Chowla - Selberg Formula

E imag. quad. field, $d := \mathrm{disc}(E)$, $h := \#\mathrm{Cl}(E)$, $w := \mathcal{O}_E^\times$, $\chi : (\mathbb{Z}/d)^\times \to \{\pm 1\}$. ← quadratic char.

<u>Thm</u> (C-S): $\prod_{i=1}^{h} \Delta(a_i)\Delta(a_i^{-1}) = \left(\frac{1}{2\pi d}\right)^{12h} \prod_{0<a<d} \Gamma(a/d)^{6w\chi(a)}$ where Δ is Ramanujan's modular discriminant, ← weight 12 modular form

a_i complete set of rep.'s for distinct equiv. classes in $\mathrm{Cl}(E)$.

Goal is to evaluate logarithmic derivative in two ways.

[Shouldn't there be a pole?]

<u>Pf</u> (Sketch): $Z_{\mathcal{O}_E} = \sum_{I \trianglelefteq \mathcal{O}_E} (NI)^{-s}$ zeta function for $\mathcal{O}_E$. This is $\sum_{i=1}^{h} Z_{\mathcal{O}_E}(s;a_i)$ for

$$Z_{\mathcal{O}_E}(s;a_i) := \sum_{I \equiv a_i} (NI)^{-s}.$$

<u>Fact</u> (Kronecker limit formula): $Z_{\mathcal{O}_E}(s;a_i) = -\frac{1}{w} - \frac{1}{12w}\log(\Delta(a_i)\Delta(a_i^{-1}))s + O(s^2)$.

$$\Rightarrow Z_{\mathcal{O}_E}(s) = -\frac{h}{w} - \frac{1}{12w}\sum_{i=1}^{h}\log(\Delta(a_i)\Delta(a_i^{-1}))s + O(s^2).$$ This gives

$$\left.\frac{Z'_{\mathcal{O}_E}}{Z_{\mathcal{O}_E}}\right|_{s=0} = \frac{1}{12h}\sum_{i=1}^{h}\log(\Delta(a_i)\Delta(a_i^{-1})).$$ At the same time, $Z_{\mathcal{O}_E}(s) = \zeta(s)L(s,\chi)$ and so

$$\left.\frac{Z'_{\mathcal{O}_E}}{Z_{\mathcal{O}_E}}\right|_{s=0} = \underbrace{\left.\frac{\zeta'(s)}{\zeta(s)}\right|_{s=0}}_{=-\log 2\pi} + \underbrace{\left.\frac{L'(s,\chi)}{L(s,\chi)}\right|_{s=0}}_{\text{compute this...}}.$$ $\square$

Arithmetic Intersection

M locally integral Deligne-Mumford (DM) stack of finite type / Spec $\mathbb{Z}$ s.t. $M_{\mathbb{Q}}$ is smooth.

<u>Def</u>: Z Cartier divisor (on M). <u>Green's function</u> for Z is smooth function $\Phi : M(\mathbb{C}) \setminus Z(\mathbb{C}) \to \mathbb{R}$ s.t.

(1) [logarithmic singularity] $\mathcal{U} \subseteq M(\mathbb{C})$ holomorphic orbifold chart, f rational function on $\mathcal{O}_{M(\mathbb{C})}|_{\mathcal{U}}$ s.t.

$\mathrm{div}(f)|_{\mathcal{U}} = Z(\mathbb{C})|_{\mathcal{U}} \Rightarrow \Phi|_{\mathcal{U}} + 2\log|f|$ extends to $\mathcal{U}$.

Def: Arithmetic divisor $\hat{Z} := (Z, \Phi)$ is pair w/ Z Cartier divisor and Φ Green's function for Z. $\hat{Z}$ is principal if

$\hat{Z} = (\mathrm{div} f, -2\log|f|)$ for some rational f on M.

Def: Metrized line bundle is pair $\hat{\mathcal{L}} = (\mathcal{L}, \|\cdot\|_{\mathcal{L}})$ w/ $\mathcal{L}$ line bundle$/M$, $\|\cdot\|_{\mathcal{L}}$ smooth family of Hermitian

metrics on $M(\mathbb{C})$.

[These are huge in general.]

$\widehat{CH}'(\cdot)$ = arithmetic divisors/principal, $\widehat{Pic}(\cdot)$ = $\otimes$-group of metrized line bundles / isom.

Fact: $\widehat{Pic}(M) \cong \widehat{CH}(M)$ via choosing sections.

[Morally, Green would have taken his Green functions to be harmonic, at least in the curve case.]

Y "stacky" curve, so regular DM stack finite flat / Spec $\mathbb{Z}$. Over $\mathbb{Q}$ we just get some finite set of pts.

$$\hat{Z} = (0, \Phi) + \sum_{\text{finite}} m_i(Z_i, 0), \quad \widehat{\deg}(0, \Phi) = \frac{1}{2}\sum_{y \in Y(\mathbb{C})} \frac{\Phi(y)}{\#\mathrm{Aut}(y)}, \quad \widehat{\deg}(Z_i, 0) = \sum_{y \in Z(\overline{\mathbb{F}}_p)} \frac{\log p}{\#\mathrm{Aut}(y)}$$

$p :=$ char of Z_i

$\rightsquigarrow \widehat{\deg} : \widehat{CH}'(Y) \to \mathbb{R}$. (extend things out linearly)

[arithmetic intersection against Y]

$\deg(\cdot \cap Y) : \widehat{CH}'(M) \to \mathbb{R}$ given by $\widehat{CH}'(M) \xrightarrow{\sim} \widehat{Pic}(M) \xrightarrow{\text{pullback}} \widehat{Pic}(Y) \simeq \widehat{CH}'(Y) \xrightarrow{\widehat{\deg}} \mathbb{R}$.

Let now M be DM stack of elliptic curves, $\pi : A \to M$ universal elliptic curve. We have Hodge bundle

$\overbrace{\pi_* \Omega^1_{A/M}}^{=:\omega}$ $(g=1)$. We get Faltings metric $\|\eta\|_y^{\mathrm{Falt}} := \left|\int_{A_y(\mathbb{C})} \eta \wedge \bar{\eta}\right|^{1/2}$, $\eta \in \omega|_y = \Gamma(A_y, \Omega^1_{A_y/\mathbb{C}})$

$y \in M(\mathbb{C})$.

$\rightsquigarrow \hat{\omega} := (\omega, \|\cdot\|^{\mathrm{Falt}})$. This is isom. to $(0, -2\log\|\Delta\|^{\mathrm{Falt}})$.

This allows us to work w/ Faltings height, noting that Δ is a certain section of $\omega^{\otimes 12}$.

[Kodaira-Spencer isom. is lurking in the background of the calculations.]

We can recover the C-S formula when we have CM.

Colmez ~~[illegible]~~ conjectures that Faltings height (which he showed only depends on the CM type) can be described entirely in terms of Galois-theoretic data (?)