More Comparison

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I want to begin by unpacking the proof of a result mentioned last time. Recall that we are concerned with p-complete objects $A(-) \in \mathsf{CAlg}(D(\mathsf{Fun}(\mathsf{CAlg}^{\mathrm{reg}}_{\mathbb{F}_p}, \mathsf{Ab})))$ equipped with $u_0 : \Omega_{(-)} \xrightarrow{\sim} \mathbb{F}_p \otimes^{\mathbb{L}} A(-)$ an isomorphism of commutative algebra objects "mod p." Recall as well that such $A(-)$ admits a left Kan extension $\overline{A} : \mathsf{CAlg}^{\mathrm{qsyn}}_{\mathbb{F}_p} \to \mathcal{D}_p(\mathbb{Z})$ which is itself a commutative algebra object, where we have implicitly viewed A as an object of the ∞-category $\mathsf{Fun}(\mathsf{CAlg}^{\mathrm{reg}}_{\mathbb{F}_p}, \mathcal{D}_p(\mathbb{Z}))$.

Lemma 1. *The functor $\overline{A} : \mathsf{CAlg}^{\mathrm{qsyn}}_{\mathbb{F}_p} \to \mathcal{D}_p(\mathbb{Z})$ is a right Kan extension of its restriction to $\mathsf{CAlg}^{\mathrm{qrsp}}_{\mathbb{F}_p}$, with the latter by definition the category of quasiregular semiperfect $\mathbb{F}_p$-algebras.*

Where does such a result come from? Certainly, this makes precise the vague idea that quasiregular semiperfect $\mathbb{F}_p$-algebras are relatively abundant among all quasisyntomic $\mathbb{F}_p$-algebras. Our key input is the following result from BMS II.

Theorem 2. *Let $S \in \mathsf{CRing}$ be a base ring, $n \geq 0$, and $f : B \to C$ a faithfully flat map of S-algebras. Then, the natural map $\wedge^n \mathbb{L}_{B/S} \to \mathrm{Tot}(\wedge^n \mathbb{L}_{C^\bullet/S})$ is an isomorphism in $\mathcal{D}(S)$.*[1]

What does this mean? By considering the tensor powers $C, C \otimes_B C, C \otimes_B C \otimes_B C$, etc. we may associate to C a cosimplicial S-algebra $C^\bullet$ which can be precisely identified with the Cech nerve of $f : B \to C$ (Isn't this backwards?). From this we obtain a cosimplicial object $\wedge^n \mathbb{L}_{C^\bullet/S}$ whose (homotopy) limit we denote by $\mathrm{Tot}(\wedge^n \mathbb{L}_{C^\bullet/S})$. Let's apply this to our context of interest. Let $R \in \mathsf{CAlg}^{\mathrm{qsyn}}_{\mathbb{F}_p}$ and choose a set $\{x_i\}_{i \in I}$ of $\mathbb{F}_p$-algebra generators for R. To this we may associate

$$R^0 := R \otimes_{\mathbb{F}_p[\{x_i\}]} \mathbb{F}_p[\{x_i\}]_{\mathrm{perf}},$$

which is weakly initial in $\mathsf{CAlg}^{\mathrm{qrsp}}_R$ (the failure of uniqueness stems from the fact that we could have chosen different generators).[2] In the above setup, we may take $S = \mathbb{F}_p, B = R, C = R^0$ and form $R^\bullet$ from R-tensor powers of R^0 as above. Given $n \geq 0$, we conclude by the theorem that the natural map $\wedge^n \mathbb{L}_{R/\mathbb{F}_p} \to \mathrm{Tot}(\wedge^n \mathbb{L}_{R^\bullet/\mathbb{F}_p})$ is an isomorphism in $\mathcal{D}(\mathbb{F}_p)$. By assumption we have $\Omega_{(-)} \simeq \mathbb{F}_p \otimes^{\mathbb{L}} A(-)$ and so may upgrade this to obtain $\mathbb{L}\Omega_{(-)} \simeq \mathbb{F}_p \otimes^{\mathbb{L}} \overline{A}(-)$. We want to conclude that the corresponding natural map $\overline{A}(R) \to \mathrm{Tot}(\overline{A}(R^\bullet))$ is a qis. There are several components to this.

- $\overline{A}(-)$ is p-complete.

[1] Said another way, the functor $B \mapsto \wedge^n_B \mathbb{L}_{B/S}$ is an (∞-categorical) fpqc sheaf.

[2] The notation $(\cdot)_{\mathrm{perf}}$ indicates taking the perfection – i.e., the filtered colimit over the power map $(\cdot)^p$. For example, given k a perfect field of characteristic p, we have $k[x]_{\mathrm{perf}} \cong k[x^{1/p^\infty}]$.

- Looking at the conjugate filtration we have $\mathrm{gr}_n(\mathrm{Fil}^{\mathrm{conj}}_\bullet \mathbb{L}\Omega_R) \simeq (\wedge^n \mathbb{L}_{R^{(1)}/\mathbb{F}_p})[-n]$.
- We know that $H^n(\overline{A}(R))$ vanishes for $n<0$ and is p-torsion-free for $n=0$.

Why does this to establish the final property that $\overline{A}$ is supposed to have as a right Kan extension?

Our next item of business is establishing the following result.

Proposition 3. *Let $\theta : \Omega_{(-)} \to \Omega_{(-)}$ be a morphism of commutative algebra objects in $\mathcal{D}(\mathsf{Fun}(\mathsf{CAlg}^{\mathrm{reg}}_{\mathbb{F}_p}, \mathsf{Vect}_{\mathbb{F}_p})) \simeq \mathsf{Fun}(\mathsf{CAlg}^{\mathrm{reg}}_{\mathbb{F}_p}, \mathcal{D}(\mathbb{F}_p))$ such that*

$$\begin{tikzcd} \Omega_{(-)} \arrow[rr, "\theta"] \arrow[dr, "\epsilon^{\mathrm{dR}}"'] & & \Omega_{(-)} \arrow[dl, "\epsilon^{\mathrm{dR}}"] \\ & \mathrm{id} & \end{tikzcd}$$

commutes. Then, $\theta = \mathrm{id}$.

We begin by extending θ to get $\overline{\theta} : \mathbb{L}\Omega_{(-)} \to \mathbb{L}\Omega_{(-)}$ an endomorphism of commutative algebra objects in $\mathsf{Fun}(\mathsf{CAlg}^{\mathrm{qsyn}}_{\mathbb{F}_p}, \mathcal{D}(\mathbb{F}_p))$. Inside $\mathsf{CAlg}_{\mathbb{F}_p}$ we may consider the full subcategory $\mathsf{CAlg}^{\mathrm{std}}_{\mathbb{F}_p}$ of standard $\mathbb{F}_p$-algebras, defined by the fact that they are finite tensor products over $\mathbb{F}_p$ of algebras of the form $\mathbb{F}_p[x]_{\mathrm{perf}}$ or $\mathbb{F}_p[x]_{\mathrm{perf}}/(x)$. Such algebras are quasiregular semiperfect, and working as before one can show that the restriction of $\mathbb{L}\Omega_{(-)}$ to $\mathsf{CAlg}^{\mathrm{poly}}_{\mathbb{F}_p}$ is a right Kan extension of the restriction to $\mathsf{CAlg}^{\mathrm{std}}_{\mathbb{F}_p}$. Assuming that $\overline{\theta}$ restricts to id on $\mathsf{CAlg}^{\mathrm{std}}_{\mathbb{F}_p}$, we conclude that $\overline{\theta}$ restricts to id on $\mathsf{CAlg}^{\mathrm{poly}}_{\mathbb{F}_p}$ (by the right Kan extension property) and thus that it restricts to id on $\mathsf{CAlg}^{\mathrm{qsyn}}_{\mathbb{F}_p}$ (by the left Kan extension property). Hence, θ restricts to id as well! So, we need to prove the first restriction claim for $\overline{\theta}$. Let $R \in \mathsf{CAlg}^{\mathrm{std}}_{\mathbb{F}_p}$ and consider $\overline{\theta}_R : \mathbb{L}\Omega_R \to \mathbb{L}\Omega_R$, which fits into a commutative diagram

$$\begin{tikzcd} \mathbb{L}\Omega_R \arrow[rr, "\overline{\theta}_R"] \arrow[dr, "\epsilon^{\mathrm{dR}}_R"'] & & \mathbb{L}\Omega_R \arrow[dl, "\epsilon^{\mathrm{dR}}_R"] \\ & R & \end{tikzcd}$$

We may work with R one tensor factor at a time, allowing us to immediately dispose of the case $R = \mathbb{F}_p[x]_{\mathrm{perf}}$ and thus only need to deal with the case $R = \mathbb{F}_p[x]_{\mathrm{perf}}/(x)$. Assume for now that we have the following result.

Lemma 4. *The endomorphism $\theta_R : \Omega_R \to \Omega_R$ is an isomorphism on cohomology for any $R \in \mathsf{CAlg}^{\mathrm{reg}}_{\mathbb{F}_p}$.*

Our aim is to decompose $\mathbb{L}\Omega_R$ in terms of the associated graded pieces of its conjugate filtration. To do this, we first need to enlarge $\mathbb{F}_p$ to k perfect containing t not algebraic over $\mathbb{F}_p$ (this modification is harmless because of faithfully flat extension of scalars). So, we assume $R = k[x]_{\mathrm{perf}}/(x)$. The k-algebra automorphism $x \mapsto tx$ of $k[x]$ extends to an automorphism of $k[x]_{\mathrm{perf}}$ sending the ideal (x) to itself and so induces an automorphism τ of $\mathbb{L}\Omega_R$. By the lemma, $\overline{\theta}_R$ preserves the conjugate filtration on $\mathbb{L}\Omega_R$ and induces the identity on associated graded terms. Moreover, the automorphism τ acts semisimply on the associated graded terms of $\mathbb{L}\Omega_R$ with disjoint eigenvalues for each term.

We thereby obtain a canonical splitting

$$\mathbb{L}\Omega_R \simeq \bigoplus_{n\geq 0} \mathrm{gr}_n(\mathrm{Fil}^{\mathrm{conj}}_\bullet \mathbb{L}\Omega_R)$$

and conclude that $\overline{\theta}_R$ is id. This leaves three things to be established.

(1) The underlying linear algebra result.

(2) The semisimplicity of the action of τ.

(3) The proof of the lemma.