$X \in Sch$, $\Lambda \in CRing$ $\rightsquigarrow$ motivic sheaves $DA^{ét}(X;\Lambda)$

$$\begin{array}{ll} DA^{ét}(X;\Lambda) & \\ \uparrow & \\ DM^{ét}(X;\Lambda) & \text{Voevodsky} \\ \uparrow & \\ SH(X) & \text{Morel-Voevodsky stable homotopy cat.} \end{array}$$

Classical derived categories :

- Constructible sheaves $D(X(\mathbb{C});\Lambda)$
- ℓ-adic sheaves $D(X_{ét};\mathbb{Q}_\ell)$
- Holonomic D-modules $D(Mod(\mathcal{D}_X))$
- Saito mixed Hodge modules $D(MHM(X))$

Step 1 : Linearization

$\mathcal{U} \in Sm/X \rightsquigarrow \mathcal{U}\otimes\Lambda$ presheaf sending $V \in Sm/X$ to free Λ-module w/ basis $Hom_X(V,\mathcal{U})$.

$\rightsquigarrow$ étale sheaf $\Lambda_{ét}(\mathcal{U}) \rightsquigarrow$ functor $\Lambda_{ét}(\cdot) : Sm/X \to Shv(Sm/X;\Lambda)$.

Step 2: $\mathbb{A}^1$- Localization

[derived cat. of étale sheaves on Sm/X]

Idea is to "identify" $\Lambda_{ét}(\mathcal{U})$ and $\Lambda_{ét}(\mathbb{A}^1\times\mathcal{U})$ for $\mathcal{U}\in Sm/X$. We pass to $D(Shv(Sm/X;\Lambda))$.

$\mathcal{T}_{\mathbb{A}^1} \subseteq D(Shv(Sm/X;\Lambda))$ smallest full subcat. closed under arbitrary direct sums, extensions, suspensions, desuspensions, and containing complexes $[\cdots \to 0 \to \underbrace{\Lambda_{ét}(\mathcal{U}) \to \Lambda_{ét}(\mathbb{A}^1\times\mathcal{U})}_{\text{induced by zero section of } \mathbb{A}^1} \to 0 \to \cdots]$

Def: Effective motivic sheaves (effective X-motives) given by Verdier quotient $DA^{eff,ét}(X;\Lambda) := D(Shv(Sm/X;\Lambda))/\mathcal{T}_{\mathbb{A}^1}$.

Really, this is inverting morphisms whose cones belong to $\mathcal{T}_{\mathbb{A}^1}$.

Def: Smooth X-scheme $\mathcal{U}$ has homological effective motive $M^{eff}(\mathcal{U}) \in DA^{eff,ét}(X;\Lambda)$ w/ underlying complex $\Lambda_{ét}(\mathcal{U})[0]$.

Step 3 : Stabilization

We obtain $DA^{ét}(X;\Lambda)$ from $DA^{eff,ét}(X;\Lambda)$ by inverting the so-called Tate object (best done using symm. spectra).

Definition: **A point of inflection** (or **inflection point**) is a place where a continuous function changes concavity, either concave down to concave up or concave up to concave down.

Does $f(x) = 2x^3 - 9x^2 + 12x - 4$ have any inflection points?

f^* has: right adj. f_* ⇝ Grothendieck-Verdier duality
left adj. $f_\#$ (when f is smooth)

Example 3: Does $y = x^3$ have any inflection points? Does $y = x^4$ have any inflection points? Does $y = \frac{1}{x}$ have any inflection points?

<u>Thm</u> (Ayoub, PhD thesis): $f_!$ is well-defined.
↗
(Maybe also $f^!$...)

<u>Proper base change</u>: Cartesian diagram of schemes

$$\begin{tikzcd} Y' \arrow[r, "g'"] \arrow[d, "f'"'] & Y \arrow[d, "f"] \\ X' \arrow[r, "g"'] & X \end{tikzcd}$$

w/ f proper $\Rightarrow \exists$ canon. isom.

$g^* f_* \simeq f'_* g'^*$.

Second Derivative Test:
Suppose $f''(x)$ is continuous near $x = c$.

- If $f'(c) = 0$ and $f''(c) > 0$, then f has a local minimum at c.
- If $f'(c) = 0$ and $f''(c) < 0$, then f has a local maximum at c.
- If $f'(c) = 0$ and $f''(c) = 0$, then this test tells you nothing and you should use a different test (like the First Derivative Test).

Example 4: Find the critical numbers of

$$g(x) = 3x^4 - 4x^3 - 36x^2 - 10$$

and classify them (if possible) as local extrema using the Second Derivative Test. If the Second Derivative Test fails, use the First Derivative Test to classify them as local extrema or say if they aren't local extrema.

Example 5: Find the critical numbers of

$$h(x) = x^4(x + 1)$$

and classify them (if possible) as local extrema using the Second Derivative Test. If the Second Derivative Test fails, use the First Derivative Test to classify them as local extrema or say if they aren't local extrema.

<u>Def</u>: k field, X, Y k-varieties. <u>Finite correspondence</u> from X to Y is linear combination of integral subvar.'s $Z \subseteq X \times Y$ which are finite surj. on conn. component of X. We get a group $\mathrm{Cor}(X, Y)$. Altogether we get additive cat. SmCor/k. Looking at graphs gives inclusion $\mathrm{Sm}/k \hookrightarrow \mathrm{SmCor}/k$.

There is also a notion of transfer. This matters because it allows us to do explicit $\mathbb{A}^1$-localization.

Big Conjectures : Hodge, Tate, Grothendieck and Kontsevich-Zagier on periods

Other Conjectures : Conservativity, Existence of Motivic t-Structure

$\Downarrow$ (Conservativity) Bloch conjecture on surfaces of genus 0

$\Downarrow$ (Existence of Motivic t-Structure) Grothendieck Standard Conjectures