Siegel half space is nice example of a symmetric space.

<u>Symmetric Spaces</u> (Helgeson(?) is canonical text)

(M,g) smooth Riemannian mfld. $\mathrm{Iso}(M,g)$ isometry gp.

<u>Def</u>: $s_x \in \mathrm{Iso}(M,g)$ is <u>symmetry</u> at $x \in M$ if $s_x(x) = x$ and one the following equiv. conditions hold:

(1) $ds_x : T_xM \to T_xM$ is mult. by -1.

(2) $s_x^2 = id$ and s_x reverses geodesics through x – i.e., $\gamma : (-\epsilon, \epsilon) \to M$ geodesic w/ $\gamma(0) = x \Rightarrow s_x(\gamma(t)) = \gamma(-t)$.

(3) $s_x^2 = id$ and $\exists$ open nbhd $x \in \mathcal{U} \subseteq M$ s.t. x is only fixed pt. of s_x in $\mathcal{U}$.

<u>Remark</u>: Symmetries are unique if they exist. (isometries between conn. Riemannian mflds determined by their behavior on tangent spaces).

<u>Prop/Def</u>: (M,g) is <u>symmetric space</u> if M is conn. and one of the following equiv. conditions hold:

(1) Every $x \in M$ admits symmetry $s_x \in \mathrm{Iso}(M,g)$.

(2) $\mathrm{Iso}(M,g) \curvearrowright M$ transitively and some $x \in M$ admits a symmetry.

Q: How (or is) this stuff related to Coxeter groups?

<u>Example</u>: Look at S^n for all $n \geq 1$.

<u>Pf</u>: Need geodesically complete $\Rightarrow$ complete (as a metric space). [Hopf - Rinow] $\square$

<u>Prop</u>: (M,g) symm. space.

(1) $\mathrm{Iso}(M,g)$ has unique structure of smooth Lie gp. equipped w/ compact-open top.

(2) $\mathrm{Iso}(M,g)^+$ (:= conn. component of id_M) acts trans. on M.

(3) $x \in M \Rightarrow \mathrm{stab}(x) \leq \mathrm{Iso}(M,g)$ is compact.

<u>Remark</u>: (3) is easy. $\mathrm{stab}(x)$ acts on T_xM preserving g_x. We get $\mathrm{stab}(x) \hookrightarrow O(T_xM)$, w/ the latter compact.

M has presentation as G/K w/ $G = \mathrm{Iso}(M,g)^+$ and $K = \mathrm{stab}(x)$ for $x \in M$. Symmetry $s_x \in \mathrm{Iso}(M,g)$

$\rightsquigarrow$ involution $\sigma \in \mathrm{Aut}(G)$ via $\sigma(g) := s_x \circ g \circ s_x$.

<u>Fact</u>: $k \in K \Rightarrow \sigma(k) = k$ and $(G^\sigma)^+ \subseteq K \subseteq G^\sigma$.

Let's go the other way. Start w/ conn. real Lie gp. G and closed $K \leq G$.

- Adjoint (conjugation) action $\mathrm{Ad} : G \to GL(\mathrm{Lie}(G))$.

<u>Def</u>: (G, K) is <u>Riemannian symmetric pair</u> if

(1) image of ~~[illegible]~~ $\mathrm{Ad}: G \to GL(\mathrm{Lie}(G))$ is compact (i.e., $K/K \cap Z(G)$ is compact)

(2) $\exists\, \sigma \in \mathrm{Aut}(G)$ s.t. $\sigma \neq \mathrm{id}$, $\sigma^2 = \mathrm{id}$, and $(G^\sigma)^+ \subseteq K \subseteq G^\sigma$.

<u>Thm</u>: Let (G, K) be Riemannian symm. pair.

(1) G/K admits G-inv. Riemannian metric.

(2) Any such metric makes G/K a symm. space and $G \xrightarrow{\sigma} G$ descends to symm. $G/K \xrightarrow{\sigma} G/K$ at $\mathrm{id} \in G/K$.

<u>Example</u>: $(SL_n(\mathbb{R}), SO_n(\mathbb{R}))$ is Riemannian symm. pair $\Rightarrow$ $SL_n(\mathbb{R})/SO_n(\mathbb{R})$ is symm. space.

For $n = 2$ this is $SL_2(\mathbb{R})/SO_2(\mathbb{R}) \cong \underbrace{\mathcal{H}^+}_{\text{upper-half-space}}$, $g \mapsto g \cdot i$. We get the same symm. space from $(GL_n(\mathbb{R})^+, SO_n(\mathbb{R})\mathbb{R}^+)$

<u>Remark</u>: Image of $\mathrm{Ad}_{\mathfrak{g}}$ is <u>adjoint group</u>, and this is adjoint (as a Lie grp.!).