Section 11 Math 2202

Triple Integrals: Changing Order of Integration and Changing Coordinates

1. **Warm Up** Set up an iterated integral for

Don't need to compute!

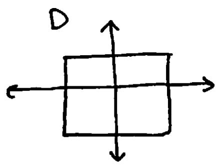
$$\iiint_E x^2 e^y \, dV$$

where E is bounded by the parabolic cylinder $z = 1 - y^2$ and the planes $z = 0, x = 1$ and $x = -1$.

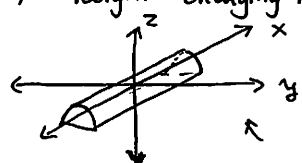
Draw two pictures. One should be a "good enough" picture of E and the other a picture of the projection of E onto the coordinate plane corresponding to the order of integration you chose.

Conditions tell us $-1 \le x \le 1$ and $0 \le z \le 1 - y^2 \Rightarrow y^2 \le 1 \Rightarrow -1 \le y \le 1$.

It's then natural to consider $\iint_D \left(\int_0^{1-y^2} x^2 e^y \, dz \right) dA$, where D is the square $\{(x,y) : -1 \le x, y \le 1\}$. We get triple integral $\int_{-1}^{1}\int_{-1}^{1}\int_0^{1-y^2} x^2 e^y \, dz\,dx\,dy = \int_{-1}^{1}\int_{-1}^{1}\int_0^{1-y^2} x^2 e^y \, dz\,dy\,dx$.



E looks like half-cylinder over D, w/ "height" changing in y-direction but not x-direction.

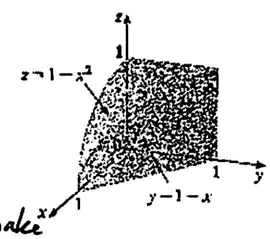


x-axis should run through the center directly.

2. **Changing Order of Integration in Triple Integrals**

The figure on page 881 in the text shows the region of integration for the triple integral

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x, y, z) \, dy \, dz \, dx.$$



(a) Rewrite the integral in the order $dy\,dx\,dz$.

We have $0 \le x \le 1$ and $0 \le z \le 1 - x^2$. We want to make z the independent variable. So, $0 \le z \le 1$ and $0 \le x \le \sqrt{1-z}$. So, we get the integral

$$\int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x,y,z) \, dy\,dx\,dz.$$

(b) Rewrite the integral in the order $dz\,dy\,dx$.

Bounds only depend on x, so we just swap the order. $\int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x,y,z) \, dz\,dy\,dx$.

(c) How many integrals are needed if you project the solid into the yz-plane?

Projection onto yz-plane is square $\{(y,z) : 0 \le y, z \le 1\}$. Notice that there are two "pieces" we must account for, giving two integrals. Integrating in this setting means looking at $dx\,dy\,dz$ or $dx\,dz\,dy$. ~~We~~ We want to reach these by swapping pairs of variables.

Solid satisfies : $z \geq \sqrt{x^2+y^2}$ (above the cone)

$x^2+y^2+z^2 \leq 4$ (below the sphere)

3. Set up a triple integral to find the volume of the solid bounded by $z = \sqrt{x^2+y^2}$ and $x^2+y^2+z^2 = 4$.

(a) In spherical coordinates Volume element $dV = \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta$ (can change order).

(boundary of cone)

ϕ is measured from positive z-axis, so cone gives $0 \leq \phi \leq \frac{\pi}{4}$.

sphere gives $0 \leq \rho^2 \leq 4 \Rightarrow 0 \leq \rho \leq 2$. θ satisfies $0 \leq \theta \leq 2\pi$. So, we want

$$\int_0^{2\pi}\int_0^{\pi/4}\int_0^2 \rho^2 \sin\phi \, d\rho \, d\phi \, d\theta = 2\pi \cdot (1 - \tfrac{\sqrt{2}}{2}) \cdot \tfrac{8}{3}$$

(b) In cylindrical coordinates Volume element $dV = r \, dr \, d\theta \, dz$ (can change order).

We have $z \geq r \geq 0$ and $r^2+z^2 \leq 4$. So, $0 \leq r \leq 2$ and $r \leq z \leq \sqrt{4-r^2}$.

We get $$\int_0^{2\pi}\int_0^{2}\int_r^{\sqrt{4-r^2}} r \, dz \, dr \, d\theta = 2\pi \int_0^2 (\sqrt{4-r^2} - r) \, r \, dr = \tfrac{8}{3}\pi(2-\sqrt{2}).$$

(c) In rectangular coordinates[1] Volume element $dV = dx \, dy \, dz$ (can change order).

We have $2 \geq z \geq \sqrt{x^2+y^2}$ and $x^2+y^2+z^2 \leq 4$.

Get two pieces, w/ $0 \leq y \leq \sqrt{2-x^2}$ and $0 \geq y \geq -\sqrt{2-x^2}$ on the other piece

$$\int_0^{\sqrt{2}}\int_0^{\sqrt{2-x^2}}\int_{\sqrt{x^2+y^2}}^{2} dz \, dy \, dx + \int_0^{\sqrt{2}}\int_{-\sqrt{2-x^2}}^{0}\int_{\sqrt{x^2+y^2}}^{2} dz \, dy \, dx.$$

[Might want to check this one in Wolfram!]

[1]How does this integral compare with the same question from section last week, where you used a double integral?

$E = \{(x,y) : x^2+y^2 \leq 2\}$

$$\iint_E \left(\sqrt{4-x^2-y^2} - \sqrt{x^2+y^2}\right) dA = \int_0^{2\pi}\int_0^{\sqrt{2}} (\sqrt{4-r^2} - r) \, r \, dr \, d\theta = 2\pi \cdot \tfrac{4}{3}(2-\sqrt{2})$$

As above we have the $dz\,dx\,dy$ integral given by:

$$\underbrace{\int_0^1 \int_0^{1-y} \int_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy}_{=\frac{5}{12} \text{ if } f(x,y,z)=1}.$$

$dy\,dx\,dz \rightsquigarrow dx\,dy\,dz$

$dz\,dx\,dy \rightsquigarrow dx\,dz\,dy$

$2y-y^2 = 1-(1-y)^2$

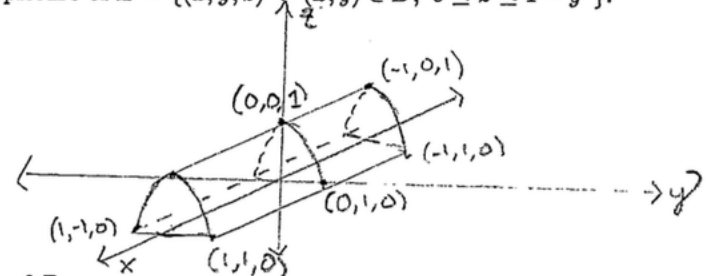
Making the swaps gives us

$$\underbrace{\int_0^1 \int_0^{2y-y^2} \int_0^{1-y} f(x,y,z)\,dx\,dz\,dy}_{=\frac{1}{4} \text{ if } f(x,y,z)=1} + \underbrace{\int_0^1 \int_{2y-y^2}^{1} \int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dz\,dy}_{=\frac{1}{6} \text{ if } f(x,y,z)=1}.$$

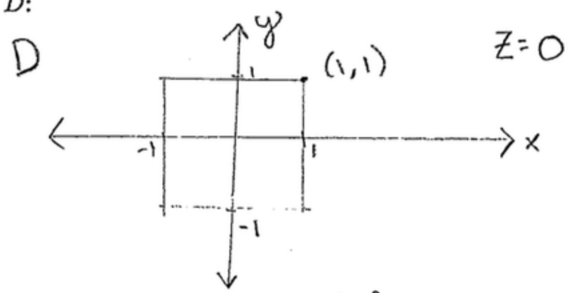
$\left[\frac{1}{4} + \frac{1}{6} = \frac{1}{12} \right]$

We also get $\displaystyle \int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dy\,dz + \int_0^1 \int_{1-\sqrt{1-z}}^{1} \int_0^{1-y} f(x,y,z)\,dx\,dy\,dz.$

Let D be the projection of E into the xy-plane.
"Good enough" picture of $E = \{(x, y, z) \; ; \; (x, y) \in D, \; 0 \le z \le 1 - y^2\}$:



Here is a picture of D:



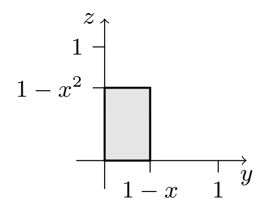
Then,

$$\iiint_E x^2 e^y dV = \iint_D \int_0^{1-y^2} x^2 e^y dz \; dA$$
$$= \int_{-1}^{1} \int_{-1}^{1} \int_0^{1-y^2} x^2 e^y dz \; dy \; dx$$

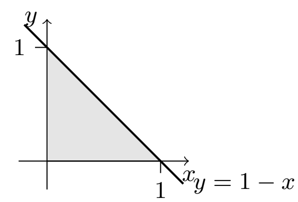
Solution: We do this by switching in pairs, with a sketch for each switch. We otherwise proceed without comment:

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\,dy\,dz\,dx$$

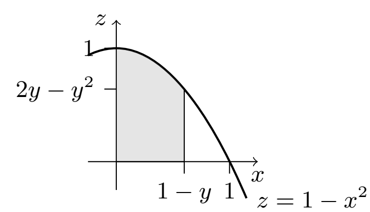
$$= \int_0^1 \int_0^{1-x} \int_0^{1-x^2} f(x,y,z)\,dz\,dy\,dx$$



$$= \int_0^1 \int_0^{1-y} \int_0^{1-x^2} f(x,y,z)\,dz\,dx\,dy$$



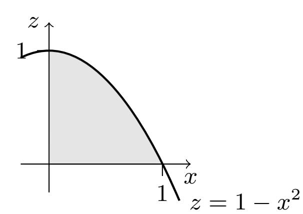
$$= \int_0^1 \int_0^{2y-y^2} \int_0^{1-y} f(x,y,z)\,dx\,dz\,dy + \int_0^1 \int_{2y-y^2}^1 \int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dz\,dy$$



To get to $dx\,dy\,dz$ and $dy\,dx\,dz$, we start again with the initial integral:

$$\int_0^1 \int_0^{1-x^2} \int_0^{1-x} f(x,y,z)\,dy\,dz\,dx$$

$$= \int_0^1 \int_0^{\sqrt{1-z}} \int_0^{1-x} f(x,y,z)\,dy\,dx\,dz$$



$$= \int_0^1 \int_0^{1-\sqrt{1-z}} \int_0^{\sqrt{1-z}} f(x,y,z)\,dx\,dy\,dz + \int_0^1 \int_{1-\sqrt{1-z}}^1 \int_0^{1-y} f(x,y,z)\,dx\,dy\,dz$$