Given field ext. $k/\mathbb{Q}$, define $\mathcal{X}(k) := G(k)\backslash \mathrm{Hom}(\mathbb{G}_{m,k}, G_k)$. Picking some $x \in X$ gives μ_x which determines $\mu \in \mathcal{X}(\mathbb{C})$ independent of x.

<u>Observation</u>: $k \hookrightarrow k'$ determines $\mathcal{X}(k) \to \mathcal{X}(k')$. So, $\mathrm{Aut}(\mathbb{C})$ acts on $\mathcal{X}(\mathbb{C})$.

<u>Def</u>: Let $H \leq \mathrm{Aut}(\mathbb{C})$ be stabilizer of $\mu \in \mathcal{X}(\mathbb{C})$ and define <u>reflex field</u> $E(G,X) := \mathbb{C}^H \subseteq \mathbb{C}$.

So, $\sigma \in \mathrm{Aut}(\mathbb{C})$ trivial on $E(G,X)$ iff $\forall x \in X$ μ_x and μ_x^σ are $G(\mathbb{C})$-conjugate.

<u>Warning</u>: There may be no $x \in X$ s.t. μ_x is defined over $E(G,X)$.

We seek "canon. model" of $Sh_K(G,X) = G(\mathbb{Q})\backslash X \times G(\mathbb{A}_f)/K$ over $E(G,X)$. [Should <u>not</u> depend on K, so we can vary the level structure.]

<u>Def</u>: Torus T is <u>split</u> if $T \cong \mathbb{G}_m \times \cdots \times \mathbb{G}_m$. Reductive grp. G is <u>split</u> if it contains maximal torus which is split.

<u>Example</u>:
- GL_n, Sp_{2n}, GSp_{2n}, SL_n split over any field.
- E/F quadratic ext. w/ V hermitian space $\Rightarrow U(V)$, $SU(V)$, $GU(V)$ non-split.
- V quadratic space over $F \Rightarrow SO(V)$ is reductive group over F. Define n by $\dim V = \begin{cases} 2n, & \dim V \text{ even}, \\ 2n+1, & \dim V \text{ odd}. \end{cases}$

  Then, $SO(V)$ split iff V contains totally isotropic subspace of dim n.

<u>Prop</u>: (1) Every torus splits over a finite ext.

(2) T split torus over F and $F \hookrightarrow F'$ $\Rightarrow \mathrm{Hom}(\mathbb{G}_m, T) \xrightarrow{\sim} \mathrm{Hom}(\mathbb{G}_{m,F'}, T')$. [$T_{F'}$]

(3) G split over F and $T \subseteq G$ split maximal torus $\Rightarrow W_T \backslash \mathrm{Hom}(\mathbb{G}_m, T) \xrightarrow{\sim} G(F)\backslash \mathrm{Hom}(\mathbb{G}_m, G)$.

W_T = Weyl group = $N_G(T)/C_G(T)$ [normalizer mod centralizer].

<u>Cor</u>: (G,X) Shimura datum and $F \subseteq \mathbb{C}$ s.t. G_F split $\Rightarrow E(G,X) \subseteq F$. In particular, $E(G,X)$ is # field and

G split (e.g., $G = GSp$) $\Rightarrow E(G,X) = \mathbb{Q}$.

Pf: let $T \subseteq G_F$ split maximal torus. $W_{T_{\mathbb{C}}} \backslash \mathrm{Hom}(\mathbb{G}_{m,\mathbb{C}}, T_{\mathbb{C}}) \xrightarrow{\sim} G(\mathbb{C}) \backslash \mathrm{Hom}(\mathbb{G}_{m,\mathbb{C}}, G_{\mathbb{C}}) = X(\mathbb{C})$

$\cong$

$W_T \backslash \mathrm{Hom}(\mathbb{G}_{m,F}, T)$ are all $\mathrm{Aut}(\mathbb{C}/F)$-equivariant. $\square$

(Done since $\mathrm{Aut}(\mathbb{C}/F)$ acts trivially on $\mathrm{Hom}(\mathbb{G}_{m,F}, T)$.)

Example: Let (V,Q) quad. space$/\mathbb{Q}$ of signature $(n,2)$. Recall:

$X := \{$ isotropic lines $\mathbb{C}v \subseteq V_{\mathbb{C}} : [v,\bar{v}] < 0\}$. Set $G := SO(V)$. We need to realize these in terms of a Shimura datum. $X \subseteq \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$ as follows. Given $\mathbb{C}v \in X$, weight 0 Hodge structure on V given by

$$V_{\mathbb{C}} = \underbrace{V^{(1,-1)}}_{\mathbb{C}v} \oplus \underbrace{V^{(0,0)}}_{(\mathbb{C}v \oplus \mathbb{C}\bar{v})^{\perp}} \oplus \underbrace{V^{(-1,1)}}_{\mathbb{C}\bar{v}}$$

. This determines $\mathbb{S}(\mathbb{C}) = \mathbb{C}^\times \times \mathbb{C}^\times \xrightarrow{h} SO(V_{\mathbb{C}})$ via

$$h(z,w) := \begin{cases} z/w, & V^{(1,-1)}, \\ 1, & V^{(0,0)}, \\ w/z, & V^{(-1,1)}. \end{cases}$$

Hodge cochar. is $\mu : \mathbb{C}^\times \to SO(V_{\mathbb{C}})$, $z \mapsto \begin{cases} z, & \mathbb{C}v, \\ 1, & (\mathbb{C}v \oplus \mathbb{C}\bar{v})^{\perp}, \\ z^{-1}, & \mathbb{C}\bar{v}. \end{cases}$

Let $\sigma \in \mathrm{Aut}(\mathbb{C})$ (whose action commutes w/ $\perp$). $\mu^\sigma(z) = \begin{cases} z, & (\mathbb{C}v)^\sigma, \\ 1, & (\mathbb{C}v \oplus \mathbb{C}\bar{v})^{\perp\sigma}, \\ z^{-1}, & (\mathbb{C}\bar{v})^\sigma. \end{cases}$

[$(\mathbb{C}v)^\sigma$ still a line because we can just realize the action via $\mu^\sigma(z) = \sigma(\mu(\sigma^{-1}(z)))$.]

Now check by hand: all decompositions $V_{\mathbb{C}} = \ell \oplus W \oplus \ell'$ w/ ℓ, ℓ' isotropic lines, $[\ell,\ell'] \neq 0$, $W = (\ell+\ell')^\perp$ form a single $SO(V_{\mathbb{C}})$-orbit.