Canon. Models for 0-dim Shimura Varieties

Fix 0-dim Shimura datum $(T, \{h\})$, so T torus$/\mathbb{Q}$ and $h : \mathbb{S} \to T_{\mathbb{R}}$. We have Hodge cochar.

$\mu : \mathbb{G}_{m,\mathbb{C}} \to T_{\mathbb{C}}$ via

$$\begin{tikzcd} \mathbb{C}^\times = \mathbb{G}_m(\mathbb{C}) \arrow[d, "z \mapsto (z,1)"'] \arrow[dr, "\mu"] & \\ \mathbb{C}^\times \times \mathbb{C}^\times = \mathbb{S}(\mathbb{C}) \arrow[r, "h"'] & T(\mathbb{C}) \end{tikzcd}$$

We defined reflex field $E = E(T, \{h\})$ as field of definition of $\mu \in \mathrm{Hom}(\mathbb{G}_{m,\mathbb{C}}, T_{\mathbb{C}})$. Given suff. small $K \le T(\mathbb{A}_f)$ we want to realize finite set $\underbrace{T(\mathbb{Q}) \backslash \{h\} \times T(\mathbb{A}_f)/K}_{(*)}$ as $\mathbb{C}$-pts. of finite étale E-scheme. All we need is suitable action of $\mathrm{Gal}(\bar{E}/E)$. This will factor through $\mathrm{Gal}(\bar{E}/E) \to \mathrm{Gal}(E^{ab}/E)$. [class field theory!]

Recall: $\mathbb{A}_E = \underbrace{E_\infty}_{\text{infinite part}} \times \underbrace{\mathbb{A}_{E,f}}_{\text{finite part}}$, $E_\infty = E \otimes_{\mathbb{Q}} \mathbb{R}$, $\mathbb{A}_{E,f} = E \otimes_{\mathbb{Q}} \mathbb{A}_f$. $E_\infty^\circ \subseteq E_\infty^{\times}$ conn. component of 1.

[Artin map, in Milne's notation]

Class field theory $\Rightarrow \exists$ canon. isom. $\mathrm{art} : E^\times \backslash \mathbb{A}_E^\times / E_\infty^\circ \xrightarrow{\sim} \mathrm{Gal}(E^{ab}/E)$. In particular, we have

(not a typo)

$\mathrm{Gal}(\bar{E}/E) \to \mathrm{Gal}(E^{ab}/E) \to E^\times \backslash \mathbb{A}_E^\times / E_\infty^\times \cong E^\times \backslash \mathbb{A}_{E,f}^\times$. So, just need action of $E^\times \backslash \mathbb{A}_{E,f}^\times$ on $(*)$, which we take to be $T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/K$ for simplicity. Apply $\mathrm{Res}_{E/\mathbb{Q}}$ to $\mu : \mathbb{G}_{m,E} \to T_E$ to get

(expected norm formula)

$\underbrace{\mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_{m,E} \xrightarrow{\mu} \mathrm{Res}_{E/\mathbb{Q}} T_E \xrightarrow{N_{E/\mathbb{Q}}} T}$. On $\mathbb{Q}$-pts., this is $E^\times \xrightarrow{\mu} T(E) \to T(\mathbb{Q}) \subseteq T(\bar{\mathbb{Q}})$.

Composition is r_h (by definition). Looking at $\mathbb{A}_f$-pts. gives

$$\begin{tikzcd} \mathbb{A}_{E,f}^\times \arrow[r, "r_h"] \arrow[d] & T(\mathbb{A}_f) \arrow[d] \\ E^\times \backslash \mathbb{A}_{E,f}^\times \arrow[r, dashed, "r_h"'] & T(\mathbb{Q}) \backslash T(\mathbb{A}_f) \end{tikzcd}$$

So, $E^\times \backslash \mathbb{A}_{E,f}^\times$ acts on $T(\mathbb{Q}) \backslash \{h\} \times T(\mathbb{A}_f)/K$ via $a \cdot [h, t] = [h, \mathrm{art}(a) t]$. So, $(*)$ has canonical model, defined to be the induced E-scheme structure. So, $\exists!$ finite étale $M_K \to \mathrm{Spec}\, E$ s.t. $(*) \cong M_K(\mathbb{C}) = M_K(E^{ab})$ and action is $\mathrm{Gal}(E^{ab}/E)$-equivariant.

(not same E!)

<u>Thm</u>: If $(T, \{h\})$ comes from CM pair (E, Φ) then canon. model M_K of (*) (defined over reflex field of (*)) agrees w/ model that comes from moduli interpretation in terms of CM abelian varieties.

This is essentially the Shimura–Taniyama Main Thm of CM (very deep result). We'll come back to this...

The reflex field in general

(G,X) Shimura datum. Any $x \in X$ is a map $h_x : \mathbb{S} \to G_{\mathbb{R}}$, which defines weight cochar. $w_x : \mathbb{G}_{m,\mathbb{R}} \to G_{\mathbb{R}}$ and Hodge cochar. $\mu_x : \mathbb{G}_{m,\mathbb{C}} \to G_{\mathbb{C}}$.

<u>Q</u>: How do these vary w/ $x \in X$?

All $x \in X$ lie in same $G(\mathbb{R})$-orbit so all things (h_x, w_x, μ_x) are $G(\mathbb{R})$-conjugate.

<u>Prop</u>: $\forall x \in X$: w_x takes values in center of $G_{\mathbb{R}}$ and so w_x does not depend on choice of x.

<u>Pf</u>: Recall relevant axiom for (G,X): in rep. $\mathbb{C}^\times \xrightarrow{h_x} G(\mathbb{R}) \xrightarrow{Ad} GL(\mathfrak{g}_{\mathbb{R}}) \hookrightarrow GL(\mathfrak{g}_{\mathbb{C}})$ only char.'s that appear are $z/\bar{z}, 1, \bar{z}/z$. So, restriction of (*) to $\mathbb{R}^\times \subseteq \mathbb{C}^\times$ is trivial. Hence, $\mathbb{R}^\times \xrightarrow{w_x} G(\mathbb{R})$ takes values in $\ker(Ad) = Z(G(\mathbb{R}))$. $\square$

Potential topics for talks

(1) Isogeny classes vs. isom. classes

(2) Complex mult. for elliptic curves (Silverman Vol. II. Ch. II) ← Could be done by multiple people...

(3) Automorphic vector bundles and sheaves (reference is several papers of Michael Harris, though might be hard to read...)