Assignment 4

due midnight (Eastern Time), Friday, October 9, 2020

Prove all statements in the questions below. You must TeX your solutions and submit your writeup as a pdf at www.gradescope.com.
You can submit it any time until the deadline (try not to spend your Friday evening working on this!).

Question 1. (Folland 2.2.14) If $f \in L^+$, let $\lambda(E) = \int_E f \, d\mu$ for $E \in \mathcal{M}$. Then λ is a measure on $\mathcal{M}$, and for any $g \in L^+$, $\int g \, d\lambda = \int fg \, d\mu$. (First suppose that g is simple.)

Question 2. (Folland 2.2.17) Assume Fatou's lemma and deduce the monotone convergence theorem from it.

Question 3. (Folland 2.2.16) If $f \in L^+$ and $\int f < \infty$, for every $\epsilon > 0$ there exists $E \in \mathcal{M}$ such that $\mu(E) < \infty$ and $\int_E f > (\int f) - \epsilon$.

Question 4. (Folland 2.3.20) If $f_n, g_n, f, g \in L^1$, $f_n \to f$ and $g_n \to g$ a.e. $|f_n| < g_n$ and $\int g_n \to \int g$, then $\int f_n \to \int f$. (Rework the proof of the dominated convergence theorem).

Question 5. (Folland 2.3.22) Let μ be counting measure on $\mathbb{N}$. Interpret Fatou's lemma and the monotone and dominated convergence theorems as statements about infinite series.

Question 6. (Folland 2.4.34) Suppose $|f_n| \le g \in L^1$ and $f_n \to f$ in measure.

1. $\int f = \lim \int f_n$.
2. $f_n \to f$ in L^1.

Question 7. (Folland 2.4.37) Suppose that f_n and f are measurable complex-valued functions and $\phi : \mathbb{C} \to \mathbb{C}$.

1. If ϕ is continuous and $f_n \to f$ a.e., then $\phi \circ f_n \to \phi \circ f$ a.e.
2. If ϕ is uniformly continuous and $f_n \to f$ uniformly, almost uniformly, or in measure, then $\phi \circ f_n \to \phi \circ f$ uniformly, almost uniformly, or in measure, respectively.

3. There are counterexamples when the continuity assumptions on ϕ are not satisfied.

Question 8. (Folland 2.4.44) (Lusin's Theorem). If $f : [a,b] \to \mathbb{C}$ is Lebesgue measurable and $\epsilon > 0$, there is a compact set $E \subset [a,b]$ such that $\mu(E^c) < \epsilon$ and $f|_E$ is continuous.
(Use Egoroff's theorem and Theorem 2.26).

Question 9. (Folland 2.5.46) Let $X = Y = [0,1]$, $\mathcal{M} = \mathcal{N} = \mathcal{B}_{[0,1]}$, $\mu =$ Lebesgue measure, and $v =$ counting measure. If $D = \{(x,x) : x \in [0,1]\}$ is the diagonal in $X \times Y$, then

$$\iint \chi_D \; d\mu d\nu, \quad \iint \chi_D \; d\nu d\mu \quad \text{and} \quad \int \chi_D \; d(\mu \times \nu)$$

are all unequal.
(To compute $\int \chi_D \; d(\mu \times \nu) = \mu \times \nu(D)$, go back to the definition of $\mu \times \nu$.)

Question 10. (Folland 2.5.50) Suppose $(X, \mathcal{M}, \mu)$ is a σ-finite measure space and $f \in L^+(X)$. Let

$$G_f := \{(x,y) \in X \times [0,\infty] : y \leq f(x)\}.$$

Then G_f is $(\mathcal{M} \times \mathcal{B}_{\mathbb{R}})$-measurable and $(\mu \times m)(G_f) = \int f d\mu$; the same is also true if the inequality $y \leq f(x)$ is replaced by $y < f$.

(To show measurability of G_f, note that the map $(x,y) \mapsto f(x) - y$ is the composition of $(x,y) \mapsto (f(x), y)$ and $(z,y) \mapsto z - y$.)
(This is the definitive statement of the familiar theorem from calculus, "the integral of a function is the area under its graph.")