BLM

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1 Some Calculations

Let's apply our theory to the torus $\mathbb{G}_{m,\mathbb{Z}_p}^n$.[1] With that in mind, let $R := \mathbb{Z}_p[x_1^{\pm 1}, \ldots, x_n^{\pm 1}]$. Then, we have a cdga isomorphism $\Omega_R^\bullet \cong \bigwedge_R^\bullet[d\log x_1, \ldots, d\log x_n]$, where the generators are in degree 1 and

$$d\log x_i := \frac{dx_i}{x_i}, \qquad d(x_i^a) = a x_i^a \cdot d\log x_i, \qquad d(d\log x_i) = 0.$$

The p-torsion-free ring R comes equipped with a mod p Frobenius lift φ which is the identity on $\mathbb{Z}_p$ and satisfies $\varphi(x_i) = x_i^p$. It follows that (R, φ) is a good ring and so there is a natural way to extend φ to all of $\Omega_R^\bullet$ making the latter a Dieudonné algebra. Using the above isomorphism we may transport the DA structure from $\Omega_R^\bullet$ to $\bigwedge_R^\bullet[d\log x_1, \ldots, d\log x_n]$, which is characterized by

$$F(x_i) = x_i^p, \qquad F(d\log x_i) = d\log x_i.$$

We have

$$R[\varphi^{-1}] \cong \mathbb{Z}_p[x_1^{\pm 1/p^\infty}, \ldots, x_n^{\pm 1/p^\infty}] =: R_\infty$$

and so

$$\Omega_R^\bullet[F^{-1}] \cong \textstyle\bigwedge_{R_\infty}^\bullet[d\log x_i, \ldots, d\log x_n]$$

since F fixes the generators $d\log x_i$.

Let $M \in \mathsf{DC}$ be p-torsion-free with F injective on M. We may describe the iterated décalage $\eta_p^r M \subseteq M[p^{-1}]$ by

$$(\eta_p^r M)^n = \{x \in p^{rn} M^n : dx \in p^{r(n+1)} M^{n+1}\}.$$

Recall that

$$M[F^{-1}] := \operatorname{colim}(M \xrightarrow{F} M \xrightarrow{F} \cdots).$$

We want to identify $\operatorname{Sat}(M)$ inside of $M[F^{-1}]$.

Remark 1. *Intuitively, $M[F^{-1}]$ should be thought of as the union $\bigcup_{r \geq 0} F^{-r} M$. Interesting phenomena only arise at the "infinite" level since* $\operatorname{colim}(M \xrightarrow{F} M) \cong M$*. This isomorphism arises from the fact that the data of a commutative diagram*

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[dr] & M \arrow[d] \\ & N \end{tikzcd}$$

[1] You might think it's more universal to look at $\mathbb{G}_{m,\mathbb{Z}}^n$, but as we shall see replacing $\mathbb{Z}$ by $\mathbb{Z}_p$ makes no material difference. This is perhaps not so surprising since our theory is local to p.

is equivalent to the data of just a map $M \to N$.

We have a commutative diagram of graded abelian groups

$$\begin{tikzcd}
M \arrow[r, "\alpha_F"] \arrow[d, equal] & \eta_p M \arrow[r, "\eta_p\alpha_F"] \arrow[d, hook, "\theta_1"] & \eta_p^2 M \arrow[r, "\eta_p^2\alpha_F"] \arrow[d, hook, "\theta_2"] & \cdots \\
M \arrow[r, hook, "F"] \arrow[d, equal] & M \arrow[r, hook, "F"] \arrow[d, hook, "F^{-1}"] & M \arrow[r, hook, "F"] \arrow[d, hook, "F^{-2}"] & \cdots \\
M \arrow[r, hook] & F^{-1}M \arrow[r, hook] & F^{-2}M \arrow[r, hook] & \cdots
\end{tikzcd}$$

with

$$\theta_r : (\eta_p^r M)^n \hookrightarrow M^n, \qquad x \mapsto p^{-rn}x.$$

It follows that there is an induced injection $\theta : \mathrm{Sat}(M) \hookrightarrow M[F^{-1}]$.[2] What is the image of θ? We have

$$(\operatorname{im}\theta)^n = \bigcup_{r\geq 0} (\operatorname{im}\theta_r)^n \cap F^{-r}M^n$$

with

$$(\operatorname{im}\theta_r)^n \cap F^{-r}M^n = \{x \in F^{-r}M^n : d(F^r x) \in p^r M^{n+1}\}.$$

It follows that may identify $\mathrm{Sat}(M)$ as a graded abelian group via

$$\mathrm{Sat}(M) \cong \{x \in M^\bullet[F^{-1}] : d(F^r x) \in p^r M^{\bullet+1} \text{ for } r \gg 0\}.$$

In this setup the differential d on $\mathrm{Sat}(M)$ is described by

$$x \mapsto p^{-r}F^{-r}d(F^r x)$$

for $r \gg 0$.[3] Viewed another way, we obtain $\widetilde{d} : M^\bullet[F^{-1}] \to M^{\bullet+1}[F^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$ using the same formula and $\mathrm{Sat}(M)$ is precisely the stuff stable under $\widetilde{d}$. Let's now apply this to understand $\Sigma := \mathrm{Sat}(\Omega_R^\bullet)$. For simplicity, let $y_i := d\log x_i$. While it suffices to describe each $\Omega_R^m[F^{-1}]$ entirely in terms of homogeneous forms, some care must be taken for Σ^m. Let's first compute $\widetilde{d} : \Omega_R^\bullet[F^{-1}] \to \Omega_R^{\bullet+1}[F^{-1}] \otimes_{\mathbb{Z}} \mathbb{Z}[p^{-1}]$. We obtain $\Omega_R^\bullet[F^{-1}]$ as the $\mathbb{Z}_p$-linear span of homogeneous forms

$$\omega = x_1^{a_1}\cdots x_n^{a_n} \underbrace{y_{k_1} \wedge \cdots \wedge y_{k_m}}_{=:\eta},$$

where we could have $m = 0$. Then,

$$\begin{aligned}
\widetilde{d}\omega &= p^{-r}F^{-r}d(F^r\omega) \\
&= p^{-r}F^{-r}d(x_1^{p^r a_1}\cdots x_n^{p^r a_n}\eta) \\
&= p^{-r}F^{-r}d(p^r x_1^{p^r a_1}\cdots x_n^{p^r a_n}(a_1y_1 + \cdots + a_ny_n) \wedge \eta) \\
&= x_1^{a_1}\cdots x_n^{a_n}(a_1y_1 + \cdots + a_ny_n) \wedge \eta \\
&= (a_1y_1 + \cdots + a_ny_n) \wedge \omega.
\end{aligned}$$

[2] The natural induced map is injective because filtered colimits commute with finite limits.

[3] The ordering of p^{-r} and F^{-r} doesn't matter if we first invert p and extend F. However, it does matter if we work more simply.

Now we turn our attention to Σ^0. Of course,

$$\Omega_R^0[F^{-1}] \cong R_\infty = \mathrm{Span}_{\mathbb{Z}_p}\{x_1^{a_1} \cdots x_n^{a_n} : a_i \in \mathbb{Z}[p^{-1}]\}.$$

Let $\alpha := \lambda x_1^{a_1} \cdots x_n^{a_n}$ and $\beta := \gamma x_1^{b_1} \cdots x_n^{b_n}$ be elements of R_∞. Then, the polynomial coefficient for y_i in $\widetilde{d}(\alpha+\beta)$ is $a_i\alpha + b_i\beta$ and so it suffices to consider just homogeneous elements for constructing Σ^0. From the above we get

$$\begin{aligned}\Sigma^0 &= \mathrm{Span}_{\mathbb{Z}_p}\{\lambda x_1^{a_1} \cdots x_n^{a_n} \in R_\infty : \lambda a_1, \ldots, \lambda a_n \in \mathbb{Z}_p\} \\ &= \mathrm{Span}_{\mathbb{Z}_p}\{\lambda x_1^{a_1} \cdots x_n^{a_n} \in R_\infty : v_p(\lambda) + \min\{v_p(a_i) : 1 \le i \le n\} \ge 0\}.\end{aligned}$$

As before,

$$\Omega_R^1[F^{-1}] \cong \mathrm{Span}_{\mathbb{Z}_p}\{x_1^{a_1} \cdots x_n^{a_n} y_k : a_i \in \mathbb{Z}[p^{-1}], 1 \le k \le n\}.$$

After performing a calculation similar to before, we conclude that the right "shape" of elements to consider for describing Σ^1 is

$$x_1^{a_1} \cdots x_n^{a_n}(\lambda_1 y_1 + \cdots + \lambda_n y_n).$$

Hitting an element like this with $\widetilde{d}$ gives

$$x_1^{a_1} \cdots x_n^{a_n}(a_1 y_1 + \cdots + a_n y_n) \wedge (\lambda_1 y_1 + \cdots + \lambda_n y_n) = x_1^{a_1} \cdots x_n^{a_n} \sum_{i<j} (a_i\lambda_j - a_j\lambda_i) y_i \wedge y_j$$

and so

$$\Sigma^1 = \mathrm{Span}_{\mathbb{Z}_p}\{x_1^{a_1} \cdots x_n^{a_n}(\lambda_1 y_1 + \cdots + \lambda_n y_n) : v_p(a_i\lambda_j - a_j\lambda_i) \ge 0 \text{ for } i < j\}.$$

The same sort of calculation goes through to describe all Σ^m for $m \ge 1$. Our next goal is to compute the completion $\mathcal{W}\Sigma$. For this we need the verschiebung map $V : \Sigma \to \Sigma$ characterized by $FV = p = VF$. A little thought gives

$$V(x_1^{a_1} \cdots x_n^{a_n} y_{k_1} \wedge \cdots \wedge y_{k_m}) = p x_1^{a_1/p} \cdots x_n^{a_n/p} y_{k_1} \wedge \cdots \wedge y_{k_m}$$

and V inherits $\mathbb{Z}_p$-linearity from F. Note that V is not an algebra map for $n > 1$ since

$$V(y_1 \wedge y_2) = p(y_1 \wedge y_2) \neq p^2(y_1 \wedge y_2) = Vy_1 \wedge Vy_2.$$

Note that d and V don't commute just as d and F don't commute. Indeed,

$$d(V\omega) = p^{-1}V(d\omega) \implies d(V^r\omega) = p^{-r}V^r(d\omega).$$

By definition,

$$\mathcal{W}\Sigma := \varprojlim_{r \ge 0} \mathcal{W}_r\Sigma, \qquad \mathcal{W}_r\Sigma := \Sigma/(\mathrm{im}(V^r) + \mathrm{im}(dV^r)).$$

We seek to understand $\mathcal{W}_r\Sigma$, so let's start with the simplest case $(\mathcal{W}_r\Sigma)^0 = \Sigma^0/V^r\Sigma^0$. We have

$$\begin{aligned}V^r\Sigma^0 &= \mathrm{Span}_{\mathbb{Z}_p}\{\lambda x_1^{a_1} \cdots x_n^{a_n} \in \Sigma^0 : v_p(\lambda) \ge r\} \\ &= \mathrm{Span}_{\mathbb{Z}_p}\{\lambda x_1^{a_1} \cdots x_n^{a_n} \in R_\infty : v_p(\lambda) \ge r, v_p(\lambda) + \min\{v_p(a_i) : 1 \le i \le n\} \ge 0\}.\end{aligned}$$

By contrast,

$$p^r\Sigma^0 = \mathrm{Span}_{\mathbb{Z}_p}\{\lambda x_1^{a_1} \cdots x_n^{a_n} \in R_\infty : v_p(\lambda) \ge r, v_p(\lambda) + \min\{v_p(a_i) : 1 \le i \le n\} \ge r\}$$

and so the V-adic completion of Σ^0 is actually much easier to describe than the p-adic completion.[4] We have

$$(\mathcal{W}\Sigma)^0 \cong \mathrm{Span}_{\mathbb{Z}_p} \left\{ \sum_{r \geq 0} \lambda^{(r)} x_1^{a_1^{(r)}} \cdots x_n^{a_n^{(r)}} : \lambda^{(r)} x_1^{a_1^{(r)}} \cdots x_n^{a_n^{(r)}} \in V^r \Sigma^0 \right\}.$$

The coefficients satisfy $v_p(\lambda^{(r)}) \to \infty$ as $r \to \infty$. We obtain $V^r\Sigma^1$ as the $\mathbb{Z}_p$-linear span of

$$x_1^{a_1} \cdots x_n^{a_n} (\lambda_1 y_1 + \cdots + \lambda_n y_n)$$

subject to $v_p(a_i\lambda_j - a_j\lambda_i) \geq 0$ for $i < j$ and $v_p(\lambda_i) \geq r$ for all i. At the same time, we obtain $dV^r\Sigma^0$ as the $\mathbb{Z}_p$-linear span of

$$\lambda x_1^{a_1} \cdots x_n^{a_n} (a_1 y_1 + \cdots + a_n y_n)$$

subject to $v_p(\lambda) + \min\{v_p(a_i) : 1 \leq i \leq n\} \geq 0$ and $v_p(\lambda) \geq r$.

[4] Just a precautionary note, beware that $\mathbb{Z}_p[t]$ is not p-adically complete. The same applies if you throw in more variables.