Quiz 2 Math 2202

Guidelines

- This quiz[1] is for you to test yourself on what we've been studying recently.
- You have 10 minutes. As a section, we will go over the quiz (or part of it). Solutions will be posted online as well.

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1. Let L be the line passing through the point $(3, 1, -1)$ and parallel to the vector $\langle 4, -3, 1\rangle$.

   For each vector, decide if it is parallel to the line L.

   If not, imagine the vector anchored at the point $(3, 1, -1)$ and determine the angle between the vector and the line L, and whether it is acute ($< \pi/2$), obtuse $> \pi/2$ or neither. (You can leave the angle value in terms of inverse sine or cosine.)

   (a) $\langle 3, 2, -6\rangle$

   (b) $\langle 3, 2, 6\rangle$

   (c) $\langle -8, 6, -2\rangle$

   Solution: Let $\mathbf{v} = \langle 4, -3, 1\rangle$, and let L be the line passing through the point $(3, 1, -1)$. The first thing to notice for this problem is that the angle between a vector anchored at the point $(3, 1, -1)$ and the line L is the same as the angle between that vector and $\mathbf{v}$, if they were both anchored at the origin. This is useful, because we can take the dot product of two vectors to learn about the angle between them.

   (a) $\langle 3, 2, -6\rangle$
   Let $\mathbf{a} = \langle 3, 2, -6\rangle$, and remember that $\mathbf{v} = \langle 4, -3, 1\rangle$.
   Since

   $$\begin{aligned} \mathbf{a} \bullet \mathbf{v} &= 3 \cdot 4 + 2 \cdot (-3) + (-6) \cdot 1 \\ &= 0 \end{aligned}$$

   Now we know that $\mathbf{a}$ and $\mathbf{v}$ are perpendicular (two vectors are perpedincular exactly when their dot product is zero). So, the line L and the vector $\mathbf{a}$ are perpendicular as well.

[1] Why? While there is much we don't know about how we learn, researchers in cognitive science have identified some principles that support learning of certain things. One such principle is frequent testing, also known as *retrieval practice*. Testing out your brain to see what you can remember without the aid of notes or other people and then getting corrective feedback can actually help you remember more of what you're trying to learn. For more, see for example `https://en.wikipedia.org/wiki/Testing_effect` and the book *Make It Stick* by Brown, Roediger III and McDaniel.

(b) $\langle 3, 2, 6\rangle$

Let $\mathbf{b} = \langle 3, 2, 6\rangle$, and remember that $\mathbf{v} = \langle 4, -3, 1\rangle$.

Remember the formula:

$$\mathbf{b} \bullet \mathbf{v} = |\mathbf{b}||\mathbf{v}| \cos(\theta)$$

Where θ is the angle between $\mathbf{b}$ and $\mathbf{v}$, the angle we're looking for. Then, solve for θ:

$$\theta = \cos^{-1}\left(\frac{b \bullet v}{|b||v|}\right)$$

Compute the dot product of $\mathbf{b}$ and $\mathbf{v}$:

$$\begin{aligned}\mathbf{b} \bullet \mathbf{v} &= 3 \cdot 4 + 2 \cdot (-3) + 6 \cdot 1 \\ &= 12\end{aligned}$$

Also compute the length of $\mathbf{b}$ and $\mathbf{v}$:

$$|\mathbf{b}| = \sqrt{3^2 + 2^2 + 6^2} = \sqrt{49} = 7$$

$$|\mathbf{v}| = \sqrt{4^2 + (-3)^2 + 1^2} = \sqrt{26}$$

Now substitute these values into the formula to solve for θ, the angle between $\mathbf{b}$ and $\mathbf{v}$, which is also the angle between $\mathbf{b}$ and L.

$$\begin{aligned}\theta &= \cos^{-1}\left(\frac{b \bullet v}{|b||v|}\right) \\ &= \cos^{-1}\left(\frac{12}{7\sqrt{26}}\right)\end{aligned}$$

This will be an acute angle, since $\cos(\theta) > 0$.

(c) $\langle -8, 6, -2\rangle$

Let $\mathbf{c} = \langle -8, 6, -2\rangle$, and remember that $\mathbf{v} = \langle 4, -3, 1\rangle$.

Just like for part b), we can solve the dot product equation for the angle between $\mathbf{c}$ and $\mathbf{v}$ (you will get π), but there's a faster way to do this problem. Notice:

$$c = \langle -8, 6, -2\rangle = -2 \cdot \langle 4, -3, 1\rangle$$

Since $\mathbf{c}$ is a scalar multiple of $\mathbf{v}$, $\mathbf{c}$ and $\mathbf{v}$ are parallel, so $\mathbf{c}$ and L are also parallel. The angle between them is then 0 or π, depending on whether the vectors are in the same or opposite direction. Since $\mathbf{c}$ is in the opposite direction $\mathbf{c} = -2\mathbf{v}$, the angle is π.

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2. Find a vector equation for the line L, the line passing through the point $(3, 1, -1)$ and parallel to the vector $\langle 4, -3, 1\rangle$. Then write parametric equations for this line.

Solution:

A vector equation for L is given by:

$$\mathbf{r}(t) = \langle 3, 1, -1\rangle + t\langle 4, -3, 1\rangle$$

Where the line is traced out by the tip of the vector $\mathbf{r}(t)$ as t varies through the real numbers.

We can find parametric equations:

$$\mathbf{r}(t) = \langle 3, 1, -1\rangle + t\langle 4, -3, 1\rangle = \langle 3 + 4t, 1 - 3t, -1 + t\rangle$$

So,

$$x(t) = 3 + 4t$$
$$y(t) = 1 - 3t$$
$$z(t) = -1 + t$$

3. Write an equation for the plane $\mathcal{P}$ parallel to the xz-plane and containing the point $(0, -3, 0)$. Sketch this plane.

Solution:

Remember that the xz-plane can be seen as the solutions to $y = 0$ in three-dimensional space, which can be written as the set $S = \{(x, 0, z) : x \text{ and } z \text{ are real numbers}\}$.

Every plane parallel to the xz-plane is the set of solutions in three dimensional space to an equation $y = a$ for some fixed real number a. Seen as a set, this plane is $\{(x, a, z) : x$ and z are real numbers$\}$.

But, if we're looking for a plane that is both parallel to the xz-plane and contains the point $(0, -3, 0)$, we need the plane given by the equation $y = -3$, which can also be viewed as the set $\{(x, -3, z) : x$ and z are real numbers$\}$.

4. (Extra) Does L intersect the plane $\mathcal{P}$ in #3? If so, where? If not, why not?

Solution:

The set of points (x, y, z) that are on the line L are exactly the solutions to the parametric equations:

$$x(t) = 3 + 4t$$
$$y(t) = 1 - 3t$$
$$z(t) = -1 + t$$

from question 2.

The points (x, y, z) on the plane #3 are exactly the solutions to:

$$y = -3$$

So, L intersects the plane in #3 precisely at the solutions to the set of simultaneous equations:

$$x(t) = 3 + 4t$$
$$y(t) = 1 - 3t$$
$$z(t) = -1 + t$$
$$y = -3$$

Since $y = -3$ and $y(t) = 1 - 3t$, if there is a solution (x, y, z) to these equations, then

$$-3 = y = 1 - 3t, \text{ so } t = \frac{4}{3}$$

.

When $t = \frac{4}{3}$,

$$x = 3 + 4(\frac{4}{3}) = \frac{25}{3}$$
$$y = 1 - 3\frac{4}{3} = -3$$
$$z = -1 + \frac{4}{3} = \frac{1}{3}$$

Since the point $(\frac{25}{3}, -3, \frac{1}{3})$ is on the line L and has $y = -3$, this is the intersection point of the line L and the plane in #3.