Assignment 7

due midnight (Eastern Time), Monday, November 9, 2020

Prove all statements in the questions below. You must TeX your solutions and submit your writeup as a pdf at www.gradescope.com.
You can submit it any time until the deadline.

Question 1. (Folland 5.1.8) Let $(X, \mathcal{M})$ be a measurable space, and let $M(X)$ be the space of complex measures on $(X, \mathcal{M})$. Then $||\mu|| = |\mu|(X)$ is a norm on $\mathcal{M}(X)$ that makes $M(X)$ into a Banach space. (Use Theorem 5.1.).

Question 2. (Folland 5.1.11) If $0 < \alpha \leq 1$, let $\Lambda_\alpha([0,1])$ be the space of Hölder continuous functions of exponent α on $[0,1]$. That is, $f \in \Lambda_\alpha([0,1])$ iff $||f||_{\Lambda_\alpha} < \infty$, where

$$||f||_{\Lambda_\alpha} = |f(0)| + \sup_{x,y\in[0,1],\ x\neq y} \frac{|f(x)-f(y)|}{|x-y|^\alpha}.$$

1. $||\cdot||_{\Lambda_\alpha}$ is a norm that makes $\Lambda_\alpha([0,1])$ into a Banach space.
2. Let $\lambda_\alpha([0,1])$ be the set of all $f \in \Lambda_\alpha([0,1])$ such that
$$\frac{|f(x)-f(y)|}{|x-y|^\alpha} \to 0 \text{ as } x \to y, \text{ for all } y \in [0,1].$$
If $\alpha < 1$, $\lambda_\alpha([0,1])$ is an infinite-dimensional closed subspace of $\Lambda_\alpha([0,1])$. If $\alpha = 1$, $\lambda_\alpha([0,1])$ contains only constant functions.

Question 3. (Folland 5.2.20) If $\mathcal{M}$ is a finite-dimensional subspace of a normed vector space $\mathcal{X}$, there is a closed subspace $\mathcal{N}$ such that $\mathcal{M} \cap \mathcal{N} = \{0\}$ and $\mathcal{M} + \mathcal{N} = \mathcal{X}$.

Question 4. (Folland 5.2.24) Suppose that $\mathcal{X}$ is a Banach space.

1. Let $\widehat{\mathcal{X}}$, $(\mathcal{X}^*)^\wedge$ be the natural images of $\mathcal{X}$, $\mathcal{X}^*$ in $\mathcal{X}^{**}$, $\mathcal{X}^{***}$ and let
$$\widehat{\mathcal{X}}^0 = \{F \in \mathcal{X}^{***} : F|\widehat{X} = 0\}.$$
Then
$$(\mathcal{X}^*)^\wedge \cap \widehat{\mathcal{X}}^0 = \{0\}$$
and
$$(\mathcal{X}^*)^\wedge + \widehat{\mathcal{X}}^0 = \mathcal{X}^{***}.$$
2. $\mathcal{X}$ is reflexive iff $\mathcal{X}^*$ is reflexive.