Arithmetic Intersection Theory I

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Ingredients for an Intersection Theory

- A class $\mathcal{C}$ of (geometric) objects which is closed under fiber product.
- A notion of rational equivalence $\sim$ of cycles formed by the above objects.
- A pairing $(\alpha, \beta) \mapsto \alpha.\beta$ on the equivalence classes of cycles that has the following features...

The features

- The pairing makes $A(X)$ (or $A(X)_{\mathbb{Q}}$) into a commutative graded ring for every $X \in \mathcal{C}$; in particular, the pairing should be compatible with $\sim$.
- A notion of "pullback" (along admissible morphisms).
- A notion of "pushforward" (along admissible morphisms).
- Projection formula.
- Normalization in codimension one.
- Reduction to the diagonal
- Local computability

What kind of geometric objects are we considering?

Arithmetic Varieties

Definition

Let $S = \operatorname{Spec} \mathcal{O}_K$ be an affine scheme of a number ring $\mathcal{O}_K$. An **arithmetic variety** is a flat quasi-projective morphism $f : \mathcal{X} \to S$ of schemes such that the generic fiber $X \to \operatorname{Spec} K$ is smooth.

So, the base change $X_{\mathbb{C}}^{\sigma} = X \otimes_K^{\sigma} \mathbb{C}$ admits a structure of *complex manifold* ($\sigma : K \hookrightarrow \mathbb{C}$ embedding).

$$X(\mathbb{C}) = \coprod_{\sigma : K \hookrightarrow \mathbb{C}} X_{\mathbb{C}}^{\sigma}(\mathbb{C}).$$

What are the cycles?

Arithmetic cycles

Let $\mathcal{X}$ be an arithmetic variety. Let $\hat{Z}^p(\mathcal{X}) := \{(Z, g_Z)\}_Z$, where

- $Z \in Z^p(\mathcal{X})$ is an algebraic cycle;
- g_Z is a Green current for $Z(\mathbb{C})$ modulo $\mathrm{im}(\partial) + \mathrm{im}(\bar{\partial})$.

Define the addition componentwise.

Definition

We call $\hat{Z}^p(\mathcal{X})$ the abelian group of **arithmetic cycles** (of codim. p).

What is a Green current?

$$dd^c g_Z + \delta_Z = \omega_Z,$$

with ω_Z some smooth form on $\mathcal{X}(\mathbb{C}) = X(\mathbb{C})$, $dd^c := \frac{i}{2\pi}\partial\overline{\partial}$ (weak derivatives) and δ_Z current of integration over $Z(\mathbb{C})$ (need resolution of singularities to justify).

(Modelled by Poincaré-Lelong formula)

Rational equivalence

Let $\hat{R}^p(\mathcal{X}) \subset \hat{Z}^p(\mathcal{X})$ be the subgroup generated by elements of the form $\widehat{\mathrm{div}}(f) := (\mathrm{div}(f), [-\log|f|^2])$, where $f \in \kappa(W)^\times$ and $W \subset \mathcal{X}$ is an integral closed subscheme of codimension $p-1$ (W varies), and $[-\log|f|^2]$ denotes the class of the current associated to the L^1-function $-\log|f|^2$ on $W(\mathbb{C})$.

That $\widehat{\mathrm{div}}(f) \in \hat{Z}(\mathcal{X})$ is a formulation of Poincaré-Lelong.

Arithmetic Chow group

Definition

The quotient $\widehat{\mathrm{CH}}^p(\mathcal{X}) := \hat{Z}^p(\mathcal{X})/\hat{R}^p(\mathcal{X})$ is called the **arithmetic Chow group**.

Remark

We can switch the grading because $\mathcal{X}$ is an excellent scheme (it is locally of finite type over an excellent ring $\mathcal{O}_K$).

Another representation

Let $\mathcal{X} \to \mathrm{Spec}(\mathcal{O}_F)$ be an arithmetic variety, $X = \mathcal{X}_F$ generic fiber. Let

$$Z^p_{\text{fin}}(\mathcal{X}) := \{Z \in Z^p(\mathcal{X});\ |Z| \cap X = \emptyset\}$$

and

$$\mathrm{CH}^p_{\text{fin}}(\mathcal{X}) := Z^p_{\text{fin}}/\langle \mathrm{div}(f)\rangle$$

$(y \in X^{(p-1)} \setminus X, f \in \kappa(y)^\times)$.

Then we have an exact sequence

$$\bigoplus_{y \in \mathcal{X}_F^{(p-1)}} \kappa(y)^\times \xrightarrow{\widehat{\mathrm{div}}} \hat{Z}^p(\mathcal{X}_F) \oplus \mathrm{CH}^p_{\mathrm{fin}}(\mathcal{X}) \to \widehat{\mathrm{CH}}^p(\mathcal{X}) \to 0$$

Just notice that every cycle on $\mathcal{X}$ can be decomposed uniquely into a finite part and a generic part, and $\widehat{\mathrm{div}}(f) = (\mathrm{div}(f), 0)$ if $f \in \kappa(y)^\times$ with $y \in \mathcal{X}^{(p-1)} \setminus \mathcal{X}_F$.

Let $Y \subset \mathcal{X}$ be a closed subscheme with $\operatorname{codim}_X(Y_F) = p$. Then the natural map

$$Z^p_Y(\mathcal{X}) \to Z^p_{\mathrm{fin}}(\mathcal{X}) \oplus Z^p_{Y_F}(\mathcal{X}_F)$$

induces a map

$$\mathrm{CH}^p_Y(\mathcal{X}) \to \mathrm{CH}^p_{\mathrm{fin}}(\mathcal{X}) \oplus \mathrm{CH}^p_{Y_F}(\mathcal{X}_F). \tag{1}$$

(Observe $Z^p_{Y_F}(\mathcal{X}_F) = \mathrm{CH}^p_{Y_F}(\mathcal{X}_F)$.)

Convention

Starting from this slice: all arithmetic varieties are assumed to be *regular*, i.e. the $\mathcal{X}$ is a regular scheme.

Let's define the pairing now!

The intersection pairing

Let $[(Y, g_Y)] \in \widehat{\mathrm{CH}}^p(\mathcal{X})$, $[(Z, g_Z)] \in \widehat{\mathrm{CH}}^q(\mathcal{X})$.

- May assume Y, Z are irreducible.
- Assume for a moment that Y, Z intersect properly on the generic fiber X, i.e.

$$\mathrm{codim}_X(Y_F \cap Z_F) = p + q$$

(so $Y \cap Z \cap \mathcal{X}^{(p)} = \emptyset$). Then there is a well-define pairing

$$[Y].[Z] \in \mathrm{CH}^{p+q}_{Y \cap Z}(\mathcal{X}).$$

Denote the image of $[Y].[Z]$ under

$$\mathrm{CH}^{p+q}_{Y\cap Z}(\mathcal{X})_{\mathbb{Q}} \to \mathrm{CH}^{p+q}_{\mathrm{fin}}(\mathcal{X})_{\mathbb{Q}} \oplus Z^{p+q}_{Y_F\cap Z_F}(\mathcal{X}_F)_{\mathbb{Q}}$$

(cf. the map (1)) also by $[Y].[Z]$.

Finally...

We put

$$[(Y, g_Y)].[(Z, g_Z)] := [([Y].[Z], g_Y * g_Z)]$$

which is an element in

$$\frac{\mathrm{CH}^{p+q}_{\mathrm{fin}}(\mathcal{X})_{\mathbb{Q}} \oplus Z^{p+q}_{Y_F \cap Z_F}(\mathcal{X}_F)_{\mathbb{Q}}}{\langle \widehat{\mathrm{div}}(f) \rangle} \cong \widehat{\mathrm{CH}}^{p+q}(\mathcal{X})_{\mathbb{Q}}.$$

What if Y, Z do not intersect properly on the generic fiber X?

Moving Lemma

First, the classical moving lemma [Roberts, 1972] tells us that we can find $f_y \in \kappa(y)^\times$ with $y \in X^{(p-1)}$ such that $Y + \sum_y \mathrm{div}(f_y)$ and Z intersect properly on the generic fiber X.

If f'_y are another choice, we want to show that

$$\sum_y \widehat{\mathrm{div}}(f_y - f'_y).(Z, g_Z) \in \hat{R}^{p+q}(\mathcal{X})_{\mathbb{Q}}.$$

K_1-chains

For a scheme X, we let

$$\mathrm{R}_p^i(X) := \bigoplus_{x \in X^{(i)}} \mathrm{K}_{p-i}(\kappa(x))$$

(algebraic K-groups).
Observe

$$\mathrm{R}_p^p(X) = \bigoplus_{x \in X^{(p)}} \mathrm{K}_0(\kappa(x)) = Z^p(X);$$

$$\mathrm{R}_p^{p-1}(X) = \bigoplus_{x \in X^{(p-1)}} \mathrm{K}_1(\kappa(x)) = \bigoplus_{x \in X^{(p-1)}} \kappa(x)^\times.$$

Definition

A K_1-chain on X is an element in $\mathrm{R}_p^{p-1}(X)$.

Proposition

Let $d : \mathrm{R}_p^i(X) \to R_p^{i+1}(X)$ *be the (boundary) maps. Then*

1. $(R_p^{p-1}(X) \xrightarrow{d} R_p^p(X)) = \mathrm{div}.$
2. $(\mathrm{R}_p^{p-2}(X) \to \mathrm{R}_p^{p-1}(X) \to \mathrm{R}_p^p(X)) = 0.$

Back to the arithmetic case. Using the proposition on the last slice, we want to solve the following problem. Let f be a K_1-chain on $\mathcal{X}$. Construct a K_1-chain, denoted $f.Z$, such that

$$\widehat{\mathrm{div}}(f.Z) = \widehat{\mathrm{div}}(f).(Z, g_Z) \quad \text{in } \hat{R}^{p+q}(\mathcal{X})_{\mathbb{Q}}.$$

(suffices to find one modulo $\mathrm{R}_p^{p-2}(\mathcal{X}) \to \mathrm{R}_p^{p-1}(\mathcal{X})$)

Solution: [Gubler, 2002].

With this solution we are done with the construction of the intersection pairing!

Theorem

The above intersection pairing makes $\bigoplus_{p\geq 0} \widehat{\mathrm{CH}}^p(\mathcal{X})_{\mathbb{Q}}$ into a commutative graded ring.

Applying the Algebraic Intersection Theory, we only need to check the corresponding properties for Green currents...

When does the pairing assume values in $\widehat{\mathrm{CH}}^{\bullet}(\mathcal{X})$?

(without $\otimes\mathbb{Q}$)

... When $p = 1$ (or $q = 1$), because we have a moving lemma on $\mathcal{X}$: if $[Y]$ is represented by a divisor, then we can find a divisor $Y' \sim Y$ such that $|Y'|$ avoid finitely many codim. q points (corresponding to prime cycles in Z).

Indeed, the divisor Y is locally principal by the following fact from Commutative Algebra:

- A Noetherian integral domain is a UFD iff. every height 1 prime ideal is principal.
- Every regular (semilocal) ring is a UFD.

On the analytic site, we have

$$\widehat{\mathrm{div}}(f).(Z, g_Z) = \widehat{\mathrm{div}}(f|_Z)$$

for every rational function f.

Another situation where the pairing happens in $\widehat{\mathrm{CH}}^{\bullet}(\mathcal{X})$ is
- $\mathcal{X}$ is smooth over $\mathcal{O}_F$ and
- $X = \mathcal{X}_F$ is projective over F.

Because we can use Fulton's approach [Fulton, §20], [GS90, §4.5] to define the pairing.

Pullback

Let $f : \mathcal{X}' \to \mathcal{X}$ be a morphism of arithmetic varieties over $\mathcal{O}_F$ ($\mathbb{Z}$-morphism). Then there is a functorial pullback map

$$f^* : \widehat{\mathrm{CH}}^p(\mathcal{X}) \to \widehat{\mathrm{CH}}^p(\mathcal{X}')_{\mathbb{Q}}$$

(for every p) defined as follows.
Let $[(Z, g_Z)] \in \widehat{\mathrm{CH}}^p(\mathcal{X})$. As usual we may assume Z is irreducible.
Assume for a moment that

$$\mathrm{codim}_{X'}(f^{-1}(Z)_F) = p.$$

Define $f^*[Z] \in \mathrm{CH}^p_{f^{-1}(Z)}(\mathcal{X}')_{\mathbb{Q}}$ by K-theory.
Denote the image of $f^*[Z]$ under

$$\mathrm{CH}^p_{f^{-1}(Z)}(\mathcal{X}')_{\mathbb{Q}} \to \mathrm{CH}^p_{\mathrm{fin}}(\mathcal{X}')_{\mathbb{Q}} \oplus Z^P_{f^{-1}(Z)_F}(X')_{\mathbb{Q}}$$

(cf. (1)) also by $f^*[Z]$.
Then we set

$$f^*[(Z, g_Z)] := [(f^*[Z], f^* g_Z)] \in \widehat{\mathrm{CH}}^p(\mathcal{X}')_{\mathbb{Q}}$$

(need to make sense of $f^* g_Z$).

- We can resolve the assumption $\mathrm{codim}_{X'}(f^{-1}(Z)_F) = p$. by the classical moving lemma.
- The definition is independent of the representative/compatible with the rational equivalence ([GS90, 4.4.2]).

Remark

If $f; \mathcal{X}' \to \mathcal{X}$ is flat and $f_F : X' \to X$ is smooth, then we have a pullback map

$$f^* : \widehat{\mathrm{CH}}^p(\mathcal{X}) \to \widehat{\mathrm{CH}}^p(\mathcal{X}')$$

which is "easier" to describe ([Fulton, §1.7]) and induces the above pullback. In this case we don't need regularity of $\mathcal{X}$ and $\mathcal{X}'$.

Pushforward

Let $f : \mathcal{X} \to \mathcal{X}'$ be a morphism of arithmetic varieties over $\mathcal{O}_F$ ($\mathbb{Z}$-morphism). Assume
- $\mathcal{X}$ and $\mathcal{X}'$ are equidimensional, $r := \dim \mathcal{X} - \dim \mathcal{X}'$.
- f is proper (e.g. if both $\mathcal{X}$ and $\mathcal{X}'$ are projective over $\mathcal{O}_F$).
- f_F is smooth.

Then we have a functorial pushforward map

$$f_* : \widehat{\mathrm{CH}}^p(\mathcal{X}) \to \widehat{\mathrm{CH}}^{p-r}(\mathcal{X}')$$

defined as follows. (In this case we don't need regularity of $\mathcal{X}$ and $\mathcal{X}'$.)

Let $[(Z, g_Z)] \in \widehat{\mathrm{CH}}^p(\mathcal{X})$, on cycle classes, we have

$$f_*[Z] = \begin{cases} [\kappa(Z) : \kappa(f(Z))] \cdot [f(Z)], & \text{if } f \text{ is finite} \\ 0, & \text{otherwise} \end{cases}.$$

For the analytic component, can check

$$f_* \delta_{Z(\mathbb{C})} = \delta_{f_* Z(\mathbb{C})} = \begin{cases} \deg(f) \delta_{f(Z(\mathbb{C}))}, & \text{if } f \text{ is finite} \\ 0, & \text{otherwise} \end{cases}.$$

Moreover, we have an equality of current modulo $\partial + \mathrm{im}(\overline{\partial})$

$$dd^c g_Z + \delta_{f_* Z} = [f_* \omega_Z]$$

($[\omega]$ current associated to a smooth differential form ω; possible to pushforward forms because $f_{\mathbb{C}}$ is a submersion).

Finally, one can show that

$$f_* \widehat{\mathrm{div}}(h) = \widehat{\mathrm{div}}(\mathrm{Norm}_{\kappa(W)/\kappa(f(W))}(h))$$

for rational function $h \in \kappa(W)^\times$, $W \subset \mathcal{X}$ codimension $p-1$ integral closed subscheme.

We thus obtain a well-defined cycle class

$$f_*[(Z, g_Z)] := [(f_*[Z], f_* g_Z)] \in \widehat{\mathrm{CH}}^{p-r}(\mathcal{X}').$$

Proposition

Let $f : \mathcal{X} \to \mathcal{X}'$ be a morphism of arithmetic varieties. With the assumptions on f in defining f^* and f_*, we have a projection formula

$$f_*(f^*\alpha.\beta) = \alpha.f_*\beta \in \widehat{\mathrm{CH}}^{p+q-r}(\mathcal{X}')_{\mathbb{Q}}$$

for $\alpha \in \widehat{\mathrm{CH}}^p(\mathcal{X}')$, $\beta \in \widehat{\mathrm{CH}}^q(\mathcal{X})$, where $r := \dim \mathcal{X} - \dim \mathcal{X}'$.

Proposition

We have the expected functoriality:

- $(g \circ f)^* = f^* g^*$.
- $(g \circ f)_* = g_* f_*$.

(whenever these expressions make sense)

Proposition

The pullback f^* is a ring homomorphism.

$\widehat{\mathrm{CH}}^1(\mathcal{X})$

Assume that $\mathcal{X}$ is projective over $\mathcal{O}_F$ (and $\mathcal{X}$ is regular). Let $\widehat{\mathrm{Pic}}(\mathcal{X})$ be the abelian group of *isometric* classes of hermitian metrized line bundles on $\mathcal{X}$ under tensor product $\otimes$.

Proposition

Then we have an isomorphism

$$\widehat{\mathrm{Pic}}(\mathcal{X}) \to \widehat{\mathrm{CH}}^1(\mathcal{X}),\ \overline{L} := (L, ||\ ||) \mapsto (\mathrm{div}(s), [-\log ||s||^2]) =: \hat{c}_1(\overline{L}),$$

where s is a nonzero rational section of L.

Proof.

The map is well-defined as two meromorphic sections differ by a rational function. Note we have an algebraic isomorphism

$$\mathrm{Pic}(\mathcal{X}) \to \mathrm{CH}^1(\mathcal{X}), L \mapsto [\mathrm{div}(s)],$$

the inverse is given by $[D] \mapsto \mathcal{O}_X(D)$. This suggests the following definition for the analytic inverse:

We map $[(Z, g_Z)] \in \widehat{\mathrm{CH}}^1(\mathcal{X})$ to the class of $(\mathcal{O}_X(Z), || \, ||)$, where the metric $|| \, ||$ is determined by the formula

$$||f||^2 = |f|^2 \exp(-g_Z), \quad f \text{ rational function.}$$

It is a smooth metric because g_Z is a Green current: $g_Z - \log |f|^2$ is smooth. (Note that g_Z is a smooth 0-form, i.e. smooth function on $\mathcal{X}(\mathbb{C}) \setminus Z(\mathbb{C})$.) □

Remark

Assume that the arithmetic variety $\pi : \mathcal{X} \to \operatorname{Spec}\mathbb{Z}$ is projective of relative dimension n. There is a degree map

$$\widehat{\deg} := \pi_* : \widehat{\mathrm{CH}}^{n+1}(\mathcal{X}) \to \widehat{\mathrm{CH}}^{1}(\operatorname{Spec}\mathbb{Z}) \cong \mathbb{R}$$

This is related to the height function in Diophantine Geometry by considering the degree of the (arithmetic) intersection of metrized line bundles on $\mathcal{X}$ using the isomorphism $\hat{c}_1$. See [Moriwaki, §9], [Sou+,§3.6], and more profoundly [GS90, §4.3].

Key exact sequences

Let $\mathcal{X}$ be an arithmetic variety (not necessarily regular). For every $p \geq 0$ there are exact sequences of abelian groups [Sou+, §3.1], [GS90, §3.3.5]

$$\mathrm{CH}^{p-1,p}(\mathcal{X}) \to \tilde{A}^{p-1,p-1}(\mathcal{X}(\mathbb{C})) \to \widehat{\mathrm{CH}}^p(\mathcal{X}) \to \mathrm{CH}^p(\mathcal{X}) \to 0$$

and

$$\mathrm{CH}^{p-1,p}(\mathcal{X}) \to H^{p-1,p-1}(\mathcal{X}(\mathbb{C})) \to \widehat{\mathrm{CH}}^p(\mathcal{X}) \to$$
$$\to \mathrm{CH}^p(\mathcal{X}) \oplus Z^{p,p}(\mathcal{X}(\mathbb{C})) \to H^{p,p}(\mathcal{X}(\mathbb{C})) \to 0,$$

where $Z^{p,p}(\mathcal{X}(\mathbb{C})) \subset A^{p,p}(\mathcal{X}(\mathbb{C}))$ is the subspace of closed forms. The group $\mathrm{CH}^{p-1,p}(\mathcal{X})$ is closely related to **Beilinson regulators**, see [GS90, §3.5].

References

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Thank you!