I. Classical Siegel-Weil

(V, Q) quadratic space / $\mathbb{Q}$, $L \subseteq V$ $\mathbb{Z}$-lattice w/ $Q(L) \subseteq \mathbb{Z}$, $x = (x_1, \dots, x_d) \in V^d \rightsquigarrow Q(x) := \left(\frac{[x_i, x_j]}{2} \right) \in \mathrm{Sym}_d(\mathbb{Q})$

$\swarrow$ $[\cdot,\cdot]$ assoc. pairing pos. def.

Theta series $\Theta_L(\tau) = \sum_{T \in \mathrm{Sym}_d(\mathbb{Q})} r_L(T) q^T$, $q^T = e^{2\pi i \operatorname{tr}(\tau T)}$, $r_L(T) = \# \{ x \in L^d : Q(x) = T \}$.

$\swarrow$ (Siegel modular form of weight $\frac{\dim(V)}{2}$)

$\uparrow$ $\tau \in$ Siegel half space $\mathcal{H}_d$

<u>Thm</u> (Siegel-Weil): $\sum_{L' \in \mathrm{Gen}(L)} \frac{\Theta_{L'}(\tau)}{\# \mathrm{Aut}(L')}$ = some Eisenstein series...

$\uparrow$

$\{ L' \subseteq V$ everywhere locally isom. to $L \} / O(V)$

II. Unitary Groups

K imag. quad. field, (V, h) n-dim Hermitian space / K, $H = U(V)$, $G = U(n,n)$

(really only need something unitary and quasi-split)

We have Weil representation $\omega : G(\mathbb{A}) \times H(\mathbb{A}) \to \mathrm{Aut}(S(V_{\mathbb{A}}^n))$

$\swarrow$ (Schwartz space)

- Obvious action of $H(\mathbb{A})$
- Non-obvious action of $G(\mathbb{A})$

$\uparrow$ (requires some choice, a certain sort of a certain character)

$\varphi \in S(V_{\mathbb{A}}^n) \rightsquigarrow$ theta kernel $\Theta(g, h, \varphi) := \sum_{x \in V^n} (\omega(g,h)\varphi)(x)$. This defines automorphic form on $G(\mathbb{A}) \times H(\mathbb{A})$.

How is this related to classical theta as above? We've basically "adelized" the picture.

<u>Thm</u> (Siegel-Weil): $\int_{H(\mathbb{Q}) \backslash H(\mathbb{A})} \Theta(g, h, \varphi)\, dh = \underbrace{E(g, 0, \varphi)}_{\text{explicit Eisenstein series}}$.

$\varphi \in S(V_{\mathbb{A}}^n)$, $g \in G(\mathbb{A}) \subseteq GL_{2n}(\mathbb{A} \otimes K)$ has Iwasawa decomposition $\begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & 0 \\ 0 & {}^t\bar{a}^{-1} \end{pmatrix} k$.

$\Downarrow$ maximal compact

$\Phi(g, s) := (\omega(g,1)\varphi)(0) \, |\det a|^s$. We have parabolic $P = \left\{ \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \right\} \subseteq G$.

$E(g, s, \varphi) := \sum_{\gamma \in P(\mathbb{Q}) \backslash G(\mathbb{Q})} \Phi(\gamma g, s)$. $\leftarrow$ Not convergent at $s = 0$! But there is analytic continuation...

III. Central Derivative

Hermitian space V of signature $(n-1,1) \rightsquigarrow$ "incoherent" Hermitian space $\mathbb{V} := \prod_{p \le \infty} \mathbb{V}_p$ over $\mathbb{A} \otimes k$ w/

$\mathbb{V}_p := V_p$ for $p < \infty$ and $\mathbb{V}_\infty$ of signature $(n,0)$. We have Weil rep. of $G(\mathbb{A})$ on $S(\mathbb{V}^n)$.

Assume $\exists$ self-dual $\mathcal{O}_k$-lattice $L \subseteq V$. Let $\varphi = \varphi_f \varphi_\infty$.

$\varphi_f :=$ char. function of $\hat{L} \subseteq \prod_{p<\infty} \mathbb{V}_p$; $\varphi_\infty := (x \mapsto e^{-2\pi \operatorname{tr}(h(x,x))})$ for $x \in \mathbb{V}_\infty^n$.

(Fourier expansion)

$E'(g,0,\varphi)$ "de-adelizes" to (nonholomorphic) Hermitian modular form. $E'(\tau,0,\varphi) = \sum_{T \in \operatorname{Herm}_n(k)} a(T,v) q^T$.

This is defined on $\mathcal{H}_n := \{\tau = u + iv \in M_n(\mathbb{C}) : u,v \text{ Hermitian w/ } v > 0\}$.

[Note: U and GU Shimura varieties play different roles, confusingly]

Fact: $T > 0 \Rightarrow a(T,v)$ independent of v (hence $a(T)$).

Idea of Kudla-Rapoport: On integral model $\mathcal{M} \to \operatorname{Spec} \mathcal{O}_k$ (n-dim) of $H = U(V)$-Shimura variety there should be

"special" 0-cycles $Z(T) \to \mathcal{M}$ (for $T > 0$) s.t. $\deg Z(T) = a(T)$. [more generally result involving arithmetic Chow gps.]

$\mathcal{M}$ is not of PEL type. Let $\mathcal{M}_{(1,0)} \to \operatorname{Spec}\mathcal{O}_k$ moduli space of elliptic curves $E \to S$ ($\mathcal{O}_k$-scheme) w/ $\mathcal{O}_k \to \operatorname{End}(E)$ s.t.

$\mathcal{O}_k \to \operatorname{End}(\operatorname{Lie}(E)) \underset{(\text{locally})}{\cong} \mathcal{O}_S$ via structure map. Let $\mathcal{M}_{(n-1,1)} \to \operatorname{Spec}\mathcal{O}_k$ moduli space of abelian n-folds

$A \to S$ w/ appropriate rigidifying data. The $\mathcal{M}$ we want should be a quotient of finitely many conn. components of

$\mathcal{M}_{(1,0)} \times \mathcal{M}_{(n-1,1)}$. Just take $\mathcal{M} \equiv \mathcal{M}_{(1,0)} \times \mathcal{M}_{(n-1,1)}$ for simplicity. Let's define $Z(T)$.

$(E,A) \in \mathcal{M}$ w/ $x,y \in \operatorname{Hom}_{\mathcal{O}_k}(E,A) \rightsquigarrow$ pos. def. Hermitian form $E \xrightarrow{x} A \cong A^\vee \xrightarrow{y^\vee} E^\vee = E$, $h(x,y) \in \operatorname{End}_{\mathcal{O}_k}(E) = \mathcal{O}_k$

(FYZ give us function field analogue of this)

$Z(T)(S) \to \mathcal{M}(S)$ given by (E,A,x) w/ $h(x,x) = T$. This wants to be dim 0. Kudla-Rapoport tell us when

the dim is 0, and show $\deg Z(T) = a(T)$ in such case.

Note: $Z(T)(S)$ is either $\emptyset$ or supported in only one nonzero char., always as long as $T > 0$.