Automorphic Vector Bundles

[compact dual]

To a Shimura datum (G,X) we will associate a compact dual symm. Hermitian space $\check{X}$. Given $x \in X$, μ_x induces decreasing filtration $\mathrm{Filt}(\mu_x)$ of $\mathrm{Rep}_{\mathbb{C}}(G)$. [We have functorial grading on objects inducing filtration on objects.]

Remark: Here we use Tannakian formalism, but we could avoid a lot of abstraction if we simply fix a faithful representation.

Def: $\check{X}$ is $G(\mathbb{C})$-conj. class of ~~fixed faith~~ filtrations of fixed faithful rep. [quotient of $G(\mathbb{C})$ by a parabolic]

We have Borel emb. $X \overset{\beta}{\hookrightarrow} \check{X}$, embedding X as open complex submfld.

[Useful example comes from looking at SL_2.]

Conjugation of Shimura Varieties

$\tau \in \mathrm{Aut}(\mathbb{C})$, $x \in X$ special $\rightsquigarrow$ $\tau\mathrm{Sh}(G,X) := \mathrm{Sh}(\overbrace{{}^{\tau,x}G, {}^{\tau,x}X}^{\text{inner twist}})$ which has certain characteristic properties. In particular, it is independent of the choice of x (in a way which we can describe explicitly, w/ conjugation involved when we change our choice).

Take an (alg.) G-vector bundle J over $\check{X}$ and pullback by β, which we require to have a $G(\mathbb{R})$-action.

For K suff. small, $V_K(J) := G(\mathbb{Q}) \backslash \beta^*(J) \times G(\mathbb{A})_f / K$ is an <u>automorphic vector bundle</u> (which is algebraic!).

We want to vary the choice of K, and get $V(J)$ independent of any choice.

Key: We can descend from an alg. vector bundle on $\check{X}$ to alg. var. on a Shimura var., which is surprising since we pass through the non-alg. X! Getting things defined over a # field here is mostly hard work of Michael Harris.

$GL_2 \rightsquigarrow$ compact dual $\mathbb{P}^1$ $(/\mathbb{C})$ – sections of exterior powers of tautological bundle correspond to (Siegel) modular forms.

Universal Lie algebra gives us an automorphic vector bundle