Deformation to the normal bundle

Jeroen Hekking

KTH Royal Institute of Technology

hekking@kth.se

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Convention: Everything derived

Overview

Cotangent complex

$M \in QC(X)$ is almost connective if $\forall\, \operatorname{Spec} A \to X$. $\exists n$ st $M(A)[n]$ is connective

Definition

Let $f : X \to Y$ be a morphism of stacks. Then f *admits a cotangent complex* if there is an almost connective $L_{X/Y} \in \mathrm{QCoh}(X)$ such that, for each $\eta : \operatorname{Spec} C \to X$ and each connective C-module M, we have an equivalence of spaces (natural in η)

Derived cat. of q.coh modules

$$\mathrm{Mod}_C(\eta^* L_{X/Y}, M) \simeq \mathrm{Fib}(X(C \oplus M) \to X(C) \times_{Y(C)} Y(C \oplus M)) \quad \circledast$$

If such $L_{X/Y}$ exists, we define the *normal sheaf* as $N_{X/Y} := L_{X/Y}[-1]$.

$\circledast$ Says:

$$\mathrm{Mod}_C(\eta^* L_{X/Y}, M) = \left\{ \begin{tikzcd} \operatorname{Spec} C \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} C \oplus M \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \right\}$$

Relation to deformation theory

$$\mathrm{Mod}_B(L_{B/A}, M[1]) \simeq \left\{ \begin{tikzcd} B \arrow[r, equal] & B \\ B\oplus M[1] \arrow[u] & A \arrow[l] \arrow[u] \arrow[ul, dashed] \end{tikzcd} \right\}$$

1s 12

Remark

Let $A \to B$ be a surjection in $\mathcal{A}\mathrm{lg}$.

- For $M \in (\mathcal{M}\mathrm{od}_B)_{\geq 0}$, have $\mathcal{M}\mathrm{od}_B(N_{B/A}, M) \simeq \mathcal{A}\mathrm{lg}_{A/B}(B, B \oplus M[1])$. These are *small extension of B by M*.
- If $A \to B$ is discrete, then a small extension $A \to \widetilde{B} \to B$ by a discrete B-module I is exactly a square-zero extension of $A \to B$ with ideal I.
- Upshot: $N_{Z/X} = L_{Z/X}[-1]$ classifies small / square-zero extension.

For a section $s: B \to B\oplus M[1]$, the small extension is def. as

$$\begin{tikzcd} \widetilde{B} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & B \arrow[d, "0"] \\ B \arrow[r, "s"] & B\oplus M[1] \end{tikzcd}$$

gives fiber sequence $M \to \widetilde{B} \to B$. Thus a sol. to the deformation problem $M \to ? \to B$.

Relation to deformation theory

Example

Let $X = \operatorname{Spec} A$ be discrete, and let $x : \operatorname{Spec} k \to X$ be a point, with ideal $I = \mathfrak{m}_x$.

- Sections of $\mathbb{V}(N_{X/k}) \to \operatorname{Spec} k$ are classified by $\mathcal{M}\mathrm{od}_k(N_{X/k}, k) \simeq \mathcal{M}\mathrm{od}_k(\pi_0(N_{X/k}), k) \simeq \pi_0 N^\vee_{X/k}$.
- We have $\mathcal{M}\mathrm{od}_k(N_{X/k}, k) \simeq \mathcal{A}\mathrm{lg}_{A/k}(, k, k \oplus k[1])$ is also classified by square-zero extensions of k over A. Since k is field, these are extensions of the form $A \to k[\epsilon] \to k$.
- Hence $\pi_0 N^\vee_{X/k}$ is the Zariski tangent space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$.

$$\begin{tikzcd} k[\epsilon] \arrow[r] & X \arrow[d] \\ k \arrow[u] \arrow[r] & k \oplus k[1] \end{tikzcd}$$

So $N_{X/k}$ controls def$^{\underline{s}}$
$$\begin{tikzcd} \operatorname{Spec} k \arrow[r] \arrow[d] & X \\ \operatorname{Spec} k[\epsilon] \arrow[ur, dashed] & \end{tikzcd}$$
, i.e. infinitesimal deformations.

Cotangent complexes for algebraic stacks

Proposition

Let $f : X \to Y$ be n-algebraic. Then $L_{X/Y} \in \mathrm{QCoh}(X)$ exists.

Using the universal property of the cotangent complex, one can reduce to Y is affine, say $Y = \operatorname{Spec} A$.
Now X is n-algebraic, hence we can take an $(n-1)$-smooth epimorphism $g : U \to X$, where U is a scheme. Let $\eta : \operatorname{Spec} A \to X$ be given. We want to construct $\eta^* L_{X/Y}$ with the desired universal property.
Since $\mathrm{QCoh}(X) = \lim_{\operatorname{Spec} A \to X} \mathrm{Mod}_A$, where the limit is taken over all smooth maps $\operatorname{Spec} A \to X$, we can assume that there is a factorization of η through some $f : \operatorname{Spec} A \to U$.
Then we let $\eta^* L_{X/Y}$ be the fiber

exists, since U, Y schemes

exists by induction

$$\eta^* L_{X/Y} \to f^* L_{U/Y} \to f^* L_{U/X}$$

$$\begin{tikzcd} & \operatorname{Spec} A \arrow[d, "\eta"] \arrow[dl, dashed, "\exists f"'] \\ U \arrow[r] & X \end{tikzcd}$$

Normal cones and normal bundles

Let $f : Z \to X$ be a closed immersion of classical schemes, with ideal I.

- The normal cone is $\mathrm{Spec}_Z(\bigoplus_n I^n/I^{n+1})$.
- The normal bundle is $\mathbb{V}_Z(I/I^2) = \mathrm{Spec}_Z(\mathrm{Sym}\, I/I^2)$.
- We always have a closed immersion from the normal cone into the normal bundle, which is an isomorphism if f is regular.
- It holds $I/I^2 \simeq \pi_1(L_{Z/X}) \simeq \pi_0(N_{Z/X})$.

later:

Classically:

$$\begin{tikzcd} \mathbb{P}_Z(\bigoplus I^n/I^{n+1}) \arrow[r] \arrow[d] & \mathrm{Bl}^{cl}_Z X \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

Derived:

$$\begin{tikzcd} \mathbb{P}_Z(N_{Z/X}) \arrow[r] \arrow[d] & \mathrm{Bl}_Z X \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

: universal VCD

But if $Z \to X$ is q.smooth, Z, X discrete, then these two agree

Normal cones and normal bundles

Definition

Let X be a stack, and $M \in \mathrm{QCoh}(X)$. Write $\mathbb{V}_X(M) = \mathbb{V}(M)$ for the stack over X, defined on points $f : T \to X$ as

$$\mathbb{V}(M)(T) = \mathrm{Map}(f^* M, \mathcal{O}_T)$$

Have $\mathbb{G}_m$-action on $\mathbb{V}(M)$

Definition

For $f : X \to Y$ a morphism between algebraic stacks, the *normal bundle* is defined as the stack $\mathbb{V}_X(N_{X/Y})$ over X.

Ex If M is connective, then $\mathbb{V}(M) = \mathrm{Spec}\,\mathrm{Sym}(M)$

Indeed: $\mathrm{Spec}\,\mathrm{Sym}(M)(T) \simeq \mathrm{Map}(f^*\mathrm{Sym}(M), \mathcal{O}_T)$

$\simeq \mathrm{Map}(f^*M, \mathcal{O}_T)$

the $\mathbb{G}_m$-action on $\mathbb{V}(M)$ agrees w/ grading on $\mathrm{Sym}(M)$ (M connective)

Weil restriction

Let $f : X \to Y$ be an affine morphism of stacks. Then the pullback functor

$$f^* : \mathcal{S}\mathrm{t}_Y \to \mathcal{S}\mathrm{t}_X$$

has a right adjoint, written Res_f.

Definition

For $Z \to X$, we call $\mathrm{Res}_f(Z)$ the *Weil restriction* of Z along f.

This classifies local sections:

$$\begin{tikzcd}
& T_X \arrow[r] \arrow[d] \arrow[dl, dashed, bend right] \arrow[dr, phantom, "\lrcorner", very near start] & T \arrow[d] \arrow[dr, dashed, bend left] & \\
Z \arrow[r] & X \arrow[r] & Y & \mathrm{Res}_f(Z) \arrow[l]
\end{tikzcd}$$

The deformation space

Let $X \to Y$ be a morphism of stacks.

Definition

The *deformation space* $\mathcal{D}_{X/Y}$ of f is the Weil restriction of

$$X \times B\mathbb{G}_m \to Y \times B\mathbb{G}_m$$

along the zero section $Y \times B\mathbb{G}_m \to Y \times [\mathbb{A}^1/\mathbb{G}_m]$.

$$\begin{tikzcd} X \times BG \arrow[d] \arrow[r, phantom, "\text{w. restr.} \rightsquigarrow"] & \mathcal{D}_{X/Y} \arrow[d] \\ Y \times BG \arrow[r] & Y \times [\mathbb{A}^1/G] \end{tikzcd}$$

So:

$$\mathrm{Map}_{Y \times [\mathbb{A}^1/G]}(T, \mathcal{D}_{X/Y}) \simeq \mathrm{Map}_{Y \times BG}(T \times_{[\mathbb{A}^1/G]} BG, X \times BG)$$

Virtual Cartier divisors

Definition

A *virtual Cartier divisor over $X \to Y$* is a commutative diagram

N.B. different terminology from [KR]

$$\begin{tikzcd} D \arrow[r] \arrow[d] & T \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Local on T: ∃

$$\begin{tikzcd} D \arrow[r] \arrow[d, hook] & T \arrow[d] \\ \{0\} \arrow[r] & \mathbb{A}^1 \end{tikzcd}$$

in $\mathcal{S}\mathrm{t}$, such that $D \to T$ is a virtual Cartier divisor.

Lemma

The map $B\mathbb{G}_m \to [\mathbb{A}^1/\mathbb{G}_m]$ classifies virtual Cartier divisors.

i.e. $\{D \to T \text{ VCD's}\} \simeq \left\{ \begin{tikzcd} D \arrow[r] \arrow[d, hook] & T \arrow[d] \\ BG \arrow[r] & [\mathbb{A}^1/G] \end{tikzcd} \right\}$

Virtual Cartier divisors

Suppose $f : X \to Y$ has a cotangent complex. Put $\mathcal{N}_{X/Y} := [N_{X/Y}/\mathbb{G}_m]$.

Fundamental Lemma

For any T over Y, we have

$$\left\{\begin{tikzcd} N_{X/Y} \arrow[d] & T \arrow[l, dashed] \arrow[d, dashed] \\ X \arrow[r] & Y \end{tikzcd}\right\} \simeq \mathcal{S}t_Y(T, N_{X/Y}) \simeq \mathcal{S}t_Y(\mathbb{V}_T(\mathcal{O}_T[1]), X) \simeq \left\{\begin{tikzcd} & \mathbb{V}_T(\mathcal{O}_T[1]) \arrow[d] \arrow[ddl, dashed] \\ & T \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}\right\}$$

Let's check on fibers over $St_Y(T,X)$. So let $f: T \to X$ be given

$$St_X(T, N_{X/Y}) \simeq Map(f^*L_{X/Y}, \mathcal{O}_T[1]) = \left\{\begin{tikzcd} T \arrow[r] \arrow[d] & X \arrow[d] \\ \mathbb{V}_T(\mathcal{O}_T[1]) \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}\right\}$$

$$\begin{tikzcd} \{\ldots\} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & St_Y(\mathbb{V}_T(\mathcal{O}_T[1]), X) \arrow[d] \\ \{f\} \arrow[r] & St_Y(T, X) \end{tikzcd}$$

Lem: $\mathcal{L} \in Pic(T)$: $Spec(\mathcal{O}_T \oplus \mathcal{L}[1]) \simeq \mathbb{V}(\mathcal{L}[1])$

locally: $T = Spec\,A$, $\mathcal{L} = A$. $A \oplus A[1] = A/(0)$

get
$$\begin{tikzcd} \mathbb{V}(A[1]) \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & T \arrow[d] \\ T \arrow[r] & \mathbb{V}(A) = \mathbb{A}^1_T \end{tikzcd}$$

Virtual Cartier divisors

Corollary

We have a Cartesian diagram

$\mathfrak{D}_{X/Y}$ is defined as:

$$
\begin{tikzcd}
N_{X/Y} \arrow[r] \arrow[d] & D_{X/Y} \arrow[r] \arrow[d] & \mathfrak{D}_{X/Y} \arrow[d] \\
Y \arrow[r] & Y \times \mathbb{A}^1 \arrow[r] & Y \times [\mathbb{A}^1/\mathbb{G}]
\end{tikzcd}
$$

$D_{X/Y}$ is also Weil restr.:

$$
\begin{tikzcd}
X \times \{0\} \arrow[d] & & D_{X/Y} \arrow[d] \\
Y \times \{0\} \arrow[rr] & \rightsquigarrow & Y \times \mathbb{A}^1
\end{tikzcd}
$$

Hence $[D_{X/Y}/\mathbb{G}] \simeq \mathfrak{D}_{X/Y}$

Let $T \to Y$ be given. Then:

$$
\begin{aligned}
\mathrm{Map}_{Y \times \mathbb{A}^1}(T, D_{X/Y}) &\simeq \mathrm{Map}_Y(T \times_{\mathbb{A}^1} \{0\}, X) \simeq \mathrm{Map}_Y(T \times_{\mathbb{A}^1_T} T, X) \\
&\simeq \mathrm{Map}_Y(\mathbb{V}(\mathcal{O}_T(n)), X) \\
&\simeq \mathrm{Map}_Y(T, N_{X/Y})
\end{aligned}
$$

Virtual Cartier divisors

Remark

We also have a $\mathbb{G}_m$-equivariant version of the fundamental lemma. This gives us a Cartesian square

$$\begin{tikzcd} \mathcal{N}_{X/Y} \arrow[r] \arrow[d] & \mathcal{D}_{X/Y} \arrow[d] \\ Y \times B\mathbb{G}_m \arrow[r] & Y \times [\mathbb{A}^1/\mathbb{G}_m] \end{tikzcd}$$

$\mathcal{N}_{X/Y} \to \mathcal{D}_{X/Y}$ is then the universal virtual Cartier divisor over $X \to Y$.

i.e.: any VCD $\begin{tikzcd} D \arrow[r] \arrow[d] & T \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$ is pullback

$$\begin{tikzcd} D \arrow[r] \arrow[d] & T \arrow[d] \\ N_{X/Y} \arrow[r] \arrow[d] & D_{X/Y} \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

Idea $\mathrm{Map}_{Y \times [\mathbb{A}^1/\mathbb{G}]}(T, D_{X/Y}) \simeq \mathrm{Map}_{Y \times B\mathbb{G}}(T \times_{[\mathbb{A}^1/\mathbb{G}]} B\mathbb{G}, X \times B\mathbb{G}) \simeq \mathrm{Map}_Y(D, X)$

$$\begin{tikzcd} & D \arrow[r] \arrow[d] \arrow[dl, dashed] & T \arrow[d] \\ X \times B\mathbb{G} \arrow[r] \arrow[d] & Y \times B\mathbb{G} \arrow[r] \arrow[d] & Y \times [\mathbb{A}^1/\mathbb{G}] \\ X \arrow[r] & Y & \end{tikzcd}$$

$\rightsquigarrow \left\{ \begin{tikzcd} D \arrow[r, "\text{VCD}"] \arrow[d] & T \arrow[d] \\ X \arrow[r] & Y \end{tikzcd} \right\}$

In terms of blow-ups

Proposition

Let $Z \to X$ be a closed immersion. Then

$$D_{Z/X} \simeq \mathrm{Bl}_{Z\times\{0\}}(X \times \mathbb{A}^1) \setminus \mathrm{Bl}_{Z\times\{0\}}(X\times\{0\})$$

i.e., Have $D_{Z/X} \xhookrightarrow{f} \mathrm{Bl}_{Z\times\{0\}}(X\times\mathbb{A}^1) \hookleftarrow \mathrm{Bl}_{Z\times\{0\}}(X\times\{0\})$

open complement of

construction of f:

$$\mathrm{Map}_{X\times\mathbb{A}^1}(T, D_{Z/X}) = \left\{ \begin{tikzcd} T_X \arrow[r] \arrow[d, dashed] & T \arrow[dd] \\ Z\times\{0\} \arrow[d] & \\ X\times\{0\} \arrow[r] & X\times\mathbb{A}^1 \end{tikzcd} \right\} \mapsto \left(\begin{tikzcd} T_X \arrow[r] \arrow[d] & T \arrow[d] \\ Z \arrow[r] & X\times\mathbb{A}^1 \end{tikzcd} \right) \in \mathrm{Bl}_Z(X\times\mathbb{A}^1)(T)$$

Naturality of $D_{(-)/(-)}$

Proposition

The functor

Arrow cat. $\mathrm{Fun}(\Delta[1], \mathcal{S}t)$

$$\mathcal{S}\mathrm{t}_{[\mathbb{A}^1/\mathbb{G}_m]} \to \mathrm{Ar}(\mathcal{S}\mathrm{t})$$
$$T \mapsto (T \times_{[\mathbb{A}^1/\mathbb{G}_m]} B\mathbb{G}_m \to T)$$

has a right adjoint, which sends $X \to Y$ to $\mathcal{D}_{X/Y}$.

IDEA

$$\left\{ \begin{tikzcd} T \times_{[A^1/G]} BG \arrow[r, dashed] \arrow[d] & X \arrow[d] \\ T \arrow[r, dashed] & Y \end{tikzcd} \right\} \simeq \mathcal{S}t_{[A^1/G]}(T, D_{X/Y})$$

By: $D_{X/Y}$ classifies VCDS

The deformation diagram

Let $f : X \to Y$ be a morphism of algebraic stacks. Then we have a diagram of Cartesian squares

$D_{X/X} \simeq X \times \mathbb{A}^1$

since

$D_{(-)/X} : St_X \to St_{X \times \mathbb{A}^1}$

is right adjoint

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X \times \mathbb{A}^1 \arrow[d] & X \times \mathbb{G}_m \arrow[l] \arrow[d] \\
\mathbb{V}(N_{X/Y}) \arrow[r] \arrow[d] & D_{X/Y} \arrow[d] & Y \times \mathbb{G}_m \arrow[l] \arrow[d] \\
Y \arrow[r] & Y \times \mathbb{A}^1 & Y \times \mathbb{G}_m \arrow[l]
\end{tikzcd}$$

→ by fund. lemma

$$\begin{array}{c} D_{X/X} \\ \downarrow \\ D_{X/Y} \\ \downarrow \\ D_{Y/Y} \end{array}$$

P.B.) For $T \to Y \times BG$:

$$\mathrm{Map}_{Y \times \mathbb{A}^1}(T, D_{X/Y}) \simeq \mathrm{Map}_Y(T \times_{\mathbb{A}^1} \{0\}, X) \simeq \mathrm{Map}_Y(\emptyset, X) \simeq *$$

Quasi-smooth morphisms

Definition

Let $f : X \to Y$ be a morphism of algebraic stack.

- Recall that $A \to B$ in $\mathcal{A}\mathrm{lg}$ is *locally of finite presentation* if $\mathcal{A}\mathrm{lg}_A(B, -)$ commutes with filtered colimit.

  $X \longrightarrow Y$

  sm sm

- Now f is *locally of finite presentation* if for all $\mathrm{Spec}\, B \to \mathrm{Spec}\, A$, smooth over f, $A \to B$ is locally of finite presentation.
- A module $M \in \mathrm{Q}\mathcal{C}\mathrm{oh}(X)$ is *of Tor-amplitude* $[n, m]$ if for all discrete $E \in \mathrm{Q}\mathcal{C}\mathrm{oh}(M)$ it holds that $\pi_i(M \otimes E) = 0$ for i outside $[n, m]$
- Now f is *quasi-smooth* if it is locally of finite presentation and $L_{X/Y}$ is of Tor-amplitude $[-\infty, 1]$.

Quasi-smooth morphisms

We say $A \to B$ is *finitely presented* if B can be obtained from A by a finite number of cell attachments. Now the following are equivalent:

- $A \to B$ is locally of finite presentation
- B is a retract of a finitely presented A-algebra
- $\pi_0 A \to \pi_0 B$ is finitely presented, and $L_{B/A}$ is perfect (=compact).

$\mathrm{Map}(L_{B/A}, -)$ preserves filt. colims

Example

The map $k[\epsilon] \to k$ is locally of finite presentation but not finitely presented. Can see this by cot. complex, which is $k[1] \oplus k[2] \dots$ Also have

$$\begin{tikzcd} k \arrow[r] \arrow[rr, bend left, "id"] & k[\epsilon]/\epsilon[1] \arrow[r] & k \end{tikzcd}$$

Example

A closed immersion $Z \to X$ of derived schemes is quasi-smooth if and only if, Zariksi locally on X, it is of the form $\mathrm{Spec}\, A/(f_1, \dots, f_n) \to \mathrm{Spec}\, A$.

Deformation space in the quasi-smooth case

Proposition

Suppose $f : X \to Y$ is quasi-smooth.

1 • *The structure map $D_{X/Y} \to Y \times \mathbb{A}^1$ is quasi-smooth*

2 • *The map $X \times \mathbb{A}^1 \to D_{X/Y}$ is quasi-smooth.*

• *Hence all maps in the deformation diagram are quasi-smooth.*

Let's do $Z \hookrightarrow X$

1
$$\begin{tikzcd} D_{Z/X} \arrow[r, hook, "q.sm"] \arrow[d] & Bl_{Z \times \{0\}}(X \times \mathbb{A}^1) \arrow[dl, "q.sm"] \\ X \times \mathbb{A}^1 & \end{tikzcd}$$

2 local, so assume $Z \to X$ corr. to $A \to B = A/(f_1 \ldots f_n)$. Then

$Z \times \mathbb{A}^1 \to D_{Z/X}$ corr. to

$$R^{ext}_{B/A} = \frac{A[t^{-1}, v_1 \ldots v_n]}{(v_i t^{-1} - f_i)_i} \longrightarrow \frac{R^{ext}_{B/A}}{(v_1 \ldots v_n)} \cong B[t^{-1}]$$

Hence q. sm.

Rees algebras

Recall that we defined the extended Rees algebra of a closed immersion $Z \to X$ as the $\mathbb{Z}$-graded $\mathcal{O}_X[t^{-1}]$-algebra $R^{\text{ext}}_{Z/X}$ such that

$$D_{Z/X} = \operatorname{Spec} R^{\text{ext}}_{Z/X}$$

Lemma

Suppose we have a commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ X' \arrow[r] \arrow[ur] & V \end{tikzcd}$$

in $\mathcal{S}\mathrm{t}$. *Then naturality of* $\mathcal{D}_{(-)/(-)}$ *gives*

$$D_{Z/X} \simeq D_{X'/X} \times_{D_{X'/V}} D_{Z/V}$$

Since:

$$\begin{tikzcd} (Z \to X) \arrow[r] \arrow[d] & (X' \to X) \arrow[d] \\ (Z \to V) \arrow[r] & (X' \to V) \end{tikzcd}$$

in $Ar(\mathcal{S}t)$

Example 1

Let $Z \to X$ correspond to $A \to B = A/(f_1, \dots, f_n)$. Then

$$R^{\text{ext}}_{B/A} = \frac{A[t^{-1}, v_1, \dots, v_n]}{(v_1 t^{-1} - f_1, \dots, v_n t^{-1} - f_n)} \qquad N_{B/A} = B^{\oplus n}$$

$$L_{B/A} \simeq L_{\mathbb{Z}/\mathbb{Z}[T_1,\dots,T_n]} \otimes_{\mathbb{Z}} B \simeq (\mathbb{Z}^{\oplus n} \otimes_{\mathbb{Z}} B)(1) \simeq (B(1))^{\oplus n}$$

$$\begin{tikzcd}
B & B[t^{-1}] \arrow[l] \arrow[r] & B[t^{\pm 1}] \\
\mathrm{Sym}(N_{B/A}) \arrow[u] \arrow[dr, phantom, "1"] & R^{\text{ext}}_{B/A} \arrow[l] \arrow[r] \arrow[u] \arrow[dr, phantom, "2"] & A[t^{\pm 1}] \arrow[u] \\
A \arrow[u] & A[t^{-1}] \arrow[l] \arrow[r] \arrow[u] & A[t^{\pm 1}] \arrow[u]
\end{tikzcd}$$

(1) $$\frac{A[t^{-1}, \underline{v}]}{(\underline{v}t^{-1} - \underline{f})} \Big/ (t^{-1}) \simeq \frac{A[\underline{v}]}{(\underline{f})} \simeq B[\underline{v}] \simeq \mathrm{Sym}(B^{\oplus n})$$

(2) $$\left(\frac{A[t^{-1}, \underline{v}]}{(\underline{v}t^{-1} - \underline{f})}\right)_t \simeq \frac{A[t^{\pm 1}, \underline{v}]}{(\underline{v} - \underline{f}t)} \simeq A[t^{\pm 1}]$$

since $vt^{-1} - f = t^{-1}(v - ft)$, t^{-1} invertible

Example 2

Let $Z \to X$ correspond to $\operatorname{Spec} k \to \operatorname{Spec} k[\epsilon]$. Then

$$R^{\mathrm{ext}}_{Z/X} \simeq \pi_0\left(\frac{k[\epsilon, t^{-1}, v]}{(vt^{-1} - \epsilon, \epsilon v)}\right) \qquad \mathbb{V}(N_{Z/X}) \simeq \mathbb{V}(k \oplus k[1])$$

Put $V = \mathbb{A}^1_u$ $= \operatorname{Spec} k[T]$ then

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ X \arrow[r] \arrow[ur, phantom, "//"] & U \end{tikzcd}$$

gives $D_{Z/X} \simeq D_{X/X} \times_{D_{X/U}} D_{Z/U}$

So

$$\begin{tikzcd} \mathbb{Z}[u,p] \arrow[r, "p \mapsto T^2 - ut^{-1}"] \arrow[d] & k[T,t^{-1},u] \arrow[d] & \\ \mathbb{Z}[u] \arrow[r] \arrow[d] & \frac{k[T,t^{-1},u]}{(t^{-1}u - T^2)} \arrow[r, "u \mapsto vT"] \arrow[d, "u \mapsto 0"] & \frac{k[T,t^{-1},v]}{(vt^{-1}-T)} \arrow[d] \\ \mathbb{Z} \arrow[r] & k[\varepsilon, t^{-1}] \arrow[r] & R^{ext}_{Z/X} \end{tikzcd}$$

$D_{X/U}$, $D_{Z/U}$, $D_{X/X}$

So:

$$R^{ext}_{Z/X} = \frac{k[T,t^{-1},v]}{(vt^{-1}-T, vT)}$$

$$R^{ext}_{Z/X}/(t^{-1}) \simeq \frac{k[T,v]}{(T,vT)} \simeq \frac{k[v]}{(0)}$$

$$\simeq \operatorname{Sym}(k \oplus k[1])$$

Example 3

Let $Z \to X$ correspond to $\operatorname{Spec} k \to \operatorname{Spec} \underbrace{k[x,y]/(xy^2, yx^2)}_{B}$. Put $I = (x,y) \subset \underbrace{k[x,y]}_{A}$. Then

$$R^{\text{ext}}_{Z/X} \simeq \frac{k[x,y][It, t^{-1}]}{(xy^2t, yx^2t)} \qquad \mathbb{V}(N_{Z/X}) \simeq \mathbb{V}(k/(0,0)[u,v])$$

$$R^{ext}_{Z/X}/(t^{-1}) = \frac{A[u,v,t^{-1}]}{(ut^{-1}-x, vt^{-1}-y, uy^2, vx^2)} \Big/ (t^{-1})$$

$$\simeq \frac{A[u,v]}{(x,y,uy^2,vx^2)} \simeq \frac{k[u,v]}{(0,0)} \simeq k/(0,0)\,[u,v]$$

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Thank you!