coordinates of the point on L which is closest to Q?