—INTRODUCTION—

The topic of this learning group is what we have decided to call derived arithmetic intersection theory (DAIT). DAIT is a relatively new idea, with several of its foundational components not yet known. Our aim is to get introduced to DAIT, explore the motivations behind it, and hopefully work toward resolving some of the gaps in our understanding. Some of our chief goals are thus the following.

- Understand key tools in derived intersection theory, especially virtual fundamental classes.
- Grapple with combining derived methods and arithmetic intersection theory.
- Explore arithmetic and geometric applications.

—OVERVIEW AND CONTEXT—

The first part of this seminar will focus on the recently developed technique of derived deformation to the normal cone and its role in the construction of virtual fundamental classes. The quasi-smooth case of this deformation technique was developed in [Kha3], building off of ideas in [KR] on virtual Cartier divisors and blow-ups, while the more general approach via Weil restriction is currently being developed in [HKR]. Our treatment of virtual fundamental classes will largely follow [Kha2], with some classical motivation coming from [KP]. We hope that the use of K-theory and G-theory, rather than more general Borel-Moore homology theories as in [Kha3], will make things more accessible. We cap off the first part with a discussion of *derived* Grothendieck-Riemann-Roch.

The second part of this seminar will focus on arithmetic intersection theory (AIT), of which we assume the audience knows nothing. AIT has its roots in the work of Arakelov, Gillet, Soulé, and others, with good references [GS2] and [Sou+]. The main goal of AIT is to establish an analog of algebraic intersection theory for appropriately defined *arithmetic* schemes X, which we compactify using complex analytic techniques on $X(\mathbb{C})$. The classical theory uses arithmetic cycles, which are the data of algebraic cycles together with Green currents, to build *arithmetic* Chow groups. K-theoretic techniques are also used as a replacement for tools like the Moving Lemma. If time allows, we will discuss some formulations of *arithmetic* Grothendieck-Riemann-Roch and higher AIT as developed in [GRR].

The third part of this seminar will focus on horizons. Virtual fundamental classes have already found arithmetic application in [FYZ2], and [Kha2, §6.5] offers some speculation on building a better Chow cohomology for singular schemes that incorporates K-theoretic techniques. AIT has many applications to arithmetic geometry, surveyed wonderfully in [PR], one of which is the study of Shimura varieties. The work of [GS1], which features in the Kudla program, gives a systematic treatment of currents that may admit a powerful derived analog. Recent work of Toën and Vezzosi ([Toë], [TV1], [TV2]) may prove instrumental in building a theory of *derived* currents, considering that it gives us access to derived foliations. Finally, the motivic Arakelov cohomology of [HS] and [Sch] could become an essential feature of our understanding of AIT and thus DAIT. Which of these we choose to address, as well as when the seminar ends, will depend on audience interest.

—CONTACT AND LOGISTICAL INFO—

We meet Tuesdays 15:15-17:15 UTC via Zoom. The organizers are Jeroen Hekking and Zachary Gardner.

- Zoom: https://bccte.zoom.us/j/92868180493
- Discord: https://discord.gg/wFfPFFbrtm
- Jeroen Email: hekking@kth.se
- Zachary Email: zachary.gardner@bc.edu

The following are some additional resources that people might find valuable.

- Adeel Khan Minicourses: https://www.preschema.com/lect.html
- Algebraic K-Theory and DAG: https://ktheoryanddag.wordpress.com/
- DAG Learning Group: https://dag-learning.gitlab.io/

—SCHEDULE—

1. Feb 1 – Review and derived Artin stacks
2. Feb 8 – Deformation to the normal cone
    a. Normal cone
    b. Weil restriction
3. Feb 15 – Introduction to (abstract) K-theory
    a. Waldhausen S_* construction
    b. Bass construction
    c. Localization property
    d. Universal property
    e. Connection to Grothendieck groups and Quillen's construction
4. Mar 1 – Introduction to the K-theory and G-theory of derived stacks
    a. Perfect and coherent complexes
    b. Resolution properties
    c. Perfect stacks
    d. Cartan map
5. Mar 29 – Key operations and examples in K-theory and G-theory
    a. Inverse image
    b. Gysin map
    c. Localization
    d. Excision
    e. Coniveau and γ-filtrations
6. Apr 5 – Virtual Cartier divisors and derived blow-ups
    a. Quasi-smoothness and virtual codimension
    b. Virtual Cartier divisors
    c. Derived blow-ups
    d. Projective bundles
7. Apr 12 – K-theory and G-theory of projective bundles
    a. Semi-orthogonal decompositions
    b. Projective bundle and blow-up formulas
    c. Homotopy invariance

8. Apr 26 – Virtual intersection theories
    a. Specialization homomorphisms
    b. Refined Gysin pullbacks
    c. Intersection products
9. May 3 - Derived virtual fundamental classes
    a. Virtual fundamental classes
10. May 10 – Excess intersection and Grothendieck-Riemann-Roch
    a. Excess intersection formula
    b. Grothendieck-Riemann-Roch formula
11. May 17 – Introduction to arithmetic intersection theory (AIT)
    a. Arithmetic cycles and arithmetic Chow groups
    b. Intersection pairing on arithmetic Chow groups
    c. Pullback, pushforward, and their properties
    d. The Beilinson regulator and key exact sequences (more on this later…)
    e. Poincaré-Lelong formula
    f. Arithmetic Picard groups
12. May 24 – The analytic side of AIT: Green currents and *-products
    a. Green currents
    b. Green forms of logarithmic type
    c. Definition and properties of *-products
    d. *-products for proper intersections
13. Jun 7 – Recap on AIT, Green currents, and *-products
14. Arithmetic characteristic classes and Quillen's approach
    a. Chern forms
    b. Superconnections
    c. Bott-Chern classes
    d. Arithmetic Chern characters
    e. Explicit arithmetic cycles
15. Toward derived currents: arithmetic deformation to the normal bundle
    a. Hu's approach and Howard's application
16. Toward derived currents: derived analytic geometry
    a. Derived complex manifolds and complex analytic spaces
    b. Quasi-smooth derived analytic subschemes
    c. Derived analytic de Rham complexes
    d. Analytic cotangent complexes and deformation theory
    e. "Formal" logarithmic (and log-log) expansions

Topics still to be scheduled:

- Deligne, Deligne-Beilinson, and absolute Hodge cohomology theories
    - Real and complex versions
- Higher arithmetic Chow groups
    - Higher Chow groups of Bloch and Kresch
    - Motivic approach of Khan to algebraic case

    - Comparison of arithmetic approaches of Gillet-Soulé, Goncharov, Burgos Gil-Feliu, and Burgos Gil-Goswami
- Arakelov motivic cohomology
- Arithmetic K-theory
    - Takeda's construction
    - Thoughts on emulating the algebraic K-theory construction of Bass-Thomason-Trobaugh
    - Inputs from Holmstrom and Scholbach
- Arithmetic Grothendieck-Riemann-Roch theorem

—CONTENT—

Click hyperlinks to access talk notes.

—Lecture 1—

Topic: Review and derived Artin stacks

Speaker: Zachary Gardner

Required Reading: [Kha2, §1.1-3]

Supplemental Reading: Too many to list…

—Lecture 2—

Topic: Deformation to the normal cone (annotated, unannotated)

Speaker: Jeroen Hekking

Required Reading: [HKR]

Supplemental Reading:

—Lecture 3—

Topic: Introduction to (abstract) K-theory

Speaker: Dirk van Bree

Required Reading:

Supplemental Reading: [BGT]

—Lecture 4—

Topic: Introduction to the K-theory and G-theory of derived stacks

Speaker: N/A

Required Reading: [Kha2, §1.2&7-9, §2.1, §3.1]

Supplemental Reading:

—Lecture 5—

Topic: Key operations and examples in K-theory and G-theory

Speaker: Jeroen Hekking

Required Reading: [Kha2, §2.2&4-5, §3.2-3]

Supplemental Reading: [Kha2, §1.4-5], [Kha4, lecture 9], [Lan]

—Lecture 6—

Topic: Virtual Cartier divisors and derived blow-ups
Speaker: Jeroen Hekking
Required Reading: [KR, §2, §3.1, §4]
Supplemental Reading: [Vak, §22]

—Lecture 7—
Topic: K-theory and G-theory of projective bundles (annotated, unannotated)
Speaker: Jeroen Hekking
Required Reading: [Kha §2.2, §3-4], [Kha2, §2.3, §3.4-5]
Supplemental Reading: [Kha4, lectures 7-8]

—Lecture 8—
Topic: Virtual intersection theories (recording)
Speaker: Zachary Gardner
Required Reading: [Kha2, §6.1], [KP, §1-2]
Supplemental Reading: [Ful]

—Lecture 9—
Topic: Derived virtual fundamental classes
Speaker: Niklas Kipp
Required Reading: [Kha2, §6.2]
Supplemental Reading: [Kha3]

—Lecture 10—
Topic: Excess intersection and Grothendieck-Riemann-Roch
Speaker: William Dudarov
Required Reading: [Kha2, §6.3-5], [Kha4, lecture 10], [Kha5]
Supplemental Reading:

—Lecture 11—
Topic: Introduction to arithmetic intersection theory (AIT)
Speaker: Zhelun Chen
Required Reading: [GS2, Introduction, §3-4], [Sou+, §II.1, §III.1-4],
Supplemental Reading: [Sou+, §0]

—Lecture 12—
Topic: The analytic side of AIT: Green currents and *-products
Speaker: Zhelun Chen
Required Reading: [GS2, §1-2], [Sou+, §II.2-3]
Supplemental Reading:

—Lecture 13—
Topic: Recap on AIT, Green currents, and *-products
Speaker: Zachary Gardner

Required Reading:
Supplemental Reading:

—Lecture 13—
Topic: Arithmetic characteristic classes and Quillen's approach
Speaker: ???
Required Reading: [Qui], [Sou+, §IV, §VII.1-3]
Supplemental Reading:

—Lecture 14—
Topic: Toward derived currents: arithmetic deformation to the normal bundle
Speaker: ???
Required Reading: [GS1], [How], [Hu]
Supplemental Reading:

—Lecture 15—
Topic: Toward derived currents: derived analytic geometry
Speaker: ???
Required Reading:
Supplemental Reading:

—REFERENCES—

- [BGG] Burgos Gil and Goswami - Higher Arithmetic Intersection Theory
- [BGT] Blumberg, Gepner, and Tabuada - A universal characterization of higher algebraic K-theory
- [Fal] Faltings - Lectures on the arithmetic Riemann-Roch theorem
- [Ful] Fulton - Intersection Theory
- [FYZ1] Feng, Yun, and Zhang - Higher Siegel-Weil formula for unitary groups: the non-singular terms
- [FYZ2] Feng, Yun, and Zhang - Higher theta series for unitary groups over function fields
- [GS1] Garcia and Sankaran - Green Forms and the Arithmetic Siegel-Weil Formula
- [GS2] Gillet and Soulé - Arithmetic intersection theory
- [HKR] Hekking, Khan, and Rydh - Deformation to the normal cone and blow-ups via derived Weil restrictions (*in preparation*)
- [How] Howard - Pullback formulas for arithmetic cycles (*in preparation*)
- [HS] Holmstrom and Scholbach - Arakelov motivic cohomology I
- [Hu] Hu - Deformation to the normal bundle in arithmetic geometry
- [Kha] Khan - Algebraic K-theory of quasi-smooth blow-ups and cdh descent
- [Kha1] Khan - Cohomological intersection theory and derived algebraic geometry
- [Kha2] Khan - K-theory and G-theory of derived algebraic stacks
- [Kha3] Khan - Virtual fundamental classes of derived stacks I
- [Kha4] Khan - Grothendieck-Riemann-Roch theorem
- [Kha5] Khan - Virtual excess intersection theory

- [KP] Kiem and Park - Virtual intersection theories
- [KR] Khan and Rydh - Virtual Cartier divisors and blow-ups
- [Lan] Landesman - Some Basics of Algebraic K-Theory
- [Lur] Lurie - Derived Algebraic Geometry IX: Closed Immersions
- [Por] Porta - The derived Riemann-Hilbert correspondence
- [PR] Peyre and Rémond - Arakelov Geometry and Diophantine Applications
- [Qui] Quillen - Superconnections and the Chern Character
- [Sch] Scholbach - Arakelov motivic cohomology II
- [Sou+] Soulé, et al. - Lectures on Arakelov Geometry
- [Tan] Tang - Localization theorem for higher arithmetic K-theory
- [Toë] Toën - Classes caractéristiques des schémas feuilletés
- [TV1] Toën and Vezzosi - Algebraic foliations and derived geometry: the Riemann-Hilbert correspondence
- [TV2] Toën and Vezzosi - Algebraic foliations and derived geometry II: the Grothendieck-Riemann-Roch theorem
- [Vak] Vakil - The Rising Sea: Foundations of Algebraic Geometry