Quiz 7 Math 2202

---

1. Compute the directional derivative of $f(x,y) = 3x^2 - 2e^{xy}$ at the point $(1,0)$ in the direction of $\langle -3, 5\rangle$.

   What does this number represent?

   Solution:

   To find the directional derivative of $f(x,y)$ in the direction of $\mathbf{v} = \langle -3, 5\rangle$, we need the gradient vector $\nabla f(1,0)$ and a unit vector in the direction of $\mathbf{v}$. First, we divide $\mathbf{v}$ be the length of $\mathbf{v}$ to find a unit vector in the direction of $\mathbf{v}$:

$$\begin{aligned}\mathbf{u} &= \frac{\mathbf{v}}{|\mathbf{v}|}\\ &= \frac{1}{\sqrt{(-3)^2 + (5)^2}} \cdot \langle -3, 5\rangle\\ &= \langle \frac{-3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \rangle\end{aligned}$$

   To compute the gradient vector of f at $(1,0)$, we need to find the partial derivatives of f with respect to x and y, and evaluate them at the point $(1,0)$:

$$\begin{aligned}f_x(x,y) &= 6x - 2ye^{xy}\\ f_x(1,0) &= 6 \cdot 1 - 2 \cdot 0e^{1\cdot 0} = 6\\ f_y(x,y) &= -2exe^{xy}\\ f_y(1,0) &= -2 \cdot 1e^{1\cdot 0} = -2\end{aligned}$$

   So,

$$\nabla f(1,0) = \langle f_x(1,0), f_y(1,0)\rangle = \langle 6, -2\rangle$$

   Finally, we compute the directional derivative of f in the direction of $\mathbf{v}$ at $(1,0)$:

$$\begin{aligned}D_{\mathbf{u}} f(1,0) &= \nabla f(1,0) \cdot \mathbf{u}\\ &= \langle 6, -2\rangle \cdot \langle \frac{-3}{\sqrt{34}}, \frac{5}{\sqrt{34}} \rangle\\ &= \frac{-18}{\sqrt{34}} + \frac{-10}{\sqrt{34}}\\ &= \frac{-28}{\sqrt{34}}\end{aligned}$$

   This number represents the instantaneous rate of change of the function $f(x,y)$ at the point $(1,0)$ in the direction $\langle -3, 5\rangle$. That is, at the point $(1,0)$, the function f is changing at a rate of $\frac{-28}{\sqrt{34}}$ units of f per one unit change in the domain, in the direction $\langle -3, 5\rangle$. In particular, $f(x,y)$ is decreasing at the point $(1,0)$ as the x and y inputs change a small amount in the $\langle -3, 5\rangle$ direction.

2. **Which of the following are true?** Choose all that are true.

   Consider again $f(x, y) = 3x^2 - 2e^{xy}$.

   (a) The gradient $\nabla f(1, 0)$ is a vector in $\mathbf{R}^3$ representing how $f(x, y)$ is changing most quickly.

   (b) The vector $-\nabla f(1, 0)$ is the direction of greatest decrease of $f(x, y)$ at the point $(1, 0)$.

   (c) If $\mathbf{u}$ is a unit vector in the direction of $\nabla f(1, 0)$, then $D_{\mathbf{u}} f(1, 0) = \nabla f(1, 0)$.

   (d) The gradient $\nabla f(1, 0)$ is perpendicular to the graph of $f(x, y)$ at $(1, 0)$.

Solution:

(a) False. The gradient $\nabla f(1, 0)$ is a vector in $\mathbf{R}^2$ representing the direction to move in the domain in order for $f(x, y)$ to increase most quickly.

One misconception is that the gradient includes the z-direction information, in other words is a tangent vector to the curve on the surface $z = f(x, y)$created by a vertical plane cutting along that direction at that point.

(A tangent vector to that curve is $\langle f_x/|\nabla f(1, 0)|, f_y/|\nabla f(1, 0)|, |\nabla f(1, 0)|\rangle$ which comes from thinking about a tangent vector projected into the xy-plane. It must lie on the line with direction vector given by the gradient vector. We know $D_{\mathbf{u}} f(1, 0) = |\nabla f(1, 0)|$, so the rate of change of z is $|\nabla f(1, 0)|$ per unit change in distance.)

(b) True. This is because if $\mathbf{u}$ is a unit vector, the rate of change in direction $\mathbf{u}$ at the point $(1, 0)$ is $D_{\mathbf{u}} f(1, 0) = \nabla f(1, 0) \cdot \mathbf{u} = |\nabla f(1, 0)||\mathbf{u}| \cos \theta$. This is smallest (i.e. f has the greatest decrease) when $\theta = \pi$, or in other words when $\mathbf{u}$ is in the opposite direction as the gradient.

For (b), the image of a skier on a hill is useful here. The negative gradient gives the compass direction (direction in the xy-plane) in which to move in order to go downhill most quickly.

(c) False. $D_{\mathbf{u}} f(1, 0)$ is a number representing the rate of change of f at $(1, 0)$ in the direction of $\mathbf{u}$, where as $\nabla f(1, 0)$ is a vector. If $\mathbf{u}$ is a unit vector in the direction of $\nabla f(1, 0)$, then $D_{\mathbf{u}} f(1, 0) = |\nabla f(1, 0)|$. In other words, the rate of change of f at $(1, 0)$ in the direction of the gradient vector is the magnitude of that vector.

(d) False. The gradient $\nabla f(1, 0)$ is perpendicular to the graph of the level curve $f(x, y) = f(1, 0) = 1$ at $(1, 0)$.

Think about it... Show that every plane that is tangent to the cone $x^2 + y^2 = z^2$ passes through the origin. (Start by creating the tangent plane to a generic point (x_0, y_0, z_0) on the cone, thinking of the cone as a level surface of $F(x, y, z) = x^2 + y^2 - z^2$.)).