Section 11 Math 2202

Triple Integrals: Changing Order of Integration and Changing Coordinates

Comments for Facilitator:

- Do the Warm Up as a Group Quiz (5 minutes on own, then talk with neighbor while you circulate) Spend time on the drawings and set up of integrals only, not the computations.

  For #1, I expect them to integrate with respect to z first. That works out nicely enough. Then you can ask what happens if you go into the other regions. For example, you could integrate with respect to x first, projecting into the zy-plane.

- (15 min) Changing the order of integration: Go from $dydzdx$ to $dydxdz$. Have them focus on the outer double integral, sketch the region in the xz plane and imagine slicing the region perp. to z now, not x. (Connect this to the coordinate plane where we're thinking of projecting the solid.)

  Now go from $dydzdx$ to $dzdydx$ There are two ways - I prefer the first which is to go back to the start and think about projecting into the yx plane.

  (The other way is to focus on the inner double integral. This is harder to think about, because the region for this double integral depends on the other variable x. You can prompt them to think of x as fixed.)

  As a challenge, some can try doing some of the other orders. Warning some require two integrals.

  I'm including the worked solutions, as well as the solutions to the homework problem they'll do like this, so you can get a sense of strategies to help them with it.

- Changing to Cylindrical Spherical Coordinates: The grader said there was issues with polar coordinates so please go over the cylindrical one carefully.

1. **Warm Up** Set up an iterated integral for

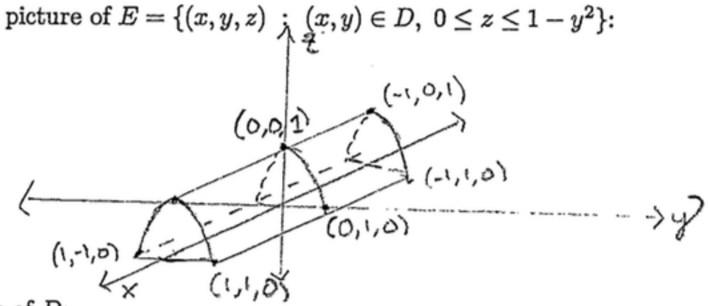
$$\int\int\int_E x^2 e^y \, dV$$

where E is bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $z = 0, x = 1$ and $x = -1$.

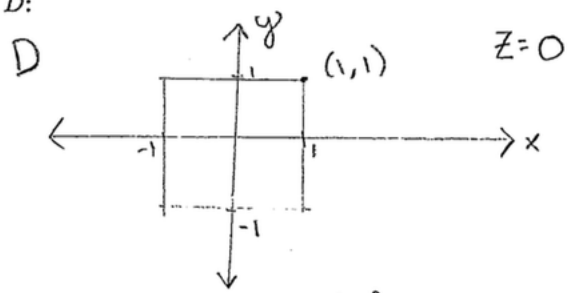
Draw two pictures. One should be a "good enough" picture of E and the other a picture of the projection of E onto the coordinate plane corresponding to the order of integration you chose.

Solution:

Let D be the projection of E into the xy-plane.
"Good enough" picture of $E = \{(x, y, z) \; ; \; (x, y) \in D, \; 0 \leq z \leq 1 - y^2\}$:



Here is a picture of D:



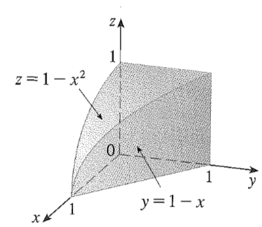
Then,

$$\iiint_E x^2 e^y \, dV = \iint_D \int_0^{1-y^2} x^2 e^y \, dz \; dA$$
$$= \int_{-1}^{1} \int_{-1}^{1} \int_0^{1-y^2} x^2 e^y \, dz \, dy \, dx$$

2. **Changing Order of Integration in Triple Integrals**

The figure on page 881 in the text shows the region of integration for the triple integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z) \, dy \, dz \, dx.$$



(a) Rewrite the integral in the order $dy \, dx \, dz$.

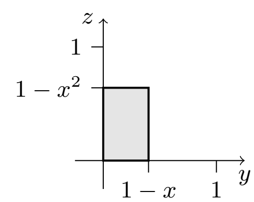
(b) Rewrite the integral in the order $dz \, dy \, dx$.

(c) How many integrals are needed if you project the solid into the yz-plane?

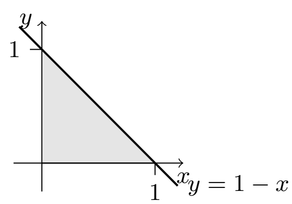
Solution: We do this by switching in pairs, with a sketch for each switch. We otherwise proceed without comment:

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\,dy\,dz\,dx$$

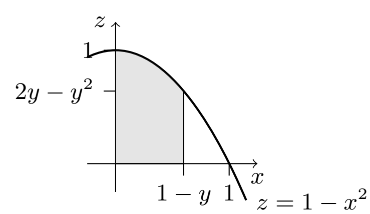
$$= \int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x,y,z)\,dz\,dy\,dx$$



$$= \int_0^1 \int_0^{1-y} \int_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy$$



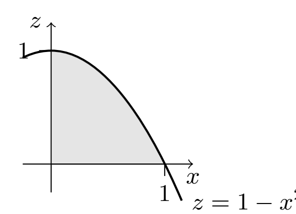
$$= \int_0^1 \int_0^{2y-y^2} \int_0^{1-y} f(x,y,z)\,dx\,dz\,dy + \int_0^1 \int_{2y-y^2}^1 \int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dz\,dy$$



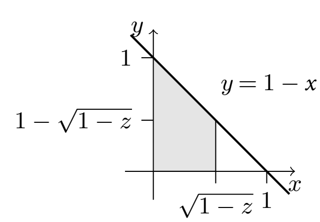
To get to $dx\,dy\,dz$ and $dy\,dx\,dz$, we start again with the initial integral:

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\,dy\,dz\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x,y,z)\,dy\,dx\,dz$$



$$= \int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dy\,dz + \int_0^1 \int_{1-\sqrt{1-z}}^1 \int_0^{1-y} f(x,y,z)\,dx\,dy\,dz$$

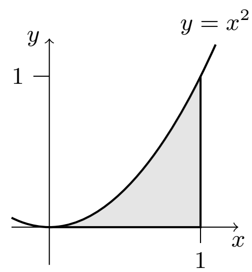


Comments for Facilitator:
(*Stewart 12.7 #36*) Write five other iterated integrals that are equal to the given iterated integral:

$$\int_0^1 \int_0^{x^2} \int_0^y f(x,y,z)\,dz\,dy\,dx.$$

Before jumping in, think about a strategy to generate all the other five iterated integrals by switching order of integration in pairs.
For example, first think about the outer limits $dy\,dx$, determine what region you are working with there and then seeing what happens if you switch them.

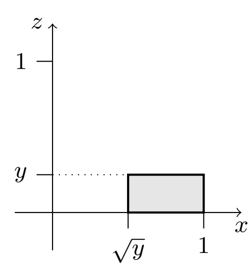
Solution: We switch the order of integration in pairs, starting with $dz\,dy\,dx$: The original outer limits for $dy\,dx$ gave us the region



in the xy-plane. We write this outer pair of integrals as $dx\,dy$ by re-writing $y = x^2$ as $x = \sqrt{y}$. We get

$$\int_0^1 \int_{\sqrt{y}}^1 \int_0^y f(x,y,z)\,dz\,dx\,dy. \tag{1}$$

Now we do an example of switching the order of integration for the inner two integrals. In (1), the inner two integrals are evaluated over a region D_y that depends on the value of y. This region is

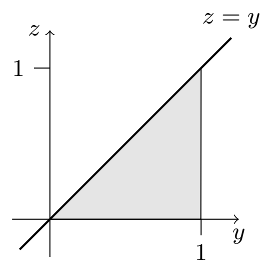


and so the triple integral can be written as

$$\int_0^1 \int_0^y \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dz\,dy. \tag{2}$$

Comments for Facilitator:

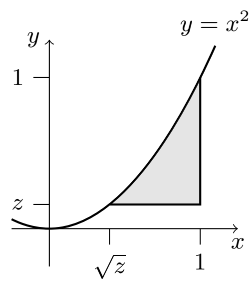
Solution: Now let's switch the outer integrals, aiming for the order $dx\,dy\,dz$. This involves the region



in the yz-plane, and we get the triple integral

$$\int_0^1 \int_z^1 \int_{\sqrt{y}}^1 f(x,y,z)\,dx\,dy\,dz. \tag{3}$$

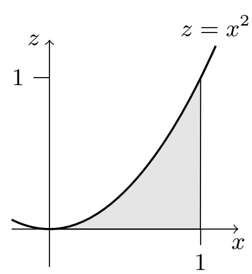
To get the order $dy\,dx\,dz$, we'll switch the inner integrals. For fixed z, the region in the xy-plane is



This gives us the triple integral

$$\int_0^1 \int_{\sqrt{z}}^1 \int_z^{x^2} f(x,y,z)\,dy\,dx\,dz. \tag{4}$$

Finally, the last order is $dy\,dz\,dx$, which we get by switching the outer integrals in (4). This gives us the region



in the xz-plane, which in turn gives us the integral

$$\int_0^1 \int_0^{x^2} \int_z^{x^2} f(x,y,z)\,dy\,dz\,dx. \tag{5}$$

3. Set up a triple integral to find the volume of the solid bounded by $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 + z^2 = 4$.

   (a) In spherical coordinates

   (b) In cylindrical coordinates

   (c) In rectangular coordinates[1]

[1] How does this integral compare with the same question from section last week, where you used a double integral?