K-theory and G-theory of projective bundles and derived blow-ups (plus miscellany)

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Overview

break?

1. Finiteness conditions
2. Operations in K-theory
3. Operations in G-theory
4. Blow-ups
5. K- and G-theory of blow-ups
6. End

The Noetherian assumption

Convention: everything derived, $(\mathcal{S}\mathrm{p}, \otimes)$ is the symmetric monoidal category of spectra.

Definition

A ring $A \in \mathrm{s}\mathcal{R}\mathrm{ing}$ is *Noetherian* is $\pi_0 A$ is Noetherian and each $\pi_n A$ is finitely generated ($=$ finitely presented) over $\pi_0 A$.

Definition

An algebraic stack X is *Noetherian* if it is qcqs and if for any smooth map $\operatorname{Spec} A \to X$, the ring A is Noetherian.

Throughout, we assume all algebraic stacks to be Noetherian, hence all rings to be Noetherian.

Perfect modules

Definition

Let $A \in \mathrm{s}\mathcal{R}\mathrm{ing}$.

- The *category of finitely presented modules* $\mathcal{M}\mathrm{od}_A^{fp}$ is the smallest stable subcategory of $\mathcal{M}\mathrm{od}_A$ which contains A.
- The *category of perfect modules* is the closure of $\mathcal{M}\mathrm{od}_A^{fp}$ under extensions in $\mathcal{M}\mathrm{od}_A$.

Lemma

$M \in \mathcal{M}\mathrm{od}_A$ is finitely presented if and only if it is obtained from 0 by a finite number of cell attachments.

Perfect modules

Lemma

$M \in \mathcal{M}\mathrm{od}_A$ *is perfect iff it is compact (iff it is dualizable)*

let $M \to K \to M$, w/ $M \in \mathrm{Perf}_A$, $K \in \mathrm{Mod}_A^{fp}$.

let $N = \mathrm{colim}\, N_\alpha$ filt. Then:

⊛: filt. colims commute past fin. lims.

$$\begin{tikzcd}
\mathrm{Mod}_R(M,N) \arrow[r] \arrow[rr, bend left, "\mathrm{id}"] & \mathrm{Mod}_R(K,N) \arrow[r] & \mathrm{Mod}_R(M,N) \\
\mathrm{colim}\, \mathrm{Mod}_R(M,N_\alpha) \arrow[r] \arrow[u] & \mathrm{colim}\, \mathrm{Mod}_R(K,N_\alpha) \arrow[r] \arrow[u, "\cong\ \circledast"] & \mathrm{colim}\, \mathrm{Mod}_R(M,N_\alpha) \arrow[u] \\
F \arrow[r] \arrow[u] & 0 \arrow[r] \arrow[u] & F \arrow[u]
\end{tikzcd}$$

$\therefore M$ compact

M dualizable iff $N \mapsto N \otimes M$ pres. lims.

use $\underline{\mathrm{Map}}(M,-) \cong M^\vee \otimes (-)$

Conversely: M compact, $M = \mathrm{colim}\, M_\alpha$ (filtered, M_α fin. pres.)

$\mathrm{id} \in \mathrm{Mod}_R(M,M) \cong \mathrm{colim}\, \mathrm{Mod}_R(M,M_\alpha)$ $\rightsquigarrow$
$$\begin{tikzcd}
& M_\alpha \arrow[dr, dashed] & \\
M \arrow[ur, dashed] \arrow[rr, "\mathrm{id}"'] & & M
\end{tikzcd}$$
$\rightsquigarrow$ M perfect

Coherent modules

Definition

fin. pres.

$M \in \mathcal{M}\mathrm{od}_A$ is *coherent* if $\pi_n M$ if ~~finitely generated~~ over $\pi_0 A$ for all n, and M has bounded homotopy. Notation: $\mathcal{C}\mathrm{oh}(A)$.[a]

[a] Lurie does not demand the boundedness assumption (e.g. in SAG). We want this due to the Eilenberg-Mazur swindle.

$$\left[\bigoplus_{n\in\mathbb{Z}} A[2n]\right] = \left[\bigoplus_{n\in\mathbb{Z}} A[2n]\right] + [A]$$

$$\therefore [A] = 0$$

Coherent modules

Definition

$M \in \mathcal{M}\mathrm{od}_A$ is *coherent* if $\pi_n M$ if finitely generated over $\pi_0 A$ for all n, and M has bounded homotopy. Notation: $\mathcal{C}\mathrm{oh}(A)$.

Lemma

If $R \in \mathcal{C}\mathrm{oh}(A)$, then $\mathcal{P}\mathrm{erf}(A) \subset \mathcal{C}\mathrm{oh}(A)$.

- $R \in \mathrm{Coh}(A) \Rightarrow \mathrm{Mod}_A^{perf} \subseteq \mathrm{Coh}(A)$ (stability)
- suffices: $\mathrm{Coh}(A)$ stable under retracts.

$$K \in \mathrm{Coh}(A) \overset{?}{\Rightarrow} M \in \mathrm{Coh}(A)$$

$$\begin{tikzcd} M \arrow[r] \arrow[rr, bend left, "id"] & K \arrow[r] & M \end{tikzcd}$$

"M retract of K"

- Now each $\pi_n M$ is fin. gen., hence fin. pres., / $\pi_0 A$
- $\pi_n K = 0 \Rightarrow \pi_n M = 0$

Global versions & K-theory (once more)

Write $\mathcal{A}\mathrm{rt}$ for the category of algebraic stack. Fix $X \in \mathcal{A}\mathrm{rt}$.

- $\mathcal{M} \in \mathrm{Q}\mathcal{C}\mathrm{oh}(X)$ is *coherent* or *perfect* if it is so smooth-locally.
- Notation: $\mathcal{C}\mathrm{oh}(X)$ and $\mathcal{P}\mathrm{erf}(X)$.
- If X has bounded structure sheaf, then $\mathcal{P}\mathrm{erf}(X) \subset \mathcal{C}\mathrm{oh}(X)$.
- The *K-theory space* of X is $K(X) := K(\mathcal{P}\mathrm{erf}(X))$ (resp. the *spectrum* is $K^B(X) := K^B(\mathcal{P}\mathrm{erf}(X))$).
- The *G-theory space* of X is $G(X) := K(\mathcal{C}\mathrm{oh}(X))$ (resp. the *spectrum* is $G^B(X) := K^B(\mathcal{C}\mathrm{oh}(X))$).

different construction
same result

Recall, $K^B(\mathcal{C})$ is roughly (equivalent to the spectrum defined) as follows:

- Define $\mathcal{C} \subset F\mathcal{C}$ such that $K(F\mathcal{C}) = 0$, and put $\Sigma\mathcal{C} := F\mathcal{C}/\mathcal{C}$
- Then $\mathcal{C} \to F\mathcal{C} \to \Sigma\mathcal{C}$ is (strict?) exact, so $K_{n+1}(\Sigma\mathcal{C}) = K_n(\mathcal{C})$
- Put $K^B(\mathcal{C}) := \mathrm{colim}_n \Omega^n K(\Sigma^n \mathcal{C})$.
- Note $\pi_n \Omega^m K(\Sigma^m \mathcal{C}) = \pi_{n+m} K(\Sigma^m \mathcal{C}) (= \pi_n K(\mathcal{C}))$ $\rightsquigarrow$ n can be negative

Cup product

Lemma

A biexact functor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ induces $K^B(\mathcal{C}) \otimes K^B(\mathcal{D}) \to K^B(\mathcal{E})$, which induces maps $K_n(\mathcal{C}) \times K_m(\mathcal{D}) \to K_{n+m}(\mathcal{E})$.

Now $S^n \to K(\mathcal{C}),\ S^m \to K(\mathcal{D}) \rightsquigarrow S^n \otimes S^m = S^{n+m} \to K(\mathcal{C}) \otimes K(\mathcal{D}) \to K(\mathcal{E})$

$$\mathcal{P}\mathrm{erf}(X) \times \mathcal{P}\mathrm{erf}(X) \xrightarrow{(-)\otimes(-)} \mathcal{P}\mathrm{erf}(X)$$

is biexact, which gives us a map

$$\cup : K^B(X) \otimes K^B(X) \to K^B(X)$$

called the *cup product*. This makes $K^B(X)$ into an $\mathbb{E}_\infty$-ring spectrum. (Reason: use naturality in multilinear functors and symmetric monoidal structure on $\mathcal{P}\mathrm{erf}(X)$?)

Pullback & Gysin map

For $f : X \to Y$ in $\mathcal{A}\mathrm{rt}$, the exact, symmetric monoidal functor $f^* : \mathcal{P}\mathrm{erf}(Y) \to \mathcal{P}\mathrm{erf}(X)$ induces a map of $\mathbb{E}_\infty$-ring spectra

$$f^* : K^B(Y) \to K^B(X)$$

Definition

If $f_* : \mathrm{Q}\mathcal{C}\mathrm{oh}(X) \to \mathrm{Q}\mathcal{C}\mathrm{oh}(Y)$ preserves perfect complexes, then we have the *Gysin map*

$$f_* : K^B(X) \to K^B(Y)$$

Remark

In [K21], certain technical conditions are given to ensure the Gysin map exists and interacts nicely with the cup product. I will highlight one.

Finite cohomological dimension

Definition

Let $f : X \to Y$ in $\mathcal{A}\mathrm{rt}$.

- f is of *finite cohomological dimension* (fcd) if there is $n \geq 0$ such that $f_*(\mathrm{Q}\mathcal{C}\mathrm{oh}(X)_{\geq 0}) \subset \mathrm{Q}\mathcal{C}\mathrm{oh}(Y)_{\geq -n}$.
- f is *universally of fcd* if for all qcqs Y' over Y, the base change $X' \to Y'$ is of fcd

Now consider a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g_2"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g_1"] & Y \end{tikzcd}$$

$$f_* \xrightarrow{\text{unit}} f_*(g_2)_* g_2^* \simeq (g_1)_* f'_* g_2^*$$

This gives a natural map

$$\varphi : g_1^* f_* \to f'_* g_2^*$$

If f is universally of fcd, it satisfies base-change, i.e., φ is an equivalence.

Finite cohomological dimension

Proposition

If $f : X \to Y$ is universally of fcd, then $f_* : \mathrm{Q}\mathcal{C}\mathrm{oh}(X) \to \mathrm{Q}\mathcal{C}\mathrm{oh}(Y)$ preserves perfect complexes iff it does so smooth-locally.

Suppose it does so smooth-locally

let $M \in \mathrm{QCoh}(X)$ be perfect, $\operatorname{Spec} A \xrightarrow{g_1} Y$ smooth

look at:
$$\begin{tikzcd} X_A \arrow[r, "g_2"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ \operatorname{Spec} A \arrow[r, "g_1"'] & Y \end{tikzcd}$$

$g_1^* f_*(M) = f'_* g_2^*(M)$ is perfect

Projection formula

Lemma

If $f : X \to Y$ is universally of fcd, then it satisfies the projection formula, stating that

$$f_*(M) \otimes N \to f_*(M \otimes f^* N)$$

is an equivalence, for all $M \in \mathrm{Q}\mathcal{C}\mathrm{oh}(X), N \in \mathrm{Q}\mathcal{C}\mathrm{oh}(Y)$.

Proposition

If $f : X \to Y$ is universally of fcd such that f_* preserves perfect complexes, then

$$f_*(m) \cup y \simeq f_*(m \cup f^*(y))$$

for all $m \in K^B(X), y \in K^B(Y)$. (see next slide)

Projection formula

Proposition

If $f : X \to Y$ is universally of fcd such that f_* preserves perfect complexes, then

$$f_*(m) \cup y \simeq f_*(m \cup f^*(y))$$

for all $m \in K^B(X), y \in K^B(Y)$.

$$\begin{tikzcd}
\mathrm{Perf}(X) \times \mathrm{Perf}(Y) \arrow[rr, "f_* \times \mathrm{id}"] \arrow[d, "\mathrm{id} \times f^*"] & & \mathrm{Perf}(Y) \times \mathrm{Perf}(Y) \arrow[d, "(-)\otimes(-)"] \\
\mathrm{Perf}(X) \times \mathrm{Perf}(X) \arrow[dr, "(-)\otimes(-)"'] \arrow[rr, phantom, "\text{commutes by projection formula}"] & & \mathrm{Perf}(Y) \\
& \mathrm{Perf}(X) \arrow[ur, "f_*"'] &
\end{tikzcd}$$

Absolute perfection

Definition

Let $X \in \mathcal{A}\mathrm{rt}$.

- X is *perfect* if the canonical map $\mathrm{Ind}(\mathcal{P}\mathrm{erf}(X)) \to \mathrm{Q}\mathcal{C}\mathrm{oh}(X)$ is an equivalence.
- For $Z \subset |X|$ closed, write $\mathrm{Q}\mathcal{C}\mathrm{oh}(X \text{ on } Z)$ for the full subcategory of $F \in \mathrm{Q}\mathcal{C}\mathrm{oh}(X)$ supported on Z. Similarly for $\mathcal{P}\mathrm{erf}(X \text{ on } Z)$.
- Now X is *absolutely perfect* if $F|_{X \setminus Z} = 0$

$$\mathrm{Ind}(\mathcal{P}\mathrm{erf}(X \text{ on } Z)) \xrightarrow{\simeq} \mathrm{Q}\mathcal{C}\mathrm{oh}(X \text{ on } Z)$$

  for al cocompact closed $Z \subset |X|$.

Note: if X is perfect then $\mathcal{P}\mathrm{erf}(X) = \mathrm{Q}\mathcal{C}\mathrm{oh}(X)^{\omega}$. (so globalization)

compacts

Localization

Put $K^B(X \text{ on } Z) := K^B(\mathcal{P}\mathrm{erf}(X \text{ on } Z))$.

Proposition

If X is absolutely perfect, then for every cocompact $Z \subset |X|$, we have an exact triangle

$$K^B(X \ on\ Z) \to K^B(X) \xrightarrow{j^*} K^B(X \setminus Z)$$

Idempotent complete stable ∞-cats

Sequence

$$\mathrm{QCoh}(X \text{ on } Z) \to \mathrm{QCoh}(X) \xrightarrow{j^*} \mathrm{QCoh}(X \setminus Z) \ : \ \text{exact in } \mathrm{Cat}^{\mathrm{perf}}$$

$\Downarrow$ (def. $\circledast$)

(abs. perf.)$^{(*)}$:

$$\mathrm{Perf}(X \text{ on } Z) \to \mathrm{Perf}(X) \to \mathrm{Perf}(X \setminus Z) \ : \ \text{exact in } \mathrm{Cat}^{\mathrm{ex}}$$

also: $j^* \dashv j_*$, j_* is f.f.:

$$\begin{tikzcd} U \arrow[r, "id"] \arrow[d, "id"'] & U \arrow[d, "j"] \\ U \arrow[r, hook, "j"'] & X \end{tikzcd}$$

$j^* j_* = id_* id^* = id$

Now apply loc. thm for $K^B(-)$

The G-spectrum is the G-space

Proposition

The canonical map $G(X) \to G^B(X)$ is an equivalence.

Roughly:

- The *theorem of the heart* says that if $\mathcal{C}$ has bounded t-structure, then $K(\mathcal{C}) \simeq K(\mathcal{C}^\heartsuit)$.
- An abelian category is *noetherian* if all objects are noetherian.

  no inf. desc. chains of $\subset$ subobjects
- If $\mathcal{C}$ has bounded t-structure and the heart is noetherian, then $K(\mathcal{C}) \simeq K^B(\mathcal{C})$.
- Since $\mathcal{C}\mathrm{oh}(X)$ has bounded t-structure and $\mathcal{C}\mathrm{oh}(X)^\heartsuit$ is noetherian, the claim follows.

Cap product

Observe that

$$\mathcal{P}\mathrm{erf}(X) \times \mathcal{C}\mathrm{oh}(X) \xrightarrow{(-)\otimes(-)} \mathrm{Q}\mathcal{C}\mathrm{oh}(X)$$

lands in $\mathcal{C}\mathrm{oh}(X)$. Indeed, for $\operatorname{Spec} A \to X$,

$$\mathcal{M}\mathrm{od}_A^{fp} \times \mathcal{C}\mathrm{oh}(A) \xrightarrow{(-)\otimes(-)} \mathcal{M}\mathrm{od}_A$$

lands in $\mathcal{C}\mathrm{oh}(A)$ since $A \otimes M = M$. Now use that $\mathcal{C}\mathrm{oh}(A)$ is stable under retracts.

Definition

The functor $\mathcal{P}\mathrm{erf}(X) \times \mathcal{C}\mathrm{oh}(X) \xrightarrow{(-)\otimes(-)} \mathcal{C}\mathrm{oh}(X)$ induces the *cap product*

$$\cap : K^B(X) \otimes G(X) \to G(X)$$

making $G(X)$ a $K^B(X)$-module.

Gysin map

Suppose that $f : X \to Y$ is of finite Tor-amplitude n. Then f^* restricts to a functor $\mathrm{QCoh}(Y)_{\leq 0} \to \mathrm{QCoh}(X)_{\leq n}$, and therefore gives a functor

If M has fin. gen. htpy groups, so does f^*M

$$f^* : \mathrm{Coh}(Y) \to \mathrm{Coh}(X)$$

Indeed, by shifting, enough to check $\pi_0(M \otimes_A B)$

Definition

For f of finite Tor-amplitude, pulling back induces the *Gysin map*

$$f^* : G(Y) \to G(X)$$

Now let $M \in \mathrm{QCoh}(Y)_{\leq 0}$ Take $K \in \mathrm{QCoh}(X)_{\geq n+1}$

then $\mathrm{QCoh}(f^*M, K) \simeq \mathrm{QCoh}(M, \underbrace{f_*M}_{\mathrm{QCoh}_{\geq 1}}) \simeq 0$

So f^* preserves bounded complexes (always right t-exact)

Projection formula

Suppose $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ preserves coherent complexes. Then we have a *direct image map*

$$f_* : G(X) \to G(Y)$$

If moreover f is of min. finite cohomological dimension, then

$$y \cap f_*(x) \simeq f_*(f^*(y) \cap x)$$

for all $x \in G(X), y \in K^B(Y)$. Moreover, base-change holds against maps of finite Tor amplitude.

$$\begin{tikzcd} X' \arrow[r, "g_2"] \arrow[d, "f'"'] & X \arrow[d, "f"] \\ Y' \arrow[r, "g_1"] & Y \end{tikzcd} \implies g_1^* f_* \simeq f'_* g_2^* : G(X) \to G(Y')$$

fin. Tor ampl.

reasoning same as before

Localization

(more explicit than $K^B(X)$, since Coh(X) "smaller")

Since $\mathcal{C}\mathrm{oh}(X)$ has bounded t-structure, the theorem of the heart says that

$$G(X) \simeq K(\mathcal{C}\mathrm{oh}(X)^{\heartsuit}) \simeq K(\mathcal{C}\mathrm{oh}(X_{\mathrm{cl}})^{\heartsuit}) \simeq G(X_{\mathrm{cl}})$$

From here on, sometimes assume X classical...

Lemma

Let $i : Z \to X$ be a closed immersion with open complement $j : U \to X$. Then we have an exact triangle

$$G(Z) \xrightarrow{i_*} G(X) \xrightarrow{j^*} G(U)$$

As before, we have an exact sequence

$$\mathcal{C}\mathrm{oh}(X \text{ on } Z) \to \mathcal{C}\mathrm{oh}(X) \xrightarrow{j^*} \mathcal{C}\mathrm{oh}(U)$$

relate to coh(Z), need K-theoretic tool

Dévissage for closed immersions

Lemma

Let $\mathcal{A} \subset \mathcal{B}$ be an inclusion of abelian categories, such that $\mathcal{A}$ is closed under subobjects and quotients, and each $B \in \mathcal{B}$ has a filtration

$$0 = B_r \subset B_{r-1} \subset \cdots \subset B_1 \subset B_0 = B$$

such that all B_i/B_{i-1} lie in $\mathcal{A}$. Then $K(\mathcal{B}) \simeq K(\mathcal{A})$.

By proper pushforward, we have $\mathrm{Coh}(Z) \to \mathrm{Coh}(X \text{ on } Z)$. We will show this induces an equivalence on K-theory. locally: $(Z \to X) = (V(f_1..f_n) \to \mathrm{Spec}\, A)$
$B := A/(f_1..f_n)$. Put $I = (f_1, .., f_n)$. Let $M \in \mathrm{Coh}(X \text{ on } Z)$, $U := X \setminus Z$
then $M|_U = 0 \Rightarrow \forall f_i \in I: M_{f_i} = 0$, i.e. $\exists n_i: f_i^{n_i} M = 0$
$\Rightarrow \exists n: I^n M = 0$. Now have filtration

$$M \supset IM \supset I^2 M \supset \ldots \supset I^n M = 0$$

with quotients in $\mathrm{Coh}(Z)$. then statement follows by dévissage

Nil invariance

Corollary

Let $Z \to X$ be a surjective closed immersion. Then

$$i_* : G(Z) \to G(X)$$

is an equivalence.

Have exact: $G(Z) \to G(X) \to G(X \setminus Z) = G(\emptyset) = 0$

Étale excision

Let $j : U \to X$ be an open immersion with closed complement $Z \subset |X|$. Let $X' \to X$ be étale ($\Rightarrow$ finite Tor-amplitude) such that $f^{-1}(Z)_{\mathrm{red}} \cong Z_{\mathrm{red}}$. Then the following induced square is cartesian

$$\begin{tikzcd}
G(Z_{red}) \arrow[r] \arrow[d, equal] & G(X) \arrow[r] \arrow[d, "f^*"] & G(U) \arrow[d, "f^*"] & \text{: exact} \\
G(f^{-1}Z_{red}) \arrow[r] & G(X') \arrow[r] & G(f^{-1}U) & \text{: exact}
\end{tikzcd}$$

Now

$$\begin{tikzcd}
A' \arrow[r] \arrow[d] & B' \arrow[d] \\
A \arrow[r] & B
\end{tikzcd}$$

cart. iff equiv. on fibers

Quasi-smoothness and virtual codimension

Let $f : X \to Y$ in $\mathcal{A}\mathrm{rt}$.

- f is *quasi-smooth* if it is locally of finite presentation and $L_{X/Y}$ has Tor-amplitude $[-\infty, 1]$.
- If f is a closed immersion of schemes, then it is quasi-smooth iff Zariski-locally on the target it is of the form $V(f_1, \ldots, f_n) \to Y$ for sections f_i on Y.
- If f is a closed immersion of algebraic stack, then it is quasi smooth iff it has a smooth atlas of schemes which is a quasi-smooth closed immersion.
- The *virtual codimension* of a quasi-smooth closed immersion is the number of sections being cut out.
- Equivalently, $N_{X/Y} := L_{X/Y}[-1]$ is smooth-locally of finite presentation with rank the virtual codimension.

Derived blow-ups

Let $Z \to X$ be a closed immersion in $\mathcal{A}\mathrm{rt}$. A *virtual Cartier divisor* is a quasi-smooth closed immersion $D \to T$ of virtual codimenion 1.

Definition

The *blow-up* of X in Z is the space

$$\mathrm{Bl}_Z X(T) := \left\{ \begin{array}{c|l} \begin{tikzcd} D \arrow[r, "i_D"] \arrow[d, "g"] & T \arrow[d] \\ Z \arrow[r] & X \end{tikzcd} & \begin{array}{l} \bullet\ i_D \text{ is a virtual Cartier divisor} \\ \bullet\ D_{\mathrm{cl}} \cong (T \times_X Z)_{\mathrm{cl}} \\ \bullet\ g^* N_{Z/X} \to N_{D/T} \text{ surjective} \end{array} \end{array} \right\}$$

Proposition

The stack $\mathrm{Bl}_Z X$ is algebraic. If Z, X are schemes, then so is $\mathrm{Bl}_Z X$.

Projective bundles

Definition

Let $X \in \mathcal{A}\mathrm{rt}$ and $\mathcal{E} \in \mathrm{Q}\mathcal{C}\mathrm{oh}(X)$ locally free of finite rank. Then the *projective bundle* of $\mathcal{E}$ is the stack $\pi : \mathbb{P}(\mathcal{E}) \to X$ such that

think: $K \subseteq f^*(\mathcal{E}) \twoheadrightarrow \mathcal{O}_T$

$$\mathbb{P}(\mathcal{E})(f : T \to X) := \{(\mathcal{L}, u) \mid \mathcal{L} \in \mathcal{P}\mathrm{ic}(T), u : f^*(\mathcal{E}) \twoheadrightarrow \mathcal{L}\}$$

Since line bundles on X are defined smooth-locally, the data $(\mathcal{L}, u)$ glue into an invertible sheaf $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{E})$ and a surjection $\pi^*(\mathcal{E}) \to \mathcal{O}(1)$.

Universal virtual Cartier divisor

The identity map $\mathrm{Bl}_Z X \to \mathrm{Bl}_Z X$ corresponds to the square

$$\begin{tikzcd}
\mathbb{P}_Z(N_{Z/X}) \arrow[r, "i_D"] \arrow[d, "g"] & \mathrm{Bl}_Z X \arrow[d] \\
Z \arrow[r] & X
\end{tikzcd}$$

which is the universal square such that

- i_D is a virtual Cartier divisor
- It is cartesian on $(-)_{\mathrm{cl}}$
- $g^* N_{Z/X} \to N_{\mathbb{P}_Z(N_{Z/X})/\mathrm{Bl}_Z X}$ surjective

Semi-orthogonal decompositions

Definition

Let $\mathcal{C}$ be a stable category with full stable subcategory $\mathcal{D}$.

- The category of *left orthogonals to* $\mathcal{D}$ is the full subcategory

$$^{\perp}\mathcal{D} := \{x \in \mathcal{C} \mid \forall d \in \mathcal{D} : \mathcal{C}(x, d) \simeq *\}$$

Definition

Let $\mathcal{C}$ be stable. A *semi-orthogonal decomposition* of $\mathcal{C}$ is a sequence $\mathcal{C}(0), \ldots, \mathcal{C}(-n)$ of full stable subcategories such that

- For all integers $i > j$ it holds $\mathcal{C}(i) \subset {}^{\perp}\mathcal{C}(j)$;
- $\mathcal{C}$ is generated by $\mathcal{C}(0), \ldots, \mathcal{C}(-n)$ under finite limits and finite colimits.

Lemma

$\mathcal{C}_{\leq -m}$

Let $\mathcal{C}$ be stable, with semi-orthogonal decompostion $(\mathcal{C}(0), \ldots, \mathcal{C}(-n))$. For $0 \leq m \leq n$, define $\mathcal{C}_{\leq -m} := \operatorname{span}(\mathcal{C}(-m) \cup \cdots \cup \mathcal{C}(-n))$ and put $\mathcal{C}_{\leq -n-1} := \{0\}$. Then there are split short exact sequences

IDEA: tfae
- SOD $(\mathcal{C}(0), \mathcal{C}(-1))$
- split s.e.s. $\mathcal{C}(0) \to \mathcal{C} \to \mathcal{C}(-1)$

$$\mathcal{C}(-m) \to \mathcal{C}_{\leq -m} \to \mathcal{C}_{\leq -m-1}$$

Now $(\mathcal{C}(-m), \mathcal{C}_{\leq -m-1})$ is S.O.D. on $\mathcal{C}_{\leq -m}$

for each $0 \leq m \leq n$.

Lemma ('Generalized additivity theorem')

Let $\mathcal{C}$ be stable, with semi-orthogonal decompostion $(\mathcal{C}(0), \ldots, \mathcal{C}(-n))$. For E an additive invariant (= exact on split exact sequences), it holds

$$E(\mathcal{C}) \simeq \bigoplus_{0 \leq m \leq n} E(\mathcal{C}(-m))$$

Semi-orthogonal decomposition on $\mathrm{QCoh}(\mathbb{P}(\mathcal{E}))$

Let $\mathcal{E}$ be locally free of rank $n+1$, and consider $\pi : \mathbb{P}(\mathcal{E}) \to X$.

Lemma

For each $0 \leq k \leq n$ we have a fully faithful functor

$$\mathrm{QCoh}(X) \to \mathrm{QCoh}(\mathbb{P}(\mathcal{E})) : \mathcal{F} \mapsto \pi^*\mathcal{F} \otimes \mathcal{O}(-k)$$

Definition

For any $-k$, let $\mathcal{C}(-k)$ be the essential image of the functor in $\mathcal{F} \mapsto \pi^*\mathcal{F} \otimes \mathcal{O}(-k)$.

IDEA: Suffices $k=0$. TS: $\pi_*\pi^*\mathcal{F} \simeq \mathcal{F}$. Is local: $X = \operatorname{Spec} R$, $\mathcal{E} = R^{\oplus n+1}$

then π_*, π^* both commute colims. So assume $\mathcal{F} = \mathcal{O}_X$

then $\pi_*\pi^*\mathcal{O}_X \simeq \pi_*\mathcal{O}(0) = \mathcal{O}_X$ $\Big(\mathcal{O}(k)$ = "hom. polys of deg k" " Serre formula$\Big)$

Semi-orthogonal decomposition on $\mathrm{QCoh}(\mathbb{P}(\mathcal{E})), \mathrm{Perf}(\mathbb{P}(\mathcal{E})), \mathrm{Coh}(\mathbb{P}(\mathcal{E}))$

Proposition

The categories $\mathcal{C}(0), \ldots, \mathcal{C}(-n)$ *form a semi-orthogonal decomposition of* $\mathrm{QCoh}(\mathbb{P}(\mathcal{E}))$. *These restrict to* $\mathrm{Perf}(\mathbb{P}(\mathcal{E})), \mathrm{Coh}(\mathbb{P}(\mathcal{E}))$.

let's show $\mathcal{C}(i) \subseteq {}^{\perp}\mathcal{C}(j)$ for $0 > i > j \Rightarrow 0 > j-i$

look at $\pi^*F \otimes \mathcal{O}(i) \in \mathcal{C}(i)$, $\pi^*G \otimes \mathcal{O}(j) \in \mathcal{C}(j)$

Then $\mathrm{QCoh}(\mathbb{P}(\mathcal{E}))(\pi^*F \otimes \mathcal{O}(i), \pi^*G \otimes \mathcal{O}(j))$

$\simeq \mathrm{QCoh}(\mathbb{P}(\mathcal{E}))(\pi^*F, \pi^*G \otimes \mathcal{O}(j-i))$

$\simeq \mathrm{QCoh}(X)(F, \pi_*(\pi^*G \otimes \mathcal{O}(j-i)))$

Projection formula

$\simeq \mathrm{QCoh}(X)(F, G \otimes \pi_*(\mathcal{O}(j-i))) \simeq \mathrm{QCoh}(X)(F, 0) \simeq$

$\pi_*(\mathcal{O}(j-i)) = 0$ (Serre, SAG 5.4.2.6)

Projective bundle formulae

Theorem

Let $\mathcal{E}$ be a locally free complex of rank $n+1$ on X. Then

$$K^B(\mathbb{P}(\mathcal{E})) \simeq \bigoplus_{0\le k\le n} K^B(X)$$

$$G(\mathbb{P}(\mathcal{E})) \simeq \bigoplus_{0\le k\le n} K^B(X)$$

Look @ S.O.D. $\mathcal{O}(0), \dots, \mathcal{O}(-n)$ on $\mathrm{Perf}(\mathbb{P}(\mathcal{E}))$ $\mathrm{Coh}(\mathbb{P}(\mathcal{E}))$

this gives $\underset{G}{K^B}(\mathbb{P}(\mathcal{E})) \simeq \bigoplus_{k=0}^{-n} \underset{G}{K^B}(\mathcal{O}(k))$

$$\simeq \bigoplus_{k=0}^{-n} \underset{G}{K^B}(\mathrm{Perf}(X)) \simeq \bigoplus_{0\le k\le n} \underset{G}{K^B}(X)$$

Blow-up formulas

Similar but more complicated. Sketch:

Let $Z \to X$ be a quasi-smooth closed immersion of virtual codimension n, write $\pi : \mathrm{Bl}_Z X \to X$ and $p : \mathbb{P}_Z(N_{Z/X}) \to Z$.

- $\pi^* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(\mathrm{Bl}_Z X)$ is fully faithful. Write image as $\mathcal{D}(0)$.
- For all $1 \leq k \leq n-1$, the composition

$$\mathrm{QCoh}(Z) \xrightarrow{p^*(-)\otimes\mathcal{O}(-k)} \mathrm{QCoh}(\mathbb{P}(N_{Z/X})) \xrightarrow{i_*} \mathrm{QCoh}(\mathrm{Bl}_Z X)$$

  is fully faithful. Write image as $\mathcal{D}(-k)$
- Now $\mathcal{D}(0), \ldots, \mathcal{D}(-n+1)$ forms a semi-orthogonal decomposition on $\mathrm{QCoh}(\mathrm{Bl}_Z X)$.
- This restricts to perfect and coherent complexes.
- We thus have

$$K^B(\mathrm{Bl}_Z X) \simeq \underbrace{K^B(X)}_{\text{}}{}^{\mathcal{D}(0)} \oplus \bigoplus_{1\leq k\leq n-1} K^B(Z)^{\,\mathcal{D}(-k)}$$

$$G(\mathrm{Bl}_Z X) \simeq G(X) \oplus \bigoplus_{1\leq k\leq n-1} G(Z)$$

Vector bundles (goal: htpy invariance)

Let $\mathcal{E}$ be a locally free sheaf of finite rank on $X \in \mathcal{A}\mathrm{rt}$.

$$\begin{tikzcd} \mathcal{E} \arrow[r] \arrow[d] & \mathcal{E} \oplus \mathcal{O}_X \arrow[dl, dashed, "\exists"] \\ \mathcal{O}_X & \end{tikzcd}$$

- The canonical map $h : \mathcal{E} \to \mathcal{E} \oplus \mathcal{O}_X$ induces a surjection $h^\vee : (\mathcal{E} \oplus \mathcal{O}_X)^\vee \to \mathcal{E}^\vee$ $\quad (f^*\mathcal{E}^\vee \twoheadrightarrow \mathcal{L}) \mapsto (f^*(\mathcal{E}\oplus\mathcal{O}_X)^\vee \twoheadrightarrow \mathcal{L})$
- We thus have a closed immersion $j : \mathbb{P}(\mathcal{E}^\vee) \to \mathbb{P}((\mathcal{E} \oplus \mathcal{O}_X)^\vee)$
- Let $\mathbb{V}(\mathcal{E}^\vee)$ be the vector bundle of sections of $\mathcal{E}$, i.e.

$$\mathbb{V}(\mathcal{E}^\vee)(f : T \to X) := \{v : f^*\mathcal{E}^\vee \to \mathcal{O}_T\}$$

$$(f^*\mathcal{E}^\vee \to \mathcal{O}_T) \longmapsto (f^*\mathcal{E}^\vee \oplus \mathcal{O}_T \twoheadrightarrow \mathcal{O}_T)$$

- We have an obvious map $i : \mathbb{V}(\mathcal{E}^\vee) \to \mathbb{P}((\mathcal{E} \oplus \mathcal{O}_X)^\vee)$
- The map i is the open complement of j. think: $\mathbb{P}^n \setminus \mathbb{P}^{n-1} \cong \mathbb{A}^n$ (i.e. affine charts)

Let $\operatorname{Spec} K \xrightarrow{\varphi} \mathbb{P}((\mathcal{E}\oplus\mathcal{O}_X)^\vee)$ be given

$$\begin{tikzcd} f^*\mathcal{E}^\vee \arrow[r] \arrow[dr, "\psi"'] & f^*\mathcal{E}^\vee \oplus K \arrow[d, "\Phi"] \\ & K \end{tikzcd}$$

Now $\varphi \notin j(\mathbb{P}(\mathcal{E}^\vee)) \Leftrightarrow \psi$ not surj.
$\Leftrightarrow \psi = 0 \Leftrightarrow \Phi$ is projection
$\Leftrightarrow \varphi \in i(\mathbb{V}(\mathcal{E}^\vee))$

Homotopy invariance

Proposition

For $\mathcal{E}$ locally free of finite rank on $X \in \mathcal{A}\mathrm{rt}$, the map since $\mathcal{E}^{\vee\vee}$,

$$\pi^* : G(X) \to G(\mathbb{V}(\mathcal{E}))$$

suffices to show for $\mathcal{E}^\vee$

induced by $\pi : \mathbb{V}(\mathcal{E}) \to X$, is invertible.

look at $\mathbb{P}(\mathcal{E}^\vee) \xrightarrow{j} \mathbb{P}((\mathcal{E} \oplus \mathcal{O}_X)^\vee) \hookleftarrow \mathbb{V}(\mathcal{E}^\vee)$

then:

localization: $G(\mathbb{P}(\mathcal{E}^\vee)) \to G(\mathbb{P}(\mathcal{E} \oplus \mathcal{O}_X)^\vee) \to G(\mathbb{V}(\mathcal{E}^\vee))$

bundle form: $\bigoplus_{0 \le k \le n} G(X) \to \bigoplus_{0 \le k \le n+1} G(X) \to G(X)$

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Thank you!