Chain Complexes

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Our goal in these notes is to describe the category of (left) R-modules (and related categories) for R a fixed commutative ring from various homotopical perspectives. (Our goal isn't to beat the homotopy theorists, who've honestly had us beat since before we even started.) Unless otherwise stated, $\mathcal{C}$ denotes a category and $\mathrm{Map}(\mathcal{C})$ the category whose objects are morphisms in $\mathcal{C}$ and morphisms are commutative squares in $\mathcal{C}$.

1 Generalities for Model Categories

Definition 1. *A morphism $f \in \mathrm{Map}(\mathcal{C})$ is a **retract** of $g \in \mathrm{Map}(\mathcal{C})$ if there exists a commutative diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"] \arrow[rr, bend left, "\mathrm{id}_A"] & C \arrow[r] \arrow[d, "g"] & A \arrow[d, "f"] \\ B \arrow[r] \arrow[rr, bend right, "\mathrm{id}_B"'] & D \arrow[r] & B \end{tikzcd}$$

*A **functorial factorization** relative to $\mathcal{C}$ is the data of a pair of functors $\alpha, \beta : \mathrm{Map}(\mathcal{C}) \to \mathrm{Map}(\mathcal{C})$ such that $f = \beta(f) \circ \alpha(f)$ for every $f \in \mathrm{Map}(\mathcal{C})$.*

Definition 2. *Let $i : A \to B$ and $p : X \to Y$ be morphisms in $\mathcal{C}$. We say that i has the **left lifting property** (LLP) with respect to p or, equivalently, p has the **right lifting property** (RLP) with respect to i if we may complete every commutative diagram*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i"'] & X \arrow[d, "p"] \\ B \arrow[r] \arrow[ur, dotted, "\exists"] & Y \end{tikzcd}$$

Definition 3. *A **model structure** on $\mathcal{C}$ is the data of subcategories W, C, F (whose morphisms are respectively called **weak equivalences**, **cofibrations**, and **fibrations**) and functorial factorizations (α, β) and (γ, δ) such that*

- *W satisfies the 2-out-of-3 property for pairs of composable morphisms;*
- *W, C, F are closed under retracts;*

- *trivial*[1] *cofibrations have the LLP with respect to fibrations;*
- *cofibrations have the RLP with respect to trivial fibrations;*
- *given any* $f \in \operatorname{Map}(\mathcal{C})$,

$$\alpha(f) \in C, \qquad \beta(f) \in F \cap W, \qquad \gamma(f) \in C \cap W, \qquad \delta(f) \in F.$$

*A **model category** is a small, complete, cocomplete category $\mathcal{C}$ equipped with a model structure.*

Unless otherwise stated, we will from here on out take $\mathcal{C}$ to be a model category. By assumption, $\mathcal{C}$ has an initial object 0 and a final object 1.[2] We say an object $X \in \mathcal{C}$ is **cofibrant** if the canonical map $0 \to X$ is a cofibration and **fibrant** if the canonical map $X \to 1$ is a fibration. Applying (α, β) to $0 \to X$ yields a **cofibrant replacement** functor $Q : \mathcal{C} \to C \subseteq \mathcal{C}$ equipped with a natural transformation $q : Q \to \mathrm{id}_{\mathcal{C}}$ such that $q_X : QX \to X$ is a trivial fibration for every $X \in \mathcal{C}$. Similarly, applying (γ, δ) to $X \to 1$ yields a **fibrant replacement** functor $R : \mathcal{C} \to F \subseteq \mathcal{C}$ equipped with a natural transformation $r : R \to \mathrm{id}_{\mathcal{C}}$ such that $r_X : RX \to X$ is a trivial cofibration for every $X \in \mathcal{C}$.

Lemma 4. *Let $f \in \operatorname{Map}(\mathcal{C})$. Then, f is a cofibration (resp., trivial cofibration) if and only if it has the LLP with respect to all trivial fibrations (resp., fibrations). Dually, f is a fibration (resp., trivial fibration) if and only if it has the RLP with respect to all trivial cofibrations (resp., cofibrations).*

Since $\mathcal{C}^{\mathrm{op}}$ is naturally a model category and $(\mathcal{C}^{\mathrm{op}})^{\mathrm{op}} = \mathcal{C}$, many statements about model categories have a corresponding dual statement. We will most often omit such statements.

Corollary 5. *Both C and $C \cap W$ are closed under pushouts.*

Just to make sure we know how to play the dualizing game, the dual of the above statment is that F and $F \cap W$ are closed under pullbacks.

Lemma 6 (Brown)**.** *Let $\mathcal{D}$ be a category with a subcategory of weak equivalences satisfying the* 2*-out-of-*3 *property for pairs of composable morphisms. Let $F : \mathcal{C} \to \mathcal{D}$ be a functor sending trivial cofibrations between cofibrant objects to weak equivalences. Then, F sends all weak equivalences between cofibrant objects to weak equivalences.*

What is the natural notion of "morphism" between model categories?[3]

Definition 7. *A functor $F : \mathcal{C} \to \mathcal{D}$ between model categories is a **left Quillen functor** if F is a left adjoint preserving both cofibrations and trivial cofibrations (there is a corresponding dual notion of **right Quillen functor**). A **Quillen adjunction** is an adjunction (F, U, φ) (so $\varphi : \mathcal{D}(FA, B) \xrightarrow{\sim} \mathcal{C}(A, UB)$ functorial in $A \in \mathcal{C}$ and $B \in \mathcal{D}$) such that F is a left Quillen functor.*[4]

[1] The word "trivial" here denotes that the morphism in question is also a weak equivalence.

[2] The notation here is meant to be reminiscent of intervals. It's also common to see the initial object denoted $\emptyset$ and the final object denoted $*$.

[3] We can away with the quotes here by working with 2-categories.

[4] Note that there is no ambiguity in this definition since any right adjoint to F is isomorphic to U via a unique natural isomorphism.

We think of a "morphism" between model categories as a Quillen adjunction. This has the advantage of being "self-dual" in the sense that, given (F, U, φ) an adjunction, F is a left Quillen functor if and only if U is a right Quillen functor. We "compose" adjunctions $(F, U, \varphi) : \mathcal{C} \to \mathcal{D}$ and $(F', U', \varphi') : \mathcal{D} \to \mathcal{E}$ by taking $(F' \circ F, U \circ U', \varphi \circ \varphi') : \mathcal{C} \to \mathcal{E}$ with $\varphi \circ \varphi'$ defined by the composition

$$\mathcal{E}(F'FA, B) \xrightarrow{\varphi'} \mathcal{D}(FA, U'B) \xrightarrow{\varphi} \mathcal{C}(A, UU'B).$$

Using this, we can define a **Quillen equivalence** to be a Quillen adjunction with an inverse Quillen adjunction under composition.

2 Chain Complexes

Let $\mathsf{Ch}(R)$ denote the abelian category of chain complexes of (left) R-modules (such modules themselves form a category denoted $\mathsf{Mod}(R)$). Note that Hom-sets in $\mathsf{Ch}(R)$ are naturally enriched over $\mathsf{Mod}(R)$ but not over $\mathsf{Ch}(R)$. We can partially overcome this shortcoming by working with internal Hom characterized by

$$\underline{\mathrm{Hom}}(X, Y)_n := \prod_{i \in \mathbb{Z}} \mathrm{Hom}_R(X_i, Y_{i+n})$$

with differential $df := d_Y \circ f - (-1)^n f \circ d_X$.[5] This should not be confused with $[X, Y]$, the set of chain homotopy classes of maps from X to Y.

In what follows, we will describe two model structures on $\mathsf{Ch}(R)$ that we choose to call the *projective* and *injective* model structures. Let's first start with the projective model structure. Given $n \in \mathbb{Z}$, define functors $S^n, D^n : \mathsf{Mod}(R) \to \mathsf{Ch}(R)$ by taking $S^n(M)$ to be M concentrated in degree n and $D^n(M)$ to be two copies of M in degrees $n, n-1$ with the identity map between them. Note that there is a natural injection $S^{n-1}(M) \hookrightarrow D^n(M)$, giving rise to a collection

$$I := \{S^{n-1} \hookrightarrow D^n\}_{n \in \mathbb{Z}}$$

with $S^{n-1} := S^{n-1}(R)$ and $D^n := D^n(R)$. We also make note of the collection $J := \{0 \to D^n\}_{n \in \mathbb{Z}}$.

Definition 8. *Fix $f \in \mathrm{Map}(\mathsf{Ch}(R))$. We say that f is a **weak equivalence** if it is a qis. We say that f is a **fibration** if it is J**-inj** – i.e., f has the RLP with respect to every morphism in J. We say that f is a **cofibration** if it is I**-cof** – i.e., f is (I-inj)-proj in the sense that it has the LLP with respect to every I-inj morphism. These notions together define the **projective model structure** on $\mathsf{Ch}(R)$.*

Of course, we still need to check that the above actually defines a model structure. We begin by examining each type of morphism in more detail, starting with the fibrations.

Proposition 9. *A morphism $f \in \mathrm{Map}(\mathsf{Ch}(R))$ is a fibration if and only if it is levelwise surjective, and a trivial fibration if and only if it is I-inj.*

Cofibrations are a bit trickier to work with. We first handle the cofibrant objects.

[5] Motivating this definition is a rabbit hole well worth the effort to explore.

Proposition 10. *Let $A \in \mathsf{Ch}(R)$. Suppose A is cofibrant. Then, A_n is projective for every $n \in \mathbb{Z}$. As a partial converse (the full converse is false), suppose that A is bounded below and A_n is projective for every $n \in \mathbb{Z}$. Then, A is cofibrant.*

Lemma 11. *Let $C, K \in \mathsf{Ch}(R)$ with C cofibrant and K acyclic. Then, $[C, K] = 0$.*

Proposition 12. *A morphism $i \in \mathrm{Map}(\mathsf{Ch}(R))$ is a cofibration if and only if it is levelwise split injective with cofibrant cokernel.*

Proposition 13. *A morphism $i \in \mathrm{Map}(\mathsf{Ch}(R))$ is J-cof if and only if it is levelwise injective with cokernel projective as a complex. Hence, every element of J-cof is a trivial cofibration.*

How are monomorphisms in $\mathsf{Ch}(R)$ related to levelwise injections? How are epimorphisms in $\mathsf{Ch}(R)$ linked to levelwise surjections? How are projective complexes linked to complexes of projectives? What role does boundedness play in all of this (and why)?

The caveat now is that this is enough to *generate* a model structure. What this means precisely is a bit subtle and requires some set theoretic considerations. This generating business implies that we have at this point only described "basic" weak equivalences, cofibrations, and fibrations. What about "general" morphisms in each class?

For the injective model structure, we say a morphism in $\mathsf{Ch}(R)$ is an **injective fibration** if it has the RLP with respect to all levelwise injective qis's. We call the fibrant objects in this setup **inj-fibrant** for emphasis (TO DO: compare this with fibrant objects in the projective setup.). As before we take weak equivalences to be qis's.

Theorem 14. *The above conventions give rise to a cofibrantly generated model structure on $\mathsf{Ch}(R)$, called the **injective module structure**. Moreover, given $f \in \mathrm{Map}(\mathsf{Ch}(R))$ and $A \in \mathsf{Ch}(R)$,*

- *f is an injective fibration if and only if it is levelwise surjective with inj-fibrant cokernel;*
- *if A is inj-fibrant then A_n is injective for every $n \in \mathbb{Z}$;*
- *if A is bounded above and A_n is injective for every $n \in \mathbb{Z}$ then A is inj-fibrant;*
- *f is a trivial injective fibration if and only if it is levelwise surjective with injective cokernel;*
- *A is injective if and only if it is inj-fibrant and acyclic.*

Let's now compare the projective and injective model structures, using $\mathsf{Ch}(R)^{\mathrm{proj}}$ and $\mathsf{Ch}(R)^{\mathrm{inj}}$ to distinguish between the associated model categories. TO DO: Finish this!!!