The Cotangent Complex (& quasi-smoothness?)

Talk @ the online DAG-learning seminar

Nov. 30 2021

Recap

$\mathcal{S}$ = Anima: spaces / ∞-groupoids / anima

$\mathrm{SCR} = \mathcal{P}_\Sigma(\mathrm{Poly})$ $\qquad$ $\mathrm{Poly} = \{\text{fin. gen. poly rings} / \mathbb{Z}\}$

For any ∞-cat $\mathcal{C}$ w/ fin. coproducts

$$\mathcal{P}_\Sigma(\mathcal{C}) = \{\mathcal{C}^{op} \to \mathcal{S} : \text{pres. fin. products}\}$$

$$\begin{tikzcd} \mathcal{C} \arrow[r] \arrow[d] & \mathcal{D} \\ \mathcal{P}_\Sigma(\mathcal{C}) \arrow[ur, dashed] & \end{tikzcd}$$

$$R : \mathrm{Poly}^{op} \to \mathcal{S}$$

$$\mathbb{Z}[x] \longmapsto R$$

$$\begin{matrix} x \mapsto u+v \\ \mathbb{Z}[x] \to \mathbb{Z}[u,v] \end{matrix} \rightsquigarrow \left(R \times R \xrightarrow{+} R\right)$$

$$\begin{tikzcd} & \mathrm{Poly}^{op} \arrow[dl, shift right, "S"'] \arrow[dl, shift left, "R"] \arrow[dr, shift right, "\pi_0 R"'] \arrow[dr, shift left, bend left, "S"] & \\ \mathcal{S} \arrow[rr, shift left, "\pi_0"] & & \mathrm{Set} \arrow[ll, shift left, "\iota"] \end{tikzcd}$$

$\mathrm{Mod}_R^{cn} \subseteq \mathrm{Mod}_R$: cat of (connective) R-modules

$$\begin{array}{lll} St & : & \text{stacks } Aff^{op} \to \mathcal{S} \\ \cup\! (& & \\ Sch & : & \text{d. schemes: stacks st } \exists \text{ Zar. open cover of affines} \end{array}$$

$$St \ni X : \quad QCoh(X) = \lim_{Spec A \to X} Mod_A.$$

$$\neq$$

§ Mono's & epi's in ∞-cat

(see e.g. Klas, Rezk SAG)

Def in any ∞-cat $\mathcal{C}$, $i : A \to B$ is <u>mono</u> if $\forall C$:

$$Fib(q) \to \mathcal{C}(C,A) \to \mathcal{C}(C,B) \ni q$$

is empty or contractible. "$C \rightrightarrows A \to B$"

Lem. i is mono iff $\Delta : A \xrightarrow{\simeq} A \times_B A$

Pf If i is mono, look at

$$\begin{tikzcd} A \arrow[dr, "h"] & & \\ & A \times_B A \arrow[r, "p"] \arrow[d, "q"] & A \arrow[d] \\ & A \arrow[r] & B \end{tikzcd}$$

$$\begin{tikzcd} & & A \\ A \times_B A \arrow[r, "p"] \arrow[urr, bend left, "q"] \arrow[rr, bend right, "id"'] & A \arrow[r, "h"] & A \times_B A \arrow[u, "q"] \arrow[d, "p"] \\ & & {} \end{tikzcd}$$

$\rightsquigarrow$ TS $p \simeq q$

$$p \searrow p \qquad \mathcal{C}(A\times_B A, A) \to \mathcal{C}(A\times_B A, B)$$

$$\begin{array}{ccc} \simeq \; p & \longmapsto & fp \; \simeq \\ q & \longmapsto & fq \end{array}$$

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Def $f: X\to Y$ in an ∞-topos is a effective epi if $\quad \operatorname{colim} \check{C}(Y/X) \xrightarrow{\simeq} Y$

Prop Given $U \xrightarrow{f} V$ in an ∞-topos, we have a factorization

$$U \xrightarrow{p} \operatorname{im}(f) \xrightarrow{i} V,$$

- p is an effective epi
- i is mono
- $\operatorname{im}(f) = \operatorname{colim} \check{C}(V/U)$.

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Ex If $U \xrightarrow{f} V$ is in $\mathcal{S}$.

let $I :=$ Subspace of V spanned by

$$\pi_0 f(\pi_0 U) \subseteq \pi_0 V.$$

then $\qquad U \to I \to V \qquad$ is the factorization

- $U\to I$ is sur. on π_0
- $I \to V$ is inj. on π_0 & iso on π_n ($\forall n > 0$)

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§ Cotangent Complex

Square-zero extensions.

Def. For $A \in SCR^{\heartsuit}$, $M \in Mod^{\heartsuit}$, have

$A \oplus M \in SCR^{\heartsuit}$, w/

$$(a,m)\cdot(a',m') := (aa', am'+a'm)$$

"Square zero extension"

$$\begin{tikzcd}
 & (A,M) \arrow[r, mapsto] & A\oplus M & \\
\cdot & PolyMod \arrow[r] \arrow[d] & SCR & R\oplus M \\
 & SCRMod^{cn} = \mathcal{P}_{\Sigma}(PolyMod) \arrow[ur, dashed] & (R,M) \arrow[ur, mapsto] &
\end{tikzcd}$$

PolyMod = (R,M)

$R \in Poly$

M : f.g. free R-module

$\simeq \{(R,M) \mid R \in SCR, M \in Mod_R^{cn}\}$

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Def. $StMod^{cn}$:= cat of pairs (X,F), w/

X : st, $F \in QC(X)^{cn}$

morphisms: $(X,F) \to (X',F')$ is

$$(f: X \to X', \varphi: f^*F' \to F).$$

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For $(X,F) \in StMod^{cn}$, have $\mathcal{O}_X \oplus F \in SCR(X) = \lim_{Spec A \to X} SCR_A$

by: the htpy coherent system

$$\{A \oplus F(A) \mid Spec A \to X\}$$

$\cdot \quad Aff_{/X} \longrightarrow SCRMod^{cn} \longrightarrow SCR$

$(Spec A \to X) \longmapsto (Spec A, F(A)) \longmapsto (A \oplus F(A))$

get: $\mathrm{StMod}^{cn} \to \mathrm{StMod}^{cn}$, have $F \to 0$
$(X,F) \mapsto (X, \mathcal{O}_X \oplus F)$ $\rightsquigarrow \mathcal{O}_X \oplus F \to \mathcal{O}_X$

so: $\mathrm{StMod}^{cn} \longrightarrow \mathrm{Arr}(\mathrm{St})$
$(X,F) \longmapsto (\underbrace{\mathrm{Spec}_X(\mathcal{O}_X \oplus F)}_{\#} \to \mathrm{Spec}_X(\mathcal{O}_X) = X)$

Def For $X \in \mathrm{St}$, let $\mathrm{StMod}_X^{cn} = \{(T,M) \mid {T \to X \atop M \in QC(T)^{\dots}}\}$.

get: $\mathrm{StMod}_X^{cn} \to \mathrm{StMod}_X^{cn} \longrightarrow \mathrm{St}_X$
$(T,M) \mapsto (T, \mathcal{O}_T \oplus M) \longmapsto (\mathrm{Spec}_T(\mathcal{O}_T \oplus M) \to X)$

let $T(M) := \underbrace{\mathrm{Spec}_T(\mathcal{O}_T \oplus M)}_{\#}$.

§ Definition

Def Let $f: X \to Y$ in St be given

- $\mathrm{Der}: (\mathrm{StMod}_X^{cn})^{op} \longrightarrow S$
$(T,M) \mapsto \mathrm{St}_{T/Y}(T(M), X)$

- For $N \in QC(X)$, consider restricted yoneda.

- $h_N: (\mathrm{StMod}_X^{cn})^{op} \longrightarrow S$
$(T,M) \longmapsto \mathrm{StMod}_X((T,M),(X,N))$.

· The morphism f admits a cotangent complex if $\exists$ an almost connective $L_{X/Y} \in QC(X)$ together w/ an equivalence

$$d: h_{L_{X/Y}} \xrightarrow{\simeq} Der$$

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Rem Suppose f admits a cotangent complex. $\forall\, T \xrightarrow{g} X$: $M \in QC(T)^{cn}$, have:

$$St_{T/Y}(T(M), X) \simeq StRel_X((T,M),(X,L_{X/Y}))$$

$$\begin{tikzcd} & T \arrow[dl] \arrow[dr, "g"] & \\ T(M) \arrow[rr, dashed] \arrow[dr] & & X \arrow[dl, "f"] \\ T \arrow[u] \arrow[dr] & Y & \\ & X \arrow[u] & \end{tikzcd}$$

$$= \left\{ \begin{tikzcd} T \arrow[r, "g"] \arrow[dr, "g"'] & X \\ & X \arrow[u, "id"'] \end{tikzcd} ,\ g^* L_{X/Y} \to M \right\}$$

$$= QC(T)(g^* L_{X/Y}, M)$$

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For $u: \operatorname{Spec} C \to X$, $M \in Mod_C^{cn}$, get

$$Mod_C(u^* L_{X/Y}, M) = \left\{ \begin{tikzcd} \operatorname{Spec} C \arrow[r, "u"] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} C\oplus M \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd} \right\}$$

$$= Fib\left(X(C\oplus M) \to X(C) \times_{Y(C)} Y(C\oplus M) \right)$$

Cor If f is of the form $\operatorname{Spec} B \to \operatorname{Spec} A$ then $\forall\, B \to C$, $M \in \mathrm{Mod}_C^{cn}$, have

$$\mathrm{SCR}_{A/C}(B, C \oplus M) \simeq \mathrm{Mod}_C(L_{B/A} \otimes_B C, M)$$

§ Existence (I).

thm If $X \to Y$ is a map of d. schemes, then admits a cot. complex (also works for alg. stacks)

1) The case $\operatorname{Spec} B \to \operatorname{Spec} \mathbb{Z}$

. Consider

$$F : \mathrm{Mod}_B^{cn} \longrightarrow \mathcal{S}$$
$$M \longmapsto \mathrm{SCR}_{/A}(A, A \oplus M)$$

- preserves lims (?)
- pres filt. colims: $M \mapsto A \oplus M$ pres filt. colims.

A is fin. pres $/A$

$\mathrm{SCR}_{/A}(A, -)$ pres filt. colims

$\rightsquigarrow$ F is corepresentable, say by $L_B \in \mathrm{Mod}_B^{cn}$

st $\mathrm{Mod}_B^{cn}(L_B, M) \simeq \mathrm{SCR}_{/A}(A, A \oplus M)$

Now let $\operatorname{Spec} C \to \operatorname{Spec} B$, $M \in \mathrm{Mod}_C^{cn}$ be given, $\varphi : B \to C$

then $$\mathrm{SCR}_{\mathbb{Z}/C}(B, C \oplus M) \overset{?}{\simeq} \mathrm{Mod}_C(L_B \otimes_B C, M)$$

well $\mathrm{Mod}_C(L_B \otimes_B C, M) \simeq \mathrm{Mod}_B(L_B, \varphi^* M)$

$\simeq \mathrm{SCR}_{/B}(B, B \oplus \varphi^* M)$

have:
$$\begin{tikzcd} B \oplus \varphi^* M \arrow[r, hook] \arrow[d] & C \oplus M \arrow[d] \\ B \arrow[r] & C \end{tikzcd}$$

so $\mathrm{SCR}_{/B}(B, B \oplus \varphi^* M) \simeq \mathrm{SCR}_{/C}(B, C \oplus M)$

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For $T \to X$ in St, $X = \mathrm{spec}\, B.$, $M \in \mathrm{QC}(T)^{\wedge}$

$$T(M) = \underset{\mathrm{Spec} \to T}{\mathrm{colim}}\ \mathrm{Spec}(C \oplus M|_C) \quad \ldots$$

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② For the case $\mathrm{Spec}\, B \to \mathrm{spec}\, A$, $\varphi : A \to B$

I want to define $L_{B/A}$ as the cofiber of a map

$$L_A \otimes_A B \xrightarrow{\alpha} L_B$$

$\mathrm{Mod}_B(L_A \otimes_A B, L_B) \simeq \mathrm{Mod}_A(L_A, \varphi^* L_B)$

$\simeq \mathrm{SCR}_{/A}(A, A \oplus \varphi^* L_B)$

$\simeq \mathrm{SCR}_{/B}(B, B \oplus L_B)$

$\simeq \mathrm{Mod}_B(L_B, L_B) \ni \mathrm{id}$

Define $L_{B/A} = \mathrm{Cofiber}(\psi)$

Then, for $\mathrm{Spec}\, C \to \mathrm{Spec}\, B$, $M \in \mathrm{Mod}_C^{cn}$, $\varphi : B \to C$

$$SCR_{A/C}(B, C\oplus M) \overset{?}{\simeq} \mathrm{Mod}_C(L_{B/A}\otimes_B C, M)$$

$$\mathrm{Mod}_C(L_{B/A}\otimes_B C, M) \simeq \mathrm{Mod}_B(L_{B/A}, \varphi^* M)$$

have $L_A \otimes_A B \to L_B \to L_{B/A}$

$$\begin{array}{ccccc}
\mathrm{Mod}_B(L_{B/A}, \varphi^*M) & \to & \mathrm{Mod}_B(L_B, \varphi^*M) & \to & \mathrm{Mod}_B(L_A\otimes_A B, \varphi^*M) \\
 & & \wr\| & & \wr\| \\
 & & SCR_{/B}(B, B\oplus\varphi^*M) & \to & SCR_{/B}(A, B\oplus\varphi^*M) \\
 & & \wr\| & & \wr\| \\
\text{fiber?} & \to & SCR_{/C}(B, C\oplus M) & \to & SCR_{/C}(A, C\oplus M) \\
 & & & & \ni *
\end{array}$$

$$\text{fiber} = \left\{ \begin{tikzcd} A \arrow[r] \arrow[d] & C\oplus M \arrow[d] \\ B \arrow[r] \arrow[ur, dashed] & C \end{tikzcd} \right\} = SCR_{A/C}(B, C\oplus M)$$

$\neq$

For general case, need some properties.

§ Main properties

1) Given

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "g"'] & Y' \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

in st, $\exists\, L_{X/Y}, L_{X'/Y'}$: $g^* L_{X/Y} \simeq L_{X'/Y'}$

2) For $X \xrightarrow{f} Y \xrightarrow{g} Z$ in St

- Suppose that g admits cotangent complex then (f admits cot. compl.) $\Leftrightarrow$ ($g \circ f$ admits one)

- If so, we have exact sequence

$$f^* L_{Y/Z} \to L_{X/Z} \to L_{X/Y}.$$

§ Existence $f : X \to Y$: map of d. schemes

$$X \to Y \to \operatorname{Spec} \mathbb{Z}$$

if $\exists\, L_X, L_Y$, $f^* L_Y \to L_X \to L_{X/Y}$

so suffices: X admits a cot. complex $(/\mathbb{Z})$

Cor Let $j : U \to X$: open immersion

then $L_{U/X} \simeq 0$. Suffices: $L_{U/X}(R) = 0$

$(\forall \operatorname{Spec} R \xrightarrow{x} X)$

$$\begin{tikzcd}
\operatorname{Spec} R \arrow[r] \arrow[d] \arrow[dd, bend right, "id"'] & U \arrow[d] \\
\operatorname{Spec}(R \oplus M) \arrow[r] \arrow[ur, dashed] \arrow[d] & X \\
\operatorname{Spec} R \arrow[r] \arrow[ur] & U \arrow[u]
\end{tikzcd}$$

this tells us that $\operatorname{Spec}(R \oplus M) \to X$ lands in U.

& since $U \to X$ is mono, this lift exists uniquely.

$$\operatorname{Mod}_R(x^* L_{X/Y}, M) = \operatorname{Fib}\left(U(R \oplus M) \to U(R) \times_{X(R)} X(R \oplus M)\right)$$

$$= \lambda * \xi.$$

$$\Rightarrow \quad x^* L_{X/Y} = 0 \quad \#$$

Now construct L_X on points $\operatorname{Spec} A \xrightarrow{x} X$

$$x^* L_X \to L_A \to L_{A/X} \simeq 0$$

take $x^* L_X := L_A$

$$\begin{tikzcd} \operatorname{Spec} A \arrow[rr, "f"] \arrow[dr, "x"'] & & \operatorname{Spec} B \arrow[dl, "y"] \\ & X & \end{tikzcd}$$

$$f^* y^* L_X \simeq f^* L_B \xrightarrow{\simeq} L_A \to 0$$

$\Rightarrow$ htpy coherent ssh

$$L_X \in QC(X) = \lim_{\operatorname{Spec} A \to X} \operatorname{Mod} A$$

§ Examples

1) Consider $\mathbb{Z}[x_1, \dots, x_n] = A$

let $\Omega_A :=$ Kähler differentials: $\bigoplus A \cdot dx_n$

Consider
$$d : A \to A \oplus \Omega$$
$$f \mapsto (f, \sum \tfrac{\partial f}{\partial x_n} \cdot dx_n)$$

let $M \in \operatorname{Mod}_A^{\heartsuit}$

$$\begin{tikzcd} A \arrow[r, dashed] \arrow[dr, "\sim"'] & A \oplus M \arrow[d] \\ & A \end{tikzcd}$$

$$M^{\oplus n} = \operatorname{Mod}_A(\Omega_A, M) \xrightarrow{\simeq} \operatorname{Scr}_{/A}(A, A \oplus M) = M^{\times n}$$

$$(\Omega_A \xrightarrow{\varphi} M) \longmapsto (A \xrightarrow{d} A \oplus \Omega \xrightarrow{\varphi} A \oplus M)$$

$$\begin{tikzcd}
M^n = \{*\} \times_{A^n} A^n \times M^n = \mathrm{SCR}_{/A}(A, A\oplus M) \arrow[r] \arrow[d] & \mathrm{SCR}(A, A\oplus M) = (A\oplus M)^{\times n} \arrow[d] = A^n \times M^n \\
\{id\} \arrow[r] & \mathrm{SCR}(A,A) = A^n \\
\{*\} \arrow[r, mapsto] & (x_1 \dots x_n)
\end{tikzcd}$$

$$L_A = \Omega_A$$

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2) In general, for $A \in \mathrm{SCR}$, $M \in \mathrm{Mod}_A^{cn}$

$B := \mathrm{Sym}_A(M)$. Have $L_{B/A} = B \otimes_A M$

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2) Consider $A = \mathbb{Z}[x_1 \dots x_n] \to B = \mathbb{Z}$

$$\begin{array}{ccccc}
L_A \otimes_A B & \to & L_B & \to & L_{B/A} = \mathbb{Z}^{\oplus n}[1] \\
\| & & \| & & \\
A^{\oplus n} \otimes_A B & & 0 & & \\
\| & & & & \\
\mathbb{Z}^{\oplus n} & & & &
\end{array}$$

3) If $Z \xrightarrow{I} X$ is a reg. closed immersion of classical schemes, then $L_{Z/X} = I/I^2[1]$ (Stacks project)

4) Consider $K[T] \xrightarrow{(T^2)} K[\varepsilon] \to K$

$$\underbrace{L_{k[\varepsilon]/k[T]} \otimes_{k[\varepsilon]} k}_{(T^2)/(T^4)[1]} \xrightarrow{\varphi} \underbrace{L_{k/k[T]}}_{(T)/(T^2)[1]} \to L_{k/k[\varepsilon]}$$

$k[\varepsilon] \curvearrowright (T^2)/(T^4)$: $(a+b\varepsilon)(\lambda T^2 + \mu T^3) = (a\lambda T^2 + (a\mu + b\lambda)T^3)$

$(a+b\varepsilon)(\lambda + \mu\varepsilon) = (a\lambda + (a\mu + b\lambda)\varepsilon)$

so $(T^2)/(T^4) = k[\varepsilon]$ (as $k[\varepsilon]$-module)

$$\left((T^2)/(T^4)[1]\right) \otimes_{k[\varepsilon]} k = k[1]$$

the map φ is induced by $(T^2)/(T^4) \xrightarrow{0} (T)/(T^2)$, so $\varphi = 0$, so we get exact sequence

$$k[1] \xrightarrow{0} k[1] \to L_{k/k[\varepsilon]}$$

$$\begin{array}{ccccc} k[1] & \to & 0 & \to & k[1] \\ \downarrow & & \downarrow & & \downarrow \\ 0 & \to & k[2] & \to & k[1]\oplus k[2]. \end{array}$$

$\therefore L_{k/k[\varepsilon]} = k[1] \oplus k[2]$.

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§ quasi-smoothness

<u>Def</u>. A closed immersion $Z \to X$ is <u>quasi-smooth</u> if locally on X

$$\begin{array}{ccc} Z & \to & X \\ \downarrow & & \downarrow \\ \{0\} & \to & \mathbb{A}^n \end{array}$$

(i.e. $\operatorname{Spec} \underbrace{A/(f_1, \dots, f_n)}_{\substack{A \otimes \mathbb{Z} \\ \mathbb{Z}[t_1 \dots t_n]}} \to \operatorname{Spec} A$, locally)

Next time:

- formal smoothness / étale / unramified
- relation to deformation theory

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