__Motivation__ $q = e^{2\pi i z}$

$$\theta(z) := \sum_{n \geq 0} \#\{n : n = a^2+b^2+c^2+d^2\} q^n = \sum_{\substack{\lambda \in (\mathbb{Z}^4, \langle \cdot, \cdot \rangle) \\ \lambda = (a,b,c,d)}} q^{\langle \lambda, \lambda \rangle}$$

"orthogonal variant"

__Key Properties__

(1) modular form (proved by Poisson summation)

(2) Siegel-Weil formula = explicit Eisenstein series

In this case: $\#\{n : n = a^2+b^2+c^2+d^2\} = 8 \sum_{\substack{d \mid n \\ 4 \nmid d}} d$.

__Function Fields__ (char. $\neq 2$)

X smooth proj. geom. conn. $/\mathbb{F}_q$. We want "unitary variant". $\iota : X' \to X$ finite étale double cover.

$\mathcal{F} \in \mathrm{Vec}(X')$ w/ $h : \mathcal{F} \xrightarrow{\sim} \sigma^* \mathcal{F}^\vee$ s.t. $\sigma^* h^\vee = h$. $\mathcal{F}^\vee := \mathrm{Hom}(\mathcal{F}, \omega_{X'})$

"$\theta \sim$ count global sections of $\mathcal{F}$" (of which there are not many).

Consider $\mathcal{L} \in \mathrm{Pic}(X')$. Consider $\mathrm{Hom}(\mathcal{L}, \mathcal{F}) \ni t$. $\rightsquigarrow$ $\mathcal{L} \xrightarrow{t} \mathcal{F} \xrightarrow{h} \sigma^* \mathcal{F}^\vee \xrightarrow{\sigma^* t^\vee} \sigma^* \mathcal{L}^\vee$, with composite $T : \mathcal{L} \to \sigma^*\mathcal{L}^\vee$.

$T \in \mathrm{Hom}_H(\mathcal{L}, \sigma^* \mathcal{L}^\vee) \cong \mathrm{Ext}^1_H(\sigma^* \mathcal{L}^\vee, \mathcal{L})$ by Serre duality

This should give $\sum_{t \in \mathrm{Hom}(\mathcal{L}, \mathcal{F})}$ "q^T". What should this be? Fix additive char. $\psi : \mathbb{F}_q \to \mathbb{C}^\times$.

($H \sim$ "hermitian condition")

Function on $\mathrm{Ext}^1_H(\sigma^* \mathcal{L}^\vee, \mathcal{L}) \ni [\mathcal{M}]$.

$= \{\underset{\text{Lagrangian}}{\mathcal{L}} \to \underset{\text{hermitian structure}}{\mathcal{M}} \to \sigma^* \mathcal{L}^\vee\}$

__Key Properties__

(1) Modularity: independent of $\mathcal{L}$ (only depends on $\mathcal{M}$)

$$\theta(\mathcal{L} \subseteq \mathcal{M}) = \sum_{t \in \mathrm{Hom}(\mathcal{L}, \mathcal{F})} \psi(\langle T, [\mathcal{M}] \rangle).$$

(2) Some analogue of Siegel-Weil formula (in terms of explicit Eisenstein series)

__Weil's Uniformization__ $\uparrow [\mathrm{Bun}_{U_m}]$

$G(F) \backslash G(\mathbb{A}_F) / G(\hat{\mathcal{O}}_F) \simeq \mathrm{Bun}_G(\mathbb{F}_q)$.

$\{(\mathcal{L} \subseteq \mathcal{M})\} = \mathrm{Bun}_{P_{2m}}(\mathbb{F}_q)$ (w/ $m = 1$)

$\downarrow$

$\mathrm{Bun}_{U(2m)}(\mathbb{F}_q)$.

P_{2m} Siegel parabolic of $U(2m)$

[m comes from vector bundles of higher rank, which we can consider.]

__Remark__: We can prove this using Riemann-Roch, which is secretly the same as Poisson summation.

Want "higher" Θ-functions Θ^r : generating series w/ coeffs. cycle classes $\in CH_\bullet(\mathrm{Sht}^r_{U(n)})$

$\swarrow$ ["special cycles"] $\qquad \downarrow (X')^r. \qquad \uparrow$ (some kind of shtuka)

. coeffs. $[Z^r_{\mathcal{E}}(T)]$ generalizing $\{t : \mathcal{E} \to \mathcal{F}$ lying over $T\}$.

$\swarrow$ (rank $\mathcal{E} = n$)

Conjecture: "higher" modularity and "higher" Siegel-Weil

$$\deg [Z^r_{\mathcal{E}}(T)] \sim \frac{\partial^r}{\partial s^r}\Big|_{s=0} \mathrm{Eis}(s).$$

$r=0$: classical story

$r=1$: # field analogue ~~of~~ on $\mathrm{Sht}^r_{U(n)}$

$r>1$: new!

$\uparrow$ (work of Kudla, Rapoport, Howard, Madapusi Pera, etc.)

Towards $\mathrm{Sht}^r_{U(n)}$

$$\mathrm{Hk}^r_{U(n)} = \text{moduli of } \left\{ \begin{array}{l} x_1', \dots, x_r' \in X' \\ (\mathcal{F}_0, h_0), \dots, (\mathcal{F}_r, h_r) \in \mathrm{Bun}_{U(r)} \\ \text{"modification" } \mathcal{F}_{i-1}|_{X' - x_i' - \sigma x_i'} \xrightarrow{\sim} \mathcal{F}_i|_{X' - x_i' - \sigma x_i'} \end{array} \right\}$$

$\leftarrow$ (don't want absolutely all such isoms. ...)

$\swarrow p_0 \downarrow \dots \searrow p_r$

$(X')^r \quad \mathrm{Bun}_{U(n)} \quad \mathrm{Bun}_{U(n)}$

Remark: Taking rational pts. recovers "old story" of Hecke operators.

$$\begin{tikzcd} \mathrm{Sht}^r_{U(n)} \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & \mathrm{Hk}^r_{U(n)} \arrow[d, "{(p_0, p_r)}"] \\ \mathrm{Bun}_{U(n)} \arrow[r, "{(\mathrm{id}, \mathrm{Frob})}"] & \mathrm{Bun}_{U(n)} \times \mathrm{Bun}_{U(n)} \end{tikzcd}$$

$\downarrow$(projection maps) moduli of $\{\mathcal{F}_0 \overset{x_1'}{\dashrightarrow} \cdots \overset{x_r'}{\dashrightarrow} \mathcal{F}_r \xrightarrow{\sim} \mathrm{Frob}^*\mathcal{F}_0\}$

Example: $(r=0)$ $\{\mathcal{F}_0 \xrightarrow{\sim} \mathrm{Frob}^*\mathcal{F}_0\} \simeq \mathrm{Bun}_{U(n)}(\mathbb{F}_q)$.

Special Cycles

Will define $Z^r_{\mathcal{E}}(T)$. $\mathcal{E}$ rank n vector bundle / X'. $\mathrm{Sht}^r \leftarrow Z^r_{\mathcal{E}} = \left\{ \begin{array}{ccccccccc} \mathcal{E} & = & \mathcal{E} & = & \cdots & = & \mathcal{E} & = & \mathrm{Frob}^*\mathcal{E} \\ {\scriptstyle t_0}\downarrow & & {\scriptstyle t_1}\downarrow & & & & {\scriptstyle t_r}\downarrow & & \downarrow{\scriptstyle \mathrm{Frob}^* t_0} \\ \mathcal{F}_0 & \dashrightarrow & \cdots & & \cdots & \dashrightarrow & \mathcal{F}_r & \xrightarrow{\sim} & \mathrm{Frob}^*\mathcal{F}_0 \end{array} \right\}$

The role of T here is similar to before. We say T is "nonsingular" if it is injective.

• $Z^{r}_{\mathcal{E}}(T)$ has "virtual codim" mr, $m = \mathrm{rank}(\mathcal{E})$: $[Z^r_{\mathcal{E}}(T)] \in CH^{mr}(\mathrm{Sht}^r)$

This is not the actual codim, unfortunately.

• $m = 1$, T nonsingular $\Rightarrow$ $Z^r_{\mathcal{E}}(T)$ is LCI of codim r $\Rightarrow$ $[Z^r_{\mathcal{E}}(T)]$ can be taken as naïve fundamental class.

$$\mathcal{E} = \mathcal{L}_1 \oplus \mathcal{L}_2 \text{ w/ } \mathcal{L}_1, \mathcal{L}_2 \text{ rank } 1 \Rightarrow \underbrace{Z^r_{\mathcal{L}_1}(T_1) \times_{\mathrm{Sht}^r} Z^r_{\mathcal{L}_2}(T_2) = \text{``}\bigcup_{T = \begin{pmatrix} T_1 & * \\ * & T_2 \end{pmatrix}}\text{''}\; Z^r_{\mathcal{E}}(T).}_{}$$

This is evidence that the "higher" θ^r is modular. $\to$ called "linear invariance", after an idea of Ben Howard

Look at top Chern classes of Hodge bundles...