1 Introduction

Our goal is to describe different types of Dieudonné complexes in terms of fixed points. This is relatively easy to accomplish for saturated complexes but requires a bit more work for strict complexes. As a reminder, recall that $M \in \mathsf{DC}$ is saturated if it is p-torsion-free and

$$\alpha_F : M \to \eta_p M, \qquad x \mapsto p^n F(x)$$

is an isomorphism of Dieudonné complexes (where $x \in M^n$). Note that the data of a map of complexes $M \to \eta_p M$ is equivalent to a choice of Frobenius on M making it into a Dieudonné complex. In more detail, $\alpha : M \to \eta_p M$ induces

$$F_\alpha : M \to M, \qquad x \mapsto p^{-n}\alpha(x)$$

a map of graded abelian groups (where $x \in M^n$). Recall also that $M \in \mathsf{DC}_{\text{sat}}$ is strict if the canonical map $\rho_F : M \to \mathcal{W}M$ is an isomorphism, noting that $\mathcal{W}M$ is always strict.

2 Décalage

As we already know, the décalage process determines an endofunctor $\eta_p : \mathsf{Ch}(\mathbb{Z})^{\text{tf}} \to \mathsf{Ch}(\mathbb{Z})^{\text{tf}}$. One of the key properties of décalage is that it kills off p-torsion in cohomology – given $M \in \mathsf{Ch}(\mathbb{Z})^{\text{tf}}$, there is a canonical isomorphism

$$H^\bullet(M)/H^\bullet(M)[p] \xrightarrow{\sim} H^\bullet(\eta_p M)$$

of graded abelian groups.[1] Hence, η_p sends qis's to qis's and we obtain the following result.

Proposition 1. *There is an essentially unique functor $L\eta_p : D(\mathbb{Z}) \to D(\mathbb{Z})$ such that*

$$\begin{tikzcd} \mathsf{Ch}(\mathbb{Z})^{\text{tf}} \arrow[r, "\eta_p"] \arrow[d] & \mathsf{Ch}(\mathbb{Z})^{\text{tf}} \arrow[d] \\ D(\mathbb{Z}) \arrow[r, dotted, "\exists!\, L\eta_p"'] & D(\mathbb{Z}) \end{tikzcd}$$

commutes up to natural isomorphism.

Thinking of $D(\mathbb{Z})$ as $\mathsf{Ch}(\mathbb{Z})^{\text{tf}}$ with qis's inverted, we obtain $L\eta_p X$ for $X \in D(\mathbb{Z})$ by choosing a representative for X in $\mathsf{Ch}(\mathbb{Z})^{\text{tf}}$, applying η_p, and taking the corresponding qis class in $D(\mathbb{Z})$.[2] Similar comments apply if one were to choose a different model of $D(\mathbb{Z})$, in particular the homotopical model $h\,\mathsf{Ch}(\mathbb{Z})^{\text{free}}$.

3 Completion

Definition 2. ***Classical*** p***-completion*** *is the functor*

$$\widehat{\cdot} : \mathsf{Mod}_{\mathbb{Z}} \to \mathsf{Mod}_{\mathbb{Z}}, \qquad X \mapsto \varprojlim_{n \geq 1} X/p^n X.$$

[1] We can upgrade this to an isomorphism of complexes if the RHS is equipped with the differential induced by the Bockstein operator.

[2] By using the term 'class' here I don't mean to suggest that we are performing some kind of quotient process. Instead, I mean that the result is well-defined up to qis (which is isomorphism in $D(\mathbb{Z})$). In particular, passing to the skeleton of $D(\mathbb{Z})$ gives something unique (I think).

*We say $X \in \mathsf{Mod}_{\mathbb{Z}}$ is **classically** p**-complete** if the natural map $X \to \widehat{X}$ is an isomorphism. On a somewhat related note, $X \in D(\mathbb{Z})$ is **derived** p**-complete** if $\mathrm{Hom}_{D(\mathbb{Z})}(Y, X) = 0$ for every $Y \in D(\mathbb{Z})$ such that $p : Y \xrightarrow{\sim} Y$. Such objects span a full subcategory $D_p(\mathbb{Z}) \subseteq D(\mathbb{Z})$.*

Proposition 3. *The inclusion $D_p(\mathbb{Z}) \hookrightarrow D(\mathbb{Z})$ admits a left adjoint $\widehat{\ } : D(\mathbb{Z}) \to D_p(\mathbb{Z})$ called the **derived** p**-completion** given by choosing a representative in $\mathsf{Ch}(\mathbb{Z})^{\mathrm{tf}}$ and applying classical p-completion in each degree.*[3]

In line with the above, we extend derived notions to $\mathsf{Ch}(\mathbb{Z})$ by passing to qis classes. This in turn allows us to extend derived notions to $\mathsf{Mod}_{\mathbb{Z}}$ by thinking of abelian groups as complexes concentrated in degree 0. Given $X \in \mathsf{Mod}_{\mathbb{Z}}$, the classical p-completion of X represents the derived p-completion of X and so we may identify the two. In this simple case, to check that X is derived p-complete we need only verify that $\mathrm{Hom}_{D(\mathbb{Z})}(\mathbb{Z}[p^{-1}], X) = 0$.

Proposition 4. *Let $X \in \mathsf{Mod}_{\mathbb{Z}}$. Then, X is **pro-free** (i.e., the p-completion of a free abelian group) if and only if it is derived p-complete and p-torsion-free.*

Complexes of pro-free abelian groups span a full subcategory $\mathsf{Ch}(\mathbb{Z})^{\text{pro-free}} \subseteq \mathsf{Ch}(\mathbb{Z})$. This category is clearly linked to $D_p(\mathbb{Z})$ by the above, and in fact the connection is strong.

Theorem 5. *The functor $\mathsf{Ch}(\mathbb{Z})^{\text{pro-free}} \to D(\mathbb{Z})$ obtained by passing to qis classes has essential image $D_p(\mathbb{Z})$ and induces an equivalence $h\,\mathsf{Ch}(\mathbb{Z})^{\text{pro-free}} \xrightarrow{\sim} D_p(\mathbb{Z})$. In more detail, given $X, Y \in \mathsf{Ch}(\mathbb{Z})^{\text{pro-free}}$, $\mathrm{Hom}_{\mathsf{Ch}(\mathbb{Z})}(X, Y) \twoheadrightarrow \mathrm{Hom}_{D(\mathbb{Z})}(X, Y)$ and $f, g \in \mathrm{Hom}_{\mathsf{Ch}(\mathbb{Z})}(X, Y)$ have the same image if and only if $f \simeq g$.*

Before discussing fixed points, we mention two supplementary results that will be important soon. The first result concerns compatibility of décalage and p-completion.

Proposition 6. *Suppose that $M \to N$ in $D(\mathbb{Z})$ exhibits N as a derived p-completion of M. Then, the induced map $L\eta_p M \to L\eta_p N$ exhibits $L\eta_p N$ as a derived p-completion of $L\eta_p M$. Hence, $L\eta_p$ restricts to an endofunctor of $D_p(\mathbb{Z})$.*

The second result concerns completion of Dieudonné complexes.

Proposition 7. *Given $M \in \mathsf{DC}_{\mathrm{sat}}$, the canonical map $\rho_F : M \to \mathcal{W}M$ exhibits $\mathcal{W}M$ as a derived p-completion of M. Moreover, ρ_F is a qis if and only if M is derived p-complete.*

4 Fixed Points

Definition 8. *Let $\mathcal{C}$ be a category and $T : \mathcal{C} \to \mathcal{C}$ an endofunctor. The **fixed point** category $\mathcal{C}^T$ of $\mathcal{C}$ with respect to T is the category whose objects are pairs (X, φ) with $X \in \mathcal{C}$ and $\varphi \in \mathrm{Isom}_{\mathcal{C}}(X, TX)$. The data of a morphism $f : (X, \varphi) \to (X', \varphi')$ is $f \in \mathrm{Hom}_{\mathcal{C}}(X, X')$ such that*

[3] Part of the content of this result is that the choice of representative does not matter (up to qis). In particular, $D_p(\mathbb{Z})$ is invariant under this process.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & X' \arrow[d, "\varphi'"] \\ TX \arrow[r, "Tf"'] & TX' \end{tikzcd}$$

commutes.

Remark 9. *Let $(\mathcal{C}, T)$ and $(\mathcal{C}', T')$ be categories equipped with endofunctors that are intertwined in the sense that there is a functor $\mathcal{C} \to \mathcal{C}'$ intertwining T and T' up to specified natural isomorphism. Then, there is a natural induced functor $\mathcal{C}^T \to (\mathcal{C}')^{T'}$.*

Basically by definition, we immediately see that there is an equivalence

$$\mathsf{DC}_{\mathrm{sat}} \xrightarrow{\sim} (\mathsf{Ch}(\mathbb{Z})^{\mathrm{tf}})^{\eta_p}, \qquad (M, F) \mapsto (M, \alpha_F).$$

Because of the earlier commutative diagram for décalage, we obtain a functor θ via

$$\mathsf{DC}_{\mathrm{str}} \hookrightarrow \mathsf{DC}_{\mathrm{sat}} \xrightarrow{\sim} (\mathsf{Ch}(\mathbb{Z})^{\mathrm{tf}})^{\eta_p} \to D(\mathbb{Z})^{L\eta_p}.$$

Theorem 10. *The composite functor $\theta : \mathsf{DC}_{\mathrm{str}} \to D(\mathbb{Z})^{L\eta_p}$ induces an equivalence $\mathsf{DC}_{\mathrm{str}} \xrightarrow{\sim} D_p(\mathbb{Z})^{L\eta_p}$.*

We begin by showing that the essential image of θ is $D_p(\mathbb{Z})^{L\eta_p}$. To that end, choose an object of $D_p(\mathbb{Z})^{L\eta_p}$. On the level of representatives, this amounts to choosing $X \in \mathsf{Ch}(\mathbb{Z})^{\mathrm{tf}}$ and a qis $\alpha : X \to \eta_p X$. Using α, we endow X with the structure of a Dieudonné module as discussed earlier. Each of the arrows in the diagram

$$X \xrightarrow{\alpha} \eta_p X \xrightarrow{\eta_p \alpha} \eta_p^2 X \xrightarrow{\eta_p^2 \alpha} \cdots$$

is a qis and so the induced map $X \to \mathrm{Sat}(X)$ is a qis. Since X is derived p-complete, $\mathrm{Sat}(X)$ is derived p-complete and so the canonical map $\mathrm{Sat}(X) \to \mathcal{W}\mathrm{Sat}(X)$ is a qis. Hence, the completed saturation map $X \to \mathcal{W}\mathrm{Sat}(X)$ is a qis. This fits into a commutative diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "\alpha"'] & \mathcal{W}\mathrm{Sat}(X) \arrow[d, "\mathcal{W}\mathrm{Sat}(\alpha)"] \\ \eta_p X \arrow[r] & \eta_p \mathcal{W}\mathrm{Sat}(X) \end{tikzcd}$$

The right vertical map is an isomorphism since $\mathcal{W}\mathrm{Sat}(X)$ is saturated. It follows that (X, α) and $(\mathcal{W}\mathrm{Sat}(X), \mathcal{W}\mathrm{Sat}(\alpha))$ represent isomorphic objects in $D_p(\mathbb{Z})^{L\eta_p}$ and so (X, α) lies in the essential image of $\mathcal{W}\mathrm{Sat}(X) \in \mathsf{DC}_{\mathrm{str}}$ under θ. To finish seeing that the essential image of θ is $D_p(\mathbb{Z})^{L\eta_p}$, note that $X \in \mathsf{DC}_{\mathrm{str}}$ satisfies $X \cong \mathcal{W}X$ and the latter is derived p-complete (which means θ factors through $D_p(\mathbb{Z})^{L\eta_p}$).

Our aim now is to show that θ is fully faithful. To that end, choose $X, Y \in \mathsf{DC}_{\mathrm{str}}$ (from which we get that X, Y are both pro-free by earlier comments) and consider the natural map

$$\Theta_{X,Y} : \mathrm{Hom}_{\mathsf{DC}_{\mathrm{str}}}(X, Y) \to \mathrm{Hom}_{D_p(\mathbb{Z})^{L\eta_p}}(X, Y)$$

induced by θ. Recall from earlier that we have an equivalence $h\,\mathsf{Ch}(\mathbb{Z})^{\text{pro-free}} \xrightarrow{\sim} D_p(\mathbb{Z})$ obtained by passing to qis classes. We wish to understand what $\mathrm{Hom}_{D_p(\mathbb{Z})^{L\eta_p}}(X, Y)$ looks like under this equivalence. With this in mind, we introduce the following definition.

Definition 11. *Suppose $X, Y \in \mathsf{Ch}(\mathbb{Z})^{\mathrm{tf}}$ are equipped with the structure of Dieudonné modules and $f \in \mathrm{Hom}_{\mathsf{Ch}(\mathbb{Z})}(X, Y)$. We say that f is **weakly F-compatible** if the diagram*

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\alpha_F"'] & Y \arrow[d, "\alpha_F"] \\ \eta_p X \arrow[r, "\eta_p(f)"'] & \eta_p Y \end{tikzcd}$$

commutes up to homotopy. In the case that Y is saturated this is the same as requiring that

$$\alpha_F^{-1} \circ \eta_p f \circ \alpha_F = F^{-1} \circ f \circ F \simeq f$$

as maps of complexes. This notion clearly extends to homotopy classes of maps in $[X, Y]$.

It follows that the (functorial) bijection $\mathrm{Hom}_{D_p(\mathbb{Z})}(X, Y) \longleftrightarrow [X, Y]$ induces a (functorial) bijection

$$\mathrm{Hom}_{D_p(\mathbb{Z})^{L\eta_p}}(X, Y) \longleftrightarrow \{f \in [X, Y] : f \text{ is weakly } F\text{-compatible}\}.$$

The matter of whether $\Theta_{X,Y}$ is bijective therefore boils down to the following lemma.

Lemma 12. *Let $X, Y \in \mathsf{Ch}(\mathbb{Z})^{\mathrm{tf}}$ equipped with the structure of Dieudonné modules such that Y is strict. Let $f \in \mathrm{Hom}_{\mathsf{Ch}(\mathbb{Z})}(X, Y)$ be weakly F-compatible. Then, there exists a unique natural choice of $\widetilde{f} \in \mathrm{Hom}_{\mathsf{DC}}(X, Y)$ such that $\widetilde{f} \simeq f$.*

Proof. We first prove existence. By hypothesis there exists a map $h : X^\bullet \to Y^{\bullet-1}$ of graded abelian groups such that $F^{-1} \circ f \circ F = f + dh + hd$. We seek a homotopy $u : X^\bullet \to Y^{\bullet-1}$ such that taking $\widetilde{f} := f + du + ud$ gives $F^{-1} \circ \widetilde{f} \circ F = \widetilde{f}$. No matter how we choose u, the identity $FdV = d$ gives

$$F^{-1} \circ (du + ud) \circ F = d(VuF) + (VuF)d$$

and so

$$\begin{aligned} F^{-1} \circ \widetilde{f} \circ F &= F^{-1} \circ f \circ F + F^{-1} \circ \widetilde{f} \circ F \\ &= f + dh + hd + d(VuF) + (VuF)d \\ &= f + d(h + VuF) + (h + VuF)d. \end{aligned}$$

Thus, the condition we want is $u = h + VuF$ and so we take

$$u := \sum_{r \geq 0} V^r u F^r.$$

Now to prove uniqueness. Let $g \in \mathrm{Hom}_{\mathsf{DC}}(X, Y)$ such that $g \simeq 0$, so $g = dh + hd$ for some homotopy $h : X^\bullet \to Y^{\bullet-1}$. Given $r \geq 0$,

$$\begin{aligned} g &= F^{-r} \circ g \circ F^r \\ &= F^{-r}(dh + hd) \circ F^r \\ &= d(V^r h F^r) + V^r(hF^r d). \end{aligned}$$

Hence, the composition

$$X \xrightarrow{g} Y \longrightarrow \mathcal{W}_r Y$$

vanishes and so g vanishes since Y is strict. $\square$