More Comparison

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I want to begin by unpacking the proof of a result mentioned last time. Recall that we are concerned with p-complete objects $A(-) \in \mathsf{CAlg}(D(\mathsf{Fun}(\mathsf{CAlg}^{\mathrm{reg}}_{\mathbb{F}_p},\mathsf{Ab})))$ equipped with $u_0 : \Omega_{(-)} \xrightarrow{\sim} \mathbb{F}_p \otimes^{\mathbb{L}} A(-)$ an isomorphism of commutative algebra objects "mod p." Recall as well that such A(-) admits a left Kan extension $\overline{A} : \mathsf{CAlg}^{\mathrm{qsyn}}_{\mathbb{F}_p} \to \mathcal{D}_p(\mathbb{Z})$ which is itself a commutative algebra object, where we have implicitly viewed A as an object of the ∞ -category $\mathsf{Fun}(\mathsf{CAlg}^{\mathrm{reg}}_{\mathbb{F}_p}, \mathcal{D}_p(\mathbb{Z}))$.

Lemma 1. The functor $\overline{A}: \mathsf{CAlg}^{\mathrm{qsyn}}_{\mathbb{F}_p} \to \mathcal{D}_p(\mathbb{Z})$ is a right Kan extension of its restriction to $\mathsf{CAlg}^{\mathrm{qrsp}}_{\mathbb{F}_p}$, with the latter by definition the category of quasiregular semiperfect \mathbb{F}_p -algebras.

Where does such a result come from? Certainly, this makes precise the vague idea that quasiregular semiperfect \mathbb{F}_p -algebras are relatively abundant among all quasisyntomic \mathbb{F}_p -algebras. Our key input is the following result from BMS II.

Theorem 2. Let $S \in \mathsf{CRing}$ be a base ring, $n \geq 0$, and $f : B \to C$ a faithfully flat map of S-algebras. Then, the natural map $\wedge^n \mathbb{L}_{B/S} \to \mathsf{Tot}(\wedge^n \mathbb{L}_{C^{\bullet}/S})$ is an isomorphism in $\mathcal{D}(S)$.

What does this mean? By considering the tensor powers $C, C \otimes_B C, C \otimes_B C \otimes_B C$, etc. we may associate to C a cosimplicial S-algebra C^{\bullet} which can be precisely identified with the Cech nerve of $f: B \to C$ (Isn't this backwards?). From this we obtain a cosimplicial object $\bigwedge^n \mathbb{L}_{C^{\bullet}/S}$ whose (homotopy) limit we denote by $\operatorname{Tot}(\bigwedge^n \mathbb{L}_{C^{\bullet}/S})$. Let's apply this to our context of interest. Let $R \in \mathsf{CAlg}_{\mathbb{F}_p}^{\mathsf{qsyn}}$ and choose a set $\{x_i\}_{i \in I}$ of \mathbb{F}_p -algebra generators for R. To this we may associate

$$R^0 := R \otimes_{\mathbb{F}_p[\{x_i\}]} \mathbb{F}_p[\{x_i\}]_{\text{perf}},$$

which is weakly initial in $\mathsf{CAlg}_R^{\mathsf{qrsp}}$ (the failure of uniqueness stems from the fact that we could have chosen different generators).² In the above setup, we may take $S = \mathbb{F}_p, B = R, C = R^0$ and form R^{\bullet} from R-tensor powers of R^0 as above. Given $n \geq 0$, we conclude by the theorem that the natural map $\wedge^n \mathbb{L}_{R/\mathbb{F}_p} \to \mathsf{Tot}(\wedge^n \mathbb{L}_{R^{\bullet}/\mathbb{F}_p})$ is an isomorphism in $\mathcal{D}(\mathbb{F}_p)$. By assumption we have $\Omega_{(-)} \simeq \mathbb{F}_p \otimes^{\mathbb{L}} A(-)$ and so may upgrade this to obtain $\mathbb{L}\Omega_{(-)} \simeq \mathbb{F}_p \otimes^{\mathbb{L}} \overline{A}(-)$. We want to conclude that the corresponding natural map $\overline{A}(R) \to \mathsf{Tot}(\overline{A}(R^{\bullet}))$ is a qis. There are several components to this.

• $\overline{A}(-)$ is *p*-complete.

¹Said another way, the functor $B \mapsto \Lambda_B^n \mathbb{L}_{B/S}$ is an (∞ -categorical) fpqc sheaf.

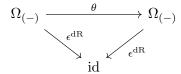
²The notation $(\cdot)_{perf}$ indicates taking the perfection – i.e., the filtered colimit over the power map $(\cdot)^p$. For example, given k a perfect field of characteristic p, we have $k[x]_{perf} \cong k[x^{1/p^{\infty}}]$.

- Looking at the conjugate filtration we have $\operatorname{gr}_n(\operatorname{Fil}^{\operatorname{conj}}_{\bullet} \mathbb{L}\Omega_R) \simeq (\wedge^n \mathbb{L}_{R^{(1)}/\mathbb{F}_n})[-n].$
- We know that $H^n(\overline{A}(R))$ vanishes for n < 0 and is p-torsion-free for n = 0.

Why does this to establish the final property that \overline{A} is supposed to have as a right Kan extension?

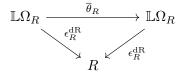
Our next item of business is establishing the following result.

Proposition 3. Let $\theta: \Omega_{(-)} \to \Omega_{(-)}$ be a morphism of commutative algebra objects in $\mathcal{D}(\mathsf{Fun}(\mathsf{CAlg}^{\mathrm{reg}}_{\mathbb{F}_p}, \mathsf{Vect}_{\mathbb{F}_p})) \simeq \mathsf{Fun}(\mathsf{CAlg}^{\mathrm{reg}}_{\mathbb{F}_p}, \mathcal{D}(\mathbb{F}_p))$ such that



commutes. Then, $\theta = id$.

We begin by extending θ to get $\overline{\theta}: \mathbb{L}\Omega_{(-)} \to \mathbb{L}\Omega_{(-)}$ an endomorphism of commutative algebra objects in $\mathsf{Fun}(\mathsf{CAlg}_{\mathbb{F}_p}^{\mathsf{qsyn}}, \mathcal{D}(\mathbb{F}_p))$. Inside $\mathsf{CAlg}_{\mathbb{F}_p}$ we may consider the full subcategory $\mathsf{CAlg}_{\mathbb{F}_p}^{\mathsf{std}}$ of standard \mathbb{F}_p -algebras, defined by the fact that they are finite tensor products over \mathbb{F}_p of algebras of the form $\mathbb{F}_p[x]_{\mathsf{perf}}$ or $\mathbb{F}_p[x]_{\mathsf{perf}}/(x)$. Such algebras are quasiregular semiperfect, and working as before one can show that the restriction of $\mathbb{L}\Omega_{(-)}$ to $\mathsf{CAlg}_{\mathbb{F}_p}^{\mathsf{poly}}$ is a right Kan extension of the restriction to $\mathsf{CAlg}_{\mathbb{F}_p}^{\mathsf{std}}$. Assuming that $\overline{\theta}$ restricts to id on $\mathsf{CAlg}_{\mathbb{F}_p}^{\mathsf{std}}$, we conclude that $\overline{\theta}$ restricts to id on $\mathsf{CAlg}_{\mathbb{F}_p}^{\mathsf{poly}}$ (by the right Kan extension property) and thus that it restricts to id on $\mathsf{CAlg}_{\mathbb{F}_p}^{\mathsf{qsyn}}$ (by the left Kan extension property). Hence, θ restricts to id as well! So, we need to prove the first restriction claim for $\overline{\theta}$. Let $R \in \mathsf{CAlg}_{\mathbb{F}_p}^{\mathsf{std}}$ and consider $\overline{\theta}_R : \mathbb{L}\Omega_R \to \mathbb{L}\Omega_R$, which fits into a commutative diagram



We may work with R one tensor factor at a time, allowing us to immediately dispose of the case $R = \mathbb{F}_p[x]_{perf}$ and thus only need to deal with the case $R = \mathbb{F}_p[x]_{perf}/(x)$. Assume for now that we have the following result.

Lemma 4. The endomorphism $\theta_R:\Omega_R\to\Omega_R$ is an isomorphism on cohomology for any $R\in\mathsf{CAlg}^{\mathrm{reg}}_{\mathbb{F}_p}$.

Our aim is to decompose $\mathbb{L}\Omega_R$ in terms of the associated graded pieces of its conjugate filtration. To do this, we first need to enlarge \mathbb{F}_p to k perfect containing t not algebraic over \mathbb{F}_p (this modification is harmless because of faithfully flat extension of scalars). So, we assume $R = k[x]_{perf}/(x)$. The k-algebra automorphism $x \mapsto tx$ of k[x] extends to an automorphism of $k[x]_{perf}$ sending the ideal (x) to itself and so induces an automorphism τ of $\mathbb{L}\Omega_R$. By the lemma, $\overline{\theta}_R$ preserves the conjugate filtration on $\mathbb{L}\Omega_R$ and induces the identity on associated graded terms. Moreover, the automorphism τ acts semisimply on the associated graded terms of $\mathbb{L}\Omega_R$ with disjoint eigenvalues for each term.

We thereby obtain a canonical splitting

$$\mathbb{L}\Omega_R \simeq \bigoplus_{n \geq 0} \operatorname{gr}_n(\operatorname{Fil}^{\operatorname{conj}}_{\bullet} \mathbb{L}\Omega_R)$$

and conclude that $\overline{\theta}_R$ is id. This leaves three things to be established.

- (1) The underlying linear algebra result.
- (2) The semisimplicity of the action of τ .
- (3) The proof of the lemma.