Schemes

Last time: We discussed (Zariski) sheaves.

"descent" = sheaf conditions holds (something is "sheafy")

Def: A scheme is a space which is a Zariski sheaf and has an open covering by affine schemes. We get $Sch \subseteq Space = Fun(CRing, Set)$.

Exercise: The Zariski sheaf condition is preserved by open and closed embeddings.

Thm: Let $X \in AffSch$. Then, $X \in Sch$. $\Rightarrow AffSch \subseteq Sch$.

Slogan: "Think geometrically, prove algebraically."

Exercise: $\mathbb{A}^n = (\mathbb{A}^1)^{\times n}$ has a closed subspace

$0 := \operatorname{Spec} \mathbb{Z}[t_1, \dots, t_n]/(t_1, \dots, t_n) \hookrightarrow \operatorname{Spec} \mathbb{Z}[t_1, \dots, t_n] = \mathbb{A}^n$.

$\rightsquigarrow$ open embedding $\mathbb{A}^n \setminus 0 \hookrightarrow \mathbb{A}^n$.

1(a) $\mathbb{A}^1 \setminus 0$ is affine. $\operatorname{Spec} \mathbb{Z}[t, t^{-1}] = \operatorname{Spec} \mathbb{Z}[t^{\pm 1}]$

(b) $\mathbb{A}^n \setminus 0$ is not affine for $n > 1$.

(c) Given $A \in \mathrm{CRing}$, $(\mathbb{A}^n \setminus 0)(A)$ looks like

$$\{ (a_1, \dots, a_n) \in A^n : \sum_{i=1}^n a_i x_i = 1 \text{ has soln.} \}.$$

$$\begin{tikzcd} \mathrm{Spec}\, O = \varnothing \arrow[r] \arrow[d] & O \arrow[d, hook] \\ \mathrm{Spec}\, A \arrow[r] & \mathbb{A}^n \end{tikzcd} \quad \rightsquigarrow \quad \begin{tikzcd} O & \mathbb{Z}[t_1, \dots, t_n]/I \arrow[l] \\ A \arrow[u] & \mathbb{Z}[t_1, \dots, t_n] \arrow[l] \arrow[u, two heads] \end{tikzcd}$$

$$O \cong A \otimes_{\mathbb{Z}[t_1, \dots, t_n]} \mathbb{Z}[t_1, \dots, t_n]/I$$

$$\cong A/IA \iff \text{images of } t_1, \dots, t_n \text{ generate } A \text{ (as an } A\text{-module)}$$

The key to proving our theorem is to show that $\mathbb{A}^1$ is a sheaf hence a scheme. To do this, let's introduce a new notion.

<u>Def</u>: Given $X \in \mathrm{Space}$, let $\mathrm{Func}(X) := \mathrm{Hom}_{\mathrm{Space}}(X, \mathbb{A}^1)$.

These are the <u>functions</u> on X.

$$\mathrm{Func}(\mathrm{Spec}\, A) = \mathrm{Hom}_{\ }(\mathrm{Spec}\, A, \mathbb{A}^1)$$

$$\begin{aligned} \mathrm{Func}(\mathrm{Spec}\,A) &= \mathrm{Hom}_{\mathrm{Space}}(\mathrm{Spec}\,A, \mathbb{A}^1) \\ &\cong \mathrm{Hom}_{\mathrm{Space}}(\mathrm{Spec}\,A, \mathrm{Spec}\,\mathbb{Z}[t]) \\ &\cong \mathrm{Hom}_{\mathrm{CRing}}(\mathbb{Z}[t], A) \\ &\cong A \end{aligned}$$

<u>Remark</u>: This should be viewed geometrically - e.g., we should be able to "evaluate" functions.

<u>Exercise</u>: Fix $A \in \mathrm{CRing}$, $f \in A$. (Think of this as $f \in \mathrm{Func}(\mathrm{Spec}\,A)$). $f : \mathrm{Spec}\,A \to \mathbb{A}^1$.

Then, $\mathrm{Spec}\,A \times_{\mathbb{A}^1} (\mathbb{A}^1 \setminus 0) \cong D(f)$.

<u>Pf</u>:

$$\begin{tikzcd} D(f) \arrow[r, "*"] \arrow[d, "*"'] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Spec}\,A \arrow[d, "f"] \\ \mathbb{A}^1 \setminus 0 \arrow[r, hook] & \mathbb{A}^1 \end{tikzcd}$$

$(t \mapsto f)$

$$\mathbb{A}^1 \setminus 0 \times_{\mathbb{A}^1} \mathrm{Spec}\,A \cong \mathrm{Spec}\,\mathbb{Z}[t^{\pm 1}] \otimes_{\mathbb{Z}[t], f} A$$

$$\Rightarrow (\mathbb{A}^1 \setminus 0 \times_{\mathbb{A}^1, f} \operatorname{Spec} A)(B) = (\mathbb{A}^1 \setminus 0)(B) \times_{\mathbb{A}^1(B), f} (\operatorname{Spec} A)(B)$$

$$\cong B^\times \times_{B, f} \operatorname{Hom}_{CRing}(A, B)$$

$$\cong \{ (b, \varphi) \in B^\times \times \operatorname{Hom}_{CRing}(A, B) : \varphi(f) = b \}$$

$$\cong \{ \varphi \in \operatorname{Hom}_{CRing}(A, B) : \varphi(f) \in B^\times \}$$

$$\cong D(f)(B)$$

Exercise: Let $A \in CRing$, $I \trianglelefteq A$ ideal.

$\Rightarrow \{ D(f) : f \in I \text{ nonzero} \}$ is open covering of $D(I)$.

Need to construct the open embeddings $D(f) \hookrightarrow D(I)$

Prop: Let $S = \operatorname{Spec} A \in AffSch$ and $\mathcal{U} \in Cov(S)$.
Then, $\exists$ big principal open covering of S refining $\mathcal{U}$.

Pf: Each $U \in \mathcal{U}$ looks like $D(I_U)$ for some

$I_{\mathcal{U}} \trianglelefteq A$. Hence, $\sum_{\mathcal{U} \in \mathcal{U}} I_{\mathcal{U}} = A$. Consider

$$\mathcal{V} := \bigcup_{\mathcal{U} \in \mathcal{U}} \{ D(f) : f \in I_{\mathcal{U}} \text{ nonzero} \}. \text{ Done.} \quad \square$$

<u>Exercise</u>: $S \subseteq A$ mult. subset $\Rightarrow$ localizing at S is exact on Mod_A.

$\Leftrightarrow$ $S^{-1}A$ is a flat A-module because $S^{-1}M \cong S^{-1}A \otimes_A M$.

<u>Upshot</u>: This lets us check the sheaf condition directly ("by hand").

Big principal open cov. of $\underset{\displaystyle S}{\overset{\displaystyle \shortparallel}{\mathrm{Spec}\, A}}$ looks like $\{D(f_i)\}_{i \in T}$.

$$S \xrightarrow{\sim} \mathrm{eq}\left(\prod_{i \in T} D(f_i) \rightrightarrows \prod_{i,j \in T} (D(f_i) \cap D(f_j)) \right)$$

$$\boxed{A \xrightarrow{\sim} \mathrm{eq}\left(\prod_{i \in T} A_{f_i} \rightrightarrows \prod_{i,j \in T} A_{f_i f_j} \right)}$$

Show this!!!