Remark: For genus 1 curve $/K$ we know $\exists$ semi-stable model after base change: take minimal regular model to get either elliptic curve for special fiber or Néron n-gon}.

Remark: Mumford curve $X_\Gamma = \Gamma \backslash \Omega_\Gamma$ admits stable model s.t. every irred. component of special fiber has normalization $\mathbb{P}^1$

Thm: X sm. proj. curve $/K$ of genus $g \geq 2$ admitting flat proper model $/K^\circ$ w/ special fiber s.t. every irred. component has normalization $\mathbb{P}^1 \Rightarrow X$ is isom. to Mumford curve after a base change.

Example: Fix distinct pts. $s_1, \dots, s_{2n} \in K$. These determine hyperelliptic curve X via $y^2 = (z-s_1)\cdots(z-s_{2n})$ of genus $g(X) = n-1$. This has degree 2 map $\phi: X \to \mathbb{P}^1$, $(y,z) \mapsto z$. View $S := \{s_1, \dots, s_{2n}\} \subseteq \mathbb{P}^1$. We get reduction $R_S: \mathbb{P}^1 \to Z_S$ (finite union of $\mathbb{P}^1$'s crossing w/ ordinary double pts.). Define graph whose vertices are irred. components of Z_S and edges are intersection pts. Each edge determines open affine subscheme $Z_S(e) \subseteq Z_S$ w/ two components of Z_S corresponding to endpts. of e w/ all simple pts. removed, except the crossing pt. corresponding to e. Then, $\mathcal{U}(e) := R_S^{-1}(Z_S(e)) \subseteq \mathbb{P}^1$ is affinoid and $\mathbb{P}^1 = \bigcup_e \mathcal{U}(e)$ is pure affinoid cover. Let $X(e) := \phi^{-1}(\mathcal{U}(e))$. Then, $X = \bigcup_e X(e)$ is pure affinoid cover defining semistable model of X.

NB: This is not explained well in the textbook. The explanation there is more scant than the paper it's taken from!

Thm: Assume K alg. closed. Every proper smooth curve $/K$ admits semistable model.

Pf: Start by taking pure affinoid cov. of X. Let $Red: X \to Z$ be the associated reduction. (We want to do some kind of resolution of singularities...)

Basic idea: for singular pt. $q \in Z$, choose open affine $q \in \overline{V} \subseteq Z$ s.t. $V = Red^{-1}(\overline{V}) \subseteq X$ is affinoid and its canonical reduction is $\overline{V}$.

(uniqueness here is key!)

Lemma: $\exists!$ proj. completion $\overline{V} \hookrightarrow \hat{\overline{V}}$ s.t. $\hat{\overline{V}} \setminus \overline{V}$ consists of finitely many smooth pts. of $\hat{\overline{V}}$. Define the genus $g(\overline{V}) :=$ arithmetic genus of $\hat{\overline{V}} = \dim H^1(\hat{\overline{V}}, \mathcal{O}_{\hat{\overline{V}}})$.

Lemma: Can complete the diagram

$$\begin{tikzcd} V \arrow[r, dashed, "\text{open imm.}"] \arrow[d, "Red"'] & \hat{V} \arrow[d, dashed, "\widehat{Red}"'] \arrow[d, phantom, shift left=6, "\text{reduction for some pure affinoid cover}"] \\ \overline{V} \arrow[r, hook, "\text{open imm.}"'] & \hat{\overline{V}} \end{tikzcd}$$

s.t. $\hat{V}$ = analytification of smooth proj. curve$/k$ of genus $g(\overline{V})$.

$\widehat{Red}^{-1}(\overline{V}) = V$

(Base case is genus 0 or 1, if you prefer)

We care about this because it lets us ~~do~~ some inductive bootstrapping. If $\underset{\text{genus}(\hat{V})}{\text{genus}(\overline{V})} < \text{genus}(X)$ then induction hypothesis $\Rightarrow$ we can find pure affinoid cov. of $\hat{V}$ giving semistable model of $\hat{V}$. Intersect that cover w/ $V \subseteq \hat{V}$ to get pure affinoid cov. of V giving new reduction.

$$\begin{tikzcd} V \arrow[r] \arrow[dr] & Z_V \arrow[d, "\rho"] & \supseteq Z_V \setminus \rho^{-1}(q) \\ & \overline{V} & \supseteq \overline{V} \setminus \{q\} \subseteq Z\setminus\{q\} \end{tikzcd}$$

$Z_V \setminus \rho^{-1}(q) \cong \overline{V}\setminus\{q\}$ (modifying the fiber above q...)

$\overline{V}\setminus\{q\} \subseteq Z\setminus\{q\}$ (open affine)

Note: Z_V has ~~only~~ (at worst) ordinary double pts. (open subscheme of reduction of semistable model of $\hat{V}$)

Glue Z_V onto $Z\setminus\{q\}$ along common open affine $Z_V \setminus \rho^{-1}(q) \cong \overline{V}\setminus\{q\}$. We get scheme Z' w/ function

$$\begin{tikzcd} X \arrow[r, "Red'"] \arrow[dr, "Red"'] & Z' \arrow[d, "\rho"] \\ & Z \end{tikzcd}$$

where Red' is defined by: for $x \in V$ use reduction to Z_V
for $x \in X\setminus V$ use reduction to $Z\setminus\{q\}$

$Z' \setminus \rho^{-1}(q) \cong Z\setminus\{q\}$ $\Rightarrow$ Z' may have more singularities but has fewer non-ordinary double pts.

Def: Singular pt. $q \in Z$ has small genus if $\exists$ open affine $q \in \overline{V} \subseteq Z$ as above s.t. $\text{genus}(\overline{V}) < \text{genus}(X)$.

We can resolve these singularities, making them manageable.

Def: $q \in Z$ is ordinary multiple point if $\hat{\mathcal{O}}_{Z,q} \cong k[[x_1,\dots,x_n]]/(x_i x_j)_{i<j}$ for some $n>2$. (transverse intersections of coord. axes)

If q is ordinary multiple pt. $\Rightarrow$ can find open affine $q \in \bar{V} \subseteq Z$ s.t. $V = \mathrm{Red}^{-1}(X)$ is affinoid, and $\exists\, t \in \mathcal{O}(V)$

s.t. $V = \{|t| \leq \varepsilon\} \cup \{|t| \geq \varepsilon\}$ (ε some positive #).

$\uparrow$

$\mathbb{P}^1$ - (finite union of open disks)

<u>Remark</u>: Proof? Forget about it!

$\downarrow$

Now choose pure affinoid cov. of $\{|t| \leq \varepsilon\}$ giving semistable model. Use this to construct pure affinoid cov. of V giving rise to semistable model. Can also resolve the ordinary multiple pt. singularities.

<u>Prop</u>: X admits a reduction $\mathrm{Red}: X \to Z$ s.t. every singular pt. of Z either has small genus or is an ordinary multiple pt.

Drinfeld Half-Space

Boutot-Carayol has all the details in full glory (Cameron Franc English translation).

Assume k discretely valued (e.g., $k = \mathbb{Q}_p$). Fix uniformizer $\pi \in k^\circ$. $q := \# k^\circ/(\pi)$ finite. $\Rightarrow |\pi| = q^{-1}$

$C := \widehat{k^{alg}}$.

Goal: Make $\Omega := \mathbb{P}^1(C) \setminus \mathbb{P}^1(k)$ into C-rigid space.

As before, a lattice is $M \subseteq k^2$ free rank 2 k°-module s.t. $M \otimes_{k^\circ} k \xrightarrow{\sim} k^2$. Let I be graph whose vertices are homothety classes $s = [M]$ of lattices. To vertices $s = [M]$, $s' = [M']$ are connected by an edge if $\pi M \subseteq M' \subseteq M$. [This is a tree!]

Remark: $\{$edges passing through $[M]\} \leftrightarrow \{\bar{k}$-lines in $M/\pi M\}$, w/ both sides having cardinality $q+1$.

Let $I_{\mathbb{R}}$ be topological realization of I. We want to make this explicit. We will look at nonarch. norms on k^2, up to scaling. Given vertex $[M]$, choose k°-basis $e_1, e_2 \in M$. For $x = a_1e_1 + a_2e_2 \in k^2$ set $|x|_M := \max\{|a_1|, |a_2|\}$. We get $M = \{x \in k^2 : |x|_M \leq 1\}$. For $[M], [N]$ connected by an edge, choose representatives w/ $\pi M \subseteq N \subseteq M$ and basis $e_1, e_2 \in M$ s.t. $e_1, \pi e_2$ is basis for N. Given $x = a_1e_1 + a_2e_2$,

$|x|_M = \max\{|a_1|, |a_2|\}$, $|x|_N = \max\{|a_1|, |a_2/\pi|\} = \max\{|a_1|, q|a_2|\}$. [NB: Choice of basis does kind of matter here...]

Define $|x|_t := \max\{|a_1|, q^t|a_2|\}$ for $0 \leq t \leq 1$. [We chose to parametrize one "direction" but could parametrize the other "direction" instead]. This construction gives bijection $I_{\mathbb{R}} \leftrightarrow \{$all homothety norms on $k^2\}$, which describes the topological realization.

Define $\rho : \Omega \to I_{\mathbb{R}}$. A point $\ell \in \Omega = \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(k)$ is a line $\ell \subseteq \mathbb{C}^2$ s.t.

$k^2 \hookrightarrow \mathbb{C}^2 \to \mathbb{C}^2/\ell \cong \mathbb{C}$ is injective. Restriction of $|\cdot|$ on $\mathbb{C}$ is a norm on k^2 and this defines $\rho(\ell)$.

<u>Note</u>: $PGL_2(k) \curvearrowright \Omega$ and on $I_{\mathbb{R}}$ we have that $\rho : \Omega \to I_{\mathbb{R}}$ is $PGL_2(k)$-equi-~~invariant~~variant.

$\rho^{-1}(\text{vertex})$ = closed disk - (union of finitely many open disks in $\mathbb{C}$). Same for $\rho^{-1}(\text{closed edge})$.

$\rho^{-1}(\text{open edge})$ = open annulus. The point is that we get affinoid preimages. What is the $PGL_2(k)$-action

on $I_{\mathbb{R}}$?
- Transitive on edges.
- Each edge meets only finitely many open edges.

$(\cdots)$

<u>Shimura Curves</u>

Fix B indefinite quaternion alg. / $\mathbb{Q}$ ("unramified at ∞") - i.e., $B \otimes_{\mathbb{Q}} \mathbb{R} \cong M_2(\mathbb{R})$ splitting condition. Fix maximal order

$\mathcal{O}_B \subseteq B$ (unique up to conjugation). Consider moduli space $X_B(N)(\mathbb{C})$ parametrizing the following data:

(<u>NB</u>: Drinfeld tells us we don't need to worry about polarizations.)

- Abelian surface $A / \mathbb{C}$
- Action $\mathcal{O}_B \to \mathrm{End}(A)$
- $\mathcal{O}_B$-linear isom. $A[N] \xrightarrow{\sim} \mathcal{O}_B/(N)$

$X_B(N)(\mathbb{C}) \cong G(\mathbb{Q}) \setminus \mathfrak{X} \times G(\mathbb{A}_f) / U(N)$. G alg. grp. $/\mathbb{Q}$ w/ $G(\mathbb{R}) = (B \otimes_{\mathbb{Q}} \mathbb{R})^\times$.

$U(N) = \{ g \in \hat{\mathcal{O}}_B^\times : g \equiv 1 \text{ in } (\hat{\mathcal{O}}_B/(N))^\times \} \subseteq \hat{B}^\times = G(\mathbb{A}_f)$.

$\mathfrak{X} \subseteq \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$ is $G(\mathbb{R})$-conj. class of $\mathbb{R}$-alg. map $\mathbb{C} \to M_2(\mathbb{R}) = B \otimes_{\mathbb{Q}} \mathbb{R}$ restricted to $\mathbb{C}^\times \to G(\mathbb{R})$.

Moduli interpretation gives smooth projective (if $B \neq M_2(\mathbb{Q})$) curve $X_B(N) \to \operatorname{Spec} \mathbb{Q}$. [$B = M_2(\mathbb{Q})$ gives modular curve, which is not projective!]

Can form the $\mathbb{C}_p$-rigid space $X_B(N)^{an}_{\mathbb{C}_p}$.

Fix $p \nmid N$ at which B is ramified. $\exists!$ quaternion alg. $\overline{B}$ obtained by "switching invariants" at p and ∞ - i.e.,

$\overline{B} \otimes_{\mathbb{Q}} \mathbb{Q}_\ell = B \otimes_{\mathbb{Q}} \mathbb{Q}_\ell$ for $\ell \notin \{p, \infty\}$, $\overline{B} \otimes_{\mathbb{Q}} \mathbb{Q}_p \cong M_2(\mathbb{Q}_p)$, $\overline{B} \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{H}$.

Define $\overline{G}(R) = (\overline{B} \otimes R)^\times \Rightarrow \overline{G}(\mathbb{Q}_p) \cong GL_2(\mathbb{Q}_p)$ acts on $\Omega = \mathbb{P}^1(\mathbb{C}_p) \setminus \mathbb{P}^1(\mathbb{Q}_p)$.

<u>Thm</u> (Cerednik-Drinfeld): $X_B(N)^{an}_{\mathbb{C}_p} \cong \overline{G}(\mathbb{Q}_p) \backslash (\Omega \times Z(N))$ for $Z(N) := U(N)^p \backslash \overline{G}(\mathbb{A}_f) / \overline{G}(\mathbb{Q})$

and $U(N)^p \subseteq G(\mathbb{A}_f^p) \cong \overline{G}(\mathbb{A}_f^p)$.

<u>NB</u>: Cerednik did this "by hand." Drinfeld basically did an argument w/ formal schemes, p-div. grps., Rapoport-Zink spaces.