The Magic of Projective Modules

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Unless otherwise stated, R is a commutative unital ring.

P is a **projective** R-module if we always have a commutative diagram of R-module homomorphisms:

$$\begin{tikzcd} & P \arrow[dl, dashed] \arrow[d] \\ M \arrow[r, two heads] & N \end{tikzcd}$$

In the above diagram, the dashed arrow indicates a morphism induced by the other morphisms. Let F be a free R-module. It follows almost immediately that F is projective, the key point being that maps out of F are entirely determined by their action on a choice of basis for F. Almost the same argument shows that direct summands of free modules are projective. This goes the other way too – projective modules are direct summands of free modules. To see this, let P be projective. Let F be a free R-module with associated epimorphism $\varphi : F \twoheadrightarrow P$ (Why does this exist?). We then have a commutative diagram

$$\begin{tikzcd} & P \arrow[dl, dashed, "\psi"'] \arrow[d, "\mathrm{id}_P"] \\ F \arrow[r, two heads, "\varphi"'] & F \end{tikzcd}$$

Since $\varphi\psi = \mathrm{id}_P$, ψ is injective and so P is isomorphic to its image under ψ (and so we can think of P as a submodule of F). By the First Isomorphism Theorem, $P = \operatorname{im}\varphi \cong F/\ker\varphi$ and a little work gives $F \cong P \oplus \ker\varphi$. With this information, we can produce a wealth of examples of projective modules.

- Let A, B be nontrivial rings and $R = A \times B$. Since

  $$(A \times 0) \oplus (0 \times B) = A \times B = R,$$

  the R-modules $A \times 0$ and $0 \times B$ are projective but not free.

- Here's an example for those who have some experience with representation theory. Let G be a finite group and k a field such that $|G| \neq 0$ in k. Consider the diagonal map $\Delta : k \to kG$ from k to the group ring kG given by

  $$\Delta(x) = \sum_{g \in G} xg.$$

  Δ is a monomorphism of kG-modules and so we can identify k as a kG-submodule of kG. By Maschke's Theorem, k has a direct sum complement in kG and so k is a projective kG-module by the above.

- Let P, P' be projective R-modules. By the above, there are R-modules Q, Q', F, F' with F, F' free such that $P \oplus Q = F$ and $P' \oplus Q' = F'$. Then,

$$(P \otimes_R P') \oplus (Q \otimes_R Q') \cong (P \oplus Q) \otimes_R (P' \oplus Q') = F \otimes_R F'$$

  and so $P \otimes_R P'$ is projective since $F \otimes_R F'$ is free (with basis given by tensoring chosen bases for F and F').

There are two more equivalent characterizations of projective modules that are very useful in practice. The first uses what we talked about last week, and the second is best viewed from the perspective of basic category theory. Without further ado:

(1) P is projective if and only if every short exact sequence

$$0 \longrightarrow A \longrightarrow B \xrightarrow{f} P \longrightarrow 0$$

splits – i.e., there is a commutative diagram

$$\begin{tikzcd} 0 \arrow[r] & A \arrow[r] \arrow[d, equal] & B \arrow[r, "f"] \arrow[d, "\cong"] & P \arrow[r] \arrow[d, equal] & 0 \\ 0 \arrow[r] & A \arrow[r] & A \oplus P \arrow[r] & P \arrow[r] & 0 \end{tikzcd}$$

This is the case if and only if the above short exact sequence has a right section – i.e., an R-module homomorphism $g : P \to B$ such that $fg = \mathrm{id}_P$.[1]

(2) Given an R-module homomorphism $f : M \to N$, there is an associated R-module homomorphism $f_* : \mathrm{Hom}_R(P, M) \to \mathrm{Hom}_R(P, N)$ given by $f_*(\varphi) = f\varphi$. If f is injective then f_* is necessarily also injective (exercise!). However, f_* need not be surjective even if f is surjective. When is f_* surjective for every choice of M, N, f? If P is projective then f_* is certainly surjective since we always have the completed commutative diagram

$$\begin{tikzcd} & P \arrow[dl, dashed, "\psi"'] \arrow[d, "\varphi"] \\ M \arrow[r, two heads, "f"'] & N \end{tikzcd}$$

That is, given any $\varphi \in \mathrm{Hom}_R(P, N)$, the element $\psi \in \mathrm{Hom}_R(P, M)$ defined as above is a lift of φ via f_*. This goes the other way too. Stated in the language of category theory, P is projective if and only if the (covariant) functor $\mathrm{Hom}_R(P, \bullet) : \mathsf{Mod}_R \to \mathsf{Mod}_R$ is exact – i.e., $\mathrm{Hom}_R(P, \bullet)$ carries short exact sequences of R-modules to short exact sequences of R-modules.

This information immediately give us another characterization of projective modules that helps further explain the terminology. Let $M_n(R)$ denote the ring of $n \times n$ matrices over R. An element of $M_n(R)$ is called a **projection** if $P^2 = P$.[2] Given an R-module M, M is projective if and only if it is the image of a projection. This characterization is actually extremely useful for proving an amazing result known as Swan's Theorem, which we discuss below.[3]

Here are some questions worth exploring.

[1]This choice of the term section makes more sense if you know something about vector bundles.
[2]This is geometric terminology. For the more algebraically minded, the term is **idempotent**.
[3]The result has been significantly generalized over time due in part to the work of J. P. Serre.

- **Why are projective modules useful?**

  We describe one major use of projective modules – namely, projective modules allow us to define (classical) left derived functors like Tor_*^R and are also useful for computing certain right derived functors like Ext_R^*.[4] These derived functors find application in algebraic geometry, algebraic number theory, algebraic topology, commutative algebra, and functional analysis (particularly in connection to operator algebras). You can learn about derived functors in any text on homological algebra.

- **When is a projective module free?**

  Finitely generated projective modules over PIDs are free by structure theory. In more detail, a finitely generated module over a PID decomposes as a direct sum of a free part and a torsion part. Projective modules are torsion-free (since they are direct summands of free modules over PIDs, which are themselves torsion-free) and so the corresponding torsion part must vanish. In general, Kaplansky and others have shown that *all* projective modules over certain nice rings (e.g., PIDs and local rings) are always free.

- **How can we view projective modules geometrically, both in terms of algebraic geometry and K-theory?**

  An R-module M is called **locally free** if, for every prime ideal $\mathfrak{p}$ of R, the localized module $M_\mathfrak{p}$ is a free $R_\mathfrak{p}$-module. It turns out that finitely generated projective modules are precisely those which are finitely presented and locally free.[5] We can give this statement a more geometric interpretation. Let $\operatorname{Spec} R$ denote the set of prime ideals of R. This set has a natural topology, called the **Zariski topology**, defined by taking the closed sets to be

  $$V(I) = \{\mathfrak{p} \in \operatorname{Spec} R : I \subset \mathfrak{p}\}$$

  for $I \subset R$ an ideal. Given a finitely generated projective module Q, we obtain a rank function

  $$r(Q) : \operatorname{Spec} R \to \mathbb{N}, \quad \mathfrak{p} \mapsto \operatorname{rank}_{R_\mathfrak{p}} Q_\mathfrak{p}.$$

  If we imbue $\operatorname{Spec} R$ with the Zariski topology and $\mathbb{N}$ with the discrete topology then $r(Q)$ is a locally constant continuous function. This rank function has lots of interesting properties that can be given geometric interpretations. Throughout the following, P and Q are projective R-modules.

  - Let P, Q be finitely generated. Then, $P \otimes_R Q$ is finitely generated projective and

    $$r(P \otimes_R Q) = r(P)r(Q).$$

  - $\operatorname{Spec} R$ is connected in the Zariski topology if and only if every finitely generated projective R-module has constant rank if and only if R has no nontrivial idempotents.[6]
  - Let R be projective of constant rank. Then, P is finitely generated.
  - Let P, Q be finitely generated and $I \subset R$ a radical ideal such that $P/IP \cong Q/IQ$. Then, $P \cong Q$.[7]

[4] Specifically, projective modules allow us to define left derived functors of right exact covariant functors and right derived functors of left exact contravariant functors.

[5] See the end of these notes for a proof.

[6] An **idempotent** of R is $e \in R$ such that $e^2 = e$, which is **nontrivial** if $e \notin \{0, 1\}$.

[7] Note that radical ideals are tied to the notion of nilpotency. Geometrically, nilpotents correspond to infinitesimals.

- Let P be finitely generated and $f : R \to S$ a morphism of commutative unital rings. Let $f^* : \operatorname{Spec} S \to \operatorname{Spec} R$ be the map induced by pullback – i.e., $f^*(\mathfrak{q}) = f^{-1}(\mathfrak{q})$ for $\mathfrak{q} \in \operatorname{Spec} S$. Then, $P \otimes_R S$ is a finitely generated projective S-module and the diagram

$$\begin{tikzcd} \operatorname{Spec} S \arrow[d, "f^*"'] \arrow[dr, "r(P\otimes_R S)"] & \\ \operatorname{Spec} R \arrow[r, "r(P)"'] & \mathbb{N} \end{tikzcd}$$

  commutes.

- Let P be finitely generated and $P^\vee = \operatorname{Hom}_R(P, R)$ its dual. Then, $P^\vee$ is a finitely generated projective module and $r(P) = r(P^\vee)$ as functions from $\operatorname{Spec} R$ to $\mathbb{N}$. Define the **trace** of P, written $\operatorname{tr}(P)$, to be the image of the natural map $P^\vee \otimes_R P \to R$. Then, $\operatorname{tr}(P)^2 = \operatorname{tr}(P)$ and, given $\mathfrak{p} \in \operatorname{Spec} R$, $P_\mathfrak{p} = 0$ if and only if $\operatorname{tr}(P) \subset \mathfrak{p}$.

Depending on the flavor and strength of the Serre-Swan Theorem that you are willing to accept, in nice settings vector bundles and/or coherent sheaves correspond to their finitely generated projective modules of sections. The above facts definitely suggest such a correspondence. What follows is a hopefully concrete correspondence, insofar as it avoids discussion of sheaves.

Let X be a topological space. Let K denote either the field $\mathbb{R}$ or $\mathbb{C}$ and $C(X)$ the K-algebra of continuous functions from X to K. A (topological K-)**vector bundle** E over X is a topological space E together with the following data:

- a continuous surjection $\pi : E \to X$;
- for every $x \in X$, a **fiber** $E_x = \pi^{-1}(x)$ over x with the structure of a K-vector space;
- for every $x \in X$, an open neighborhood U of x in X and a homeomorphism

$$f : \pi^{-1}(U) \to U \times K^n$$

  with $n \in \mathbb{N}$ such that, for every $y \in U$, f induces a K-linear isomorphism $E_y \to \{y\} \times K^n$ (we call this last piece of data a **local trivialization** of X at x).

With a little work, we obtain a nice additive category $\mathsf{Vect}_K(X)$ of topological K-vector bundles over X. A **global section** of the vector bundle $\pi : E \to X$ is a continuous map $\sigma : X \to E$ such that $\pi\sigma = \mathrm{id}_X$. The set $\Gamma(E)$ of global sections naturally has the structure of a $C(X)$-module. With a little work, one can show that Γ is an additive functor from $\mathsf{Vect}_K(X)$ to $\mathsf{Mod}_{C(X)}$. Swan's Theorem asserts that Γ is an equivalence of categories between $\mathsf{Vect}_K(X)$ and the additive category of finitely generated projective $C(X)$-modules when X is a compact Hausdorff space.

Endnote

Our aim is to show that an R-module P is finitely generated projective if and only if it is finitely presented and locally free. We begin by showing that P projective is finitely generated if and only if it is finitely presented. Finite presentation is a stronger condition than finite generation, so we need only show that P is finitely presented if it is finitely generated. Choose F a free R-module of finite rank and $\varphi : F \to P$ an epimorphism. We have a short exact sequence

$$0 \longrightarrow \ker\varphi \longrightarrow F \xrightarrow{\varphi} P \longrightarrow 0$$

which splits since P is projective and so $F \cong \ker\varphi \oplus P$. Thus, F surjects onto $\ker\varphi$ and so $\ker\varphi$ is finitely generated since F is finitely generated. It follows that P is finitely presented.

Next, we show that P finitely generated projective is free if $(R, \mathfrak{m})$ is a local ring. By assumption, P has a generating set of size $d < \infty$ and so there is an epimorphism $\varphi : R^d \to P$ giving a short exact sequence

$$0 \longrightarrow \ker\varphi \longrightarrow R^d \xrightarrow{\varphi} P \longrightarrow 0$$

Since P is projective, this sequence splits and so $P \oplus Q \cong R^d$ for $Q = \ker\varphi$ (note that Q is finitely generated). Since $\mathfrak{m}$ is a maximal ideal of R, $R/\mathfrak{m}$ is a field. We have a natural isomorphism $R/\mathfrak{m} \otimes_R P \cong P/\mathfrak{m}P$ of R-modules, which gives an $R/\mathfrak{m}$-vector space structure on $P/\mathfrak{m}P$ induced by the natural $R/\mathfrak{m}$-vector space structure on $R/\mathfrak{m} \otimes_R P$. Hence, $P/\mathfrak{m}P$ has an $R/\mathfrak{m}$-vector space basis of size $d' \leq d$. By Nakayama's Lemma, this basis lifts to an R-module generating set for P and so we may assume without loss of generality that $\dim_{R/\mathfrak{m}} P/\mathfrak{m}P = d' = d$. A similar argument shows that $R^d/\mathfrak{m}R^d \cong R/\mathfrak{m} \otimes_R R^d$ has dimension d as an $R/\mathfrak{m}$-vector space. Since

$$R/\mathfrak{m} \otimes_R R^d \cong R/\mathfrak{m} \otimes_R (P \oplus Q) \cong (R/\mathfrak{m} \otimes_R P) \oplus (R/\mathfrak{m} \otimes_R Q)$$

and $R/\mathfrak{m} \otimes_R Q \cong Q/\mathfrak{m}Q$, we have

$$d = \dim_{R/\mathfrak{m}} R/\mathfrak{m} \otimes_R R^d = \dim_{R/\mathfrak{m}} R/\mathfrak{m} \otimes_R P + \dim_{R/\mathfrak{m}} R/\mathfrak{m} \otimes_R Q = d + \dim_{R/\mathfrak{m}} Q/\mathfrak{m}Q$$

and so $\dim_{R/\mathfrak{m}} Q/\mathfrak{m}Q = 0 \implies Q = \mathfrak{m}Q$. By Nakayama's Lemma, $Q = 0$ and so $P \cong R^d$.

Finally, we show that we can reduce to the local case. Suppose first that P is projective. Since $R_\mathfrak{p}$ is a local ring with maximal ideal $\mathfrak{p}R_\mathfrak{p}$ for $\mathfrak{p} \in \operatorname{Spec} R$, it suffices by the above to show that each $P_\mathfrak{p}$ is a projective $R_\mathfrak{p}$-module. As above, $P \oplus Q \cong R^d$ for some R-module Q and $d \geq 0$. Since localization commutes with direct sums, we have a sequence of isomorphisms of $R_\mathfrak{p}$-modules

$$P_\mathfrak{p} \oplus Q_\mathfrak{p} \cong (P \oplus Q)_\mathfrak{p} \cong (R^d)_\mathfrak{p} \cong (R_\mathfrak{p})^d$$

and so $P_\mathfrak{p}$ is projective.

Conversely, suppose that P is locally free. It suffices to show that $f_* : \operatorname{Hom}_R(P, A) \to \operatorname{Hom}_R(P, B)$ is surjective given any R-module epimorphism $f : A \twoheadrightarrow B$. Since localization is right exact (and so commutes with cokernels), it suffices to show that the localized map

$$(f_*)_\mathfrak{p} : (\operatorname{Hom}_R(P, A))_\mathfrak{p} \to (\operatorname{Hom}_R(P, B))_\mathfrak{p}$$

is an epimorphism for every $\mathfrak{p} \in \operatorname{Spec} R$. Given any R-module M, there is a natural map

$$\Phi : (\operatorname{Hom}_R(P, M))_\mathfrak{p} \to \operatorname{Hom}_{R_\mathfrak{p}}(P_\mathfrak{p}, M_\mathfrak{p}).$$

We claim that Φ is an isomorphism. Since P is finitely presented, there is a short exact sequence

$$0 \longrightarrow R^m \longrightarrow R^n \longrightarrow P \longrightarrow 0$$

with $m, n < \infty$. This induces a commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & (\operatorname{Hom}_R(P, M))_{\mathfrak{p}} \arrow[r] \arrow[d, "\Phi"] & (\operatorname{Hom}_R(R^n, M))_{\mathfrak{p}} \arrow[r] \arrow[d] & (\operatorname{Hom}_R(R^m, M))_{\mathfrak{p}} \arrow[d] \\
0 \arrow[r] & \operatorname{Hom}_{R_{\mathfrak{p}}}(P_{\mathfrak{p}}, M_{\mathfrak{p}}) \arrow[r] & \operatorname{Hom}_{R_{\mathfrak{p}}}((R^n)_{\mathfrak{p}}, M_{\mathfrak{p}}) \arrow[r] & \operatorname{Hom}_{R_{\mathfrak{p}}}((R^m)_{\mathfrak{p}}, M_{\mathfrak{p}})
\end{tikzcd}$$

with exact rows. The two right-hand vertical maps are isomorphisms by inspection. Adjoining 0's to the left of each row and applying the Five Lemma gives that Φ is an isomorphism. Thus, we have a commutative diagram

$$\begin{tikzcd}
(\operatorname{Hom}_R(P, A))_{\mathfrak{p}} \arrow[r, "(f_*)_{\mathfrak{p}}"] \arrow[d, "\cong"'] & (\operatorname{Hom}_R(P, B))_{\mathfrak{p}} \arrow[d, "\cong"] \\
\operatorname{Hom}_{R_{\mathfrak{p}}}(P_{\mathfrak{p}}, A_{\mathfrak{p}}) \arrow[r, "(f_{\mathfrak{p}})_*"'] & \operatorname{Hom}_{R_{\mathfrak{p}}}(P_{\mathfrak{p}}, B_{\mathfrak{p}})
\end{tikzcd}$$

Since $f : A \twoheadrightarrow B$ and localization is right exact, $f_{\mathfrak{p}} : A_{\mathfrak{p}} \to B_{\mathfrak{p}}$ is surjective. Since $P_{\mathfrak{p}}$ is free hence projective, $(f_{\mathfrak{p}})_*$ is therefore also surjective. By the commutativity of the above diagram, $(f_*)_{\mathfrak{p}}$ is surjective. Thus, f_* is surjective and so P is projective.