$\zeta(s) := \prod_p (1-p^{-s})^{-1} = \sum_{n\geq 1} n^{-s}$ $[\mathrm{Re}(s)>1]$

$\Gamma(s) := \int_0^\infty \underbrace{e^{-t}}_{(1)} \underbrace{t^s}_{(2)} \underbrace{\frac{dt}{t}}_{(3)}$ $[\mathrm{Re}(s)>1]$

(1) add. char. $\mathbb{R} \to \mathbb{C}^\times$
(2) mult. char. $\mathbb{R}^\times \to \mathbb{C}^\times$
(3) Haar meas. on $\mathbb{R}^\times$

Thm (Riemann):
(AC) $\zeta(s)$ extends meromorphically to all of $\mathbb{C}$ and is holomorphic except for simple pole at $s=1$.
(FE) $\xi(s) := \pi^{-s/2}\Gamma(s/2)\zeta(s)$ satisfies $\xi(s) = \xi(1-s)$.

Schwartz space: $\mathcal{S} = \mathcal{S}(\mathbb{R}) := \{ f : \mathbb{R} \to \mathbb{C} \text{ smooth} \mid f^{(r)} \to 0 \text{ rapidly } \forall r \geq 0 \}$.

Fourier transform: $\hat{\ } : \mathcal{S} \to \mathcal{S}$, $f \mapsto \left(y \mapsto \int_{\mathbb{R}} f(x) e^{-2\pi i x y}\, dx\right)$

Fourier inversion: $f \in \mathcal{S} \Rightarrow f(x) = \int_{\mathbb{R}} \hat{f}(y) e^{2\pi i x y}\, dy$ – i.e., $\hat{\hat{f}}(x) = f(-x)$.

Poisson summation: $f \in \mathcal{S} \Rightarrow \sum_{n\in\mathbb{Z}} f(n) = \sum_{n\in\mathbb{Z}} \hat{f}(n)$.

Theta function: $[t \in \mathbb{R}^{>0}]$ $\Theta(t) := \sum_{n\in\mathbb{Z}} e^{-\pi n^2 t}$. Satisfies $\Theta(t) = t^{-1/2}\Theta(1/t)$.

LCA := { loc. compact Hausdorff ab. top. grp.'s }, $U(1) = \{ z \in \mathbb{C} : |z| = 1 \}$

Given $G \in$ LCA, char. of G is cont. homo. $\chi : G \to \mathbb{C}^\times$ [set of these denoted $X(G)$]
unitary char. of G is cont. homo. $\chi : G \to U(1)$

Pontryagin dual $\hat{G} := \mathrm{Hom}_{\mathrm{cont}}(G, U(1)) = \{$ unitary char.'s of $G \}$ ⇝ functor $\hat{\ } : \mathrm{LCA} \to \mathrm{LCA}$

Thm: (a) $\hat{\ }$ is exact.
(b) [Pontryagin Duality] $G \to \hat{\hat{G}}$, $g \mapsto (\chi \mapsto \chi(g))$ is isom. of LCA grps.

Think of this in terms of pairing $\langle \cdot,\cdot \rangle : G \times \hat{G} \to U(1)$, $(g, \chi) \mapsto \chi(g)$.

$H \leq G$ ⇝ $H^\perp := \{ \chi \in \hat{G} : \chi|_H = 1 \}$.

G	$\hat{G}$
$\mathbb{R}$	$\mathbb{R}$
$\mathbb{Q}_p$	$\mathbb{Q}_p$
$\mathbb{A}$	$\mathbb{A}$
$\mathbb{Z}$	$\mathbb{R}/\mathbb{Z}$
$\mathbb{Z}_p$	$\mathbb{Q}_p/\mathbb{Z}_p$
K	$\mathbb{A}/K$
finite	finite
discrete	compact
discrete torsion	profinite

Self-duality of $\mathbb{R}$ exhibited by $\mathbb{R}\times\mathbb{R} \to U(1)$, $(x,y) \mapsto e^{2\pi i x y}$.

General Fourier transform: Given $f \in L^1(G)$, we have $\hat{f} : \hat{G} \to \mathbb{C}$ defined by

$\hat{f}(\chi) := \int_G f(g)\overline{\chi(g)}\, dg$. [Fact: $\hat{f} : \hat{G} \to \mathbb{C}$ is cont.]

General Fourier inversion: $G \in LCA$, dg Haar meas. on $G \Rightarrow \exists !$ Haar meas. $d\chi$ on $\hat{G}$ (dual/Plancherel meas.) s.t. $f \in L^1(G)$ w/ $\hat{f} \in L^1(\hat{G}) \Rightarrow f(g) = \int_{\hat{G}} \hat{f}(\chi)\chi(g)\,d\chi$ for a.e. $g \in G$. If g cont. then we can remove the "a.e."

[Cor: $\hat{}$ extends to all of $L^2(G)$ and is isom. of Hilbert spaces]

Plancherel Thm: Let (G, dg) and $(\hat{G}, d\chi)$ be as above and $f \in L^1(G) \cap L^2(G)$. Then, $\|f\|_2 = \|\hat{f}\|_2$.

Additive char. on F is nontriv. unitary char. $\psi : (F,+) \to U(1)$. Given ψ and $a \in F$, we get ψ_a another add. char. defined by $\psi_a(x) := \psi(ax)$. [Each local field F admits some standard add. char.]

Thm: $\Psi : F \to \hat{F}$, $a \mapsto \psi_a$ is LCA grp. isom. (for ψ fixed add. char.).

Def: Let F be nonarch. and ψ an add. char. Choosing minimal $m \in \mathbb{Z}$ s.t. $\psi|_{\mathfrak{p}^m} = 1$ yields the fractional ideal $\mathfrak{p}^m$ called the conductor (w/ $\mathfrak{p}^0 = \mathcal{O}$).

We extend notion of $\mathcal{S} = \mathcal{S}(F)$ to F nonarch. local by taking it to be compactly supp. loc. constant functions.

Def: F local field, ψ add. char., dx Haar meas. $\rightsquigarrow \hat{} : \mathcal{S} \to \mathcal{S}$ via $f \mapsto (y \mapsto \int_F f(x)\psi(xy)\,dx)$.

Prop: Unique Haar meas. dx self-dual (rel. to standard ψ) is described explicitly by

- $F = \mathbb{R}$: dx = Lebesgue meas.
- $F = \mathbb{C}$: $dx = 2 \cdot$ Lebesgue meas.
- F nonarch: dx is Haar meas. s.t. $\mathcal{O}$ gets meas. $(N\mathcal{D})^{-1/2}$

F_0 "fund." local field
$\mathcal{D}$ different of F/F_0
$N\mathcal{D} := \#(\mathcal{O}/\mathcal{D})$

Given F local, define $\mathcal{U} := \{x \in F^\times : |x| = 1\} = \begin{cases} \{\pm 1\}, & F = \mathbb{R} \\ U(1), & F = \mathbb{C} \\ \mathcal{O}^\times, & F \text{ nonarch.} \end{cases}$ $\quad \underbrace{|F^\times|}_{\text{value grp.}} := \{|x| : x \in F^\times\} = \begin{cases} \mathbb{R}^{>0}, & \text{arch} \\ q^{\mathbb{Z}}, & \text{non-arch} \end{cases}$

LCA SES $1 \to \mathcal{U} \to F^\times \xrightarrow{|\cdot|} |F^\times| \to 1$.

Def: $\chi \in X(F^\times)$ is unram. if $\chi|_{\mathcal{U}} = 1$. [Motivation comes from inertia grps. and local Artin homo.]

Prop: $\chi : F^\times \to \mathbb{C}^\times$ mult. char. TFAE:
(i) χ unram.
(ii) χ factors through $|F^\times|$.
(iii) $\chi = |\cdot|^s$ for some $s \in \mathbb{C}$.

Cor: Unram. char.'s of $F^\times$ form subgrp. of $X(F^\times)$ isom. to $\begin{cases} \mathbb{C}, & F \text{ arch.} \\ \mathbb{C}/(\mathbb{Z} \cdot 2\pi i/\log q), & F \text{ nonarch.} \end{cases}$

[This puts RS structure on $X(F^\times)$ w/ ∞'ly many conn. components.]

Prop: χ char. of $F^\times \Rightarrow \chi = \eta \cdot |\cdot|^s$ for η unitary char. and $s \in \mathbb{C}$.

Cor: $\chi \in X(F^\times) \Rightarrow |\chi| = |\cdot|^\sigma$ for unique $\sigma \in \mathbb{R}$. [exponent of χ]

Pf: $\chi = \eta \cdot |\cdot|^s$ uniquely $\Rightarrow |\chi| = |\eta| \cdot |\cdot|^{Re(s)} = |\cdot|^{Re(s)}$. $\square$

Def: F nonarch., $\chi \in X(F^\times) \Rightarrow$ we may chose $m \in \mathbb{Z}^{\geq 0}$ minimal s.t. $\chi|_{1+\mathfrak{p}^m} = 1$ (w/ $1+\mathfrak{p}^0 := \mathcal{O}^\times = \mathcal{U}$). $\mathfrak{p}^m$ is the conductor of χ and measures ramification.

Cor: (a) Char. of $\mathbb{R}^\times$ has form $\chi_{a,s}(x) := x^{-a}|x|^s$ for $a \in \{0,1\}, s \in \mathbb{C}$.

(b) Char. of $\mathbb{C}^\times$ has form $\chi_{a,b,s}(z) := z^{-a}\bar{z}^{-b}\|z\|^s$ for $a,b \in \mathbb{Z}$ w/ $\min\{a,b\} = 0$ and $s \in \mathbb{C}$

Note: This is normalized so that sign char. $\mathrm{sgn}(x) := x^{-1}|x|$ has nice notation for L-factors.

We want to think about L-factors and zeta integrals not as functions of $s \in \mathbb{C}$ ~~[illegible]~~ but of $\chi \in X(F^\times)$.

The twisted dual $\chi^\vee := \chi^{-1} \cdot |\cdot|$ has property that $|\cdot|^s \mapsto |\cdot|^{1-s}$ (FE!)

Def: F nonarch., $\chi \in X(F^\times) \rightsquigarrow L(\chi) := \begin{cases} (1-\chi(\varpi))^{-1}, & \chi \text{ unram.} \\ 1, & \chi \text{ ram.} \end{cases}$ [Local L-factors]

$L(\chi_{a,s}) \equiv \Gamma_{\mathbb{R}}(s) := \pi^{-s/2}\Gamma(s/2)$. [look at induced rep.'s and Weil grps.] NB: $L(\chi)$ is meromorphic function of χ w/ no zeros.

$L(\chi_{a,b,s}) \equiv \Gamma_{\mathbb{C}}(s) := \Gamma_{\mathbb{R}}(s)\Gamma_{\mathbb{R}}(s+1) = 2(2\pi)^{-s}\Gamma(s)$.

Fix Haar meas. $d^\times x$ on $F^\times$. [Then, $d^\times x = c\frac{dx}{|x|}$ for some $c \in \mathbb{R}^{>0}$ in the sense that $\forall f \in C_c(F^\times)$: $\int_{F^\times} f(x)\, d^\times x = \int_F f(x)\frac{c}{|x|}\,dx$.] Given $f \in \mathcal{S}$ and $\chi \in X(F^\times)$, we get

local zeta integral $Z(f,\chi) := \int_{F^\times} f(x)\chi(x)\,d^\times x$.

Thm (MC+FE for local zeta integrals): For (a)-(c), let $f \in \mathcal{S}$.

(a) $Z(f,\chi)$ conv. for χ of exponent $\sigma > 0$.
(b) $Z(f,\chi)$ extends to meromorphic function on $X(F^\times)$.
(c) Meromorphic function $Z(f,\chi)/L(\chi)$ on $X(F^\times)$ is actually holomorphic.
(d) $X_0 \in \pi_0(X(F^\times)) \Rightarrow \exists f \in \mathcal{S}$ s.t. $Z(f,\chi)/L(\chi)$ is non-vanishing on X_0.

In fact: F nonarch., $X_0 = \{|\cdot|^s : s \in \mathbb{C}\}$, $f = \mathbb{1}_{\mathcal{O}} \Rightarrow Z(f,\chi)/L(\chi) \equiv 1$.

(e) Given ψ, dx, $\exists\, \varepsilon(\chi,\psi,dx)$ non-vanishing holomorphic function of $\chi \in X(F^\times)$ s.t. [$\forall f \in \mathcal{S}$]
$$\frac{Z(\hat{f},\chi^\vee)}{L(\chi^\vee)} = \underbrace{\varepsilon(\chi,\psi,dx)}_{\text{local } \varepsilon\text{-factor}}\frac{Z(f,\chi)}{L(\chi)}$$

- $\chi = \eta|\cdot|^s$ for η fixed $\Rightarrow \varepsilon(\chi,\psi,dx) = Ae^{Bs}$ for $A,B \in \mathbb{C}$.
- F nonarch., ψ conductor $\mathfrak{p}^0$, $\int_{\mathcal{O}} dx = 1 \Rightarrow \varepsilon(|\cdot|^s,\psi,dx) = 1\ \forall s \in \mathbb{C}$.

Choose a uniformizer π with associated normalized valuation v (i.e., $v(\pi) = 1$) and ramification index $e := v(p)$. Suppose that $e < p - 1$.

(a) *Let G_0 be a finite (flat) commutative K-group scheme killed by some power of p. Then, G_0 admits at most one* prolongation *over R – i.e., at most one finite flat commutative R-group scheme G such that $G_K \cong G_0$. In particular, any finite flat commutative R-group scheme is the unique prolongation of its generic fiber.*

(b) *The generic fiber functor from the category of finite flat commutative R-group schemes to the category of finite flat commutative K-group schemes is fully faithful with (essential) image stable under taking sub-objects and quotients.*

We say G_0 in the above theorem has **unique prolongation** (or **UP** for short).

Remark 4.7. *The condition $e < p - 1$ is necessary as can be seen by considering a suitable finite extension $K/\mathbb{Q}_p$ containing the pth roots of unity and comparing μ_p and $\underline{\mathbb{Z}/p}$ over K and $\mathcal{O}_K$.*

The proof of this result rests on the structure theory of so-called *Raynaud F-module schemes.* The major steps in the proof are roughly as follows. ↑ What are these?

(1) If G_0 is an extension whose sub and quotient have UP then G_0 has UP, hence we need only consider simple groups.

(2) Classify the simple groups and their prolongations.

(3) Check by hand that UP holds when $e < p - 1$.

- $(F, \mathcal{O}, \mathfrak{p}, k)$ local (w/ π, q)
- $\psi: (F,+) \to U(1)$ nontriv. cont. homo.

We claim $\exists m \in \mathbb{Z}$ s.t. $\psi|_{\mathfrak{p}^m} = 1$.

Fix ψ, dx (Fourier!). $\exists\ \varepsilon(\chi, \psi, dx)$ non-vanishing hol. function of $\chi \in X(F^\times)$ s.t.

$$\frac{Z(\hat{f}, \chi^\vee)}{L(\chi^\vee)} = \varepsilon(\chi, \psi, dx)\frac{Z(f,\chi)}{L(\chi)} \quad \forall f \in \mathcal{S}.$$

$X_0 \in \pi_0(X(F^\times)) \Rightarrow \exists f \in \mathcal{S}$ s.t. $Z(f,\chi)/L(\chi)$ non-vanishing on X_0

F nonarch., $X_0 = \{1 \cdot 1^s : s \in \mathbb{C}\}$, $f = \mathbb{1}_{\mathcal{O}} \Rightarrow Z(f,\chi)/L(\chi) \equiv 1$

Unitary char.'s $\omega: \mathcal{O}^\times \to U(1)$, $\psi: \mathcal{O} \to U(1) \rightsquigarrow$ <u>Gauss sum</u> $g(\omega,\psi) := \int_{\mathcal{O}^\times} \omega(x)\psi(x)\, d^\times x$.

Suppose ω has conductor $\mathfrak{p}^n$ for $n > 0$ and ψ has conductor $\mathfrak{p}^m$. Then, $m \neq n \Rightarrow g(\omega,\psi) = 0$. $m = n \Rightarrow |g(\omega,\psi)|^2 = q^{-m}$.

Fix a field k. <u>CSA/k</u> is fin. dim. assoc. k-alg. A which is simple and satisfies $Z(A) \cong k$.