Affine Schemes

Classical $\to$ Semi-Modern $\to$ Modern

Classical: Let k be a field. $\mathbb{A}^n_k = \mathbb{A}^n := k^n$.

Coordinate ring $I(\mathbb{A}^n) := k[t_1, \dots, t_n]$.

Vanishing locus $V(I) := \{ \alpha \in \mathbb{A}^n : f(\alpha) = 0 \ \forall f \in I \}$

$I \trianglelefteq k[t_1, \dots, t_n]$

We get a topology by considering $V(I)$.

$V(0) = \mathbb{A}^n$ $\qquad$ $V(IJ) = V(I) \cup V(J)$

$V(k[t_1, \dots, t_n]) = \emptyset$ $\qquad$ $V(I+J) = V(I) \cap V(J)$

Semi-Modern: Let $A \in \mathrm{CRing}$. The spectrum $\operatorname{Spec} A$ is the collection of prime ideals of A.

Example: Let k be a field.

- $\operatorname{Spec} k = \{(0)\}$
- $\operatorname{Spec} \mathbb{Z} = \{(0), (2), (3), \dots\}$

$\operatorname{Spec} k[t]/(t^2) = \{(t)\}$

- $\operatorname{Spec} \mathbb{Z}_p = \{(0), (p)\}$

$\operatorname{Spec} k[t]/(t^2) = \{(t)\}$ $\cdot \operatorname{Spec} \mathbb{Z}_{(p)} = \{(0), (p)\}$

The Zariski topology has closed sets (for $I \trianglelefteq A$)

$V(I) := \{ \mathfrak{p} \in \operatorname{Spec} A : I \subseteq \mathfrak{p} \}$.

Given $\varphi \in \operatorname{Hom}_{\mathrm{CRing}}(A, B)$, we get a map

$\operatorname{Spec}(\varphi) : \operatorname{Spec} B \to \operatorname{Spec} A$.

Exercise: Show that $\operatorname{Spec}(\varphi)$ is cont.

Given $I \trianglelefteq A$ and $f \in A$, we get

$A \twoheadrightarrow A/I \rightsquigarrow \operatorname{Spec} A/I \hookrightarrow \operatorname{Spec} A$ w/ image $V(I)$

$A \to A_f \rightsquigarrow \operatorname{Spec} A_f \hookrightarrow \operatorname{Spec} A$ w/ image $D(f)$

Exercise: The sets $D(f)$ (called principal open subsets) are open and form a basis for the Zariski top.

$D(f) = \operatorname{Spec} A \setminus V(f)$ $\{(0), (x-a)\}_{a \in k}$

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$$\shortparallel$$

<u>Example</u>: Let $k = \bar{k}$. $\mathbb{A}^1 = \operatorname{Spec} k[t]$.

The next step is to equip $\operatorname{Spec} A$ w/ a so-called structure sheaf $\mathcal{O}_{\operatorname{Spec} A}$.

<u>Modern</u>:

<u>Def</u>: The cat. of <u>spaces</u> is the cat.

$$\text{Space} := \operatorname{Fun}(\text{CRing}, \text{Set}).$$

Cat.'s $\mathcal{C}, \mathcal{D} \rightsquigarrow \operatorname{Fun}(\mathcal{C}, \mathcal{D}) \ni F, G$

$$\eta \in \operatorname{Hom}_{\operatorname{Fun}(\mathcal{C}, \mathcal{D})}(F, G)$$

$$\Updownarrow$$

$$\{\eta_X : F(X) \to G(X)\}_{X \in \mathcal{C}} \text{ "compatible"}$$

$$\varphi : X \to Y \text{ in } \mathcal{C} \rightsquigarrow \quad F(X) \xrightarrow{\eta_X} G(X)$$

$$\varphi : X \to Y \text{ in } \mathcal{C} \rightsquigarrow \begin{tikzcd} F(X) \arrow[r] \arrow[d, "F(\varphi)"'] \arrow[dr, phantom, "\circlearrowright"] & G(X) \arrow[d, "G(\varphi)"] \\ F(Y) \arrow[r, "\eta_Y"'] & G(Y) \end{tikzcd}$$

Given $X \in \text{Space}$ and $A \in \text{CRing}$, we get the <u>A-valued pts.</u> $X(A)$. $\text{Space}^{op} \simeq \mathcal{P}(\text{CRing})$.

$$\begin{aligned} \text{Space}^{op} &= \text{Fun}(\text{CRing}, \text{Set})^{op} \\ &\simeq \text{Fun}(\text{CRing}^{op}, \text{Set}) =: \mathcal{P}(\text{CRing}) \\ &\simeq \text{Fun}(\text{CRing}, \text{Set}^{op}) \end{aligned}$$

Fix $A \in \text{CRing}$. We get <u>affine scheme</u> $\text{Spec} A \in \text{Space}$:

$$(\text{Spec} A)(B) := \text{Hom}_{\text{CRing}}(A, B).$$

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<u>Example</u>: $(\text{Spec} \mathbb{Z})(A) = \text{Hom}_{\text{CRing}}(\mathbb{Z}, A) = \{*\}$.

$$(\text{Spec} \mathbb{Z}[t])(A) = \text{Hom}_{\text{CRing}}(\mathbb{Z}[t], A) \cong A. \quad (\mathbb{G}_a)$$

$(\mathrm{Spec}\,\mathbb{Z}[t^{\pm 1}])(A) = \mathrm{Hom}_{\mathrm{CRing}}(\mathbb{Z}[t^{\pm 1}], A) \cong A^\times$. $(\mathbb{G}_m)$

Let $X \in \mathrm{Space}$ and $A \in \mathrm{CRing}$. There is a natural map

$\mathrm{Hom}_{\mathrm{Space}}(\mathrm{Spec}\,A, X) \to X(A)$, $F \mapsto F(A)(\mathrm{id}_A)$.

$$\begin{array}{c} (\mathrm{Spec}\,A)(A) \xrightarrow{F(A)} X(A) \\ \| \\ \mathrm{End}(A) \end{array}$$

<u>Claim</u>: This map is a bijection.

This follows from Yoneda's lemma. We have a nat. trans.

$\mathrm{Hom}_{\mathrm{Space}}(\mathrm{Spec}(\cdot), X) \to X$. [adjunction...]

As Ehsan said, this is equiv. to Yoneda.

$\mathrm{CRing} \hookrightarrow \mathrm{Space}$ w/ image Aff Sch

$\mathrm{Space} = \mathrm{Fun}(\mathrm{CRing}, \mathrm{Set})$ $\qquad \bullet \to \bullet \leftarrow \bullet$

"Pointwise evaluation"

$$(X \times_Z Y)(A) = X(A) \times_{Z(A)} Y(A)$$

$$\begin{tikzcd} X \times_Z Y \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ Y \arrow[r] & Z \end{tikzcd}$$

Example: Let X be a top. space. Let $\mathrm{Op}(X)$ be the cat. whose objects are open subsets of X.

$\mathcal{U} \subseteq V$

$$\mathcal{U} \times V = \mathcal{U} \cap V$$
$$\mathcal{U} \amalg V = \mathcal{U} \cup V$$

CRing : Product : $\times$

Coproduct : $\otimes = \otimes_{\mathbb{Z}}$

Final/terminal object : 0

Initial object : $\mathbb{Z}$

$\mathrm{Spec} : \mathrm{CRing}^{op} \to \mathrm{Space}$ w/ image AffSch

Space :

$$\mathrm{Spec}\, A \times \mathrm{Spec}\, B \simeq \mathrm{Spec}(A \otimes B)$$
$$\mathrm{Spec}\, A \amalg \mathrm{Spec}\, B \simeq \mathrm{Spec}(A \times B)$$

Final object : $\operatorname{Spec} \mathbb{Z}$

Initial object : $\operatorname{Spec} 0 = \emptyset$

First Isom. Thm. tells us data of $A \twoheadrightarrow A/I$ w/ $I \trianglelefteq A$

$\updownarrow$ data of $A \twoheadrightarrow B$

<u>Def</u> : A map $\operatorname{Spec} B \to \operatorname{Spec} A$ of affine schemes is a <u>closed embedding</u> (emb.) if the assoc. map $A \to B$ is surj.

A map $f : X \to Y$ of spaces is <u>affine</u> if,

$\forall\, g \in \operatorname{Hom}_{\text{Space}}(\operatorname{Spec} A, Y)$, $\operatorname{Spec} A \times_Y X$ is affine.

$$\begin{tikzcd} \operatorname{Spec} A \times_Y X \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r, "g"'] & Y \end{tikzcd}$$

$Y(A) \leftrightarrow \operatorname{Hom}_{\text{Space}}(\operatorname{Spec} A, Y)$

<u>Def</u> : A map $X \to Y$ of <u>spaces</u> is a <u>closed emb.</u> if it is affine and the induced map $\operatorname{Spec} A \times_Y X \to \operatorname{Spec} A$ is

be a closed emb.

always a closed emb.

<u>Exercise</u>: Check that this is an extension.

$$\begin{tikzcd} \operatorname{Spec} A \times_Y X \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r] & Y \end{tikzcd}$$

$$\operatorname{Spec}(A \otimes_C B) \cong$$

$$\begin{tikzcd} A \otimes_C B & B = C/I \arrow[l] \\ A \arrow[u] & C \arrow[l] \arrow[u] \end{tikzcd} \quad \leftrightsquigarrow \quad \begin{tikzcd} \operatorname{Spec} A \times_{\operatorname{Spec} C} \operatorname{Spec} B \arrow[r] \arrow[d] & \operatorname{Spec} B \arrow[d] \\ \operatorname{Spec} A \arrow[r] & \operatorname{Spec} C \end{tikzcd}$$

$X \in \text{Space} \rightsquigarrow$ subspace

$$\begin{array}{ccc} Z \hookrightarrow X & & Z \to X \\ \updownarrow & \leftsquigarrow & \updownarrow \\ \{Z(A) \hookrightarrow X(A)\} & & \{Z(A) \to X(A)\} \end{array}$$

Subspace $Z \hookrightarrow X \rightsquigarrow X \setminus Z \overset{?}{\hookrightarrow} X$

$$(X \setminus Z)(A) \overset{?}{=} X(A) \setminus Z(A)$$

$\subseteq X(A)$ ... (Spec A, X)

$(X \setminus Z)(A) =$ [illegible]

$$\subseteq X(A) \longleftrightarrow \mathrm{Hom}_{\text{space}}(\operatorname{Spec} A, X)$$

$$D_x: \quad \begin{tikzcd} \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Z \arrow[d, hook] \\ \operatorname{Spec} A \arrow[r, "x"'] & X \end{tikzcd}$$

<u>Def</u>: Open emb. is complement of closed emb

$$(X \setminus Z)(A) = \{ x \in X(A) : D_x \text{ is a fiber square} \}$$

<u>Claim</u>: $\operatorname{Spec} A \setminus \operatorname{Spec} A/f \cong \operatorname{Spec} A_f$.