# Relative Spaces

Affine schemes are schemes

Last time: Open coverings of affine schemes

Sub-canonical (for a topos) means that representable things (the analogue of affine schemes) are sheaves.

## Relative Spaces ("Relative Algebraic Geometry")

Let $S \in$ Space (often called the base space).

We get a category $\mathrm{Space}_S = \mathrm{Space}_{/S}$ of S-spaces

or spaces over S. "slice category"

- Objects: $X \to S$ maps of spaces
- Morphisms: $\begin{tikzcd} X \arrow[r] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$ comm. diagrams

1. There is a forgetful functor $\mathrm{Space}_S \to \mathrm{Space}$
2. The axioms of a category are "abstract."

$X \to Y \circ Y \to Z = X \to Y \to Z$

$$\begin{tikzcd}[column sep=small] X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd} \circ \begin{tikzcd}[column sep=small] Y \arrow[rr] \arrow[dr] & & Z \arrow[dl] \\ & S & \end{tikzcd} = \begin{tikzcd}[column sep=small] X \arrow[r] \arrow[dr] & Y \arrow[r] & Z \arrow[dl] \\ & S & \end{tikzcd}$$

Two perspectives:

(1) "Things living inside $S$"

(2) "Families of things living over $S$"

In $\mathrm{Space}_S$ the terminal object is $S$!

$$\mathrm{Space} \simeq \mathrm{Space}_{\mathrm{Spec}\,\mathbb{Z}} \qquad \begin{matrix}\mathrm{Spec}\,A \\ \downarrow \\ \mathrm{Spec}\,\mathbb{Z}\end{matrix} \leftrightsquigarrow \mathbb{Z} \to A$$

$$\mathrm{Space}_{\mathrm{Spec}A} \simeq \mathrm{Fun}(\mathrm{CAlg}_A, \mathrm{Set})$$

$$\mathrm{CAlg}_A \simeq \mathrm{CRing}_{A/}$$

<u>Base Change</u>: $T \in \mathrm{Space}_S \quad (T \to S)$

$\rightsquigarrow \quad T \underset{S}{\times} \bullet : \mathrm{Space}_S \to \mathrm{Space}_T$ base change functor

$$\mathrm{Space}_T \ni \begin{tikzcd} T \underset{S}{\times} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner", very near start] & X \arrow[d] \\ T \arrow[r] & S \end{tikzcd} \in \mathrm{Space}_S$$

$$\begin{tikzcd}[column sep=small] X \arrow[rr] \arrow[dr] & & Y \arrow[dl] \\ & S & \end{tikzcd} \rightsquigarrow \begin{tikzcd}[column sep=small] T \underset{S}{\times} X \arrow[rr] \arrow[dr] & & T \underset{S}{\times} Y \arrow[dl] \\ & T & \end{tikzcd}$$

$$\begin{tikzcd} & S & \rightsquigarrow & & T \end{tikzcd}$$

$$\begin{tikzcd}[column sep=small]
 & X \arrow[r] \arrow[d] & Y \arrow[d] & \\
T \times_S X \arrow[ur] \arrow[dr] \arrow[rrr, bend left=60, dashed, red, "\exists!"] & S \arrow[r, equal] & S & T \times_S Y \arrow[ul] \arrow[dl] \\
 & T \arrow[u] \arrow[ur] & &
\end{tikzcd}$$

$$\begin{tikzcd}
T' \arrow[r] & T \arrow[r] & S \\
T' \times_S X \arrow[u] & T \times_S X \arrow[u] &
\end{tikzcd}
\qquad
T' \times_T (T \times_S X) \simeq T' \times_S X$$

$$\mathbb{Z} \to \mathbb{Q} \quad \rightsquigarrow \quad
\begin{tikzcd}
X_{\mathbb{Q}} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner", very near start] & X \arrow[d] \\
\operatorname{Spec}\mathbb{Q} \arrow[r] & \operatorname{Spec}\mathbb{Z}
\end{tikzcd}$$

$$T \times_S X =: X_T \qquad X_{\operatorname{Spec} A} =: X_A$$

$E$ elliptic curve $\qquad E(\mathbb{Q})$ $\qquad \mathbb{Q}$-pts.

$\qquad\qquad E_{\mathbb{Q}}$ $\qquad \mathbb{Q}$-base change

$E$: cut out by equation $y^2 = x^3 + x$

$$E = \operatorname{Spec} \mathbb{Z}[x,y]/(y^2 - x^3 - x)$$

$$E_{\mathbb{Q}} = \operatorname{Spec} \mathbb{Q} \times_{\operatorname{Spec} \mathbb{Z}} E$$

$$\cong \operatorname{Spec} \mathbb{Q} \otimes_{\mathbb{Z}} \mathbb{Z}[x,y]/(y^2 - x^3 - x)$$

$$\cong \operatorname{Spec} \mathbb{Q}[x,y]/(y^2 - x^3 - x)$$

$$E(\mathbb{Q}) \overset{?}{=} \{ (a,b) \in \underset{\mathbb{Z}^2 \text{ (}\shortparallel\text{)}}{\mathbb{Q}^2} : b^2 - a^3 - a \overset{=}{\phantom{.}} 0 \}$$

$$E(\mathbb{Q}) := \operatorname{Hom}_{\mathrm{CRing}}(\mathbb{Z}[x,y]/(y^2 - x^3 - x), \mathbb{Q})$$

<u>Aside</u>: $\mathbb{RP}^n = (\mathbb{R}^{n+1} \setminus 0)/\mathbb{R}^\times$

(1) Weil
(2) Cartier

$$\mathbb{P}^n_k(k) = (k^{n+1} \setminus 0)/k^\times$$

$$\mathbb{P}^n_k = (\mathbb{A}^{n+1}_k \setminus 0)/\mathbb{G}_m$$

$f : X \to Y$ What is $\mathrm{im}(f)$?

$X \overset{\text{epic}}{\twoheadrightarrow} \mathrm{im}(f) \overset{\text{monic}}{\hookrightarrow} Y$

1. It's functorial.
2. $\mathrm{im}(f)$ is a subset of $Y$.

People distinguish between the "set-theoretic" image and the "scheme-theoretic" image.