# Ch. 4 of Laumon

$\mathcal{O}_F$ complete DVR w/ finite res. field $\mathbb{F}_q$, $F := \mathrm{Frac}(\mathcal{O}_F)$. Fix uniformizer $\pi$.

$R$-valued distributions : $\mathrm{Hom}_R(C_c^\infty(H(F), R), R)$. $\int_{H(F)} f \, d\mu := \mu(f)$.

(need same to have $R$-linear structure)

(Want $\mathbb{Q} \subseteq R \subseteq \mathbb{C}$.)

**Remark** : char. $R = 0 \Rightarrow \forall c \in R \ \exists!$ left-invariant measure $\mu_c$ on $H(F)$ s.t. $\mu_c(H(\mathcal{O}_F)) := \mu_c(\mathbb{1}_{H(\mathcal{O}_F)}) = 1$.

$H(\mathcal{O}_F) / \mathcal{U}_{H,n} \cong H(\mathcal{O}_F/\pi^n)$ is finite. $\mu_c(\mathcal{U}_{H,n}) = \dfrac{1}{[H(\mathcal{O}_F) : \mathcal{U}_{H,n}]} \underbrace{\mu_c(H(\mathcal{O}_F))}_{=c}$.

$\mu_H := \mu_1$

We get $\delta_H$ by considering the effect of right mult. on the left-invariant Haar measures.

(extend by $0$ at $0$)

$f \in C_c^\infty(F^\times, R) \Rightarrow \dfrac{f}{|\cdot|} \in C_c^\infty(F, R)$ [ this is the case of $GL_1$ ]

[Cartan decomposition]

(Just need to check $\mathbb{1}_{GL_n(\mathcal{O}_F)}$.)

Take $H = GL_n$, $f \in C_c^\infty(GL_n(F), R)$. Need to think about $GL_n(F) \subseteq M_n(F)$, being careful about top.

We can extend $\dfrac{f}{|\det(\cdot)|^n}$ smoothly by $0$ to $M_n(F) \setminus GL_n(F)$.

Prop: Let $\mu_{M_n}$ be the invariant measure s.t. $\mu_{M_n}(M_n(\mathcal{O}_F)) = 1$. Then, $f \mapsto \int_{M_n(F)} \dfrac{|f(x)|}{|\det(x)|^n} d\mu_{M_n}$ is left- and right-invariant measure on $GL_n(F)$. Hence, $GL_n(F)$ is unimodular.

Let $g \in GL_n(F)$. $L' := g M_n(\mathcal{O}_F)$, $L := M_n(\mathcal{O}_F)$. These are both additive subgps. of $M_n(F)$. We claim

(very important that residue field is finite)

$L \cap L'$ has finite index in both $L$ and $L'$, w/ $\dfrac{[L' : L \cap L']}{[L : L \cap L']} = |\det(g)|^n$. Easy when $g$ is diagonal, since we can just use the case $n = 1$. Cartan decomposition is going to let us deal w/ non-diagonal case.

$$GL_n(F) = \bigsqcup_{(\lambda_1 \dots \lambda_n) \in \mathrm{Diag}_n^+(F)} GL_n(\mathcal{O}_F) \begin{pmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{pmatrix} GL_n(\mathcal{O}_F), \quad |\lambda_1| \geq |\lambda_2| \geq \dots \geq |\lambda_n| > 0.$$

$T_n(F)$ has invariant (product) measure, $\mu_{T_n}$.

As top. spaces, $U_n(F) \cong F^{n(n-1)/2}$. Not compatible w/ grp. structure, since LHS is generally not abelian. for $n>2$

Observation: $U_n(F)$ is unimodular. The key is that we can (sort of) transport invariant measures along the above isom. [ $U_n(F)$ is nilpotent as high enough commutators vanish. ]

Now, $B_n = U_n \rtimes T_n$. More generally, suppose $H = H_1 \rtimes H_2$ and we have invariant measures $\mu_{H_1}$ and $\mu_{H_2}$ on $H_1$ and $H_2$.

Lemma: $\varphi \in C_c^\infty(H, R)$.

(i) $\varphi_{H_2} : h_1 \in H_1(F) \mapsto \int_{H_2(F)} \varphi(h_2 h_1)\, d\mu_{H_2}$ ($h_2 \in H_2(F)$) is an elt. of $C_c^\infty(H_1(F), R)$.

(ii) $\varphi \mapsto \int_{H_1(F)} \varphi_{H_2}\, d\mu_{H_1}$ is left invariant measure on $H(F)$.

Pf: (i) Just need to check for $\varphi = \mathbb{1}_{U_{n,H} h}$.

$\varphi_{H_2} : h_1 \mapsto \int_{H_2(F)} \mathbb{1}_{U_{n,H} h}(h_2 h_1)\, d\mu_{H_2} = \mu_{H_2}(H_2(F) h_1 \cap U_{n,H} h)$. Need to check this is "smooth" (finite image) w/ compact support. The key is that the underlying top. on $H(F)$ is the product top. arising from the topologies on $H_1(F)$ and $H_2(F)$. So, $U_{n,H}$ is built from $U_{n,H_1}$ and $U_{n,H_2}$.

(ii) Need $\int_{H_1(F)} (\ell_{h_1}\varphi)_{H_2}\, d\mu_{H_1} \overset{(1)}{=} \int_{H_1(F)} \varphi_{H_2}\, d\mu_{H_1} \overset{(2)}{=} \int_{H_1(F)} (\ell_{h_2}\varphi)_{H_2}\, d\mu_{H_2}$ $\forall h_1 \in H_1(F)$, $h_2 \in H_2(F)$.

(2) is clear. (1) is where order matters. The reason is that only one ~~on~~ of $H_1, H_2$ is a priori normal in $H$, and we want to simplify an ~~a~~ appropriate conjugation. $\square$

[Something is wrong, possibly w/ the order of ~~integration~~ multiplication/integration ]

$h_2 \in H_2(F)$, $f \in C_c^\infty(H_1(F), R) \rightsquigarrow h_2 :_c f$ via $h_1 \mapsto f(h_2^{-1} h_1 h_2)$.

Left invariant measure on $H_1(F)$: $f \mapsto \int_{H_1(F)} (h_2 :_c f)\, d\mu_{H_1}$. So, we can find $\theta(h_2) \in q^{\mathbb{Z}}$ s.t.

$$\theta(h_2) \int_{H_1(F)} (h_2 :_c f)\, d\mu_{H_1} = \int_{H_1(F)} f \, d\mu_{H_1}.$$

Unsurprisingly, this $\theta$ guy is closely related to the modulus.

$$\begin{tikzcd} H(F) \arrow[r, two heads] \arrow[rr, bend right, "\delta_H"'] & H_2(F) \arrow[r, "\theta"] & q^{\mathbb{Z}} \end{tikzcd}.$$

__Remark__: This is the case $\delta_{H_1} \equiv 1$ and $\delta_{H_2} \equiv 1$. We can get a more general formula.