Siegel Moduli Space

Fix dim $g > 0$ and $D = (d_1, \ldots, d_g)$ ("type") w/ $d_1 | d_2 | \cdots | d_g$ pos. integers. Define a symplectic form

$E_D : \mathbb{Z}^{2g} \times \mathbb{Z}^{2g} \to \mathbb{Z}$, $(x,y) \mapsto {}^t x \begin{pmatrix} & d_1 \ldots d_g \\ -d_1 \ldots d_g & \end{pmatrix} y$ $\quad$ (${}^t x$ = transpose).

<u>Fact</u>: $\mathcal{U}$ free $\mathbb{Z}$-mod. of rank $2g$, $E : \mathcal{U} \times \mathcal{U} \to \mathbb{Z}$ nondeg. symplectic form $\Rightarrow \exists\, \mathbb{Z}^{2g} \xrightarrow{\sim} \mathcal{U}$ (symplectic basis) identifying E_D w/ E

for D as above uniquely determined by $\mathcal{U}$ and E.

$X = V/\mathcal{U}$ complex torus of dim g w/ polarization $E : \mathcal{U} \times \mathcal{U} \to \mathbb{Z}$ of type D. <u>This just says</u> $(x,y) \mapsto E(ix, y)$ is

symm. pos. def. $\mathbb{R}$-bilinear on $\mathbb{R}$-vec. space V. $\quad \underset{\underset{H_1(X;\mathbb{C})}{\parallel}}{\mathcal{U} \otimes_{\mathbb{Z}} \mathbb{C}} = \underset{\circlearrowleft\, i \quad \circlearrowleft\, I}{V \otimes_{\mathbb{R}} \mathbb{C}} = \underbrace{\overset{\overset{F^0 H_1(X)}{\parallel}}{V^{0,-1}}}_{i = -I} \oplus \underbrace{\overset{\overset{\overline{F^0 H_1(X)}}{\parallel}}{V^{-1,0}}}_{i = I}$.

We have isom. of $\mathbb{C}$-vec. spaces $V \hookrightarrow V \otimes_{\mathbb{R}} \mathbb{C} \to V \otimes_{\mathbb{R}} \mathbb{C} / V^{0,-1}$.

$V \otimes_{\mathbb{R}} \mathbb{C} / V^{0,-1} \cong V^{-1,0}$ has unambiguous complex structure.

$X = \mathcal{U} \backslash V \cong H_1(X;\mathbb{Z}) \backslash H_1(X;\mathbb{C}) / F^0 H_1(X)$.

<u>Lemma</u>: If we extend $E : \mathcal{U} \times \mathcal{U} \to \mathbb{Z}$ $\mathbb{C}$-lin. to alt. form on $\mathcal{U} \otimes_{\mathbb{Z}} \mathbb{C}$ then $E(V^{0,-1}, V^{0,-1}) = 0$, $E(V^{-1,0}, V^{-1,0}) = 0$. (summands are totally isotropic)

(Pf: $E(x,y) = -E(x,y)\ \forall x,y \in V^{-1,0}$.)

Given $(\mathbb{Z}^{2g}, E_D)$ as before, define $\mathcal{H}_{g,D}$ to be (the set of) decompositions $\mathbb{C}^{2g} = F^0 \oplus \overline{F^0}$ s.t.

(1) both summands are tot. isotropic and (2) $E_D(ix, y)$ on $\mathbb{R}^{2g}$ is symm. pos. def.

<u>NB</u>: $\mathbb{C}$-structure on $\mathbb{R}^{2g}$ comes from isom. $\mathbb{R}^{2g} \hookrightarrow \mathbb{C}^{2g} \to \mathbb{C}^{2g}/F^0$. [Exercise: symm. is automatic here!]

To Deligne, this is {Hodge structures $h : \mathbb{S} \to GSp(\mathbb{R}^{2g}, E_D)$ of type $(0,-1), (-1,0)$ s.t. symm. $\mathbb{R}$-bilinear form $E(h(i)x, y)$ on $\mathbb{R}^{2g}$ is pos. def.}. $h(i)$ = image of i under $\mathbb{C}^\times \cong \mathbb{S}(\mathbb{R}) \xrightarrow{h(\mathbb{R})} GSp(\mathbb{R}^{2g}, E_D)$. ← [This perspective generalizes, letting us work w/ other reductive gps.]

<u>Remark</u>: $\mathcal{H}_{g,D}$ is open subset of {maximal isotropic subspaces of $(\mathbb{C}^{2g}, E_D)$} so nat. sits inside some flag var.

<u>Remark</u>: $Sp(\mathbb{R}^{2g}, E_D)$ acts trans. on $\mathcal{H}_{g,D}$.

<u>Remark</u>: We can replace "pos. def." by "pos./neg. def." to get action of $GSp(\mathbb{R}^{2g}, E_D)$ ("similitudes"). $\mathcal{H}^+ \cup \mathcal{H}^-$ is single $GSp(\mathbb{R}^{2g}, E_D)$-conj. class in $\mathrm{Hom}(\mathbb{S}, GSp(\mathbb{R}^{2g}, E_D))$.

__Prop__: Bij. $\{\text{polarized complex tori } (X,E) \text{ w/ symplectic basis } \mathbb{Z}^{2g} \to H_1(X;\mathbb{Z})\} \xrightarrow{\sim} \mathcal{H}_{g,D}$.

$$E_D \leftrightarrow E$$

__Pf__: $\mathbb{C}^{2g} = F^0 \oplus \overline{F^0} \in \mathcal{H}_{g,D} \mapsto X := \mathbb{Z}^{2g} \backslash \mathbb{C}^{2g}/F^0$ w/ polarization E_D on $\mathbb{Z}^{2g} = H_1(X;\mathbb{Z})$.

other direction... $\square$

__Cor__: $\{\text{polarized complex tori of type } D \text{ and } \dim g\} \xrightarrow{\sim} Sp(\mathbb{Z}^{2g}, E_D) \backslash \mathcal{H}_{g,D}$.

(non-smooth) ("generic" stabilizer is $\mathbb{Z}/2 \cong \{\pm 1\}$)

__Thm (Cartan)__: RHS has nat. structure of quasi-proj. var. / $\mathbb{C}$.

This requires producing enough (Siegel) modular forms on $Sp(\mathbb{Z}^{2g}, E_D) \backslash \mathcal{H}_{g,D}$.

__Classical Description__

Let (D, E_D) as before. Let $u_1, \dots, u_g, v_1, \dots, v_g$ be standard basis. $\mathbb{C}^{2g} = F^0 \oplus \overline{F^0} \in \mathcal{H}_{g,D}$

$\rightsquigarrow \mathbb{R}^{2g} \cong \mathbb{C}^{2g}/F^0$ w/ basis $e_1 = \frac{1}{d_1} v_1, \dots, e_g = \frac{1}{d_g} v_g$.

$\Rightarrow e_1, \dots, e_g, ie_1, \dots, ie_g \in \mathbb{R}^{2g}$ is $\mathbb{R}$-basis. In this basis $\mathbb{Z}^{2g} \hookrightarrow \mathbb{R}^{2g}$ looks like

$$\Pi_{\mathbb{R}} = \left(\begin{array}{c|c} \Pi_{11} & d_1 \dots d_g \\ \hline \Pi_{21} & 0 \end{array}\right) \in Mat_{2g}(\mathbb{R}). \quad Z := \Pi_{11} + i\Pi_{21} \in Mat_g(\mathbb{C}).$$

__Prop__: This construction identifies $\mathcal{H}_{g,D}$ w/ $\{Z \in Mat_g(\mathbb{C}) : {}^tZ = Z \text{ and } \operatorname{Im} Z \text{ is pos. def.}\}$. Moreover,

nat. action of $Sp(\mathbb{R}^{2g}, E_D)$ on $\mathcal{H}_{g,D}$ becomes $\left(\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right) Z = (AZ+B)(CZ+D)^{-1}$.

$\mathcal{H}_{g,D} \cong \{\text{polarized } (X,E) \dots\}$ becomes $Z \mapsto X = \mathbb{C}^g/\mathcal{U}_Z$ where $\mathcal{U}_Z = \operatorname{im}([Z \mid d_1 \dots d_g] : \mathbb{Z}^{2g} \to \mathbb{C}^g)$ and

symplectic basis $H_1(X;\mathbb{Z}) = \mathcal{U}_Z \cong \mathbb{Z}^{2g}$ is the obvious one.

(Field of definition for Shimura datum = reflex field)