K discretely valued field.

__Recall__ : For A ab. var. $/K$, __Néron model__ of A is ext. of A to smooth gp. scheme $\mathcal{A} \to \operatorname{Spec} K^\circ$ s.t. if $\mathcal{S} \to \operatorname{Spec} K^\circ$ is smooth scheme w/ generic fiber $S \to \operatorname{Spec} K$ then

$$\begin{tikzcd} S \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d] \\ \mathcal{S} \arrow[r, dashed, "\exists!"'] & \mathcal{A} \end{tikzcd}$$

__Thm__ (Néron) : These exist and are unique.

__Thm__ : C smooth proj. curve $/K$.

(__NB__: Look at Qing Liu's wonderful book.)

(1) (Abhyankar, Lipman) $\exists\, \underbrace{\text{regular}}_{\text{(real content!)}}$ proper flat scheme $\mathcal{C} \to \operatorname{Spec} K^\circ$ w/ generic fiber $C \to \operatorname{Spec} K$.

(2) (Lichtenbaum, Shafarevich) Among all regular, proper, flat models as in (1), there is minimal $\mathcal{C}^{min}$ in the sense, that, given any $\mathcal{C}$ as in (1) and any isom. $\mathcal{C}_K \xrightarrow{\sim} \mathcal{C}^{min}_K$, there is unique ext. $\mathcal{C} \to \mathcal{C}^{min}$ (which may not be isom.!).

(may not be regular...)

__Thm__ : Suppose E is elliptic curve $/K$ w/ $\mathcal{E}^{min}$ minimal proper regular model. (__not__ the same as minimal Weierstrass model)

(1) There is complete classification (Kodaira-Néron) of all possible special fibers $\mathcal{E}^{min}_k$. Algorithm of Tate takes Weierstrass eq. for E and then actually produces $\mathcal{E}^{min}_k$. "Typical" special fibers : either $\mathcal{E}^{min}_k$ is elliptic curve or union of finitely many $\mathbb{P}^1$'s crossing transversely ("Néron polygon"). "Typical" here means that we always have one of these reduction types after some finite ext. of K.

(2) $\exists$ finitely many pts. of $\mathcal{E}^{min}$ at which $\mathcal{E}^{min} \to \operatorname{Spec} K^\circ$ is not smooth. Let $\mathcal{E} \subseteq \mathcal{E}^{min}$ be their complement, which satisfies $\mathcal{E}_K = \mathcal{E}^{min}_K = E$. Gp. law $E \times E \to E$ extends uniquely to $\mathcal{E} \times \mathcal{E} \to \mathcal{E}$.

(3) $\mathcal{E}$ is the Néron model of E.

__Remark__ : We just dropped a bunch of big results.

Consider the Tate curve $T = \mathbb{G}_m^{an}/\langle q\rangle \cong E^{an}$. Let $\pi \in k^\circ$ be uniformizer and write $q = u\pi^n$ w/ $u \in (k^\circ)^\times$, $n \geq 1$.

We will in fact assume $n \geq 2$. Define $A_0 := \{ t \in \mathbb{G}_m^{an} : |\pi| \leq |z(t)| \leq 1 \}$, $A_1 := \{ t \in \mathbb{G}_m^{an} : |\pi|^2 \leq |z(t)| \leq |\pi| \}$, $\ldots$, $A_{n-1} := \{ t \in \mathbb{G}_m^{an} : |\pi|^n \leq |z(t)| \leq |\pi|^{n-1} \}$. Each is isom. to its image under $\mathbb{G}_m^{an} \xrightarrow{pr} T$ and these images form a pure affinoid cover of T : $\mathfrak{U} = \{ pr(A_0), \ldots, pr(A_{n-1}) \}$. We have reduction $\overline{(T, \mathfrak{U})} = \bigcup_{i=0}^{n-1} \overline{A_i}$. What does this look like?

$$A_0 = \mathrm{Sp}\, k\langle z, \pi/z\rangle = \mathrm{Sp}\, k\langle s_0, t_0\rangle / (s_0 t_0 - \pi),$$

$$A_1 = \mathrm{Sp}\, k\langle z/\pi, \pi^2/z\rangle = \mathrm{Sp}\, k\langle s_1, t_1\rangle / (s_i t_i - \pi),$$

$$\vdots$$

$$A_i = \mathrm{Sp}\, k\langle z/\pi^{i}, \pi^{i+1}/z\rangle = \mathrm{Sp}\, k\langle s_i, t_i\rangle / (s_i t_i - \pi).$$

NB: They all have the same "shape" but we have conveniently accounted for shifts w/ our indexing.

$$A_0 \cap A_1 = \{ t \in \mathbb{G}_m^{an} : |z(t)| = |\pi| \} = \mathrm{Sp}\, k\langle z/\pi, \pi/z\rangle = \mathrm{Sp}\, k\langle s_1, t_0\rangle / (s_1 t_0 - 1),$$

$$\vdots$$

$$A_{i-1} \cap A_i = \cancel{\{} \mathrm{Sp}\, k\langle z/\pi^i, \pi^i/z\rangle = \mathrm{Sp}\, k\langle s_i, t_{i-1}\rangle / (s_i t_{i-1} - 1).$$

We get reductions

$$\begin{tikzcd} & \overline{A_{i-1}} = \mathrm{Spec}\, \tilde{k}[s_{i-1}, t_{i-1}] / (s_{i-1} t_{i-1}) \\ \overline{A_{i-1} \cap A_i} = \mathrm{Spec}\, \tilde{k}[s_i, t_{i-1}] / (s_i t_{i-1} - 1) \arrow[ur] \arrow[dr] & \\ & \overline{A_i} = \mathrm{Spec}\, \tilde{k}[s_i, t_i] / (s_i t_i) \end{tikzcd}$$

From this it's clear that $\overline{(T, \mathfrak{U})}$ is a Néron n-gon. From this we get flat proj. integral model $\mathcal{E}^{min} \to \mathrm{Spec}\, k^\circ$, whose special fiber is Néron n-gon.

Remark: Every non-smooth pt. of $\mathcal{E}^{min}$ is the crossing of two $\mathbb{P}^1$'s and completed local ring at such pt. is $k^\circ[\![s_i, t_i]\!] / (s_i t_i - \pi)$ which is regular! (next time: smooth locus recovers Néron model)

<u>Lemma A</u>: $k^\circ \subseteq R$ unramified ext. of DVR's (ie., maximal ideal of R is πR). Then, we have extensions

$$\begin{tikzcd} \operatorname{Spec} \operatorname{Frac}(R) \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & E \arrow[d] \\ \operatorname{Spec} R \arrow[r, dashed] & \mathcal{E} \end{tikzcd}$$

<u>Pf</u>: $\mathcal{E}^{min}$ is proper so valuative criterion shows
$$\begin{tikzcd} \operatorname{Spec} \operatorname{Frac}(R) \arrow[r] \arrow[d] & E \arrow[d] \\ \operatorname{Spec} R \arrow[r, dashed] & \mathcal{E}^{min} \end{tikzcd}$$
. Need to show dotted arrow factors through $\mathcal{E}$.

If not then closed pt. of $\operatorname{Spec} R$ maps to singular pt. $Q \in \mathcal{E}^{min}$. Dotted arrow factors as
$$\begin{tikzcd} \operatorname{Spec} R \arrow[r] \arrow[dr] & \operatorname{Spec} \mathcal{O}_{\mathcal{E}^{min},Q} \arrow[d] \\ & \mathcal{E}^{min} \end{tikzcd}$$
.

(pass to π-adic completions)

This gives $k^\circ[[x,y]]/(xy-\pi) \cong \hat{\mathcal{O}}_{\mathcal{E}^{min},Q} \to \hat{R}$. This introduces ramification! (since now π has nontriv. factorization in $\hat{R}$) $\square$

<u>Lemma B</u>: Given $a \in E(k)$, the translation map $t_a : E \to E$ extends to $\mathcal{E} \to \mathcal{E}$.

<u>Pf</u>: View a as a pt. of $T = \mathbb{G}_m^{an}/\langle q \rangle$ w/ res. field $k_a = k$. $(a \in k^\times/\langle q \rangle)$ Factor $a = u\pi^\ell$ w/ $\ell \in \mathbb{Z}$, $u \in (k^\circ)^\times$. Suffices to check that translations by u and π preserve the pure affinoid cover of T from before.

But this is clear. $\square$

<u>Prop</u>: Still assuming $n \geq 2$. Smooth locus $\mathcal{E} \subseteq \mathcal{E}^{min}$ is the Néron model of E. $(\Rightarrow \mathcal{E}_{\bar{k}} \cong \mathbb{G}_m \times \mathbb{Z}/n\mathbb{Z})$.

<u>Pf</u>: Assume WLOG $\bar{k}$ is alg. closed. Let $\mathcal{E}^{Néron} \to \operatorname{Spec} k^\circ$ be the honest Néron model. We have $\mathcal{E}_k = E = \mathcal{E}_k^{Néron}$.

$$\begin{tikzcd} \mathcal{E}_k \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{E}_k^{Néron} \arrow[d] \\ \mathcal{E} \arrow[r, dashed] & \mathcal{E}^{Néron} \end{tikzcd}$$

The dotted arrow exists by Néron ext. property. We seek an inverse.

<u>Step 1</u>: $\exists$ maximal open subscheme $\mathcal{U} \subseteq \mathcal{E}^{Néron}$ containing $\mathcal{E}_k^{Néron}$ s.t.
$$\begin{tikzcd} \mathcal{E}_k^{Néron} \arrow[r, "\sim"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{E}_k \arrow[d] \\ \mathcal{U} \arrow[r] & \mathcal{E} \end{tikzcd}$$
.

(Mostly pure algebro-geometric fact.)

<u>Intuition</u>: Lemma A $\Rightarrow$ $\mathcal{U}$ is "pretty big." Lemma B $\Rightarrow$ we can translate around to get homogeneity hence $\mathcal{U}$ is everything.

<u>Step 2</u>: Show closed subset $\mathcal{E}^{\text{Néron}} \setminus \mathcal{U}$ has codim ≥ 2 so is finite set of pts. in special fiber of $\mathcal{E}^{\text{Néron}}$.

Let $\xi \in \mathcal{E}^{\text{Néron}}_{\bar{k}}$ be generic pt. of some irred. component of $\mathcal{E}^{\text{Néron}}_{\bar{k}}$. The local ring $\mathcal{O}_{\mathcal{E}^{\text{Néron}},\xi}$ has dim 1 so is DVR by smoothness of the Néron model. This contains k° and has maximal ideal gen. by π. By Lemma A, $\operatorname{Spec}\operatorname{Frac}(\mathcal{O}_{\mathcal{E}^{\text{Néron}},\xi}) \to \mathcal{E}^{\text{Néron}}_k$ extends to $\operatorname{Spec}\mathcal{O}_{\mathcal{E}^{\text{Néron}},\xi} \to \mathcal{E}^{\text{Néron}}$. So, $\exists$ open nbhd $\xi \in V_\xi \subseteq \mathcal{E}^{\text{Néron}}$ s.t.

$$\begin{tikzcd} \mathcal{E}^{\text{Néron}}_k \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & \mathcal{E}_k \arrow[d] \\ V_\xi \arrow[r] & \mathcal{E}^{\text{Néron}} \end{tikzcd}$$

Hence, $\mathcal{U}$ contains generic pt. of every component of $\mathcal{E}^{\text{Néron}}_{\bar{k}}$ and so $\mathcal{E}^{\text{Néron}} \setminus \mathcal{U}$ is finite set of pts. in $\mathcal{E}^{\text{Néron}}_{\bar{k}}$.

<u>Step 3</u>: $\bar{a} \in \mathcal{E}^{\text{Néron}}(\bar{k}) \Rightarrow \mathcal{U}_{\bar{k}} \subseteq \mathcal{E}^{\text{Néron}}_{\bar{k}}$ is stable under translation $t_{\bar{a}}: \mathcal{E}^{\text{Néron}}_{\bar{k}} \to \mathcal{E}^{\text{Néron}}_{\bar{k}}$.

(Maybe this isn't necessary...)

Smoothness of $\mathcal{E}^{\text{Néron}}$ $\Rightarrow$ we have lift $a \in \mathcal{E}^{\text{Néron}}(k^\circ)$ of $\bar{a}$. But, $\mathcal{E}^{\text{Néron}}(k^\circ) = \mathcal{E}^{\text{Néron}}(k)$. $t_a: E \to E$ extends to $t_a: \mathcal{E}^{\text{Néron}} \to \mathcal{E}^{\text{Néron}}$ and to $t_a: \mathcal{E} \to \mathcal{E}$ by Lemma B. Definition of $\mathcal{U}$ $\Rightarrow$ we have ext.

$$\begin{tikzcd} \mathcal{E}^{\text{Néron}}_k \arrow[r, "\phi"] \arrow[d] & \mathcal{E}_k \arrow[d] \\ \mathcal{U} \arrow[r, "\phi"'] & \mathcal{E} \end{tikzcd}$$

hence ext.

$$\begin{tikzcd} \mathcal{E}^{\text{Néron}}_k \arrow[rrr, "\phi"] \arrow[d] & & & \mathcal{E}_k \arrow[d] \\ t_a(\mathcal{U}) \arrow[r, "t_a^{-1}"'] & \mathcal{U} \arrow[r, "\phi"'] & \mathcal{E} \arrow[r, "t_a"'] & \mathcal{E} \end{tikzcd}$$

So, $t_a(\mathcal{U}) \subseteq \mathcal{U}$ by maximality.

Hence, $t_{\bar{a}}(\mathcal{U}_{\bar{k}}) \subseteq \mathcal{U}_{\bar{k}}$.

<u>Step 4</u>: $\bar{k}$ alg. closed (by assumption) $\Rightarrow \mathcal{E}^{\text{Néron}}_{\bar{k}}(\bar{k}) \subseteq \mathcal{E}^{\text{Néron}}_{\bar{k}}$ is Zariski dense. Nonempty open set $\mathcal{U}_{\bar{k}} \subseteq \mathcal{E}^{\text{Néron}}_{\bar{k}}$

is stable under translation by $\mathcal{E}^{\text{Néron}}_{\bar{k}}(\bar{k}) \Rightarrow \mathcal{U}_{\bar{k}} = \mathcal{E}^{\text{Néron}}_{\bar{k}} \Rightarrow \mathcal{U} = \mathcal{E}^{\text{Néron}}$. So we get our inverse!

<u>NB</u>: For $n=1$ work w/ minimal Weierstrass model.

(*NB*: Valuation here is discrete since we are working w/ Néron models.)

<u>Cor</u>: E elliptic curve $/k$.

(1) $|j(E)| \leq 1 \Rightarrow \exists$ finite ext. k'/k s.t. $E' = E_{k'}$ extends to elliptic curve over k'°.

(2) $|j(E)| > 1 \Rightarrow \exists$ finite ext. k'/k s.t. Néron model of $E' = E_{k'}$ has special fiber $\mathbb{G}_m \times \mathbb{Z}/n\mathbb{Z}$ w/ $n = -\mathrm{ord}_{k'}(j(E))$.

Moreover, $\exists$ ext. of E' to smooth grp. scheme $/ k'^{\circ}$ w/ special fiber $\mathbb{G}_m$.

<u>Pf</u>: (1) This is in Silverman - one works w/ Weierstrass equations to get model w/ discriminant a unit.

(2) We have proved $\exists$ finite ext. k'/k and $q \in (k')^{\times}$ w/ $0 < |q| < 1$ s.t. $E' \cong E_q$. Here, E_q satisfies $E_q^{an} \cong \mathbb{G}_{m,k'}^{an}/\langle q \rangle$. Now use the construction of the Néron model of the Tate curve. For the (moreover), let $\mathcal{E}'$ be Néron model of E'. Delete all non-identity components from special fiber of $\mathcal{E}'$. This is smooth grp. scheme ext. as desired. □

<u>Applications</u>

Assume char $k = 0$, and valuation is discrete. E_q (assc. elliptic curve as above) = Tate curve defined by $q \in k^{\times}$ (Tate parameter) w/ $0 < |q| < 1$.

(Serre) <u>Prop</u>: ℓ prime w/ $\ell \nmid \mathrm{ord}_k(q) \Rightarrow \exists \sigma \in \mathrm{Gal}(k^{alg}/k^{unr})$ acting on $E_q(k^{alg})[\ell] \cong \mathbb{Z}/\ell\mathbb{Z} \times \mathbb{Z}/\ell\mathbb{Z}$ via $\sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$.

<u>Pf</u>: First assume $\mu_\ell \subseteq k$. (helps explain char. 0 condition) Choose $Q \in k^{alg}$ s.t. $Q^\ell = q$. Let $L := k(Q)$. L/k is totally ramified of degree ℓ. This is because $\ell\, \mathrm{ord}_L(Q) = \mathrm{ord}_L(q) = e_{L/k}\, \mathrm{ord}_k(q) \Rightarrow \ell \mid e_{L/k} \Rightarrow \ell = e_{L/k}$. Since $\mu_\ell \subseteq k$, L/k is Galois and every automorphism satisfies $Q \mapsto \zeta Q$ for ζ ℓth root of unity. Hence, $\mathrm{Gal}(L/k) \cong \mu_\ell$ by Kummer theory.

Fix nontrivial $\sigma \in \mathrm{Gal}(L/k)$ and thus $\zeta \in \mu_\ell$ via $\sigma(Q) = \zeta Q$. We get that $\zeta, Q \in L^{\times}/\langle q \rangle = E_q(L)$ give basis for $E_q(k^{alg})[\ell]$ (since we have two lin. ind. elts. of order ℓ each). $\sigma(\zeta) = \zeta$ and $\sigma(Q) = \zeta Q$ so matrix is $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ in this basis. By Galois theory we have lift

$$\begin{tikzcd}[column sep=small]
 & k^{alg} \arrow[d, dash] \arrow[d, bend left=60, dash, "\sigma"] & \\
 & Lk^{unr} \arrow[dl, dash] \arrow[dr, dash, "\sigma"] & \\
L \arrow[dr, dash, "\sigma"'] & & k^{unr} \arrow[dl, dash] \\
 & k &
\end{tikzcd}$$

Now for the general case. Set $L := k(\mu_\ell)$. $[L:k]$ divides $\ell - 1$ so ramification degree $e_{L/k}$ is prime to ℓ (as $e_{L/k} \mid [L:k] \mid (\ell-1)$ and $\gcd(\ell-1, \ell) = 1$. We get $\sigma \in \mathrm{Gal}(k^{alg}/L^{unr}) \subseteq \mathrm{Gal}(k^{alg}/k^{unr})$.

Cor: $\mathrm{End}(E_q) = \mathbb{Z}$ (so no CM).

Pf: Suppose not. char $k = 0 \Rightarrow \mathrm{End}(E_q) = \mathcal{O}$ order in quad. imag. field. Pick prime ℓ s.t.

(1) $\ell \nmid \mathrm{ord}_k(q)$; (2) ℓ is inert in $\mathrm{Frac}(\mathcal{O})$; (3) $\mathcal{O} \otimes_{\mathbb{Z}} \mathbb{Z}_\ell$ is maximal order.

- (1, 3) exclude fin. many primes
- (2) excludes ~1/2 of primes

Thus, $\mathcal{O}/\ell \cong \mathbb{F}_{\ell^2}$ and $\mathbb{F}_{\ell^2} \cong \mathcal{O}/\ell \to \mathrm{End}_{\mathrm{Gal}(k^{alg}/k^{unr})}(E_q(k^{alg})[\ell])$. In particular,

$\mathbb{F}_{\ell^2} \hookrightarrow \mathrm{End}(E_q(k^{alg})[\ell]) \cong \mathrm{Mat}_2(\mathbb{F}_\ell)$ w/ image commuting w/ $\sigma = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$. The image of any emb.

$\mathbb{F}_{\ell^2} \hookrightarrow \mathrm{Mat}_2(\mathbb{F}_\ell)$ ~~[illegible]~~ is equal to its own centralizer. Contradiction because $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ has order ℓ while $\mathbb{F}_{\ell^2}^{\times}$ has order $\ell^2 - 1$. □

Let's now give another proof. Let $\mathcal{E}_q$ be Néron model. $\mathcal{E}_q^\circ$ is $\mathcal{E}_q$ w/ all non-identity components from special fiber removed. This is smooth grp. scheme / k° w/ generic fiber E_q and special fiber $\mathbb{G}_m$. Reduction gives ring homomorphism

$\mathrm{End}(E_q) \to \mathrm{End}(\mathcal{E}_q) \to \mathrm{End}(\mathcal{E}_q^\circ) \to \mathrm{End}(\mathbb{G}_m) \cong \mathbb{Z}$. This rules out all non-$\mathbb{Z}$ possibilities.

Cor: E/k elliptic curve w/ CM $\Rightarrow j(E) \in k^\circ$ (hence E has potentially good reduction).

Pf: $|j(E)| > 1 \Rightarrow E$ becomes Tate curve after some finite ext. This would have CM, impossible by the above. □

So elliptic curves / # fields have integral j-inv. and (globally) potentially good reduction. □

<u>Mumford Curves</u>

Can view Tate curve as quotient of $\mathbb{G}_m^{an} \subseteq \mathbb{P}^{an}$ by $\left\langle \begin{pmatrix} q & \\ & 1 \end{pmatrix} \right\rangle \subseteq PGL_2(k)$. Mumford seeks other examples of subsets

$\Omega \subseteq \mathbb{P}^{an}$ stable under action of $\Gamma \leq PGL_2(k)$ s.t. $\Gamma \backslash \Omega$ has natural rigid structure.

NB: We go about this by looking at "affinoid fundamental domains" rather than some theory of abstract quotients.

Γ, Ω have to be chosen "very carefully."