Meditations on Gross-Keating

- Very recently, Li-Zhang (building off of Cho-Yamauchi and others) proved the local Kudla-Rapoport conjecture.

This relates:

- arithmetic intersection #'s of special cycles on unitary Rapoport-Zink spaces; and
- derivatives of local representation densities of Hermitian forms.

Combining this w/ work of Liu and Garcia-Sankaran proves the global Kudla-Rapoport conjecture. This circle of ideas is closely related to arithmetic Siegel-Weil (whose higher forms interest me). Since the range of ideas here is vast, we will focus narrowly on earlier work of Gross and Keating. We begin by considering even earlier work of Kronecker and Hurwitz, starting w/ some basics on quad. spaces. Given a ring R, a <u>quadratic space</u> is the data of a pair (L,Q) w/ L a finitely gen. free R-mod. and Q a quad. form on L. This has assoc. bilin. form $(x,y) := Q(x+y) - Q(x) - Q(y)$. We define $\det(Q)$ to be the elt. of $R^\times$ given by the determinant of the matrix $((b_i,b_j))_{i,j}$ for $\{b_i\}$ basis of L (this # is well-defined up to $(R^\times)^2$). For fixed basis $\{b_i\}$ and n the rank of L, Q has <u>diagonal</u> $\mathrm{diag}(Q) := (Q(b_1),\dots,Q(b_n))$. Given quad. space (R^m, F), we have the <u>representation number</u> $R_L(F) := \#\{\text{isometries } (R^m,F) \to (L,Q)\}$. If $R=\mathbb{Z}$ and Q is pos. def. then this # is finite. Given $D \in \mathbb{Z}_{>0}$, let $H(D)$ be the Hurwitz class #, which counts $SL_2(\mathbb{Z})$-equiv. classes of pos. def. bin. quad forms over $\mathbb{Z}$ w/ determinant D, w/ multiples of $x_1^2+x_2^2$ + $x_1^2+x_1x_2+x_2^2$ receiving respective weights $1/2$ and $1/3$. For $m \in \mathbb{Z}_{>0}$ <u>not</u> a perfect square, define $G(m) := \sum_{t\in\mathbb{Z},\, t^2 \le 4m} H(4m-t^2)$. Define $T_m := V(\varphi_m) \subseteq \mathbb{A}^2_{\mathbb{C}}$. Here, φ_m is the classical modular polynomial which has coeffs. in $\mathbb{Z}$ and detects existence of isogenies of degree m between elliptic curves.

Thm (Hurwitz): $T_{m_1,\mathbb{C}}, T_{m_2,\mathbb{C}}$ intersect properly iff $m := m_1 m_2$ is <u>not</u> a perfect square. Moreover,

$$(T_{m_1,\mathbb{C}} \cdot T_{m_2,\mathbb{C}}) = \sum_{t\in\mathbb{Z},\, t^2<4m} \ \sum_{d \mid \gcd(m_1,m_2,t)} d\, H\left(\frac{4m-t^2}{d^2}\right) = \sum_{n \mid \gcd(m_1,m_2)} n\, G(m/n^2).$$

Pf: Skipping to the second claim, let $(j_0, j_0') \in \mathbb{C}^2$ corresponding to pair of elliptic curves (E,E'). Let $u_E := \frac{1}{2}\#\mathrm{Aut}(E)$, $u_{E'} := \frac{1}{2}\#\mathrm{Aut}(E')$. The key is that, in the (j,j')-plane, we have local [This gives one concrete way of doing computations...]

$$\text{intersection } \# \quad \underbrace{(T_{m_1,\mathbb{C}} \cdot T_{m_2,\mathbb{C}})_{(j_0,j_0')}}_{\text{Denoted } (T_{m_1,\mathbb{C}} \cdot T_{m_2,\mathbb{C}})_{(E,E')} \text{ more intrinsically.}} = \underbrace{\frac{1}{4u_E u_{E'}} \#\{(f_1,f_2) \in \mathrm{Hom}(E,E') : \deg f_i = m_i\}}_{\#\{\text{branches of } T_{m_1}\}\,\#\{\text{branches of } T_{m_2}\}}.$$

[branches are nonsing. and intersect transversely for m_1, m_2]

Pairs $(f_1,f_2) \leftrightarrow$ representations of pos. def. quad. forms $Q(x_1,x_2) = \deg(x_1 f_1 + x_2 f_2)$ [often called a degree form]

$$\Rightarrow \#\{(f_1,f_2) \in \mathrm{Hom}(E,E') : \deg f_i = m_i\} = \sum_{\substack{Q>0 \\ \mathrm{diag}(Q) = (m_1,m_2)}} R_{\mathrm{Hom}(E,E')}(Q).$$ Hence,

$$(T_{m_1,\mathbb{C}} \cdot T_{m_2,\mathbb{C}}) = \sum_{(E,E')} \frac{1}{4u_E u_{E'}} \sum_{\substack{Q>0 \\ \mathrm{diag}(Q)=(m_1,m_2)}} R_{\mathrm{Hom}(E,E')}(Q)$$

$$= \frac{1}{4} \sum_{\substack{Q>0 \\ \mathrm{diag}(Q)=(m_1,m_2)}} \sum_{(E,E')} \frac{R_{\mathrm{Hom}(E,E')}(Q)}{u_E u_{E'}}$$

Remark: ratios appearing here are first examples of representation densities.

$$= \frac{1}{4} \sum_{\substack{Q>0 \\ \mathrm{diag}(Q)=(m_1,m_2)}} 4 \sum_{d \mid e(Q)} d\, H(\det(Q)/d^2)$$

$$= \sum_{t\in\mathbb{Z},\, t^2<4m} \ \sum_{d \mid \gcd(m_1,m_2,t)} d\, H\left(\frac{4m-t^2}{d^2}\right).$$

$Q(x_1,x_2) = m_1 x_1^2 + t x_1 x_2 + m_2 x_2^2$

pos. def. bin. quad. form / $\mathbb{Z}$

$\Rightarrow 4m_1 m_2 - t^2 = \det(Q) > 0$

content of $Q = e(Q) := \gcd(m_1, m_2, t)$

$\square$

Cor (Kronecker, Hurwitz): m <u>not</u> perfect square $\Rightarrow$ $G(m) = \sum_{ad=m} \max\{a,d\}$.

key: $(T_{m,\mathbb{C}} \cdot T_{1,\mathbb{C}}) = \deg \varphi_m(j,j)$.

using "j-inv. coords."

We now move to the work of Gross and Keating.

Let $m \in \mathbb{Z}_{>0}$ and consider the Deligne-Mumford stacks (over $\mathbb{Z}$) $\mathcal{M}$ of elliptic curves and $\mathcal{J}_m$ of isogenies of elliptic curves of degree m. In many ~~cases~~ instances we could work instead w/ coarse moduli spaces, but we will see some reason for this perspective.

$\mathcal{J}_m$ is of course closely related to the divisor T_m defined by φ_m (over $\mathbb{Z}$ instead of $\mathbb{C}$), w/ geometric pts. of T_m corresponding to pairs (E,E') of elliptic curves s.t. $\exists$ degree m isogeny $E \to E'$. Gross and Keating are interested in triples of such "divisors." So, let $m_1, m_2, m_3 \in \mathbb{Z}_{>0}$. Let $\tilde{S} := \mathcal{M} \underset{(\operatorname{Spec}\mathbb{Z})}{\times} \mathcal{M}$, which has coarse moduli space S describing pairs of elliptic curves. We are interested in the "intersection"

$$\mathcal{X} := \mathcal{J}_{m_1} \times_{\tilde{S}} \mathcal{J}_{m_2} \times_{\tilde{S}} \mathcal{J}_{m_3},$$

which does not have $T_{m_1} \times_S T_{m_2} \times_S T_{m_3}$ as coarse moduli space.

[Remark: $S = \operatorname{Spec}\mathbb{Z}[j,j']$ thinking in terms of j-invariants.]

Prop: Define $(T_{m_1} \cdot T_{m_2} \cdot T_{m_3}) := \log \# \underbrace{\mathbb{Z}[j,j']/(\varphi_{m_1}, \varphi_{m_2}, \varphi_{m_3})}_{(\text{finite quotient})}$. Then,

$$(T_{m_1} \cdot T_{m_2} \cdot T_{m_3}) = \frac{1}{2} \sum_p \log(p) \sum_{x \in \mathcal{X}(\overline{\mathbb{F}}_p)} \frac{1}{\# \operatorname{Aut}_{\mathcal{X}}(x)} \lg \hat{\mathcal{O}}_{\mathcal{X},x}.$$

$\llcorner$ length term

We will omit the proof as our time today is limited (this is a deformation-theoretic statement about lifting isogenies). Note that we may assume $T_{m_1} \cap T_{m_2} \cap T_{m_3}$ is finite since otherwise both sides of the above are infinite. When is this intersection finite? This is related to what we have seen before.

Prop: $T_{m_1}, T_{m_2}, T_{m_3}$ intersect in dim 0 iff m_1, m_2, m_3 are not simultaneously represented by pos. def. bin. quad. form/$\mathbb{Z}$. The latter holds iff every pos. semi-def. half-integral symm. matrix T w/ $\operatorname{diag}(T) = (m_1, m_2, m_3)$ (← entries land in $\mathbb{Z}[\tfrac{1}{2}]$) is nondeg. [This reformulates things in terms of ternary forms.] [Notation: $\operatorname{Sym}_3(\mathbb{Z})^\vee_{>0}$]

Assuming $T_{m_1}, T_{m_2}, T_{m_3}$ intersect in dim 0, Gross and Keating calculate that the coeff. $n(p)$ appearing in front of $\log(p)$ is 0 for $p > 4m_1m_2m_3$ and for smaller p satisfies

$$n(p) = \frac{1}{8} \sum_{(E,E')\text{ s.s.}} \sum_{Q} \frac{R_{\operatorname{Hom}(E,E')}(Q)}{u_E u_{E'}} \alpha_p(Q) = \frac{1}{2} \sum_Q \Big(\prod_{\substack{\ell \mid \Delta \\ \ell \neq p}} \beta_\ell(Q) \Big) \alpha_p(Q).$$

(anisotropic for $\ell \neq p$)

$[\Delta := \tfrac{1}{2}\det(Q) \in \mathbb{Z}]$

What's going on here? Earlier we gave a "stacky" interpretation of $n(p)$. This # can be viewed as the sum of intersection multiplicities of pts. (E,E') which are pairs of supersingular elliptic curves in char. p. ($\simeq$ maximal rank endo. ring / $\overline{\mathbb{F}}_p$)

The statement is simple to verify. Beyond that, what we are ~~also~~ doing is ~~deforming~~ a triple of isogenies $f_i : E \to E'$ of degrees m_i $(i=1,2,3)$. We can do this up to a factor of ± 1 because of rigidity results (which is where the $1/8$ factor comes from, to deal w/ over-counting). The term $\alpha_p(Q)$ is a length factor obtained from f_1, f_2, f_3, which actually only depends on the isom. class of (the degree form) Q (as a ternary quad. form).
$[(f_1,f_2,f_3) \mapsto \deg(m_1 f_1 + m_2 f_2 + m_3 f_3)]$

This isn't surprising given our "stacky" formula, but the key is that we can actually compute $\alpha_p(Q)$ using formal grps. via Serre-Tate theory [IDK how this works...]. The term $\beta_\ell(Q)$ is a so-called local representation density.

This is great (although we really haven't said much!) but actually we can bring Eisenstein series into the picture. Namely, $(T_{m_1} \cdot T_{m_2} \cdot T_{m_3})$ agrees (up to a constant) w/ a Fourier coeff. of the restriction of the derivative at $s=0$ of a Siegel-Eisenstein series of genus 3 and weight 2. Let (E,E') be a pair of elliptic curves (over S) and $f_i \in \mathcal{H}om(E,E')$ isogenies of degree m_i $(i=1,2,3)$. Letting Q denote the degree quad. form on $\mathrm{Hom}(E,E')$,

$$\mathcal{T}_{m_1} \underset{\tilde{S}}{\times} \mathcal{T}_{m_2} \underset{\tilde{S}}{\times} \mathcal{T}_{m_3} = \coprod_{T \in \mathrm{Sym}_3(\mathbb{Z})^\vee_{>0}} \mathcal{T}_T \quad \text{w/}$$

$\mathcal{T}_T(S) = \{\vec{f} \in \mathrm{Hom}(E,E')^3 : \frac{1}{2}(\vec{f},\vec{f}) = T\}$ and $(\vec{f},\vec{f})$ the matrix w/ terms $a_{i,j} = (f_i, f_j) = Q(f_i + f_j) - Q(f_i) - Q(f_j)$. $(1 \le i,j \le 3)$. To such T we may associate

$$\widehat{\deg}(\mathcal{T}_T) := \Bigg(\underbrace{\sum_{\xi \in \mathcal{T}_T(\overline{\mathbb{F}}_p)} \underbrace{\frac{1}{\#\mathrm{Aut}(\xi)}}_{=:\, e_\xi} \lg(\mathcal{O}_{\mathcal{T}_T,\xi})}_{=:\, \lg(\mathcal{T}_T)} \Bigg) \log(p)$$

, which should look familiar (as a "piecemeal" version of the earlier story). The prime p here is the unique prime s.t. $\mathcal{T}_T$ has support in the supersingular locus of $(\mathcal{M} \times \mathcal{M}) \otimes_{\mathbb{Z}} \overline{\mathbb{F}}_p$.

(Probably won't get to talking about these terms specifically.)

Let $\mathcal{H}_3 = \{ \tau = x+iy \in \mathrm{Sym}_3(\mathbb{C}) : \mathrm{im}(\tau) = y > 0 \}$ denote the Siegel upper half space. Consider the classical Siegel-Eisenstein series $\underset{\tau \in \mathcal{H}_3}{E_{\text{class}}(\tau,s)} := \det(y)^{s/2} \sum_{(c,d)} \underbrace{\det(c\tau+d)^{-2}}_{\text{weight } 2} |\det(c\tau+d)|^{-s}$ which

admits a Fourier expansion $E_{\text{class}}(\tau,s) = \sum_{T \in \mathrm{Sym}_3(\mathbb{Z})^\vee} c(T,y,s) q^T$ w/ $q^T := \exp(2\pi i\, \mathrm{tr}(T\tau))$.

<u>Thm</u>: Let $T \in \mathrm{Sym}_3(\mathbb{Z})^\vee_{>0}$.

(1) $c'(T) := \frac{\partial}{\partial s}\Big|_{s=0} c(T,y,s)$ is independent of y.

(2) $\mathrm{Diff}(T,V) = \{p\} \Rightarrow \mathcal{Z}_T$ has support in char. p and $\widehat{\deg}(\mathcal{Z}_T) = K c'(T)$ for K negative constant independent of T.

(standard sympectic matrix $\begin{pmatrix} 0 & I_3 \\ -I_3 & 0 \end{pmatrix}$)

Lurking in the background here is an $Sp_6(\mathbb{Q})$. Let V be quad. space given by quaternion alg. $B/\mathbb{Q}$ equipped w/ its norm form Q. We have interest in Siegel-Weil (e.g., relating $O(V)$). We have $\mathrm{Diff}(T,V) := \{ p \text{ prime} : T \text{ is } \underline{\text{not}} \text{ represented by } V(\mathbb{Q}_p) \}$. One of the key properties of this set is that it can detect identical vanishing of local Whittaker factors. This ties into another notion of Eisenstein series. We have a nice class $\mathcal{S}(V(\mathbb{A})^3)$ of Schwartz functions. To $\varphi \in \mathcal{S}(V(\mathbb{A})^3)$ we ~~can may~~ associate the Eisenstein series $E(g,s,\Phi) := \sum_{\gamma \in P(\mathbb{Q}) \backslash Sp_6(\mathbb{Q})} \Phi(\gamma g, s)$ w/ $P \leq Sp_6$ the standard Siegel parabolic. We can chose Φ so that this series is <u>incoherent</u>, which in particular implies that $E(g,0,\Phi) \equiv 0$. We have Fourier expansion $E(g,s,\Phi) = \sum_{T \in \mathrm{Sym}_3(\mathbb{Q})} E_T(g,s,\Phi)$ w/ $E_T(g,s,\Phi) = \prod_v W_{T,v}(g_v, s, \Phi_v)$ a product of local Whittaker factors for $T \in \mathrm{Sym}_3(\mathbb{Q})$ s.t. $\det(T) \neq 0$.

Prop: (1) $p \in \mathrm{Diff}(T,V) \Rightarrow W_{T,p}(g_p, 0, \Phi_p) \equiv 0$.

(2) $W_{T,\infty}(g_\infty, 0, \Phi_\infty) \neq 0$ and so $\operatorname{ord}_{s=0} E_T(g,s,\Phi) \geq \#\mathrm{Diff}(T,V)$.

(3) $E_T'(g,0,\Phi) \neq 0 \Rightarrow \mathrm{Diff}(T,V) = \{p\}$ for unique p.

To get our earlier thm we must first match up our two notions of Eisenstein series. This can be done, up to a determinant factor. Using the Siegel-Weil formula, one proves the following result.

Thm: Given $z \in \mathcal{J}_T(\bar{\mathbb{F}}_p)$, $\lg(\mathcal{O}_{\mathcal{J}_T, z}) = -\dfrac{2}{(p-1)^2} \cdot \dfrac{W_{T,p}'(e,0,\Phi_p)}{W_{T,p}(e,0,\tilde{\Phi}_p)} \cdot (\log(p))^{-1}$. [indep. of z]

$\uparrow$

($\Phi_p, \tilde{\Phi}_p$ appropriately chosen)