Zariski Sheaves

$$\{(\mathcal{U}, i_{\mathcal{U}} : \mathcal{U} \hookrightarrow S)\}$$
$$\shortparallel$$

Let $X, S \in Space = Fun(CRing, Set)$. Let $\mathfrak{U}$ be an open covering of S. Let $\mathcal{U}, V \in \mathfrak{U}$.

$$\begin{tikzcd}
\mathcal{U} \cap V \arrow[r, "i_{V,\mathcal{U}}"] \arrow[d, "i_{\mathcal{U},V}"'] \arrow[dr, phantom, "\ulcorner", very near start] & \mathcal{U} \arrow[d, hook, "i_{\mathcal{U}}"] \\
V \arrow[r, hook, "i_V"'] & S
\end{tikzcd}$$

Both $i_{\mathcal{U},V}$, $i_{V,\mathcal{U}}$ are monomorphisms

$\Rightarrow i_{\mathcal{U}} \times i_V := i_{\mathcal{U}} \circ i_{V,\mathcal{U}} = i_V \circ i_{\mathcal{U},V} : \mathcal{U} \cap V \to S$ is monic.

Suppose $f_{\mathcal{U}} : \mathcal{U} \to X$ and $f_V : V \to X$ are arbitrary maps of spaces.

$$f_{\mathcal{U}}|_{\mathcal{U}\cap V} := \mathcal{U} \cap V \xhookrightarrow{i_{V,\mathcal{U}}} \mathcal{U} \xrightarrow{f_{\mathcal{U}}} X$$

$$f_V|_{\mathcal{U}\cap V} := \mathcal{U} \cap V \xhookrightarrow{i_{\mathcal{U},V}} V \xrightarrow{f_V} X$$

<u>Intuition</u>: X is a mfld, $\mathfrak{U}$ atlas of coord. charts, S is a "manageable chunk" of X.

$$\Psi(S, \mathfrak{U}, X) := \left\{ \{ f_{\mathcal{U}} \in Hom_{Space}(\mathcal{U}, X) \}_{\mathcal{U} \in \mathfrak{U}} : f_{\mathcal{U}}|_{\mathcal{U}\cap V} = f_V|_{\mathcal{U}\cap V} \ \forall\, \mathcal{U}, V \in \mathfrak{U} \right\}$$

$$Hom_{Space}(S, X) \to \left\{ \{ f_{\mathcal{U}} \in Hom_{Space}(\mathcal{U}, X) \}_{\mathcal{U} \in \mathfrak{U}} \right\}$$

"section of $\mathcal{X}$ over $\mathcal{U}$"

space

"section of f over $\mathcal{U}$"

$$f \mapsto \{ f|_{\mathcal{U}} = f \circ i_{\mathcal{U}} \}_{\mathcal{U} \in \mathfrak{U}}$$

is natural in X and factors through $\Psi(S, \mathfrak{U}, X)$:

$$\begin{aligned} (f|_{\mathcal{U}})|_{\mathcal{U}\cap V} &= f \circ i_{\mathcal{U}} \circ i_{V,\mathcal{U}} \\ &= f \circ i_V \circ i_{\mathcal{U},V} \\ &= (f|_V)|_{\mathcal{U}\cap V}. \end{aligned}$$

Remark: $\{\mathcal{U}_i \hookrightarrow \mathcal{U}\}_{i\in I}$ covering, $\mathcal{F}$ presheaf

Underlying "stuff"

Site $(\mathcal{C}, \mathrm{Cov}(\mathcal{C}))$

$\mathcal{E}$ target category

$\mathcal{F} \in \mathrm{Fun}(\mathcal{C}^{op}, \mathcal{E})$

Usually: $\mathcal{E} = \mathrm{Set}$

or

$\mathcal{E} = \mathrm{Ab}$

$$\mathcal{U} = \bigcup_{i\in I} \mathcal{U}_i = \coprod_{i\in I} \mathcal{U}_i \,/\!\sim$$

$$\mathcal{U} \leftarrow \coprod_{i\in I} \mathcal{U}_i \leftleftarrows \coprod_{i,j\in I} \mathcal{U}_i \cap \mathcal{U}_j$$

Intuition

$$\mathcal{F}(\mathcal{U}) \to \prod_{i\in I} \mathcal{F}(\mathcal{U}_i) \rightrightarrows \prod_{i,j\in I} \mathcal{F}(\mathcal{U}_i \cap \mathcal{U}_j)$$

Natural

Sheaf Condition: $\mathcal{F}(\mathcal{U}) \xrightarrow{\sim} \mathrm{eq}\Big(\underbrace{\prod_{i\in I} \mathcal{F}(\mathcal{U}_i)}_{A} \overset{f}{\underset{g}{\rightrightarrows}} \underbrace{\prod_{i,j\in I} \mathcal{F}(\mathcal{U}_{ij})}_{B} \Big)$ General

$$0 \to \mathcal{F}(\mathcal{U}) \to A \xrightarrow{f-g} B \to \mathrm{coker}(f-g) \to 0$$ In Ab

Exercise: $X \in \mathrm{Sch}$. $\mathfrak{U}$ collection of open subspaces of X.

$\mathfrak{U}$ is an open covering iff $X(T) = \bigcup_{U \in \mathfrak{U}} U(T)$ $\forall T \in \mathrm{CRing}$ local.

Naive Zariski topology ~ set-theoretic $\rightsquigarrow Sch'$

Zariski topology ~ Grothendieck topology (notion of covering) $\rightsquigarrow Shv_{Zar}$

Space + Zariski covering $\rightsquigarrow Sch \subseteq Space$

$AffSch' \simeq AffSch$

$Sch' \simeq Sch$

equiv. of cat.'s

$Shv_{Zar}(Sch') \simeq Shv_{Zar}(Sch)$

equiv. of cat.'s and of topoi

$$\Psi(S, \mathfrak{U}, X) := \{ \{ f_U \in \mathrm{Hom}_{Space}(U, X) \}_{U \in \mathfrak{U}} : f_U|_{U \cap V} = f_V|_{U \cap V} \ \forall U, V \in \mathfrak{U} \}$$

$$\mathrm{Hom}_{Space}(S, X) \to \{ \{ f_U \in \mathrm{Hom}_{Space}(U, X) \}_{U \in \mathfrak{U}} \}$$

"section of f over U"

$$f \mapsto \{ f|_U = f \circ i_U \}_{U \in \mathfrak{U}}$$

is natural in X and factors through $\Psi(S, \mathfrak{U}, X)$:

Def: If this natural map is a bijection then we say

X satisfies the Zariski sheaf condition w.r.t. $\mathfrak{U}$ (and S)

What about all $\mathcal{U}$ (and all S)?

<u>Def</u>: X is a <u>Zariski sheaf</u> if it satisfies the Zariski sheaf condition for every open cov. of every affine scheme mapping to X.

This gives $\mathrm{Shv}_{Zar} \subseteq \mathrm{Space}$.

<u>Example</u>: $X, Y \in \mathrm{Shv}_{Zar} \rightsquigarrow X \times Y \in \mathrm{Shv}_{Zar}$.

Any product of sheaves is a sheaf.

<u>Example</u>: $\mathbb{A}^1 = \mathrm{Spec}\, \mathbb{Z}[t]$.

<u>Example</u>: $\mathbb{A}^n := (\mathbb{A}^1)^{\times n} \cong \mathrm{Spec}\, \mathbb{Z}[t_1, \ldots, t_n]$.

<u>Non-Example</u>: Consider $X \in \mathrm{Space}$ given by

$X(A) := \{ f \in A : f \in A^\times \text{ or } 1-f \in A^\times \}$.

[Question: What is the "sheafification" of X?]

We want to compare different coverings. Fix a base $S \in \mathrm{Space}$.

To this we can associate a cat. $\mathrm{Cov}(S)$ of coverings of S.

$$\mathcal{U} = \{ (U, i_U : U \hookrightarrow S) \}, \quad \mathcal{V} = \{ (V, j_V : \underset{i_V\mathcal{U}}{V} \hookrightarrow S) \} \in \mathrm{Cov}(S)$$

$$\mathcal{U} = \{(U, i_U : U \hookrightarrow S)\}, \quad \mathcal{V} = \{(V, j_V : V \hookrightarrow S)\} \in \mathrm{Cov}(S)$$

$\mathcal{V} \leq \mathcal{U}$ if $\forall\, U \in \mathcal{U}$:

$$\begin{tikzcd} U \cap V \arrow[r, "j_{V,U}"] \arrow[d, "j_{U,V}"'] & U \arrow[d, hook, "i_U"] \\ V \arrow[r, hook, "j_V"'] & S \end{tikzcd}$$

$$\mathcal{V}_U := \{(U \cap V, j_{V,U} : U \cap V \to U)\} \in \mathrm{Cov}(U).$$

Exercise: Fix $\mathcal{U}, \mathcal{V} \in \mathrm{Cov}(S)$.

(0) Show that refinements (as above) make $\mathrm{Cov}(S)$ a category.

(a) What is $\mathrm{Hom}_{\mathrm{Cov}(S)}(\mathcal{U}, \mathcal{V})$?

(b) Construct the product $\mathcal{U} \times \mathcal{V}$ in $\mathrm{Cov}(S)$.

(c) Suppose that $\mathcal{U}$ and $\mathcal{V}$ each refine each other.
Is this a natural notion of isomorphism?

Lemma 47. *Let $X \in \mathsf{Space}$ and $\mathscr{U}, \mathscr{V} \in \mathsf{Cov}(S)$ with $\mathscr{V}$ refining $\mathscr{U}$. Suppose that X satisfies the Zariski sheaf condition with respect to $\mathscr{V}$. Suppose further that every $U \in \mathscr{U}$ satisfies the Zariski sheaf condition with respect to $\mathscr{V}_U$. Then, X satisfies the Zariski sheaf condition with respect to $\cancel{\mathscr{V}}$ $\mathcal{U}$.*

Explicitly, if $\mathrm{Hom}_{\mathsf{Space}}(S, X) \xrightarrow{\sim} \Psi(S, \mathscr{V}, X)$ and $\mathrm{Hom}_{\mathsf{Space}}(S, U) \xrightarrow{\sim} \Psi(S, \mathscr{V}_U, U)$ for every $U \in \mathscr{U}$ then $\mathrm{Hom}_{\mathsf{Space}}(S, X) \xrightarrow{\sim} \Psi(S, \mathscr{U}, X)$.

Exercise 48. *Prove the lemma!*

Corollary 49. *Let $X \in \mathsf{Shv}_{\mathrm{Zar}}$. Then, X satisfies the Zariksi sheaf condition with respect to every $\mathscr{U} \in \mathsf{Cov}(S)$ for every $S \in \mathsf{Space}$.*