Prop (Mittag-Leffler Decomposition): let $D_1, \dots, D_r \subseteq \mathbb{P}$ be disjoint open disks and $F := \bigcap_{i=1}^{r} (\mathbb{P} \setminus D_i)$ [recall every conn. affinoid can be assumed to look like this] Then, every $f \in \mathcal{O}(F)$ can be written as $f_1 + \dots + f_r$ for $f_i \in \mathcal{O}(\mathbb{P} \setminus D_i)$.
Moreover, fixing q and requiring $f(q) = 0 = f_i(q)\ \forall i$, we get uniqueness of the f_i.

Pf: Assume $q = \infty \in F$. Assume first $f \in \mathrm{Rat}(F)$. Subtract off poles until we get a constant.

$f(z) = \sum_i \frac{1}{(z-a_i)^{n_i}} + f(\infty)$. Now just collect up the poles depending on where they are located.

$\square$

For general f one needs to work carefully w/ sequences.

Lemma: $\mathcal{F}$ presheaf for weak top. on $\mathbb{P}$.

(a) $\mathcal{U} = \mathcal{U}_1 \sqcup \dots \sqcup \mathcal{U}_n$ disjoint union of conn. affinoids $\Rightarrow \mathcal{F}(\mathcal{U}) = \mathcal{F}(\mathcal{U}_1) \oplus \dots \oplus \mathcal{F}(\mathcal{U}_n)$.

(b) $\mathcal{U}_1, \mathcal{U}_2$ conn. affinoid $\Rightarrow 0 \to \mathcal{F}(\mathcal{U}_1 \cup \mathcal{U}_2) \to \mathcal{F}(\mathcal{U}_1) \oplus \mathcal{F}(\mathcal{U}_2) \to \mathcal{F}(\mathcal{U}_1 \cap \mathcal{U}_2)$ exact.

(b') (b) + exactness w/ $(\to 0)$ on the right.

(a) + (b) $\Rightarrow \mathcal{F}$ is a sheaf. (a) + (b') $\Rightarrow \check{H}^n(\mathfrak{U}, \mathcal{F}) = \ \forall n > 0$ and $\mathfrak{U}$ admissible cover of some adm. open.

Pf (of Tate's thm): We need to show that $\mathcal{U}, V \subseteq \mathbb{P}$ conn. affinoid $\Rightarrow$ $(*)\ 0 \to \mathcal{O}(\mathcal{U} \cup V) \xrightarrow{\alpha} \mathcal{O}(\mathcal{U}) \oplus \mathcal{O}(V) \xrightarrow{\beta} \mathcal{O}(\mathcal{U} \cap V) \to 0$ exact. Injectivity of α obvious. Assume WLOG $\mathcal{U} \cap V \neq \emptyset$ w/ $\infty \in \mathcal{U} \cap V$. For affinoid $F \subseteq \mathbb{P}$ w/ $\infty \in F$ let $\mathcal{O}(F)_+ := \{ f \in \mathcal{O}(F) : f(\infty) = 0 \}$. So, $\mathcal{O}(F) = \mathcal{O}(F)_+ \oplus k$. Suffices to check exactness for "+-version" of $(*)$.

Write $\mathcal{U} = \mathbb{P} \setminus (D_1 \sqcup \dots \sqcup D_s)$, $V = \mathbb{P} \setminus (E_1 \sqcup \dots \sqcup E_t)$ for open disks D_i, E_j. None of these disks contains ∞ since $\infty \in \mathcal{U} \cap V$. Any two disks either don't meet or one is contained in the other. Assume WLOG

$D_1 \subseteq E_1, \dots, D_a \subseteq E_a,\ D_{a+1} \supseteq E_{a+1}, \dots, D_b \supseteq E_b$ and $E_i \cap D_i = \emptyset$ for $i > b$.

$\mathcal{U} \cup V = \mathbb{P} \setminus (D_1 \sqcup \dots \sqcup D_a \sqcup E_{a+1} \sqcup \dots \sqcup E_b)$. Other fairly explicit stuff...

$\square$

Rigid Spaces in $\mathbb{P}$

Let $\Omega \subseteq \mathbb{P}$ be open (for canonical metric top.). Ω has its own weak G-top.

Ex: Take $K = \widehat{\mathbb{Q}_p^{alg}}$ and $\Omega := \mathbb{P}(K) \setminus \mathbb{P}(\mathbb{Q}_p)$ [Drinfeld half-plane]

Remark: $\Omega = \bigcup_{F \in \mathfrak{X}} F$ where $\mathfrak{X} := \{ \text{affinoids } F \subseteq \Omega \}$.

More generally, start w/ nonempty collection $\mathfrak{X}$ of affinoids in $\mathbb{P}$ s.t.

(1) $X_1, X_2 \in \mathfrak{X} \Rightarrow X_1 \cup X_2 \in \mathfrak{X}$ or $X_1 \cup X_2 = \mathbb{P}$;

(2) $X_1 \subseteq X_2$ affinoid and $X_2 \in \mathfrak{X} \Rightarrow X_1 \in \mathfrak{X}$.

Define $\Omega(\mathfrak{X}) := \bigcup_{F \in \mathfrak{X}} F$. We can give $\Omega(\mathfrak{X})$ a weak G-top., which depends on $\mathfrak{X}$!

- admissible opens $\mathfrak{X} \cup \{\emptyset, \cancel{\mathbb{P}}\, \Omega(\mathfrak{X})\}$
- covering condition suitably modified
- $\mathcal{O}$ defined as expected w/ $\mathcal{O}(\Omega(\mathfrak{X})) := \varprojlim_{F \in \mathfrak{X}} \mathcal{O}(F)$.

Fact: Take $\Omega \subseteq \mathbb{P}$ open and let $\mathfrak{X} := \{ \text{(all) affinoids in } \Omega \}$. Then, $\Omega = \Omega(\mathfrak{X})$ as G-top. spaces.

Def: K is <u>spherically complete</u> if every seq. $D_1 \supset D_2 \supset \cdots$ of nonempty open disks satisfies $\bigcap D_i \neq \emptyset$.

(**Remark**: $\widehat{\mathbb{Q}_p^{alg}}$ is <u>not</u> spherically complete)

Take K spherically complete and choose such D_i. Let $\mathfrak{X} := \{ \text{affinoids in } \mathbb{P} \text{ contained in some } F_i = \mathbb{P} \setminus D_i \}$.

Then, $\Omega(\mathfrak{X}) = \mathbb{P}$. Weak G-topologies are different!