Fact: G-torsor $X \to Y$ is trivial iff $\exists$ section $X \underset{\curvearrowleft}{\to} Y$

Example: Let $A \to S$ be ab. scheme of dim g and $m \in \mathbb{Z}$ w/ $m \in \mathcal{O}_S^\times$. Define

$\mathrm{Level}_m : \mathrm{Sch}_S \to \mathrm{Set}$, $T \mapsto \{$ isom.'s $(\mathbb{Z}/m\mathbb{Z})^{2g}_T \xrightarrow{\sim} A_T[m] \}$. This is representable by finite étale S-scheme,

which is $GL_{2g}(\mathbb{Z}/m\mathbb{Z})$-torsor for the étale top.

Example: $Y \in \mathrm{Sch}$, $L \in \mathrm{Pic}(Y) \rightsquigarrow \underline{\mathrm{Iso}}(L, \mathcal{O}_Y) : \mathrm{Sch}_Y \to \mathrm{Set}$, $T \mapsto \{$ isoms.'s of line bundles $L_T \cong \mathcal{O}_T \}$.

This is represented by a scheme $\underline{\mathrm{Iso}}(L, \mathcal{O}_Y) \to Y$. $\mathbb{G}_m(T) = \mathcal{O}_T^\times$ acts on $\underline{\mathrm{Iso}}(L, \mathcal{O}_Y)(T)$ and makes

$\underline{\mathrm{Iso}}(L, \mathcal{O}_Y)$ into Zariski $\mathbb{G}_m$-torsor. This gives equiv. of cat.'s

$\mathrm{Pic}(Y) \simeq$ (Zariski, étale, fppf) $\mathbb{G}_m \to Y$ torsors

[later we will talk about automorphic vectors bundles living over Shimura varieties]

Example: $E \in \mathrm{Vect}_r(Y) \Rightarrow \underline{\mathrm{Iso}}(E, \mathcal{O}_Y^{\oplus r}) \to Y$ is GL_r-torsor. We get equiv. of cat.'s GL_r-torsors $\leftrightarrow$ rank r vector bundles.

Example: $E \in \mathrm{Vect}_r(Y) \rightsquigarrow \mathrm{LRig}_E : \mathrm{Sch}_Y \to \mathrm{Set}$, $T \mapsto \{$ isom.'s of T-schemes $\mathbb{P}(E_T) \cong \mathbb{P}(\mathcal{O}_T^{\oplus r}) = \mathbb{P}^{r-1}_T \}$.

This is (Zariski) PGL_r-torsor.

Prop: $X \to Y$ G-torsor $\Rightarrow G \backslash X := Y$ is categorical G-quotient. Moreover, $G(k) \backslash X(k) \cong Y(k)$ assuming k is a field.
- Alg. closed for fppf top.
- Sep. closed for étale top.

Moduli of Polarized Abelian Schemes

Fix $g, d, n \geq 1$. Let $\mathcal{A}_{g,d,n} : \mathrm{Sch}_{\mathbb{Z}[1/n]} \to \mathrm{Set}$, $S \mapsto \{$ isom. classes of ab. schemes $A \to S$ of dim g w/ $\lambda : A \to A^\vee$ degree d^2 polarization and level-n structure$\}$

If this is representable then universal (A, λ) defines vector bundle $M := (\mathrm{id} \times \lambda)^* \mathcal{P}_A^{\otimes 3}$ on $\mathcal{A}_{g,d,n}$ of rank $6^g d$.

Moreover, $\mathcal{H}_{g,d,n} \to \mathcal{A}_{g,d,n}$ must be a Zariski $PGL_{6^g d}$-torsor.

Thm (Mumford): $n > 6^g d \sqrt{g!} \Rightarrow \mathcal{A}_{g,d,n}$ representable by quasi-proj. $\mathbb{Z}[1/n]$-scheme. (Can improve to $n \geq 3$.)

Very rough idea: First prove general result of the following form. Given
$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ & & \downarrow q' \\ & & Y' \end{array}$$
schemes over $\mathbb{Z}[1/n]$

and reductive grp. G over $\mathbb{Z}[1/n]$ acting on X and X' s.t. ϕ is G-equivariant and q' is Zariski G-torsor.

Then, $\exists$ Cartesian diagram
$$\begin{array}{ccc} X & \xrightarrow{\phi} & X' \\ q \downarrow & & \downarrow q' \\ Y & \to & Y' \end{array}$$
w/ q Zariski G-torsor. [This is not very deep - result of faithfully flat descent.]

For ease of notation, $\mathbb{P}_m := \mathbb{P}^m$. $\quad m = 6^g d - 1$

Start w/ universal object

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr] & \mathbb{P}_m \times \mathcal{H}_{g,d,n} \arrow[d] \\ & \mathcal{H}_{g,d,n} \arrow[ul, dashed, bend left, "e"] \end{tikzcd}$$

. Universal level structure

$(\mathbb{Z}/n\mathbb{Z})^{2g} \to \underline{(\mathbb{Z}/n\mathbb{Z})}^{2g}(\mathcal{H}_{g,d,n}) \cong A[n](\mathcal{H}_{g,d,n})$. We get n^{2g} sections $\alpha_1, \ldots, \alpha_{n^{2g}} : \mathcal{H}_{g,d,n} \to A[n]$.

(this is $A[n] \to \mathbb{P}_m \times \mathcal{H}_{g,d,n} \xrightarrow{pr_1} \mathbb{P}_m$)

Each determines a map $\mathcal{H}_{g,d,n} \xrightarrow{\alpha_i} A[n] \to \mathbb{P}_m$, so we get $\mathcal{H}_{g,d,n} \xrightarrow{(*)} \mathbb{P}_m^{n^{2g}}$. Each α_i is $(PGL_{m-1} \to \operatorname{Spec} \mathbb{Z}[1/m])$ - equivariant. Thus, so is $(*)$. [This is where things get messy.]

__Very__ technical calculation : $n > 6^g d \sqrt{g!} \Rightarrow$ image of $(*)$ lands in open subset $V \subseteq \mathbb{P}_m^{n^{2g}}$ of points $(x_1, \ldots, x_{n^{2g}}) \in \mathbb{P}_m^{n^{2g}}$ s.t. $\forall$ linear subspaces $L \subseteq \mathbb{P}_m$:

$$\frac{\#\ \text{coords. } (x_1, \ldots, x_{n^{2g}}) \text{ lying in } L}{n^{2g}} < \frac{1 + \dim L}{m+1}.$$

__Remark__ : This is some kind of stability condition probably.

__Hard part__ : $\exists$ open cover of V by V_i s.t. each V_i is PGL_{m+1} - stable and has form $V_i \cong Z_i \times PGL_{m+1}$.

Glue the Z_i together. [We choose this open cov. using very explicit determinant calculations/conditions.]

__Remark__ : We can move from n really large to n small by handling finite grp. scheme actions.

__Remark__ : The uniform rank result for Mumford bundles is something of a miracle. An important and similar miracle occurs when thinking about moduli of curves (I think elliptic).