MODULI OF TRUNCATED PRISMATIC (G, μ)-DISPLAYS AND CONJECTURES OF DRINFELD

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Abstract. We use the newly developed stacky prismatic technology of Drinfeld and Bhatt-Lurie to give a uniform, group-theoretic construction of smooth stacks of truncated (G, μ)-displays associated with a smooth $\mathbb{Z}_p$-group scheme G and 1-bounded cocharacter μ, verifying a recent conjecture of Drinfeld. This is the first step towards a purely group-theoretic (but still somewhat explicit) construction of integral models for local Shimura varieties or for Rapoport-Zink spaces, without appeal to the theory of p-divisible groups. The proofs use derived algebraic geometry and Lurie's amplification of the classical Artin representability theorem, combined with an animated variant of Lau's theory of higher frames and displays, and actually show representability of a wide range of stacks whose tang

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1. Introduction

The goal of this paper is to prove two recent conjectures of Drinfeld [15]. The first of these has to do with a Dieudonné theory for p-divisible groups over arbitrary p-adic formal schemes; that is, we aim to describe p-divisible groups, or more generally truncated p-divisible groups or Barsotti-Tate groups, in terms of linear algebraic data. For the purposes of this paper, this last phrase means a subcategory of vector bundles on a certain formal algebraic stack, though it has historically taken the form of a description in terms of modules equipped with a Frobenius semi-linear map along with certain additional structures. See Remark 1 below.

The stacks we consider here arose in recent work of Bhatt-Lurie [7, 8, 5] and Drinfeld [16]. They have shown that one can associate with every p-adic formal scheme X a p-adic formal stack[1] X^{syn}, its *syntomification*, whose coherent cohomology computes the p-adic syntomic cohomology of X. If $X = \operatorname{Spf} R$ is affine, we will also denote this by R^{syn}. Vector bundles of rank h on this stack and its mod-p^n fibers—which are examples of objects known as *F-gauges over R*—have a natural h-tuple of locally constant integer valued functions associated with them: these are the *Hodge-Tate weights*.

[1] This is actually a *derived* formal stack that is in general not a classical object. We will attempt to ignore this fact in this introduction.

Suppose that X is covered by p-adic formal affine schemes $\operatorname{Spf} R$ with $\Omega^1_{R/pR}$ finitely generated over R. For instance, X can be formally of finite type over $\mathbb{Z}_p$. We prove:

dieudonne **Theorem A.** *Let $\mathcal{BT}_n(X)$ be the category of truncated Barsotti-Tate groups over X of level n [23], and let $\mathrm{Vect}_{[0,1]}(X^{\mathrm{syn}})$ be the category of vector bundles on $X^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ with Hodge-Tate weights in $\{0,1\}$. Then there is a canonical equivalence of categories*

$$\mathcal{G} : \mathrm{Vect}_{[0,1]}(X^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\simeq} \mathcal{BT}_n(X)$$

compatible with Cartier duality.

As a consequence of our construction, one finds that this equivalence is functorial in X and satisfies p-quasisyntomic descent.

istorical *Remark* 1. Here are some historical remarks. We will write $\mathcal{BT}(X)$ for the category of p-divisible groups over X:

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- A general construction of a *crystalline* Dieudonné functor was given in [3]
- When $X = \operatorname{Spec} R$ with R a perfect $\mathbb{F}_p$-algebra, a form of Theorem A is due to Gabber and Lau [32]: One can show that $\mathrm{Vect}_{[0,1]}(X^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})$ is equivalent to a category of finite locally free $W_n(R)$-modules equipped with certain additional structures appearing in *loc. cit.*
- When X is quasisyntomic, Anschütz and Le Bras demonstrated in [1] an equivalence of categories between $\mathcal{BT}(X)$ and a certain category of *admissible* φ-modules over a sheaf of rings $\mathcal{O}^{\mathrm{pris}}$ obtained using prismatic cohomology. One can once again reformulate their result as proving Theorem A for such rings. See also the recent paper of Mondal [41], where a similar connection is made.

Remark 2. As a prior footnote observed, X^{syn} is not in general a classical object, and, correspondingly the category of vector bundles on X^{syn} is in general an ∞-*category* that is not classical. However, the theorem shows that the subcategory spanned by the objects with Hodge-Tate weights in $\{0,1\}$ *is* classical.

Remark 3. The compatibility with Cartier duality takes the following shape: There is a canonical object $\mathcal{O}_n^{\mathrm{syn}}\{1\}$ in $\mathrm{Vect}_{[0,1]}(X^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})$ of rank 1, the *Breuil-Kisin twist*, which we can tensor with any vector bundle $\mathcal{M}$ over $X^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ to obtain the twist $\mathcal{M}\{1\}$. If $\mathcal{M}$ has Hodge-Tate weights $0,1$, then so does $\mathcal{M}^\vee\{1\}$, and we now have a canonical isomorphism of truncated Barsotti-Tate groups

$$\mathcal{G}(\mathcal{M}^\vee\{1\}) \xrightarrow{\simeq} \mathcal{G}(\mathcal{M})^*,$$

where the right hand side is the Cartier dual of $\mathcal{G}(\mathcal{M})$.

syn_cohom *Remark* 4. One striking feature of the theorem to those familiar with Dieudonné theory hitherto is the natural direction of the functor realizing the equivalence. Usually, one associates linear algebraic objects with p-divisible groups and their truncations. Here, our functor $\mathcal{G}$ goes in the *other* direction and associates truncated Barsotti-Tate groups with objects that are more linear algebraic in nature. Its definition is in terms of syntomic cohomology: That is, for any map $f : \operatorname{Spf} C \to X$, we set

$$\mathcal{G}(\mathcal{M})(C) = \tau^{\leq 0} R\Gamma(C^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}, (f^{\mathrm{syn}})^*\mathcal{M}).$$

As such it is completely canonical and compatible with arbitrary base-change.

1.1. **Method of proof.** Our proof is geometric in nature, and is essentially independent of the previous works referenced above. Its starting point is the fundamental result of Grothendieck that the stack BT_n of n-truncated Barsotti-Tate groups is a smooth Artin stack [23]. Our proof begins by showing the following analogue of Grothendiek's theorem:

ct_repble **Theorem B.** *The assignment*[2]

$$X \mapsto \mathrm{Vect}_{[0,1]}(X^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})^{\simeq}$$

is represented by a smooth p-adic formal Artin stack over $\mathbb{Z}_p$.

[2]We write $\mathcal{C}^{\simeq}$ for the underlying groupoid of any category $\mathcal{C}$.

Theorem A can be reduced to the assertion that this formal Artin stack—which we will denote for the purposes of this introduction by $\mathrm{Vect}^{\mathrm{syn}}_{[0,1],n}$—is canonically isomorphic to BT_n. To construct this isomorphism, we need another representability result (see Remark 4).

ns_repble **Theorem C.** *For any $\mathcal{M}$ in $\mathrm{Vect}_{[0,1]}(X^{\mathrm{syn}})$ the functor $\mathcal{G}(\mathcal{M})$ on formal schemes over X given for $f : \mathrm{Spf}\, C \to X$ by*

$$\mathcal{G}(\mathcal{M})(C) = \tau^{\leq 0} R\Gamma(C^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}, (f^{\mathrm{syn}})^*\mathcal{M})$$

is represented by a truncated Barsotti-Tate group scheme over X.

Theorems B and C together now give us a map of smooth p-adic formal Artin stacks

$$\mathcal{G} : \mathrm{Vect}^{\mathrm{syn}}_{[0,1],n} \to \mathrm{BT}_n.$$

To get a map in the other direction, by the smoothness of the stacks involved, and quasisyntomic descent, it suffices to define a canonical map $\mathcal{M} : \mathrm{BT}_n(X) \to \mathrm{Vect}^{\mathrm{syn}}_{[0,1],n}(X)$ when $X = \mathrm{Spf}\, R$ with R quasiregular semiperfectoid (qrsp). For this, we use the functor defined by Mondal [41], though with a bit more work we could have gotten away with only using the Dieudonné functor of Berthelot-Breen-Messing [3] for characteristic p qrsp inputs; see Remark 11.23.

With the functor $\mathcal{M}$ in hand, the verification that it is indeed an inverse proceeds via a direct and quite simple argument that comes down to a computation of Bhatt-Lurie showing that we have a canonical isomorphism $\mathcal{G}(\mathcal{O}^{\mathrm{syn}}_n\{1\}) \simeq \mu_{p^n}$.

trunc_Gmu 1.2. **Truncated (G,μ)-displays.** The representability result in Theorem B is a special case of a more general result that proves another conjecture of Drinfeld from [15]. Here is the setup for this: We start with a smooth affine group scheme G over $\mathbb{Z}_p$ and a cocharacter $\mu : \mathbb{G}_m \to \underline{\mathrm{Aut}}(G)$ defined over the ring of integers $\mathcal{O}$ of a finite unramified extension of $\mathbb{Q}_p$ that is **1-bounded** in the sense of Lau [30], so that the weights of the adjoint action of μ on $\mathfrak{g}$ are bounded above by 1. For example, if G is connected, reductive, then μ will simply be a minuscule cocharacter of G^{ad}. A standard example, for non-negative integers $d \leq h$ with $h > 0$, is $G = \mathrm{GL}_h$ with $\mu = \mu_d$ given by $z \mapsto \mathrm{diag}(\underbrace{z,\ldots,z}_{d}, 1, \ldots, 1)$.

Drinfeld has given a definition for a stack $\mathrm{BT}^{G,\mu}_n$ associated with the pair (G,μ) that specializes to an open substack of $\mathrm{Vect}^{\mathrm{syn}}_{[0,1],n}$ when $(G,\mu) = (\mathrm{GL}_h, \mu_d)$. He conjectured that this should be representable by smooth 0-dimensional p-adic formal Artin stack over $\mathcal{O}$.

Here's a somewhat quick overview of the definition: We begin with a cartoon of how the syntomification is obtained. For any p-complete commutative ring R, the stack R^{syn} is obtained as follows. We have (derived) p-adic formal stacks $R^{\Delta}, R^{\mathcal{N}}$: These are the *prismatization* of R and the *(Nygaard) filtered prismatization* of R, respectively. The second of these is a *filtered stack*: it lives naturally over $\mathbb{A}^1/\mathbb{G}_m$. The open locus lying over the point $\mathbb{G}_m/\mathbb{G}_m$ can be identified with R^{Δ}: this is the *de Rham* embedding of R^{Δ} into $R^{\mathcal{N}}$. There is another open immersion of R^{Δ} into $R^{\mathcal{N}}$, called the *Hodge-Tate* embedding, that is physically disjoint from the de Rham embedding. The syntomification is obtained by gluing these two copies of R^{Δ} together.

fect_case *Remark* 5. When R is a perfect $\mathbb{F}_p$-algebra, we can identify R^{Δ} with $\mathrm{Spf}\, W(R)$ and describe $R^{\mathcal{N}}$ via the Rees construction applied to the p-adic filtration on $W(R)$: this yields a stack isomorphic to $[\mathrm{Spf}\, W(R)[u,t]/(ut-p)/\mathbb{G}_m]$. Here, u has degree 1 and t has degree -1, and the de Rham and Hodge-Tate embeddings correspond respectively to the loci $\{t \neq 0\}$ and $\{u \neq 0\}$ (though the latter appears with a Frobenius twist). Objects over the mod-p fiber of the syntomification can be interpreted as giving two filtrations on objects over R—a decreasing Hodge filtration and an increasing conjugate filtration—along with an identification of their associated gradeds up to Frobenius twist. In other words, vector bundles over this stack are the F-zips of Moonen-Pink-Wedhorn-Ziegler [43].

Let us return to the question of defining $\mathrm{BT}^{G,\mu}_n$. Using the Breuil-Kisin twist $\mathcal{O}^{\mathrm{syn}}_n\{1\}$ and the cocharacter μ, we can twist G to a group stack $G\{\mu\}$ over $\mathcal{O}^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$.

We now define $\mathrm{BT}_n^{G,\mu}$ as the groupoid-valued functor on a certain full subcategory $\mathrm{CRing}^{f,p\text{-comp}}_{\heartsuit,\mathcal{O}/}$ of the category of p-complete commutative $\mathcal{O}$-algebras[3] For $\mathcal{O}$-algebras R in this subcategory, $\mathrm{BT}_n^{G,\mu}(R)$ is the (∞)-groupoid of flat $G\{\mu\}$-torsors on $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ whose restriction to $R^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is trivial *flat locally on* $\mathrm{Spec}\, R$.

Remark 6. This local triviality condition should be viewed as an analogue of the possibly familiar Kottwitz signature condition appearing in the moduli description of Shimura varieties of PEL type: When $G = \mathrm{GL}_h$, the definition is essentially concerned with vector bundles on $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. Any such F-gauge gives rise to a filtered vector bundle over $\mathrm{Spec}\, R$ equipped with a *Hodge filtration*. The triviality condition imposed here fixes the type of this filtration.

The next theorem proves [15, Conjecture C.3.1].[4]

othm:main **Theorem D.** *The formal prestack $\mathrm{BT}_n^{G,\mu}$ is represented by a zero-dimensional quasi-compact smooth Artin formal stack over $\mathcal{O}$ with affine diagonal. Moreover, the natural map $\mathrm{BT}_{n+1}^{G,\mu} \to \mathrm{BT}_n^{G,\mu}$ is smooth.*

As far as we are aware, the work of Bültel-Pappas [11] was the first to attempt to construct such stacks in generality. However, their construction—which involves working with a more direct generalization of the perfect case explained in Remark 5 using Witt vectors—has the expected properties only when restricted to what the authors there call the 'adjoint nilpotent' locus. When considering the stack of p-divisible groups, this amounts to working only with the connected ones.

We should also make note of the work of K. Ito [25], where a somewhat different definition of (G, μ)-displays is given, using the prismatic site of Bhatt-Scholze. This agrees with ours for qrsp and hence also for smooth or even quasisyntomic inputs; see Section 7 of *loc. cit.*

parahoric *Remark* 7. One should formulate and prove versions of the Theorem D 'with coefficients' (see for instance [25] or [40]), allowing smooth group schemes over the ring of integers of finite extensions of $\mathbb{Q}_p$. This will be considered in work of the first author, Z. G.

There is also the interesting question of allowing the cocharacter to be defined over a *ramified* extension of the field of coefficients, and, relatedly, to find the correct analogues of $\mathrm{BT}_n^{G,\mu}$ associated with parahoric group schemes, but this appears to require a genuinely new idea.

Let us now record some other results about $\mathrm{BT}_n^{G,\mu}$ that are of independent interest, and give some idea of the proof of Theorem D along the way.

Following Drinfeld, we first obtain a somewhat explicit description of the mod-p fiber $\mathrm{BT}_1^{G,\mu} \otimes \mathbb{F}_p$. To explain this, recall that we can associate with the pair (G, μ) the algebraic stack $F\mathrm{Zip}^{G,\mu}$ of **F-zips with G-structure and type** μ; see [43]: it is a smooth zero-dimensional Artin stack over k with affine diagonal. We now have:

to_f-zips **Theorem E.** *There is a natural map $\mathrm{BT}_1^{G,\mu} \otimes \mathbb{F}_p \to F\mathrm{Zip}^{G,\mu}$ that is a relatively representable by a smooth zero-dimensional Artin stack with relatively affine diagonal: in fact, it is a gerbe banded by a finite flat commutative p-group scheme of height one, the* Lau group scheme. *In particular, $\mathrm{BT}_1^{G,\mu} \otimes \mathbb{F}_p$ is a smooth zero-dimensional Artin stack over k with affine diagonal.*

Remark 8. When restricted to smooth inputs, this result is due to Drinfeld [15]. We verify here that his description continues to hold in general.

With Theorem E in hand, the rest of the proof of Theorem D comes down to a double bootstrapping argument. First, we inductively establish representability for $\mathrm{BT}_n^{G,\mu} \otimes \mathbb{F}_p$ for $n \geq 1$. For this, note that, given an object $\mathcal{P} \in \mathrm{BT}_n^{G,\mu}(R)$, we can twist the adjoint representation on $\mathfrak{g}$ by $\mathcal{P}$ to obtain a vector bundle $(\mathfrak{g})_{\mathcal{P}}$ over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. It is not difficult now to see that the fibers of $\mathrm{BT}_{n+1}^{G,\mu} \to \mathrm{BT}_n^{G,\mu}$ over $\mathcal{P}$ are controlled by the syntomic cohomology of this F-gauge. The main property that makes this F-gauge tractable is that it has Hodge-Tate weights bounded

[3]Explicitly, R is in $\mathrm{CRing}^{f,p\text{-comp}}_{\heartsuit}$ if it is p-complete and is such that $\Omega^1_{\pi_0(R)/p\pi_0(R))/\mathbb{F}_p}$ is finitely generated over $\pi_0(R)$. This condition ensures that $\mathrm{Spf}\, R$ admits a p-quasisyntomic cover by the formal spectrum of a *semiperfectoid* algebra.

[4]Drinfeld's definition imposes a seemingly weaker local triviality condition than ours, but we verify in the body of this paper that it in fact yields a notion equivalent to ours here.

by 1: this is a direct consequence of the fact that μ is 1-bounded. The inductive argument therefore comes down to a special case of the following theorem, which is also an input into the proof of Theorem C:

:f_gauges **Theorem F.** *Suppose that $R\in\mathrm{CRing}^{f,p\text{-}comp}$ and suppose that $\mathcal{M}$ is an F-gauge over R corresponding to a perfect complex on $R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}$ with Tor amplitude $[-r,\infty)$ and Hodge-Tate weights bounded by 1. Then the assignment on p-complete R-algebras given by*

$$C\mapsto\tau^{\leq 0}R\Gamma(C^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z},\mathcal{M}|_{C^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}})$$

is represented by a locally finitely presented p-adic formal derived algebraic r-stack over R.

In fact, we only need the case where R is an $\mathbb{F}_p$-algebra and where $\mathcal{M}$ is a vector bundle over $R^{\mathrm{syn}}\otimes\mathbb{F}_p$ (and this is also an input into a similar bootstrapping argument in the proof of the above theorem).

The second bootstrapping argument involves a derived descent statement, encapsulated by:

d_descent **Proposition 1.** *The natural map*

$$\mathrm{BT}_n^{G,\mu}(R)\to\mathrm{Tot}\left(\mathrm{BT}_n^{G,\mu}(R\otimes^{\mathbb{L}}\mathbb{F}_p^{\otimes^{\mathbb{L}}_{\mathbb{Z}}(\bullet+1)})\right)$$

is an equivalence.

Note that even to state this result, one needs to be working with *animated* commutative rings. We will do so systematically in the body of the paper.

To make full use of the proposition, we also need some finer control of the deformation theory of $\mathrm{BT}_n^{G,\mu}$. This involves an interesting (and in a sense elementary) technical tool: Weil restriction from $\mathbb{Z}/p^n\mathbb{Z}$ to $\mathbb{Z}$, an operation that is only fully sensible in the derived realm. This yields, for any p-adic formal Artin stack X, a new *derived* formal Artin stack $X^{(n)}$, whose values are characterized by

$$X^{(n)}(R)=X(R\otimes^{\mathbb{L}}\mathbb{Z}/p^n\mathbb{Z}).$$

Using this, for any animated divided power thickening $(R'\twoheadrightarrow R,\gamma)$ of k-algebras in $\mathrm{CRing}^f_{k/}$ we can write down a canonical commuting diagram

ng_square (1.2.1)
$$\begin{tikzcd}
\mathrm{BT}_n^{G,\mu}(R') \arrow[r] \arrow[d] & BP_\mu^{-,(n)}(R') \arrow[d] \\
\mathrm{BT}_n^{G,\mu}(R) \arrow[r] & BP_\mu^{-,(n)}(R)\times_{BG^{(n)}(R)}BG^{(n)}(R')
\end{tikzcd}.$$

Here, $P_\mu^-\subset G_{\mathcal{O}}$ is the parabolic subgroup associated with the non-negative eigenspaces of the adjoint action of μ, and BH for any group scheme H denotes its classifying stack. The obstruction theory for $\mathrm{BT}_n^{G,\mu}$ is now captured by the following result:

roth_mess **Theorem G** (Grothendieck-Messing theory)**.** *The above commuting square is* Cartesian *when the divided powers are* nilpotent.

This should be viewed as a truncated analogue of classical Grothendieck-Messing theory, which classifies liftings of p-divisible groups across classical nilpotent divided power thickenings in terms of lifts of the Hodge filtration on its crystalline realization. We first prove this when R' is an $\mathbb{F}_p$-algebra, and then lift it to general inputs using Proposition 1. It is now not hard to deduce the general case of Theorem D from its mod-p version (at least when $p>2$) by applying Theorem G to the canonical nilpotent divided power thickening $R\to R/^{\mathbb{L}}p$. A very slightly more involved argument also works when $p=2$.

1.3. **Further remarks on the proofs.** From a technical perspective, all the results above are special cases of theorems about objects that we call 1-*bounded stacks*. Given such an object $\mathcal{X}$ over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$, we can define a functor on animated p-complete R-algebras given by:

$$\Gamma_{\mathrm{syn}}(\mathcal{X}) : \mathrm{Map}_{/R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}}(C^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}).$$

Much of this paper is devoted to finding the correct conditions on $\mathcal{X}$ and the mapping functor that ensure that $\Gamma_{\mathrm{syn}}(\mathcal{X})$ is representable. Essentially, these conditions ensure that the arguments sketched in (1.2) go through when applied to $\Gamma_{\mathrm{syn}}(\mathcal{X})$. Examples of 1-bounded stacks include:

- The stack over $\mathcal{O}^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ parameterizing $G\{\mu\}$-torsors that are trivial when restricted to $B\mathbb{G}_m \times \mathrm{Spec}\, k$ for any algebraically closed field k: this is of course relevant for Theorem D;
- The total spaces of vector bundles (and more generally perfect complexes) with Hodge-Tate weights bounded by 1: this is relevant for Theorem C.

The general representability result boils down to knowing the representability of certain Artin-Milne type cohomology groups, generalizing the fppf cohomology of finite flat group schemes of height one; see Section 7. For this, we use representability results of Bragg and Olsson [10].

This level of generality will be required for future applications, including, for instance, the construction of spaces of *isogenies* between objects in $\mathrm{BT}_\infty^{G,\mu}$ [33], leading to a general construction of Rapoport-Zink spaces *without* the direct involvement of p-divisible groups. One can also hope that it will help address some of the difficulty in constructing the correct analogues of $\mathrm{BT}_n^{G,\mu}$ when G is a parahoric, non-reductive group scheme; see Remark 7.

For now, as a consequence of these general results, we are also able to obtain an extension of Theorem B to perfect F-gauges.

f_fgauges **Theorem H.** *The prestack* $\mathrm{Perf}^{\mathrm{syn}}_{[0,1],n}$ *assigning to every p-complete ring R the ∞-groupoid of perfect complexes on $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ with Hodge-Tate weights in* $\{0,1\}$ *is represented by a locally finitely presented p-adic derived Artin stack over $\mathbb{Z}_p$.*

1.4. **Application to Shimura varieties.** Theorem D also has a global application, which was the main motivation for one of us (K.M.) to pursue the work here. Suppose that (G, X) is a Shimura datum of abelian type with reflex field E. Suppose that G is unramified at p with reductive model $G_{\mathbb{Z}_p}$: this implies in particular that E is unramified over p. Fix a place $v \mid p$ of E, and choose an $\mathscr{O}_{E_v}$-rational representative $\mu^{-1} : \mathbb{G}_m \to G_{\mathscr{O}_{E_v}}$ for the (inverse of the) conjugacy class of Shimura cocharacters underlying X. Then, for any level subgroup $K \subset G(\mathbb{A}_f)$ with $K_p = G_{\mathbb{Z}_p}(\mathbb{Z}_p)$, we have the integral canonical model $\mathcal{S}_K$ over $\mathscr{O}_{E,(v)}$. Let $\mathcal{S}_K^{\mathfrak{F}}$ be its formal completion along the mod-v fiber. Combining the results here with those of Imai-Kato-Youcis in [24], one obtains the following theorem; when $p > 2$, it is already contained in *loc. cit.*, and the ingredients necessary to remove this condition will appear in (a revision of) [38].

m:shimura **Theorem I.** *There exists a canonical formally étale map*

$$\varpi : \mathcal{S}_K^{\mathfrak{F}} \to \mathrm{BT}_\infty^{G^c_{\mathbb{Z}_p}, \mu^{-1}}.$$

When (G, X) is of Siegel type, this agrees via the (polarized version of the) equivalence of Theorem A with the map carrying a polarized abelian variety to its corresponding polarized p-divisible group.

Here, G^c is the quotient of G by its maximal torus that is $\mathbb{R}$-split but not $\mathbb{Q}$-split, and μ is an $\mathscr{O}_{E_v}$-rational representative for the conjugacy class of the Shimura cocharacter associated with X. The map in the theorem is determined in a precise way by the canonical pro-étale $G(\mathbb{Z}_p)$-torsor over the generic fiber of the Shimura variety via a functor such as the one described in [5, §6.3] in the context of vector bundles over the syntomification.

The above theorem will be combined with Theorem F in [38] to give a general construction of special cycle classes on the integral model $\mathcal{S}_K$.

1.5. **Structure of the paper.**

- We begin in Section 3 with some background on derived stacks. We also recall the notion of derived Weil restriction, and some facts about divided powers in the animated context.

- In Section 4, we recall the story of filtered animated rings and as well as of filtered derived stacks via C. Simpson's perspective of viewing such gadgets as objects over $\mathbb{A}^1/\mathbb{G}_m$. We give an account of our notion of a 1-bounded stack, give examples of such objects and prove some general facts about them.
- In Section 5, we present Lau's theory of higher frames and displays from [30] in an animated context, and use this to prove an abstract version of Grothendieck-Messing theory for 1-bounded stacks that is employed later.
- In Section 6, we review the stack-theoretic constructions of Drinfeld and Bhatt-Lurie from [5], [7], [8] and [16]. In particular, we recall the *affineness* of the various stacks when working with *semiperfectoid* rings, where the stacks of Drinfeld and Bhatt-Lurie are now obtained—via the Rees construction—from Nygaard filtered prismatic cohomology.
- Section 7 recalls a result of Bragg-Olsson on the representability of derived stacks that parameterize the fppf cohomology of certain 'perfect complexes' of finite flat group schemes of height one.
- We then prove our general representability theorems for stacks of sections associated with 1-bounded stacks: this takes up Section 8. We follow the strategy sketched above: Representability on the level of F-zips is first lifted to representability of the stack of sections over the mod-p syntomification of $\mathbb{F}_p$-algebras. This is then bootstrapped to representability over the syntomification of $\mathbb{F}_p$-algebras, followed by a further bootstrapping up to arbitrary p-nilpotent algebras. We give some applications of our general representability results for stacks of F-gauges, and prove Theorems F and H.
- Section 9 is where we recall Drinfeld's definition of the stacks $\mathrm{BT}_n^{G,\mu}$ and prove Theorems D, E and G.
- In Section 10, we use deformation theory to give explicit descriptions of the points of $\mathrm{BT}_n^{G,\mu}$ valued in certain regular complete local Noetherian rings, and show that the deformation rings defined by Faltings in [18, §7] are in fact providing explicit coordinates for the complete local rings of $\mathrm{BT}_\infty^{G,\mu} = \varprojlim_n \mathrm{BT}_n^{G,\mu}$.
- Finally, in Section 11, we gather our results together to prove Theorem A.

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:notation

2. Notation and other conventions

(1) We adopt a resolutely ∞-categorical approach. This means that all operations, including (but not limited to) limits, colimits, tensor products, exterior powers etc. are always to be understood in a derived sense, unless otherwise stated.

(2) We will use Spc to denote the ∞-category of spaces, anima, or homotopy types: roughly speaking, this is the localization of the Quillen model category of simplicial sets with respect to homotopy equivalences.

(3) A map $X \to Y$ in Spc is **surjective** if the induced map $\pi_0(X) \to \pi_0(Y)$ is a surjective map of sets; we will denote surjective maps with $\twoheadrightarrow$.

(4) For any ∞-category $\mathcal{C}$ and an object c of $\mathcal{C}$, we will write $\mathcal{C}_{c/}$ (resp. $\mathcal{C}_{/c}$) for the comma ∞-categories of arrows $c \to d$ (resp. $d \to c$).

(5) We will in a few places make reference to the process of **animation**, as described say in [39, Appendix A]. This is a systematic way to get well-behaved ∞-categories and functors between them, starting from 'nice' classical categories $\mathcal{C}$ with a set $\mathcal{C}_0$ of compact, projective generators. The animation of such a category is the ∞-category $\mathcal{P}_\Sigma(\mathcal{C}_0)$ of presheaves of spaces on $\mathcal{C}_0$ that preserve finite products.

(6) We will denote by CRing the ∞-category of **animated commutative rings**, obtained via the process of animation from the usual category of commutative rings. Objects here can be viewed as being simplicial commutative rings up to homotopy equivalence.

(7) We will follow homological notation for CRing: For any $n \in \mathbb{Z}_{\geq 0}$, $\mathrm{CRing}_{\leq n}$ will be the subcategory of CRing spanned by those objects R with $\pi_k(R) = 0$ for $k > n$; that is, by the n**-truncated objects**. If $n = 0$, we will write $\mathrm{CRing}^\heartsuit$ instead of $\mathrm{CRing}_{\leq 0}$: its objects are the **discrete** or classical commutative rings, and the category can be identified with the usual category of commutative rings.

(8) Any animated commutative ring R admits a **Postnikov tower** $\{\tau_{\leq n}R\}_{n\in\mathbb{Z}_{\geq 0}}$ where $R \to \tau_{\leq n}R$ is the universal arrow from R into $\mathrm{CRing}_{\leq n}$ and the natural map $R \to \varprojlim_n \tau_{\leq n}R$ is an equivalence.
(9) We will also need the notion of a **stable ∞-category** from [35]: this is the ∞-category analogue of a triangulated category. The basic example is the ∞-category Mod_R, the derived ∞-category of R-modules. We will use *cohomological* conventions for these objects and so will write for instance $H^{-1}(M)$ instead of $\pi_1(M)$.
(10) An important feature of a stable ∞-category $\mathcal{C}$ is that it has an initial and final object 0, and, for any map $f : X \to Y$ in $\mathcal{C}$, we have the **homotopy cokernel** $\mathrm{hcoker}(f)$ defined as the pushout of $0 \to Y$ along f. We will sometimes abuse notation and write Y/X for this object.
(11) If $R \in \mathrm{CRing}^{\heartsuit}$ is a classical commutative ring, $M \in \mathrm{Mod}_R$ is a complex of R-modules, and $a_1, \ldots, a_m \in R$ form a regular sequence, we will write $M/^{\mathbb{L}}(a_1, \ldots, a_m)$ for the derived tensor product

$$M \otimes^{\mathbb{L}}_R R/(a_1, \ldots, a_m).$$

(12) In any stable ∞-category $\mathcal{C}$ and an object X in $\mathcal{C}$, we set $X[1] = \mathrm{hcoker}(X \to 0)$: this gives a shift functor $\mathcal{C} \to \mathcal{C}$ with inverse $X \mapsto X[-1]$, and we set $\mathrm{hker}(f : X \to Y) = \mathrm{hcoker}(f)[-1]$.
(13) Given an animated commutative ring R, we will write $\mathrm{Mod}^{\mathrm{cn}}_R$ for the sub ∞-category spanned by the connective objects (that is, objects with no cohomology in positive degrees), and $\mathrm{Perf}(R)$ for the sub ∞-category spanned by the perfect complexes.
(14) We have a truncation operator $\tau^{\leq 0} : \mathrm{Mod}_R \to \mathrm{Mod}^{\mathrm{cn}}_R$ defined as the right adjoint to the natural functor in the other direction. This leads to truncation operators $\tau^{\leq n}$ and cotruncation operators $\tau^{\geq n}$ for any $n \in \mathbb{Z}$ in the usual way.
(15) If $f : X \to Y$ is a map in $\mathrm{Mod}^{\mathrm{cn}}_R$, we set $\mathrm{hker}^{\mathrm{cn}}(f) = \tau^{\leq 0}\,\mathrm{hker}(f)$: this is the **connective (homotopy) kernel**.
(16) For any stable ∞-category $\mathcal{C}$, the mapping spaces $\mathrm{Map}_{\mathcal{C}}(X, Y)$ between any two objects have canonical lifts to the ∞-category of connective spectra. We will be interested in stable ∞-categories like Mod_R, which are $\mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}}$-modules, in the sense that the mapping spaces have canonical lifts to $\mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}}$. In this case, we can extend the mapping spaces $\mathrm{Map}_{\mathcal{C}}(X, Y)$ from $\mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}}$ to objects $\underline{\mathrm{Map}}_{\mathcal{C}}(X, Y)$ in $\mathrm{Mod}_{\mathbb{Z}}$ by taking

$$\underline{\mathrm{Map}}_{\mathcal{C}}(X, Y) = \mathrm{colim}_{k\geq 0}\,\mathrm{Map}_{\mathcal{C}}(X, Y[k])[-k] \in \mathrm{Mod}_{\mathbb{Z}}\,.$$

When $\mathcal{C} = \mathrm{Mod}_R$ for an animated commutative ring R, this lifts to the **internal Hom** in Mod_R.
(17) We will write $\mathbf{\Delta}$ for the usual **simplex** category with objects the sets $\{0, 1, \ldots, n\}$ and morphisms given by the non-decreasing functions between them.
(18) A **cosimplicial object** $S^{(\bullet)}$ in an ∞-category $\mathcal{C}$ is a functor

$$\mathbf{\Delta} \mapsto \mathcal{C}$$
$$[n] \mapsto S^{(n)}.$$

If $\mathcal{C}$ admits limits, we will write $\mathrm{Tot}S^{(\bullet)}$ for the limit of the corresponding functor: this is the **totalization** of $S^{(\bullet)}$.
(19) Given any ∞-category $\mathcal{C}$ with finite coproducts, and any object S in $\mathcal{C}$ there is a canonical cosimplicial object $S^{(\bullet)}$ in $\mathcal{C}$, the **Čech conerve** with $S^{(n)} = \bigsqcup_{i\in[n]} S$.
(20) If X is a (derived) stack (resp. an object of Mod_R for some R), and $N \in \mathbb{Z}\backslash\{0\}$, we will write $X[N^{-1}]$ for the base change $\mathrm{Spec}\,\mathbb{Z}[N^{-1}] \times X \to \mathrm{Spec}\,\mathbb{Z}[N^{-1}]$ (resp. for the base change $\mathbb{Z}[N^{-1}] \otimes_{\mathbb{Z}} X$ in $\mathrm{Mod}_{\mathbb{Z}[N^{-1}]\otimes_{\mathbb{Z}}R}$). On the rare occasions when these notations collide, context will make the usage clear.

3. Stacks and other preliminaries

3.1. **Square-zero extensions and differential conditions.** Given a pair (R, M) with $R \in \mathrm{CRing}$ and $M \in \mathrm{Mod}^{\mathrm{cn}}_R$, we have a canonical object $R \oplus M \in \mathrm{CRing}_{R//R}$, the **trivial square-zero extension** of R by M: This is obtained by animating the construction on such pairs with R a polynomial algebra and M a finite free R-module to the usual square-zero extension $R \oplus M$.

If $R \in \mathrm{CRing}_{A/}$, we set

$$\mathrm{Der}_A(R,M) = \mathrm{Map}_{A//R}(R, R \oplus M).$$

This is the space of A-**derivations** of R **valued in** M. We always have the trivial A-derivation $d_{\mathrm{triv}} = (\mathrm{id}, 0)$.

A **square-zero extension** of R by M in $\mathrm{CRing}_{A/}$ is a surjective map $R' \twoheadrightarrow R$ in $\mathrm{CRing}_{A/}$ such that there exists an A-derivation $d : R \to R \oplus M[1]$ and an equivalence of A-algebras

$$R' \xrightarrow{\simeq} R \times_{d, R\oplus M[1], d_{\mathrm{triv}}} R.$$

We have the **cotangent complex** $\mathbb{L}_{R/A} \in \mathrm{Mod}_R^{\mathrm{cn}}$: this is obtained by animating the functor taking maps $S \to S'$ of polynomial rings over $\mathbb{Z}$ in finitely many variables to the module of differentials $\Omega^1_{S'/S}$, and is characterized by the property that, for any trivial square zero extension $R \oplus M \twoheadrightarrow R$, there is a canonical equivalence

$$\mathrm{Map}_{\mathrm{Mod}_R}(\mathbb{L}_{R/A}, M) \xrightarrow{\simeq} \mathrm{Der}_A(R,M).$$

onditions **Definition 3.2.** An R-algebra $C \in \mathrm{CRing}_{R/}$ is **finitely presented** (over R) if the functor $S \mapsto \mathrm{Map}_{\mathrm{CRing}_{R/}}(C,S)$ respects filtered colimits. For any such finitely presented C, the cotangent complex $\mathbb{L}_{C/R} \in \mathrm{Mod}_C^{\mathrm{cn}}$ is perfect; see [35, p. 17.4.3.18].

If in addition $\mathbb{L}_{C/R}$ is 1-connective, we say that C is **unramified** over R; if $\mathbb{L}_{C/R} \simeq 0$, we say that C is **étale** over R.

We say that a finitely presented $C \in \mathrm{CRing}_{R/}$ is **smooth** over R if $\mathbb{L}_{C/R} \in \mathrm{Mod}_C^{\mathrm{cn}}$ is locally free of finite rank. It is **quasi-smooth** if $\mathbb{L}_{C/R}$ is perfect with Tor amplitude $[-1,0]$.

3.3. **Derived (pre)stacks.** Suppose that $\mathcal{C}$ is an ∞-category admitting all finite and sequential limits, totalizations of cosimplicial objects, and filtered colimits. A $\mathcal{C}$**-valued prestack over** $R \in \mathrm{CRing}$ is a functor

$$F : \mathrm{CRing}_{R/} \to \mathcal{C}.$$

If $\mathcal{C} = \mathrm{Spc}$, we will simply call F a **prestack over** R. Such objects organize into an ∞-category PStk_R.

We view such prestacks as presheaves on the ∞-category of derived affine schemes $\operatorname{Spec} R'$ over R (by definition opposite to $\mathrm{CRing}_{R/}$), and we can consider the subcategory of prestacks that are fpqc (resp. étale) sheaves—that is, presheaves satisfying descent along faithfully flat (resp. faithfully flat and étale) maps $\operatorname{Spec} R' \to \operatorname{Spec} R$.

in_stacks **Definition 3.4.** Following Toën-Vezzosi [47], we will say that F is 0-**geometric** if we have $F \simeq \operatorname{Spec} R'$ for some $R' \in \mathrm{CRing}_{R/}$, and, inductively, that it is an n-**geometric derived Artin stack** over R for an integer $n \geq 1$ if it is an étale sheaf and admits a surjective cover $f : U = \sqcup_{i\in I} \operatorname{Spec} R'_i \to F$ of étale sheaves with $R'_i \in \mathrm{CRing}_{R/}$ satisfying the following condition: For every $S \in \mathrm{CRing}_{R/}$ and $x \in F(S)$, the base-change $U \times_{f,F,x} \operatorname{Spec} S \to \operatorname{Spec} S$ is represented by an $(n-1)$-geometric derived Artin stack over S.

Following Lurie [34, §5], we will say that F is a **derived Artin** n**-stack** over R if it is m-geometric for some m and is such that $F(R')$ is n-truncated for all discrete $R' \in \mathrm{CRing}_{R/,\heartsuit}$. A derived Artin 0-stack will be called a **derived algebraic space** over R.

A **derived Artin stack** over R is a prestack F that is a derived Artin n-stack for some $n \geq 0$.

A map of $X \to Y$ of prestacks over R is a **relative derived Artin stack** if, for every R-algebra C and every $y \in Y(C)$, the base-change $X_y \to \operatorname{Spec} C$ is a derived Artin stack over C.

almost_fp **Definition 3.5.** A prestack F over R is **locally of finite presentation** or **locally finitely presented** if for every filtered system $\{C_i\}_{i\in I}$ in $\mathrm{CRing}_{R/}$ with colimit $C \in \mathrm{CRing}_{R/}$, the natural map the natural map

$$\mathrm{colim}_{i\in I} F(C_i) \to F(C)$$

ly_smooth **Definition 3.6.** A prestack F over R is **formally smooth** if for every square-zero extension $C' \twoheadrightarrow C$ in $\mathrm{CRing}_{R/}$, the map $F(C') \to F(C)$ is surjective.

A derived Artin stack over R is **smooth** if is locally finitely presented and formally smooth.

Definition 3.7. A prestack F over $A \in \mathrm{CRing}$ that is an fpqc sheaf is **classical** if it is equivalent as an fpqc sheaf to the left Kan extension to $\mathrm{CRing}_{A/}$ of its classical truncation $F_{\mathrm{cl}} : \mathrm{CRing}_{\pi_0(A)/} \to \mathrm{Spc}$: That is, it is a colimit of derived affine schemes $\operatorname{Spec} B$ with $B \in \mathrm{CRing}_{\pi_0(A)/}$ in the ∞-category of fpqc sheaves on $\mathrm{CRing}^{\mathrm{op}}_{A/}$. The functor $F \mapsto F^{\mathrm{cl}}$ is fully faithful when restricted to classical prestacks.

t_complex

3.8. For any prestack $F \in \mathrm{PStk}_R$, we have an ∞-category $\mathrm{QCoh}(F)$ of **quasi-coherent sheaves on** F. The precise definition can be found in [37, §6.2.2]: roughly speaking, it is obtained by right Kan extension of the contravariant functor sending $S \in \mathrm{CRing}_{R/}$ to Mod_S. One can think of an object $\mathcal{M}$ in $\mathrm{QCoh}(F)$ as a way of assigning to every point $x \in F(S)$ an object $\mathcal{M}_x \in \mathrm{Mod}_S$ compatible with base-change. This ∞-category is well-behaved when F is **quasi-geometric** [37, §9.1]: this means that F is an fpqc sheaf admitting a flat cover by an affine derived scheme with quasi-affine diagonal. Most of the prestacks we will encounter in this paper will be of this nature; see Corollary 6.33.

We will say that $\mathcal{M}$ is **connective** if $\mathcal{M}_x$ belongs to $\mathrm{Mod}^{\mathrm{cn}}_S$ for each $x \in F(S)$ as above. We will say that it is **almost connective** if, for every $x \in F(S)$, there exists $n \in \mathbb{Z}_{\geq 0}$ such that $\mathcal{M}_x[n]$ is connective. We will say that it is **perfect** if, for every $x \in F(S)$, $\mathcal{M}_x$ is perfect.

Write $\mathrm{QCoh}^{\mathrm{cn}}(F)$ (resp. $\mathrm{QCoh}^{\mathrm{acn}}(F)$, resp. $\mathrm{Perf}(F)$ for the ∞-category spanned by the connective (resp. almost connective, resp. perfect) objects in $\mathrm{QCoh}(F)$.

t_complex

Definition 3.9. Following [37, §17.2.4], we will say that a morphism $f : F \to G$ in PStk_R **admits a cotangent complex** if there exists $\mathbb{L}_{F/G} \in \mathrm{QCoh}^{\mathrm{acn}}(X)$ such that, for every $C \in \mathrm{CRing}_{R/}$, every $M \in \mathrm{Mod}^{\mathrm{cn}}_R$, and every $x \in F(C)$, we have a canonical equivalence

$$\mathrm{Map}_{\mathrm{Mod}_C}(\mathbb{L}_{F/G,x}, M) \xrightarrow{\simeq} \mathrm{fib}_{(f(x)[M],x)}(F(C \oplus M) \to G(C \oplus M) \times_{G(C)} F(C)).$$

Here, $f(x)[M] \in G(C \oplus M)$ is the image of $f(x)$ along the natural section $G(C) \to G(C \oplus M)$.

If $F = \operatorname{Spec} C$ and $G = \operatorname{Spec} D$, then by Yoneda, any morphism $f : F \to G$ corresponds to an arrow $D \to C$ in $\mathrm{CRing}_{R/}$, and f admits a cotangent complex, namely $\mathbb{L}_{C/D}$.

ssicality

Remark 3.10. Suppose that F is a locally finitely presented derived Artin stack over $R \in \mathrm{CRing}_{\heartsuit}$ such that the cotangent complex $\mathbb{L}_{F/R}$ is a perfect complex of *non-negative* Tor-amplitude. Then F is smooth and classical. By an argument via induction on n where F is an n-geometric derived Artin stack, this reduces to the fact that smooth R-algebras are flat over R and are hence classical; see [34, Prop. 3.4.9].

3.11. **Derived vector stacks.** We have the classical construction associating with every finite locally free R-module M the affine R-scheme $\mathbf{V}(M)$ with ring of functions $\mathrm{Sym}_R(M^\vee)$ the symmetric algebra of the R-dual $M^\vee$ of M. Its functor of points is given by $S \mapsto S \otimes_R M$.[5]

One can now consider, for any $R \in \mathrm{CRing}$ and any perfect complex $M \in \mathrm{Mod}_R$, the prestack

$$\mathrm{CRing}_{R/} \xrightarrow{S \mapsto S \otimes_R M} \mathrm{Spc}.$$

It is represented by a finitely presented derived Artin n-stack $\mathbf{V}(M)$ over R where n is such that M has Tor amplitude in $[-n, \infty)$: a somewhat droll way to see this is to reduce to the case where R is finitely presented over $\mathbb{Z}$, and then appeal to Artin-Lurie representability, Theorem **??**.

When M has Tor amplitude in $[0, \infty)$, one can simply take the derived affine R-scheme associated with the animated symmetric algebra $\mathrm{Sym}_R(M^\vee)$ of the connective perfect complex $M^\vee$.

It is easy to see from the definition that $\mathbf{V}(M)$ has cotangent complex given by

$$\mathbb{L}_{\mathbf{V}(M)/R} \simeq \mathscr{O}_{\mathbf{V}(M)} \otimes_R M^\vee.$$

al_stacks

3.12. ***p*-adic formal stacks.** Let $\mathrm{CRing}^{p\text{-nilp}}$ be the subcategory of CRing spanned by those objects R such that p is nilpotent in $\pi_0(R)$. A ***p*-adic formal prestack** over $R \in \mathrm{CRing}$ is simply a Spc-valued functur on $\mathrm{CRing}^{p\text{-nilp}}_{R/}$. We will usually extend such an object $\mathcal{Y}$ to (derived) p-complete R-algebras A by setting

$$\mathcal{Y}(A) \overset{\mathrm{defn}}{=} \varprojlim_n \mathcal{Y}(A/^{\mathbb{L}} p^n).$$

[5] Note that this is dual to Grothendieck's convention.

For any R-algebra S, the restriction of the affine scheme $\operatorname{Spec} S$ to $\mathrm{CRing}^{p\text{-nilp}}_{R/}$ yields a p-adic formal prestack, which, since p will be fixed in this paper, we will denote simply by $\operatorname{Spf} S$. This depends only on the p-completion of S.

A p-adic formal prestack is a **p-adic formal derived Artin stack** if, for each $n \geq 1$, its restriction to $\mathrm{CRing}_{(\mathbb{Z}/p^n\mathbb{Z})/}$ is represented by a derived Artin stack. Given such a formal derived Artin stack F, we will say that it is **foo**, if 'foo' is an attribute applicable to derived Artin stacks, and, if for each $n \geq 1$, the restriction of F to $\mathrm{CRing}_{(\mathbb{Z}/p^n\mathbb{Z})/}$ is an Artin stack that is foo.

Suppose that we have a surjective map $A \twoheadrightarrow \overline{A}$ in CRing with fiber J such that $\pi_0\overline{A}_{\mathrm{red}}$ is an $\mathbb{F}_p$-algebra. Then we can consider the p-adic formal prestack $\operatorname{Spf}(A, J)$ given for each $C \in \mathrm{CRing}^{p\text{-nilp}}$ by

$$\operatorname{Spf}(A,J)(C) = \mathrm{Map}_{\mathrm{CRing}}(A, C) \times_{\mathrm{Map}_{\mathrm{CRing}}(\pi_0(A), \pi_0(C)_{\mathrm{red}})} \mathrm{Map}_{\mathrm{CRing}}(\pi_0\overline{A}_{\mathrm{red}}, \pi_0(C)_{\mathrm{red}}).$$

In other words, we are looking at maps $A \to C$ such that J maps to a nilpotent ideal in $\pi_0(C)$. If J is clear from context, we will sometimes just write $\operatorname{Spf}(A)$ instead.

bsec:weil

3.13. **Weil restrictions.** A very useful product of derived geometry is the ability to construct well-behaved Weil restrictions along certain non-flat maps. This will enable us to correctly identify the local models for our stacks from Theorem D.

Given $R \in \mathrm{CRing}$, for any prestack X over $R/^{\mathbb{L}}p^n$, we will define its **Weil restriction** $\mathrm{Res}_{(\mathbb{Z}/p^n\mathbb{Z})/\mathbb{Z}_p} X$ to be the p-adic formal prestack over R given by the composition

$$\mathrm{CRing}^{p\text{-nilp}}_{R/} \xrightarrow{C \mapsto C/^{\mathbb{L}}p^n} \mathrm{CRing}_{R/^{\mathbb{L}}p^n/} \xrightarrow{X} \mathrm{Spc}.$$

If Y is a p-adic formal prestack over R, we will set

$$Y^{(n)} = \mathrm{Res}_{(\mathbb{Z}/p^n\mathbb{Z})/\mathbb{Z}_p}(Y|_{\mathrm{CRing}_{R/^{\mathbb{L}}p^n/}}).$$

There is then a canonical map $\alpha^{(n)} : Y \to Y^{(n)}$.

There is a natural functor

$$\mathrm{QCoh}(Y \otimes_{\mathbb{Z}} \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\mathcal{F} \mapsto \mathcal{F}^{(n)}} \mathrm{QCoh}(Y^{(n)}).$$

With any $y \in Y^{(n)}(C)$ corresponding to $\overline{y} \in Y(C/^{\mathbb{L}}p^n)$ it associates the module $\mathcal{F}^{(n)}_y = \mathcal{F}_{\overline{y}}[-1]$.

For any p-adic formal prestack Z, write $i_n : Z \otimes \mathbb{Z}/p^n\mathbb{Z} \to Z$ for the associated canonical closed immersion.

The following lemma is easily verified:

_pullback

Lemma 3.14. *For any $\mathcal{F} \in \mathrm{QCoh}(Y \otimes_{\mathbb{Z}} \mathbb{Z}/p^n)$, we have a natural equivalence*

$$\alpha^{(n),*}\mathcal{F}^{(n)} \simeq i_{n,*}\mathcal{F}[-1] \in \mathrm{QCoh}(Y).$$

striction

Lemma 3.15. *Suppose that we have a map of p-adic formal prestacks $Y \to Z$ such that*

$$Y \otimes \mathbb{Z}/p^n\mathbb{Z} \to Z \otimes \mathbb{Z}/p^n\mathbb{Z}$$

is a relative locally finitely presented (resp. smooth, resp. étale) derived Artin stack with cotangent complex $\mathcal{L} \overset{\mathrm{defn}}{=} \mathbb{L}_{Y \otimes \mathbb{Z}/p^n\mathbb{Z}/Z \otimes \mathbb{Z}/p^n\mathbb{Z}}$. Then $Y^{(n)} \to Z^{(n)}$ is once again a relative locally finitely presented (resp. smooth, resp. étale) p-adic formal derived Artin stack, and we have a canonical identification

$$\mathbb{L}_{Y^{(n)}/Z^{(n)}} \xrightarrow{\simeq} \mathcal{L}^{(n)}.$$

Proof. Given that this is a question of relative representability, we can assume that $Z = \operatorname{Spf} A$ is formally affine. Using the finite presentation condition, we can further reduce to the case where $A/^{\mathbb{L}}p$ is a finitely generated $\mathbb{F}_p$-algebra. Representability of $Y^{(n)}$ can now be verified by checking the criteria of Theorem **??**. See [37, §19.1] for a much more general version of this in the context of spectral algebraic geometry.

Suppose that we have $y^{(m)} \in Y^{(n)}(C)$ corresponding to $\overline{y} \in Y(C/^{\mathbb{L}}p^n) = (Y \otimes \mathbb{Z}/p^n\mathbb{Z})(C/^{\mathbb{L}}p^n)$. Then we have:

$$\begin{aligned}\mathrm{fib}_{y^{(m)}}(Y^{(n)}(C \oplus M) \to Y^{(n)}(C)) &= \mathrm{fib}_{\overline{y}}(Y(C/^{\mathbb{L}}p^n \oplus M/^{\mathbb{L}}p^n) \to Y(C/^{\mathbb{L}}p^n)) \\ &\simeq \mathrm{Map}_{\mathrm{Mod}_{C/^{\mathbb{L}}p^n}}(\mathcal{L}_{\overline{y}}, M/^{\mathbb{L}}p^n) \\ &\simeq \mathrm{Map}_{\mathrm{Mod}_{C/^{\mathbb{L}}p^n}}(\mathcal{L}_{\overline{y}}, \underline{\mathrm{Map}}_{\mathrm{Mod}_C}(C/^{\mathbb{L}}p^n, M[1])) \\ &\simeq \mathrm{Map}_{\mathrm{Mod}_C}(i_{n,*}\mathcal{L}_{\overline{y}}[-1], M).\end{aligned}$$

This proves that the cotangent complex is as claimed.

Note that, if $Y \otimes \mathbb{Z}/p^n\mathbb{Z}$ is smooth over $Z \otimes \mathbb{Z}/p^n\mathbb{Z}$, so that $\mathcal{L}$ is a perfect complex with Tor-amplitude in $[0, \infty)$, then $(i_{n,*}\mathcal{L})^{(n)}$ is also perfect with Tor-amplitude in $[0, \infty)$, showing that $Y^{(n)}$ is a smooth Artin stack over $Z^{(n)}$. The same argument shows that $Y^{(n)}$ is étale when $Y \otimes \mathbb{Z}/p^n\mathbb{Z}$ is étale over $Z \otimes \mathbb{Z}/p^n\mathbb{Z}$. □

ed_powers

3.16. **Divided powers.** We will also need the notion of animated divided powers, which is an additional structure γ on surjective maps $R' \twoheadrightarrow R$ in CRing that 'animates' the classical notion.

This can be approached in two ways. First, we have the presentation from [39, §3.2], where one obtains an ∞-category AniPDPair via the process of animation: one takes the full subcategory $\mathcal{E}^0$ of the classical category PDPair of divided power thickenings $(R' \twoheadrightarrow R, \gamma)$ spanned by those thickenings of the form

$$(D_{(Y)}\mathbb{Z}[X, Y] \twoheadrightarrow \mathbb{Z}[X], \gamma)$$

where X, Y are finite sets of variables and $D_{(Y)}\mathbb{Z}[X, Y]$ is the divided power envelope of $\mathbb{Z}[X, Y] \xrightarrow{Y \mapsto 0} \mathbb{Z}[X]$ equipped with its tautological divided powers, and then takes AniPDPair $= \mathcal{P}_\Sigma(\mathcal{E}^0)$ to be the ∞-category of finite product preserving presheaves on $\mathcal{E}^0$. The natural functor PDPair $\to$ AniPDPair obtained via the Yoneda map is fully faithful; see [39, Lemma 3.13].

There is a forgetful functor AniPDPair $\to$ AniPair that preserves all limits and finite colimits to the ∞-category AniPair of surjective maps $R' \twoheadrightarrow R$, and we can view a divided power structure γ on such a surjective map as being a lift along the forgetful functor. The forgetful functor admits a left adjoint, the **divided power envelope**, carrying $f : R' \twoheadrightarrow R$ to a surjection $D(f) \twoheadrightarrow R$ equipped with a divided power structure. For all this, see the discussion in [39, §3.2].

A second way to go about things is to first construct the animated divided power algebra $\Gamma_R(M)$ for any $R \in$ CRing and any $M \in \mathrm{Mod}_R^{\mathrm{cn}}$: this is obtained by animating the usual divided power algebra on pairs (R, M) with R a polynomial algebra in finitely many variables and M is a finite free R-module defined for instance in [4, App. A]. This is in some sense a classical construction that goes back to the seminal work of Dold-Puppe [13].

By construction, $\Gamma_R(M)$ is an object in $\mathrm{CRing}_{R/}$, equipped with a map of R-modules $M \to \Gamma_R(M)$ that satisfies a certain universal property. To explain this, write $\mathbb{G}_a^\sharp$ for the affine scheme $\mathrm{Spec}\,\Gamma_{\mathbb{Z}}(\mathbb{Z})$: this is the divided power envelope of the origin in the additive group $\mathbb{G}_a$.

wers_univ

Lemma 3.17. *For any $M \in \mathrm{Mod}_R^{\mathrm{cn}}$ and for any other R-algebra C, we have a canonical equivalence*

$$\mathrm{Map}_{\mathrm{CRing}_{R/}}(\Gamma_R(M), C) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Mod}_R}(M, \mathbb{G}_a^\sharp(C)).$$

Proof. Both sides of the purported equivalence are evaluations on M of functors $\mathrm{Mod}_R^{\mathrm{cn}} \to \mathrm{Spc}^{\mathrm{op}}$ that preserve sifted colimits. Therefore, by [36, Prop. 5.5.8.15], it suffices to construct a canonical equivalence between these functors when evaluated on free R-modules of finite rank. That is, we want to construct isomorphisms

$$\mathrm{Map}_{\mathrm{CRing}_{R/}}(\Gamma_R(R^n), C) \xrightarrow{\simeq} \mathbb{G}_a^\sharp(C)^n;$$

or in other words isomorphisms

$$\Gamma_R(R^n) \xrightarrow{\simeq} \underbrace{\Gamma_R(R) \otimes_R \otimes \cdots \otimes_R \Gamma_R(R)}_{n}$$

in $\mathrm{CRing}_{R/}$. This is classical; see for instance [4, Prop. (A2)].

□

If $R' \twoheadrightarrow R$ is in AniPair with homotopy kernel I, then a divided power structure on it is also equivalent to giving a map of R'-algebras $\Gamma_{R'}(I) \to R'$ equipped with a homotopy equivalence between the induced map $I \to \Gamma_{R'}(I) \to R'$ with the tautological one.

A **divided power thickening** is a pair $(R' \twoheadrightarrow R, \gamma)$, where γ is a divided power structure on $R' \twoheadrightarrow R$.

potent_pd

3.18. **Nilpotent divided powers.** It will be useful to know what it means for a divided power thickening of animated rings to be **nilpotent** or **trivial**. It's easiest to use the process of animation, this time applied to classical nilpotent divided power thickenings of a fixed order. That is, analogously to what we did in (3.16), we let $\mathcal{E}^{0,[n]}$ be the full subcategory of the classical category PDPair of divided power thickenings $(R' \twoheadrightarrow R, \gamma)$ spanned by those thickenings of the form

$$(D_{(Y)}\mathbb{Z}[X,Y]/D^{\geq n}_{(Y)}\mathbb{Z}[X,Y] \twoheadrightarrow \mathbb{Z}[X], \gamma)$$

where X, Y are finite sets of variables, and $D^{\geq\bullet}_{(Y)}\mathbb{Z}[X,Y]$ is the divided power filtration on $D_{(Y)}\mathbb{Z}[X,Y]$.

We now set $\mathrm{AniPDPair}^{[n]} = \mathcal{P}_\Sigma(\mathcal{E}^{0,[n]})$ and call this the ∞-category of **nilpotent animated divided power thickenings of order** $\leq n$. If $\mathrm{PDPair}^{[n]} \subset \mathrm{PDPair}$ is the full subcategory consisting of classical divided power thickenings that are nilpotent of order $\leq n$, then one checks as in [39, Lemma 3.13] that the Yoneda functor yields a fully faithful embedding $\mathrm{PDPair}^{[n]} \to \mathrm{AniPDPair}^{[n]}$.

_faithful

Lemma 3.19. *The natural quotient functor* $\mathcal{E}^0 \to \mathcal{E}^{0,[n]}$ *induces a fully faithful functor* $\mathrm{AniPDPair}^{[n]} \to \mathrm{AniPDPair}$.

We now say that a divided power thickening $(R' \twoheadrightarrow R, \gamma)$ in AniPDPair is **nilpotent** if it is in the image of $\mathrm{AniPDPair}^{[n]}$ for some $n \geq 1$.

nilpotent

Remark 3.20. Suppose that $(R' \twoheadrightarrow R, \gamma)$ is a nilpotent divided power thickening of order $\leq n$. Then it admits a canonical triangle

$$\begin{tikzcd}
(R' \twoheadrightarrow R, \gamma) \arrow[r] \arrow[dr] & (R' \twoheadrightarrow R_0, \gamma_0') \arrow[d] \\
& (R \twoheadrightarrow R_0, \gamma_0)
\end{tikzcd}$$

in AniPDPair, where $(R' \twoheadrightarrow R_0, \gamma_0')$ is nilpotent of order $\leq (n-1)$ and $(R \twoheadrightarrow R_0, \gamma_0)$ is nilpotent of order ≤ 2. This follows by animating the triangle

$$\begin{tikzcd}
(D_{(Y)}\mathbb{Z}[X,Y]/D^{\geq n}_{(Y)}\mathbb{Z}[X,Y] \twoheadrightarrow \mathbb{Z}[X], \gamma) \arrow[r] \arrow[dr] & (D_{(Y)}\mathbb{Z}[X,Y]/D^{\geq n}_{(Y)}\mathbb{Z}[X,Y] \twoheadrightarrow D_{(Y)}\mathbb{Z}[X,Y]/D^{\geq 2}_{(Y)}\mathbb{Z}[X,Y], \gamma_0') \arrow[d] \\
& (D_{(Y)}\mathbb{Z}[X,Y]/D^{\geq 2}_{(Y)}\mathbb{Z}[X,Y] \to \mathbb{Z}[X], \gamma_0).
\end{tikzcd}$$

d_trivial

Remark 3.21. When $n = 2$, the natural map

$$\mathbb{Z}[X,Y]/(Y^2) \to D_{(Y)}\mathbb{Z}[X,Y]/D^{\geq 2}_{(Y)}\mathbb{Z}[X,Y]$$

is an isomorphism, and the divided powers on the right hand are *trivial*, in the sense that all divided powers of order ≥ 2 are zero. In this case, one checks that the forgetful functor

$$\mathrm{AniPDPair}^{[2]} \to \mathrm{AniPair}$$

is fully faithful. We will call any object $R' \twoheadrightarrow R$ in its image a **trivial** divided power thickening. Note that any such object is a square-zero extension as in (3.1), since this is clearly true for all the generating objects. Conversely, it follows from [44, Lemma 1.4] that any square-zero extension is in this image.

gs_stacks

4. Filtered abstractions

The purpose of this rather technical section is to r ec:graded

4.1. **Graded rings and modules.** As usual, a graded ring or module can be viewed as a $\mathbb{G}_m$-equivariant object. Therefore, given $R \in \mathrm{CRing}$, we will define the ∞-category of **graded animated commutative R-algebras** to be the opposite to the ∞-category of relatively affine map derived stack X over $B\mathbb{G}_m \times \operatorname{Spec} R$. Let $\mathcal{O}\{1\}$ be the inverse tautological bundle over $B\mathbb{G}_m$, and set $\mathcal{O}\{i\} = \mathcal{O}\{1\}^{\otimes i}$. Then, given a relatively affine map $X \to B\mathbb{G}_m \times \operatorname{Spec} R$, we will denote the coresponding graded animated ring symbolically by $B_\bullet = \oplus_i B_i$, where $B_i = R\Gamma(X, \mathcal{O}\{i\})$, so that $X = (\operatorname{Spec} B_\bullet)/\mathbb{G}_m$.

The ∞-category $\mathrm{GrMod}_{B_\bullet}$ of **graded $B_\bullet$-modules** is the category $\mathrm{QCoh}(X)$. Symbolically, if $\mathcal{F}$ is a quasicoherent sheaf over X, we can write the associated graded module in the form $M_\bullet = \oplus_i M_i$, where $M_i = R\Gamma(X, \mathcal{F} \otimes \mathcal{O}\{i\})$. Note that by construction this is a symmetric monoidal ∞-category, and we denote the associated graded tensor product of $M_\bullet$ and $N_\bullet$ in $\mathrm{GrMod}_{B_\bullet}$ by

$$M_\bullet \otimes_{B_\bullet} N_\bullet.$$

We will usually write $\mathrm{Map}_{B_\bullet}(M_\bullet, N_\bullet)$ for mapping spaces in this category.

Given any graded module $M_\bullet$ over $B_\bullet$ and an integer i, we obtain the **i-shifted module** $M_\bullet\{i\}$: If $M_\bullet$ is associated with a quasicoherent sheaf $\mathcal{F}$ over $(\operatorname{Spec} B_\bullet)/\mathbb{G}_m$, $M_\bullet\{i\}$ is associated with $\mathcal{F} \otimes \mathcal{O}\{i\}$ and satisfies $(M_\bullet\{i\})_m = M_{m+i}$.

Note that we can use this optic to speak of **graded perfect $B_\bullet$**-modules and **graded vector bundles** over $B_\bullet$: they will correpond to perfect complexes (resp. vector bundles) over X.

For any R-algebra C, the relatively affine map $B\mathbb{G}_m \times \operatorname{Spec} C \to B\mathbb{G}_m \times \operatorname{Spec} R$ corresponds to C with its *trivial* grading. In this case, we can speak simply of graded C-modules, etc.

struction

4.2. **Filtered objects via the Rees construction.** We will make frequent use of the quotient stack $\mathbb{A}^1/\mathbb{G}_m$, where we view $\mathbb{G}_m$ as acting on the affine line via $(t, z) \mapsto tz^{-1}$. Explicitly, this stack parameterizes line bundles $\mathcal{L}$ equipped with a *co*section $t : \mathcal{L} \to \mathscr{O}$.

This stack gives a geometric method for dealing with filtered objects [42]. More precisely, for any $R \in \mathrm{CRing}$, there is a canonical equivalence

$$\mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R) \xrightarrow{\simeq} \mathrm{FilMod}_R,$$

where the right hand side is the stable ∞-category of filtered objects in Mod_R: classically, if $R = \pi_0(R)$ is discrete, then its associated triangulated derived category is the usual filtered derived category.

Symbolically, under this equivalence, a filtered module $\mathrm{Fil}^\bullet M$ on the right is associated with the $\mathbb{G}_m$-equivariant $R[t]$-module

$$\mathrm{Rees}(\mathrm{Fil}^\bullet M) = \bigoplus_{i \in \mathbb{Z}} \mathrm{Fil}^i M \cdot t^{-i}.$$

Our convention is that t lives in graded degree 1. For the functor in the other direction, note that we have a canonical family of line bundles $\mathcal{O}\{n\} = \mathcal{O}\{1\}^{\otimes n}$ over $\mathbb{A}^1/\mathbb{G}_m$ indexed by integers $n \in \mathbb{Z}$: Here, $\mathcal{O}\{1\}$ is the inverse tautological line bundle $\mathcal{L}^{\otimes -1}$. Note that we have canonical maps $t : \mathcal{O}\{i\} \to \mathcal{O}\{i+1\}$. Given a quasi-coherent sheaf $\mathcal{F}$ over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$, we now obtain a filtered module $\mathrm{Fil}^\bullet M$ by setting $\mathrm{Fil}^i M = R\Gamma(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R, \mathcal{F} \otimes \mathcal{O}\{-i\})$ with the transition maps given by t.

Any R-module M, viewed as a quasi-coherent sheaf on $\operatorname{Spec} R$ pulls back to a quasi-coherent sheaf on $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$, and this yields a filtered R-module $\mathrm{Fil}^\bullet_{\mathrm{triv}} M$ with underlying R-module M. This filtration is just the **trivial filtration** with $\mathrm{Fil}^i_{\mathrm{triv}} M = M$ if $i \leq 0$ and 0 otherwise.

A **filtered stack over** R is an R-stack X equipped with a map to $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$; we will view it as a filtration on the R-stack $X_{(t \neq 0)}$ with associated graded $X_{(t=0)} \to B\mathbb{G}_m$.

4.3. **Filtered animated commutative rings and filtered modules.** The Rees equivalence also gives us a compact way of defining **filtered animated commutative R-algebras**: these correspond to relatively *affine* stacks over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$. Symbolically, given a filtered R-algebra $\mathrm{Fil}^\bullet S$, the $\mathbb{G}_m$-equivariant $R[t]$-module $\mathrm{Rees}(\mathrm{Fil}^\bullet S)$ has a canonical $\mathbb{G}_m$-equivariant structure of an animated commutative $R[t]$-algebra, and taking the quotient of the associated affine scheme over $\mathbb{A}^1 \times \operatorname{Spec} R$ yields the associated affine morphism

$$\mathcal{R}(\mathrm{Fil}^\bullet S) \to \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R.$$

We will call the source of this map the associated **Rees stack**. Note that the fiber of this stack over the open point $\mathbb{G}_m/\mathbb{G}_m$ is canonically isomorphic to $\operatorname{Spec} S$, and its fiber over $B\mathbb{G}_m$ is the affine stack associated with the graded ring $\bigoplus_i \operatorname{gr}^{-i} S$.

Note that we can now give a precise meaning to the ∞-category of filtered commutative algebras over the filtered ring $\operatorname{Fil}^\bullet S$: it is opposite to the category of relatively affine stacks over $\mathcal{R}(\operatorname{Fil}^\bullet S)$.

A **filtered module** over $\operatorname{Fil}^\bullet S$ is now just a quasi-coherent sheaf $\mathcal{F}$ over the associated Rees stack. Once again, concretely one can write it in the form $\operatorname{Fil}^\bullet M$ where the S-modules $\operatorname{Fil}^i M$ are obtained as global sections of suitable twists of $\mathcal{F}$. Write $\operatorname{FilMod}_{\operatorname{Fil} S}$ for the associated ∞-category. We will write mapping spaces in this category in the form $\operatorname{Map}_{\operatorname{Fil}^\bullet S}(\_,\_)$. If $\operatorname{Fil}^\bullet_{\mathrm{triv}} S$ is the *trivial* filtration, then we will also write $\operatorname{Map}_{\operatorname{FilMod}_S}(\_,\_)$ for this mapping space.

Note that this gives us a symmetrical monoidal ∞-category by definition, where the tensor product corresponds to that of quasicoherent sheaves on the Rees stack. We will denote the associated product between filtered $\operatorname{Fil}^\bullet S$-modules $\operatorname{Fil}^\bullet M$ and $\operatorname{Fil}^\bullet N$ by

$$\operatorname{Fil}^\bullet M \otimes_{\operatorname{Fil}^\bullet S} \operatorname{Fil}^\bullet N.$$

Using this optic, we can also systematically talk about **filtered perfect** complexes as well as **filtered vector bundles** over $\operatorname{Fil}^\bullet S$: these correspond to perfect complexes (resp. vector bundles) on the associated Rees stacks.

Pullback from $\mathcal{R}(\operatorname{Fil}^\bullet S)$ to the closed substack $\mathcal{R}(\operatorname{Fil}^\bullet S)_{(t=0)}$ yields a symmetric monoidal functor from $\operatorname{FilMod}_{\operatorname{Fil}^\bullet S}$ to $\operatorname{GrMod}_{\operatorname{gr}^\bullet S}$: this is just the functor taking a filtered module to its associated graded.

4.4. **Increasing filtrations.** There is a variant of the above that looks at objects over the stack $\mathbb{A}^1_+/\mathbb{G}_m$ classifying *sections* of line bundles $u : \mathscr{O} \to L$: this corresponds to the 'usual' action of $\mathbb{G}_m$ on $\mathbb{A}^1$. Quasi-coherent sheaves over this stack are now equivalent to *increasingly* filtered modules $\operatorname{Fil}_\bullet M$, and relatively affine schemes over it are now equivalent to increasingly filtered animated commutative rings $\operatorname{Fil}_\bullet S$. We will denote the corresponding Rees construction by $\mathcal{R}_+(\operatorname{Fil}_\bullet S)$.

uare-zero

4.5. **Filtered deformation theory.** Every filtered animated commutative algebra $\operatorname{Fil}^\bullet S$ over a filtered animated commutative ring $\operatorname{Fil}^\bullet R$ admits a filtered cotangent complex $\mathbb{L}_{\operatorname{Fil}^\bullet S/\operatorname{Fil}^\bullet R}$: this is a filtered $\operatorname{Fil}^\bullet S$-module corresponding to the cotangent complex of the associated Rees stacks. This controls the filtered deformation theory as follows:

A map of filtered animated commutative rings $\operatorname{Fil}^\bullet S' \to \operatorname{Fil}^\bullet S$ is a **filtered square-zero extension** if the corresponding map of $\mathbb{G}_m$-equivarint affine schemes over $\mathbb{A}^1/\mathbb{G}_m$ is a square zero thickening. In this case the fiber of the map of filtered rings is a filtered $\operatorname{Fil}^\bullet S$-module $\operatorname{Fil}^\bullet M$.

Given a connective filtered module $\operatorname{Fil}^\bullet M$ over $\operatorname{Fil}^\bullet S$, we can consider the *trivial* square-zero extension $\operatorname{Fil}^\bullet S \oplus \operatorname{Fil}^\bullet M$. We then have a canonical equivalence:

$$\operatorname{Map}_{\operatorname{Fil}^\bullet R/}(\operatorname{Fil}^\bullet S, \operatorname{Fil}^\bullet S \oplus \operatorname{Fil}^\bullet M) \simeq \operatorname{Map}_{\operatorname{Fil}^\bullet S}(\mathbb{L}_{\operatorname{Fil}^\bullet S/\operatorname{Fil}^\bullet R}, \operatorname{Fil}^\bullet M).$$

Sections of either equivalent space will be called $\operatorname{Fil}^\bullet R$**-derivations** from $\operatorname{Fil}^\bullet S$ to $\operatorname{Fil}^\bullet M$.

One way to obtain square-zero thickenings with fiber $\operatorname{Fil}^\bullet M$ therefore is as the left vertical arrow of a Cartesian diagram of the form

$$\begin{tikzcd}
\operatorname{Fil}^\bullet S' \arrow[r] \arrow[d] & \operatorname{Fil}^\bullet S \arrow[d, "d_{\mathrm{triv}}"] \\
\operatorname{Fil}^\bullet S \arrow[r, "d"'] & \operatorname{Fil}^\bullet S \oplus \operatorname{Fil}^\bullet M[1]
\end{tikzcd}$$

where the right vertical arrow is the trivial map and the horizontal one on the bottom is a $\operatorname{Fil}^\bullet R$-derivation.

Now suppose that $X \to \mathcal{R}(\operatorname{Fil}^\bullet S)$ is a prestack admitting a relative cotangent complex $\mathbb{L}_X \overset{\mathrm{defn}}{=} \mathbb{L}_{X/\mathcal{R}(\operatorname{Fil}^\bullet S)}$. For any filtered $\operatorname{Fil}^\bullet S$-algebra $\operatorname{Fil}^\bullet A$, set

$$X(\operatorname{Fil}^\bullet A) = \operatorname{Map}_{/\mathcal{R}(\operatorname{Fil}^\bullet S)}(\mathcal{R}(\operatorname{Fil}^\bullet A), X).$$

Then we obtain a Cartesian diagram

$$\begin{tikzcd} X(\mathrm{Fil}^\bullet S') \arrow[r] \arrow[d] & X(\mathrm{Fil}^\bullet S) \arrow[d, "d_{\mathrm{triv}}"] \\ X(\mathrm{Fil}^\bullet S) \arrow[r, "d"'] & X(\mathrm{Fil}^\bullet S \oplus \mathrm{Fil}^\bullet M[1]). \end{tikzcd}$$

Moreover, for any $x \in X(\mathrm{Fil}^\bullet S)$, pulling $\mathbb{L}_X$ along x yields a filtered module $\mathrm{Fil}^\bullet \mathbb{L}_{X,x}$ over $\mathrm{Fil}^\bullet S$, and we have a canonical equivalence:

$$\mathrm{fib}_x(X(\mathrm{Fil}^\bullet S \oplus \mathrm{Fil}^\bullet M[1]) \to X(\mathrm{Fil}^\bullet S)) \xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Fil}^\bullet S}(\mathrm{Fil}^\bullet \mathbb{L}_{X,x}, \mathrm{Fil}^\bullet M[1]).$$

or_stacks

4.6. **Attractor stacks.** If we have a prestack $\mathcal{Y} \to B\mathbb{G}_m$, its associated **fixed point prestack** is the functor X^0 on R-algebras given by

$$Y^0(C) = \mathrm{Map}_{B\mathbb{G}_m \times \mathrm{Spec}\, R}(B\mathbb{G}_m \times \mathrm{Spec}\, C, \mathcal{Y}).$$

Suppose that we have a prestack $\mathcal{X} \to \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R$; its associated **attractor prestack** is the functor X^- on R-algebras given by:

$$X^-(C) = \mathrm{Map}_{/\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R}(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, C, \mathcal{X}).$$

We define its fixed point prestack X^0 to be that of the restriction $\mathcal{X}_{(t=0)}$ of $\mathcal{X}$ over the closed substack $B\mathbb{G}_m \times \mathrm{Spec}\, R$.

In other words, X^- (resp. X^0) is the Weil restriction of $\mathcal{X}$ (resp. $\mathcal{X}_{(t=0)}$) from $\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R$ (resp. $B\mathbb{G}_m \times \mathrm{Spec}\, R$) down to $\mathrm{Spec}\, R$. Note that the sequence of natural maps

$$B\mathbb{G}_m \times \mathrm{Spec}\, R \hookrightarrow \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R \to B\mathbb{G}_m \times \mathrm{Spec}\, R$$

yields maps

$$X^0 \leftarrow X^- \leftarrow X^0$$

whose composition is the identity.

Replacing $\mathbb{A}^1/\mathbb{G}_m$ with $\mathbb{A}^1_+/\mathbb{G}_m$ gives us the **repeller prestack** X^+ associated with $\mathcal{X}$, which also admits maps $X^0 \to X^+ \to X^0$ whose composition is the identity.

Remark 4.7. If $\mathcal{X}$ is the pullback of an algebraic space over $B\mathbb{G}_m$, these notions are studied (with perhaps a sign difference for X^-) by Drinfeld in [14].

4.8. Suppose that $\mathcal{X}$ is locally finitely presented over $\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R$ and that it admits a perfect relative cotangent complex. Note that, over X^-, we have a canonical filtered perfect complex $\mathrm{Fil}^\bullet \mathbb{L}^-_{\mathcal{X}}$: This associates with every $x \in X^-(C)$ the filtered module corresponding to the pullback of the cotangent complex $\mathbb{L}_{\mathcal{X}/(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R)}$ to $\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, C$ along x.

Similarly, over X^0, we have a canonical graded perfect complex $\mathbb{L}^0_{\mathcal{X},\bullet}$: it is isomorphic to the associated graded of the restriction of $\mathrm{Fil}^\bullet \mathbb{L}^-_{\mathcal{X}}$ along $X^0 \to X^-$.

We have:

t_complex

Lemma 4.9. *The prestack X^- admits a perfect cotangent complex over R; we have*

$$\mathbb{L}_{X^-/R} = \mathbb{L}^-_{\mathcal{X}} / \mathrm{Fil}^1 \mathbb{L}^-_{\mathcal{X}}.$$

Similarly, the prestack X^0 admits a perfect cotangent complex over X with

$$\mathbb{L}_{X^0/R} = \mathbb{L}^0_{\mathcal{X},0}.$$

Proof. From the discussion in (4.5), we see that, for $C \in \mathrm{CRing}_{R/}$, $M \in \mathrm{Mod}^{\mathrm{cn}}_C$, and $x \in X^-(C)$, we have

$$\mathrm{fib}_x(X^-(C \oplus M) \to X^-(C)) \simeq \mathrm{Map}_{\mathrm{FilMod}_C}(\mathrm{Fil}^\bullet \mathbb{L}^-_{\mathcal{X},x}, \mathrm{Fil}^\bullet_{\mathrm{triv}} M) \simeq \mathrm{Map}_{\mathrm{Mod}_C}(\mathbb{L}^-_{\mathcal{X},x} / \mathrm{Fil}^1 \mathbb{L}^-_{\mathcal{X},x}, M).$$

This proves the first part of the lemma. The proof of the second is entirely analogous. ∎

We will now give a general criterion for representability of X^-, X^+ and X^0 essentially due to Halpern-Leistner and Preygel [22, Example 1.2.2].

le_repble **Proposition 4.10.** *Suppose that $\pi_0(R)$ is a G-ring and that $\mathcal{X} \to \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$ is a relative locally finitely presented derived Artin r-stack $\mathcal{Y} \to \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$ with quasi-affine (resp. affine) diagonal. Then X^-, X^0, X^+ are locally finitely presented derived Artin r-stacks over R, and if $\mathcal{X}$ is* flat *over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$, then X^-, X^0, X^+ all have quasi-affine (resp. affine) diagonal.*

Proof. We recall some key points of the proof, which uses Lurie's derived generalization of Artin's representability theorem [34, Theorem 7.1.6].

It is straightforward to see that X^-, X^0, X^+ are all étale sheaves that are locally finitely presented, nilcomplete and infinitesimally cohesive. We have already seen that they admit perfect cotangent complexes, and it is clear that their classical truncations are valued in r-truncated spaces.

The main difficulty now is to show that they are *integrable* (condition (3) in *loc. cit.*. For X^0 this is once again immediate. For X^- (the argument for X^+ is identical), one needs the following assertion: Suppose that C is a complete local Noetherian R-algebra with maximal ideal $\mathfrak{m}$. Then the map

ility_map
$$\operatorname{Map}_{/\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R}(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} C, \mathcal{X}) \to \varprojlim_m \operatorname{Map}_{/\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R}(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} C/\mathfrak{m}^m, \mathcal{X}). \tag{4.10.1}$$

is an isomorphism.

For this, one first finds that—via reduction to Tannaka duality as formulated in [6]—for any $B \in \mathrm{CRing}_{R/}$ with $\pi_0(B)$ Noetherian, we have

$$\operatorname{Map}_{/\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R}(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} B, \mathcal{X}) \xrightarrow{\simeq} \varprojlim_n \operatorname{Map}_{/\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R}((\mathbb{A}^1/\mathbb{G}_m)_{(t^n=0)} \times \operatorname{Spec} B, \mathcal{X}).$$

This follows from [22, Proposition 5.1.13]. To apply this result, we need the additional observation that the symmetric monoidal ∞-category of almost perfect complexes on $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} B$ is equivalent to the inverse limit over the corresponding categories of almost perfect complexes on $(\mathbb{A}^1/\mathbb{G}_m)_{(t^n=0)} \times \operatorname{Spec} B$. This boils down to the fact that filtered almost perfect complexes over B are *complete* for their filtration.

Via filtered deformation theory, the desired integrability now reduces to the already known assertion for X^0.

It remains only to check the assertion about the diagonal, which is [22, Proposition 5.1.15]. □

ec:cochar

4.11. **Cocharacters of group schemes and twisted group stacks.** This following discussion is essentially from [15, §2.3]. Suppose that G is a smooth affine group scheme over a classical commutatie ring R and let $\mu : \mathbb{G}_{m,R} \to \underline{\mathrm{Aut}}(G)$ be a cocharacter of its automorphism group.

An important example arises when μ lifts to a cocharacter $\mathbb{G}_{m,R} \to G$, which we once again denote by μ, and the associated action of $\mathbb{G}_m$ on G is the *adjoint action* given by

$$G \times \mathbb{G}_m \xrightarrow{(g,z)\mapsto \mu(z)g\mu(z)^{-1}} G$$

The fpqc quotient of G by the action of μ yields a group stack $G\{\mu\}$ over $B\mathbb{G}_{m,R}$.

We have subgroups $U^{\pm}_\mu \subset P^{\pm}_\mu \subset G$ with $P^{\pm}_\mu/U^{\pm}_\mu \simeq M_\mu$ independent of sign: Namely, P^-_μ (resp. P^+_μ) is the attractor stack (resp. repeller stack) of the basechange of $G\{\mu\}$ over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$, and M_μ is the fixed point stack of $G\{\mu\}$.

Explicitly, given an R-algebra S, P^-_μ

$$P^-_\mu(S) = \operatorname{Map}_{B\mathbb{G}_{m,R}}\left(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} S, G\{\mu\}\right)\ ;\ P^+_\mu(S) = \operatorname{Map}_{B\mathbb{G}_{m,R}}\left(\mathbb{A}^1_+/\mathbb{G}_m \times \operatorname{Spec} S, G\{\mu\}\right);$$
$$M_\mu(S) = \operatorname{Map}_{B\mathbb{G}_{m,R}}\left(B\mathbb{G}_m \times \operatorname{Spec} S, G\{\mu\}\right)\ ;$$

Restriction to the open point $\mathbb{G}_m/\mathbb{G}_m \subset \mathbb{A}^1/\mathbb{G}_m$ now gives a closed immersion of group schemes $P^{\pm}_\mu \hookrightarrow G$. The section $M_\mu \to P^{\pm}_\mu$ exhibits it as the centralizer in $P^{\pm}_\mu$ (and in G) of μ.

The subgroup $U^{\pm}_\mu \subset P^{\pm}_\mu$ is the kernel of the map to M_μ.

In terms of Lie algebras, the action of μ gives us a grading of $\mathfrak{g} \overset{\text{defn}}{=} \operatorname{Lie} G$:

$$\mathfrak{g} = \bigoplus_{i\in\mathbb{Z}} \mathfrak{g}_i,$$

where $\mathbb{G}_m$ acts on $\mathfrak{g}_i$ via $z \mapsto z^{-i}$. We now have:[6]

$$\operatorname{Lie} P^{\pm}_{\mu} = \bigoplus_{\pm i \geq 0} \mathfrak{g}_i \; ; \; \operatorname{Lie} U^{\pm}_{\mu} = \bigoplus_{\pm i > 0} \mathfrak{g}_i \; ; \; \operatorname{Lie} M_{\mu} = \mathfrak{g}_0.$$

When G is reductive and μ lifts to a cocharacter of G, what we have defined here are the parabolic and unipotent subgroups of G associated with μ.

c:BGmhphi

4.12. **Descent over the stack $B\mathbb{G}^{\phi}_{m,\mathcal{O}}$.** Suppose now that $R = \mathcal{O}$ is the ring of integers in a finite unramified extension of $\mathbb{Q}_p$ with residue field k. Suppose also that our group scheme is the base-change over $\mathcal{O}$ of a $\mathbb{Z}_p$-group scheme G.

For the purposes of defining the main objects of study in this paper, we will need a certain quotient stack $B\mathbb{G}^{\phi}_{m,\mathcal{O}}$ of $B\mathbb{G}_{m,\mathcal{O}}$ obtained as the coequalizer of the two maps

$$\operatorname{Spf} \mathcal{O} \to B\mathbb{G}_{m,\mathcal{O}} \; ; \; \operatorname{Spf} \mathcal{O} \xrightarrow{\varphi} \operatorname{Spf} \mathcal{O} \to B\mathbb{G}_{m,\mathcal{O}}$$

where the first is the canonical quotient map, and the second is its Frobenius twist. Giving a quasicoherent sheaf over $B\mathbb{G}^{\phi}_{m,\mathcal{O}}$ is equivalent to giving a graded $\mathcal{O}$-module $M = \oplus_i M_i$ along with an isomorphism $\varphi^* M \xrightarrow{\simeq} M$ of (ungraded) $\mathcal{O}$-modules. If M is perfect, one can show that such an isomorphism underlies a canonical descent datum, and so one can view the perfect complexes on $B\mathbb{G}^{\phi}_{m,\mathcal{O}}$ as being perfect complexes over $\mathbb{Z}_p$ equipped with a grading when base-changed over $\mathcal{O}$.

Since G is defined over $\mathbb{Z}_p$, we see that $G\{\mu\}$ descends to a group stack over $B\mathbb{G}^{\phi}_{m,\mathcal{O}}$, which we still denote by $G\{\mu\}$.

The cocharacter μ also gives a $\mathbb{G}_m$-action on the classifying stack $BG_{\mathcal{O}}$, and so gives us a descent $BG\{\mu\} \to B\mathbb{G}_{m,\mathcal{O}}$. The same argument as before shows that this actually admits a further descent over $B\mathbb{G}^{\phi}_{m,\mathcal{O}}$, and we denote the resulting object once again by $BG\{\mu\}$.

For any $B\mathbb{G}^{\phi}_{m,\mathcal{O}}$-stack Y, a $G\{\mu\}$-**torsor over** Y is a map—the classifying map for the torsor—from Y to $BG\{\mu\}$ over $B\mathbb{G}^{\phi}_{m,\mathcal{O}}$. There is a canonical map $B\mathbb{G}^{\phi}_{m,\mathcal{O}} \to BG\{\mu\}$ classifying the *trivial* $G\{\mu\}$-torsor, and we will call a $G\{\mu\}$-torsor over Y **trivial** if its classifying map admits a lift to $B\mathbb{G}^{\phi}_{m,\mathcal{O}}$.

ed_points

4.13. 1**-bounded fixed points.** Suppose that $\mathcal{Z} \to B\mathbb{G}_m \times \operatorname{Spec} R$ is a relative locally finitely presented Artin stack and let $Z^0 \to \operatorname{Spec} R$ be the corresponding fixed point stack. As observed in the previous subsection, over Z^0 we have the canonical graded perfect complex $\mathbb{L}^0_{\mathcal{Z},\bullet}$. Let $\mathbb{T}^0_{\mathcal{Z},\bullet}$ be its graded dual. The locus $Z^0_{\text{1-b}}$ of 1**-bounded fixed points** is the *open* substack of Z^0 where we have $\mathbb{T}^0_{\mathcal{Z},i} \simeq 0$ for $i > 1$.

complexes

Example 4.14. Suppose that we have $\mathcal{M} \in \operatorname{Perf}(B\mathbb{G}_m \times \operatorname{Spec} R)$ corresponding to a graded R-module $M_\bullet$. Then we can take $\mathcal{Z} = \mathbf{V}(\mathcal{M}) \to B\mathbb{G}_m \times \operatorname{Spec} R$ to be the associated vector stack.

One checks that the corresponding fixed point stack Z^0 now is just the vector stack $\mathbf{V}(M_0) \to \operatorname{Spec} R$, while the graded perfect complex $\mathbb{T}^0_{\mathcal{Z},\bullet}$ corresponds simply to the restriction of $\mathcal{M}$ to $B\mathbb{G}_m \times \mathbf{V}(M_0)$. This implies that

$$Z^0_{\text{1-b}} = \mathbf{V}(M_0) \times_{\operatorname{Spec} R} (\operatorname{Spec} R)_{\text{1-b}},$$

where $(\operatorname{Spec} R)_{\text{1-b}} \subset \operatorname{Spec} R$ is the open locus M_i becomes nullhomotopic for $i > 1$.

Let us call $\mathcal{M}$ 1**-bounded** if $M_i \simeq 0$ for $i > 1$ already over R. Then we see that, for such 1-bounded perfect complexes, we have

$$Z^0_{\text{1-b}} = Z^0 = \mathbf{V}(M_0).$$

_HTwts_01

Example 4.15. Consider the stack $\mathcal{P} : R \mapsto \operatorname{Perf}(R)^{\simeq}$ on CRing: this is represented by a locally finitely presented derived Artin stack over $\mathbb{Z}$; see [46, § 3].

Now take $\mathcal{Z} = \mathcal{P} \times B\mathbb{G}_m \to B\mathbb{G}_m$: the fixed point stack Z^0 associates with every $R \in$ CRing the ∞-groupoid of graded perfect R-modules.

[6] We are following the sign conventions from [30]

The tangent complex of $\mathcal{P}$ is $M^\vee_{\mathrm{taut}} \otimes M_{\mathrm{taut}}$, where $M_{\mathrm{taut}} \in \mathrm{Perf}(\mathcal{P})$ is the tautological perfect complex. From this, one finds that the graded perfect complex $\mathbb{T}^0_{\mathcal{Z},\bullet}$ is $M^\vee_{\mathrm{taut},\bullet} \otimes M_{\mathrm{taut},\bullet}$, where $M_{\mathrm{taut},\bullet}$ is the tautological graded perfect complex over Z^0.

Now, $Z^0_{\text{1-b}}$ is precisely the locus where $M_{\mathrm{taut},i} \otimes M_{\mathrm{taut},j} \simeq 0$ for all $i, j \in \mathbb{Z}$ with $j - i > 1$.

In particular, the locus where $M_{\mathrm{taut},i} \simeq 0$ for $i \neq 0, 1$ is an open substack of $Z^0_{\text{1-b}}$.

nded_case

4.16. 1-**bounded cocharacters.** Here is the main example where 1-bounded fixed points will play a key role.

Let us put ourselves in the situation of (4.12). Following [30], we will say that μ is 1-**bounded**, if under the adjoint action of μ, $\mathfrak{g}_{\mathcal{O}}$, we have $\mathfrak{g}_i = 0$ for $i > 1$.[7] In this case, we will set $\mathfrak{g}^+_\mu = \mathfrak{g}_1$.

unded_exp

Lemma 4.17. *If μ is 1-bounded, the exponential map induces an equivalence:*

$$\mathbf{V}(\mathfrak{g}^+_\mu) \xrightarrow{\simeq} U^+_\mu.$$

In particular, for any $R \in \mathrm{CRing}_{\mathcal{O}/}$, we have an equivalence

$$\mathfrak{g}^+_\mu \otimes_{\mathcal{O}} (R/^{\mathbb{L}}p^n) \xrightarrow[\simeq]{\exp} U^{+,(n)}_\mu(R).$$

Proof. See [30, Lemma 6.3.2]. □

ints_BGmu

Remark 4.18. Consider the stack $\mathcal{Z} \overset{\mathrm{defn}}{=} BG\{\mu\} \to B\mathbb{G}_m \times \operatorname{Spec} \mathcal{O}$ with associated fixed point stack Z^0 over $\mathcal{O}$. Then, unwinding definitions, one finds that, for any $R \in \mathrm{CRing}_{\mathcal{O}/}$, $Z^0(R)$ is the ∞-groupoid of the following equivalent kinds of objects:

- $G\{\mu\}$-torsors over $B\mathbb{G}_m \times \operatorname{Spec} R$;
- $G \rtimes_\mu \mathbb{G}_m$-equivariant[8] schemes $\mathcal{P} \to \operatorname{Spec} R$ such that the underlying G action presents $\mathcal{P}$ as a G-torsor over R.

In particular, for every $\mathcal{O}$-rational representation W of $G \rtimes_\mu \mathbb{G}_m$, the usual twisting process by the G-torsor $\mathcal{P}$ yields a canonical graded vector bundle $\mathcal{M}_\bullet(W)_{\mathcal{P}}$ over $\operatorname{Spec} R$.

In this way, we obtain a graded vector bundle $\mathcal{M}_\bullet(W)$ over Z^0. Now, the graded perfect complex $\mathbb{T}^0_{\mathcal{Z},\bullet}$ over Z^0 is simply $\mathcal{M}_\bullet(\mathfrak{g}_{\mathcal{O}})[-1]$. The 1-bounded locus $Z^0_{\text{1-b}}$ is the open and closed locus over which we have $\mathcal{M}_i(\mathfrak{g}_{\mathcal{O}}) \simeq 0$ for $i > 1$.

We now isolate a particular open substack of $Z^0_{\text{1-b}}$:

ial_locus

Lemma 4.19. *There is a canonical open and closed immersion $BM_\mu \to Z^0_{\text{1-}b}$ mapping isomorphically onto the locus of $G\{\mu\}$-torsors $\mathcal{Q} \to B\mathbb{G}_m \times \operatorname{Spec} R$ satisfying the following equivalent conditions when* $\operatorname{Spec} R$ *is connected:*

1. *There exists an étale cover $R \to R'$ such that the restriction of $\mathcal{Q}$ over $B\mathbb{G}_m \times \operatorname{Spec} R'$ is trivial;*
2. *For every geometric point $R \to \kappa$ of* $\operatorname{Spec} R$*, the $G\{\mu\}$-torsor $x^*\mathcal{Q}$ over $B\mathbb{G}_m \times \operatorname{Spec} \kappa$ is trivial;*
3. *For* some *geometric point $R \to \kappa$ of* $\operatorname{Spec} R$*, the $G\{\mu\}$-torsor $x^*\mathcal{Q}$ over $B\mathbb{G}_m \times \operatorname{Spec} \kappa$ is trivial.*

Proof. The map $BM_\mu \to Z^0$ associates with each M_μ-torsor $\mathcal{P}^0$ the G-torsor obtained via pushforward along the map $M_\mu \to G_{\mathcal{O}}$: Such a G-torsor is equipped with a canonical extension to an action of $G \rtimes_\mu \mathbb{G}_m$.

Over BM_μ, for each $i \in \mathbb{Z}$, we have the vector bundles $\mathcal{M}^0(\mathfrak{g}_i)$ obtained by twisting the representation $\mathfrak{g}_i$ by the universal M_μ-torsor. The restriction of $\mathbb{T}^0_{\mathcal{Z},\bullet}$ to BM_μ is then seen to be isomorphic to $\bigoplus_{i \in \mathbb{Z}} \mathcal{M}^0(\mathfrak{g}_i)\{-i\}[-1]$. In particular, the 1-boundedness of μ ensures exactly that this is a 1-bounded complex. That is, we have defined a map $BM_\mu \to Z^0_{\text{1-b}}$.

Let us now show that this yields an isomorphism of $BM_\mu(R)$ with the space of $G\{\mu\}$-torsors over $B\mathbb{G}_m \times \operatorname{Spec} R$ satisfying any of the three given conditions (1), (2) and (3). For condition (1), it is easy: Giving such an object over $B\mathbb{G}_m \times \operatorname{Spec} R$ is the same as giving an étale torsor over R for the fixed point group scheme of $G\{\mu\}$, which is of course M_μ.

To finish, it is enough to see that BM_μ is an open and closed substack of $Z^0_{\text{1-b}}$. The quickest way to see this is to observe, as Drinfeld does in [15, §C.2.3] that we have discrete invariants on Z^0 given by G-conjugacy classes of

[7]Recall that this is the μ-eigenspace for the characters $z \mapsto z^{-i}$.

[8]This is the semidirect product for the action of $\mathbb{G}_m$ on G via μ.

cocharacters $\mathbb{G}_m \to G \rtimes_\mu \mathbb{G}_m$ lifting the identity map of $\mathbb{G}_m$. Now, BM_μ is the open and closed substack of Z^0 associated with the trivial such lift. $\square$

mu_simple

Remark 4.20. Let Z^- be the attractor stack (of the base change over $\mathbb{A}^1/\mathbb{G}_m$ of) $\mathcal{Z}$. Then we find that $Z^-(R) \times_{Z^0(R)} BM_\mu(R)$ is spanned by $G\{\mu\}$-torsors over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$ satisfying the following condition: There exists an étale cover $R \to R'$ such that the restriction of $\mathcal{Q}$ over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R'$ is trivial. Indeed, this amounts to checking that a $G\{\mu\}$-torsor $\mathcal{Q}$ over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$ with trivial restriction over $B\mathbb{G}_m \times \operatorname{Spec} R$ is itself trivial. This is because $\mathcal{Q}$ is smooth over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$, and we have

$$\operatorname{Map}_{\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R}(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R, \mathcal{Q}) \xrightarrow{\simeq} \varprojlim_n \operatorname{Map}_{\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R}((\mathbb{A}^1/\mathbb{G}_m)_{(t^m=0)} \times \operatorname{Spec} R, \mathcal{Q}).$$

See the proof of Proposition 8.9.

In particular, $Z^- \times_{Z^0} BM_\mu$ is isomorphic to the stack of étale torsors for the attractor group scheme associated with $G\{\mu\}$, which is of course P^-_μ. In other words, we have $Z^- \times_{Z^0} BM_\mu \simeq BP^-_\mu$.

ed_stacks

4.21. 1-**bounded stacks.** A **pointed graded prestack** is a prestack $\mathcal{Y} \to B\mathbb{G}_m$ equipped with a morphism $\iota : B\mathbb{G}_m \times \operatorname{Spec} R \to \mathcal{Y}$ of graded prestacks. In particular, for such a prestack, any relative derived Artin stack $\mathcal{Z} \to \mathcal{Y}$ has an associated fixed point stack $Z^0 \to \operatorname{Spec} R$ obtained from the base-change of $\mathcal{Z}$ over $B\mathbb{G}_m \times \operatorname{Spec} R$.

We now introduce a key technical device. A 1-**bounded stack** $\mathcal{X} = (\mathcal{X}^\diamond, X^0) \to (\mathcal{Y}, \iota)$ over $(\mathcal{Y}, \iota)$ (or simply $\mathcal{Y}$ if ι is clear from context) consists of the following data:

(1) A relative locally finitely presented derived Artin stack $\mathcal{X}^\diamond \to \mathcal{Y}$;
(2) An open substack $X^0 \subset X^{\diamond,0}_{\text{1-b}}$ of the 1-bounded locus of the fixed point stack of $\mathcal{X}^\diamond$, which we will refer to as the **fixed point stack** of $\mathcal{X}$.

One can speak of maps between 1-bounded stacks over $\mathcal{Y}$ in the obvious way.

Suppose that $\mathcal{Y} \to \mathbb{A}^1/\mathbb{G}_m$ is in fact a filtered prestack, and that ι lifts to a map of filtered stacks $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R \to \mathcal{Y}$. Then we will associate with any 1-bounded stack $\mathcal{X} \to \mathcal{Y}$ its **attractor stack** $X^- \to \operatorname{Spec} R$ by setting $X^- \overset{\text{defn}}{=} X^{\diamond,-} \times_{X^{\diamond,0}} X^0$. Here $X^{\diamond,-}$ is the attractor stack of the base-change of $\mathcal{X}^\diamond$ over $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$. Analogously, given a lift $\mathbb{A}^1_+/\mathbb{G}_m \times \operatorname{Spec} R \to \mathcal{Y}$, we can define an associated repeller stack X^+.

If $(\mathcal{Z}, \eta) \to (\mathcal{Y}, \iota)$ is a map of pointed graded prestacks with $\eta : B\mathbb{G}_m \times \operatorname{Spec} C \to \mathcal{Z}$ and $\mathcal{X} = (\mathcal{X}^\diamond, X^0)$ is a 1-bounded stack over $\mathcal{Y}$, we set

$$\operatorname{Map}_{/(\mathcal{Y},\iota)}((\mathcal{Z}, \iota), \mathcal{X}) = \operatorname{Map}_{/\mathcal{Y}}(\mathcal{Z}, \mathcal{X}^\diamond) \times_{X^{\diamond,0}(C)} X^0(C).$$

Here, the map

$$\operatorname{Map}_{/\mathcal{Y}}(\mathcal{Z}, \mathcal{X}^\diamond) \to X^{\diamond,0}(C) = \operatorname{Map}_{/\mathcal{Y}}(B\mathbb{G}_m \times \operatorname{Spec} C, \mathcal{X}^\diamond)$$

is obtained via restriction along η.

If η and ι are clear from context, we will simply write $\operatorname{Map}_{/\mathcal{Y}}(\mathcal{Z}, \mathcal{X})$ for this space.

Here are our main examples:

xes_stack

Example 4.22. Example 4.14 shows that, if $\mathcal{M} \in \operatorname{Perf}(\mathcal{Y})$ is a perfect complex whose restriction over $B\mathbb{G}_m \times \operatorname{Spec} R$ is 1-bounded, then the stack $\mathcal{X}^\diamond \overset{\text{defn}}{=} \mathbf{V}(\mathcal{M}) \to \mathcal{Y}$ underlies a 1-bounded stack over $\mathcal{Y}$ with $X^0 = X^{\diamond,0}$.

_01_stack

Example 4.23. Example 4.15 shows that, when $\mathcal{Y} = B\mathbb{G}_m$ (viewed as a pointed graded stack in the tautological sense), then we obtain a 1-bounded stack $\mathcal{P}_{[0,1]} \to B\mathbb{G}_m$ with $\mathcal{P}^\diamond_{[0,1]} = \mathcal{P} \times B\mathbb{G}_m$, and $P^0_{[0,1]} \subset P^{\diamond,0}_{[0,1]}$ is the open substack parameterizing graded perfect complex $M_\bullet$ with $M_i \simeq 0$ for $i \neq 0, 1$.

For any pointed graded stack $(\mathcal{Z}, \eta) \to B\mathbb{G}_m$, the space

$$\operatorname{Perf}_{[0,1]}((\mathcal{Z}, \eta)) \overset{\text{defn}}{=} \operatorname{Map}_{/B\mathbb{G}_m}(\mathcal{Z}, \mathcal{P}_{[0,1]})$$

is the ∞-groupoid of perfect complexes on $\mathcal{Z}$ whose restriction along η is 1-bounded. We will refer to such objects simply as 1-**bounded perfect complexes** over $(\mathcal{Z}, \eta)$.

Once again, if η is clear from context, we will denote this space simply by $\operatorname{Perf}_{[0,1]}(\mathcal{Z})$, and refer to its sections as 1-bounded perfect complexes over Z.

If we take $\mathbb{A}^1/\mathbb{G}_m \to B\mathbb{G}_m$ to be the canonical map, then the associated attractor stack is the stack

$$R \mapsto \mathrm{Perf}_{[0,1]}(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R)$$

of filtered perfect complexes $\mathrm{Fil}^\bullet M$ with $\mathrm{gr}^i M \simeq 0$ for $i \neq 0,-1$.

Similarly, the repeller stack associated with $\mathbb{A}^1_+/\mathbb{G}_m \to B\mathbb{G}_m$ is the stack of ascendingly filtered perfect complexes $\mathrm{Fil}_\bullet M$ with $\mathrm{gr}_i M \simeq 0$ for $i \neq 0,1$.

_01_stack **Example 4.24.** We have an 'open substack' $\mathcal{V}_{[0,1]}$ of $\mathcal{P}_{[0,1]}$ by restricting to the open locus $\mathcal{V}^\diamond_{[0,1]} \subset \mathcal{P}^\diamond_{[0,1]}$, where the tautological perfect complex is in fact a vector bundle.

For any pair of non-negative integers $d \leq h$, we can further refine this to the 1-bounded stack $\mathcal{V}^{h,d}_{[0,1]} = (\mathcal{V}^\diamond_{[0,1]}, V^{0,h,d}_{[0,1]})$, where $V^{0,h,d}_{[0,1]}$ is the open and closed substack of the fixed point stack parameterizing graded vector bundles $M_\bullet$ such that $M_i \simeq 0$ for $i \neq 0,1$, and such that M_1 is a vector bundle of rank d and M_0 is a vector bundle of rank $h-d$.

The attractor stack $V^{-,h,d}_{[0,1]}$ is the stack of filtered vector bundles $\mathrm{Fil}^\bullet V$ where: V has rank h; $\mathrm{gr}^i V \simeq 0$ for $i \neq 0,-1$; and $\mathrm{gr}^{-1} V$ has rank d.

ded_stack **Example 4.25.** Suppose that we are in the situation of (4.16). Lemma 4.19 then shows that we have a 1-bounded stack $\mathcal{B}(G,\mu)$ over the pointed graded stack $B\mathbb{G}^\phi_{m,\mathcal{O}}$ given by the pair $(BG\{\mu\}, BM_\mu)$.

Remark 4.20 shows that the attractor stack $B(G,\mu)^-$ is simply BP^-_μ, and an analogous argument shows that the repeller stack is BP^+_μ.

fixed_pts 4.26. **Deformations of 1-bounded fixed points.** Suppose that $\mathcal{Y} = \operatorname{Spec}(B_\bullet)/\mathbb{G}_m$ for a non-positively graded animated commutative ring $B_\bullet$: this is pointed via the map $B\mathbb{G}_m \times \operatorname{Spec} B_0 \to \mathcal{Y}$. For any $m \geq 0$, we have the animated commutative graded 'quotient' $B_\bullet \to B_{\geq -m}$ with underlying B_0-module $\bigoplus_{i \geq -m} B_i$. This can be constructed as follows:

formation **Proposition 4.27.** *Let $\mathcal{X} \to \mathcal{Y}$ be a 1-bounded stack such that $\mathcal{X}^\diamond$ has quasi-affine diagonal over $\mathcal{Y}$. Then the natural map*

$$\mathrm{Map}_{/\mathcal{Y}}(\mathcal{Y},\mathcal{X}) \to \mathrm{Map}_{/\mathcal{Y}}((\operatorname{Spec} B_{\geq -1})/\mathbb{G}_m, \mathcal{X})$$

is an equivalence.

Proof. Set $\mathcal{Y}_m = \operatorname{Spec}(B_{\geq -m})/\mathbb{G}_m$, so that $\mathcal{Y}_0 = B\mathbb{G}_m \times \operatorname{Spec} B_0$. Note that $\mathcal{Y}_m$ inherits the pointed structure from $\mathcal{Y}$.

By the graded analogue of the discussion in (4.5), one sees that, for $m \geq 1$, the fiber of the map

$$\mathrm{Map}_{/\mathcal{Y}}(\mathcal{Y}_m, \mathcal{X}^\diamond) \to \mathrm{Map}_{/\mathcal{Y}}(\mathcal{Y}_{m-1}, \mathcal{X}^\diamond)$$

over a section x_{m-1} is equivalent to $\mathrm{Map}_{B_{\geq -m+1}}(\mathbb{L}_{\mathcal{X}^\diamond, x_{m-1},\bullet}, B_{-m}\{m\})$. Here, $\mathbb{L}_{\mathcal{X}^\diamond,x,\bullet}$ is the graded $B_{\geq -m+1}$-module obtained via pulling the relative cotangent complex of $\mathcal{X}^\diamond$ over $\mathcal{Y}$ along x. If $\mathbb{T}_{\mathcal{X}^\diamond,x_{m-1},\bullet}$ is the dual graded perfect module, then we see that the fiber in question is equivalent to

$$(\mathbb{T}_{\mathcal{X}^\diamond,x_{m-1},\bullet} \otimes_{B_{\geq -m+1}} B_{-m}\{m\})_0,$$

where the subscript denotes the degree 0 component.

Let $x_0 \in \mathrm{Map}_{/\mathcal{Y}}(\mathcal{Y}_0, \mathcal{X}^\diamond) = X^{\diamond,0}(B_0)$ be the image of x_{m-1}. Then Lemma 4.28 gives us a canonical filtration $\mathrm{Fil}^\bullet_{\mathrm{wt}} \mathbb{T}_{\mathcal{X}^\diamond,x_{m-1},\bullet}$ with

$$\mathrm{gr}^i_{\mathrm{wt}} \mathbb{T}_{\mathcal{X}^\diamond,x_{m-1},\bullet} \simeq B_{\geq -m+1}\{-i\} \otimes_{B_0} \mathbb{T}_{\mathcal{X}^\diamond,x_0,i},$$

and we have

$$(B_{-m}\{m\} \otimes_{B_{\geq -m+1}} \mathrm{gr}^i_{\mathrm{wt}} \mathbb{T}_{\mathcal{X}^\diamond,x_{m-1},\bullet})_0 \simeq \left(B_{\geq -m+1}\{m-i\} \otimes_{B_0} \mathbb{T}_{\mathcal{X}^\diamond,x_0,i}\right)_0.$$

The right hand side here is equivalent to $B_{m-i} \otimes_{B_0} \mathbb{T}_{\mathcal{X}^\diamond,x_0,i}$, which is non-trivial only when $m \leq i$ and $\mathbb{T}_{\mathcal{X}^\diamond,x_0,i}$ is not nullhomotopic.

Now, if $x_0 \in X^0(B_0)$, then it is a 1-bounded fixed point, and so we see that the fiber over x_{m-1} is contractible as soon as $m > 1$. Therefore, we deduce that, for $m > 1$, the map

$$\mathrm{Map}_{/\mathcal{Y}}(\mathcal{Y}_m, \mathcal{X}) \to \mathrm{Map}_{/\mathcal{Y}}(\mathcal{Y}_{m-1}, \mathcal{X})$$

is an equivalence.

To finish, it is now enough to know that the natural map

$$\mathrm{Map}_{/\mathcal{Y}}(\mathcal{Y}, \mathcal{X}^{\diamond}) \to \varprojlim_m \mathrm{Map}_{/\mathcal{Y}}(\mathcal{Y}_m, \mathcal{X}^{\diamond})$$

is an equivalence. □

iltration

Lemma 4.28. *Let $M_\bullet$ be a graded perfect module over a non-positively graded animated commutative ring $B_\bullet$. Write*

$$\overline{M}_\bullet = B_0 \otimes_{B_\bullet} M_\bullet$$

for the graded base change of $M_\bullet$. Then $M_\bullet$ admits a canonical decreasing filtration $\mathrm{Fil}^\bullet_{\mathrm{wt}} M_\bullet$ in $\mathrm{GrMod}_{B_\bullet}$ with

$$\mathrm{gr}^i_{\mathrm{wt}} M_\bullet \simeq B_\bullet\{-i\} \otimes_{B_0} \overline{M}_i,$$

where $B_\bullet\{-i\}$ is the shifted finite free graded $B_\bullet$-module of rank 1 satisfying $B_\bullet\{-i\}_m = B_{m-i}$.

Proof. There exist integers $a < b$ such that the graded perfect module $\overline{M}_\bullet$ satisfies $\overline{M}_i \simeq 0$ for $i \notin [a, b]$. By shifting the grading, we can assume that $a = 0$. The construction of the filtration is by induction on b.

We first claim that the natural map $M_b \to \overline{M}_b$ is an equivalence. To see this, observe that we have a fiber sequence

$$B_{\le -1} \otimes_{B_\bullet} M_\bullet \to M_\bullet \to \overline{M}_\bullet,$$

and note that the left hand side is a graded module supported in degrees $\le b - 1$.

We can use the equivalence $M_b \xrightarrow{\simeq} \overline{M}_b$ to obtain a canonical map

$$B_\bullet \otimes_{B_0} \overline{M}_b\,\{-b\} \xrightarrow{\simeq} B_\bullet \otimes_{B_0} M_b\,\{-b\} \to M_\bullet$$

whose cofiber $M'_\bullet$ is supported in degrees $< b$. Graded base-change to B_0 yields a cofiber sequence

$$\overline{M}_b\,\{-b\} \to \overline{M} \to \overline{M}',$$

which shows that $\overline{M}_i \xrightarrow{\simeq} \overline{M}'_i$ for $i \le b-1$ and $\overline{M}'_i \simeq 0$ for $i > b-1$.

By our inductive hypothesis, $M'_\bullet$ admits a filtration $\mathrm{Fil}^\bullet_{\mathrm{wt}} M'_\bullet$ with

$$\mathrm{gr}^i_{\mathrm{wt}} M'_\bullet \simeq B_\bullet \otimes_{B_0} \overline{M}'_i\,\{-i\} \simeq B_\bullet \otimes_{B_0} \overline{M}_i\,\{-i\}$$

for $i \le b - 1$.

We now obtain our desired filtration on $M_\bullet$ by setting

$$\mathrm{Fil}^i_{\mathrm{wt}} M_\bullet = \begin{cases} 0 & \text{if } i > b \\ \mathrm{Fil}^i_{\mathrm{wt}} M'_\bullet \times_{M'_\bullet} M_\bullet & \text{if } i \le b. \end{cases}$$

□

4.29. **A useful cartesian square.** Suppose that $\mathrm{Fil}^\bullet S$ is a non-negatively filtered animated commutative ring, and set $\overline{S} = \mathrm{gr}^0 S$. The map $S \to \overline{S}$ underlies an arrow $\mathrm{Fil}^\bullet S \to \mathrm{Fil}^\bullet_{\mathrm{triv}} \overline{S}$ of filtered animated commutative rings corresponding to a map of stacks

$$\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\,\overline{S} \to \mathcal{R}(\mathrm{Fil}^\bullet S)$$

whose restriction over the open point of $\mathbb{A}^1/\mathbb{G}_m$ is the closed immersion $\mathrm{Spec}\,\overline{S} \to \mathrm{Spec}\,S$.

We will view $\mathcal{Y} \overset{\mathrm{defn}}{=} \mathcal{R}(\mathrm{Fil}^\bullet S)$ as a pointed graded stack via the composition

$$B\mathbb{G}_m \times \mathrm{Spec}\,\overline{S} \to \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\,\overline{S} \to \mathcal{Y}.$$

Let $\mathcal{X} = (\mathcal{X}^\diamond, X^0) \to \mathcal{Y}$ be a 1-bounded stack, and let $X^- \to \operatorname{Spec} \overline{S}$ be its associated attractor stack. We now have a commutative diagram

mm_square (4.29.1)
$$\begin{tikzcd} \mathrm{Map}_{/\mathcal{Y}}(\mathcal{Y}, \mathcal{X}) \arrow[r] \arrow[d] & \mathrm{Map}_{/\mathcal{Y}}(\operatorname{Spec} S, \mathcal{X}^\diamond) \arrow[d] \\ X^-(S) = \mathrm{Map}_{/\mathcal{Y}}(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} \overline{S}, \mathcal{X}) \arrow[r] & \mathrm{Map}_{/\mathcal{Y}}(\operatorname{Spec} \overline{S}, \mathcal{X}^\diamond). \end{tikzcd}$$

cartesian **Proposition 4.30.** *Suppose that the kernel of the map*

$$\pi_0(S) \to \pi_0(\overline{S})$$

is nilpotent. Then (4.29.1) *is a Cartesian square.*

Proof. Let $\mathrm{Fil}^\bullet_{(S\twoheadrightarrow\overline{S})} S$ be the non-negatively filtered animated commutative ring with $\mathrm{Fil}^0_{(S\twoheadrightarrow\overline{S})} S \simeq S$ and

$$\mathrm{gr}^i_{(S\twoheadrightarrow\overline{S})} S \simeq \begin{cases} \overline{S} & \text{if } i = 0; \\ 0 & \text{otherwise.} \end{cases}$$

The associated Rees algebra sits in a Cartesian square of graded animated commutative rings

struction (4.30.1)
$$\begin{tikzcd} \bigoplus_i \mathrm{Fil}^i_{(S\twoheadrightarrow\overline{S})} S \cdot t^{-i} \arrow[r] \arrow[d] & S[t, t^{-1}] \arrow[d] \\ \overline{S}[t] \arrow[r] & \overline{S}[t, t^{-1}]. \end{tikzcd}$$

One can obtain this construction for instance by animating the obvious one for surjections of polynomial algebras over $\mathbb{Z}$.

For simplicity, for any non-negatively filtered animated commutative ring $\mathrm{Fil}^\bullet A$ set

$$\mathrm{Fil}^\bullet A' = \mathrm{Fil}^\bullet_{(A\twoheadrightarrow \mathrm{gr}^0(A))} A,$$

where the right hand side is defined as above.

The diagram (4.29.1) is Cartesian with $\mathcal{R}(\mathrm{Fil}^\bullet S)$ replaced by $\mathcal{R}(\mathrm{Fil}^\bullet S')$. This is clear when $\mathcal{X}^\diamond \to \mathcal{Y}$ is relatively affine, given the Cartesian nature of (4.30.1). The general case can be deduced from this via flat descent.

Note that there is a natural map $\mathcal{R}(\mathrm{Fil}^\bullet S') \to \mathcal{R}(\mathrm{Fil}^\bullet S)$. To complete the proof of the proposition it now suffices to show that the corresponding map

$$\mathcal{X}(\mathrm{Fil}^\bullet S) \overset{\mathrm{defn}}{=} \mathrm{Map}_{/\mathcal{Y}}(\mathcal{R}(\mathrm{Fil}^\bullet S), \mathcal{X}) \to \mathrm{Map}_{/\mathcal{Y}}(\mathcal{R}(\mathrm{Fil}^\bullet S'), \mathcal{X}) = \mathcal{X}(\mathrm{Fil}^\bullet S')$$

is an equivalence. We will prove this using filtered deformation theory.

Suppose quite generally that $\mathrm{Fil}^\bullet B \to \mathrm{Fil}^\bullet A$ is a square-zero extension of non-negatively filtered $\mathrm{Fil}^\bullet S$-algebras with fiber $\mathrm{Fil}^\bullet K$. Then, by the discussion in (4.5), we obtain a Cartesian square

$$\begin{tikzcd} \mathcal{X}(\mathrm{Fil}^\bullet B) \arrow[r] \arrow[d] & \mathcal{X}(\mathrm{Fil}^\bullet A) \arrow[d] \\ \mathcal{X}(\mathrm{Fil}^\bullet A) \arrow[r] & \mathcal{X}(\mathrm{Fil}^\bullet A \oplus \mathrm{Fil}^\bullet K[1]). \end{tikzcd}$$

Moreover, if $\mathbb{L}_{\mathcal{X}^\diamond/\mathcal{Y}}$ is the relative cotangent complex, then for any $x \in \mathcal{X}(\mathrm{Fil}^\bullet A)$, we obtain a filtered $\mathrm{Fil}^\bullet A$-module $\mathrm{Fil}^\bullet \mathbb{L}_{\mathcal{X}^\diamond/\mathcal{Y},x}$, and we have a canonical equivalence:

$$\begin{aligned} \mathrm{fib}_x(\mathcal{X}(\mathrm{Fil}^\bullet B) \to \mathcal{X}(\mathrm{Fil}^\bullet A)) &\simeq \mathrm{Map}_{\mathrm{Fil}^\bullet A}(\mathrm{Fil}^\bullet \mathbb{L}_{\mathcal{X}^\diamond/\mathcal{Y},x}, \mathrm{Fil}^\bullet K[1]) \\ &\simeq \tau^{\le 0} \mathrm{Fil}^0 \left(\mathrm{Fil}^\bullet \mathbb{T}_{\mathcal{X}^\diamond/\mathcal{Y},x} \otimes_{\mathrm{Fil}^\bullet A} \mathrm{Fil}^\bullet K[1]\right). \end{aligned}$$

Now, the associated graded of the filtered tensor product on the right is $\mathrm{gr}^\bullet \mathbb{T}_{\mathcal{X}^\diamond/\mathcal{Y},x} \otimes_{\mathrm{gr}^\bullet A} \mathrm{gr}^\bullet K[1]$. By Lemma 4.28, it admits a further filtration with associated gradeds isomorphic to

$$\mathrm{gr}^i \overline{\mathbb{T}}_{\mathcal{X}^\diamond/\mathcal{Y},x}\{i\} \otimes_{\mathrm{gr}^0 A} \mathrm{gr}^\bullet K[1],$$

where the corresponding degree j component is $\mathrm{gr}^i \overline{\mathbb{T}}_{\mathcal{X}/\mathcal{Y},x} \otimes_{\mathrm{gr}^0 A} \mathrm{gr}^{j-i} K[1]$. Our hypothesis of 1-boundedness now tells us that the first factor in this tensor product is zero for $i < -1$, while the second factor is zero for $j < i$. Using this, one finds that there is a fiber sequence

$$\mathrm{Fil}^0 \left(\mathrm{Fil}^\bullet \mathbb{T}_{\mathcal{X}^\diamond/\mathcal{Y},x} \otimes_{\mathrm{Fil}^\bullet A} \mathrm{Fil}^\bullet K[1]\right) \to \mathbb{T}_{\mathcal{X}^\diamond/\mathcal{Y},x} \otimes_A K[1] \to \mathrm{gr}^{-1} \overline{\mathbb{T}}_{\mathcal{X}^\diamond/\mathcal{Y},x} \otimes_{\mathrm{gr}^0 A} \mathrm{gr}^0 K[k+1].$$

In particular, this tells us that the object on the left hand side depends only on $\mathrm{Fil}^1 A \to A$. Thus, if $\mathcal{X}(\mathrm{Fil}^\bullet A) \to \mathcal{X}(\mathrm{Fil}^\bullet A')$ is an isomorphism, then so is $\mathcal{X}(\mathrm{Fil}^\bullet B) \to \mathcal{X}(\mathrm{Fil}^\bullet B')$.

For every k, we have the truncated filtered animated commutative ring $\tau_{\leq k}(\mathrm{Fil}^\bullet S)$ obtained by taking the corresponding truncation for the associated Rees algebra: this is a square-zero extension of $\tau_{\leq (k-1)} \mathrm{Fil}^\bullet S$ by a filtered module $\pi_k(\mathrm{Fil}^\bullet S)[k]$. Via the deformation argument from the previous paragraph, induction on k, and using nilcompleteness, one therefore reduces to showing that the map

$$\mathcal{X}(\pi_0(\mathrm{Fil}^\bullet S)) \to \mathcal{X}(\pi_0(\mathrm{Fil}^\bullet S'))$$

restricts to an isomorphism on 1-bounded sections.

Thus, we can assume that $\mathrm{Fil}^\bullet S$ is discrete filtered commutative ring. For $m \geq 1$, let $\mathrm{Fil}^\bullet S_m$ be the quotient filtration on the ring $S_m = S/(\mathrm{Fil}^1 S)^m$: if $m = 1$, we of course get $\mathrm{Fil}^\bullet_{\mathrm{triv}} \overline{S}$. Since $\mathrm{Fil}^1 S$ is nilpotent by hypothesis, we also have $\mathrm{Fil}^\bullet S = \mathrm{Fil}^\bullet S_m$ for m sufficiently large. Once again, the deformation argument reduces us to the case $m = 1$, where the desired assertion is trivial. □

Remark 4.31. The above result can be viewed as a generalization of [30, Remark 6.3.3].

er_frames

5. Animated higher frames and displays

The purpose of this section is to give an account of the theory of [30] in an animated context. This will used in (9.4) to give a somewhat concrete description of the values of the stacks of interest in this paper on semiperfectoid inputs. We will also give an account of deformation theory in this context, which will later lead to an efficient proof of Theorem G; this is in turn essential for bootstrapping representability up from characteristic p.

5.1. **Generalized Cartier divisors.** Recall that a **generalized Cartier divisor** for an animated commutative ring R is a surjective map $R \twoheadrightarrow \overline{R}$ whose homotopy kernel I is an invertible R-module. By abuse of notation we will refer to such an object via the cosection $s : I \to R$, which is the same as a map $s : \mathrm{Spec}\, R \to \mathbb{A}^1/\mathbb{G}_m$.

Any generalized Cartier divisor lifts R to a filtered animated commutative ring $\mathrm{Fil}^\bullet_I R$ where the filtration is the I-adic one given by

$$\mathrm{Fil}^k_I R = \begin{cases} I^{\otimes k} & \text{if } k \geq 0 \\ R & \text{if } k < 0, \end{cases}$$

and the transition maps are the identity for $k \leq 0$ and given by

$$I^{\otimes k} \simeq I \otimes_R I^{\otimes (k-1)} \xrightarrow{s \otimes 1} R \otimes_R I^{\otimes (k-1)} \simeq I^{\otimes (k-1)}$$

for $k > 0$. We will also have occasion to considered the **two-sided I-adic filtration** given by $\mathrm{Fil}^k_{I,\pm} R = I^{\otimes k}$ for all $k \in \mathbb{Z}$, which once again underlies a filtered animated commutative ring with

$$\mathcal{R}(\mathrm{Fil}^\bullet_{I,\pm} R) \simeq \mathrm{Spec}\, R.$$

To verify the assertions in the previous paragraph, using the classifying map $s : \mathrm{Spec}\, R \to \mathbb{A}^1/\mathbb{G}_m$, one reduces everything to the case where $R = \mathbb{Z}[x]$ with $I = x\mathbb{Z}[x]$, and here everything is clear.

ck_points *Remark* 5.2. This also gives a concrete way of thinking of a point $\mathrm{Spec}\, R \to \mathcal{R}(\mathrm{Fil}^\bullet S)$ of the Rees stack corresponding to a filtered commutative ring $\mathrm{Fil}^\bullet S$: it is equivalent to giving a generalized Cartier divisor $I \to R$, along with a map of filtered animated commutative rings $\mathrm{Fil}^\bullet S \to \mathrm{Fil}^\bullet_{I,\pm} R$.

For any $M \in \mathrm{Mod}_R$, we will set $M[I^{-1}] = \mathrm{colim}_{k\geq 0} I^{-k} \otimes_R M$, where the transition maps are induced by s.

When we have an isomorphism $R \xrightarrow{\simeq} I$ of R-modules given by a section ξ of I, we will write $\mathrm{Fil}^\bullet_\xi R$ and $\mathrm{Fil}^\bullet_{\xi,\pm} R$ for these filtered rings.

For any R-module M, we will write $M/^{\mathbb{L}}(p,I)$ for $M/^{\mathbb{L}}p \otimes_R \overline{R}$.

If $R' \in \mathrm{CRing}_{R/}$ is an R-algebra, and $s' : I' = R' \otimes_R I \xrightarrow{1\otimes s} R'$, then we will sometimes also denote the I'-adic filtrations on R' by $\mathrm{Fil}^\bullet_I R'$ and $\mathrm{Fil}^\bullet_{I,\pm} R'$.

rmal_rees

5.3. **Formal Rees stacks.** Suppose that $I \to A$ is a generalized Cartier divisor with p-complete quotient $\overline{A}$, and suppose that A underlies a filtered animated commutative ring $\mathrm{Fil}^\bullet A$.

The **formal Rees stack** associated with this datum is the one associating with each $R \in \mathrm{CRing}^{p\text{-nilp}}$ the space of generalized Cartier divisors $J \to R$ along with maps $\mathrm{Fil}^\bullet A \to \mathrm{Fil}^\bullet_{J,\pm} R$ (see Remark 5.2) such that the underlying map $A \to R$ is in $\mathrm{Spf}(A,I)(R)$.

In the sequel, the Rees construction will only be appealed to in this formal context. Therefore, by abuse of notation, we will denote this formal stack once again by $\mathcal{R}(\mathrm{Fil}^\bullet A)$.

In particular, we have $\mathcal{R}(\mathrm{Fil}^\bullet_{I,\pm} A) \simeq \mathrm{Spf}(A,I)$.

5.4. **Witt vectors, δ-rings and prisms.** We recall the notion of an animated δ-ring from [8, App. A]: First, one defines for any animated commutative ring R the level-2 Witt ring $W_2(R)$ with underlying space R^2 such that the projection onto the first coordinate is a map of animated commutative rings $W_2(R) \twoheadrightarrow R$. This amounts to the observation that the functor $C \mapsto W_2(C)$ on discrete commutative rings is represented by a smooth ring scheme, and so extends canonically to an animated ring scheme equipped with a map to $\mathbb{G}_a$.

Now, a **δ-structure** on an animated ring R is a section of the natural map $W_2(R) \to R$. Such a section is determined completely by its projection onto the second coordinate, which yields an operator $\delta : R \to R$ satisfying certain properties. When R is flat over $\mathbb{Z}_{(p)}$, giving such a δ is equivalent to giving a lift $\varphi : R \to R$ of the mod-p Frobenius satisfying $\varphi(r) = r^p + p\delta(r)$ for any $r \in R$. This interpretation (though perhaps not the explicit formula) is valid for all R if we treat Frobenius as an endomorphism of $R/^{\mathbb{L}}p$.

An (animated) **δ-ring** is an animated ring R equipped with a δ-structure. We obtain an ∞-category CRing_δ of animated δ-rings in the usual fashion.

Following [8, Def. 2.4], we define a(n animated) **prism** to be an animated δ-ring A equipped with a generalized Cartier divisor $I \to A$ with quotient $\overline{A}$ such that the following conditions hold:

(1) A is (p,I)-complete.
(2) Given a perfect field k of characteristic p and a map $A \to W(k)$ of δ-rings, we have $W(k) \otimes_A \overline{A} \simeq k$.

ed_frames

5.5. **Animated higher frames.** An (animated higher) **frame** $\underline{A}$ is a tuple $(\mathrm{Fil}^\bullet A, I, J, \Phi)$, where:

(1) $\mathrm{Fil}^\bullet A$ is a non-negatively filtered animated commutative ring;
(2) (A,I) is a prism with underlying generalized Cartier divisor $s : I \to A$ with quotient $\overline{A}$;
(3) J is an invertible A-module equipped with an isomorphism $\varphi^* J \xrightarrow{\simeq} I \otimes_A J$;
(4) $\Phi : \mathrm{Fil}^\bullet A \to \mathrm{Fil}^\bullet_I A$ is a map of filtered animated commutative rings underlying the Frobenius lift $\varphi : A \to A$.

Frames organize into an ∞-category in the obvious way: A map $\underline{A} \to \underline{A}'$ is given by a map $\mathrm{Fil}^\bullet A \to \mathrm{Fil}^\bullet A'$ of filtered animated commutative rings, along with isomorphisms $(A', A' \otimes_A I) \xrightarrow{\simeq} (A', I')$ of prisms, and isomorphisms $A' \otimes_A J \xrightarrow{\simeq} J'$ that are compatible with all the additional structure.

Given a frame $\underline{A}$, we will write R_A for the p-complete animated ring $\mathrm{gr}^0 A$.

Let $\mathrm{Fil}^\bullet_{I,\pm} A$ be the two-sided I-adic filtration on A; then we obtain a map

$$\Phi_\pm : \mathrm{Fil}^\bullet A \to \mathrm{Fil}^\bullet_{I,\pm} A,$$

which restricts to Φ in non-negative degrees, and which in filtered degree $-i$ (for $i \in \mathbb{Z}_{>0}$) is given by $s^{-i} \circ \varphi$.

Let $\mathrm{Spf}\, A \overset{\mathrm{defn}}{=} \mathrm{Spf}(A,I)$ be the p-adic formal scheme obtained from A with its (p,I)-adic topology.

If $\mathcal{R}(\mathrm{Fil}^\bullet A) \to \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spf}\, \mathbb{Z}_p$ is the associated formal Rees stack over $\mathrm{Spf}\, \mathbb{Z}_p$ as in (5.3), we obtain two maps

$$\tau, \sigma : \mathrm{Spf}\, A \hookrightarrow \mathcal{R}(\mathrm{Fil}^\bullet A)$$

as follows:

- τ is obtained by pulling back the open point

$$\mathbb{G}_m/\mathbb{G}_m \times \operatorname{Spf}\mathbb{Z}_p \hookrightarrow \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spf}\mathbb{Z}_p.$$

  It is in particular, an open immersion.
- σ is obtained as the composition

$$\operatorname{Spf} A \xrightarrow{\simeq} \mathcal{R}(\mathrm{Fil}^\bullet_{I,\pm} A) \xrightarrow{\mathcal{R}(\Phi_\pm)} \mathcal{R}(\mathrm{Fil}^\bullet A).$$

cture_map

Remark 5.6. We also have a canonical map $\pi : \mathcal{R}(\mathrm{Fil}^\bullet A) \to \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spf} A$ arising from the map of filtered rings $\mathrm{Fil}^\bullet_{\mathrm{triv}} A \to \mathrm{Fil}^\bullet A$. The composition $\pi \circ \tau$ is the natural open immersion of $\operatorname{Spf} A$ into the open point in $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spf} A$, while the composition $\pi \circ \sigma$ is the pre-composition of this immersion with the Frobenius lift on $\operatorname{Spf} A$.

ham_point

Remark 5.7. The restriction of the map τ to $\operatorname{Spec} R_A$ extends along the open immersion

$$\mathbb{G}_m/\mathbb{G}_m \times \operatorname{Spec} R_A \hookrightarrow \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R_A$$

to a map $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R_A \to \mathcal{R}(\mathrm{Fil}^\bullet A)$. On the level of filtered rings this corresponds to the map $\mathrm{Fil}^\bullet A \to \mathrm{Fil}^\bullet_{\mathrm{triv}} R_A$.

ic_frames

Definition 5.8. If $I = (p)$, so that (A, I) is a *crystalline prism*, we will say that $\underline{A}$ is a p**-adic frame**. In this case, we will take $J = A$ to be trivial. Note that R_A is now an $\mathbb{F}_p$-algebra. Also, the p-adic filtration $\mathrm{Fil}^\bullet_p A$ factors through $\mathrm{Fil}^\bullet A$, and in particular the Frobenius endomorphism of $\overline{A} = A/^{\mathbb{L}}p$ factors through R_A.

Before we state the next result, recall the following:

Definition 5.9. We will say that a map $B \to C$ in CRing is **Henselian** if $\pi_0(B) \to \pi_0(C)$ is surjective, and if $(\pi_0(B), \ker(\pi_0(B) \to \pi_0(C)))$ is a Henselian pair as defined for instance in [45, Tag 09XD].

The next result follows from [17, Thm. A.0.4].

lian_lift

Proposition 5.10. *Suppose that $B \to C$ is a Henselian map in $\mathrm{CRing}_{\mathbb{Z}_p/}$. Then, for any quasi-compact smooth algebraic stack X over $\mathbb{Z}_p$ with affine diagonal, the map $X(B) \to X(C)$ has non-empty fibers.*

ps_hensel

Proposition 5.11. *Suppose that $\underline{A}$ is a frame. Then $A \twoheadrightarrow R_A$ is Henselian. Moreover, every p-completely étale map $R_A \to R_{A'}$ lifts uniquely to a (p, I)-completely étale map $A \to A'$[9], where A' underlies a frame $\underline{A}'$ uniquely determined by the fact that $\mathrm{Fil}^\bullet A' = \mathrm{Fil}^\bullet A \otimes_A A'$.*

Proof. This is an animated variant of [30, Lemma 4.2.3].

Let us check that $A \twoheadrightarrow R_A$ is Henselian. We follow the argument from [1, Lemma 4.1.28]. Since A is (p, I)-adically complete, it is enough to check that $\pi_0(A/^{\mathbb{L}}(p, I)) \to \pi_0(R_A/^{\mathbb{L}}(p, I))$ is Henselian, which is true since its kernel is locally nilpotent; indeed, our hypotheses imply that it is annihilated by the p-power Frobenius.

In fact, this argument also proves the assertion on lifting p-completely étale maps to (p, I)-completely étale ones; see [45, Tag 0ALI].

We can interpret δ-ring structures on A' as sections $A' \to W_2(A')$, and the (p, I)-complete étaleness of A' over A guarantees that there exists a unique (up to homotopy) such section lifting the corresponding one for A, and hence a Frobenius lift $\varphi' : A' \to A'$. The corresponding filtered map $\Phi' : \mathrm{Fil}^\bullet A' \to \mathrm{Fil}^\bullet_I A'$ is now given by

$$\mathrm{Fil}^\bullet A' \simeq A' \otimes_A \mathrm{Fil}^\bullet A' \xrightarrow{\varphi' \otimes \Phi} A' \otimes_A \mathrm{Fil}^\bullet_I A' \simeq \mathrm{Fil}^\bullet_I A'.$$

∎

[9]By this, we mean that A' is (p, I)-complete, and $A/^{\mathbb{L}}(p, I) \to A'/^{\mathbb{L}}(p, I)$ is étale.

_displays

5.12. **(G,μ)-displays over higher frames.** Now, suppose that we are in the situation of (4.11), so that G is a smooth group scheme over $\mathbb{Z}_p$, $\mathcal{O}$ is the ring of integers in a finite unramified extension of $\mathbb{Z}_p$ and $\mu : \mathbb{G}_{m,\mathcal{O}} \to \underline{\mathrm{Aut}}(G_{\mathcal{O}})$ is a cocharacter.

Let k be the residue field of $\mathcal{O}$. If $\underline{A}$ is a frame such that R_A lifts to $\mathrm{CRing}_{k/}$, then A lifts canonically to $\mathrm{CRing}_{\mathcal{O}/}$: Lift the map $\mathcal{O} \to R_A/^{\mathbb{L}}(p,I)$ first to $\mathcal{O} \to A/^{\mathbb{L}}(p,I)$ using local nilpotence and the formal étaleness of $\mathcal{O}$, and then to A by (p,I)-completeness.

We will view $\mathcal{R}(\mathrm{Fil}^\bullet A)$ as living over the stack $B\mathbb{G}_m$ via the line bundle $\mathcal{O}\{1\}$ associated with the filtered module $\mathrm{Fil}^\bullet A\{1\} \otimes_A J$. Note that the restriction of $\mathcal{O}\{1\}$ along τ corresponds to the A-module J, while that along σ corresponds to $I \otimes_A \varphi^* J \simeq J$. Therefore, if we take the structure map $\mathrm{Spf}\, A \to B\mathbb{G}_m$ classifying the line bundle associated with J, both σ, τ can be viewed as maps of stacks over $B\mathbb{G}_m$.

_is_etale

Proposition 5.13. *The following are equivalent for an fpqc $G\{\mu\}$-torsor $\mathcal{Q}$ over $\mathcal{R}(\mathrm{Fil}^\bullet A) \otimes \mathbb{Z}/p^n\mathbb{Z}$.*

(1) $\mathcal{Q}$ is trivial étale-locally on $\mathrm{Spf}\, R_A$. That is, there exists a p-completely étale map $R_A \to R_{A'}$ such that the restriction of $\mathcal{Q}$ over $\mathcal{R}(\mathrm{Fil}^\bullet A') \otimes \mathbb{Z}/p^n\mathbb{Z}$ is trivial.

(2) The restriction of $\mathcal{Q}$ to $\mathcal{R}(\mathrm{Fil}^\bullet A)_{(t=0)} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is trivial étale-locally on $\mathrm{Spf}\, R_A$.

(3a) The restriction of $\mathcal{Q}$ over $\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, R_A/^{\mathbb{L}}p^n$ is trivial étale-locally on $\mathrm{Spf}\, R_A$.

(3b) For every geometric point $R_A \to \kappa$ of $\mathrm{Spf}\, R_A$, the restriction of $\mathcal{Q}$ over $\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\,\kappa$ is trivial.

(4a) The restriction of $\mathcal{Q}$ over $B\mathbb{G}_m \times \mathrm{Spec}\, R_A/^{\mathbb{L}}p^n$ is trivial étale-locally on $\mathrm{Spf}\, R_A$.

(4b) For every geometric point $R_A \to \kappa$ of $\mathrm{Spf}\, R_A$, the restriction of $\mathcal{Q}$ over $B\mathbb{G}_m \times \mathrm{Spec}\,\kappa$ is trivial.

Proof. The equivalence of (3a), (3b), (4a) and (4b) follows from Lemma 4.19 and Remark 4.20. We can finish by showing (3a)$\Rightarrow$(1), but this comes down to the assertion that the surjective map of graded animated commutative rings

$$\bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^i A/^{\mathbb{L}}(p,I)\cdot t^{-i} \to \bigoplus_{i\in\mathbb{Z}} \mathrm{Fil}^i_{\mathrm{triv}} R_A/^{\mathbb{L}}(p,I)\cdot t^{-i}$$

is Henselian. This in turn is clear, since the kernel of its classical truncation is locally nilpotent, as observed in the proof of Proposition 5.11. $\square$

_quotient

Definition 5.14. $\mathrm{Disp}^{G,\mu}_{n,\underline{A}}(R_A)$ is the ∞-groupoid of $G\{\mu\}$-torsors $\mathcal{Q}$ over $\mathcal{R}(\mathrm{Fil}^\bullet A) \otimes \mathbb{Z}/p^n\mathbb{Z}$ equipped with an isomorphism of G-torsors $\sigma^*\mathcal{Q} \xrightarrow{\simeq} \tau^*\mathcal{Q}$, and satisfying the following equivalent conditions:

(1) $\mathcal{Q}$ is trivial étale locally on $\mathrm{Spf}\, R_A$.

(2) For every geometric point $R_A \to k$, the restriction of $\mathcal{Q}$ over $B\mathbb{G}_m \times \mathrm{Spec}\, k$ is trivial.

We will refer to the objects of this ∞-groupoid as **abstract (G,μ)-displays** over $\underline{A}$ **of level** n.

_abstract

Remark 5.15. Note that σ is only φ-semilinear as a map of stacks over $B\mathbb{G}_{m,\mathcal{O}}$; but σ and τ can both be viewed as maps of stacks over $B\mathbb{G}^{\phi}_{m,\mathcal{O}}$, the stack introduced in (4.12). In particular, the coequalizer $\mathsf{Syn}(\underline{A}) \otimes \mathbb{Z}/p^n\mathbb{Z}$ of the two maps

$$\sigma, \tau : \mathrm{Spf}\, A/^{\mathbb{L}}p^n \to \mathcal{R}(\mathrm{Fil}^\bullet A) \otimes \mathbb{Z}/p^n\mathbb{Z}$$

admits a canonical map of pointed graded stacks to $B\mathbb{G}^{\phi}_{m,\mathcal{O}}$.

Now, $\mathrm{Disp}^{G,\mu}_{n,\underline{A}}(R_A)$ is simply the space

$$\mathrm{Map}_{/B\mathbb{G}^{\phi}_{m,\mathcal{O}}}(\mathsf{Syn}(\underline{A}) \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{B}(G,\mu)),$$

where $\mathcal{B}(G,\mu) \to B\mathbb{G}^{\phi}_{m,\mathcal{O}}$ is the 1-bounded stack from Example 4.25.

ient_desc

Remark 5.16. Here is a slightly different perspective on the definition, closer to the treatment in [30, 11]. Let $L^+_{\underline{A}}G^{(n)}$ and $L^+_{\underline{A}}G^{(n)}\{\mu\}$ be the étale sheaves on $(R_A)_{\mathrm{\acute{e}t}}$ of group-like spaces given by

$$L^+_{\underline{A}}G^{(n)}(R_{A'}) = G^{(n)}(R_{A'}) = G(A'/^{\mathbb{L}}p^n)\ ;\ L^+_{\underline{A}}G^{(n)}\{\mu\}(R_{A'}) = \mathrm{Map}_{B\mathbb{G}_{m,\mathcal{O}}}(\mathcal{R}(\mathrm{Fil}^\bullet A') \otimes \mathbb{Z}/p^n\mathbb{Z}, G\{\mu\}).$$

Pullback along σ and τ gives two maps

$$\sigma^*, \tau^* : L^+_{\underline{A}} G^{(n)}\{\mu\} \to L^+_{\underline{A}} G^{(n)}$$

We now have

$$\mathrm{Disp}^{G,\mu}_{n,\underline{A}} = [L^+_{\underline{A}} G^{(n)} \ {}_{\sigma^*}\!/\!\!/_{\tau^*} \ L^+_{\underline{A}} G^{(n)}\{\mu\}],$$

where the right hand side indicates the quotient by the action given symbolically by

$$L^+_{\underline{A}} G^{(n)}\{\mu\} \times L^+_{\underline{A}} G^{(n)} \xrightarrow{(h,g)\mapsto \sigma^*(h)gh^{-1}} L^+_{\underline{A}} G^{(n)}.$$

:A_gauges

5.17. **$\underline{A}$-gauges and the case of GL_h.** Here, we explicate what the above definitions specialize to for $G = \mathrm{GL}_h$ for some $h \geq 1$. An **$\underline{A}$-gauge of level** n is a quasicoherent sheaf $\mathcal{M} \in \mathrm{QCoh}(\mathcal{R}(\mathrm{Fil}^\bullet A) \otimes \mathbb{Z}/p^n\mathbb{Z})$ equipped with an isomorphism $\xi : \sigma^*\mathcal{M} \xrightarrow{\simeq} \tau^*\mathcal{M}$ in $\mathrm{QCoh}(\mathrm{Spf}\, A/^{\mathbb{L}}p^n)$.

star_expl

Remark 5.18. Explicitly, we can view $\mathcal{M}$ as a derived I-complete filtered module $\mathrm{Fil}^\bullet \mathsf{M}$ over $\mathrm{Fil}^\bullet A/^{\mathbb{L}}p^n$. Base-change along $\Phi_\pm : \mathrm{Fil}^\bullet A \to \mathrm{Fil}^\bullet_{I,\pm} A$ yields a filtered module $\Phi^*_\pm \mathrm{Fil}^\bullet \mathsf{M}$ over $\mathrm{Fil}^\bullet_{I,\pm} A/^{\mathbb{L}}p^n$ whose degree-0 filtered piece is an $A/^{\mathbb{L}}p^n$-module M_σ corresponding to $\sigma^*\mathcal{M}$. The isomorphism ξ now corresponds to an isomorphism $\mathsf{M}_\sigma \xrightarrow{\simeq} \mathsf{M}$ in $\mathrm{Mod}_{A/^{\mathbb{L}}p^n}$.

$\underline{A}$-gauges of level n organize into a symmetric monoidal stable ∞-category $\underline{A}\mathrm{-}\mathsf{gauge}_n$, where the unit object is the structure sheaf $\mathscr{O}$ of $\mathcal{R}(\mathrm{Fil}^\bullet A) \otimes \mathbb{Z}/p^n\mathbb{Z}$ equipped with the canonical isomorphism ξ_0 between its pullbacks via σ^* and τ^*.

For any map $\underline{A} \to \underline{B}$ of frames, there is a natural base-change map

$$\underline{A}\mathrm{-}\mathsf{gauge}_n \xrightarrow{(\mathcal{M},\xi)\mapsto \underline{B}\otimes_{\underline{A}}(\mathcal{M},\xi)} \underline{B}\mathrm{-}\mathsf{gauge}_n.$$

The proof of the next result is as in [30, Example 5.3.5].

auge_gl_n

Proposition 5.19. *Suppose that $\mu : \mathbb{G}_m \to \mathrm{GL}_h$ is the cocharacter given by*

$$z \mapsto \mathrm{diag}(z^{m_1}, z^{m_2}, \ldots, z^{m_h})$$

for $m_1, m_2, \ldots, m_n \in \mathbb{Z}$. Then $\mathrm{Disp}^{\mathrm{GL}_h,\mu}_{n,\underline{A}}$ is equivalent to the ∞-groupoid of $\underline{A}$-gauges $\mathcal{M}$ of level n such that the underlying filtered module $\mathrm{Fil}^\bullet \mathsf{M}$ satisfies the following condition: There exists an étale cover $R_A \to R_{A'}$ and an isomorphism

$$\mathrm{Fil}^\bullet A' \otimes_{\mathrm{Fil}^\bullet A} \mathrm{Fil}^\bullet \mathsf{M} \xrightarrow{\simeq} \bigoplus_{i=1}^{h} \mathrm{Fil}^\bullet A'\{m_i\}/^{\mathbb{L}}p^n,$$

where $\mathrm{Fil}^\bullet A'\{m_i\}$ is the invertible filtered $\mathrm{Fil}^\bullet A'$-module with

$$\mathrm{Fil}^m A'\{m_i\} = J^{\otimes m_i} \otimes_A \mathrm{Fil}^{m+m_i} A'.$$

ubsec:alb

5.20. **Relationship with the definitions of Anschütz-Le Bras.** The purpose of this subsection is to connect the definitions here with those of [1].

Suppose that $0 \leq d \leq h$ is such that $m_i = 1$ for $0 \leq i < d$ and $m_i = 0$ for $i \geq d$, so that μ is *minuscule*. Let

$$\mathrm{Disp}^{\mathrm{GL}_h,\mu}_{\infty,\underline{A}}(R_A) = \varprojlim_n \mathrm{Disp}^{\mathrm{GL}_h,\mu}_{n,\underline{A}}(R_A)$$

be the inverse limit of ∞-groupoids.

Suppose also that $\underline{A}$ is *classical*: by this, we mean that $\mathrm{Fil}^\bullet A$ is a discrete filtered commutative ring filtered by A-submodules $\mathrm{Fil}^i A \subset A$, and also that $I \subset A$ is a locally principal ideal.

Suppose that we have $\mathrm{Fil}^1 A = \varphi^{-1}(I) \subset A$; equivalently, we are assuming that the composition $A \xrightarrow{\varphi} A \to \overline{A}$ factors through an *injective* map $R_A \to \overline{A}$ allowing us to view $\overline{A}$ as an R_A-algebra.

We now claim that $\mathrm{Disp}^{\mathrm{GL}_h,\mu}_{\infty,\underline{A}}(R_A)$ is equivalent to the (classical) category of pairs $(\mathsf{N}, \varphi_{\mathsf{N}})$ where:

(1) N is a finite locally free A-module of rank h;
(2) $\varphi_{\mathsf{N}} : \mathsf{N} \to \mathsf{N}$ is a φ-linear map such that the cokernel of the linearization $\varphi^*\mathsf{N} \to \mathsf{N}$ is killed by I;

(3) The image of the composition $\mathsf{N} \xrightarrow{\varphi_{\mathsf{N}}} \mathsf{N} \to \mathsf{N}/I\mathsf{N}$ is a locally free R_A-module F_{N} of rank d such that the induced map $\overline{A} \otimes_{R_A} F_{\mathsf{N}} \to \mathsf{N}/I\mathsf{N}$ is injective.

In harmony with [1, Definition 4.1.10], we will call such pairs **admissible $\underline{A}$-Dieudonné modules**.

To verify this, first note that, by Proposition 5.19, $\mathrm{Disp}^{\mathrm{GL}_h,\mu}_{\infty,\underline{A}}(R_A)$ is equivalent to the category of pairs $(\mathrm{Fil}^\bullet \mathsf{M}, \xi)$ where:

(1) $\mathrm{Fil}^\bullet \mathsf{M}$ is a filtered $\mathrm{Fil}^\bullet A$-module such that for some étale cover $R_A \to R_{A'}$, we have an isomorphism

$$\mathrm{Fil}^\bullet A' \otimes_{\mathrm{Fil}^\bullet A} \mathrm{Fil}^\bullet \mathsf{M} \xrightarrow{\simeq} \mathrm{Fil}^\bullet A'\{1\}^{\oplus d} \oplus (\mathrm{Fil}^\bullet A')^{\oplus(h-d)}.$$

(2) $\Psi : \mathrm{Fil}^0 \Phi^*_{\pm} \mathrm{Fil}^\bullet \mathsf{M} \xrightarrow{\simeq} \mathsf{M}$ is an isomorphism of A-modules.

We claim that giving the filtered module $\mathrm{Fil}^\bullet \mathsf{M}$ is equivalent to giving a pair $(\mathsf{M}, \mathrm{Fil}^0 \mathsf{M})$ with M a finite locally free A-module of rank h and $\mathrm{Fil}^0 \mathsf{M} \subset \mathsf{M}$ a submodule such that $\mathsf{M}/\mathrm{Fil}^0 \mathsf{M}$ is locally free over R_A of rank d. Indeed, suppose we are given the latter data. Set $M = R_A \otimes_A \mathsf{M}$ and $P = \mathrm{hcoker}(\mathrm{Fil}^0 \mathsf{M} \to \mathsf{M})$, so that we have a quotient map $M \twoheadrightarrow P$ of locally free R_A-modules. Choose a splitting $M \simeq P \oplus Q$, and lift it to a splitting $\mathsf{M} \simeq \mathsf{P} \oplus \mathsf{Q}$ (possible by the Henselianness of $A \twoheadrightarrow R_A$; see [21]). Then the filtered module $\mathrm{Fil}^\bullet \mathsf{M}$ is given by[10]

minuscule

$$\mathrm{Fil}^\bullet \mathsf{M} = (\mathrm{Fil}^\bullet A\{1\} \otimes_A (J^\vee \otimes_A \mathsf{P})) \oplus (\mathrm{Fil}^\bullet A \otimes_A \mathsf{Q}). \tag{5.20.1}$$

It is not hard to see that this is independent of the choice of splitting and yields an inverse to the obvious forgetful functor in the other direction.

Set $\mathsf{M}_\sigma = \mathrm{Fil}^0 \Phi^*_{\pm} \mathrm{Fil}^\bullet \mathsf{M}$. Using the identity (5.20.1), we find that we have

$$\mathrm{Fil}^k(\Phi^*_{\pm} \mathrm{Fil}^\bullet \mathsf{M}) = (I^{\otimes(k+1)} \otimes_A \varphi^*\mathsf{P}) \oplus (I^{\otimes k} \otimes_A \varphi^*\mathsf{Q}),$$

and hence that

mal_forms

$$\mathsf{M}_\sigma = (I \otimes_A \varphi^*\mathsf{P}) \oplus \varphi^*\mathsf{Q} \subset \varphi^*\mathsf{P} \oplus \varphi^*\mathsf{Q} = \varphi^*\mathsf{M} \tag{5.20.2}$$

is the kernel of the map $\varphi^*\mathsf{M} \to \overline{A} \otimes_{R_A} P \simeq \varphi^*\mathsf{P}/I\varphi^*\mathsf{P}$. Note in particular that we have $I\varphi^*\mathsf{M} \subset \mathsf{M}_\sigma$.

Now, set $\mathsf{N} = J \otimes_A \mathsf{M}$, so that we have

$$\varphi^*\mathsf{N} \simeq \varphi^*J \otimes_A \varphi^*\mathsf{M} \simeq J \otimes_A I\varphi^*\mathsf{M} \subset J \otimes_A \mathsf{M}_\sigma.$$

Consider the composition

$$\varphi_{\mathsf{N}} : \mathsf{N} \xrightarrow{m \mapsto \varphi^* m} \varphi^*\mathsf{N} \to J \otimes_A \mathsf{M}_\sigma \xrightarrow[\simeq]{\mathrm{id}\otimes\Psi} J \otimes_A \mathsf{M} = \mathsf{N}.$$

We claim the pair $(\mathsf{N}, \varphi_{\mathsf{N}})$ is an admissible $\underline{A}$-Dieudonné module. Indeed, one checks using (5.20.2) that we have $\varphi_{\mathsf{N}}^{-1}(I\mathsf{N}) = J \otimes_A \mathrm{Fil}^0 \mathsf{M}$. Hence the composition of the projection onto $\mathsf{N}/I\mathsf{N}$ with φ_{N} has image $F_{\mathsf{N}} \simeq J \otimes_A \mathsf{M}/\mathrm{Fil}^0 \mathsf{M}$. Similarly, one finds that the base-change along $R_A \to \overline{A}$ of F_{N} maps isomorphically onto a direct summand of $\mathsf{N}/I\mathsf{N}$.

Conversely, given such a pair, we set

$$\mathsf{M} = J^{\otimes -1} \otimes_A \mathsf{N}\ ;\ \mathrm{Fil}^0 \mathsf{M} = J^{\otimes -1} \otimes_A \varphi_{\mathsf{N}}^{-1}(I\mathsf{N}).$$

Then condition (2) for being an admissible $\underline{A}$-Dieudonné module tells us that $\mathsf{M}/\mathrm{Fil}^0 \mathsf{M}$ is locally free over R_A of rank d. Therefore, we can attach to this data a filtered module $\mathrm{Fil}^\bullet \mathsf{M}$ of type μ. To obtain the isomorphism Ψ, we first observe that we have

$$J \otimes_A I\mathsf{M}_\sigma = (1 \otimes \varphi_{\mathsf{N}})^{-1}(I\mathsf{N}) \subset \varphi^*\mathsf{N}.$$

Since $J \otimes_A I\mathsf{M}_\sigma = \ker(\varphi^*\mathsf{N} \to \overline{A} \otimes_{R_A} F_{\mathsf{N}})$, this is equivalent to condition (3), which asserts that $\overline{A} \otimes_{R_A} F_{\mathsf{N}}$ maps injectively into $\mathsf{N}/I\mathsf{N}$. Now, tensoring this equality with $(J \otimes_A I)^{\otimes -1}$ gives us the desired isomorphism $\mathsf{M}_\sigma \simeq \mathsf{M}$.

[10] The appearance of $J^\vee$ here is necessitated by our definition of the twist $\mathrm{Fil}^\bullet A\{1\}$ in the general situation.

ef_theory

5.21. **Abstract deformation theory for 1-bounded stacks.** In this subsection, we will assume that all our frames are p-adic as in Remark 5.8.

Suppose that $\mathcal{X} = (\mathcal{X}^\diamond, X^0) \to \mathcal{R}(\mathrm{Fil}^\bullet A) \otimes \mathbb{Z}/p^n\mathbb{Z}$ is a 1-bounded stack, equipped with an isomorphism $\xi : \sigma^*\mathcal{X}^\diamond \xrightarrow{\simeq} \tau^*\mathcal{X}^\diamond$ of stacks over $A/^{\mathbb{L}}p^n$. Then for any p-completely étale R_A-algebra $R_{A'}$ we will set

$$\Gamma_{\underline{A}}(\mathcal{X},\xi)(R_{A'}) \overset{\mathrm{defn}}{=} \mathrm{eq}\left(\mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet A') \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}) \underset{\sigma^*}{\overset{\xi\circ\tau^*}{\rightrightarrows}} \mathcal{X}^\diamond(A'/^{\mathbb{L}}p^n)\right).$$

All mapping spaces here are over $\mathcal{R}(\mathrm{Fil}^\bullet A) \otimes \mathbb{Z}/p^n\mathbb{Z}$.

ependence

Remark 5.22. By Proposition 4.30, the source and the target in the equalizer diagram depend only on $\mathrm{Fil}^1 A' \to A'$.

Suppose that we have a map of p-adic frames $q : \underline{B} \to \underline{A}$ such that the associated map $R_B \to R_A$ is a locally nilpotent thickening—that is, $\pi_0(R_B) \to \pi_0(R_A)$ is a surjection with locally nilpotent kernel. Suppose also that the pair $(\mathcal{X}, \xi)$ lifts over $\underline{B}$, and denote this lift by the same symbol.

We now have a canonical equivalence of small étale sites $(R_B)_{\text{ét}} \xrightarrow{\simeq} (R_A)_{\text{ét}}$, and thus a natural base change functor

$$q_* : \Gamma_{\underline{B}}(\mathcal{X},\xi) \to \Gamma_{\underline{A}}(\mathcal{X},\xi)$$

of sheaves on $(R_B)_{\text{ét}}$.

re_frames

Remark 5.23. Suppose that $B \twoheadrightarrow A$ is surjective, and set $K = \mathrm{hker}(B \twoheadrightarrow A)$. Suppose in addition that the map $B \to R_B$ factors through a map $\pi : A \to R_B$ lifting $A \to R_A$; equivalently, $K \to B$ factors through a map $\mathrm{Fil}^1 B \to B$. Note that this is trivially the case if either $B \xrightarrow{\simeq} A$ or $R_B \xrightarrow{\simeq} R_A$.

Let $X^{-,(n)}$ be the Weil restricted attractor stack on R_B-algebras given by

$$C \mapsto \mathrm{Map}(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, C/^{\mathbb{L}}p^n, \mathcal{X}),$$

and let $X^{(n)}$ be the stack $C \mapsto \mathcal{X}^\diamond(C/^{\mathbb{L}}p^n)$.

Then we have a commuting diagram

$$\begin{tikzcd} \Gamma_{\underline{B}}(\mathcal{X},\xi)(R_B) \arrow[r] \arrow[d, "q_*"'] & X^{-,(n)}(R_B) \arrow[d] \\ \Gamma_{\underline{A}}(\mathcal{X},\xi)(R_A) \arrow[r] & X^{-,(n)}(R_A) \times_{X^{(n)}(R_A)} X^{(n)}(R_B) \end{tikzcd}.$$

where the top arrow arises from pullback along the map from Remark 5.7 (for $\underline{B}$), while that on the bottom is from this map (for $\underline{A}$) combined with pullback along π.

There is a trivial situation in which the square from Remark 5.23 is Cartesian.

s_trivial

Proposition 5.24. *Suppose that $q : B \xrightarrow{\simeq} A$, and the lift $A \xrightarrow{\simeq} B \to R_B$ is the obvious one. Then the square in Remark 5.23 obtained via the map $A \xrightarrow{\simeq} B \to R_B$ is Cartesian.*

Proof. For any étale map $R_B \to R_{B'}$ reducing to $R_A \to R_{A'}$, first note that our hypothesis says that $X(B') \xrightarrow{\simeq} X(A')$.

Proposition 4.30 now shows that we have:

$$\Gamma_{\underline{B}}(\mathcal{X},\xi)(R_{B'}) \simeq X^{-,(n)}(R_{B'}) \times_{X^{(n)}(R_{B'})} X^{(n)}(B');$$
$$\Gamma_{\underline{A}}(\mathcal{X},\xi)(R_{A'}) \simeq X^{-,(n)}(R_{A'}) \times_{X^{(n)}(R_{A'})} X^{(n)}(B');.$$

The proposition follows quite formally now from the two previous paragraphs. $\square$

hi1_lifts *Remark* 5.25. With the hypotheses of Remark 5.23, set $\widetilde{\mathrm{Fil}}^1 A = \mathrm{hker}(\pi : A \to R_B)$, which factors through $\mathrm{Fil}^1 A \to A$ via a map $v : \widetilde{\mathrm{Fil}}^1 A \to \mathrm{Fil}^1 A$. This gives us a canonical map $\dot{\varphi}_1 : K \to K$ such that we have a commuting diagram with exact rows:

$$\begin{tikzcd} K \arrow[r] \arrow[d, "\dot{\varphi}_1"'] & \mathrm{Fil}^1 B \arrow[r] \arrow[d, "\mathrm{Fil}^1 \Phi"'] & \widetilde{\mathrm{Fil}}^1 A \arrow[d, "(\mathrm{Fil}^1 \Phi)\circ v"] \\ K \arrow[r] & B \arrow[r] & A. \end{tikzcd}$$

Here is our key technical result, which is simply an animated elaboration of arguments of Lau [30] and Bültel-Pappas [11].

ry_frames **Proposition 5.26.** *With the hypotheses and notation of Remark 5.25, suppose that the following additional conditions hold:*

(1) $\pi_0(\mathrm{Fil}^1 B) \twoheadrightarrow \pi_0(\mathrm{Fil}^1 A)$ *is surjective; equivalently,*

$$\mathrm{Fil}^1 K = \mathrm{hker}(\mathrm{Fil}^1 B \to \mathrm{Fil}^1 A)$$

is connective.

(2) The map $K/^{\mathbb{L}}p \to K/^{\mathbb{L}}p$ *induced by* $\dot{\varphi}_1$ *is locally nilpotent.*

Then the commuting square in Remark 5.23 obtained from condition (2) is Cartesian.

ry_frames **Corollary 5.27.** *With the hypotheses as in Proposition 5.26, if* $R_B \xrightarrow{\simeq} R_A$*, then* q_* *is an equivalence.*

The proof of the proposition will be given below. We begin with a couple of remarks on the conditions involved in its statement.

urjective *Remark* 5.28. By p-completeness, to see that $B \to A$ is surjective, it is enough to know that $B/^{\mathbb{L}}p \to A/^{\mathbb{L}}p$ is so. Moreover, in the situation of Remark 5.23, we have a commuting diagram with exact rows

$$\begin{tikzcd} K \arrow[r] \arrow[d] & \mathrm{Fil}^1 B \arrow[r] \arrow[d] & \widetilde{\mathrm{Fil}}^1 A \arrow[d] \\ \mathrm{Fil}^1 K \arrow[r] \arrow[d] & \mathrm{Fil}^1 B \arrow[r] \arrow[d] & \mathrm{Fil}^1 A \arrow[d] \\ K \arrow[r] & B \arrow[r] & A \end{tikzcd}$$

where the composition of the vertical arrows on the left is isomorphic to the identity on K. In other words, we have a section $K \to \mathrm{Fil}^1 K$ splitting the fiber sequence

$$\overline{K}[-1] \to \mathrm{Fil}^1 K \to K,$$

and so we have $\mathrm{Fil}^1 K \xrightarrow{\simeq} K \oplus \overline{K}[-1]$. This shows in particular that condition (1) of Proposition 5.26 holds if and only if $\overline{K}$ is 1-connective.

5.29. We will assume from now on that the hypotheses of Proposition 5.26 hold. Suppose that we have a $\underline{B}$-gauge $(\mathcal{M}, \xi)$ of level 1 corresponding to a filtered module $\mathrm{Fil}^\bullet \mathsf{M}$ over $\mathrm{Fil}^\bullet B/^{\mathbb{L}}p$. The associated $\underline{A}$-gauge

$$(\mathcal{M}_A, \xi_A) \overset{\mathrm{defn}}{=} \underline{A} \otimes_{\underline{B}} (\mathcal{M}, \xi)$$

corresponds to a filtered $\mathrm{Fil}^\bullet A/^{\mathbb{L}}p$-module $\mathrm{Fil}^\bullet \mathsf{M}_A$. Set

$$R\Gamma_{\underline{B}}(\mathcal{M},\xi)(R_B) \overset{\mathrm{defn}}{=} \mathrm{hker}(\mathrm{Fil}^0 \mathsf{M} \xrightarrow{\xi\circ\sigma^*-\tau^*} \mathsf{M}),$$

and similarly for the basechange over $\underline{A}$.

nilpotent **Lemma 5.30.** *Suppose that $R_B \xrightarrow{\simeq} R_A$ and that the restriction of $\mathcal{M}$ over $B\mathbb{G}_m \times \operatorname{Spec} R_B$ is 1-bounded. Then the map*

$$q_* : R\Gamma_{\underline{B}}(\mathcal{M},\xi)(R_B) \to R\Gamma_{\underline{A}}(\mathcal{M}_A,\xi_A)(R_A)$$

is an isomorphism.

Proof. Since $\mathcal{M}$ is 1-bounded, if $\mathrm{Fil}^\bullet M$ is the filtered R_B-module obtained from $\mathrm{Fil}^\bullet \mathsf{M}$, we have

$$\mathrm{Fil}^0 \mathsf{M} \simeq \mathrm{Fil}^0 M \times_M \mathsf{M}.$$

Equivalently, the natural map $\mathrm{gr}^{-1} \mathsf{M} \to \mathrm{gr}^{-1} M$ is an equivalence. This can be seen as a special case of Proposition 4.30.

Set $\mathrm{Fil}^\bullet \mathsf{M}_K = \mathrm{hker}(\mathrm{Fil}^\bullet \mathsf{M} \to \mathrm{Fil}^\bullet \mathsf{M}_A)$. Then, since $R_B = R_A$, we find that $\mathrm{Fil}^0 \mathsf{M}_K \simeq \mathsf{M}_K \simeq K \otimes_B \mathsf{M}$, and that

$$\mathrm{hker}(q_*) \simeq \mathrm{hker}(\mathsf{M}_K \xrightarrow{\sigma^*_\xi-\mathrm{id}} \mathsf{M}_K).$$

Here, σ^*_ξ is the endomorphism of M_K obtained as the composition

$$\mathsf{M}_K \simeq \mathrm{Fil}^0 \mathsf{M}_K \to \mathrm{Fil}^0 \mathsf{M} \xrightarrow{\xi\circ\sigma^*} \mathsf{M},$$

where the final arrow is obtained via

$$\mathrm{Fil}^0 \mathsf{M} = R\Gamma(\mathcal{R}(\mathrm{Fil}^\bullet B)\otimes \mathbb{Z}/p\mathbb{Z},\mathcal{M}) \xrightarrow{\sigma^*} R\Gamma(\mathrm{Spec}(B/^{\mathbb{L}}p),\sigma^*\mathcal{M}) = \mathsf{M}_\sigma \underset{\simeq}{\xrightarrow{\xi}} R\Gamma(\mathrm{Spec}(B/^{\mathbb{L}}p),\tau^*\mathcal{M}) = \mathsf{M}.$$

The module M_σ, as described in Remark 5.18, is obtained via filtered base change along $\Phi_\pm$ and then taking the degree-0 filtered piece. Since we are in characteristic p and $I = (p)$, the map $\Phi_\pm$ factors through the associated graded algebra $\mathrm{gr}^\bullet A$, and the isomorphism ξ gives maps $\overline{\xi}_i : \mathrm{gr}^i \mathsf{M} \to \mathsf{M}$ for each i.

We now claim that σ^*_ξ is given by the composition

$$\mathsf{M}_K \simeq K \otimes_B \mathsf{M} \to K \otimes_B \mathrm{gr}^{-1} \mathsf{M} \xrightarrow{\dot{\varphi}_1\otimes\overline{\xi}_{-1}} K \otimes_B \mathsf{M} \simeq \mathsf{M}_K.$$

Assuming this, it follows that σ^*_ξ is locally nilpotent, which in turn will imply the lemma, since, for any locally nilpotent endomorphism ψ of $N \in \mathrm{Mod}_{\mathbb{Z}}$, the endomorphism $\psi - \mathrm{id}$ is a self-equivalence.

To see the local nilpotence, it is enough to show that the composition

$$K \otimes_B \mathrm{gr}^{-1} M \xrightarrow{\dot{\varphi}_1\otimes\overline{\xi}_{-1}} K \otimes_B \mathsf{M}_K \to K \otimes_B \mathrm{gr}^{-1} M$$

is locally nilpotent, which comes down to the assumed nilpotence of $\dot{\varphi}_1$ as an endomorphism of $K/^{\mathbb{L}}p$.

Let us prove the claim about σ^*_ξ. Since the map σ^*_ξ for $\mathrm{Fil}^0 \mathsf{M}$ factors through $\mathrm{gr}^0 \mathsf{M}$ and the analogous map for M_A factors through $\mathrm{gr}^0 \mathsf{M}_A$, we find that the restriction to M_K must factor through $\mathrm{gr}^0 \mathsf{M}_K$. One can check (using Lemma 4.28 for instance) that we have natural fiber sequences

$$\mathrm{gr}^1 B \otimes_{R_A} \mathrm{gr}^{-1} M \to \mathrm{gr}^0 \mathsf{M} \to \mathrm{gr}^0 M \;;\; \mathrm{gr}^1 A \otimes_{R_A} \mathrm{gr}^{-1} M \to \mathrm{gr}^0 \mathsf{M}' \to \mathrm{gr}^0 M,$$

showing that we have

$$\mathrm{gr}^0 \mathsf{M}_K \simeq \mathrm{gr}^1 K \otimes_{R_A} \mathrm{gr}^{-1} M.$$

From this, the desired claim is clear, since we have a canonical map $K \simeq \mathrm{Fil}^1 K \to \mathrm{gr}^1 K$. $\square$

We begin by proving the corollary.

Proof of Corollary 5.27. We want to show that the map $\Gamma_{\underline{B}}(\mathcal{X},\xi) \to \Gamma_{\underline{A}}(\mathcal{X},\xi)$ is an isomorphism of sheaves on $(R_B)_{\text{ét}}$.

For each $m \le n$, let $\mathcal{X}_m = (\mathcal{X}_m^\diamond, X_m^0) \to \mathcal{R}(\mathrm{Fil}^\bullet B) \otimes \mathbb{Z}/p^m\mathbb{Z}$ be the restriction of $\mathcal{X}$: it is also equipped with an isomorphism $\xi_m : \sigma^*\mathcal{X}_m^\diamond \to \tau^*\mathcal{X}_m^\diamond$.

Given $x \in \Gamma_{\underline{B}}(\mathcal{X}_1,\xi_1)(R_{B'})$, we obtain a 1-bounded $\underline{B}$-gauge $(\mathcal{M}(\mathcal{X}_1)_x, \xi_x)$ of level 1 by pulling back the tangent complex of $\mathcal{X}_1^\diamond$ along x.

For $m < n$, the filtered deformation theory from (4.5) gives a canonical Cartesian diagram of étale sheaves

$$\begin{tikzcd}
\Gamma_{\underline{B}}(\mathcal{X}_{m+1},\xi_{m+1}) \arrow[r] \arrow[d] & \Gamma_{\underline{B}}(\mathcal{X}_m,\xi_m) \arrow[d] \\
\Gamma_{\underline{B}}(\mathcal{X}_m,\xi_m) \arrow[r] & \Gamma_{\underline{B}}(\mathcal{M}(\mathcal{X}_1)[1]) \times_{\Gamma_{\underline{B}}(\mathcal{X}_1,\xi_1)} \Gamma_{\underline{B}}(\mathcal{X}_m,\xi_m)
\end{tikzcd}.$$

Here, $\Gamma_{\underline{B}}(\mathcal{M}(\mathcal{X}_1)[1])$ is the sheaf over $\Gamma_{\underline{B}}(\mathcal{X}_1,\xi_1)$ whose fiber over x is the space

$$\Gamma_{\underline{B}}(\mathbf{V}(\mathcal{M}(\mathcal{X})_x[1]), \xi_x[1]).$$

The reader can find a few more details for this kind of argument in the proof of Proposition 8.16.

Given that we have a similar diagram with $\underline{B}$ replaced with $\underline{A}$, we see, using Lemma 5.30, that we can assume $n = 1$.

We can now conclude using a similar deformation theoretic argument, first applied to the Postnikov filtrations on $\mathrm{Fil}^\bullet B/^{\mathbb{L}}p$ and $\mathrm{Fil}^\bullet A/^{\mathbb{L}}p$ to reduce us to the case where everything is discrete, and then to the case where $B/pB \to A/pA$ is itself a square-zero thickening. □

Proof of Proposition 5.26. By Remark 5.28, for condition (1) of the Proposition to hold, $\overline{K}$ must be 1-connective, and we must have $\mathrm{Fil}^1 K \xrightarrow{\simeq} K \oplus \overline{K}[-1]$. Now, set $\widetilde{\mathrm{Fil}}^1 A = \mathrm{hker}(A \twoheadrightarrow R_B)$, and also set

$$\widetilde{\mathrm{Fil}}^i A = \mathrm{Fil}^i A \times_{\mathrm{Fil}^1 A} \widetilde{\mathrm{Fil}}^1 A,$$

where the fiber product is taking place is in Mod_A.

Then the exact sequence

$$\widetilde{\mathrm{Fil}}^i A \to \mathrm{Fil}^i A \to \overline{K}$$

shows that $\widetilde{\mathrm{Fil}}^i A$ is connective, and the filtration $\widetilde{\mathrm{Fil}}^\bullet A$ with the restricted Frobenius lifts equips A with the structure of a frame $\underline{\widetilde{A}}$ so that we have a factoring of maps

$$q : \underline{B} \xrightarrow{\tilde{q}} \underline{\widetilde{A}} \xrightarrow{q'} \underline{A}.$$

Note that we have $\mathrm{gr}^0 \widetilde{A} \simeq R_B$.

Since q' is an equivalence of the underlying δ-rings, Proposition 5.24 shows we have

$$\mathrm{Disp}^{G,\mu}_{n,\underline{\widetilde{A}}}(R_B) \xrightarrow{\simeq} \mathrm{Disp}^{G,\mu}_{n,\underline{A}}(R_A) \times_{X_\mu^{(n)}(R_B \to R_A)} BP_\mu^{-,(n)}(R_B).$$

Therefore, by replacing q with $\tilde{q}$, we are reduced to showing that q_* is an equivalence when $R_B \xrightarrow{\simeq} R_A$, which is the content of the already known Corollary 5.27. □

6. The stacks of Drinfeld and Bhatt-Lurie

6.1. **Transmutation.** Suppose that we have a map $\pi : Z \to Y$ of p-adic formal prestacks such that Z is a **relative ring prestack** over Y: For us, this will mean that we have specified a lift of the associated functor

$$\mathrm{CRing}^{p\text{-nilp}}_{Y/} \xrightarrow{(C,y)\mapsto Z((C,y))} \mathrm{Spc}$$

to a presheaf valued in CRing, which we will denote by the same symbol.

Here, $\mathrm{CRing}^{p\text{-nilp}}_{/Y}$ is the ∞-category of pairs (C, y) with $C \in \mathrm{CRing}^{p\text{-nilp}}$ and $y \in Y(C)$, and $Z((C, y))$ is the fiber of $Z(C)$ over y. Then, for any $R \in \mathrm{CRing}^{p\text{-nilp}}$, its **transmutation with respect to** π is the p-adic formal prestack over Y given by

$$\begin{aligned} \mathfrak{Tr}_\pi(R) : \mathrm{CRing}^{p\text{-nilp}}_{/Y} &\to \mathrm{Spc} \\ (C, y) &\mapsto \mathrm{Map}_{\mathrm{CRing}}(R, Z((C, y))) \end{aligned}$$

This gives us a limit preserving functor

$$\mathrm{CRing}^{p\text{-nilp,op}} \xrightarrow{\mathrm{Spec}\, R \mapsto \mathfrak{Tr}_\pi(R)} \mathrm{PStk}_{/Y}.$$

6.2. **Cartier-Witt divisors and prismatizations.** Here, we quickly recall the story of (derived) absolute prismatizations from [8, §8].

To begin, we have the p-adic formal prestack $\mathbb{Z}_p^{\mathbb{\Delta}}$ (the notation is from [5]) of Cartier-Witt divisors, denoted WCart in [7]. For $R \in \mathrm{CRing}^{p\text{-nilp}}$, $\mathbb{Z}_p^{\mathbb{\Delta}}(R)$ parameterizes surjective maps $\pi : W(R) \twoheadrightarrow \overline{W(R)}$ of animated rings such that two properties hold:

- $I = \mathrm{hker}(\pi)$ is a locally free $W(R)$-module of rank 1;
- The map $\pi_0(I) \simeq I \otimes_{W(R)} W(\pi_0(R)) \to W(\pi_0(R))$ is a Cartier-Witt divisor in the sense of [7, §3.1.1].

The second condition means that, Zariski-locally on $\mathrm{Spec}\, R$, we have a $W(\pi_0(R))$-linear isomorphism $\pi_0(I) \simeq W(\pi_0(R))$ such that the composition $W(\pi_0(R)) \simeq \pi_0(I) \to W(\pi_0(R))$ is given by multiplication by a **distinguished element** $d \in W_{\mathrm{dist}}(\pi_0(R))$, given in Witt coordinates by $(d_0, d_1, \dots)$ with $d_0 \in \pi_0(R)$ nilpotent mod-p and with $d_1 \in \pi_0(R)^\times$.

In this situation, we will call the map $I \to W(R)$ a **Cartier-Witt divisor** over R.

This description shows (see [8, Proposition 8.4]):

sentation **Proposition 6.3.** *We have $\mathbb{Z}_p^{\mathbb{\Delta}} \simeq W_{\mathrm{dist}}/W^\times$, where W_{dist} is represented by the formal spectrum of $\mathbb{Z}_p[x_0, x_1^{\pm 1}, x_2, \dots]^\wedge_{(x_0,p)}$, and inherits the $W^\times$-action from the natural one on W, where W is represented by the formal spectrum of $\mathbb{Z}_p[x_0, x_1, \dots]^\wedge_p$. In particular, $\mathbb{Z}_p^{\mathbb{\Delta}}$ is* classical.

Over $\mathbb{Z}_p^{\mathbb{\Delta}}$, we have the tautological relative ring prestack $\mathbb{G}_a^{\mathbb{\Delta}}$ given by $(W(R) \xrightarrow{\pi} \overline{W(R)}) \mapsto \overline{W(R)}$. Transmutation with respect to this (see (6.1)) now gives a functorial assignment $R \to R^{\mathbb{\Delta}}$ from $\mathrm{CRing}^{p\text{-nilp}}$ to $\mathrm{PStk}_{/\mathbb{Z}_p^{\mathbb{\Delta}}}$. Concretely, $R^{\mathbb{\Delta}}$ associates to any $(W(C) \twoheadrightarrow \overline{W(C)}) \in \mathbb{Z}_p^{\mathbb{\Delta}}(C)$ the space $\mathrm{Map}_{\mathrm{CRing}}(R, \overline{W(C)})$. We call $R^{\mathbb{\Delta}}$ the **prismatization** of R.

ion_of_Fp *Remark* 6.4. We have a canonical equivalence $\mathrm{Spf}\, \mathbb{Z}_p \xrightarrow{\simeq} \mathbb{F}_p^{\mathbb{\Delta}}$ induced by the Cartier-Witt divisor $W(\mathbb{F}_p) = \mathbb{Z}_p \xrightarrow{p} \mathbb{Z}_p = W(\mathbb{F}_p)$.

omorphism *Remark* 6.5. There is a canonical 'Frobenius lift' $\varphi : \mathbb{Z}_p^{\mathbb{\Delta}} \to \mathbb{Z}_p^{\mathbb{\Delta}}$ arising from the map $F : W \to W$, which carries a Cartier-Witt divisor $I \to W(R)$ to $F^*I \to W(R)$

atization *Remark* 6.6. If (A, I) is an animated prism as in [8, §2], then there is a canonical map $\iota_{(A,I)} : \mathrm{Spf}\, A \to \mathbb{Z}_p^{\mathbb{\Delta}}$ associating to each p-nilpotent A-algebra C, the Cartier-Witt divisor $I \otimes_A W(C) \to W(C)$. Here, $A \to W(C)$ is the canonical lift of $A \to C$ afforded by the δ-ring structure on A. If we have a map $R \to A/I$—that is, if (A, I) lifts to an object in the (animated) prismatic site of R—then $\iota_{(A,I)}$ admits a lift to a map to $R^{\mathbb{\Delta}}$.

6.7. **The Hodge-Tate locus.** Let $\hat{\mathbb{A}}^1$ be the p-adic formal completion of $\mathbb{A}^1$, equipped with the inverse of the usual action of $\mathbb{G}_m$ (as in (4.2)). Then, $\hat{\mathbb{A}}^1/\mathbb{G}_m$ parameterizes line bundles $\mathcal{L}$ equipped with a cosection $t : \mathcal{L} \to \mathscr{O}$ that is nilpotent mod-p, meaning that for some (hence any) local trivialization of $\mathcal{L}$, t is given by a section of $\mathscr{O}$ that is nilpotent mod-p.

Then the natural map $W_{\mathrm{dist}} \to \hat{\mathbb{A}}^1$ descends to a map $\mathbb{Z}_p^{\mathbb{\Delta}} \to \hat{\mathbb{A}}^1/\mathbb{G}_m$, and the **Hodge-Tate locus** $\mathbb{Z}_p^{\mathrm{HT}}$ is defined to be the fiber product (derived or classical: both are the same in this case)

$$\mathbb{Z}_p^{\mathrm{HT}} = \mathbb{Z}_p^{\mathbb{\Delta}} \times_{\hat{\mathbb{A}}^1/\mathbb{G}_m} B\mathbb{G}_m.$$

This is a closed substack of $\mathbb{Z}_p^{\Delta}$ with locally invertible ideal sheaf, which Bhatt-Lurie make a very detailed study of in [7, §3.4]. They show that $\mathbb{Z}_p^{\mathrm{HT}}$ has a somewhat concrete description.

To explain this, first note that we have a canonical map $\mathrm{Spf}\,\mathbb{Z}_p \to \mathbb{Z}_p^{\mathrm{HT}}$ corresponding to the Cartier-Witt divisor $W(\mathbb{Z}_p) \xrightarrow{V(1)} W(\mathbb{Z}_p)$.

ate_locus **Proposition 6.8.** *The above map presents $\mathbb{Z}_p^{\mathrm{HT}}$ as the formal classifying stack $B\mathbb{G}_m^{\sharp}$ over $\mathrm{Spf}\,\mathbb{Z}_p$. In particular, it is a flat surjection. Moreover, there is a natural equivalence*

$$\mathrm{Spf}\,\mathbb{Z}_p \times_{\mathbb{Z}_p^{\Delta}} \mathbb{F}_p^{\Delta} \xrightarrow{\simeq} \mathbb{G}_m^{\sharp} \times \mathrm{Spec}\,\mathbb{F}_p.$$

Proof. The first assertion is [7, Theorem 3.4.13]; see also [16, Lemma 4.5.2].

For the second, note that the left hand side is a canonically trivial $\mathbb{G}_m^{\sharp}$-torsor over

$$\mathbb{F}_p^{\mathrm{HT}} \stackrel{\mathrm{defn}}{=} \mathbb{Z}_p^{\mathrm{HT}} \times_{\mathbb{Z}_p^{\Delta}} \mathbb{F}_p^{\Delta},$$

but this base is canonically identified with the closed subscheme $\mathrm{Spec}\,\mathbb{F}_p \subset \mathrm{Spf}\,\mathbb{Z}_p$ via the equivalence of Remark 6.4. $\square$

c:nygaard

6.9. **The Nygaard filtered prismatization.** The underived story of the Nygaard filtered prismatization is explained in Bhatt's notes [5, §5] following Drinfeld in [16, §5]. We recall this story here and briefly explain how to make sense of these constructions in the context of animated rings.

For this, we will need to review some constructions of W-modules in the ∞-category of fppf sheaves over a given $R \in \mathrm{CRing}^{p\text{-nilp}}$; we will simply call them *W-modules over R*:

- Given an invertible R-module L, we can take the divided power envelope of the identity section within the vector group scheme $\mathbf{V}(L)$ to obtain the W-module $\mathbf{V}(L)^{\sharp}$. Here, the W-action is via the map $W \to \mathbb{G}_a$.
- Given any W-module M, we obtain a new W-module F_*M via restriction along the map $F : W \to W$.
- Given an invertible $W(R)$-module I, we obtain an associated W-module $I \otimes_{W(R)} W$ that assigns to every R-algebra S the $W(S)$-module $I \otimes_{W(R)} W(S)$.
- Suppose that we are given an invertible R-module L and a section $u : R \to L$. Note that u gives rise to a map $u^{\sharp} : \mathbb{G}_{a,R}^{\sharp} \to \mathbf{V}(L)^{\sharp}$, and define $E((L,u))$ to be the cofiber sequence of W-modules at the bottom of the following diagram

$$\begin{tikzcd} \mathbb{G}_{a,R}^{\sharp} \arrow[r] \arrow[d, "u^{\sharp}"'] & W \arrow[r, "F"] \arrow[d] & F_*W \arrow[d, equal] \\ \mathbf{V}(L)^{\sharp} \arrow[r] & M((L,u)) \arrow[r] & F_*W, \end{tikzcd}$$

where the left half is a pushout square.

sible_ext *Remark* 6.10. In the last construction, note that giving a splitting for the bottom row is equivalent to giving a map of W-modules $W \to \mathbf{V}(L)^{\sharp}$ restricting to $u^{\sharp}$ on $\mathbb{G}_{a,R}^{\sharp}$. This is in turn equivalent to giving a section of $\mathbf{V}(L)^{\sharp}$ such that the restriction of the associated map $\mathbb{G}_a \to \mathbf{V}(L)^{\sharp}$ along the natural map $\mathbb{G}_a^{\sharp} \to \mathbb{G}_a$ is $u^{\sharp}$.

We can now consider $\mathbb{Z}_p^{\mathcal{N}}$, the Nygaard filtered prismatization of $\mathrm{Spf}\,\mathbb{Z}_p$. This is a stack living over $\mathbb{A}^1/\mathbb{G}_m$ and, given $R \in \mathrm{CRing}^{p\text{-nilp}}$ and $(L \xrightarrow{t} R) \in (\mathbb{A}^1/\mathbb{G}_m)(R)$, the fiber of $\mathbb{Z}_p^{\mathcal{N}}$ over $(L \xrightarrow{t} R)$ parameterizes tuples (M,d) with the following properties.

- M is an **admissible** W-module over R. That is, it sits in a cofiber sequence

$$\mathbf{V}(L)^{\sharp} \to M \to F_*M',$$

which, flat-locally on $\mathrm{Spec}\,R$, is equivalent to a fiber sequence of the form $E((L,u))$. This implies in particular that M' is equivalent to $I \otimes_{W(R)} W$ for some invertible $W(R)$-module I.

- $d : M \to W$ is a map of W-modules that sits in a diagram

$$\begin{tikzcd} \mathbf{V}(L)^\sharp \arrow[r] \arrow[d, "t^\sharp"] & M \arrow[r] \arrow[d, "d"] & F_*M' \arrow[d, "F_*d'"] \\ \mathbb{G}_a^\sharp \arrow[r] & W \arrow[r, "F"'] & F_*W \end{tikzcd}$$

where $d' : M' \to W$ is obtained from a Cartier-Witt divisor $I \to W(R)$.

We will usually write $(M \xrightarrow{d} W) \in \mathbb{Z}_p^{\mathcal{N}}(R)$ for this datum and refer to it as a **filtered Cartier-Witt divisor** over R. Over $\mathbb{Z}_p^{\mathcal{N}}$, we now have the prestack $\mathbb{G}_a^{\mathcal{N}}$ that, over $(M \xrightarrow{d} W) \in \mathbb{Z}_p^{\mathcal{N}}(R)$, parameterizes sections of the quotient $(W/_dM)(R)$, where $W/_dM$ is the cone of the map d.

classical

Proposition 6.11. *The p-adic formal prestacks $\mathbb{Z}_p^{\mathcal{N}}$ and $\mathbb{G}_a^{\mathcal{N}}$ are classical. Moreover, $\mathbb{G}_a^{\mathcal{N}}$ can be canonically lifted to a relative ring stack over $\mathbb{Z}_p^{\mathcal{N}}$.*

Proof. By construction $\mathbb{Z}_p^{\mathcal{N}}$ lives over $\mathbb{A}^1/\mathbb{G}_m \times \mathbb{Z}_p^{\Delta}$. Moreover, by definition, pulling it back along the flat cover $W_{\mathrm{dist}} \times \mathbb{A}^1 \to \mathbb{Z}_p^{\Delta} \times \mathbb{A}^1/\mathbb{G}_m$, we obtain the prestack X whose fiber over $(d', t) \in W_{\mathrm{dist}}(R) \times \mathbb{A}^1(R)$ parameterizes tuples (M, d) where we have a commuting diagram

$$\begin{tikzcd} \mathbb{G}_a^\sharp \arrow[r] \arrow[d, "t^\sharp"] & M \arrow[r] \arrow[d, "d"] & F_*W \arrow[d, "F_*d'"] \\ \mathbb{G}_a^\sharp \arrow[r] & W \arrow[r, "F"'] & F_*W \end{tikzcd}$$

where both rows are cofiber sequences, and where there exists a faithfully flat map $R \to S$ such that the restriction of M over S is equivalent to $M((S, u))$ for some $u \in S$. By Remark 6.10, the stack of such W-modules M is parameterized by $\mathbb{G}_a/\mathbb{G}_a^\sharp$. Therefore, pulling X back further along the faithfully flat map $\mathbb{G}_a \to \mathbb{G}_a/\mathbb{G}_a^\sharp$, we obtain the prestack

$$Y \to W_{\mathrm{dist}} \times \mathbb{A}^1 \times \mathbb{G}_a$$

whose fiber over $(d', t, u) \in W_{\mathrm{dist}}(R) \times R \times R$ parameterizes maps of W-modules $d : W \to W$—equivalently, sections $d \in W(R)$—such that we have a commuting diagram

$$\begin{tikzcd} \mathbb{G}_a^\sharp \arrow[r] \arrow[d, "t^\sharp \circ u^\sharp"] & W \arrow[r, "F"] \arrow[d, "d"] & F_*W \arrow[d, "F_*d'"] \\ \mathbb{G}_a^\sharp \arrow[r] & W \arrow[r, "F"'] & F_*W \end{tikzcd}$$

After all is said and done, the classicality of $\mathbb{Z}_p^{\mathcal{N}}$ is now reduced to the classicality of Y, which comes down to the assertion that the maps of formal schemes

$$W \xrightarrow{(F,\pi)} W \times \mathbb{G}_a \ ; \ W_{\mathrm{dist}} \times \mathbb{A}^1 \times \mathbb{G}_a \xrightarrow{(\mathrm{id}, m)} W \times \mathbb{G}_a$$

are p-completely Tor-independent. Here, $m : \mathbb{A}^1 \times \mathbb{G}_a \xrightarrow{(t,u) \mapsto tu} \mathbb{G}_a$ is the multiplication map.

But these are actually maps of *affine* formal schemes corresponding to maps of p-complete rings given by:

$$\alpha : \mathbb{Z}_p[T, x_0, x_1, \ldots]_p^\wedge \xrightarrow{T \mapsto x_0, x_i \mapsto F^*x_i} \mathbb{Z}_p[x_0, x_1, \ldots]_p^\wedge$$
$$\beta : \mathbb{Z}_p[T, x_0, x_1, \ldots]_p^\wedge \xrightarrow{T \mapsto ut, x_i \mapsto x_i} \mathbb{Z}_p[u, t, x_0, x_1^{\pm 1}, \ldots]_{(x_0,p)}^\wedge.$$

The first map is the composition of the map $\alpha' : T \mapsto x_0, x_i \mapsto x_i$ with the *flat* map F^*. Therefore, it is enough to check that α' and β are p-completely Tor-independent. This comes down to the concrete (and easy) assertion that the element

$$ut - x_0 \in \mathbb{F}_p[u, t, x_0, x_1^{\pm 1}, \dots]^{\wedge}_{(x_0)}$$

is a non-zero divisor.

It remains to check the assertions about $\mathbb{G}_a^{\mathcal{N}}$. We can check that it is classical after flat base change to the classical affine formal scheme $Y = \operatorname{Spf} R$. Here, we are looking at a quotient of the universal pushout $M((R,u))$—which is classical, being the pushout of classical schemes—by the classical scheme W. This of course yields a classical stack.

We now know that $\mathbb{G}_a^{\mathcal{N}}$ is the left Kan extension of $\mathbb{G}_a^{\mathcal{N},\mathrm{cl}}$. Therefore, it is enough to know that the latter admits the structure of a ring stack over $\mathbb{Z}_p^{\mathcal{N},\mathrm{cl}}$. This is the content of [5, Proposition 5.3.8], which follows [16, Lemma 3.12.12]. □

Definition 6.12. For any $R \in \mathrm{CRing}^{p\text{-nilp}}$, we now define the **Nygaard filtered prismatization** $R^{\mathcal{N}}$ to be the transmutation of $\operatorname{Spec} R$ with respect to the p-adic formal ring stack $\mathbb{G}_a^{\mathcal{N}}$ over $\mathbb{Z}_p^{\mathcal{N}}$. That is, it is the p-adic formal prestack over $\mathbb{Z}_p^{\mathcal{N}}$ that associates with each $C \in \mathrm{CRing}^{p\text{-nilp}}$ and $(M \xrightarrow{d} W) \in \mathbb{Z}_p^{\mathcal{N}}(C)$ the space $\mathrm{Map}_{\mathrm{CRing}}(R, (W/_d M)(C))$.

ion_of_Fp *Remark* 6.13. The prestack $\mathbb{F}_p^{\mathcal{N}}$ admits a quite explicit description explained in [5, Prop. 5.4.2]. Suppose that we are given a filtered Cartier-Witt divisor $(M \xrightarrow{d} W) \in \mathbb{Z}_p^{\mathcal{N}}(R)$ with M an extension of F_*M' by $\mathbf{V}(L)^\sharp$, and associated with a cosection $t : L \to R$. Then the existence of a map $\mathbb{F}_p \to (W/_d M)(R)$ implies that $F_*M' \to F_*W$ is isomorphic to $p : F_*W \to F_*W$, and also implies that M is canonically equivalent to $M((L,u))$ for some section $u : R \to L$, with $t \circ u = p$. Therefore we have

$$\mathbb{F}_p^{\mathcal{N}} \simeq Z(ut - p)/\mathbb{G}_m,$$

where $Z(ut - p) \subset \mathbb{A}^1 \times \mathbb{G}_a = \operatorname{Spf} \mathbb{Z}_p[t, u]$ is the closed formal subscheme defined by the equation $ut - p$.

6.14. **The locus $(\mathbb{Z}_p^{\mathcal{N}})_{(t=0)}$.** We will now recall some properties of $(\mathbb{Z}_p^{\mathcal{N}})_{(t=0)}$, the 'associated graded' for the filtered Cartier-Witt stack.

There is a canonical map of formal p-adic stacks $\mathbb{G}_a \to (\mathbb{Z}_p^{\mathcal{N}})_{(t=0)}$ obtained as follows. To $u \in R = \mathbb{G}_a(R)$ we associate the admissible W-module $M_u = M((R,u))$, and the composition $M_u \to F_*W \xrightarrow{V} W$ gives us a filtered Cartier-Witt divisor over R.

t=0_locus **Proposition 6.15.** *The above map presents $(\mathbb{Z}_p^{\mathcal{N}})_{(t=0)}$ as the quotient $\mathbb{G}_a/(\mathbb{G}_a^\sharp \rtimes \mathbb{G}_m)$ as stacks over $\mathbb{A}^1/\mathbb{G}_m$. In particular, $\mathbb{G}_a \to (\mathbb{Z}_p^{\mathcal{N}})_{(t=0)}$ is a flat cover. Moreover, there is a natural isomorphism of $\mathbb{G}_a$-schemes*

$$\mathbb{G}_a \times_{\mathbb{Z}_p^{\mathcal{N}}} \mathbb{F}_p^{\mathcal{N}} \xrightarrow{\simeq} (\mathbb{G}_a \times \mathbb{G}_a^\sharp) \times \operatorname{Spec} \mathbb{F}_p$$

Proof. The first assertion is [16, Lemma 5.12.4]; see also [5, Proposition 5.3.7]. The semi-direct product here arises from the natural action of $\mathbb{G}_m$ on $\mathbb{G}_a^\sharp$, and the combined action on $\mathbb{G}_a$ is the expected one.

For the last assertion, note that the left hand side is a $\mathbb{G}_a^\sharp \rtimes \mathbb{G}_m$-torsor over $(\mathbb{F}_p^{\mathcal{N}})_{(t=0)}$, which by Remark 6.13 is canonically equivalent to $\mathbb{G}_a/\mathbb{G}_m \times \operatorname{Spec} \mathbb{F}_p$.

Concretely, given an $\mathbb{F}_p$-algebra R and a section $u \in \mathbb{G}_a(R)$, the fiber of the left hand side over u is the space of automorphisms of the extension $M((R,u))$, which Remark 6.10 shows to be canonically equivalent to $\mathbb{G}_a^\sharp(R)$. □

ification

6.16. **The de Rham and Hodge-Tate embeddings and syntomification.** There are two canonical open immersions $j_{\mathrm{dR}}, j_{\mathrm{HT}} : \mathbb{Z}_p^{\Delta} \to \mathbb{Z}_p^{\mathcal{N}}$, called the **de Rham** and **Hodge-Tate** embeddings. These are described in [16, §5.3, 5.6], and in [5, §5.3].

To begin, observe that we have a canonical map $\pi : \mathbb{Z}_p^{\mathcal{N}} \to \mathbb{Z}_p^{\Delta}$ carrying a filtered Cartier-Witt divisor $(M \to W)$ to the Cartier-Witt divisor corresponding to $(M' \to W)$.

The open immersion j_{dR} is now simply the open locus $\mathbb{Z}_p^{\Delta} \times_{\mathbb{A}^1/\mathbb{G}_m} (\mathbb{G}_m/\mathbb{G}_m)$ parameterizing filtered Cartier-Witt divisors $(M \xrightarrow{d} W) \in \mathbb{Z}_p^{\Delta}(R)$, where the underlying map $t : L \to R$ is an isomorphism. This means that d is just the pullback of $(F_*M' \to F_*W)$ along $F : W \to F_*W$. This open locus maps isomorphically to $\mathbb{Z}_p^{\Delta}$ via the map π.

The open immersion j_{HT} associates with every Cartier-Witt divisor $(I \to W(R))$ over R the filtered Cartier-Witt divisor $(I \otimes_{W(R)} W \to W)$. The composition $\pi \circ j_{\mathrm{HT}}$ is canonically isomorphic to the Frobenius endomorphism $\varphi : \mathbb{Z}_p^{\mathbb{\Delta}} \to \mathbb{Z}_p^{\mathbb{\Delta}}$.

We now define the prestack $\mathbb{Z}_p^{\mathrm{syn}}$ to be the coequalizer of the immersions $j_{\mathrm{dR}}, j_{\mathrm{HT}}$. Practically, what this means is that we have

$$\mathrm{QCoh}(\mathbb{Z}_p^{\mathrm{syn}}) \xrightarrow{\simeq} \mathrm{eq}\left(\mathrm{QCoh}(\mathbb{Z}_p^{\mathcal{N}}) \underset{j_{\mathrm{HT}}^*}{\overset{j_{\mathrm{dR}}^*}{\rightrightarrows}} \mathrm{QCoh}(\mathbb{Z}_p^{\mathbb{\Delta}})\right)$$

The process of transmutation now yields for every $R \in \mathrm{CRing}^{p\text{-nilp}}$ open immersions $j_{\mathrm{dR}}, j_{\mathrm{HT}} : R^{\mathbb{\Delta}} \to R^{\mathcal{N}}$, and we now define the **syntomification** of R, R^{syn}, to be their coequalizer. By construction we have a canonical structure map $R^{\mathrm{syn}} \to \mathbb{Z}_p^{\mathrm{syn}}$.

uil-kisin

6.17. **The Breuil-Kisin twist.** Here we will describe a canonical line bundle $\mathcal{O}^{\mathrm{syn}}\{1\}$ on $\mathbb{Z}_p^{\mathrm{syn}}$ called the **Breuil-Kisin twist**. Specifying such a line bundle is equivalent to specifying a line bundle $\mathcal{O}^{\mathcal{N}}\{1\}$ on $\mathbb{Z}_p^{\mathcal{N}}$ equipped with an isomorphism

$$j_{\mathrm{dR}}^*\mathcal{O}^{\mathcal{N}}\{1\} \xrightarrow{\simeq} j_{\mathrm{HT}}^*\mathcal{O}^{\mathcal{N}}\{1\}$$

of line bundles on $\mathbb{Z}_p^{\mathbb{\Delta}}$.

We begin by considering the pullback of the tautological line bundle on $B\mathbb{G}_m$: this gives a line bundle $\mathcal{O}^{B\mathbb{G}_m}\{1\}$ over $\mathbb{Z}_p^{\mathcal{N}}$. Note that we have a canonical trivialization

rham_triv

$$j_{\mathrm{dR}}^*\mathcal{O}^{B\mathbb{G}_m}\{1\} \xrightarrow{\simeq} \mathscr{O}_{\mathbb{Z}_p^{\mathbb{\Delta}}}. \tag{6.17.1}$$

Via the Hodge-Tate embedding, we obtain a canonical equivalence

_pullback

$$j_{\mathrm{HT}}^*\mathcal{O}^{B\mathbb{G}_m}\{1\} \xrightarrow{\simeq} \mathscr{I}_{\mathbb{Z}_p^{\mathrm{HT}}} \tag{6.17.2}$$

where the right hand side is the ideal sheaf of the Hodge-Tate divisor.

Next, we consider a canonical line bundle $\mathcal{O}^{\mathbb{\Delta}}\{1\}$ on $\mathbb{Z}_p^{\mathbb{\Delta}}$ that is characterized up to isomorphism by any of the following properties:

- ([7, p. 2.2.11]) For any *transversal* prism (A, I) (see [7, §2.1]), equipped with its Frobenius lift $\varphi_A : A \to A$, the pullback of $\mathcal{O}^{\mathbb{\Delta}}\{1\}$ to $\mathrm{Spf}\, A$ under the map $\iota_{(A,I)}$ from Remark 6.6 is isomorphic to an inverse limit
$$A\{1\} = \varprojlim_k I_k/I_{k+1},$$
where $I_k = I\varphi_A^*(I)\cdots(\varphi_A^{k-1})^*(I)$, and the maps $I_k/I_{k+1} \to I_{k-1}/I_k$ are induced by dividing the natural maps by p.
- ([16, §4.8, 4.9]) We have the line bundle $\mathscr{O}(\mathbb{Z}_p^{\mathrm{HT}})$ on $\mathbb{Z}_p^{\mathbb{\Delta}}$ corresponding to the Hodge-Tate divisor. The endomorphism $\varphi^* - \mathrm{id}$ of the Picard group $\mathrm{Pic}(\mathbb{Z}_p^{\mathbb{\Delta}})$ is an equivalence, and we take $\mathcal{O}^{\mathbb{\Delta}}\{1\}$ to be the preimage of $\mathscr{O}(\mathbb{Z}_p^{\mathrm{HT}})$.
- ([7, p. 9.1.6]) Let $f : (\mathbb{P}^1_{\mathbb{Z}_p})^{\mathbb{\Delta}} \to \mathbb{Z}_p^{\mathbb{\Delta}}$ be the map of prismatizations arising from the structure morphism $\mathbb{P}^1_{\mathbb{Z}_p} \to \mathrm{Spf}\,\mathbb{Z}_p$. Then, we have $\mathcal{O}^{\mathbb{\Delta}}\{-1\} \simeq R^2 f_*\mathscr{O}$.

There is a canonical isomorphism

elta_isom

$$\varphi^*\mathcal{O}^{\mathbb{\Delta}}\{1\} \xrightarrow{\simeq} \mathscr{O}(\mathbb{Z}_p^{\mathrm{HT}}) \otimes \mathcal{O}^{\mathbb{\Delta}}\{1\} \tag{6.17.3}$$

of line bundles over $\mathbb{Z}_p^{\mathbb{\Delta}}$.

We now set

$$\mathcal{O}^{\mathcal{N}}\{1\} = \mathcal{O}^{B\mathbb{G}_m}\{1\} \otimes \pi^*\mathcal{O}^{\mathbb{\Delta}}\{1\}.$$

Since $\pi \circ j_{\mathrm{dR}} = \mathrm{id}$ and $\pi \circ j_{HT} = \varphi$, combining (6.17.3), (6.17.1) and (6.17.2) shows that this line bundle does admit a canonical descent to a line bundle $\mathscr{O}^{\mathrm{syn}}\{1\}$ over $\mathbb{Z}_p^{\mathrm{syn}}$.

_sections

6.18. **The canonical sections x_{dR} and $x_{\mathrm{dR}}^{\mathcal{N}}$.** For any $R \in \mathrm{CRing}^{p\text{-nilp}}$, we have canonical maps

$$x_{\mathrm{dR}} : \operatorname{Spec} R \to R^{\Delta} \ ; \ x_{\mathrm{dR}}^{\mathcal{N}} : \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R \to R^{\mathcal{N}}$$

with the following properties:

- The composition
$$\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R \xrightarrow{x_{\mathrm{dR}}^{\mathcal{N}}} R^{\mathcal{N}} \xrightarrow{t} \mathbb{A}^1/\mathbb{G}_m$$
is the canonical structure map.
- The restriction of $x_{dR}^{\mathcal{N}}$ to the open point $\operatorname{Spec} R$ is isomorphic to x_{dR}.
- We have a commuting square
$$\begin{tikzcd} \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R \arrow[r, "x_{dR}^{\mathcal{N}}"] \arrow[d] & R^{\mathcal{N}} \arrow[d, "\pi"] \\ \operatorname{Spec} R \arrow[r, "x_{dR}"'] & R^{\Delta} \end{tikzcd} .$$

The map x_{dR} corresponds to the Cartier-Witt divisor $(W(R) \xrightarrow{p} W(R))$ with the map $R \to W(R)/^{\mathbb{L}}p$ the one through which the Frobenius endomorphism of $W(R)/^{\mathbb{L}}p$ canonically factors. Alternatively, there are equivalences of abstract animated rings

$$W(R)/^{\mathbb{L}}p \xrightarrow{\simeq} F_*W(R)/^{\mathbb{L}}p \xrightarrow{\simeq} \mathbb{G}_a^{dR}(R),$$

and the desired map is the evaluation of the natural map $\mathbb{G}_a \to \mathbb{G}_a^{dR}$ on R.

The map $x_{dR}^{\mathcal{N}}$ associates with every cosection $t : L \to C$ of a line bundle L over an R-algebra C the filtered Cartier-Witt divisor

$$M(t) = F_*W \oplus \mathbf{V}(L)^{\sharp} \xrightarrow{d=(V,t^{\sharp})} W,$$

where the quotient $W/_dM(t)$ is also a quotient of $W/VW \simeq \mathbb{G}_a$, giving us the map

$$R \to C = \mathbb{G}_a(C) \to (W/_dM(t))(C).$$

s_lifting

6.19. **Relationship with divided powers.** The next observation will be used later to formulate abstract Grothendieck-Messing type statements.

ed_powers

Lemma 6.20. *Suppose that we have a divided power thickening $R' \twoheadrightarrow R$ in $\mathrm{CRing}^{f,p\text{-}comp}$. Then the canonical point $x_{\mathrm{dR},R'}$ from* (6.18) *factors canonically through a point $\tilde{x}_{\mathrm{dR},R'} : \operatorname{Spec} R' \to R^{\Delta}$.*

Proof. The claim here is that the canonical map $R' \to \mathbb{G}_a^{\mathrm{dR}}(R')$ admits a factoring through R that depends only on the divided powers on the fiber J of $R' \twoheadrightarrow R$. Since the constructions and the conclusion are compatible with sifted colimits, we reduce to the case where $R' \twoheadrightarrow R$ is a classical divided power thickening of flat p-complete $\mathbb{Z}_p$-algebras.

Now, it suffices to construct a canonical lift of R'-modules $J \to \mathbb{G}_a^{\sharp}(R')$. This is obtained from the following sequence of maps:

$$\begin{aligned} J &\xrightarrow{\simeq} \mathrm{Map}_{\mathrm{Mod}_{R'}}(R', J) \\ &\to \mathrm{Map}_{\mathrm{CRing}_{R'/}}(\Gamma_{R'}(R'), \Gamma_{R'}(J)) \\ &\to \mathrm{Map}_{\mathrm{CRing}_{R'/}}(\Gamma_{R'}(R'), R') \simeq \mathbb{G}_a^{\sharp}(R'), \end{aligned}$$

where in the last row we have used the map $\Gamma_{R'}(J) \to R'$ arising from the divided powers on J. $\square$

6.21. **Descent.** The stacks we are concerned with here carry p-quasisyntomic covers to covers in the fpqc topology.

Definition 6.22. A map $R \to S$ of p-complete animated commutative rings is **p-quasisyntomic** (or simply **quasisyntomic**) if it is p-completely flat (that is, $S/^{\mathbb{L}}p$ is flat over $R/^{\mathbb{L}}p$), and if $\mathbb{L}_{S/R}$ has p-complete Tor amplitude $[-1,0]$: that is, $\mathbb{L}_{S/R} \otimes \mathbb{F}_p$ has Tor amplitude $[-1,0]$ over $S/^{\mathbb{L}}p$.

We will say that $R \to S$ is **quasisyntomic cover** if it is quasisyntomic and $S/^{\mathbb{L}}p$ is faithfully flat over $R/^{\mathbb{L}}p$.

These properties are invariant under base-change via maps $R \to R'$ of p-complete animated commutative rings.

Example 6.23. The key example of a quasisyntomic map is

$$\mathbb{Z}_p[T]^\wedge \to \mathbb{Z}_p[T^{1/p^\infty}]^\wedge,$$

where the superscript $^\wedge$ denotes p-adic completion.

rjections

Proposition 6.24. *Suppose that we have $g : R \to S$ in $\mathrm{CRing}^{p\text{-}comp}$. Then the associated maps $g^{\mathcal{N}} : S^{\mathcal{N}} \to R^{\mathcal{N}}$ and $g^{\Delta} : S^{\Delta} \to R^{\Delta}$ are surjective in the p-completely flat topology under either of the following assumptions.*

1. *g is a p-completely étale cover;*
2. *More generally, when g is a quasisyntomic cover.*

Proof. In case (1), we will actually see that g^{Δ} and $g^{\mathcal{N}}$ are p-completely étale covers. For the former, this is shown in [8, Remark 3.9]. The main point is that, for any $C \in \mathrm{CRing}^{p\text{-nilp}}$ and a Cartier-Witt divisor $I \to W(C)$ with quotient $\overline{W(C)}$, the small p-completely étale site of $\overline{W(C)}$ is equivalent to that of $W(C)$, and hence to that of C. This shows that, given $R \to \overline{W(C)}$, we have

$$S \otimes_R \overline{W(C)} \simeq \overline{W(C')}$$

for a canonical étale map $C \to C'$.

Let us now look at $g^{\mathcal{N}}$: given $(M \xrightarrow{d} W)$ in $\mathbb{Z}_p^{\mathcal{N}}(C)$, $W/_d M$ is a square-zero extension of $\overline{W}$ by $\mathrm{hcoker}(t^\sharp)$, where $t^\sharp : \mathbf{V}(L)^\sharp \to \mathbb{G}_a^\sharp$ is the map associated with a cosection of a line bundle over $\mathrm{Spec}\, C$. The map

$$(W/_d M)(C) \to \overline{W(C)}$$

is a square zero extension as long as C is $\mathbb{G}_a^\sharp$-acyclic. Therefore, for such C, the same argument as in the previous paragraph shows that

$$S \otimes_R (W/_d M)(C) \simeq (W/_d M)(C')$$

for a canonical étale map $C \to C'$. In general, we can choose a faithfully flat map $C \to D$ such that D is $\mathbb{G}_a^\sharp$-acyclic, and so the conclusion from the previous sentence follows for C by fpqc descent for étale C-algebras.

In case (2), the result for prismatizations is [8, Lemma 6.3]. For the Nygaard filtered prismatizations, just as in case (1), we can assume that we are looking at a point $(M \xrightarrow{d} W, R \to (W/_d M)(C)) \in R^{\mathcal{N}}(C)$ such that C is $\mathbb{G}_a^\sharp$-cyclic, so that the corresponding animated commutative ring $(W/_d M)(C)$ is a square-zero extension of $\overline{W(C)}$ by $J \overset{\mathrm{defn}}{=} \mathrm{hcoker}(t^\sharp)(C)$ for some cosection $t : L \to C$. We would like to show that there is a flat cover $C \to C'$ such that we have a lift $S \to (W/_d M)(C')$. By the known flat surjectivity of prismatizations, we can assume that we already have a lift $\tilde{f} : S \to \overline{W(C)}$. The obstruction to lifting it to a map of R-algebras $S \to (W/_d M)(C)$ is classified by an S-linear map $o(\tilde{f}) : \mathbb{L}_{S/R} \to J[1]$. Therefore, we have to show that the lift $\tilde{f}$ can be chosen so that $o(\tilde{f})$ is nullhomotopic.

$\square$

:f-finite

6.25. We will consider the full ∞-subcategory

$$\mathrm{CRing}^f \to \mathrm{CRing}$$

of animated commutative rings satisfying the following condition: The module of differentials $\Omega^1_{(\pi_0(R)/p\pi_0(R))/\mathbb{F}_p}$ is finitely generated over $\pi_0(R)$. The following algebras lie in this subcategory:

1. Any R such that $\pi_0(R)$ is a finitely generated $\mathbb{Z}/p^m\mathbb{Z}$-algebra for some $m \geq 1$

(2) Any complete local Noetherian $\mathbb{Z}_p$-algebra R with residue field κ admitting a finite p-basis (that is, $[\kappa : \kappa^p] < \infty$); in fact, a complete local Noetherian $\mathbb{F}_p$-algebra belongs to CRing^f if and only if its residue field has finite p-basis.
(3) Any *semiperfect* $\mathbb{F}_p$-algebra R—that is, $R \in \mathrm{CRing}_{\mathbb{F}_p/}$ such that the Frobenius endomorphism of R is surjective.
(4) Any $\mathbb{Z}[1/p]$-algebra R: This, however, will be irrelevant for what follows.

erf_qsynt

Corollary 6.26. *For any $R \in \mathrm{CRing}^{f,p\text{-}comp}$, there exists a quasisyntomic cover $R \to R_\infty$ with $R_\infty^{\otimes_R m}$ semiperfectoid for all m, and such that $R_\infty^{\mathbb{\Delta}} \to R^{\mathbb{\Delta}}$ and $R_\infty^{\mathcal{N}} \to R^{\mathcal{N}}$ are surjective in the flat topology.*

Proof. This is as in the proof of [8, Theorem 7.20]: One begins by choosing a map $\mathbb{Z}_p[T_1,\dots,T_r]^\wedge \to R$ such that $\Omega^1_{\pi_0(R/pR)/\mathbb{F}_p}$ is generated by the images of $dT_1,\dots,dT_r$, and sets

$$R_\infty = \mathbb{Z}_p[T_1^{1/p^\infty},\dots,T_r^{1/p^\infty}]^\wedge \otimes_{\mathbb{Z}_p[T_1,\dots,T_r]^\wedge} R.$$

□

ohomology

6.27. **Prismatic cohomology.** We will need the relationship between the cohomology of the stacks defined above and relative (Nygaard filtered) prismatic cohomology as constructed in [9] and [7].

For this, recall that, considering $(\mathbb{Z}_p, p\mathbb{Z}_p)$ as a bounded prism, we can associate to $R \in \mathrm{CRing}_{\mathbb{F}_p/}$ its **relative prismatic cohomology** $\mathbb{\Delta}_{R/\mathbb{Z}_p}$. This can be obtained—see [7, Construction 4.1.3]—as the left Kan extension of its restriction to polynomial $\mathbb{F}_p$-algebras R, where, by [7, Theorem 4.6.1], we have an isomorphism

$$\mathbb{\Delta}_{R/\mathbb{Z}_p} \xrightarrow[\simeq]{\gamma^{\mathrm{crys}}_{\mathbb{\Delta}}} R\Gamma_{\mathrm{crys}}(R/\mathbb{Z}_p)$$

to the classical crystalline cohomology of R with respect to $\mathbb{Z}_p$.

The base change $\overline{\mathbb{\Delta}}_{R/\mathbb{Z}_p}$ over $\mathbb{F}_p$ is the **Hodge-Tate cohomology**, which is equipped with a canonical exhaustive increasing filtration $\mathrm{Fil}^{\mathrm{conj}}_\bullet \overline{\mathbb{\Delta}}_{R/\mathbb{Z}_p}$ supported in non-positive degrees, and a canonical equivalence

$$\mathrm{gr}^{\mathrm{conj}}_i \overline{\mathbb{\Delta}}_{R/\mathbb{Z}_p} \xrightarrow{\simeq} \wedge^i \mathbb{L}_{R/\mathbb{F}_p}[-i]$$

for each $i \geq 0$; see [7, Remark 4.1.7]. The identification of prismatic cohomology with crystalline cohomology in the previous paragraph gives an equivalence $\overline{\mathbb{\Delta}}_{R/\mathbb{Z}_p} \xrightarrow{\simeq} \mathrm{dR}_{R/\mathbb{F}_p}$, where the right hand side is derived de Rham cohomology.

miperfect

Remark 6.28. As explained in [7, Remark 4.6.5], the existence of the isomorphism $\gamma^{\mathrm{crys}}_{\mathbb{\Delta}}$ holds for any animated $\mathbb{F}_p$-algebra R if we replace the right hand side with *derived* crystalline cohomology, which we will denote by $\mathbb{\Delta}^{\mathrm{crys}}_{R/\mathbb{Z}_p}$.

In particular, if R is *semiperfect*, $R^\flat$ is its perfection, and $A_{\mathrm{crys}}(R) \twoheadrightarrow R$ is the p-completed animated divided power envelope of the natural map $W(R^\flat) \twoheadrightarrow R$, then we have canonical isomorphisms

atization

(6.28.1) $$\mathbb{\Delta}_{R/\mathbb{Z}_p} \xrightarrow{\simeq} \mathbb{\Delta}^{\mathrm{crys}}_{R/\mathbb{Z}_p} \xrightarrow{\simeq} \mathbb{\Delta}^{\mathrm{crys}}_{R/R^\flat} \simeq A_{\mathrm{crys}}(R).$$

Here, the third object is the relative derived crystalline cohomology of R over $R^\flat$, and the last isomorphism is an animated enhancement of a result of Illusie [39, Prop. 4.64]. The composition of these isomorphisms is φ-semilinear over $W(R^\flat)$ via the corresponding isomorphism $\mathbb{\Delta}_{R^\flat/\mathbb{Z}_p} \xrightarrow{\simeq} W(R^\flat)$.

In particular, note that $\mathbb{\Delta}_{R/\mathbb{Z}_p}$ is itself a p-complete animated $W(R)$-algebra, and if R is *quasiregular*, so that $\mathbb{L}_{R/\mathbb{F}_p}[-1]$ is flat over R, then it is actually a *classical* $W(R)$-algebra. In general, as explained in [8, §7], it is only a p-complete derived ring.

omparison

Theorem 6.29 (Comparison with prismatic cohomology and affineness)**.** *Suppose that R belongs to $\mathrm{CRing}^f_{\mathbb{F}_p/}$. Then there is a natural equivalence*

$$\mathbb{\Delta}_{R/\mathbb{Z}_p} \xrightarrow{\simeq} R\Gamma(R^{\mathbb{\Delta}}, \mathscr{O}).$$

Moreover, if R is semiperfect then the associated map $R^{\mathbb{\Delta}} \to \mathrm{Spf}\, \mathbb{\Delta}_{R/\mathbb{Z}_p}$ is an equivalence.

Proof. The first assertion is [8, Theorem 7.20(1)]. The only comment to make here is that in the notation of *loc. cit*, the stack $\mathrm{WCart}_{\mathrm{Spec}\,R/\mathbb{Z}_p}$ is identified with $R^{\mathbb{\Delta}}$: this has to do with the fact that the only Cartier-Witt divisor with $\overline{W(C)}$ an $\mathbb{F}_p$-algebra is $W(C) \xrightarrow{p} W(C)$.

The second assertion is a special case of [8, Corollary 7.18]. ◻

crys_comp

6.30. **Nygaard filtered prismatic cohomology.** We will now abbreviate $\mathbb{\Delta}_{R/\mathbb{Z}_p}$ to $\mathbb{\Delta}_R$. There exists a canonical lift $\mathbb{\Delta}_R$ to a filtered object $\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbb{\Delta}_R$—the **Nygaard filtered prismatic cohomology**—defined once again as the left Kan extension of such a lift for polynomial $\mathbb{F}_p$-algebras R. For such polynomial algebras, the filtration is obtained—see [7, p. 116]—via quasisyntomic descent from quasiregular semiperfect $\mathbb{F}_p$-algebras, where, under the isomorphisms (6.28.1), it corresponds to the filtration

$$\mathrm{Fil}^i_{\mathcal{N}} A_{\mathrm{crys}}(R) = \varphi^{-1}\left(p^i A_{\mathrm{crys}}(R)\right).$$

The Nygaard filtration is characterized by the following properties:

- The Frobenius endomorphism φ of $\mathbb{\Delta}_R$ (arising functorially from that of R) admits a canonical lift to a map
$$\mathrm{Fil}^\bullet\Phi : \mathrm{Fil}^\bullet_{\mathcal{N}}\mathbb{\Delta}_R \to \mathrm{Fil}^\bullet_p\mathbb{\Delta}_R;$$
- For any $i \in \mathbb{Z}$, the natural map
$$\mathrm{gr}^i_{\mathcal{N}}\mathbb{\Delta}_R \to \mathrm{gr}^\bullet_p\mathbb{\Delta}_R \simeq \overline{\mathbb{\Delta}}_R$$
factors through a canonical isomorphism
$$\mathrm{gr}^i_{\mathcal{N}}\mathbb{\Delta}_R \xrightarrow{\simeq} \mathrm{Fil}^{\mathrm{conj}}_i\overline{\mathbb{\Delta}}_R.$$
- If R is a finitely generated polynomial algebra over $\mathbb{F}_p$, then $\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbb{\Delta}_R$ is complete.

red_frame

Lemma 6.31. *Suppose that R is semiperfectoid. Then, in the notation of* (5.5), $\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbb{\Delta}_R$ *underlies a* frame $\underline{\mathbb{\Delta}}_R$ *with* $\mathrm{gr}^0_{\mathcal{N}}\mathbb{\Delta}_R \simeq R$.

Proof. The only thing to be checked is that $\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbb{\Delta}_R$ is a filtered animated commutative ring and that the map $\mathrm{Fil}^\bullet\Phi$ is a map of filtered animated commutative rings. This is clear when R is quasiregular, and to know this in general it is easiest to note that the construction via right Kan extension from qrsp rings followed by left Kan extension from polynomial R_0-algebras now endows $\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbb{\Delta}_R$ with the structure of a filtered *derived* commutative ring in general, which specializes when R is semiperfectoid to the structure desired. ◻

The following description of the Nygaard filtered prismatization of semiperfect rings will be very important for us.

perf_crys

Theorem 6.32. *Suppose that R is semiperfectoid. Then there is a canonical equivalence*

$$R^{\mathcal{N}} \xrightarrow{\simeq} \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbb{\Delta}_R)$$

of p-adic formal stacks over $\mathbb{Z}_p^{\mathcal{N}}$. Under this equivalence, j_{dR} and j_{HT} are identified with, in the notation of (5.5), *the maps τ and σ associated with the frame $\underline{\mathbb{\Delta}}_R$.*

Proof. When R is quasiregular, this is sketched in [5, Theorem 5.5.10], with a full proof to appear. The idea works in our general setting as well. Let us recall at least some of the details.

We begin by noting that the description of $\mathbb{F}_p^{\mathcal{N}}$ in (6.13) shows that it can be canonically identified with $\mathcal{R}(\mathrm{Fil}^\bullet_p\mathbb{Z}_p)$. We will use this to identify $\mathrm{QCoh}(\mathbb{F}_p^{\mathcal{N}})$ with the ∞-category of The first step now is to show that, if S is a polynomial algebra over $\mathbb{F}_p$, and $\pi : S^{\mathcal{N}} \to \mathbb{F}_p^{\mathcal{N}}$ is the natural map, then $\pi_*\mathscr{O}_{S^{\mathcal{N}}}$ is in $\mathrm{QCoh}(\mathbb{F}_p^{\mathcal{N}})$, and is canonically isomorphic to $\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbb{\Delta}_S$; see [5, Theorem 3.3.5].

With this in hand, noting that $R \mapsto \mathrm{Fil}^\bullet_{\mathcal{N}}\mathbb{\Delta}_R$ is left Kan extended from polynomial $\mathbb{F}_p$-algebras, we obtain for *any* $R \in \mathrm{CRing}^f_{\mathbb{F}_p/}$, a canonical arrow

$$\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbb{\Delta}_R \to \pi_*\mathscr{O}_{R^{\mathcal{N}}}$$

of filtered derived rings, which, when R is semiperfect is adjoint to a map

$$R^{\mathcal{N}} \to \mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}}\mathbb{\Delta}_R)$$

of p-adic formal stacks over $\mathbb{F}_p^{\mathcal{N}}$. Bhatt now shows that this is an isomorphism by looking at the pullback of the map over the stratification of $\mathbb{F}_p^{\mathcal{N}}$ given by

$$\left(\mathbb{F}_p^{\mathcal{N}}\right)_{t\neq 0} \simeq \mathbb{F}_p^{\Delta}\ ;\ \left(\mathbb{F}_p^{\mathcal{N}}\right)_{t=0,u\neq 0} \simeq \mathbb{F}_p^{\Delta}\times \operatorname{Spec}\mathbb{F}_p\ ;\ \left(\mathbb{F}_p^{\mathcal{N}}\right)_{t=0,u=0} \simeq B\mathbb{G}_m\times \operatorname{Spec}\mathbb{F}_p,$$

and showing that this pullback is an isomorphism over each stratum.

Over the first two strata, this essentially follows from Theorem 6.29. To verify that we have an isomorphism over the third, we can pullback along the cover $\operatorname{Spec}\mathbb{F}_p \to B\mathbb{G}_m\times\operatorname{Spec}\mathbb{F}_p$. Here, the right hand side is now the affine scheme represented by the *Hodge cohomology* $\wedge^\bullet\mathbb{L}_{R/\mathbb{F}_p}[-\bullet]$, while the left is the stack over $\operatorname{Spec} R$ representing maps of R-algebras $R\to C\oplus B\mathbb{G}_a^\sharp(C)$, where the right hand side is a trivial square-zero extension of C. By the defining property of the cotangent complex, we can identify this stack with that parameterizing maps of R-modules $\mathbb{L}_{R/\mathbb{F}_p}\to B\mathbb{G}_a^\sharp(C)$.

Now, we have the observation that the R-algebra $\wedge^\bullet\mathbb{L}_{R/\mathbb{F}_p}[-\bullet]$ is precisely the divided power algebra $\Gamma_R(\mathbb{L}_{R/\mathbb{F}_p}[-1])$, and so, by Lemma 3.17, the corresponding affine scheme over R represents maps of R-modules $\mathbb{L}_{R/\mathbb{F}_p}[-1]\to\mathbb{G}_a^\sharp(C)$ or equivalently maps $\mathbb{L}_{R/\mathbb{F}_p}\to\mathbb{G}_a^\sharp(C)[1]$. Since $\mathbb{L}_{R/\mathbb{F}_p}$ is 1-connective, this agrees with the space of maps $\mathbb{L}_{R/\mathbb{F}_p}\to B\mathbb{G}_a^\sharp(C)$. ☐

geometric

Corollary 6.33. *For any $R\in\mathrm{CRing}^{f,p\text{-}nilp}$, R^{Δ} and $R^{\mathcal{N}}$ are* quasi-geometric *formal stacks in the sense of [37, §9.1.1]. More precisely, if $R\to R_\infty$ is as in Corollary 6.26, then for any $n\geq 1$:*

(1) The maps

$$R_\infty^{\Delta}\otimes(\mathbb{Z}/p^n\mathbb{Z})\to R^{\Delta}\otimes(\mathbb{Z}/p^n\mathbb{Z})\ ;\ R_\infty^{\mathcal{N}}\otimes(\mathbb{Z}/p^n\mathbb{Z})\to R^{\mathcal{N}}\otimes(\mathbb{Z}/p^n\mathbb{Z})$$

are surjections of fpqc sheaves that are relatively affine and faithfully flat;

(2) $R_\infty^{\Delta}\otimes(\mathbb{Z}/p^n\mathbb{Z})$ is a derived formal affine scheme and $R_\infty^{\mathcal{N}}\otimes(\mathbb{Z}/p^n\mathbb{Z})$ is a derived formal Artin 1-stack over $\mathbb{Z}_p$;

Proof. Most of this follows from Proposition 6.24 and Theorem 6.32. The only thing to be checked is that

$$R_\infty^{\mathcal{N}}\otimes(\mathbb{Z}/p^n\mathbb{Z})\to R^{\mathcal{N}}\otimes(\mathbb{Z}/p^n\mathbb{Z})$$

is relatively affine and faithfully flat, which reduces to the assertion that

$$\left(\frac{\mathbb{Z}_p[T^{1/p^\infty},U^{1/p^\infty}]^\wedge}{(T-U)}\right)^{\mathcal{N}}\otimes(\mathbb{Z}/p^n\mathbb{Z})\to(\mathbb{Z}_p[T^{1/p^\infty}]^\wedge)^{\mathcal{N}}\otimes(\mathbb{Z}/p^n\mathbb{Z})$$

is a faithfully flat, relatively affine map of formal Artin stacks. The relative affineness is clear from Theorem 6.32, while the faithful flatness can be checked mod-p, where it reduces via the same theorem to the following concrete assertion:

Observation 6.33.1. *Let $S=\mathbb{F}_p[T^{1/p^\infty}]$, $\overline{S}=\mathbb{F}_p[T^{1/p^\infty}]/(T)$ and let $D\twoheadrightarrow\overline{S}$ be the divided power envelope of $S\twoheadrightarrow\overline{S}$ equipped with the divided power filtration $\mathrm{Fil}^\bullet_{\mathrm{PD}}D$. Then $\mathrm{gr}^\bullet_{\mathrm{PD}}D$ is free over $\overline{S}$.*

☐

ilpotence

6.34. **A nilpotence result.** We will need a certain nilpotence result for applications to the abstract Grothendieck-Messing theory formulated in (8.18). Suppose that $S'\twoheadrightarrow S$ is a square-zero thickening in $\mathrm{CRing}_{\mathbb{F}_p/}$; then the natural map

$$\Delta_{S'}\xrightarrow{\simeq}\Delta_{S'/\mathbb{F}_p}\xrightarrow{\simeq}\mathrm{dR}_{S'/\mathbb{F}_p}\to S'$$

factors canonically through Δ_S. This can be viewed as a special case of Lemma 6.20, where we equip the thickening with the trivial divided power structure.

Set

$$K_{S'\twoheadrightarrow S}\overset{\mathrm{defn}}{=}\mathrm{hker}\left(\Delta_{S'}\to\Delta_S\right)$$

Just as in Remark 5.25, the maps

$$\mathrm{Fil}^1\Phi:\mathrm{Fil}^1_{\mathcal{N}}\Delta_{S'}\to\Delta_{S'}\ ;\ \mathrm{Fil}^1\Phi:\mathrm{Fil}^1_{\mathcal{N}}\Delta_S\to\Delta_S$$

now give rise to a map

$$\dot{\varphi}_1 : K_{S' \twoheadrightarrow S} \to K_{S' \twoheadrightarrow S}.$$

nilpotent **Lemma 6.35.** *The map $\dot{\varphi}_1$ induces a locally nilpotent endomorphism of $K_{S' \twoheadrightarrow S}/^{\mathbb{L}} p$.*

Proof. All constructions involved here are compatible with sifted colimits, as is the conclusion. So we can reduce to the case where we have $S' = \mathbb{F}_p[X,Y]/(Y^2) \to \mathbb{F}_p[X,Y]/(Y) = S$ for some finite sets of variables X and Y. Since the construction $S \mapsto \mathrm{Fil}^\bullet_{\mathcal{N}} \mathbb{\Delta}_S$ respects finite colimits, we can further reduce to the consideration of the case of the thickening

$$\mathbb{F}_p[T]/(T^2) \twoheadrightarrow \mathbb{F}_p.$$

Here, we claim that there is a nullhomotopy $\dot{\varphi}_1^2 \simeq 0$. For this, using quasisyntomic descent along the map $\mathbb{F}_p[T] \to \mathbb{F}_p[T^{1/p^\infty}]$, we can reduce still further to the case of a thickening of the form

$$\mathbb{F}_p[T_1^{1/p^\infty}, \ldots, T_r^{1/p^\infty}]/(T_1^2, T_1 - T_i :\ i \geq 2) \twoheadrightarrow \mathbb{F}_p[T_1^{1/p^\infty}, \ldots, T_r^{1/p^\infty}]/(T_1, T_1 - T_i :\ i \geq 2),$$

where, using the compatibility with finite colimits once again, we are finally reduced to the consideration of

$$\mathbb{F}_p[T^{1/p^\infty}]/(T^2) = S' \twoheadrightarrow \mathbb{F}_p[T^{1/p^\infty}]/(T) = S$$

equipped with trivial divided powers. In this case, one now checks using Remark 6.28 that we have

'toR_expl (6.35.1)
$$\overline{\mathbb{\Delta}}_{C'} \simeq \mathbb{F}_p[T^{1/p^\infty}]\langle T^{2/p}\rangle \to \mathbb{F}_p[T^{1/p^\infty}]\langle T^{1/p}\rangle \simeq \mathbb{\Delta}_S/^{\mathbb{L}} p, \tag{6.35.1}$$

where we have set

$$\mathbb{F}_p[T^{1/p^\infty}]\langle T^{2/p}\rangle \overset{\text{defn}}{=} \mathbb{F}_p[T^{1/p^\infty}] \otimes_{T^{2/p} \leftarrow\!\shortmid T, \mathbb{F}_p[T]} \mathbb{F}_p\langle T\rangle \ ;\ \mathbb{F}_p[T^{1/p^\infty}]\langle T^{1/p}\rangle \overset{\text{defn}}{=} \mathbb{F}_p[T^{1/p^\infty}] \otimes_{T^{1/p} \leftarrow\!\shortmid T, \mathbb{F}_p[T]} \mathbb{F}_p\langle T\rangle.$$

Explicitly, we have coordinates

$$\mathbb{F}_p[T^{1/p^\infty}]\langle T^{2/p}\rangle \xrightarrow[\simeq]{\gamma_p^n(T^{2/p}) \mapsto U_n} \mathbb{F}_p[T^{1/p^\infty}][U_i : i \in \mathbb{Z}_{\geq 0}]/(U_0 - T^{2/p}, U_i^p)$$
$$\mathbb{F}_p[T^{1/p^\infty}]\langle T^{1/p}\rangle \xrightarrow[\simeq]{\gamma_p^n(T^{1/p}) \mapsto V_n} \mathbb{F}_p[T^{1/p^\infty}][V_i : i \in \mathbb{Z}_{\geq 0}]/(V_0 - T^{1/p}, V_i^p),$$

and the map (6.35.1) is given by the map of $\mathbb{F}_p[T^{1/p^\infty}]$-algebras

$$\mathbb{F}_p[T^{1/p^\infty}][U_i : i \in \mathbb{Z}_{\geq 0}]/(U_0 - T^{2/p}, U_i^p) \xrightarrow{U_i \mapsto 0} \mathbb{F}_p[T^{1/p^\infty}][V_i : i \in \mathbb{Z}_{\geq 0}]/(V_0 - T^{1/p}, V_i^p).$$

The lift $\overline{\mathbb{\Delta}}_S \to S'$ is now given by the composition

$$\mathbb{F}_p[T^{1/p^\infty}][V_i : i \in \mathbb{Z}_{\geq 0}]/(V_0 - T^{1/p}, V_i^p) \to \mathbb{F}_p[T^{1/p^\infty}][V_0]/(V_0 - T^{1/p}, V_0^2) \xrightarrow{\simeq} \mathbb{F}_p[T^{1/p^\infty}]/(T^{2/p}),$$

where the first map kills V_i for $i \geq 1$. The kernel of this map is

$$\widetilde{\mathrm{Fil}}^1 \overline{\mathbb{\Delta}}_S = (V_i :\ i \geq 1)$$

The kernel of the map $\overline{\mathbb{\Delta}}_{C'} \to S'$ is

$$\mathrm{Fil}^1 \overline{\mathbb{\Delta}}_S = (U_i :\ i \geq 0).$$

In this case, we have a fiber sequence

$$K_{S' \twoheadrightarrow S}/^{\mathbb{L}} p \to \mathrm{Fil}^1 \overline{\mathbb{\Delta}}_S \to \widetilde{\mathrm{Fil}}^1 \overline{\mathbb{\Delta}}_S,$$

and one can check that the endomorphism of $K_{S' \twoheadrightarrow S}/^{\mathbb{L}} p$ arising from $\dot{\varphi}_1$ is induced by the diagram

$$\begin{tikzcd}
\mathrm{Fil}^1 \overline{\mathbb{\Delta}}_S \arrow[r] \arrow[d, "U_0 \mapsto (p-1)! U_1"', "U_i \mapsto 0 : i \geq 1"] & \widetilde{\mathrm{Fil}}^1 \overline{\mathbb{\Delta}}_S \arrow[d, "V_i \mapsto 0:\ i \geq 1"] \\
\mathrm{Fil}^1 \overline{\mathbb{\Delta}}_S \arrow[r] & \widetilde{\mathrm{Fil}}^1 \overline{\mathbb{\Delta}}_S
\end{tikzcd}$$

and so we clearly have $\dot{\varphi}_1^2 = 0$.

∎

repbility

7. A result of Bragg-Olsson

The goal of this section is to state Theorem 7.3, which is due to Bragg-Olsson, and is an important ingredient in the general representability theorem that will appear in the next section.

:height_1

7.1. **Formulation of the result.** For $R \in \mathrm{CRing}_{\mathbb{F}_p/}$, set

$$Z^1_{\mathbb{\Delta}}(R) = \mathrm{Fil}^{\mathrm{conj}}_1 \overline{\mathbb{\Delta}}_R \times_{\overline{\mathbb{\Delta}}_R} \mathrm{Fil}^1_{\mathrm{Hdg}} \overline{\mathbb{\Delta}}_R \simeq \mathrm{hker}(\mathrm{Fil}^{\mathrm{conj}}_1(R/^{\mathbb{L}}p) \to R)$$

Then we have two maps

$$q_1, q_2 : Z^1_{\mathbb{\Delta}}(R) \to \mathrm{gr}^{\mathrm{conj}}_1 \overline{\mathbb{\Delta}}_R \simeq \mathbb{L}_{R/\mathbb{F}_p}[-1],$$

where q_1 arises from the natural map $\mathrm{Fil}^{\mathrm{conj}}_1 \overline{\mathbb{\Delta}}_C \to \mathrm{gr}^{\mathrm{conj}}_1 \overline{\mathbb{\Delta}}_R$, and q_2 arises from the natural map $\mathrm{Fil}^1_{\mathrm{Hdg}} \overline{\mathbb{\Delta}}_C \to \mathrm{gr}^1_{\mathrm{Hdg}} \overline{\mathbb{\Delta}}_R$ composed with the Cartier isomorphism.

$Z^1_{\mathbb{\Delta}}(R)$ inherits the structure of an object in Mod_R from $\mathrm{Fil}^{\mathrm{conj}}_1(R/^{\mathbb{L}}p)$. For this structure, q_1 is R-linear, while q_2 is φ-semilinear, and so corresponds to an R-linear map $1 \otimes q_2 : \varphi^* Z^1_{\mathbb{\Delta}}(R) \to \mathrm{gr}^{\mathrm{conj}}_1 \overline{\mathbb{\Delta}}_R$.

Let $\mathrm{Mod}^{\varphi}_{R/}$ be the ∞-category of pairs (N, ψ), where $N \in \mathrm{Mod}_R$, and $\psi : \varphi^* N \to N$ is a map in Mod_R. For each such pair, we obtain two maps

$$q_{1,\psi}, q_{2,\psi} : N \otimes_R Z^1_{\mathbb{\Delta}}(R) \to N \otimes_R \mathrm{gr}^{\mathrm{conj}}_1 \overline{\mathbb{\Delta}}_R.$$

The first is simply $q_{1,\psi} = \mathrm{id} \otimes q_1$, and is independent of ψ, while the second is the composition

$$N \otimes_R Z^1_{\mathbb{\Delta}}(R) \to \varphi^* N \otimes_R \varphi^* Z^1_{\mathbb{\Delta}}(R) \xrightarrow{\psi \otimes (1 \otimes q_2)} N \otimes_R \mathrm{gr}^{\mathrm{conj}}_1 \overline{\mathbb{\Delta}}_R.$$

rism_expl

Remark 7.2. Suppose that R is a smooth $\mathbb{F}_p$-algebra. Then we have

$$\mathrm{Fil}^{\mathrm{conj}}_i \overline{\mathbb{\Delta}}_R \simeq \tau^{\leq i} \Omega^{\bullet}_{R/\mathbb{F}_p},$$

and therefore $Z^1_{\mathbb{\Delta}}(R) \simeq \Omega^{1,\mathrm{cl}}_{R/\mathbb{F}_p}[-1]$ is the shifted module of closed differential forms, while $\mathrm{gr}^{\mathrm{conj}}_1 \overline{\mathbb{\Delta}}_R \simeq H^1(\Omega^{\bullet}_{R/\mathbb{F}_p})[-1]$.

The map q_1 now is the natural surjection, while q_2 is the composition

$$\Omega^{1,\mathrm{cl}}_{R/\mathbb{F}_p}[-1] \to \Omega^1_{R/\mathbb{F}_p}[-1] \xrightarrow{\simeq} H^1(\Omega^{\bullet}_{R/\mathbb{F}_p})[-1],$$

where the first map is the inclusion, and the second is the Cartier isomorphism.

When N is a vector bundle over R, there is an associated finite flat commutative group scheme $G(N, \psi)$ of height one (see (7.4) below), and the complex $R\Gamma_{\varphi}(R, (N, \psi))$ is (up to shift) precisely that of Artin-Milne appearing in [2, Prop. 2.4], which computes the flat cohomology of $G(N, \psi)$.

We can now associate with each $(N, \psi) \in \mathrm{Mod}^{\varphi}_R$ the space

$$R\Gamma_{\varphi}\,(R, (N, \psi)) = \mathrm{hker}(N \otimes_R Z^1_{\mathbb{\Delta}}(R) \xrightarrow{q_{1,\psi} - q_{2,\psi}} N \otimes_R \mathrm{gr}^{\mathrm{conj}}_1 \overline{\mathbb{\Delta}}_R).$$

This yields a prestack

$$\begin{aligned} \mathsf{S}_{(N,\psi)} : \mathrm{CRing}_{R/} &\to \mathrm{Mod}^{\mathrm{cn}}_{\mathbb{F}_p} \\ C &\mapsto \tau^{\leq 0} R\Gamma_{\varphi}\,(C, (C \otimes_R N, \mathrm{id} \otimes \psi)) \end{aligned}$$

eight_one

Theorem 7.3. *Suppose that N is perfect of Tor amplitude $[-m, n]$; then $\mathsf{S}_{(N,\psi)}$ is represented by a derived Artin m-stack over R.*

p_schemes

7.4. **Flat cohomology of height one group schemes.** We now present the proof of Bragg-Olsson. Begin by considering the following special case: Suppose that R is discrete and that N is locally free of finite rank over R. Then the pair (N, ψ) gives rise to a finite flat commutative p-group scheme $G(N, \psi)$ over R of height 1, meaning that its Frobenius endomorphism is trivial; see [20, Exp. VIIA], [12, §2]. Explicitly, the Cartier dual $G^{\vee}(N, \psi)$ is given as the kernel of the map

$$\mathbf{V}(N^{\vee}) \xrightarrow{\psi^{\vee} - F} \mathbf{V}(\varphi^* N^{\vee}) \simeq R \otimes_{\varphi, R} \mathbf{V}(N^{\vee}),$$

where F is the relative Frobenius map for $\mathbf{V}(N^{\vee})$ with respect to R.

We now have:

ohomology **Theorem 7.5.** *For any $C \in \mathrm{CRing}_{R/}$, we have a canonical isomorphism*

$$R\Gamma_{\varphi}\left(C, (C \otimes_R N, \mathrm{id} \otimes \psi)\right) \xrightarrow{\simeq} R\Gamma_{\mathrm{fppf}}(\operatorname{Spec} C, G(N, \psi)).$$

Proof. By considering the universal situation for pairs (N, ψ), one reduces to the case where R is a smooth $\mathbb{F}_p$-algebra. The arrow in question is obtained via left Kan extension from an isomorphism of functors for smooth R-algebras C, where it is a classical theorem of Artin-Milne [2, Proposition 2.4]; see Remark 7.2.

That this map is an isomorphism is a theorem of Bragg-Olsson [10, Theorem 4.8], who attribute the proof to Bhatt-Lurie. $\square$

ohomology **Corollary 7.6.** *The stack $\mathsf{S}_{(N,\psi)}$ is canonically isomorphic to the group scheme $G(N, \psi)$, and the stack $\mathsf{S}_{(N[1],\psi[1])}$ is isomorphic to the classifying stack $BG(N, \psi)$. For any $r \geq 2$, $\mathsf{S}_{(N,\psi)[r]}$ is isomorphic to $B^2G(N, \psi)[r-2]$.*

Proof. Most of this is clear from Theorem 7.5. For the last assertion, we only need to note that the left hand side of the isomorphism in the theorem is always (-2)-connective. Indeed, $\mathrm{gr}_1^{\mathrm{conj}} \overline{\mathbb{\Delta}}_C$ and $Z^1_{\mathbb{\Delta}}(C)$ are both (-1)-connective for any $C \in \mathrm{CRing}_{\mathbb{F}_p/}$. $\square$

Proof of Theorem 7.3. Suppose that we have a cofiber sequence

$$(N', \psi') \to (N, \psi) \to (N'', \psi'')$$

of φ-modules over R. Then we obtain a Cartesian diagram of prestacks

$$\begin{tikzcd} \mathsf{S}_{(N,\psi)} \arrow[r] \arrow[d] & \mathsf{S}_{(N'',\psi'')} \arrow[d, "0"'] \\ \mathsf{S}_{(N'',\psi)} \arrow[r] & \mathsf{S}_{(N',\psi')[1]}. \end{tikzcd}$$

Therefore, $\mathsf{S}_{(N,\psi)}$ is representable as soon as $\mathsf{S}_{(N'',\psi'')}$ and $\mathsf{S}_{(N',\psi')[1]}$ are representable.

By working Zariski locally on $\operatorname{Spec} R$, we can assume that N is represented by a bounded complex of locally free modules over R and that ψ lifts to a φ-semilinear map of this complex. In particular, by considering the stupid filtration for this complex, we reduce to showing that $\mathsf{S}_{(N,\psi)[r]}$ is representable when N is a locally free R-module and $r \in \mathbb{Z}$. This follows from Corollary 7.6 for $r \geq 0$. For the representability of $\mathsf{S}_{(N,\psi)[r]}$ with $r < 0$, note that we have a Cartesian square of prestacks

$$\begin{tikzcd} \mathsf{S}_{(N,\psi)[r]} \arrow[r] \arrow[d] & \operatorname{Spec} R \arrow[d, "0"] \\ \operatorname{Spec} R \arrow[r, "0"'] & \mathsf{S}_{(N,\psi)[r+1]}. \end{tikzcd}$$

$\square$

:abstract

8. Representability theorems for 1-bounded stacks

In this section, we prove our main representability theorem under somewhat general hypotheses, and record some applications to stacks obtained from F-gauges.

zip_stack

8.1. **The F-zip stack.** For $R \in \mathrm{CRing}_{\mathbb{F}_p/}$, we will now define an fpqc stack $R^{F\mathrm{Zip}}$ over $\mathbb{F}_p$ as follows: We first glue $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$ with $\mathbb{A}^1_+/\mathbb{G}_m \times \operatorname{Spec} R$ along the open substack $\operatorname{Spec} R \simeq \mathbb{G}_m/\mathbb{G}_m \times \operatorname{Spec} R$. Denote the resulting stack by $Y \times \operatorname{Spec} R$.

Consider the two maps

mbda_maps (8.1.1)
$$\lambda_+ : B\mathbb{G}_m \times \operatorname{Spec} R \to \mathbb{A}^1_+/\mathbb{G}_m \times \operatorname{Spec} R \to Y \times \operatorname{Spec} R;$$

(8.1.2)
$$\lambda_- : B\mathbb{G}_m \times \operatorname{Spec} R \xrightarrow{\mathrm{id}\otimes\varphi} B\mathbb{G}_m \times \operatorname{Spec} R \to \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R \to Y \times \operatorname{Spec} R$$

where the second-to-last arrow in each line is the canonical closed immersion of R-stacks.

We now take $R^{F\mathrm{Zip}}$ to be the coequalizer of these two maps. Explicitly, giving a map $R^{F\mathrm{Zip}} \to X$ to any other fpqc stack X over $\mathbb{F}_p$ amounts to giving two maps

$$f_+ : \mathbb{A}^1_+/\mathbb{G}_m \times \operatorname{Spec} R \to X \ ; \ f_- : \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R \to X$$

along with homotopies between the compositions

$$\mathbb{G}_m/\mathbb{G}_m \times \operatorname{Spec} R \to \mathbb{A}^1_+/\mathbb{G}_m \times \operatorname{Spec} R \xrightarrow{f_+} X \ ; \ \mathbb{G}_m/\mathbb{G}_m \times \operatorname{Spec} R \to \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R \xrightarrow{f_-} X$$

as well as between the compositions

$$B\mathbb{G}_m \times \operatorname{Spec} R \to \mathbb{A}^1_+/\mathbb{G}_m \times \operatorname{Spec} R \xrightarrow{f_+} X \ ; \ B\mathbb{G}_m \times \operatorname{Spec} R \xrightarrow{\mathrm{id}\otimes\varphi} B\mathbb{G}_m \times \operatorname{Spec} R \to \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R \xrightarrow{f_-} X$$

Let $B\mathbb{G}^{\phi}_{m,\mathcal{O}}$ be the quotient of $B\mathbb{G}_{m,\mathcal{O}}$ defined in (4.11), and let $B\mathbb{G}^{\phi}_{m,k}$ be its special fiber. Then, if R is a k-algebra, there is a canonical map

ne_bundle (8.1.3)
$$R^{F\mathrm{Zip}} \to B\mathbb{G}^{\phi}_{m,k}$$

whose restriction to $\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R$ is the natural structure map, while its restriction to $\mathbb{A}^1_+/\mathbb{G}_m \times \operatorname{Spec} R$ is the pre-composition of the natural map with the endomorphism $\mathrm{id} \times \varphi$.

sec:fzips

8.2. ***F*-zips.** We will call objects in $\mathrm{QCoh}(R^{F\mathrm{Zip}})$ **F-zips over** R.

Unpacking the definitions, one sees that giving an F-zip M is equivalent to specifying the following data:

- A decreasingly filtered module $\mathrm{Fil}^{\bullet}_{\mathrm{Hdg}} M^-$ over R obtained via pullback along $x^{\mathcal{N}}_{dR}$;
- An increasingly filtered module $\mathrm{Fil}^{\mathrm{conj}}_{\bullet} M^+$ over R obtained via pullback along $x^{\mathcal{N}}_{HT}$;
- An isomorphism $\eta : M^+ \xrightarrow{\simeq} M^-$ in Mod_R identifying both with a common R-module M;
- An isomorphism of graded R-modules

$$\alpha : \mathrm{gr}^{\mathrm{conj}}_{\bullet} M^+ \xrightarrow{\simeq} \varphi^* \mathrm{gr}^{-\bullet}_{\mathrm{Hdg}} M^-.$$

In the sequel, we will use the identification $M^+ \simeq M^- \simeq M$ to drop all superscripts.

When M is a vector bundle over $R^{F\mathrm{Zip}}$, we essentially recover the definition of Moonen-Pink-Wedhorn-Ziegler from [43]. The de Rham cohomology of any smooth projective scheme over R with degenerating Hodge-to-de Rham spectral sequence, when equipped with its decreasing Hodge filtration and its increasing conjugate filtration, yields an example of such an F-zip.

Given an F-zip M over R, we obtain a $\mathrm{Mod}_{\mathbb{F}_p}$-valued prestack over R:

$$R\Gamma_{F\mathrm{Zip}}(\mathsf{M}) : C \mapsto R\Gamma(C^{F\mathrm{Zip}}, \mathsf{M}|_{C^{F\mathrm{Zip}}}).$$

We can make this 'F-zip cohomology' quite explicit. Let $(\mathrm{Fil}^{\bullet}_{\mathrm{Hdg}} M, \mathrm{Fil}^{\mathrm{conj}}_{\bullet} M, \eta, \alpha)$ be the tuple corresponding to $\mathsf{M} \overset{\mathrm{defn}}{=} \eta^* \mathcal{M}$.

Then we obtain two maps

$$q_1, q_2 : \mathrm{Fil}^{\mathrm{conj}}_0 M \times_M \mathrm{Fil}^0_{\mathrm{Hdg}} M \to \mathrm{gr}^{\mathrm{conj}}_0 M,$$

where the first is via the map $\mathrm{Fil}^{\mathrm{conj}}_0 M \to \mathrm{gr}^{\mathrm{conj}}_0 M$, and so is R-linear, while the second is via

$$\mathrm{Fil}^0_{\mathrm{Hdg}} M \to \varphi^* \mathrm{Fil}^0_{\mathrm{Hdg}} M \to \varphi^* \mathrm{gr}^0_{\mathrm{Hdg}} M \xrightarrow{\simeq} \mathrm{gr}^{\mathrm{conj}}_0 M,$$

and so is φ-semilinear. We now have:

perf_comp (8.2.1)
$$R\Gamma_{F\mathrm{Zip}}(\operatorname{Spec} R, \mathsf{M}) \simeq \mathrm{hker}\left(\mathrm{Fil}^{\mathrm{conj}}_0 M \times_M \mathrm{Fil}^0_{\mathrm{Hdg}} M \xrightarrow{q_1 - q_2} \mathrm{gr}^{\mathrm{conj}}_0 M\right).$$

8.3. **The map to the mod-p syntomification.** There is a canonical map $\eta : R^{F\mathrm{Zip}} \to R^{\mathrm{syn}} \otimes \mathbb{F}_p$ obtained as follows. Consider first the map

$$\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R \xrightarrow{x^{\mathcal{N}}_{\mathrm{dR}}} R^{\mathcal{N}} \otimes \mathbb{F}_p \to R^{\mathrm{syn}} \otimes \mathbb{F}_p. \tag{8.3.1}$$

Next, we need a map

$$x^{\mathcal{N}}_{\mathrm{HT}} : \mathbb{A}^1_+/\mathbb{G}_m \times \operatorname{Spec} R \to R^{\mathcal{N}} \otimes \mathbb{F}_p$$

whose restriction to the open point $\operatorname{Spec} R$ factors through the Hodge-Tate embedding j^{HT} via the map $x_{\mathrm{dR}} : \operatorname{Spec} R \to R^{\mathbb{\Delta}}$.

This is constructed as follows: Given a point $(L, u : C \to L)$ of $\mathbb{A}^1_+/\mathbb{G}_m \times \operatorname{Spec} R$ over $C \in \mathrm{CRing}_{R/}$, we can associate a filtered Cartier-Witt divisor $(M((L,u)) \xrightarrow{d} W, f : R \to (W/_d M((L,u))(C))$ (notation as in (6.9)) by taking d to be the composition

$$M((L,u)) \to F_* W \xrightarrow{V} W.$$

There is a natural map $W/^{\mathbb{L}}p \to W/_d M((L,u))$, and the structure map $f : R \to (W/_d M((L,u))(C)$ is obtained from the composition

$$R \to C \to \mathbb{G}_a^{\mathrm{dR}}(C) \simeq W(C)/^{\mathbb{L}}p \to (W/_d M((L,u))(C).$$

The restriction to the open point corresponds to the natural structure map $R \to C \to W(C)/^{\mathbb{L}}p$, as desired.

Thus we have another map

$$\mathbb{A}^1_+/\mathbb{G}_m \times \operatorname{Spec} R \xrightarrow{x^{\mathcal{N}}_{\mathrm{HT}}} R^{\mathcal{N}} \otimes \mathbb{F}_p \to R^{\mathrm{syn}} \otimes \mathbb{F}_p \tag{8.3.2}$$

whose restriction to the open substack $\mathbb{G}_m/\mathbb{G}_m \times \operatorname{Spec} R$ agrees with that of (8.3.1), and thus yields a map $\tilde{\eta} : Y \times \operatorname{Spec} R \to R^{\mathrm{syn}} \otimes \mathbb{F}_p$.

Consider now the two maps

$$B\mathbb{G}_m \times \operatorname{Spec} R \to \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} R \xrightarrow{x^{\mathcal{N}}_{\mathrm{dR}}} R^{\mathcal{N}} \otimes \mathbb{F}_p;$$

$$B\mathbb{G}_m \times \operatorname{Spec} R \to \mathbb{A}^1_+/\mathbb{G}_m \times \operatorname{Spec} R \xrightarrow{x^{\mathcal{N}}_{\mathrm{HT}}} R^{\mathcal{N}} \otimes \mathbb{F}_p.$$

The composition with the map to $\mathbb{F}_p^{\mathcal{N}}$ is the same for both of these: It attaches to every line bundle L the admissible module $M((L,0)) \simeq F_* W \oplus \mathbf{V}(L)^\sharp$ and the map $d = (0, V) : M((L,0)) \to W$; however the structure map $R \to (W/_d M((L,0)))(C)$ associated with the second composition differs from that associated with the first via pre-composition with $\varphi : R \to R$. This shows that the map $\tilde{\eta}$ descends to the desired map $\eta : R^{F\mathrm{Zip}} \to R^{\mathrm{syn}} \otimes \mathbb{F}_p$.

If R is a k-algebra, one finds that η is a map of stacks over $B\mathbb{G}^{\phi}_{m,k}$.

8.4. **F-gauges and F-zips.** Suppose that $R \in \mathrm{CRing}^f_{(\mathbb{Z}/p^m\mathbb{Z})/}$. An **$F$-gauge over R of level** n is a quasi-coherent sheaf $\mathcal{M}$ over $R^{\mathrm{syn}} \otimes (\mathbb{Z}/p^n\mathbb{Z})$. Via the map $x^{\mathcal{N}}_{\mathrm{dR}}$ of (6.18), we obtain a symmetric monoidal functor of symmetric monoidal ∞-categories

$$\mathrm{QCoh}(R^{\mathrm{syn}} \otimes (\mathbb{Z}/p^n\mathbb{Z})) \to \mathrm{QCoh}(R^{\mathcal{N}} \otimes (\mathbb{Z}/p^n\mathbb{Z})) \xrightarrow{(x^{\mathcal{N}}_{\mathrm{dR}})^*} \mathrm{QCoh}(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec}(R/^{\mathbb{L}}p^n)),$$

which can be viewed as a functor $\mathcal{M} \to \mathrm{Fil}^\bullet_{\mathrm{Hdg}} M_n$ from F-gauges of level n to filtered modules over $R/^{\mathbb{L}}p^n$.

As an immediate consequence of Theorem 6.32, we find:

Proposition 8.5. *Suppose that R is semiperfectoid. Then there is a canonical symmetric monoidal equivalence of stable ∞-categories*

$$\mathrm{QCoh}(R^{\mathrm{syn}} \otimes (\mathbb{Z}/p^n\mathbb{Z})) \xrightarrow{\simeq} \mathbb{\Delta}_R\text{--}\mathsf{gauge}_n,$$

where the right hand side is the ∞-category of $\mathbb{\Delta}_R$-gauges of level n from (5.17).

From this we obtain:

s_descent **Proposition 8.6.** *Suppose that $R \to R_\infty$ is as in Corollary 6.26. Then we have an equivalence of symmetric monoidal stable ∞-categories*

$$\mathrm{QCoh}(R^{\mathrm{syn}} \otimes (\mathbb{Z}/p^n\mathbb{Z})) \xrightarrow{\simeq} \mathrm{Tot}\left(\mathbb{\Delta}_{R_\infty^{\otimes(\bullet+1)}}\text{-}\mathsf{gauge}_n\right).$$

The **Hodge-Tate weights** of $\mathcal{M}$ are the integers i such that $\mathrm{gr}^{-i}_{\mathrm{Hdg}} M_n \neq 0$. We will say that $\mathcal{M}$ is 1**-bounded** if its Hodge-Tate weights are bounded above by 1.

Associated with $\mathcal{M}$ is the $\mathrm{Mod}_{\mathbb{Z}/p^n\mathbb{Z}}$-valued prestack over R

$$R\Gamma_{\mathrm{syn}}(\mathcal{M}) : \mathrm{CRing}_{R/} \to \mathrm{Mod}_{\mathbb{Z}/p^n\mathbb{Z}}$$
$$C \mapsto R\Gamma_{\mathrm{syn}}(\mathrm{Spec}\, C, \mathcal{M}|_{C^{\mathrm{syn}}}) \overset{\mathrm{defn}}{=} R\Gamma(C^{\mathrm{syn}}, \mathcal{M}|_{C^{\mathrm{syn}}}).$$

We will set $\Gamma_{\mathrm{syn}}(\mathcal{M}) = \tau^{\leq 0} R\Gamma_{\mathrm{syn}}(\mathcal{M})$.

Suppose now that R is an $\mathbb{F}_p$-algebra. Then pullback along $\eta : R^{F\mathrm{Zip}} \to R^{\mathrm{syn}} \otimes \mathbb{F}_p$. gives a symmetric monoidal functor

$$\eta^* : \mathrm{QCoh}(R^{\mathrm{syn}} \otimes \mathbb{F}_p) \to \mathrm{QCoh}(R^{F\mathrm{Zip}}).$$

Note that there is a natural map

$$R\Gamma_{\mathrm{syn}}(\mathcal{M}) \to R\Gamma_{F\mathrm{Zip}}(\eta^*\mathcal{M})$$

for any F-gauge $\mathcal{M}$ over R of level 1.

_sections

8.7. **Prestacks of sections.** We will now use the terminology and results of (4.13).

Suppose that $R \in \mathrm{CRing}^f_{(\mathbb{Z}/p^m\mathbb{Z})/}$. Note that $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is a pointed graded stack equipped with a canonical map of graded stacks

$$B\mathbb{G}_m \times \mathrm{Spec}(R/^{\mathbb{L}}p^n) \to R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}.$$

arising via the map

$$x^{\mathcal{N}}_{\mathrm{dR},R} \otimes \mathbb{Z}/p^n\mathbb{Z} : \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}(R/^{\mathbb{L}}p^n) \to R^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}.$$

Suppose that we have a 1-bounded stack $\mathcal{X} = (\mathcal{X}^\diamond, X^0) \to R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. Define a prestack $\Gamma_{\mathrm{syn}}(\mathcal{X})$ that associates with each $C \in \mathrm{CRing}_{R/}$ the space

$$\Gamma_{\mathrm{syn}}(\mathcal{X})(C) = \mathrm{Map}_{R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(C^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}).$$

Unpacking definitions, one sees that we have

$$\Gamma_{\mathrm{syn}}(\mathcal{X})(C) = \mathrm{Map}_{R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(C^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}^\diamond) \times_{X^{\diamond,0}(C/^{\mathbb{L}}p^n)} X^0(C/^{\mathbb{L}}p^n),$$

where $X^{\diamond,0} \to \mathrm{Spec}\, R/^{\mathbb{L}}p^n$ is the fixed point stack of $\mathcal{X}^\diamond$.

This has a slightly more explicit description. First, define prestacks $\Gamma_{\mathcal{N}}(\mathcal{X})$ and $\Gamma_{\mathbb{\Delta}}(\mathcal{X})$ over R by

$$\Gamma_{\mathcal{N}}(\mathcal{X})(C) = \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(C^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X});$$
$$\Gamma_{\mathbb{\Delta}}(\mathcal{X})(C) = \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(C^{\mathbb{\Delta}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}^\diamond).$$

Restriction along the de Rham and Hodge-Tate immersions yields maps

$$j^*_{\mathrm{dR}}, j^*_{\mathrm{HT}} : \Gamma_{\mathcal{N}}(\mathcal{X}) \to \Gamma_{\mathbb{\Delta}}(\mathcal{X}),$$

and we now have

irst_desc (8.7.1)
$$\Gamma_{\mathrm{syn}}(\mathcal{X}) \xrightarrow{\simeq} \mathrm{eq}\left(\Gamma_{\mathcal{N}}(\mathcal{X}) \underset{j^*_{\mathrm{HT}}}{\overset{j^*_{\mathrm{dR}}}{\rightrightarrows}} \Gamma_{\mathbb{\Delta}}(\mathcal{X})\right).$$

Suppose now that $n = m = 1$. Then we can get a somewhat different description of $\Gamma_{\mathrm{syn}}(X)$ as follows: First, note that $R^{\mathrm{syn}} \otimes \mathbb{F}_p$ can be obtained by first gluing $R^{\mathcal{N}}_{(u=0)}$ and $R^{\mathcal{N}}_{(t=0)}$ along their common open $R^{\mathbb{\Delta}} \otimes \mathbb{F}_p$, and then gluing along the common closed substack $R^{\mathcal{N}}_{(t=u=0)}$. To do this somewhat rigorously, one can, using quasisyntomic descent, reduce the proof of this claim to the case of R semiperfect, in which case all the base stacks correspond to $\mathbb{G}_m$-equivariant (derived) affine schemes, and the desired geometric statement is easy to check.

There is now a little subtlety here in the choice of base-points for the various graded stacks involved. For any R-algebra C, as observed at the end of (8.3), we have two choices of base-points for $C^{\mathcal{N}}_{(u=t=0)}$, arising from those for $C^{\mathcal{N}}_{(u=0)}$ and $C^{\mathcal{N}}_{(t=0)}$. Denote the first by η_t and the second by η_u; then the latter is the Frobenius twist of the former.

Now, consider the prestacks $\Gamma_{(u=0)}(\mathcal{X})$, $\Gamma_{(t=0)}(\mathcal{X})$ and $\Gamma_{(u=t=0)}(\mathcal{X})$ over R given by:

$$\begin{aligned}\Gamma_{(u=0)}(\mathcal{X})(C) &= \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{F}_p}(C^{\mathcal{N}}_{(u=0)},\mathcal{X});\\ \Gamma_{(t=0)}(\mathcal{X})(C) &= \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{F}_p}(C^{\mathcal{N}}_{(t=0)},\mathcal{X});\\ \Gamma_{(u=t=0)}(\mathcal{X})(C) &= \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{F}_p}((C^{\mathcal{N}}_{(u=t=0)},\eta_t),\mathcal{X})\\ {}^{\varphi}\Gamma_{(u=t=0)}(\mathcal{X})(C) &= \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{F}_p}((C^{\mathcal{N}}_{(u=t=0)},\eta_u),\mathcal{X}).\end{aligned}$$

Explicitly, we have

$$\begin{aligned}\Gamma_{(u=t=0)}(\mathcal{X})(C) &= \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{F}_p}(C^{\mathcal{N}}_{(u=t=0)},\mathcal{X}^{\diamond})\times_{X^{\diamond,0}(C)}X^0(C);\\ {}^{\varphi}\Gamma_{(u=t=0)}(\mathcal{X})(C) &= \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{F}_p}(C^{\mathcal{N}}_{(u=t=0)},\mathcal{X}^{\diamond})\times_{X^{\diamond,0}(C)}{}^{\varphi}X^0(C),\end{aligned}$$

where${}^{\varphi}X^0$ is given by ${}^{\varphi}X^0(C)=X^0(\varphi_*C)$; in other words, it is just the base-change of X^0 along $\varphi:R\to R$.

Then we obtain two maps

$${}^{\varphi}\iota_-^*,\iota_+^*:\Gamma_{(u=0)}(\mathcal{X})\times_{\Gamma_{\Delta}(\mathcal{X})}\Gamma_{(t=0)}(\mathcal{X})\to{}^{\varphi}\Gamma_{(u=t=0)}(\mathcal{X})$$

obtained via restriction along the two closed immersions

$$\iota_-:C^{\mathcal{N}}_{(u=t=0)}\to C^{\mathcal{N}}_{(u=0)}\ ;\ \iota_+:C^{\mathcal{N}}_{(u=t=0)}\to C^{\mathcal{N}}_{(t=0)}$$

We now also have:

cond_desc

$$\Gamma_{\mathrm{syn}}(\mathcal{X})\xrightarrow{\simeq}\mathrm{eq}\left(\Gamma_{(u=0)}(\mathcal{X})\times_{\Gamma_{\Delta}(\mathcal{X})}\Gamma_{(t=0)}(\mathcal{X})\ \overset{{}^{\varphi}\iota_-^*}{\underset{\iota_+^*}{\rightrightarrows}}\ {}^{\varphi}\Gamma_{(u=t=0)}(\mathcal{X})\right).\tag{8.7.2}$$

ux_stacks

8.8. **Some auxiliary stacks.** Suppose that $\mathcal{X}$ is a stack over $R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}$. Let $X^{(n)},X^{-,(n)},X^{-,0}$ be the prestacks over R given by

$$\begin{aligned}X^{(n)}(C) &= \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(\mathrm{Spec}\,C/^{\mathbb{L}}p^n,\mathcal{X});\\ X^{-,(n)}(C) &= \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(\mathbb{A}^1/\mathbb{G}_m\times\mathrm{Spec}\,C/^{\mathbb{L}}p^n,\mathcal{X});\\ X^{0,(n)}(C) &= \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(B\mathbb{G}_m\times\mathrm{Spec}\,C/^{\mathbb{L}}p^n,\mathcal{X}).\end{aligned}$$

The last two stacks are simply the Weil restriction from $R/^{\mathbb{L}}p^n$ to R of the attractor and fixed point stacks of $(\mathcal{X}^{\diamond},X^0)$, respectively.

If R is in addition an $\mathbb{F}_p$-algebra, then also define

$$X^{+,(n)}(C)=\mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(\mathbb{A}^1_+/\mathbb{G}_m\times\mathrm{Spec}\,C/^{\mathbb{L}}p^n,\mathcal{X}).$$

This is the Weil restriction of the repeller stack associated with $\mathcal{X}$ and the map $\mathbb{A}^1_+/\mathbb{G}_m\times\mathrm{Spec}\,R/^{\mathbb{L}}p^n\to R^{\mathcal{N}}\otimes\mathbb{Z}/p^n\mathbb{Z}$.

Suppose that $n=m$, so that R is a $\mathbb{Z}/p^n\mathbb{Z}$-algebra. In this case, for any R-algebra C, we have a section $C/^{\mathbb{L}}p^n\to C$, and so we can also define prestacks X,X^-,X^0,X^+ over R (the last only when $n=m=1$) by:

$$\begin{aligned}X(C) &= \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(\mathrm{Spec}\,C,\mathcal{X}^{\diamond});\\ X^-(C) &= \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(\mathbb{A}^1/\mathbb{G}_m\times\mathrm{Spec}\,C,\mathcal{X});\\ X^0(C) &= \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(B\mathbb{G}_m\times\mathrm{Spec}\,C,\mathcal{X});\\ X^+(C) &= \mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{F}_p}(\mathbb{A}^1_+/\mathbb{G}_m\times\mathrm{Spec}\,C,\mathcal{X}).\end{aligned}$$

The last three stacks are simply the attractor, fixed point and repeller stacks for $\mathcal{X}$ base-changed from $C/^{\mathbb{L}}p^n$ to C. Note once again the subtletly about base-points: The base-point $B\mathbb{G}_m \times \operatorname{Spec} C \to \mathbb{A}^1_+/\mathbb{G}_m \times \operatorname{Spec} C$ is a lift of the Frobenius endomorphism of $B\mathbb{G}_m \times \operatorname{Spec} C$. In particular, we actually have

$$X^+(C) = \operatorname{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{F}_p}(\mathbb{A}^1_+/\mathbb{G}_m \times \operatorname{Spec} C, \mathcal{X}^\diamond) \times_{{}^\varphi X^{\diamond,0}(C)} {}^\varphi X^0(C).$$

or_repble

Proposition 8.9. *Suppose that $\mathcal{X} \to R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is a 1-bounded stack and suppose that $\pi_0(R)$ is a G-ring. Suppose also that $\mathcal{X}^\diamond$ has quasi-affine diagonal over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. Then $X^{(n)}$, $X^{-,(n)}$, $X^{0,(n)}$ and $X^{+,(n)}$ (if $m = 1$)—and, if $n = m$, the prestacks X, X^-, X^0 and X^+ (if $n = m = 1$)—are derived Artin stacks over R.*

Proof. By Lemmas 3.15 and 4.10 it follows that $X^{-,(n)}$ is representable. The same argument works to show that $X^{+,(n)}$ is also representable when $m = 1$. The argument for $X^{0,(n)}$ is simpler, since its integrability follows immediately from the fact that $\mathcal{X}$ is a relative derived Artin stack, whereas the one for $X^{(n)}$ is even simpler, since it is a mod-p^n Weil restriction of a derived Artin stack by definition.

The argument for the remaining four prestacks (when $n = m$) is the same, but doesn't involve Weil restrictions. □

_to_Fzips

8.10. **Dévissage to the F-zip stack.** For the rest of this section—except for the final subsection—we will assume that R is an $\mathbb{F}_p$-algebra. Suppose that $n = 1$, and that we have a 1-bounded stack $\mathcal{X} \to R^{\mathrm{syn}} \otimes \mathbb{F}_p$.

For any $\mathbb{F}_p$-algebra C, the stack $C^{F\mathrm{Zip}}$ is a pointed graded stack over $B\mathbb{G}_m$ equipped with a canonical map $B\mathbb{G}_m \times \operatorname{Spec} C \to C^{F\mathrm{Zip}}$. Therefore, we can define the prestack $\Gamma_{F\mathrm{Zip}}(\mathcal{X})$ over R by

$$\Gamma_{F\mathrm{Zip}}(\mathcal{X})(C) = \operatorname{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{F}_p}(C^{F\mathrm{Zip}}, \mathcal{X}).$$

This can be explicitly understood as follows: Since R is an $\mathbb{F}_p$-algebra, there are two maps ${}^\varphi\lambda^*_-, \lambda^*_+ : X^- \times_X X^+ \to {}^\varphi X^0$ induced via pullback along the maps from (8.1.1). We now have

ons_fzips

(8.10.1)
$$\Gamma_{F\mathrm{Zip}}(\mathcal{X}) \xrightarrow{\simeq} \mathrm{eq}\left(X^- \times_X X^+ \underset{\lambda^*_+}{\overset{{}^\varphi\lambda^*_-}{\rightrightarrows}} {}^\varphi X^0\right).$$

_repblity

Lemma 8.11. *Suppose that $\mathcal{X}^\diamond$ is a relative r-stack with quasi-affine diagonal; then $\Gamma_{F\mathrm{Zip}}(\mathcal{X})$ is a locally finitely presented derived Artin r-stack over R.*

Proof. Given the presentation 8.10.1, this follows from Proposition 8.9. □

Let $\mathbb{T}_{\mathcal{X}}$ be the relative tangent complex for $\mathcal{X}^\diamond$ over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. For every section $x \in \Gamma_{F\mathrm{Zip}}(\mathcal{X})(C)$, the perfect F-zip $\mathsf{M}(\mathcal{X})_x = x^*\mathbb{T}_{\mathcal{X}} \in \operatorname{Perf}(C^{F\mathrm{Zip}})$ is 1-bounded. We will view this as giving us an F-zip $\mathsf{M}(\mathcal{X})$ over $\Gamma_{\mathrm{FZip}}(\mathcal{X})$. Note that, if $\mathcal{X}^\diamond$ is a relative r-stack, then this will be a perfect F-zip with Tor amplitude in $[-r,\infty)$.

We now obtain from this a φ-module $(N(\mathsf{M}(\mathcal{X})), \psi(\mathsf{M}(\mathcal{X})))$ over $\Gamma_{F\mathrm{Zip}}(\mathcal{X})$. Explicitly this means the following: Given a section x of $\mathcal{X}$ over $C^{F\mathrm{Zip}}$, we obtain an F-zip $x^*\mathbb{T}_{\mathcal{X}}$, which corresponds to a tuple $(\operatorname{Fil}^\bullet_{\mathrm{Hdg}} M_x, \operatorname{Fil}^{\mathrm{conj}}_\bullet M_x, \eta, \alpha)$. We now have

$$N(\mathsf{M}(\mathcal{X}))_x = \operatorname{gr}^{\mathrm{conj}}_1 M_x,$$

and the linearization of $\psi(\mathsf{M}(X))_x$ is given by

efinition

(8.11.1)
$$\varphi^* \operatorname{gr}^{\mathrm{conj}}_1 M_x \xrightarrow{\simeq} \operatorname{gr}^{\mathrm{conj}}_1 M_x = \operatorname{Fil}^{\mathrm{conj}}_1 M_x \to M_x = \operatorname{Fil}^{-1}_{\mathrm{Hdg}} M_x \to \operatorname{gr}^{\mathrm{conj}}_1 M_x.$$

Using Theorem 7.3, we now obtain a relative derived Artin stack $\mathsf{S}(\mathcal{X}) \to \Gamma_{F\mathrm{Zip}}(\mathcal{X})$ associating with every x as above the stack

$$\mathsf{S}(\mathcal{X})_x \overset{\mathrm{defn}}{=} \mathsf{S}_{(N(\mathsf{M}(\mathcal{X}))_x, \psi(\mathsf{M}(\mathcal{X}))_x)}.$$

Explicitly, for any $C' \in \mathrm{CRing}^f_{C/}$, we have

$$\mathsf{S}(\mathcal{X})_x(C') \simeq \tau^{\leq 0}\,\mathrm{hker}\left(\operatorname{gr}^{\mathrm{conj}}_1 M_x \otimes_C Z^1_{\mathbb{\Delta}}(C') \xrightarrow{q_{1,x}-q_{2,x}} \operatorname{gr}^{\mathrm{conj}}_1 M_x \otimes_C \operatorname{gr}^{\mathrm{conj}}_1 \overline{\mathbb{\Delta}}_{C'}\right)$$

for certain maps $q_{1,x}, q_{2,x}$, which we will encounter again below. If $\mathcal{X}^\diamond$ is a relative r-stack, then we have already noted above that $N(\mathsf{M}(\mathcal{X}))$ has Tor amplitude in $[-r,\infty)$, and so $\mathsf{S}(\mathcal{X})$ is a relative r-stack.

1-bounded **Theorem 8.12.** *The natural map $\Gamma_{\mathrm{syn}}(\mathcal{X}) \to \Gamma_{FZip}(\mathcal{X})$ is a quasisyntomic torsor under $\mathsf{S}(\mathcal{X})$. In particular, if $\mathcal{X}^\diamond$ has quasi-affine diagonal and $\pi_0(R)$ is a G-ring, then $\Gamma_{\mathrm{syn}}(\mathcal{X})$ is represented by a locally finitely presented derived Artin r-stack over R.*

or_G_ring *Remark* 8.13. The conditions that $\pi_0(R)$ to be a G-ring and that $\mathcal{X}^\diamond$ have quasi-affien diagonal are only required for their application in Lemma 8.11. In turn, they were used in the cited lemma only for the application of Proposition 4.10 to see that $X^{-,(n)}$, $X^{+,(n)}$ and $X^{0,(n)}$ are representable. In particular, if we had an independent way of showing the representability of these three stacks, then we would obtain the last assertion of the theorem without requiring quasi-affine diagonal or that $\pi_0(R)$ is a G-ring.

Proof. The last assertion is immediate from Lemma 8.11.

For the first, we make a series of observations:

us_fibers **Observation 8.13.1.** *There is a Cartesian diagram of prestacks*

$$\begin{tikzcd} \Gamma_{(u=0)}(\mathcal{X}) \arrow[r] \arrow[d] & \Gamma_{\Delta}(\mathcal{X}) \arrow[d] \\ X^- \arrow[r] & X \end{tikzcd}$$

Proof. By quasisyntomic descent, one reduces to looking at the values of these prestacks for C semiperfect. Here, we have $C^{\Delta} \otimes \mathbb{F}_p = \operatorname{Spec} \overline{\Delta}_C$ and also

$$C^{\mathcal{N}}_{(u=0)} = \mathcal{R}(\mathrm{Fil}^\bullet_{\mathrm{Hdg}} \overline{\Delta}_C).$$

Therefore we conclude using Proposition 4.30 and the argument from Proposition 5.11. $\square$

Now, over X^0, we have a canonical graded perfect complex associating with every $x : B\mathbb{G}_m \times \operatorname{Spec} C \to \mathcal{X}^\diamond$ in $X^0(C)$, the graded perfect complex associated with $x^*\mathbb{T}_{\mathcal{X}^\diamond}$. If x lifts to a point $\tilde{x} : \mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} C \to \mathcal{X}$, then this graded complex is the associated graded $\mathrm{gr}^\bullet_{\mathrm{Hdg}} M_{\tilde{x}}$ for the filtered perfect complex corresponding to $\tilde{x}^*\mathbb{T}_{\mathcal{X}}$. With this in mind, we will suggestively denote the graded perfect complex over X^0 by $\mathrm{gr}^\bullet_{\mathrm{Hdg}} M$.

We will now define two $\mathrm{Mod}^{\mathrm{cn}}_{\mathbb{F}_p}$-valued prestacks $\mathfrak{t}(\mathcal{X})$ and $\bar{\mathfrak{t}}(\mathcal{X})$ over X^0, given by:

$$\mathfrak{t}(\mathcal{X}) : (C, x) \mapsto \tau^{\leq 0}(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M_x \otimes_C Z^1_{\Delta}(C));$$
$$\bar{\mathfrak{t}}(\mathcal{X}) : (C, x) \mapsto \tau^{\leq 0}(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M_x \otimes_C \mathrm{gr}^{\mathrm{conj}}_1 \overline{\Delta}_C).$$

These are both quasisyntomic sheaves over R.

us_kernel **Observation 8.13.2.** *The natural map $\Gamma_{(t=0)}(\mathcal{X}) \to X^+$ of ${}^\varphi X^0$-stacks is a quasisyntomic torsor under $\mathfrak{t}(\mathcal{X})$.*

Proof. Suppose that C is semiperfect. Then, using Theorem 6.32, we have a canonical isomorphism

$$C^{\mathcal{N}}_{(t=0)} \simeq \left(\operatorname{Spec} \bigoplus_{i=0}^{\infty} \mathrm{gr}^i_{\mathcal{N}} \Delta_C \cdot t^{-i}\right) / \mathbb{G}_m,$$

using which and Proposition 4.27 we find that

$$\begin{aligned} \Gamma_{(t=0)}(\mathcal{X})(C) &= \mathrm{Map}\left(\left(\operatorname{Spec} \bigoplus_{i=0}^{\infty} \mathrm{gr}^i_{\mathcal{N}} \Delta_C \cdot t^{-i}\right) / \mathbb{G}_m, \mathcal{X}\right) \\ &\simeq \mathrm{Map}\left(\operatorname{Spec}(\mathrm{Fil}^{\mathrm{conj}}_1 \overline{\Delta}_C \cdot t^{-1} \oplus C \cdot t^0)/\mathbb{G}_m, \mathcal{X}\right). \end{aligned}$$

Similarly, we have

$$X^+(C) \simeq \mathrm{Map}\left(\operatorname{Spec}(C \cdot u \oplus C \cdot u^0)/\mathbb{G}_m, \mathcal{X}\right).$$

Using graded deformation theory (see (4.5) for the filtered analogue), we find that the composition $\Gamma_{(t=0)}(\mathcal{X}) \to X^+ \to {}^{\varphi}X^0$ when restricted over the small étale site of $\operatorname{Spec} C$ is a torsor under the assignment

$$(C', x') \mapsto \operatorname{gr}^{-1}_{\mathrm{Hdg}} M_{x'} \otimes_C \operatorname{Fil}^{\mathrm{conj}}_1 \overline{\Delta}_{C'},$$

while $X^+ \to {}^{\varphi}X^0$ is a torsor under

$$(C', x') \mapsto \operatorname{gr}^{-1}_{\mathrm{Hdg}} M_{x'}.$$

From this, and the identity

$$Z^1_{\Delta}(C') \simeq \operatorname{hker}(\operatorname{Fil}^{\mathrm{conj}}_1 \overline{\Delta}_{C'} \to C'),$$

the observation now follows in the semiperfect case.

The general case follows from this via quasisyntomic descent. ☐

Completely analogously, we have:

mu_kernel **Observation 8.13.3.** *The natural map* ${}^{\varphi}\Gamma_{(u=t=0)}(\mathcal{X}) \to {}^{\varphi}X^0$ *is a quasisyntomic torsor under* $\bar{\mathfrak{t}}(\mathcal{X})$.

As a consequence of the first two observations, we find that

$$\Gamma_{(u=0)}(\mathcal{X}) \times_{\Gamma_{\Delta}(\mathcal{X})} \Gamma_{(t=0)}(\mathcal{X}) \to X^- \times_X X^+$$

is a quasisyntomic torsor under $\mathfrak{t}(\mathcal{X})$.

We now have two maps

$$(\iota_+^*, \mathrm{id}), ({}^{\varphi}\iota_-^*, \mathrm{id}) : Z' \overset{\mathrm{defn}}{=} \left(\Gamma_{(u=0)}(\mathcal{X}) \times_{\Gamma_{\Delta}(\mathcal{X})} \Gamma_{(t=0)}(\mathcal{X})\right) \times_{X^- \times_X X^+} \Gamma_{F\mathrm{Zip}}(\mathcal{X}) \to {}^{\varphi}\Gamma_{(u=t=0)}(\mathcal{X}) \times_{{}^{\varphi}X^0} \Gamma_{F\mathrm{Zip}}(\mathcal{X}) \overset{\mathrm{defn}}{=} Z.$$

Over $\Gamma_{F\mathrm{Zip}}(\mathcal{X})$, we have the two maps

$$q_1, q_2 : \mathfrak{t}(\mathcal{X})|_{\Gamma_{F\mathrm{Zip}}(\mathcal{X})} \to \bar{\mathfrak{t}}(\mathcal{X})|_{\Gamma_{F\mathrm{Zip}}(\mathcal{X})}$$

obtained as follows. The first map is actually already defined over X^0: Given $x \in X^0(C)$, it corresponds to

$$q_{1,x} : \operatorname{gr}^{-1}_{\mathrm{Hdg}} M_x \otimes_C Z^1_{\Delta}(C) \xrightarrow{\mathrm{id}\otimes q_1} \operatorname{gr}^{-1}_{\mathrm{Hdg}} M_x \otimes_C \operatorname{gr}^{\mathrm{conj}}_1 \overline{\Delta}_C.$$

For the second, suppose that we have $x \in \Gamma_{F\mathrm{Zip}}(\mathcal{X})(C)$. Pulling $\mathbb{T}_{\mathcal{X}}$ back along this point gives us the F-zip attached to a tuple $(\operatorname{Fil}^{\bullet}_{\mathrm{Hdg}} M_x, \operatorname{Fil}^{\mathrm{conj}}_{\bullet} M_x, \eta, \alpha)$. This yields a φ-semilinear endomorphism $\psi(\mathsf{M}(\mathcal{X})_x)$ of $\operatorname{gr}^{-1}_{\mathrm{Hdg}} M_x$, whose linearization is given by

$$\varphi^* \operatorname{gr}^{-1}_{\mathrm{Hdg}} M_x \xrightarrow[\simeq]{\alpha} \operatorname{Fil}^{\mathrm{conj}}_{-1} M_x \to M_x \to \operatorname{gr}^{-1}_{\mathrm{Hdg}} M_x.$$

We now set

$$q_{2,x} : \operatorname{gr}^{-1}_{\mathrm{Hdg}} M_x \otimes_C Z^1_{\Delta}(C) \xrightarrow{\psi(\mathsf{M}(\mathcal{X}))_x \otimes q_2} \operatorname{gr}^{-1}_{\mathrm{Hdg}} M_x \otimes_C \operatorname{gr}^{\mathrm{conj}}_1 \overline{\Delta}_C.$$

Given (8.7.2) and (8.10.1), the proof of the theorem is now completed by the two following observations:

ariance_1 **Observation 8.13.4.** $(\iota_+^*, \mathrm{id}) : Z' \to Z$ *is equivariant for the map* q_1

Proof. Straightforward. ☐

ariance_2 **Observation 8.13.5.** $({}^{\varphi}\iota_-^*, \mathrm{id}) : Z' \to Z$ *is equivariant for the map* q_2.

Proof. Suppose that we have a section in $\Gamma_{F\mathrm{Zip}}(\mathcal{X})(C)$ corresponding to $(x^+, x^-) \in X^+(C) \times X^-(C)$ and associated sections $x \in X(C)$ and $x^0 \in {}^{\varphi}X^0(C)$. Then we have an $(x^0)^*\mathfrak{t}(\mathcal{X})$-torsor

$$\Gamma_{(t=0)}(\mathcal{X})_{x^+} \to \operatorname{Spec} C,$$

and an $(x^0)^*\bar{\mathfrak{t}}(\mathcal{X})$-torsor

$${}^{\varphi}\Gamma_{(u=t=0)}(\mathcal{X})_{x^0} \to \operatorname{Spec} C.$$

There are maps

$$\Gamma_{(t=0)}(\mathcal{X})_{x^+} \to \Gamma_{\Delta}(\mathcal{X})_x \xrightarrow[\simeq]{\text{Obs.8.13.1}} \Gamma_{(u=0)}(\mathcal{X})_{x^-} \xrightarrow{{}^{\varphi}\iota_-^*} \Gamma_{(u=t=0)}(\mathcal{X})_{x^0}.$$

The claim being made here is that the composition of these maps is equivariant for the map q_2. ☐

□

_1_repble

Corollary 8.14. *Suppose that $\mathcal{M}$ is a 1-bounded perfect F-gauge of level 1 with Tor amplitude $[-r,s]$ over R. Then the prestack $\Gamma_{\mathrm{syn}}(\mathcal{M})$ over R is represented by a finitely presented derived Artin r-stack.*

Proof. The vector stack $\mathbf{V}(\mathcal{M}) \to R^{\mathrm{syn}} \otimes \mathbb{F}_p$ is a finitely presented r-stack, and Example 4.22 shows that it can be extended to a 1-bounded stack—denoted by the same symbol—by bringing along the entire fixed point stack.

Moreover, the associated stacks $X^{-,(n)}$, $X^{+,(n)}$ and $X^{0,(n)}$ admit the following explicit description: Associated with $\mathcal{M}$ is an F-zip giving in particular the pair $(\mathrm{Fil}^{\bullet}_{\mathrm{Hdg}} M, \mathrm{Fil}^{\mathrm{conj}}_{\bullet} M)$; then $X^{-,(n)}$ (resp. $X^{+,(n)}$, $X^{0,(n)}$) is the mod-p^n Weil restriction of the vector stack associated with $\mathrm{Fil}^0_{\mathrm{Hdg}} M$ (resp. $\mathrm{Fil}^{\mathrm{conj}}_0 M$, $\mathrm{gr}^0_{\mathrm{Hdg}} M$). In particular, all three stacks are representable.

Therefore, Theorem 8.12, along with Remark 8.13, tells us that $\Gamma_{\mathrm{syn}}(\mathbf{V}(\mathcal{M})) = \Gamma_{\mathrm{syn}}(\mathcal{M})$ is represented by a finitely presented derived Artin stack. □

fficients

8.15. **Bootstrapping from characteristic p: coefficients.** Continue to assume that R is an $\mathbb{F}_p$-algebra. Suppose now that we have a 1-bounded stack $\mathcal{X} \to R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. For $m \leq n$, write $\mathcal{X}_m$ for its restriction over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^m\mathbb{Z}$.

Over $\Gamma_{\mathrm{syn}}(\mathcal{X}_1)$, we have a 1-bounded perfect F-gauge $\mathcal{M}_1(X)$ of level 1: this associates with every $x \in \Gamma_{\mathrm{syn}}(\mathcal{X}_1)(C)$ the pullback to $C^{\mathrm{syn}} \otimes \mathbb{F}_p$ of the tangent complex of $\mathcal{X}_1^{\diamond}$ over $R^{\mathrm{syn}} \otimes \mathbb{F}_p$. In turn, for any $i \in \mathbb{Z}$, this gives us via Corollary 8.14 a $\mathrm{Mod}^{\mathrm{cn}}_{\mathbb{F}_p}$-valued relative locally finitely presented derived Artin stack $\Gamma(\mathcal{M}_1(\mathcal{X})[i]) \to \Gamma_{\mathrm{syn}}(\mathcal{X}_1)$ associating with each x the derived Artin stack

$$\Gamma(\mathcal{M}_1(\mathcal{X})[i])_x = \Gamma_{\mathrm{syn}}(\mathcal{M}_1(\mathcal{X})_x[i])$$

over C.

ng_coeffs

Proposition 8.16. *For each $m < n$, there is a canonical Cartesian square*

$$\begin{tikzcd} \Gamma_{\mathrm{syn}}(\mathcal{X}_{m+1}) \arrow[r] \arrow[d] & \Gamma_{\mathrm{syn}}(\mathcal{X}_m) \arrow[d, "0"] \\ \Gamma_{\mathrm{syn}}(\mathcal{X}_m) \arrow[r] & \Gamma(\mathcal{M}_1(\mathcal{X})[1]) \times_{\Gamma_{\mathrm{syn}}(\mathcal{X}_1)} \Gamma_{\mathrm{syn}}(\mathcal{X}_m), \end{tikzcd}$$

where the left vertical arrow is the natural map. In particular, if $\mathcal{X}$ is a relative r-stack with quasi-affine diagonal and $\pi_0(R)$ is a G-ring—or, more generally, if $X^{-,(n)}$, $X^{+,(n)}$ and $X^{0,(n)}$ are representable—then $\Gamma_{\mathrm{syn}}(\mathcal{X})$ is represented by a locally finitely presented derived Artin r-stack over R.

Proof. The second assertion follows from Theorem 8.12, Remark 8.13, and Corollary 8.14.

The first assertion is an application of deformation theory. Let $\mathcal{M}_1^{\mathcal{N}}(\mathcal{X})$ (resp. $\mathcal{M}_1^{\mathbb{\Delta}}(\mathcal{X})$) be the perfect complex over the Nygaard filtered prismatization of $\Gamma_{\mathcal{N}}(\mathcal{X}_1)$ (resp. over the prismatization of $\Gamma_{\mathbb{\Delta}}(\mathcal{X}_1)$) obtained similarly to $\mathcal{M}_1(\mathcal{X})$ by pulling back the relative tangent complex of $\mathcal{X}^{\diamond}$ along the tautological map.

From the first relative vector stack, we obtain a prestack over $\Gamma_{\mathcal{N}}(\mathcal{X}_1)$ given on pairs (C,x) with $x \in \Gamma_{\mathcal{N}}(\mathcal{X}_1)(C)$ by

$$\Gamma_{\mathcal{N}}(\mathcal{M}_1(\mathcal{X}[1])) : (C,x) \mapsto \tau^{\leq 0} R\Gamma(C^{\mathcal{N}} \otimes \mathbb{F}_p, \mathcal{M}_1^{\mathcal{N}}(\mathcal{X})_x[1]).$$

Similarly, over $\Gamma_{\mathbb{\Delta}}(\mathcal{X}_1)$, we obtain a prestack given on pairs (C,x) with $x \in \Gamma_{\mathbb{\Delta}}(\mathcal{X}_1)(C)$ by

$$\Gamma_{\mathbb{\Delta}}(\mathcal{M}_1(\mathcal{X}[1])) : (C,x) \mapsto \tau^{\leq 0} R\Gamma(C^{\mathbb{\Delta}} \otimes \mathbb{F}_p, \mathcal{M}_1^{\mathbb{\Delta}}(\mathcal{X})_x[1]).$$

Suppose that C is semiperfect to fix ideas: we can reduce to considering only such inputs by quasi-syntomic descent.

Let us look at the fibers of the map

$$\Gamma_{\mathcal{N}}(\mathcal{X}_{m+1})(C) = \mathrm{Map}(C^{\mathcal{N}} \otimes \mathbb{Z}/p^{m+1}\mathbb{Z}, \mathcal{X}) \to \mathrm{Map}(C^{\mathcal{N}} \otimes \mathbb{Z}/p^m\mathbb{Z}, \mathcal{X}) = \Gamma_{\mathcal{N}}(\mathcal{X}_m)(C)$$

Now, by Theorem 6.32, $C^{\mathcal{N}} \otimes \mathbb{Z}/p^r\mathbb{Z}$ is the Rees stack $\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}} \mathbb{\Delta}_C) \otimes \mathbb{Z}/p^r\mathbb{Z}$. Then by the discussion in (4.5) we have a canonical Cartesian square

$$\begin{tikzcd}
\Gamma_{\mathcal{N}}(\mathcal{X}_{m+1})(C) \arrow[r] \arrow[d] & \Gamma_{\mathcal{N}}(\mathcal{X}_m)(C) \arrow[d] \\
\Gamma_{\mathcal{N}}(\mathcal{X}_m)(C) \arrow[r] & \mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}} \mathbb{\Delta}_C/^{\mathbb{L}}p^m \oplus \mathrm{Fil}^\bullet_{\mathcal{N}} \overline{\mathbb{\Delta}}_C[1]), \mathcal{X}).
\end{tikzcd}$$

Moreover, the prestack over R given by

$$C \mapsto \mathrm{Map}(\mathcal{R}(\mathrm{Fil}^\bullet_{\mathcal{N}} \mathbb{\Delta}_C/^{\mathbb{L}}p^m \oplus \mathrm{Fil}^\bullet_{\mathcal{N}} \overline{\mathbb{\Delta}}_C[1]), \mathcal{X})$$

is canonically isomorphic over $\Gamma_{\mathcal{N}}(\mathcal{X}_m)$ to the pullback of the stack $\Gamma_{\mathcal{N}}(\mathcal{M}_1(\mathcal{X})[1])$, showing that we have a Cartesian diagram of prestacks over $\Gamma_{\mathcal{N}}(\mathcal{X}_m)$:

$$\begin{tikzcd}
\Gamma_{\mathcal{N}}(\mathcal{X}_{m+1}) \arrow[r] \arrow[d] & \Gamma_{\mathcal{N}}(\mathcal{X}_m) \arrow[d, "0"] \\
\Gamma_{\mathcal{N}}(\mathcal{X}_m) \arrow[r] & \Gamma_{\mathcal{N}}(\mathcal{M}_1(\mathcal{X})[1]) \times_{\Gamma_{\mathcal{N}}(\mathcal{X}_1)} \Gamma_{\mathcal{N}}(\mathcal{X}_m)
\end{tikzcd}$$

There exists an analogous Cartesian diagram with $\mathcal{N}$ replaced with $\mathbb{\Delta}$. Combining these two diagrams with (8.7.1) now proves the Proposition. □

Remark 8.17. Note that the establishment of the Cartesian diagram in the Proposition above did not use the hypothesis that R is an $\mathbb{F}_p$-algebra: One simply has to replace the word 'semiperfect' by 'semiperfectoid' in the proof.

ef_theory

8.18. **Deformation theory.** We continue with the assumptions of the previous subsection. Suppose that we have a divided power thickening $(C' \twoheadrightarrow C, \gamma)$ of R-algebras. Then, using the notation of (3.13), we have a canonical commuting square

ng_square (8.18.1)

$$\begin{tikzcd}
\Gamma_{\mathrm{syn}}(\mathcal{X})(C') \arrow[r] \arrow[d] & X^{-,(n)}(C) \arrow[d] \\
\Gamma_{\mathrm{syn}}(\mathcal{X})(C) \arrow[r] & X^{-,(n)}(C) \times_{X^{(n)}(C)} X^{(n)}(C').
\end{tikzcd}$$

This is obtained as follows: The top arrow and the first coordinate of the bottom arrow are obtained from the canonical map $\Gamma_{\mathrm{syn}}(\mathcal{X}) \to X^{-,(n)}$ obtained via pullback along the mod-p^n reduction of the map

$$x^{\mathcal{N}}_{\mathrm{dR}} : \mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, C \to C^{\mathcal{N}}$$

for every R-algebra C. The second coordinate of the bottom arrow is obtained via pullback along the (mod-p^n reduction of the) lift

$$\tilde{x}_{\mathrm{dR},C'} : \mathrm{Spec}\, C' \to C^{\mathbb{\Delta}}$$

from Lemma 6.20.

ing_mod_p

Theorem 8.19. *Suppose that $\mathcal{X}$ is 1-bounded and that $\Gamma_{\mathrm{syn}}(\mathcal{X})$ is representable*[11]*. Let $(C' \twoheadrightarrow C, \gamma)$ be a* nilpotent *divided power thickening. Then the commuting square* (8.18.1) *is* Cartesian*.*

[11] We will only need it to be infinitesimally cohesive.

Proof. Write

$$\alpha_{(C'\twoheadrightarrow C,\gamma)}:\Gamma_{\mathrm{syn}}(\mathcal{X})(C')\to\Gamma_{\mathrm{syn}}(\mathcal{X})(C)\times_{X^{-,(n)}(C)\times_{X^{(n)}(C)}X^{(n)}(C')}X^{-,(n)}(C')$$

for the natural map. We would like to show that it is an equivalence when the divided power thickening is nilpotent.

Suppose that we have a triangle

$$\begin{tikzcd} C'' \arrow[r, two heads] \arrow[dr, two heads] & C' \arrow[d, two heads] \\ & C \end{tikzcd}$$

of maps underlying a triangle of nilpotent divided power thickenings

$$\begin{tikzcd} (C''\twoheadrightarrow C',\gamma') \arrow[r] \arrow[dr] & (C''\twoheadrightarrow C,\gamma'') \arrow[d] \\ & (C'\twoheadrightarrow C,\gamma). \end{tikzcd}$$

Then one easily finds that, if $\alpha_{(C'\twoheadrightarrow C,\gamma)}$ and $\alpha_{(C''\twoheadrightarrow C',\gamma')}$ are equivalences, then so is $\alpha_{(C''\twoheadrightarrow C,\gamma'')}$.

By Remark 3.20, we can factor $C'\twoheadrightarrow C$ into a sequence of divided power thickenings

$$C'=C'_m\twoheadrightarrow C'_{m-1}\twoheadrightarrow\cdots\twoheadrightarrow C'_1\twoheadrightarrow C'_0=C,$$

where each intermediate thickening $(C'_j\to C'_{j-1},\gamma_j)$ is trivial and square-zero. Therefore, repeatedly using the discussion above, we can reduce to the case where $C'\twoheadrightarrow C$ is a square-zero extension with the trivial divided power structure.

In this case, since $\Gamma_{\mathrm{syn}}(\mathcal{X})$ is representable by Proposition 8.16, and hence infinitesimally cohesive, it suffices to verify the theorem in the situation where $C'=C\oplus M[1]$ for some $M\in\mathrm{Mod}^{\mathrm{cn}}_C$. In particular, we can assume that $\mathrm{hker}(C'\to C)$ is 1-connective.

By quasisyntomic descent, we can reduce further to the case where C' and C are semiperfect. In this case, we are in the situation of Proposition 5.26 with $\underline{B}=\underline{\mathbb{\Delta}}_{C'}$ and $\underline{A}=\underline{\mathbb{\Delta}}_C$. Therefore, by Remark 5.28, it only remains to check two things:

(i) The map $\mathbb{\Delta}_{C'}\to\mathbb{\Delta}_C$ is surjective: this follows from [7, Remark 4.1.20]
(ii) If we set

$$K=\mathrm{hker}(\underline{\mathbb{\Delta}}_{C'}\to\underline{\mathbb{\Delta}}_C),$$

then the map $\dot{\varphi}_1:K/^{\mathbb{L}}p\to K/^{\mathbb{L}}p$ is locally nilpotent: this follows from Lemma 6.35.

□

ping_base

8.20. **Bootstrapping from characteristic p: the base.** We will now take R to be in $\mathrm{CRing}^{f,p\text{-comp}}$ for some $m\geq 1$. For any p-adic formal prestack Z over R set $\mathsf{R}(Z)=Z^{(1)}$, so that $\mathsf{R}(Z)(C)=Z(C/^{\mathbb{L}}p)$ for any $C\in\mathrm{CRing}^f_{R/}$. This gives an endomorphism of the ∞-category of p-adic formal prestacks over R, and so can be iterated: We have

$$\mathsf{R}^t(Z)(C)=Z(C\otimes\mathbb{F}_p^{\otimes_{\mathbb{Z}}t}).$$

The key for us is the following systematic dévissage from characteristic p:

devissage

Proposition 8.21. *Let $\mathcal{X}\to R^{\mathrm{syn}}\otimes\mathbb{Z}/p^m\mathbb{Z}$ be a 1-bounded stack. For any $C\in\mathrm{CRing}^{f,p\text{-comp}}_{R/}$, the canonical map*

$$\Gamma_{\mathrm{syn}}(\mathcal{X})(C)\to\mathrm{Tot}\left(\Gamma_{\mathrm{syn}}(\mathcal{X})(C\otimes_{\mathbb{Z}}\mathbb{F}_p^{\otimes_{\mathbb{Z}}\bullet+1})\right)$$

is an equivalence. That is, we have an equivalence of p-adic formal prestacks

$$\Gamma_{\mathrm{syn}}(\mathcal{X})\xrightarrow{\simeq}\mathrm{Tot}\left(\mathsf{R}^{\bullet+1}(\Gamma_{\mathrm{syn}}(\mathcal{X}))\right).$$

Now, if $(C' \twoheadrightarrow C, \gamma)$ is a divided power thickening of R-algebras, we obtain the canonical commuting square (8.18.1).[12]

h_messing **Corollary 8.22** (Grothendieck-Messing). *Suppose that $\mathcal{X}$ is 1-bounded and $\Gamma_{\mathrm{syn}}(\mathcal{X}) \otimes \mathbb{F}_p$ is representable. Then, if $(C' \twoheadrightarrow C, \gamma)$ is a* nilpotent *divided power thickening, the commuting square* (8.18.1) *is* Cartesian.

Proof. Nilpotence of divided powers is preserved under arbitrary base change along maps $C' \to D'$. Therefore, for every $m \geq 1$, the map

$$C' \otimes \mathbb{F}_p^{\otimes_{\mathbb{Z}} \bullet + 1} \twoheadrightarrow C \otimes \mathbb{F}_p^{\otimes_{\mathbb{Z}} \bullet + 1}$$

of cosimplicial $R \otimes \mathbb{F}_p$-algebras canonically lifts to a cosimplicial diagram of nilpotent divided power thickenings of $R \otimes \mathbb{F}_p$-algebras. This gives us a cosimplicial diagram of commuting squares as in (8.18.1), which are all Cartesian by Theorem 8.19. We conclude by Proposition 8.21, which now shows that the commuting square the corollary is concerned with is a limit of Cartesian ones. $\square$

As an immediate consequence, we obtain:

t_complex **Corollary 8.23.** *With the hypotheses above, write $\varpi_{\mathcal{X}} : \Gamma_{\mathrm{syn}}(\mathcal{X}) \to X^{-,(n)}$ for the canonical map. Then $\Gamma_{\mathrm{syn}}(\mathcal{X})$ admits a cotangent complex over R, and we have*

$$\mathbb{L}_{\Gamma_{\mathrm{syn}}(\mathcal{X})/R} \simeq \varpi_{\mathcal{X}}^* \mathbb{L}_{X^{-,(n)}/X^{(n)}}.$$

_explicit *Remark* 8.24. We can describe $\mathbb{L}_{X^{-,(n)}/X^{(n)}}$ explicitly in terms of the relative cotangent complex $\mathbb{L}_{\mathcal{X}^\diamond}$ over R^{syn}. First, note that $X^{-,(n)}$ is the mod-p^n Weil restriction of the attractor stack X^- over $R/^{\mathbb{L}}p^n$associated with $\mathcal{X}$. Now, if $\mathrm{Fil}^\bullet \mathbb{L}_X^-$ is the associated filtered perfect complex from Lemma 4.9, we have

$$\mathbb{L}_{X^-/(R/^{\mathbb{L}}p^n)} \simeq \mathbb{L}_X^- / \mathrm{Fil}^1 \mathbb{L}_X^-.$$

Note that $\mathbb{L}_X^-$ is simply the restriction of the cotangent complex $\mathbb{L}_{X/(R/^{\mathbb{L}}p^n)}$ along the natural map $X^- \to X$.

Therefore using the fundamental fiber sequence for the cotangent complex we obtain a natural isomorphism

$$\mathbb{L}_{X^-/X} \simeq \mathrm{Fil}^1 \mathbb{L}_X^-[1]$$

Lemma 3.15 now shows that, for any point $x \in X^{-,(n)}(C)$ corresponding to a point $\overline{x} \in X^-(C/^{\mathbb{L}}p^n)$, we have

$$\mathbb{L}_{X^{-,(n)}/X^{(n)},x} \simeq \mathrm{Fil}^1 \mathbb{L}_{X,\overline{x}}^-,$$

where the right hand side is a perfect complex over $C/^{\mathbb{L}}p^n$ but viewed as a perfect complex over C.

Assuming Proposition 8.21, we can now show:

ping_base **Theorem 8.25.** *Suppose that $\mathcal{X}$ is a 1-bounded r-stack over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. Suppose that one of the following holds:*

(1) $\mathcal{X}^\diamond$ has quasi-affine diagonal and $\pi_0(R)$ is a G-ring;
(2) The p-adic formal stacks $X^{-,(n)}$, $X^{+,(n)}$ and $X^{0,(n)}$ over R are representable.

Then $\Gamma_{\mathrm{syn}}(\mathcal{X})$ is represented by a p-adic formal locally finitely presented Artin r-stack over R.

Proof. By Proposition 8.16, we know that, under our hypotheses, $\Gamma_{\mathrm{syn}}(\mathcal{X}) \otimes \mathbb{F}_p$ is represented by a locally finitely presented Artin r-stack over $R/^{\mathbb{L}}p$.

If $p > 2$, then applying that result to the natural nilpotent divided power structure on $R \twoheadrightarrow R/^{\mathbb{L}}p$, we obtain a Cartesian square of prestacks

$$\begin{tikzcd}
\Gamma_{\mathrm{syn}}(\mathcal{X}) \arrow[r] \arrow[d] & X^{-,(n)} \arrow[d] \\
\mathsf{R}(\Gamma_{\mathrm{syn}}(\mathcal{X})) \arrow[r] & X^{(n)} \times_{\mathsf{R}(X^{(n)})} \mathsf{R}(X^{-,(n)})
\end{tikzcd}.$$

[12]Strictly speaking, we had imposed the condition that R be an $\mathbb{F}_p$-algebra when we introduced this square; however, this hypothesis was not used in its construction.

The proposition now follows, since all the prestacks involved except for the one in the top left corner are known to be locally finitely presented derived Artin p-adic formal stacks over R.

If $p = 2$, then we can do something similar, by first considering the nilpotent divided power thickening $R \twoheadrightarrow R/^{\mathbb{L}}4$ to reduce to showing that $\Gamma_{\mathrm{syn}}(\mathcal{X}) \otimes \mathbb{Z}/4\mathbb{Z}$ is a smooth Artin stack over $\mathbb{Z}/4\mathbb{Z}$, and then using the trivial divided powers on the square zero extension $R \twoheadrightarrow R \otimes_{\mathbb{Z}/4\mathbb{Z}} \mathbb{F}_2$ (for $R \in \mathrm{CRing}^f_{(\mathcal{O}/4)/}$) to reduce further to the known case of Γ. ☐

si-smooth **Corollary 8.26.** *Suppose that $\mathcal{X}^\diamond$ is in addition a relative smooth scheme over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. Then $\Gamma_{\mathrm{syn}}(\mathcal{X})$ is a p-adic formal quasi-smooth derived algebraic space over R.*

Proof. Combining the descriptions of the cotangent complex given by Remark 8.24 shows that $\Gamma_{\mathrm{syn}}(\mathcal{X})$ has a perfect cotangent complex over R with Tor amplitude in $[-1, 0]$. ☐

We now proceed towards the proof of Proposition 8.21. Let us say that a map $f : Z \to Y$ of p-adic formal prestacks over R^{syn} **satisfies Tot descent for** $\mathcal{X}^\diamond$ if the natural map

$$\mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(Y \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}^\diamond) \to \mathrm{Tot}\left(\mathrm{Map}_{/R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z}}(Z^{\times_Y(\bullet+1)} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}^\diamond)\right)$$

is an equivalence. The map **satisfies universal Tot descent for** $\mathcal{X}^\diamond$ if, for any $Y' \to Y$, the base-change $Z \times_Y Y' \to Y'$ also satisfies Tot descent for $\mathcal{X}^\diamond$

t_descent *Remark* 8.27. Any flat cover satisfies universal Tot descent for $\mathcal{X}^\diamond$.

t_descent *Remark* 8.28. A composition of maps satisfying (universal) Tot descent for $\mathcal{X}^\diamond$ also satisfies (universal) Tot descent for $\mathcal{X}^\diamond$.

cent_flat *Remark* 8.29. Suppose that we have maps $Z \xrightarrow{f} Y \xrightarrow{g} V$ such that:

- $f \circ g$ satisfies Tot descent for $\mathcal{X}^\diamond$;
- f satisfies universal Tot descent for $\mathcal{X}^\diamond$

Then g also satisfies Tot descent for $\mathcal{X}^\diamond$. This follows because, from our assumption on f, we find that the map $f^{\times_V m} : Z^{\times_V m} \to Y^{\times_V m}$ also satisfies Tot descent for $\mathcal{X}^\diamond$ for all $m \geq 1$.

l_preygel *Remark* 8.30. We have the following observation of Halpern-Leistner and Preygel: Suppose that we have $A \in \mathrm{CRing}$ equipped with a map $\mathbb{Z}[T_1, \ldots, T_r] \to A$ such that A is derived J-complete, where $J = (T_1, \ldots, T_r) \subset \mathbb{Z}[T_1, \ldots, T_r]$; set $\overline{A} = A/^{\mathbb{L}}(T_1, \ldots, T_r)$. Suppose that we have a J-adic formal Artin stack $\mathcal{Y}$ over A. Then, for $R \in \mathrm{CRing}_{A/}$ derived J-complete, the map

$$\mathcal{Y}(R) \to \mathrm{Tot}\left(\mathcal{Y}(R \otimes_A \overline{A}^{\otimes^{\mathbb{L}}_A \bullet+1})\right)$$

is an equivalence. In fact, one only needs for $\mathcal{Y}$ to be nilcomplete and infinitesimally cohesive; see [22, Cor. 3.1.4]. So Proposition 8.21 is certainly implied by Theorem D. Here we will use the former to complete the proof of the latter.

_HT_locus **Lemma 8.31.** *The map $C^{\mathbb{A}} \to C^{\mathbb{A}} \times_{\mathbb{Z}_p^{\mathbb{A}}} \mathbb{Z}_p^{\mathrm{HT}}$ satisfies Tot descent for $\mathcal{X}^\diamond$.*

Proof. Via quasisyntomic descent once again, we reduce to the case where C is semiperfectoid. Let $I \xrightarrow{t} \mathbb{A}_C$ be the generalized Cartier divisor on $\mathbb{A}_C$ underlying its structure of a prism, so that we have

$$C^{\mathbb{A}} \simeq \mathrm{Spf}(\mathbb{A}_C)\ ;\ C^{\mathbb{A}} \times_{\mathbb{Z}_p^{\mathbb{A}}} \mathbb{Z}_p^{\mathrm{HT}} \simeq \mathrm{Spf}(\mathbb{A}_C)_{(t=0)}.$$

Now, $\mathbb{A}_C$ here is equipped with its I-adic topology with respect to which it is derived complete. Therefore, the lemma follows from Remark 8.30. ☐

_HT_locus **Lemma 8.32.** *The map*

$$\mathbb{Z}_p^{\mathrm{HT}} \to \mathbb{F}_p^{\mathrm{HT}}$$

satisfies universal Tot descent for $\mathcal{X}^\diamond$

Proof. We will use Proposition 6.8, which shows (via Remark 8.27) that the map

$$\mathbb{Z}_p^{\mathrm{HT}} \to \operatorname{Spf}\mathbb{Z}_p$$

saitisfies universal Tot descent for $\mathcal{X}$.

It is now enough to show (see Remark 8.29) that the map

$$\operatorname{Spf}\mathbb{Z}_p \to \operatorname{Spec}\mathbb{F}_p$$

satisfies universal Tot descent for $\mathcal{X}^\diamond$. This follows from Remark 8.30 and the fact that we are dealing with p-adic formal stacks. □

ng_needed **Lemma 8.33.** *Suppose that we have $C \in \mathrm{CRing}_{R/}^{f,p\text{-}nilp}$. Then we have a canonical Cartesian square*

$$\begin{tikzcd} \operatorname{Map}(C^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}) \arrow[r] \arrow[d] & \operatorname{Map}(C^{\Delta} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}^\diamond) \arrow[d] \\ X^-(C/^{\mathbb{L}}p^n) \arrow[r] & X(C/^{\mathbb{L}}p^n). \end{tikzcd}$$

Proof. Via quasisyntomic descent, we reduce to the case of C semiperfectoid, where this follows from Theorem 6.32 and Proposition 4.30. □

Proof of Proposition 8.21. The limit preserving property of the functors $\operatorname{Spec} C \mapsto C^{\Delta}$ and $\operatorname{Spec} C \mapsto C^{\mathcal{N}}$ shows that we have

$$C^{\Delta} \times_{\mathbb{Z}_p^{\Delta}} \underbrace{\mathbb{F}_p^{\Delta} \times_{\mathbb{Z}_p^{\Delta}} \times \cdots \times_{\mathbb{Z}_p^{\Delta}} \mathbb{F}_p^{\Delta}}_{\bullet+1} \xrightarrow{\simeq} (C \otimes \mathbb{F}_p^{\otimes \bullet+1})^{\Delta};$$

$$C^{\mathcal{N}} \times_{\mathbb{Z}_p^{\mathcal{N}}} \underbrace{\mathbb{F}_p^{\mathcal{N}} \times_{\mathbb{Z}_p^{\mathcal{N}}} \times \cdots \times_{\mathbb{Z}_p^{\mathcal{N}}} \mathbb{F}_p^{\mathcal{N}}}_{\bullet+1} \xrightarrow{\simeq} (C \otimes \mathbb{F}_p^{\otimes \bullet+1})^{\mathcal{N}}.$$

Lemmas 8.31 and 8.32 together show that the composition

$$C^{\Delta} \to C^{\Delta} \times_{\mathbb{Z}_p^{\Delta}} \mathbb{F}_p^{\Delta} \to C^{\Delta} \times_{\mathbb{Z}_p^{\Delta}} \mathbb{F}_p^{\mathrm{HT}}$$

satisfies Tot descent for $\mathcal{X}^\diamond$, while the second map satisfies universal Tot descent for $\mathcal{X}^\diamond$. Therefore, Remark 8.29 now shows that $C^{\Delta} \to C^{\Delta} \times_{\mathbb{Z}_p^{\Delta}} \mathbb{F}_p^{\Delta}$ satisfies Tot descent for $\mathcal{X}^\diamond$.

This, combined with the discussion in the first paragraph, shows that we have

$$\operatorname{Map}(C^{\Delta} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}^\diamond) \xrightarrow{\simeq} \operatorname{Tot}\operatorname{Map}\big((C \otimes \mathbb{F}_p^{\otimes \bullet+1})^{\Delta} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}^\diamond\big).$$

Now, Lemma 8.33 combined with Remark 8.30 tells us that we also have

$$\operatorname{Map}(C^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}) \xrightarrow{\simeq} \operatorname{Tot}\operatorname{Map}\big((C \otimes \mathbb{F}_p^{\otimes \bullet+1})^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}, \mathcal{X}\big).$$

The proof is now concluding by contemplating the identity (8.7.1). □

8.34. **Functoriality.** Suppose that we have a map $\mathcal{X}_1 \to \mathcal{X}_2$ of 1-bounded stacks over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ with quasi-affine diagonal. Then we obtain a map of derived stacks $\Gamma_{\mathrm{syn}}(\mathcal{X}_1) \to \Gamma_{\mathrm{syn}}(\mathcal{X}_2)$ over R. We also have the corresponding map of Weil restricted stacks $X_1^{-,(n)} \to X_2^{-,(n)}$ and $X_1^{(n)} \to X_2^{(n)}$.

The following is immediate from Corollary 8.23:

t_complex **Proposition 8.35.** *Let $\varpi_1 : \Gamma_{\mathrm{syn}}(\mathcal{X}_1) \to X_1^{-,(n)}$ be the canonical map. Then we have a natural isomorphism*

$$\mathbb{L}_{\Gamma_{\mathrm{syn}}(\mathcal{X}_1)/\Gamma_{\mathrm{syn}}(\mathcal{X}_2)} \xrightarrow{\simeq} \varpi_1^* \mathbb{L}_{X_1^{-,(n)}/(X_1^{(n)} \times_{X_2^{(n)}} X_2^{-,(n)})}.$$

ed_repble

8.36. **Sections of 1-bounded perfect F-gauges.** Suppose that $\mathcal{M}$ is a 1-bounded perfect F-gauge of level n over $R \in \mathrm{CRing}^{f,p\text{-comp}}$ and Hodge-Tate weights bounded by 1. Pulling $\mathcal{M}$ back along $x^{\mathcal{N}}_{\mathrm{dR}}$ yields an increasingly filtered perfect complex $\mathrm{Fil}^{\bullet}_{\mathrm{Hdg}} M_n$ over $R/^{\mathbb{L}}p^n$. The next theorem is immediate from Theorem 8.25 and Corollary 8.14.

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Theorem 8.37. *The prestack $\Gamma_{\mathrm{syn}}(\mathcal{M})$ is represented by a p-adic formal locally finitely presented derived Artin stack over R with cotangent complex $\mathscr{O}_{\Gamma_{\mathrm{syn}}(\mathcal{M})} \otimes_R (\mathrm{gr}^{-1}_{\mathrm{Hdg}} M_n)^{\vee}[1]$. Moreover, if $(C' \twoheadrightarrow C, \gamma)$ is a nilpotent divided power thickening in $\mathrm{CRing}^{f,p\text{-}comp}_{R/}$, then we have a Cartesian square*

$$\begin{tikzcd}
\Gamma_{\mathrm{syn}}(\mathcal{M})(C') \arrow[r] \arrow[d] & C' \otimes_R \mathrm{Fil}^0_{\mathrm{Hdg}} M_n \arrow[d] \\
\Gamma_{\mathrm{syn}}(\mathcal{M})(C) \arrow[r] & (C \otimes_R \mathrm{Fil}^0_{\mathrm{Hdg}} M_n) \times_{C \otimes_R M_n} (C' \otimes_R M_n).
\end{tikzcd}$$

Moreover, if $\mathcal{M}$ has Tor amplitude in $(-\infty, 0]$, then $\Gamma_{\mathrm{syn}}(\mathcal{M})$ is an affine *scheme over R.*

Proof. Only the last assertion still requires proof, and to show it, we can assume that R is an $\mathbb{F}_p$-algebra.

Let M be the underlying F-zip: It is a vector bundle F-zip and so, by hypothesis, and so (8.2.1) shows that $\Gamma_{F\mathrm{Zip}}(\mathsf{M})$ is an affine scheme over R.

By Theorem 8.12, $\Gamma_{\mathrm{syn}}(\mathcal{M}_1)$ is a torsor over $\Gamma_{F\mathrm{Zip}}(\mathsf{M})$ under a stack of the form $\mathsf{S}_{(N,\psi)}$ where (N, ψ) is a perfect φ-module over $\Gamma_{F\mathrm{Zip}}(\mathsf{M})$ with Tor amplitude in $[0, \infty)$.

The proof of Theorem 7.3 reduces the relative affineness of $\Gamma_{\mathrm{syn}}(\mathcal{M}_1)$ to that of $\mathsf{S}_{(N,\psi)}$ when (N, ψ) is a φ-module over R with N locally free. Here, the desired conclusion follows from Corollary 7.6, which shows that $\mathsf{S}_{(N,\psi)}$ is a finite flat group scheme over R.

We now conclude using the usual dévissage (say in the form of Proposition 8.16) that $\Gamma_{\mathrm{syn}}(\mathcal{M})$ is also affine over R. ☐

:HTwts_01

8.38. **Stacks of perfect F-zips of Hodge-Tate weights $0, 1$.** For every $n \geq 1$, let $\mathcal{X} \to \mathbb{Z}_p^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ be the 1-bounded stack obtained via base-change from $\mathcal{P}_{[0,1]} \to B\mathbb{G}_m$ as described in Example 4.23.

We will denote the associated formal prestack $\Gamma_{\mathrm{syn}}(\mathcal{X}) \to \mathbb{Z}_p$ by $\mathrm{Perf}^{\mathrm{syn}}_{[0,1],n}$. Concretely, this associates with every $R \in \mathrm{CRing}^{f,p\text{-nilp}}$ the ∞-groupoid $\mathrm{Perf}_{[0,1]}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})^{\simeq}$ of perfect F-gauges of level n over R with Hodge-Tate weights $0, 1$.

Over this prestack we have a canonical filtered perfect complex $\mathrm{Fil}^{\bullet}_{\mathrm{Hdg}} M_{\mathrm{taut}}$ obtained by viewing, for each R, the universal perfect F-gauge of level n as a perfect complex over R^{syn}, and pulling back along $x^{\mathcal{N}}_{\mathrm{dR}}$.

Since $\mathbb{Z}/p^m\mathbb{Z}$ is a G-ring for all m, we obtain the next theorem from Theorem 8.25 and the discussion in Example 4.23.

esentable

Theorem 8.39. *The prestack $\mathrm{Perf}^{\mathrm{syn}}_{[0,1],n}$ is represented by a p-adic formal locally finitely presented derived Artin stack over $\mathbb{Z}_p$ with cotangent complex $(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M_{\mathrm{taut}})^{\vee} \otimes \mathrm{Fil}^0_{\mathrm{Hdg}} M_{\mathrm{taut}}$. Moreover, if $(C' \twoheadrightarrow C, \gamma)$ is a nilpotent divided power thickening of p-complete algebras in CRing^f, then we have a Cartesian square*

$$\begin{tikzcd}
\mathrm{Perf}^{\mathrm{syn}}_{[0,1],n}(C') \arrow[r] \arrow[d] & \mathrm{Perf}_{[0,1]}(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} C'/^{\mathbb{L}}p^n) \arrow[d] \\
\mathrm{Perf}^{\mathrm{syn}}_{[0,1],n}(C) \arrow[r] & \mathrm{Perf}_{[0,1]}(\mathbb{A}^1/\mathbb{G}_m \times \operatorname{Spec} C/^{\mathbb{L}}p^n) \times_{\mathrm{Perf}(C/^{\mathbb{L}}p^n)} \mathrm{Perf}(C'/^{\mathbb{L}}p^n).
\end{tikzcd}$$

Proof. The only thing that needs still to be verified is the assertion about the cotangent complex. For this, note that we have

$$X^{-,(n)} : C \mapsto \mathrm{Perf}_{[0,1]}(\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spec}\, C/^{\mathbb{L}}p^n)$$
$$X^{(n)} : C \mapsto \mathrm{Perf}(C/^{\mathbb{L}}p^n).$$

The fiber of the map $X^{-,(n)} \to X^{(n)}$ over a perfect complex M over $C/^{\mathbb{L}}p^n$ classifies two step filtrations $\mathrm{Fil}^\bullet M$ on M with $\mathrm{gr}^i M$ perfect for all i, and $\mathrm{gr}^i M \simeq 0$ for $i \neq -1, 0$. Giving such a datum is equivalent to specifying the map $f : \mathrm{Fil}^0 M \to \mathrm{Fil}^{-1} M = M$ with $\mathrm{Fil}^0 M$ perfect over $C/^{\mathbb{L}}p^n$.

This shows that the tangent space of the map at M is canonically isomorphic to the space of maps $\mathrm{Fil}^0 M \to \mathrm{gr}^{-1} M$, which is of course the C-module $(\mathrm{Fil}^0 M)^\vee \otimes_C \mathrm{gr}^{-1} M$. Taking duals and using Corollary 8.23 now gives the desired cotangent complex. ☐

sec:main

9. Truncated displays: the main representability theorem

We can finally introduce the main protagonist of this paper. Fix a smooth affine group scheme G over $\mathbb{Z}_p$ and a 1-bounded cocharacter $\mu : \mathbb{G}_{m,\mathcal{O}} \to \underline{\mathrm{Aut}}(G_{\mathcal{O}})$ defined over the ring of integers $\mathcal{O}$ in a finite unramified extension of $\mathbb{Z}_p$ with residue field k.

sec:defns

9.1. **Definitions.** There is a canonical map $\mathbb{Z}_p^{\mathrm{syn}} \to B\mathbb{G}_m$ classifying the Breuil-Kisin twist from (6.17). The restriction to $\mathcal{O}^{\mathcal{N}}$ lifts to a map $B\mathbb{G}_{m,\mathcal{O}}$, which does not descend to $\mathcal{O}^{\mathrm{syn}}$; however, the composition with the map $B\mathbb{G}_{m,\mathcal{O}} \to B\mathbb{G}_{m,\mathcal{O}}^{\phi}$ from (4.11) *does* descend to a map

$$\mathcal{O}^{\mathrm{syn}} \to B\mathbb{G}_{m,\mathcal{O}}^{\phi}.$$

In particular, we can view the 1-bounded 'classifying stack' $\mathcal{B}(G,\mu)$ from Example 4.25 as a 1-bounded stack over $\mathcal{O}^{\mathrm{syn}}$, and hence $\mathcal{B}(G,\mu) \otimes \mathbb{Z}/p^n\mathbb{Z}$ as a 1-bounded stack over $\mathcal{O}^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$.

Definition 9.2. For any $R \in \mathrm{CRing}^{f,p\text{-comp}}_{\mathcal{O}/}$, we set $\mathrm{BT}_n^{G,\mu}(R) = \Gamma_{\mathrm{syn}}(\mathcal{B}(G,\mu) \otimes \mathbb{Z}/p^n\mathbb{Z})(R)$.

For any $R \in \mathrm{CRing}^{f,p\text{-comp}}_{\mathcal{O}/}$, an ***n*-truncated** (G,μ)**-display** over R is an object of the ∞-groupoid $\mathrm{BT}_n^{G,\mu}(R)$.

pointwise

Remark 9.3. For $R \in \mathrm{CRing}^{f}_{\mathbb{F}_p/}$, $\mathrm{BT}_n^{G,\mu}(R)$ can be described as the ∞-groupoid of $G\{\mu\}$-torsors $\mathcal{Q}$ over $R^{\mathcal{N}} \otimes (\mathbb{Z}/p^n\mathbb{Z})$ admitting flat-local sections, equipped with an equivalence $j_{dR}^*\mathcal{Q} \xrightarrow{\simeq} j_{HT}^*\mathcal{Q}$ of G-torsors over $R^{\Delta} \otimes (\mathbb{Z}/p^n\mathbb{Z})$, and satisfying the following condition: For every geometric point $R \to \kappa$ of $\mathrm{Spf}\, R$, the restriction of $(x_{dR}^{\mathcal{N}})^*\mathcal{Q}$ over $B\mathbb{G}_m \times \mathrm{Spec}\, \kappa$ is trivial.

erfectoid

9.4. **The semiperfectoid case.**

erf_etale

Lemma 9.5. *Suppose that R is semiperfectoid and that $R \to S$ is étale. Then S is also semiperfectoid. Moreover, the canonical map of frames $\underline{\Delta}_R \to \underline{\Delta}_S$ is (p, I_R)-completely étale, and we have*

$$\Delta_S \otimes_{\Delta_R} \mathrm{Fil}^\bullet_{\mathcal{N}} \Delta_R \xrightarrow{\simeq} \mathrm{Fil}^\bullet_{\mathcal{N}} \Delta_S.$$

Proof. Choose a perfectoid ring R_0 and a surjection $R_0 \twoheadrightarrow R$, lifting to a map $A_{\mathrm{inf}}(R_0) = \Delta_{R_0} \to \Delta_R$.

By [45, Tag 04D1], and by the equivalence of the étale sites of R and $\pi_0(R)$, there exists a p-completely étale map $R_0 \to R_0'$ such that $S = R_0' \otimes_{R_0} R$. We therefore reduce the first statement to showing that $\Delta_{R_0} \to \Delta_{R_0'}$ is (p, I_{R_0})-completely étale. This follows from the fact that R_0' is also perfectoid,[13] and the associated map of tilts $R_0^\flat \to R_0^{',\flat}$ is also étale, which implies that the map

$$\Delta_{R_0} = W(R_0^\flat) \to W(R_0^{',\flat}) = \Delta_{R_0'}$$

is (p, I_{R_0})-completely étale.

For the second assertion, first note that by (the proof of) assertion (1) of Proposition 6.24, $S^{\mathcal{N}} \to R^{\mathcal{N}}$ is (p, I_R)-completely étale. The proof is now completed by combining Proposition 5.11 and Theorem 6.32. ☐

[13] This is a *much* easier assertion than the almost purity theorem for perfectoid algebras over fields!

The next two results tell us that $\mathrm{BT}_n^{G,\mu}$ can be understood in terms of the abstract (G,μ)-displays from (5.12)

p:descent **Proposition 9.6** (Quasisyntomic descent). *If $R \to R_\infty$ is as in Corollary 6.26, then we have:*

$$\mathrm{BT}_n^{G,\mu}(R) \xrightarrow{\simeq} \mathrm{Tot}\left(\mathrm{BT}_n^{G,\mu}(R_\infty^{\otimes_R \bullet+1})\right)$$

Proof. This is essentially immediate from the definitions and Proposition 6.24. ☐

tack_desc **Lemma 9.7.** *If R is semiperfectoid, as sheaves on the small étale site $R_{\text{ét}}$ of $\mathrm{Spf}(R)$, we have a canonical equivalence*

$$\mathrm{Disp}^{G,\mu}_{n,{\mathbb{\Delta}}_R} \xrightarrow{\simeq} \mathrm{BT}_n^{G,\mu}|_{R_{\text{ét}}}$$

Proof. This is immediate from Lemma 9.5, Theorem 6.32 and Definition 5.14; see also Remark 5.15. ☐

onditions *Remark* 9.8. Combining Proposition 9.6 and Lemma 9.7 with Proposition 5.13, we see that $\mathrm{BT}_n^{G,\mu}(R)$ can also be described as the ∞-groupoid of $G\{\mu\}$-torsors $\mathcal{Q}$ over $R^{\mathcal{N}} \otimes (\mathbb{Z}/p^n\mathbb{Z})$ admitting flat-local sections, equipped with an equivalence $j_{dR}^*\mathcal{Q} \xrightarrow{\simeq} j_{HT}^*\mathcal{Q}$ of G-torsors over $R^{\mathbb{\Delta}} \otimes (\mathbb{Z}/p^n\mathbb{Z})$, and satisfying the following equivalent conditions:

(1) For every geometric point $R \to \kappa$ of $\mathrm{Spf}\, R$, the restriction of $(x_{\mathrm{dR}}^{\mathcal{N}})^*\mathcal{Q}$ over $B\mathbb{G}_m \times \mathrm{Spec}\,\kappa$ is trivial;
(2) The restriction of $\mathcal{Q}$ over $R^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is trivial locally in the p-quasisyntomic topology on $\mathrm{Spf}\, R$;
(3) The restriction of $\mathcal{Q}$ over $R^{\mathcal{N}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is trivial flat locally on $\mathrm{Spf}\, R$.

If $\mathrm{Spf}\, R$ is connected, then these conditions are also equivalent to: For *some* geometric point $R \to \kappa$, the restriction of $(x_{\mathrm{dR}}^{\mathcal{N}})^*\mathcal{Q}$ over $B\mathbb{G}_m \times \mathrm{Spec}\,\kappa$ is trivial.

9.9. **Representability.** If $\mathcal{X} = \mathcal{B}(G,\mu) \otimes \mathbb{Z}/p^n\mathbb{Z})$ as a 1-bounded stack over $\mathcal{O}^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$, then the associated attractor stack $X^{(n)} \to \mathrm{Spec}\,\mathcal{O}$ is $BG^{(n)}$, and the stack $X^{-,(n)}$ is the Weil restricted classifying stack $BP_\mu^{-,(n)}$.

If $\mathcal{X}_1 = \mathcal{B}(G,\mu) \otimes \mathbb{F}_p$, then $\Gamma_{F\mathrm{Zip}}(\mathcal{X}_1)$ is a quasicompact smooth 0-dimensional Artin stack over k with affine diagonal. Indeed, it is not difficult to see using Remark 4.20 and (8.10.1) that this is just the stack of ***F*-zips with *G*-structure of type** μ; that is, of tuples $\mathsf{F} = (\mathcal{F}, \mathcal{F}^+, \mathcal{F}^-, \eta^+, \eta^-, \alpha)$ where:

- $\mathcal{F}$ is a G-torsor over R;
- $\mathcal{F}^+$ is a $P^+_{\mu^\varphi}$-torsor over R;
- $\mathcal{F}^-$ is a P^-_μ-torsor over R;
- $\eta^+ : \mathcal{F}^+ \to \mathcal{F}$ is a $P^+_{\mu^\varphi}$-equivariant map;
- $\eta^- : \mathcal{F}^- \to \mathcal{F}$ is a P^-_μ-equivariant map;
- $\alpha : \mathcal{F}^+/U^+_{\mu^\varphi} \xrightarrow{\simeq} \varphi^*\left(\mathcal{F}^-/U^-_\mu\right)$ is an isomorphism of M_{μ^φ}-torsors.

The conclusion now follows from [43, §3.3]; see also the discussion in [15, §3.2], where it is denoted Disp_1^G.

The next result proves Theorems D and G

_thm_body **Theorem 9.10.** $\mathrm{BT}_n^{G,\mu}$ *is a quasicompact smooth 0-dimensional p-adic formal Artin stack over $\mathcal{O}$ with affine diagonal. For every nilpotent divided power thickening $(C' \twoheadrightarrow C, \gamma)$ of p-complete $\mathcal{O}$-algebras, we have a Cartesian square*

$$\begin{tikzcd}
\mathrm{BT}_n^{G,\mu}(C') \arrow[r] \arrow[d] & BP_\mu^{-,(n)}(C') \arrow[d] \\
\mathrm{BT}_n^{G,\mu}(C) \arrow[r] & BP_\mu^{-,(n)}(C) \times_{BG^{(n)}(C)} BG^{(n)}(C').
\end{tikzcd}$$

In particular, the transition maps $\mathrm{BT}_{n+1}^{G,\mu} \to \mathrm{BT}_n^{G,\mu}$ are smooth.

Proof. The representability of $\mathrm{BT}_n^{G,\mu}$ by a locally finitely presented p-adic formal derived Artin stack over $\mathcal{O}$ follows from Theorem 8.12 and the discussion above.

The existence of the stated Cartesian square for nilpotent divided power thickenings follows from Corollary 8.22. Since $BP_\mu^{-,(n)} \to BG^{(n)}$ is smooth, this also shows that $\mathrm{BT}_n^{G,\mu}$ is a smooth p-adic formal Artin stack over $\mathcal{O}$.

For the smoothness of the transition maps, observe that Proposition 8.35 tells us that we have

$$\mathbb{L}_{\mathrm{BT}^{G,\mu}_{n+1}/\mathrm{BT}^{G,\mu}_{n}} \simeq \varpi^*\mathbb{L}_{BP^{-,(n+1)}_{\mu}/(BG^{(n+1)}\times_{BG^{(n)}}BP^{-,(n)}_{\mu})},$$

so it suffices to know that the map

$$BP^{-,(n+1)}_{\mu} \to BG^{(n+1)} \times_{BG^{(n)}} BP^{-,(n)}_{\mu}$$

is smooth, and this is immediate from the smoothness of $BP^{-}_{\mu} \to BG_{\mathcal{O}}$.

It remains to see that $\mathrm{BT}^{G,\mu}_{n}$ is quasicompact with affine diagonal, and it is enough to do this for $\mathrm{BT}^{G,\mu}_{n}\otimes\mathbb{F}_p$.

We have already seen that $\Gamma_{F\mathrm{Zip}}(\mathcal{X}_1) = \mathrm{Disp}^{G}_{1}$ has these properties. To see that $\mathrm{BT}^{G,\mu}_{1}\otimes\mathbb{F}_p$ also does, using Theorem 8.12, it is enough to see that $\mathsf{S}(\mathcal{X}_1)$ is quasi-compact with affine diagonal. But one sees from Corollary 7.6 that $\mathsf{S}(\mathcal{X}_1)$ is the classifying stack of a certain finite flat group scheme of height one. This is nothing but the **Lau group scheme** from [15]. In particular, one sees that $\mathrm{BT}^{G,\mu}_{1}\otimes\mathbb{F}_p$ is a gerbe over $\Gamma_{F\mathrm{Zip}}(\mathcal{X}_1)$ banded by the Lau group scheme.

Over $\mathrm{BT}^{G,\mu}_{1}\otimes\mathbb{F}_p$, we now have a canonical F-gauge $\mathcal{M}_1(\mathfrak{g})$ obtained by twisting the adjoint representation of $G\rtimes\mathbb{G}_m$ by the tautological $G\{\mu\}$-torsor over $(\mathrm{BT}^{G,\mu}_{1}\otimes\mathbb{F}_p)^{\mathrm{syn}}$. Proposition 8.16 shows that $\mathrm{BT}^{G,\mu}_{n}\otimes\mathbb{F}_p \to \mathrm{BT}^{G,\mu}_{n-1}\otimes\mathbb{F}_p$ is a torsor under $\Gamma(\mathcal{M}_1(\mathfrak{g}[1]))$, so we are reduced to showing the following: For any 1-bounded vector bundle F-gauge $\mathcal{F}$ over $R^{\mathrm{syn}}\otimes\mathbb{F}_p$, the stack $\Gamma_{\mathrm{syn}}(\mathcal{F}[1]) \to \operatorname{Spec} R$ is smooth and quasi-compact with affine diagonal. The smoothness follows from the description of its cotangent complex in Theorem 8.37. The diagonal map for the stack is a torsor under $\Gamma_{\mathrm{syn}}(\mathcal{F})$, so it suffices to now observe that $\Gamma_{\mathrm{syn}}(\mathcal{F})$ is affine by the same theorem. $\square$

10. Explicit descriptions of $\mathrm{BT}^{G,\mu}_{n}(R)$

:explicit

In this section, we will see that the above theory yields explicit descriptions for $\mathrm{BT}^{G,\mu}_{n}(R)$ in certain cases, following the deformation theoretic method of Ito [25]. All objects in this subsection will be discrete, unless otherwise noted, so we are back on firm classical ground.

We will also find that the deformation rings constructed by Faltings in [18], and which play a key role in the construction of integral canonical models in [29] admit a clean interpretation as versal deformation rings for $\mathrm{BT}^{G,\mu}_{n}$.

ings_expl

10.1. **An explicit description over some classical rings.** We will put ourselves in the following situation (compare with [31, §6]):

- $(S, I=(E))$ will be an oriented prism, flat over $\mathbb{Z}_p$, with associated Frobenius lift $\varphi: S\to S$;
- $J\subset S$ will be a finitely generated ideal such that $\varphi(J)\subset J^2$;
- We will assume that that S is J-adically complete, and that E and p map to non-zero divisors in $R \overset{\mathrm{defn}}{=} S/J$.

For $m\geq 1$, set $S_m = S/J^m$. Then φ descends to an endomorphism of S_m.

We will associate with this data the following frames: For each $m\geq 1$, we define $\underline{S}_m$ to be the frame with underlying non-negatively filtered ring $\mathrm{Fil}^{\bullet}_{I} S_m$ with filtered Frobenius $\mathrm{Fil}^{\bullet}_{I} S_m \to \mathrm{Fil}^{\bullet}_{\varphi(I)} S_m$. Set $R_m \overset{\mathrm{defn}}{=} S/(J^m + I) = S_m/\mathrm{Fil}^1_I S_m$.

Repeating the above construction with S_m replaced with S gives a frame $\underline{S}$. We then have maps of frames $\underline{S}_{m+1}\to\underline{S}_m$ for each $m\geq 1$, and also

$$\underline{S} \xrightarrow{\simeq} \varprojlim_{m} \underline{S}_m.$$

ef_theory

Proposition 10.2. *There is a canonical map*

$$\mathrm{BT}^{G,\mu}_{n}(R_m) \to \mathrm{Disp}^{G,\mu}_{\underline{S}_m,n}(R_m),$$

and the square

$$\begin{tikzcd}
\mathrm{BT}^{G,\mu}_n(R_{m+1}) \arrow[r] \arrow[d] & \mathrm{Disp}^{G,\mu}_{\underline{S}_{m+1},n}(R_{m+1}) \arrow[d] \\
\mathrm{BT}^{G,\mu}_n(R_m) \arrow[r] & \mathrm{Disp}^{G,\mu}_{\underline{S}_m,n}(R_m)
\end{tikzcd}$$

is Cartesian. In particular, we have

$$\mathrm{BT}^{G,\mu}_n(R) \simeq \mathrm{BT}^{G,\mu}_n(R_1) \times_{\mathrm{Disp}^{G,\mu}_{\underline{S}_1,n}(S_1)} \mathrm{Disp}^{G,\mu}_{\underline{S},n}(R)$$

Proof. We begin by explicating the categories of displays. By Remarks 4.20, Proposition 4.30 and Proposition 5.13, giving a $G^{(n)}\{\mu\}$-torsor $\mathcal{Q}_m$ over $\mathcal{R}(\mathrm{Fil}^\bullet_E S_m)$ that is trivial étale locally on $\mathrm{Spf}\, R_m$ is equivalent to giving an étale G-torsor $\mathcal{P}_m$ over S_m/p^nS_m along with a reduction of structure group to P^-_μ of the induced G-torsor over $R_m/^{\mathbb{L}}p^n$ (this is a derived quotient). The map τ^* forgets the reduction of structure group, while the map σ^* will be described in the next couple of paragraphs.

Let $L^+_{\underline{S}_m}G^{(n)}$ and $L^+_{\underline{S}_m}G^{(n)}\{\mu\}$ be the étale sheaves of groups from Remark 5.16: Unwinding the definitions, and using Proposition 4.30 one find that, given an étale R_m-algebra R'_m, we have $L^+_{\underline{S}_m}G^{(n)}(R'_m) = G(S'_m/p^nS'_m)$, while we have

$$\sigma^*(L^+_{\underline{S}_m}G^{(n)}\{\mu\}(R'_m)) = P^-_\mu(S'_m/p^mS'_m) \times U^+_\mu(\mathrm{Fil}^1_I S'_m/p^m \mathrm{Fil}^1_I S'_m) \subset G(S'_m/p^mS'_m).$$

Here, we S'_m is the unique étale S_m-algebra lifting R'_m, and we are viewing the left hand side as a subset of the right via the multiplication map; see also [30, Remark 6.3.3]. The map σ^* corresponds to a homomorphism

$$\sigma^*_m : L^+_{\underline{S}_m}G^{(n)}\{\mu\} \to L^+_{\underline{S}_m}G^{(n)},$$

which, on the product decomposition of the left hand side, has the following description: Its restriction to $P^-_\mu(S'_m/p^nS'_m)$ is given by

$$\varphi^* \circ \mathrm{int}(\mu(p)^{-1}) : P^-_\mu(S'_m/p^nS'_m) \to (\varphi^*P^-_\mu)(S'_m/p^nS'_m) \subset G(S'_m/p^nS'_m),$$

while its restriction to $U^+_\mu(\mathrm{Fil}^1_I S'_m/p^m \mathrm{Fil}^1_I S'_m)$ is given by:

$$\begin{aligned}
U^+_\mu(\mathrm{Fil}^1_I S'_m/p^n \mathrm{Fil}^1_I S'_m) &\xleftarrow[\simeq]{\exp} \mathfrak{g}^+_\mu \otimes_{\mathcal{O}} (\mathrm{Fil}^1_I S'_m/p^n \mathrm{Fil}^1_I S'_m) \\
&\xrightarrow{\mathrm{id}\otimes\varphi(E)^{-1}\circ\varphi} \varphi^*\mathfrak{g}^+_\mu \otimes_{\mathcal{O}} S_m/p^nS_m \\
&\xrightarrow[\simeq]{\exp} (\varphi^*U^+_\mu)(S_m/p^nS_m) \subset G(S_m/p^nS_m).
\end{aligned}$$

Now, $\mathcal{Q}_m$ corresponds to an $L^+_{\underline{S}_m}G^{(n)}\{\mu\}$-torsor over R_m, and $\sigma^*\mathcal{Q}_m$ to the $L^+_{\underline{S}}G^{(n)}$-torsor obtained via pushforward along σ^*_m as described above.

Suppose that we have an object $\mathrm{Disp}^{G,\mu}_{\underline{S}_{m+1},n}(R_{m+1})$ corresponding to a $G^{(n)}\{\mu\}$-torsor $\mathcal{Q}_{m+1}$-torsor over $\mathcal{R}(\mathrm{Fil}^\bullet_E S_{m+1})$ equipped with an isomorphism $\xi_{m+1} : \sigma^*\mathcal{Q}_{m+1} \xrightarrow{\simeq} \tau^*\mathcal{Q}_{m+1}$. Let $(\mathcal{Q}_m, \xi_m)$ be the corresponding display over $\underline{S}_m$.

The hypothesis that $\varphi(J) \subset J^2$ implies that, for each m, the Frobenius lift $S_{m+1} \to S_{m+1}$ factors canonically through a map $S_m \to S_{m+1}$, and also that the map σ^*_{m+1} factors through a map

$$\overline{\sigma}^*_{m+1} : L^+_{\underline{S}_m}G^{(n)}\{\mu\} \to L^+_{\underline{S}_{m+1}}G^{(n)}.$$

Moreover, there exists a canonical identification

$$\sigma^*\mathcal{Q}_{m+1} \xrightarrow{\simeq} \overline{\sigma}^*_m\mathcal{Q}_m.$$

This shows that giving the lift $(\mathcal{Q}_{m+1}, \xi_{m+1})$ of $(\mathcal{Q}_m, \xi_m)$ is equivalent to specifying the lift $\mathcal{Q}_{m+1}$, along with an isomorphism $\tau^*\mathcal{Q}_{m+1} \xrightarrow{\simeq} \overline{\sigma}^*_m\mathcal{Q}_m$. We can further formulate this as follows: Let $\overline{P}_{m+1}$ be the G-torsor over $R_m/^{\mathbb{L}}p^n$

obtained from $\varphi_m^*\tau^*\mathcal{Q}_m$. Then giving the lift $(\mathcal{Q}_{m+1},\xi_{m+1})$ is equivalent to specifying a reduction of structure group for $\overline{P}_{m+1}$ to a P_μ^--torsor. More precisely, we have a Cartesian diagram

n_bk_disp (10.2.1)
$$\begin{tikzcd} \mathrm{Disp}^{G,\mu}_{\underline{S}_{m+1},n}(R_{m+1}) \arrow[r] \arrow[d] & BP_\mu^{-,(n)}(R_m) \arrow[d] \\ \mathrm{Disp}^{G,\mu}_{\underline{S}_m,n}(R_m) \arrow[r] & BP^{-,(n)}(R_m)\times_{BG^{(n)}(R_m)} BG_\mu^{(n)}(R_{m+1}). \end{tikzcd}$$

To finish the proof, it now suffices to construct a natural map $\mathrm{BT}_n^{G,\mu}(R_m)\to\mathrm{Disp}^{G,\mu}_{\underline{S}_m,n}(R_m)$ such that the composition to $BP^{-,(n)}(R_m)\times_{BG^{(n)}(R_m)} BG_\mu^{(n)}(R_{m+1})$ with the bottom map in the above square agrees with that appearing in the Cartesian square in Theorem 9.10.

For this, we will define a map $\mathcal{R}(\mathrm{Fil}_I^\bullet S_m)\to R_m^{\mathcal{N}}$ as follows: View the source as parameterizing, for p-nilpotent C, a generalized Cartier divisor $t: L\to C$, along with maps of filtered animated commutative rings $\mathrm{Fil}_I^\bullet S_m\to\mathrm{Fil}_L^\bullet C$ such that the underlying homomorphism $S_m\to\pi_0(C)$ maps I to a nilpotent ideal. More prosaically, we are saying that C is an S_m-algebra such that E maps to a nilpotent element of $\pi_0(C)$, and that we have maps $I\otimes_A C\xrightarrow{u} L\xrightarrow{t} C$ of S-modules such that $t\circ u$ is the base-change of the inclusion $I\to S$.

We can then give a filtered Cartier-Witt divisor over C as follows (see also the proof of [5, Prop. 5.4.2]): First, the prism structure on S gives a map $\iota: S\to W(S)$, which we can use to obtain a fiber sequence of W-modules over S:
$$\mathbb{V}(I)^\sharp\to I\otimes_S W\to I\otimes_S F_*W.$$
When restricted over C, we can pushout along the map $u^\sharp:\mathbb{V}(I\otimes_S C)^\sharp\to\mathbb{V}(L)^\sharp$ to obtain an admissible W-module $M((L,u))$ over C such that we have a commuting diagram of fppf sheaves over C:
$$\begin{tikzcd} \mathbb{V}(I\otimes_S C)^\sharp \arrow[r] \arrow[d, "u^\sharp"'] & I\otimes_S W \arrow[r] \arrow[d] & I\otimes_S F_*W \arrow[d, equal] \\ \mathbb{V}(L)^\sharp \arrow[r] \arrow[d, "t^\sharp"'] & M((L,u)) \arrow[r] \arrow[d] & I\otimes_S F_*W \arrow[d] \\ \mathbb{G}_a^\sharp \arrow[r] & W \arrow[r] & F_*W, \end{tikzcd}$$
where the composition of the middle vertical arrow is the base-change of the inclusion $I\to S$.

The cofiber of the map $M((L,u))\to W$ is a quotient of that of $\mathrm{hcoker}(I\otimes_S W\to W)\simeq R\otimes_S W$, and is naturally an R-algebra. From this, it is not hard to see that we have constructed a map

me_to_nyg (10.2.2)
$$\mathcal{R}(\mathrm{Fil}_I^\bullet S_m)\to R_m^{\mathcal{N}}.$$

The restriction of this map over the locus $t\neq 0$ is just the composition of the map $\iota_{(S_m,(E))}$ from Remark 6.6 with the de Rham embedding $j_{\mathrm{dR}}: R_m^{\mathbb{\Delta}}\to R_m^{\mathcal{N}}$. The restriction along the map
$$\mathrm{Spf}\,S_m\simeq\mathcal{R}(\mathrm{Fil}^\bullet_{\varphi(I),\pm} S_m)\to\mathcal{R}(\mathrm{Fil}_I^\bullet S_m)$$
is the composition of the map $\iota_{(S_m,(\varphi(E)))}$ with the Hodge-Tate embedding j_{HT}.

From the previous paragraph, we find that pullback along (10.2.2) yields the desired functor from $\mathrm{BT}_n^{G,\mu}(R_m)$ to $\mathrm{Disp}^{G,\mu}_{\underline{S}_m,n}(R_m)$.

To finish, one first checks that the restriction of (10.2.2) to $\mathbb{A}^1/\mathbb{G}_m\times\mathrm{Spf}\,R_m$ agrees with the map $x_{\mathrm{dR}}^{\mathcal{N}}$, and also that the lift
$$\mathrm{Spf}\,R_{m+1}\to\mathrm{Spf}\,S_{m+1}\xrightarrow{\mathrm{Spf}(\varphi_m)}\mathrm{Spf}\,S_m\xrightarrow{\iota_{(S_m,(E))}} R_m^{\mathbb{\Delta}}$$

is that of Lemma 6.20 if we equip $R_{m+1} \twoheadrightarrow R_m$ with trivial divided powers.

The last part follows by taking the limit over m on the equivalence

$$\mathrm{BT}_n^{G,\mu}(R_m) \xrightarrow{\simeq} \mathrm{BT}_n^{G,\mu}(R_1) \times_{\mathrm{Disp}^{G,\mu}_{\underline{S}_1,n}(S_1)} \mathrm{Disp}^{G,\mu}_{\underline{S}_m,n,(}(R_m).$$

That this limit yields the desired equivalence follows from [6, Corollary 1.5]. □

situation

Example 10.3. Let κ be a perfect field in characteristic p, with associated ring of Witt vectors $W(\kappa)$. We set $S = W(\kappa)[|t_1,\dots,t_r|]$ for some $n \geq 0$, and take φ to be the Frobenius lift with $\varphi(t_i) = t_i^p$. Take $J = (t_1,\dots,t_n)$, so that we have $\varphi(J) \subset J^p$. Suppose that E satisfies $\varphi(E) \equiv E^p \pmod p$, and is such that $S/(E)$ is p-torsion free.

Here, $R_m = S/((E) + J^m)$ with $R_1 = S/((E) + J) = W(\kappa)/(p) = \kappa$, and $\underline{S}_1$ is isomorphic to the frame $\underline{\Delta}_\kappa$. In particular, the map

$$\mathrm{BT}_n^{G,\mu}(\kappa) = \mathrm{BT}_n^{G,\mu}(R_1) \to \mathrm{Disp}^{G,\mu}_{\underline{S}_1,n}(\kappa)$$

is an equivalence, and so we conclude that we have

$$\mathrm{BT}_n^{G,\mu}(R) \xrightarrow{\simeq} \mathrm{Disp}^{G,\mu}_{\underline{S},n}(R).$$

Note that the argument actually shows that we have

$$\mathrm{BT}_n^{G,\mu}(R_m) \xrightarrow{\simeq} \mathrm{Disp}^{G,\mu}_{\underline{S}_m,n}(R_m)$$

for all $m \geq 1$.

The case where $n = 1$ and E is an Eisenstein polynomial is the context for the classical story of Breuil-Kisin modules. In this case, $R = S/(E)$ is a totally ramified ring of integers over $W(\kappa)$, and $R_m = R/(\pi^m)$ where $\pi \in R$ is a uniformizer with minimal polynomial E.

Example 10.4. We have a non-Noetherian analogue of the previous example by taking $\underline{S} = \underline{\Delta}_R$ for a perfectoid ring R, $E = \xi$ a generator for $\ker(\theta : \Delta_R \to R)$, and $J = ([\varpi_1], [\varpi_2], \dots, [\varpi_m])$ to be an ideal generated by Teichmüller lifts of topologically nilpotent elements $\varpi_i \in R^\flat$. We now have $S_1 = \Delta_R/J$, and $R_1 = R/\theta(J)$.

In this case, we already know that $\mathrm{BT}_n^{G,\mu}(R) \simeq \mathrm{Disp}^{G,\mu}_{\underline{S},n}$ by Lemma 9.7. But Proposition 10.2 tells us that we also have

$$\mathrm{BT}_n^{G,\mu}(R) \simeq \mathrm{BT}_n^{G,\mu}(R_1) \times_{\mathrm{Disp}^{G,\mu}_{n,\underline{S}_1}(R_1)} \mathrm{Disp}^{G,\mu}_{n,\underline{S}}(R).$$

Since $\mathrm{BT}_n^{G,\mu}(R) \to \mathrm{BT}_n^{G,\mu}(R_1)$ is an effective epimorphism, this tells us that we in fact have

$$\mathrm{BT}_n^{G,\mu}(R_1) \simeq \mathrm{Disp}^{G,\mu}_{n,\underline{S}_1}(R_1).$$

This gives a somewhat concrete description of the left hand side, and also recovers—via Theorem 11.3 below—a description of p-divisible groups over R_1 which was first observed by Ito [25, Theorem 6.3.6].

situation

Example 10.5. Let S and J be as in Example 10.3, but assume now that $E = p$, so that we have a *crystalline* prism $(S,(p))$. In this case, and we obtain an equivalence

$$\mathrm{BT}_n^{G,\mu}(\kappa[|t_1,\dots,t_r|]) \xrightarrow{\simeq} \mathrm{Disp}^{G,\mu}_{\underline{S},n}(\kappa[|t_1,\dots,t_r|]) \times_{\mathrm{Disp}^{G,\mu}_{\underline{S}_1,n}(\kappa)} \mathrm{BT}_n^{G,\mu}(\kappa).$$

But note that $\underline{S}_1$ is simply the frame $\underline{\Delta}_\kappa$, and so as in *loc. cit.* the map $\mathrm{BT}_n^{G,\mu}(\kappa) \to \mathrm{Disp}^{G,\mu}_{\underline{S}_1,n}(\kappa)$ is an equivalence. This gives us an equivalence:

$$\mathrm{BT}_n^{G,\mu}(\kappa[|t_1,\dots,t_r|]) \xrightarrow{\simeq} \mathrm{Disp}^{G,\mu}_{\underline{S},n}(\kappa[|t_1,\dots,t_r|]).$$

ion_rings 10.6. **Relationship with Faltings deformation rings.** Suppose now that $I=(p)$; for instance, this is the case in the situation of Example 10.5. We can then also define frames $\widetilde{\underline{S}}$ and $\widetilde{\underline{S}}_m$, where we take the underlying filtered commutative ring to be S (resp. S_m) with the *trivial* filtration. We then have maps of frames $\widetilde{\underline{S}}_{m+1}\to\widetilde{\underline{S}}_m$ for each $m\geq 1$, and also

$$\widetilde{\underline{S}}\xrightarrow{\simeq}\varprojlim_m\widetilde{\underline{S}}_m.$$

In this case, we have $\widetilde{S}_m/\operatorname{Fil}^1\widetilde{S}_m=S_m$ and $\widetilde{S}/\operatorname{Fil}^1\widetilde{S}=S$.

We have a functor $\mathrm{BT}_n^{G,\mu}(S_m)\to\mathrm{Disp}^{G,\mu}_{n,\widetilde{\underline{S}}_m}(S_m)$ obtained via pullback along the map

$$x^{\mathcal{N}}_{\mathrm{dR},S_m}:\mathbb{A}^1/\mathbb{G}_m\times\operatorname{Spf}S_m\to S_m^{\mathcal{N}},$$

and the further observation that the composition

$$\operatorname{Spf}S_m\xrightarrow{\simeq}\mathcal{R}(\operatorname{Fil}^\bullet_{p,\pm}S_m)\to\mathcal{R}(\operatorname{Fil}^\bullet\widetilde{S}_m)\simeq\mathbb{A}^1/\mathbb{G}_m\times\operatorname{Spf}S_m\xrightarrow{x^{\mathcal{N}}_{\mathrm{dR},S_m}}S_m^{\mathcal{N}},$$

with the second map given by the Frobenius lift, factors through the Hodge-Tate embedding of $S_m^{\mathbb{A}}$ (in fact, through $R_m^{\mathbb{A}}$.

Using this functor and arguing just as in the proof of Proposition 10.2, one finds that, for each $m\geq 1$, there is a Cartesian square:

an_square (10.6.1)
$$\begin{tikzcd}
\mathrm{BT}_n^{G,\mu}(S_{m+1}) \arrow[r] \arrow[d] & \mathrm{Disp}^{G,\mu}_{\widetilde{\underline{S}}_{m+1},n}(S_{m+1}) \arrow[d] \\
\mathrm{BT}_n^{G,\mu}(S_m) \arrow[r] & \mathrm{Disp}^{G,\mu}_{\widetilde{\underline{S}}_m,n}(S_m)
\end{tikzcd}$$

10.7. Set

$$\mathrm{BT}_\infty^{G,\mu}=\varprojlim_n\mathrm{BT}_n^{G,\mu}.$$

Let κ be a perfect field, and suppose that we have a point $x\in\mathrm{BT}_\infty^{G,\mu}(\kappa)$corresponding to a compatible sequence of points $x_n\in\mathrm{BT}_n^{G,\mu}(\kappa)$. We can then, for each n, consider the deformation problem on the usual category $\mathrm{Art}_{W(\kappa)}$ of Artin local $W(\kappa)$-algebras with residue field κ:

$$\begin{aligned}\mathrm{Def}_{x_n}:\mathrm{Art}_{W(\kappa)}&\to\mathrm{Spc}\\ A&\mapsto\mathrm{fib}_{x_n}(\mathrm{BT}_n^{G,\mu}(A)\to\mathrm{BT}_n^{G,\mu}(\kappa)).\end{aligned}$$

We will also set $\mathrm{Def}_x=\varprojlim_n\mathrm{Def}_{x_n}$.

Grothendieck-Messing theory now tells us that, if $A'\twoheadrightarrow A$ is a square-zero thickening in $\mathrm{Art}_{W(\kappa)}$, then we have a Cartesian square

$$\begin{tikzcd}
\mathrm{BT}_\infty^{G,\mu}(A') \arrow[r] \arrow[d] & BP_\mu^-(A') \arrow[d] \\
\mathrm{BT}_\infty^{G,\mu}(A) \arrow[r] & BP_\mu^-(A)\times_{BG(A)}BG(A').
\end{tikzcd}$$

Using this, we find:

ef_theory **Lemma 10.8.** *For each $A\in\mathrm{Art}_{W(\kappa)}$, $\mathrm{Def}_x(A)$ is equivalent to a set, and Def_x is prorepresented by $\operatorname{Spf}R_x$ with $R_x\simeq W(\kappa)[|t_1,\dots,t_d|]$ where $d=\dim G-\dim P_\mu^-$.*

10.9. We will now see that the co-ordinates on R_x can be described explicitly using results from the beginning of this subsection, combined with a construction of Faltings [18, §7]. Compare with the main result of Ito in [25], where one finds a version of such a result. There, however, the Faltings deformation space is only shown to have good descriptions for particular inputs from $\mathrm{Art}_{W(\kappa)}$.

We begin by reformulating Faltings's construction in the language of torsors. Choose a lift $x' \in \mathrm{BT}^{G,\mu}_\infty(W(\kappa))$, which in turn yields a display (or more precisely a compatible family of n-truncated displays) over the frame $\underline{\widetilde{S}}_1$ associated with the trivial filtration on $W(\kappa)$.

Explicitly, as we saw in the proof of Proposition 10.2, this means the following: Under the equivalence

$$\mathrm{BT}^{G,\mu}_\infty(\kappa) \xrightarrow{\simeq} \varprojlim_n \mathrm{Disp}^{G,\mu}_{n,\Delta_\kappa}(\kappa),$$

x corresponds to a $G\{\mu\}$-torsor $\mathcal{P}$ over $\mathcal{R}(\mathrm{Fil}^\bullet_p W(\kappa))$, whose restriction to $B\mathbb{G}_m \times \mathrm{Spec}\,\overline{\kappa}$ is trivial for any algebraic closure $\overline{\kappa}$ of κ, equipped with an isomorphism $\sigma^*\mathcal{P} \xrightarrow{\simeq} \tau^*\mathcal{P}$ of G-torsors over $W(\kappa)$.

In what follows, we will abuse notation a little bit and identify $G\{\mu\}$-torsors over $\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spf}\, W(\kappa)$ or $\mathbb{A}^1/\mathbb{G}_m \times \mathrm{Spf}\, R'_x$ arising from restriction of sections of $\mathrm{BT}^{G,\mu}_\infty$ with the corresponding P^-_μ-torsors over $W(\kappa)$ or R'_x. The display $\mathcal{P}_{x'}$ associated with the lift x' now amounts to refining $\mathcal{P}$ to a P^-_μ-torsor $\mathcal{P}'$ over $W(\kappa)$.

Set $G_x = \mathrm{Aut}(\tau^*\mathcal{P})$ and $P^-_x = \mathrm{Aut}(\mathcal{P}')$, so that G_x is a pure inner form over $W(\kappa)$ of G, and $P^-_x \subset G_x$ is associated with a cocharacter $\mu_x : \mathbb{G}_m \to G_x$ that is conjugate to μ, via the process explained in (4.11). For such a choice of cocharacter, we can look at the 'opposite' unipotent $U^+_x \subset G_x$: this is a commutative unipotent group scheme over $W(\kappa)$ (see Lemma 4.17). We now define R^{Fal}_x to be the complete local ring of U^+_x at the identity: this is abstractly isomorphic to $W(\kappa)[|t_1, \ldots, t_d|]$ as a $W(\kappa)$-algebra.

We equip R^{Fal}_x with the Frobenius lift φ arising from the p-power map on U^+_x, and take $J_x \subset R^{\mathrm{Fal}}_x$ to be the augmentation ideal: note that we have $\varphi(J_x) \subset J^p_x$.

We can now apply the setup from the beginning of the subsection with $(S, I) = (R^{\mathrm{Fal}}_x, (p))$ and $J = J_x$, and we find that we have:

$$\mathrm{fib}_{x'}(\mathrm{BT}^{G,\mu}_\infty(R^{\mathrm{Fal}}_x) \to \mathrm{BT}^{G,\mu}_\infty(W(\kappa))) \xrightarrow{\simeq} \mathrm{fib}_{\mathcal{P}_{x'}}(\mathrm{Disp}^{G,\mu}_{\underline{\widetilde{S}},\infty}(R^{\mathrm{Fal}}_x) \to \mathrm{Disp}^{G,\mu}_{\underline{\widetilde{S}}_1,\infty}(W(\kappa))).$$

One way to get an object on the right is as follows: Let $j : W(\kappa) \to R^{\mathrm{Fal}}_x$ be the structure map: this actually underlies a map of frames $\underline{\widetilde{S}}_1 \to \underline{\widetilde{S}}$, and so we can pull $\mathcal{P}_{x'}$ back to get the 'constant' lift $\mathcal{P}^{\mathrm{con}}_{x'}$ over $\underline{\widetilde{S}}$. More precisely, this corresponds to the P^-_μ-torsor $\mathcal{P}^{\mathrm{con}}_{x'}$ over R^{Fal}_x, along with an isomorphism of G-torsors $\xi^{\mathrm{con}}_{x'} : \sigma^*\mathcal{P}^{\mathrm{con}}_{x'} \xrightarrow{\simeq} \tau^*\mathcal{P}^{\mathrm{con}}_{x'}$. All of this data is obtained simply via pullback from the corresponding data over $W(\kappa)$.

In $U^+_x(R^{\mathrm{Fal}}_x)$, we have the tautological element g_x. We now define a new display $\mathcal{P}^{\mathrm{Fal}}_{x'}$ by keeping the P^-_μ-torsor $\mathcal{P}^{\mathrm{con}}_{x'}$, but replacing $\xi^{\mathrm{con}}_{x'}$ with the composition $\xi^{\mathrm{Fal}}_{x'} = g_x \circ \xi^{\mathrm{con}}_{x'}$.

As explained above, this yields an object $x^{\mathrm{Fal}} \in \mathrm{BT}^{G,\mu}_\infty(R^{\mathrm{Fal}}_x)$ lifting $x' \in \mathrm{BT}^{G,\mu}_\infty(W(\kappa))$, and so corresponds to a unique map $R_x \to R^{\mathrm{Fal}}_x$.

def_rings **Proposition 10.10.** *The map $R_x \to R^{\mathrm{Fal}}_x$ is an isomorphism.*

Proof. Let $\widehat{U}_x$ (resp. $\widehat{U}^{\mathrm{Fal}}_x$) be the deformation functor on $\mathrm{Art}_{W(\kappa)}$ represented by R_x (resp. R^{Fal}_x). If $\kappa[\epsilon]$ is the ring of dual numbers, we obtain maps of tangent spaces

$$\widehat{U}^{\mathrm{Fal}}_x(\kappa[\epsilon]) \to \widehat{U}_x(\kappa[\epsilon]) \xrightarrow{\simeq} \mathrm{fib}_{(\mathcal{P}_{x'}, \sigma^*\mathcal{P}_{x'})}(BP^-_\mu(\kappa[\epsilon]) \to BP^-_\mu(\kappa) \times_{BG(\kappa)} BG(\kappa[\epsilon])),$$

where the second arrow is the isomorphism from Grothendieck-Messing theory.

The source of this composition is simply $\epsilon\kappa[\epsilon] \otimes_{W(\kappa)} \mathrm{Lie}\, U^+_x$, and the proof of Proposition 10.2 shows that this composition takes a tangent vector ϵN to $\exp(-\epsilon N) \cdot \mathcal{P}_{x'}$. In particular, it is an isomorphism onto its image.

Since both complete local rings are normal of the same dimension, the proposition now follows from Nakayama's lemma. □

ification 11. The classification of truncated Barsotti-Tate groups

11.1. **The statement of the theorem.** Recall that a **truncated Barsotti-Tate group scheme of level** n over a discrete ring $R \in \mathrm{CRing}^{p\text{-nilp}}_\heartsuit$ is a finite flat commutative group scheme G over R with the following properties:

(1) G is p^n-torsion;
(2) The sequence $G \xrightarrow{p^{n-1}} G \xrightarrow{p} G$ is exact in the middle;
(3) If $n = 1$, over R/pR, we have $\ker F = \operatorname{im} V \subset G \otimes \mathbb{F}_p$, where $F : G \otimes \mathbb{F}_p \to (G \otimes \mathbb{F}_p)^{(p)}$ and $V : (G \otimes \mathbb{F}_p)^{(p)} \to G \otimes \mathbb{F}_p$ are the Frobenius and Verschiebung homomorphisms, respectively.

See for instance [23, §I].

These organize into a category $\mathcal{BT}_n(R)$, and we will write $\mathrm{BT}_n(R)$ for the underlying groupoid obtained by jettisoning the non-isomorphisms. For $1 \leq r \leq n$, sending G to $G[p^r]$ yields a functor $\mathcal{BT}_n(R) \to \mathcal{BT}_r(R)$.

An important role will be played by the following fundamental result of Grothendieck [23]:

thendieck **Theorem 11.2.** *The assignment $R \mapsto \mathrm{BT}_n(R)$ on $\mathrm{CRing}^{f,p\text{-}nilp}$ is represented by a finitely presented smooth 0-dimensional p-adic formal Artin stack with affine diagonal.*

There is a canonical involution

$$\mathcal{BT}_n(R) \xrightarrow{G \mapsto G^*} \mathcal{BT}_n(R)$$

induced by Cartier duality, with $G^* = \underline{\mathrm{Hom}}(G, \mu_{p^n})$.

Let $\mathrm{Vect}_{[0,1]}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})$ be the ∞-category of vector bundles on $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ with Hodge-Tate weights $\{0, 1\}$. There is once again a canonical involution

$$\mathrm{Vect}_{[0,1]}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\mathcal{M} \mapsto \mathcal{M}^*} \mathrm{Vect}_{[0,1]}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}),$$

with $\mathcal{M}^* = \mathcal{M}^\vee\{1\}$ is the twist of the dual vector bundle by the Breuil-Kisin twist $\mathcal{O}_n^{\mathrm{syn}}\{1\}$. In analogy with the involution on $\mathcal{BT}_n(R)$, we will refer to $\mathcal{M}^*$ as the **Cartier dual** of $\mathcal{M}$.

We can now state the main result of this section.

dieudonne **Theorem 11.3.** *Suppose that R belongs to CRing^f. Then there is a canonical equivalence of ∞-categories*

$$\mathcal{G} : \mathrm{Vect}_{[0,1]}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\simeq} \mathcal{BT}_n(R)$$

compatible with Cartier duality, so that for every $\mathcal{M} \in \mathrm{Vect}_{[0,1]}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})$, there is a canonical isomorphism

$$\mathcal{G}(\mathcal{M})^* \xrightarrow{\simeq} \mathcal{G}(\mathcal{M}^*).$$

Remark 11.4. The theorem implies in particular that $\mathrm{Vect}_{[0,1]}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})$ is a classical category. This fact is not evident from the definitions, since the derived stack $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ is in general not classical.

Remark 11.5. When R is quasisyntomic, Theorem 11.3 simply recovers the main result of Anschütz and Le Bras from [1] classifying p-divisible groups over R in terms of admissible prismatic Dieudonné crystals over R (see [1, Def. 4.1.5]).

First, when R is qrsp, Proposition 8.5 shows that $\mathrm{Vect}_{[0,1]}(R^{\mathrm{syn}})$ can be described equivalently in terms of 1-bounded vector bundle $\underline{\Delta}_R$-gauges.

The discussion in (5.20) shows that such $\underline{\Delta}_R$-gauges are canonically equivalent to the category of admissible prismatic Dieudonné modules over R as defined in [1, Def. 4.1.10].

The general quasisyntomic case now follows by descent.

One mildly interesting point: The proof of the equivalence exhibited in (5.20) shows that admissible prismatic Dieudonné modules should naturally be identified with (the negative) Breuil-Kisin twists of vector bundle F-gauges of Hodge-Tate weights $0, 1$: these of course span the category of vector bundle F-gauges of Hodge-Tate weights $-1, 0$.

Remark 11.6. When R is a totally ramified ring of integers over $W(\kappa)$ for some perfect field κ, then, combining the theorem with Example 10.3, one can recover a classification of p-divisible groups due to Kisin (the case $p > 2$ for R) [28], W. Kim (the case $p = 2$ for R) [27], and Lau (the general case of both R and R_m) [31].

series_BT *Remark* 11.7. When $R = \kappa[|x_1, \ldots, x_n|]$, using Example 10.5 and an argument such as the one used in (5.20), one recovers de Jong's description [26] of p-divisible groups over power series rings over perfect fields k in terms of certain filtered φ-modules $\mathrm{Fil}^\bullet M$ over $W(\kappa)[|x_1, \ldots, x_n|]$ equipped with the Frobenius lift satisfying $x_i \mapsto x_i^p$ (see also [18,

§7]). *A priori*, de Jong's description is in terms of F-crystals, and so also requires a topologically nilpotent integrable connection on M compatible with the φ-semilinear structure; however, as observed by Faltings [18, Theorem 10], with this choice of Frobenius, the integrable connection is actually uniquely determined by the rest of the data.

11.8. **Height and dimension.** The **height** of a truncated Barsotti-Tate group G of level n over R is the $\mathbb{Z}_{\geq 0}$-valued locally constant function on $\operatorname{Spec} R$ such that $G[p]$ has degree p^h over R. The **dimension** d is the $\mathbb{Z}_{\geq 0}$-valued locally constant function such that $\ker F \subset G[p] \otimes \mathbb{F}_p$ has degree p^d over R/pR.

These are locally constant invariants of G, and yield decompositions

$$\mathrm{BT}_n = \bigsqcup_{d \leq h} \mathrm{BT}_n^{h,d},$$

of formal Artin stacks, where h ranges over the non-negative integers, d over the non-negative integers bounded by h, and $\mathrm{BT}_n^{h,d}$ is the locus of truncated Barsotti-Tate group G of level n of height h and dimension d.

On the F-gauge side of things, we have ∞-subgroupoids

$$\mathrm{Vect}_{h,d}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}) \subset \mathrm{Vect}_{[0,1]}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})^{\simeq}$$

spanned by vector bundles $\mathcal{M}$ over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ of rank h and Hodge-Tate weights $0, 1$ such that the associated graded R-module $\mathrm{gr}^{-1}_{\mathrm{Hdg}} M$ is locally free of rank d.

Let us note the following:

r_duality **Lemma 11.9.** *Cartier duality yields equivalences*

$$\mathrm{Vect}_{h,d}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\simeq} \mathrm{Vect}_{h,h-d}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})$$

For $0 \leq d \leq h$, let $\mu_d : \mathbb{G}_m \to \mathrm{GL}_h$ be the cocharacter given by the diagonal matrix

$$\mu_d(z) = \mathrm{diag}(\underbrace{z, z, \ldots, z}_{d}, \underbrace{1, \ldots, 1}_{h-d}).$$

Associated with this, we have the smooth Artin stacks $\mathrm{BT}_n^{\mathrm{GL}_h, \mu_d}$ over $\mathbb{Z}_p$.

ect_to_BT **Proposition 11.10.** *For $R \in \mathrm{CRing}^{f,p\text{-}nilp}$, there is a canonical equivalence of groupoids*

$$\mathrm{BT}_n^{\mathrm{GL}_h, \mu_d}(R) \xrightarrow{\simeq} \mathrm{Vect}_{h,d}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}).$$

Proof. One can see this by combining Lemma 9.7 with Propositions 5.19 and 9.6. ☐

r_duality *Remark* 11.11. Via the above proposition and Lemma 11.9, we find that there is a Cartier duality equivalence

$$* : \mathrm{BT}_n^{\mathrm{GL}_h, \mu_d} \xrightarrow{\simeq} \mathrm{BT}_n^{\mathrm{GL}_h, \mu_{h-d}}.$$

We now have:

:BT_to_BT **Theorem 11.12.** *There is a canonical equivalence of smooth p-adic formal Artin stacks*

$$\mathcal{G} : \mathrm{BT}_n^{\mathrm{GL}_h, \mu_d} \xrightarrow{\simeq} \mathrm{BT}_n^{h,d}$$

such that the following diagram commutes:

$$\begin{tikzcd}
\mathrm{BT}_n^{\mathrm{GL}_h, \mu_d} \arrow[r, "\mathcal{G}", "\simeq"'] \arrow[d, "*"'] & \mathrm{BT}_n^{h,d} \arrow[d, "G \mapsto G^*"] \\
\mathrm{BT}_n^{\mathrm{GL}_h, \mu_{h-d}} \arrow[r, "\simeq", "\mathcal{G}"'] & \mathrm{BT}_n^{h,h-d}.
\end{tikzcd}$$

Assuming this theorem, we can easily deduce Theorem 11.3 via a standard argument.

Proof of Theorem 11.3. Theorem 11.12, combined with Proposition 11.10, gives us an isomorphism of ∞-groupoids

$$\mathrm{Vect}_{[0,1]}(R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z})^{\simeq}\xrightarrow{\simeq}\mathrm{BT}_n(R)$$

compatible with Cartier duality. To get an equivalence of ∞-categories, one now uses a graph construction: For $\mathcal{M}_1,\mathcal{M}_2$ in $\mathrm{Vect}_{[0,1]}(R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z})$, the space of maps $\mathcal{M}_1\to\mathcal{M}_2$ is equivalent to the space of isomorphisms $\mathcal{M}_1\oplus\mathcal{M}_2\xrightarrow{\simeq}\mathcal{M}_1\oplus\mathcal{M}_2$ that are 'upper triangular' and project onto the identity endomorphisms of $\mathcal{M}_1$ and $\mathcal{M}_2$. A similar description holds for $\mathcal{G}_1$ and $\mathcal{G}_2$ in $\mathcal{BT}_n(R)$. ☐

G_functor

11.13. **From F-gauges to Barsotti-Tate groups.** Suppose that we have $R\in\mathrm{CRing}^{f,p\text{-nilp}}$ and $\mathcal{M}$ in $\mathrm{Vect}_{[0,1]}(R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z})$. Set $\mathcal{G}(\mathcal{M})=\Gamma_{\mathrm{syn}}(\mathcal{M})$. Then by Theorem 8.37 $\mathcal{G}(\mathcal{M})$ is locally finitely presented and *quasi-smooth* over R with cotangent complex given by $\mathscr{O}_{\mathcal{G}(\mathcal{M})}\otimes_R\mathrm{gr}^{-1}_{\mathrm{Hdg}}M[1]$.

g_functor

Theorem 11.14. *Suppose that R is discrete and that $\mathcal{M}$ is in $\mathrm{Vect}_{h,d}(R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z})$, then $\mathcal{G}(\mathcal{M})$ is a relative truncated Barsotti-Tate group scheme over R of height h and dimension d. In particular, we have a canonical map of p-adic formal Artin stacks*

$$\mathcal{G}:\mathrm{BT}_n^{\mathrm{GL}_h,\mu_d}\to\mathrm{BT}_n^{h,d}.$$

Proof. Let us begin by consider the case $n=1$, and let us also suppose that R is an $\mathbb{F}_p$-algebra. Here, we can consider the F-zip M underlying $\mathcal{M}$, which is given explicitly by the following data:

(1) A locally free R-module M equipped with direct summands $\mathrm{Fil}^0_{\mathrm{Hdg}}M\subset M$ and $\mathrm{Fil}^{\mathrm{conj}}_1M\subset M$ of codimension d and $h-d$, respectively;
(2) Isomorphisms

$$\xi_1:\mathrm{Fil}^{\mathrm{conj}}_1M\xrightarrow{\simeq}\varphi^*(M/\mathrm{Fil}^0_{\mathrm{Hdg}}M)\ ;\ \xi_0:\mathrm{gr}^0_{\mathrm{conj}}M=M/\mathrm{Fil}^{\mathrm{conj}}_1M\xrightarrow{\simeq}\varphi^*\mathrm{Fil}^0_{\mathrm{Hdg}}M.$$

Now, consider the functor

$$\mathsf{G}(\mathsf{M})=\tau^{\le 0}R\Gamma_{F\mathrm{Zip}}(\mathsf{M}):\mathrm{CRing}_{R/}\to\mathrm{Mod}^{\mathrm{cn}}_{\mathbb{F}_p}$$

Unwinding definitions, one finds

$$\mathsf{G}(\mathsf{M})(C)=\{m\in C\otimes_R\mathrm{Fil}^0_{\mathrm{Hdg}}M:\ \xi_0(\overline{m})=\varphi^*m\},$$

where $M\xrightarrow{m\mapsto\overline{m}}\mathrm{gr}^{\mathrm{conj}}_0M$ is the natural quotient map. Viewing $m\mapsto\xi_0(\overline{m})$ as a map $\overline{\xi}_0:\mathrm{Fil}^0_{\mathrm{Hdg}}M\to\varphi^*\mathrm{Fil}^0_{\mathrm{Hdg}}M$, we see that we have

$$\mathsf{G}(\mathsf{M})=\ker(\mathbf{V}(\mathrm{Fil}^0_{\mathrm{Hdg}}M)\xrightarrow{\overline{\xi}_0-F}\mathbf{V}(\varphi^*\mathrm{Fil}^0_{\mathrm{Hdg}}M)),$$

so that $\mathsf{G}(\mathsf{M})$ is the Cartier dual to a height one finite flat group scheme over R of rank p^{h-d}.

By Theorem 8.37, with input from Corollary 7.6, we see that the natural map $\mathcal{G}(\mathcal{M})\to\mathsf{G}(\mathsf{M})$ presents the source as a torsor over the target under the finite flat height 1 group scheme $G(\mathrm{gr}^{-1}_{\mathrm{Hdg}}M,\psi_{\mathsf{M}})$, which has rank p^d. Therefore, we have a short exact sequence of finite flat group schemes

_sequence

$$0\to G(\mathrm{gr}^{-1}_{\mathrm{Hdg}}M,\psi_{\mathsf{M}})\to\mathcal{G}(\mathcal{M})\to\mathsf{G}(\mathsf{M})\to 0. \tag{11.14.1}$$

In particular, we see that $\mathcal{G}(\mathcal{M})$ is finite flat over R of rank p^h.

We can now consider the general case of $R\in\mathrm{CRing}^{f,p\text{-nilp}}_{\heartsuit}$ and $n\ge 1$. For $1\le r\le n$, let $\mathcal{M}_r$ be the restriction of $\mathcal{M}$ over $R^{\mathrm{syn}}\otimes\mathbb{Z}/p^r\mathbb{Z}$. By tensoring $\mathcal{M}$ with the canonical short exact sequence

$$0\to\mathbb{Z}/p^r\mathbb{Z}\xrightarrow{a\mapsto p^{n-r}a}\mathbb{Z}/p^n\mathbb{Z}\to\mathbb{Z}/p^{n-r}\mathbb{Z}\to 0,$$

we obtain a fiber sequence in $\mathrm{Perf}(R^{\mathrm{syn}}\otimes\mathbb{Z}/p^n\mathbb{Z})$:

$$\mathcal{M}_r\to\mathcal{M}\to\mathcal{M}_{n-r}.$$

Taking derived global sections yields a fiber sequence

$$R\Gamma_{\mathrm{syn}}(\mathcal{M}_r)\to R\Gamma_{\mathrm{syn}}(\mathcal{M})\to R\Gamma_{\mathrm{syn}}(\mathcal{M}_{n-r})$$

of $\mathrm{Mod}_{\mathbb{Z}/p^n\mathbb{Z}}$-valued quasisyntomic sheaves over R. Taking connective truncations shows that we have an isomorphism

devissage

$$\mathcal{G}(\mathcal{M}_r) \xrightarrow{\simeq} \mathrm{hker}^{\mathrm{cn}}\left(\mathcal{G}(\mathcal{M}) \to \mathcal{G}(\mathcal{M}_{n-r})\right) \tag{11.14.2}$$

of $\mathrm{Mod}^{\mathrm{cn}}_{\mathbb{Z}/p^n\mathbb{Z}}$-valued quasisyntomic sheaves over R.

If $R = k$ is an algebraically closed field, then this isomorphism, combined with a simple induction on r shows that $\mathcal{G}(\mathcal{M})(k)$ is a finite set. Therefore, applying the following lemma to $Y = \mathcal{G}(\mathcal{M})$ shows that $\mathcal{G}(\mathcal{M})$ is quasi-finite and flat over R

criterion

Lemma 11.15. *Suppose that Y is a finitely presented quasi-smooth derived algebraic space over $R \in \mathrm{CRing}$ of virtual codimension 0. Set $S = \mathrm{Spec}\, R$. Then the following are equivalent:*

(1) Y is flat over R;
(2) $Y \otimes_R \pi_0(R)$ is flat and classicaly lci over $\pi_0(R)$;
(3) For every $x \in S(k)$ with k algebraically closed, $x^*Y \to \mathrm{Spec}\, k$ is flat and classically lci;
(4) For every $x \in S(k)$ with k algebraically closed, $\pi_0((x^*Y)(k))$ is a finite set.

Proof. Since Y is quasi-smooth of virtual codimension 0, étale locally on Y we can present it as a derived complete intersection subscheme of some affine space $\mathbb{A}^n_R$ over R cut out as the derived zero locus of n polynomials $f_1, \dots, f_n$. If R is discrete, then such a derived zero locus is flat over R if and only if the animated ring $R[x_1, \dots, x_n]/^{\mathbb{L}}(f_1, \dots, f_n)$ has no higher homotopy groups. This is precisely equivalent to this ring being a classical lci algebra over R. Since flatness can be tested after derived basechange over the classical truncation, we see that (1) and (2) are equivalent.

If $R = k$ is an algebraically closed field, then $\pi_0((x^*Y)(k))$ is finite precisely when the classical truncation Y^{cl} is 0-dimensional and of finite type over k. We now note that $k[x_1, \dots, x_n]/(f_1, \dots, f_n)$ is 0-dimensional precisely when $f_1, \dots, f_n$ form a regular sequence. This shows the equivalence of (3) and (4).

To see the equivalence of (1) and (3), we now only have to make the additional observation that $M \in \mathrm{Mod}_R$ is flat over R if and only if its derived base-change over every algebraically closed field over R is flat. □

Now, by considering the isomorphism (11.14.2) twice, first as given, and then again with r replaced by $n-r$, we find that we have

$$\mathrm{im}\left(\mathcal{G}(\mathcal{M}) \xrightarrow{p^{n-r}} \mathcal{G}(\mathcal{M})\right) = \mathcal{G}(\mathcal{M}_r) = \ker\left(\mathcal{G}(\mathcal{M}) \xrightarrow{p^r} \mathcal{G}(\mathcal{M})\right).$$

In particular, for every r, $\mathcal{G}(\mathcal{M})$ is a $\mathcal{G}(\mathcal{M}_r)$-torsor over $\mathcal{G}(\mathcal{M}_{n-r})$. An inductive argument now shows that $\mathcal{G}(\mathcal{M})$ is a finite flat group scheme over R that is also a flat $\mathbb{Z}/p^n\mathbb{Z}$-module.

It still remains to verify that, when $n = 1$, $\mathcal{G}(\mathcal{M})$ is truncated Barsotti-Tate of height h and dimension d. From what we have just seen, to check that $\mathcal{G}(\mathcal{M})$ is a truncated Barsotti-Tate group scheme, it suffices to observe that, étale locally on $\mathrm{Spec}\, R$, $\mathcal{M}$ can be lifted to $\mathrm{Vect}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^2\mathbb{Z})$ by Theorem D and Proposition 11.12.

The last thing to check is that $\mathcal{G}(\mathcal{M})$ has dimension d. Once again, we can assume that $n = 1$ and that R is an $\mathbb{F}_p$-algebra. Let $F : \mathcal{G}(\mathcal{M}) \to \mathcal{G}(\mathcal{M})^{(p)}$ be the Frobenius map and let $V : \mathcal{G}(\mathcal{M})^{(p)} \to \mathcal{G}(\mathcal{M})$ be the Verschiebung. Then, since the Verschiebung on $\mathsf{G}(\mathsf{M})$ is identically zero (its Cartier dual is the Frobenius homomorphism for a height one group scheme), we have

$$\mathrm{im}\, V \subset G(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M, \psi_{\mathsf{M}}) \subset \ker F.$$

Since the outer subgroups are equal, it follows that they must equal the one in the middle, which has rank d over R by construction. □

11.16. **Cartier duality.** Let $\mathcal{O}_n$ be the structure sheaf of $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$. We will have use for the following result, which is due to Bhatt-Lurie:

att_lurie

Proposition 11.17. *We have canonical isomorphisms*

$$\mathcal{G}(\mathcal{O}_n) \xrightarrow{\simeq} \mathbb{Z}/p^n\mathbb{Z}\ ;\ \mathcal{G}(\mathcal{O}_n\{1\}) \xrightarrow{\simeq} \mu_{p^n}$$

in $\mathrm{BT}_n(R)$.

Proof. Unwinding definitions, the first isomorphism follows from [7, Theorem 8.1.9], while the second follows from [7, Theorem 7.5.6]

We can also give alternate proofs using the methods of this paper. As we already know from what we have seen above, $\mathcal{G}(\mathcal{O}_n)$ (resp. $\mathcal{G}(\mathcal{O}_n\{1\})$) is a truncated Barsotti-Tate group schemes over R of level n, height 1 and dimension 0 (resp. dimension 1).

We can assume that $R = \mathbb{Z}/p^m\mathbb{Z}$ for some $m \geq 1$. Note that $\mathcal{G}(\mathcal{O}_n)$ is étale over $\mathbb{Z}/p^m\mathbb{Z}$, so it suffices to give a map $\mathbb{Z}/p^n\mathbb{Z} \to \mathcal{G}(\mathcal{O}_n)$ that is an isomorphism over $\overline{\mathbb{F}}_p$. This is given by the structure map $\mathbb{Z}/p^n\mathbb{Z} \to \mathcal{O}_n$.

For the case of the Breuil-Kisin twist, note that $\mathcal{G}(\mathcal{O}_n\{1\})$ is of multiplicative type. Once again, it suffices to give a canonical map $\mu_{p^n} \to \mathcal{G}(\mathcal{O}_n\{1\})$ that is an isomorphism over $\overline{\mathbb{F}}_p$. In fact, since the scheme parameterizing such maps is finite étale over $\mathbb{Z}/p^m\mathbb{Z}$, we can assume that $m = 1$. In this case, by quasisyntomic descent, we only have to construct a canonical map

_to_Acrys (11.17.1)
$$\mu_{p^n}(C) \to (\mathrm{Fil}^1 \Delta_C/p^n)^{\varphi = p} = \mathcal{G}(\mathcal{O}_n)(C)$$

for qrsp $\mathbb{F}_p$-algebras C. This is obtained using the isomorphism $\Delta_C \xrightarrow{\simeq} A_{\mathrm{crys}}(C)$ from (6.28.1), and assigning to each $\alpha \in \mu_{p^n}(C)$ the image of the logarithm $\log([\tilde{\alpha}]^{p^n}) \in A_{\mathrm{crys}}(C)$, where $\tilde{\alpha} \in C^\flat$ is a lift of α and $[\tilde{\alpha}]$ is its Teichmüller lift; see [7, §7.1].

To finish, it is enough to know that the map (11.17.1) is *injective* for all qrsp $\mathbb{F}_p$-algebras C. In fact, it suffices to verify that, when $C = \mathbb{F}_p[x^{1/p^\infty}]/(x)$ and $n = 1$, the element $\log([1-x]) \in A_{\mathrm{crys}}(C)$ is not divisible by p. But one has (see displayed equation (37) in [7, p. 168]):

$$\log([1-x]) \equiv -\sum_{d=1}^{p} \frac{[x]^d}{d} \pmod{p}.$$

To see that the right hand side is non-zero, we only have to note that the element

$$x + \frac{x^2}{2} + \ldots + \frac{x^{p-1}}{p-1} + (p-1)! \cdot x^{[p]} \in \mathbb{F}_p\langle x\rangle$$

in the standard divided power $\mathbb{F}_p$-algebra is non-zero.

□

For every vector bundle $\mathcal{M}$ over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ with Hodge-Tate weights $0, 1$, we will now define a canonical map

ual_arrow (11.17.2)
$$\mathcal{G}(\mathcal{M}^*) \to \mathcal{G}(\mathcal{M})^*.$$

This is obtained as follows: For every $x : R \to C$, we have

$$\mathcal{G}(\mathcal{M}^*)(C) \simeq \mathrm{Map}_{\mathrm{QCoh}(C^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})}(x^*\mathcal{M}, \mathcal{O}_n\{1\}),$$

and evaluation on global sections now yields a map

$$\mathcal{G}(\mathcal{M}^*)(C) \to \mathrm{Hom}(\mathcal{G}(x^*\mathcal{M}), \mathcal{G}(\mathcal{O}_n\{1\})) \simeq \mathrm{Hom}(x^*\mathcal{G}(\mathcal{M}), \mu_{p^n}) \simeq \mathcal{G}(\mathcal{M})^*(C).$$

Here, we have used Proposition 11.17 for the penultimate isomorphism.

r_duality **Theorem 11.18.** *The map* (11.17.2) *is an isomorphism.*

Proof. It is enough to check this with $n = 1$ and R an $\mathbb{F}_p$-algebra. Here, we can use the short exact sequence (11.14.2), which gives us the following commuting diagram where the middle vertical arrow is (11.17.2) and the horizontal rows are short exact:

$$\begin{tikzcd}
0 \arrow[r] & G(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M^*, \psi_{\mathsf{M}^*}) \arrow[r] \arrow[d] & \mathcal{G}(\mathcal{M}^*) \arrow[r] \arrow[d] & \mathsf{G}(\mathsf{M}^*) \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathsf{G}(\mathsf{M})^* \arrow[r] & \mathcal{G}(\mathcal{M})^* \arrow[r] & G(\mathrm{gr}^{-1}_{\mathrm{Hdg}} M, \psi_{\mathsf{M}})^* \arrow[r] & 0
\end{tikzcd}$$

That the vertical arrows on the left and right are isomorphisms can be checked directly.

☐ Maybe add mo

M_functor

11.19. **From Barsotti-Tate groups to F-gauges.** Let R be p-complete and p-quasisyntomic, and let

$$\epsilon_n : (R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})_{\mathrm{fppf}} \to R_{\mathrm{qsyn}}$$

be the map of (classical) sites arising via the functor $C \mapsto C^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$ on p-quasisyntomic R-algebras.

We can view $\mathcal{O}_n$ and $\mathcal{O}_n\{1\}$ as quasicoherent sheaves on $(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})_{\mathrm{fppf}}$.

For $G \in \mathrm{BT}_n(R)$, set

$$\mathcal{M}(G) = \underline{\mathrm{Hom}}_{(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})_{\mathrm{fppf}}}(\epsilon_n^* G^*, \mathcal{O}_n\{1\})$$

where on the right we are considering the internal Hom sheaf over $(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})_{\mathrm{fppf}}$. Note that by construction $\mathcal{M}(G)$ is a module over $\mathcal{O}_n$.

The next result can be found in a certain form in Anschütz-Le Bras [1], though this formulation (and most of the proof—we give a different approach to Cartier duality) is due to Mondal [41]:

lb_mondal

Proposition 11.20. *(1) $\mathcal{M}(G)$ is a vector bundle over $\mathcal{O}_n$ and yields an F-gauge in* $\mathrm{Vect}_{[0,1]}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})$.

(2) The functors

$$\mathcal{M} : \mathrm{BT}_n(R) \to \mathrm{Vect}_{[0,1]}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z});$$
$$\mathcal{G} : \mathrm{Vect}_{[0,1]}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}) \xrightarrow{\textit{Theorem 11.14}} \mathrm{BT}_n(R)$$

form an adjoint pair.

(3) The unit $\mathrm{id} \to \mathcal{G} \circ \mathcal{M}$ *is an isomorphism.*

(4) There is a natural isomorphism $\mathcal{M}(G^*)^* \to \mathcal{M}(G)$.

Proof. Most of the proof that we present here can be found in [41, §3].

For claim (1), via quasisyntomic descent we reduce to the case where R is qrsp. Here, the result follows from [41, Prop. 3.16], which uses embeddings of G into abelian schemes and a computation of the syntomic cohomology of abelian schemes.

For the second claim, given $G \in \mathcal{BT}_n(R)$ and $\mathcal{M} \in \mathrm{Vect}_{[0,1]}(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})$, we find canonical isomorphisms:

$$\begin{aligned}
\mathrm{Hom}_{\mathcal{O}_n}(\mathcal{M}, \mathcal{M}(G)) &\simeq \mathrm{Hom}_{\mathcal{O}_n}\left(\mathcal{M}, \underline{\mathrm{Hom}}_{(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})_{\mathrm{fppf}}}(\epsilon_n^* G^*, \mathcal{O}_n\{1\})\right) \\
&\simeq \mathrm{Hom}_{(R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z})_{\mathrm{fppf}}}\left(\epsilon_n^* G^*, \underline{\mathrm{Hom}}_{\mathcal{O}_n}(\mathcal{M}, \mathcal{O}_n\{1\})\right) \\
&\simeq \mathrm{Hom}_{R_{\mathrm{fppf}}}\left(G^*, \epsilon_{n,*}\underline{\mathrm{Hom}}_{\mathcal{O}_n}(\mathcal{O}_n, \mathcal{M}^*)\right) \\
&\simeq \mathrm{Hom}_{R_{\mathrm{fppf}}}\left(G^*, \mathcal{G}(\mathcal{M}^*)\right) \\
&\simeq \mathrm{Hom}_{\mathcal{BT}_n(R)}(G^*, \mathcal{G}(\mathcal{M})^*) \\
&\simeq \mathrm{Hom}_{\mathcal{BT}_n(R)}(\mathcal{G}(\mathcal{M}), G).
\end{aligned}$$

Here, in the penultimate isomorphism, we have used Theorem 11.18.

For claim (3), suppose that we are given $G \in \mathrm{BT}_n(R)$. We then find:

$$\begin{aligned}
\mathcal{G}(\mathcal{M}(G))(R) &\simeq \mathrm{Hom}_{\mathcal{O}_n}(\mathcal{O}_n, \mathcal{M}(G)) \\
&\simeq \mathrm{Hom}_{\mathcal{BT}_n(R)}(\mathcal{G}(\mathcal{O}_n), G) \\
&\simeq \mathrm{Hom}_{\mathcal{BT}_n(R)}(\mathbb{Z}/p^n\mathbb{Z}, G) \\
&\simeq G(R).
\end{aligned}$$

Here, in the penultimate isomorphism, we have used Proposition 11.17. Since this isomorphism is valid with R replaced by any p-quasisyntomic R-algebra, claim (3) has been verified.

Finally, let us consider claim (4): We have

$$\begin{aligned}\mathrm{Hom}_{\mathcal{O}_n}(\mathcal{M}(G^*)^*,\mathcal{M}(G)) &\simeq \mathrm{Hom}_{\mathcal{BT}_n(R)}(\mathcal{G}(\mathcal{M}(G^*)^*),G)\\ &\simeq \mathrm{Hom}_{\mathcal{BT}_n(R)}(\mathcal{G}(\mathcal{M}(G^*))^*,G)\\ &\simeq \mathrm{Hom}_{\mathcal{BT}_n(R)}((G^*)^*,G)\\ &\simeq \mathrm{Hom}_{\mathcal{BT}_n(R)}(G,G).\end{aligned}$$

Here, the first isomorphism uses claim (2), the second uses Theorem 11.18 and the third uses claim (3). The identity endomorphism of G corresponds via these isomorphisms to the canonical arrow involved in claim (4).

That this arrow is an isomorphism is a consequence of the next lemma, applied with $\mathcal{M}_1 = \mathcal{M}(G^*)^*$ and $\mathcal{M}_2 = \mathcal{M}(G)$:

servative **Lemma 11.21.** *Suppose that $\mathcal{M}_1$ and $\mathcal{M}_2$ are two perfect F-gauges of level n over R with Hodge-Tate weights $0,1$. Set $\mathcal{N}^* = \mathcal{N}^\vee\{1\}$ for any perfect F-gauge $\mathcal{N}$: this underlies an involution on the ∞-category of perfect F-gauges of Hodge-Tate weights $0,1$. Suppose that $f:\mathcal{M}_1\to\mathcal{M}_2$ is an arrow such that*

$$\Gamma_{\mathrm{syn}}(f):\Gamma_{\mathrm{syn}}(\mathcal{M}_1)\to\Gamma_{\mathrm{syn}}(\mathcal{M}_2)\ ;\ \Gamma_{\mathrm{syn}}(f^*):\Gamma_{\mathrm{syn}}(\mathcal{M}_2^*)\to\Gamma_{\mathrm{syn}}(\mathcal{M}_1^*)$$

are equivalences of derived stacks over R. Then f is an isomorphism.

Proof. Set $\mathcal{N} = \mathrm{hker}(f)$; then we see that $\Gamma_{\mathrm{syn}}(\mathcal{N})(C) = 0$ for all $C\in\mathrm{CRing}_{R/}$. Similarly, $\Gamma_{\mathrm{syn}}(\mathcal{N}^*[-1])(C) = 0$ for all $C\in\mathrm{CRing}_{R/}$. Theorem 8.37 now tells us that, if $\mathrm{Fil}^\bullet_{\mathrm{Hdg}} N$ and $\mathrm{Fil}^\bullet_{\mathrm{Hdg}} N^*$ are the filtered perfect complexes over R obtained from $\mathcal{N}$ and $\mathcal{N}^*$ via pullback along $x^{\mathcal{N}}_{\mathrm{dR}}$, then we have

$$\mathrm{gr}^{-1}_{\mathrm{Hdg}} N \simeq 0 \simeq \mathrm{gr}^{-1}_{\mathrm{Hdg}} N^* \simeq (\mathrm{Fil}^0_{\mathrm{Hdg}} N)^\vee.$$

Since $\mathcal{N}$ has Hodge-Tate weights $0,1$, this shows that $\mathrm{gr}^i_{\mathrm{Hdg}} N\simeq 0$ for all i, and hence that $\mathrm{Fil}^\bullet_{\mathrm{Hdg}} N\simeq 0$. This implies that $\mathcal{N}\simeq 0$: To see this, we can assume that R is semiperfectoid, in which case it follows from the observation that the map

$$\pi_0(\mathrm{Fil}^\bullet_{\mathcal{N}}\, \Delta_R/^{\mathbb{L}}p^n)\to\pi_0(\mathrm{Fil}^\bullet_{\mathrm{triv}}(R/^{\mathbb{L}}p^n))$$

has locally nilpotent kernel. ☐

☐

Remark 11.22. We wonder if there is an argument for claim (1) above that does not appeal to a computation of the syntomic cohomology of abelian schemes.

We are now ready to prove Theorem 11.12

Proof of Theorem 11.12. We first note that Proposition 11.20 gives us a left inverse $\mathcal{M}:\mathrm{BT}^{h,d}_n\to\mathrm{BT}^{\mathrm{GL}_h,\mu_d}_n$ to the map $\mathcal{G}$ from Theorem 11.14. Indeed, by Theorem 11.2, $\mathrm{BT}^{h,d}_n$ is smooth, and therefore is a left Kan extension of its restriction to p-completely smooth $\mathbb{Z}_p$-algebras. This means that, to obtain the map $\mathcal{M}$ and verify that it is a left inverse to $\mathcal{G}$, it suffices to do so on such inputs, where it follows from the proposition.

From the same proposition, we find, for all $\mathcal{M}\in\mathrm{BT}^{\mathrm{GL}_h,\mu_d}_n(R)$, a canonical map of F-gauges $\mathcal{M}\to\mathcal{M}(\mathcal{G}(\mathcal{M}))$. To finish the proof of the theorem, we have to verify that this map is an isomorphism.

For this, we can assume without loss of generality that R is p-quasisyntomic. Now, we begin by observing that we also have a corresponding canonical map of Cartier dual F-gauges

$$\mathcal{M}^*\to\mathcal{M}(\mathcal{G}(\mathcal{M}^*))\simeq\mathcal{M}(\mathcal{G}(\mathcal{M})^*),$$

where the last isomorphism using Theorem 11.18. Taking Cartier duals again yields a map

$$\mathcal{M}(\mathcal{G}(\mathcal{M})^*)^*\to\mathcal{M}.$$

and the composition

$$\mathcal{M}(\mathcal{G}(\mathcal{M})^*)^*\to\mathcal{M}\to\mathcal{M}(\mathcal{G}(\mathcal{M}))$$

is the canonical isomorphism in claim (4) of Proposition 11.20 applied with $G=\mathcal{G}(\mathcal{M})$. Alternatively, instead of using Cartier duality in this form, one can argue directly using Lemma 11.21.

This shows that $\mathcal{M} \to \mathcal{M}(\mathcal{G}(\mathcal{M}))$ is an epimorphism, and since it is a map of vector bundles of the same rank over $R^{\mathrm{syn}} \otimes \mathbb{Z}/p^n\mathbb{Z}$, we conclude that it is in fact an isomorphism. $\square$

ot_needed *Remark* 11.23. In the proof above, since we already have the map $\mathcal{G}$, to verify that it is an equivalence, it suffices to do so over the special fiber. In particular, logically speaking, we only need Proposition 11.20 in the case where R is an $\mathbb{F}_p$-algebra. In this case, the verification of claim (1) of that proposition only requires the computations from [3] of the *crystalline* cohomology of abelian schemes in characteristic p. Formulating the proof this way would make it completely independent (strictly in a logical sense) of the results of [1] or [41], though it wouldn't shed much light on what the inverse functor is, away from characteristic p.

Alternatively, if one is happy to assume classical Dieudonné theory over perfect fields as explicated, say, in [19], one could also reduce to the case of power series rings over perfect fields, where we could use Grothendieck-Messing theory on both sides of the purported equivalence to further reduce to the known case of perfect fields. However, here one would have to compare our construction of finite flat group schemes here with that used by Fontaine, which involves Witt covectors.

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