Formal schemes à la EGA

(e.g., topological rings)

We can work in "alg." categories w/ topological conditions if we are careful about respecting and endowing topologies. This applies, e.g., when we write down the sheaf condition and definition of stalk.

$f : X \to Y$ cont. $\rightsquigarrow f_*$ on sheaves.

Def: $f : X \to Y$ cont., $\mathcal{F}$ sheaf of sets on X, $\mathcal{G}$ sheaf of sets on Y. An f-map $\xi : \mathcal{G} \to \mathcal{F}$ is a collection of $\xi_V : \mathcal{G}(V) \to \mathcal{F}(f^{-1}(V))$ indexed by opens $V \subseteq Y$ s.t.

$$\begin{tikzcd} \mathcal{G}(V) \arrow[r, "\xi_V"] \arrow[d, "\mathrm{res}"'] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{F}(f^{-1}(V)) \arrow[d, "\mathrm{res}"] \\ \mathcal{G}(V') \arrow[r, "\xi_{V'}"'] & \mathcal{F}(f^{-1}(V')) \end{tikzcd}$$

$\forall\ V' \subseteq V \subseteq Y$ open.

In the topological setting, we can do as above to get $\xi : \mathcal{G} \to \mathcal{F}$, requiring the maps ξ_V to be cont. We obtain $\{f \text{ maps } \mathcal{G} \to \mathcal{F}\} \leftrightarrow \mathrm{Mor}_{Sh(Y, Top)}(\mathcal{G}, f_*\mathcal{F})$.

Given f-map $\xi : \mathcal{G} \to \mathcal{F}$ and $x \in X$ w/ $y = f(x) \in Y$, we get cont. stalk map $\xi_x : \mathcal{G}_y \to \mathcal{F}_x$.

All of this gives cat. LTRS "locally topologically ringed spaces". Things work nice if X has basis of qc opens.

$\mathcal{F}$ sheaf in non-top. cat. We get sheaf in top. cat. as follows. $\mathcal{F}(\mathcal{U})$ gets discrete top. for $\mathcal{U} \subseteq X$ qc open. More generally we choose qc open cover $\mathcal{U} = \bigcup_i \mathcal{U}_i$ and give $\mathcal{F}(\mathcal{U}) = \prod_i \mathcal{F}(\mathcal{U}_i)$ the induced top.

Sheaf in top. cat. is <u>pseudo-discrete</u> if sections over qc opens are discrete. Let's recall admissibility.

A top. R-mod. is <u>linearly topologized</u> if 0 has $\overbrace{\text{fund. system. of nbhds}}^{=\text{ local base}}$ which are (top.) submodules. Let R be linearly topologized. $I \trianglelefteq R$ is <u>ideal of definition</u> if I is open and every nbhd of 0 contains some I^n ("the powers of I see the top. at 0"). R is <u>preadmissible</u> if it has such an I, and <u>admissible</u> if it is in addition complete.

Let A be top. adm. ring and $\{I_\lambda\}$ fund. system of ideals of definition. To each λ we have $\mathrm{Spec}(A/I_\lambda)$.

Given $I_\lambda \subseteq I_\mu$, we have $I_\mu^n \subseteq I_\lambda$ and so $\mathrm{Spec}(A/I_\mu) \to \mathrm{Spec}(A/I_\lambda)$ is thickening.

$\mathrm{Spec}(A/I_\lambda) \to \mathrm{Spec}(A)$ is homeo. onto set of open prime ideals. We call this image $\mathrm{Spf}(A)$.

$\mathcal{O}_{\mathrm{Spec}(A/I_\lambda)}$ is sheaf on $\mathrm{Spf}(A)$ and gives $\mathcal{O}_\lambda$ pseudo-discrete (in top. cat.) sheaf. Define

$\mathcal{O}_{\mathrm{Spf}(A)} := \lim\limits_{(\text{top. cat.})} \mathcal{O}_\lambda$. $(\mathrm{Spf}(A), \mathcal{O}_{\mathrm{Spf}(A)})$ is <u>formal spectrum</u>.