$X = \{\mathbb{R}\text{-alg. maps } h : \mathbb{C} \to M_2(\mathbb{R})\}$. $G = GL_{2,\mathbb{Q}}$. Can identify X as $G(\mathbb{R})$-conj. class $X \subseteq \mathrm{Hom}(\mathbb{C}^\times, G(\mathbb{R}))$.

We saw last time that, given $k \leq GL_2(\hat{\mathbb{Z}})$ compact open, [If we instead had $GL_2(\mathbb{A}_f)$ how would we even know there is an action as desired?!]

$G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/k \xrightarrow{\sim} \{$isom. classes of elliptic curves $E/\mathbb{C}$ w/ k-orbit $[\eta] = k\cdot\eta \subseteq \mathrm{Isom}_{\mathbb{A}_f}(\hat{T}E, \hat{\mathbb{Z}}^2)\}$.

<u>Digression on Level Structure</u>

E elliptic curve / field F, $N \in \mathbb{Z}^+$ coprime to char(F). Recall that a level N structure is $E[N] \cong (\mathbb{Z}/N)^2$.

Same as $E[N](F) \cong (\mathbb{Z}/N)^2$ (by coprime condition). Same as $\mathrm{Gal}(F^{sep}/F)$-inv. isom. $E[N](F^{sep}) \cong (\mathbb{Z}/N)^2$ (must have trivial action!). Assume now char$(F) = 0$ (so no funny business w/ adèlic Tate module). $E' := E_{F^{sep}}$. Isom. $E[N](F^{sep}) \cong (\mathbb{Z}/N)^2 \leftrightsquigarrow k(N) := \ker(GL_2(\hat{\mathbb{Z}}) \to GL_2(\mathbb{Z}/N))$-orbit of isom.'s

$$\begin{array}{ccc} \hat{T}E' & \cong & \hat{\mathbb{Z}}^2 \\ \downarrow & & \downarrow \\ E[N](F^{sep}) & \cong & (\mathbb{Z}/N)^2 \end{array}$$

The Galois invariance condition says correspond $k(N)$-orbit $\hat{T}E' \cong \hat{\mathbb{Z}}^2$ is $\mathrm{Gal}(F^{sep}/F)$-stable.

$S \in \mathrm{Sch}$ w/ geom. pt. s. We get $\pi_1^{ét}(S,s)$. This generalizes the abs. Galois grp. (to geom. setting).

Suppose $E \to S$ elliptic curve w/ S conn. Let p prime w/ $p \in \mathcal{O}_S^\times$ (so inv. in the base).

(1) $s \in S$ geom. pt. $\rightsquigarrow$ monodromy action of $\pi_1^{ét}(S,s)$ on T_pE_s (p-adic Tate module rel. to s).

(2) $s' \in S$ another geom. pt. $\rightsquigarrow$ distinguished set of isoms. in $\mathrm{Isom}_{\mathbb{Z}_p}(T_pE_s, T_pE_{s'})$ which is single orbit for induced actions of both $\pi_1^{ét}(S,s)$ and $\pi_1^{ét}(S,s')$.

<u>Def</u>: $k \leq GL_2(\hat{\mathbb{Z}})$ compact open, $E \to S$ elliptic curve / conn. $\mathbb{Q}$-scheme, $s \to S$ geom. pt. <u>Level-k-structure</u> on E is k-orbit in $\mathrm{Isom}_{\hat{\mathbb{Z}}}(\hat{T}E_s, \hat{\mathbb{Z}})$ that is stable under $\pi_1^{ét}(S,s)$-action.

<u>Remark</u>: In this setting, giving the data at some geom. pt. uniquely gives data at every geom. pt. So, choice of geom. pt. does not matter.

<u>Thm</u>: $k \leq GL_2(\hat{\mathbb{Z}})$ suff. small compact open (e.g. $k \leq k(N)$ w/ $N \geq 3$). The functor

$Y_k : \mathrm{Sch}_{\mathbb{Q}} \to \mathrm{Set}$, $S \mapsto \{$isom. classes of elliptic curves $E \to S$ w/ k-level-structure$\}$ is rep. by smooth quasi-proj. $\mathbb{Q}$-scheme w/ $Y_k(\mathbb{C}) \cong GL_2(\mathbb{Q}) \backslash X \times GL_2(\mathbb{A}_f)/k$.

Pf: Choose $N \geq 3$ s.t. $K(N) \leq K$. Level-$K(N)$-structure is same as level-N-structure. So, $Y_{K(N)} = Y(N)$ is representable by earlier work. $GL_2(\mathbb{Z}/N)$ acts on $Y(N)$ (by changing level structures), hence $K/K(N) \leq GL_2(\mathbb{Z}/N)$ acts as well. Check that quotient of $Y(N)$ by $K/K(N)$ represents Y_K. $\square$

($= \mathbb{A}_f = \hat{\mathbb{Z}} \otimes_{\mathbb{Z}} \mathbb{Q}$)

Notation: E elliptic curve / alg. closed field of char. 0 has $\hat{V}E := \hat{T}E \otimes_{\hat{\mathbb{Z}}} \hat{\mathbb{Q}}$.

$K \leq GL_2(\mathbb{A}_f)$ compact open $\rightsquigarrow$ K-level-structure on E is K-orbit in $\mathrm{Iso}_{\hat{\mathbb{Q}}}(\hat{V}E, \hat{\mathbb{Q}}^2)$. Extend to $E \to S$ as before ("smear out the fibers").

Thm: Y_K defined as before is representable as before. [Isomorphism gives way to isogeny...]

[$Sh_K(G,X)$]

Paradox: We have two! moduli interpretations for $K \leq GL_2(\hat{\mathbb{Z}})$ compact open. We are classifying both isom. classes and isogeny classes. What's going on?! [Potential student topic for the end of the semester.]