Recall: $k$-alg. $A$ is affinoid if $\exists\ \overset{\text{Tate alg.}}{\overbrace{T_n}} \to A$ making $A$ into fin. gen. $T_n$-module.

Fact: $T_n \to A$ can be chosen inj. or surj. Surj. $T_n \overset{q}{\twoheadrightarrow} A \rightsquigarrow$ norm $\|a\| := \inf\{\|f\| : q(f) = a\}$ making $A$ into $k$-Banach alg. All norms on $A$ making it into Banach $k$-alg. are equivalent. Notions of "bdd", "power-bdd", etc. are preserved by all maps $A \to B$. Spectral seminorm on $A$ is $\|f\|_{sp} := \sup\{|f(x)| : x \in \overset{\text{(set of maximal ideals)}}{\overbrace{Sp(A)}}\}$.

$\|\cdot\|_{sp}$ is not mult. but does satisfy $\|f^k\|_{sp} = \|f\|_{sp}^k$.

Prop: (1) $\|\cdot\|$ norm on $A$ making it Banach $k$-alg. $\Rightarrow \|\cdot\|_{sp} \leq \|\cdot\|$.

(2) $A = T_n \Rightarrow \|\cdot\|_{sp}$ is the Gauss norm $\|\cdot\|$.

Pf: (1) Assume WLOG $\|f\| = 1$. Then, $\|f^n\| \leq \|f\|^n = 1\ \forall n$. Given $x \in Sp(A)$, $A \to k_x = A/x$ is a ~~map~~ map preserving power-bdd. elts. $\Rightarrow |f(x)|^n \leq c$ indep. of $n$ $\Rightarrow |f(x)| \leq 1 \Rightarrow \|f\|_{sp} \leq 1 = \|f\|$.

(2) Fix ~~$x \in Sp(T_n)$~~ $f \in \overset{k\langle z_1, \ldots, z_n\rangle}{\overset{\|}{T_n}}$ w/ $\|f\| = 1$. So, $\|f\|_{sp} \leq 1$. $\bar{f} \in \bar{k}[z_1, \ldots, z_n]$ is nonzero. $\exists$ fin. ext. $\bar{L}/\bar{k}$ and $(\bar{x}_1, \ldots, \bar{x}_n) \in \bar{L}^n$ s.t. $\bar{f}(\bar{x}_1, \ldots, \bar{x}_n) \neq 0$. Find fin. ext. $L/k$ s.t. $\bar{L} = L^\circ / L^{\circ\circ}$. Lift $\bar{x}_1, \ldots, \bar{x}_n$ to $x_1, \ldots, x_n \in L$, inducing $k\langle z_1, \ldots, z_n\rangle \to L$ via $z_i \mapsto x_i$. Now, $|f(x_1, \ldots, x_n)| = 1$. $x = \ker(f) \in Sp(T_n)$ is a point at which $|f(x)| = 1$. So, $\|f\|_{sp} \geq 1 \Rightarrow \|f\|_{sp} = 1 = \|f\|$. $\square$

Exercise: For $f \in A$ TFAE:
(i) $\inf\{|f(x)| : x \in Sp(A)\} > 0$;
(ii) $f(x) \neq 0\ \forall x \in Sp(A)$;
(iii) $f \in A^\times$.

For $A$ affinoid, $A^\circ := \{f \in A : \|f\|_{sp} \leq 1\}$, $A^{\circ\circ} := \{f \in A : \|f\|_{sp} < 1\}$.

Prop: $A$ affinoid.
(1) $A$ is Jacobson.
(2) Spectral seminorm is attained at some pt.
(3) $A^\circ = \{\text{power-bdd. elts.}\}$, $A^{\circ\circ} = \{f \in A : \overset{\text{(top. nilpotent)}}{\overbrace{f^n \to 0}}\}$.

Remark: Proof is long and messy.

Cor: Morphism of affinoids $A \to B$ sends $A^\circ$ to $B^\circ$ and $A^{\circ\circ}$ to $B^{\circ\circ}$.

Cor: $\|\cdot\|$ making $A$ into Banach $k$-alg. $\Rightarrow \forall f \in A : \|f\|_{sp} = \lim_{n\to\infty} \|f^n\|^{1/n}$.

For $A$ affinoid recall we can make sense of $A\langle z_1, \dots, z_n\rangle$. This has expected univ. property, w.r.t.

sending variables to power-bdd. elts.

Let $\bar{A} := A^\circ / A^{\circ\circ}$ w/ $\mathrm{Max}(\bar{A})$ set of maximal ideals. We have reduction map $r : Sp(A) \to \mathrm{Max}(\bar{A})$ given by

$x \in Sp(A) \mapsto \ker(\bar{A} \to \bar{k}_x)$ induced by $A \xrightarrow{x} k_x$. [the map and the ideal are basically synonymous]

Prop: (1) $\phi : A \to B$ map of affinoids $\Rightarrow$ TFAE: (i) $\phi$ is finite; (ii) $\phi^\circ : A^\circ \to B^\circ$ is integral; (iii) $\bar{\Phi} : \bar{A} \to \bar{B}$ is finite. Remark: Proof is long and messy.

(2) Reduction map $r : Sp(A) \to \mathrm{Max}(\bar{A})$ is surj.

Pf of (2): (Case 1) First assume $A = \overset{T_n}{\overset{\shortparallel}{k\langle z_1, \dots, z_n\rangle}}$. Pick $\bar{m} \in \mathrm{Max}(\bar{T}_n) = \mathrm{Max}(\bar{k}[z_1, \dots, z_n]) \rightsquigarrow \bar{l} = \bar{T}_n / \bar{m}$ fin. ext. of $\bar{k}$.

(w/ $L^\circ / L^{\circ\circ} = \bar{l}$)

Let $\bar{b}_1, \dots, \bar{b}_n \in \bar{l}$ be images of $z_1, \dots, z_n \in \bar{T}_n$. Lift to some $L$. Lift to get $b_1, \dots, b_n \in L^\circ$. $\exists!\, T_n \to L$ w/

$z_i \mapsto b_i$. Now look at the kernel to conclude surjectivity.

(Case 2) Now assume $A$ sits in a diagram

$$\begin{array}{ccc} A & \to & \mathrm{Frac}(A) = \mathcal{L} \\ \uparrow & & \uparrow \text{finite Galois} \\ T_n & \to & \mathrm{Frac}(T_n) = \mathcal{T}_n \end{array}$$

s.t. $A$ is integral closure of $T_n$ in $\mathcal{L}$.

[implicit here that $A$ is integral domain]

So, $T_n \to A$ finite $\Rightarrow \bar{T}_n \to \bar{A}$ finite. By Going Up Thm, (+ Case 1)

$$\begin{array}{ccc} Sp(A) & \longrightarrow & \mathrm{Max}(\bar{A}) \\ \downarrow & & \downarrow \\ Sp(T_n) & \twoheadrightarrow & \mathrm{Max}(\bar{T}_n) \end{array}.$$

**Lemma:** $G = \mathrm{Gal}(\mathcal{A}/\mathcal{T}_n) \subseteq \mathrm{Aut}(A/T_n) \subseteq \mathrm{Aut}(A^\circ/T_n^\circ)$ acts transitively on

- all maximal ideals of $A^\circ$ above given maximal ideal of $T_n^\circ$;
- all maximal ideals of $\bar{A}$ above given maximal ideal of $\bar{T}_n$.

(Case 3) $A$ is any integral domain. Still have

$$\begin{tikzcd} A \arrow[r] & \mathcal{A} \\ T_n \arrow[u] \arrow[r] & \mathcal{T}_n \arrow[u] \end{tikzcd}$$

$\mathcal{B} :=$ Galois closure of $\mathcal{A}/\mathcal{T}_n$

$B :=$ integral closure of $\cancel{A}$ $T_n$ in $\mathcal{B}$

Going Up Thm + Case 2 gives

$$\begin{tikzcd} \mathrm{Sp}(B) \arrow[r, two heads] \arrow[d] & \mathrm{Max}(\bar{B}) \arrow[d] \\ \mathrm{Sp}(A) \arrow[r] & \mathrm{Max}(\bar{A}) \end{tikzcd}$$

, forcing $\mathrm{Sp}(A) \twoheadrightarrow \mathrm{Max}(\bar{A})$.

(Case 4) $A$ arbitrary. Since $A$ is Noetherian, it has fin. many minimal primes. $B := \bigoplus_{\substack{\mathfrak{p} \trianglelefteq A \\ \text{minimal prime}}} A/\mathfrak{p}$ is direct sum of integral affinoids. Now $A \to B$ is finite and apply Going Up Thm + Case 3. $\square$

**Derivations**

For technical reasons assume $k$ is perfect. For affinoid $A$ form $\Omega^1_{A/k}$.

**Prop:** $\exists$ fin. gen. $A$-module $\Omega^f_{A/k}$ (<u>finite differentials</u>) w/ derivation universal for fin. gen. $A$-modules.

**Remark:** We are not claiming that such a thing exists for $A$ <u>not</u> affinoid.

We will discuss this more next time.

Pf: Suppose first $A = k\langle z_1, \dots, z_n \rangle$. Define $\Omega^f_{T_n/k} := T_n dz_1 \oplus \cdots \oplus T_n dz_n$ and $df := \frac{\partial f}{\partial z_1} dz_1 + \cdots + \frac{\partial f_n}{\partial z_n} dz_n$.

($M \in \mathrm{Mod}_{T_n}$)

key is showing that a derivation $E: T_n \to M$ w/ $E(z_i) = 0 \ \forall i \Rightarrow E = 0$. Clearly $E(k[z_1, \dots, z_n]) = 0$.

Let $m \trianglelefteq T_n$ maximal. We have $k[z_1, \dots, z_n] \twoheadrightarrow T_n/m^s$ $(\forall s > 0)$

(1) Use induction on $s$ to show $T_n/m^s$ is fin. dim$/k$.

(2) Show image is closed and dense.

So, $T_n = k[z_1, \dots, z_n] + m^s \Rightarrow E(T_n) = E(m^s) \ \forall s > 0$. Leibniz rule $\Rightarrow E(m^{2s}) \subseteq m^s E(m^s)$.

Hence, $E(T_n) \subseteq m^s M$. Krull Intersection Thm $\Rightarrow \exists\, a \in T_n$ w/ $a \equiv 1 \bmod m$ s.t. $a E(T_n) = 0$.

So, localization of $E(T_n)$ at $m$ is $0$. But, vanishing at localization of every maximal ideal is same as vanishing.

For general $A$ write $A = T_n / (f_1, \dots, f_s)$. Define $\Omega^f_{A/k} := A \otimes_{T_n} \Omega^f_{T_n/k} / (T_n df_1 + \cdots + T_n df_s)$.

$\square$

This works (and doesn't depend on lifts).

Prop: Assume $A$ affinoid w/ no zero divisors (AKA integral domain) and $\mathcal{A} := \mathrm{Frac}(A)$. Then, krull dim of $A = \dim_{\mathcal{A}} \mathcal{A} \otimes_A \Omega^f_{A/k}$. Moreover,

For maximal $m \trianglelefteq A$, TFAE:

(i) Localization $A_m$ is regular.

(ii) Localization of $\Omega^f_{A/k}$ is free over $A_m$.

(iii) $A/m$-dim of $A/m \otimes_A \Omega^f_{A/k}$ = krull dim of $A_m$.

## Rigid Spaces

Analog of metric top.

Fix affinoid $X = Sp(A)$. Canonical top. is gen. by $X(f) := \{x \in X : |f(x)| \le 1\}$ as $f$ varies.

Remark: Zariski top. here is gen. by $\mathcal{U}_f := \{x \in X : f(x) \neq 0\}$ as $f$ varies.

Lemma: $f \in A$, $\varepsilon > 0 \rightsquigarrow X(f;\varepsilon) := \{x \in X : |f(x)| \le \varepsilon\}$. This is open and $\exists\, g \in A$ s.t. $X(g) \subseteq X(f;\varepsilon)$.

In particular, $\{x \in X : |f(x)| < \varepsilon\} = \bigcup_{\delta < \varepsilon} X(f;\delta)$ is open.

(So, $X(g)$ is open nbhd of $x$ on which $|f|$ is constant.)

Prop: Fix $f \in A$ and $x \in X$ s.t. $f(x) \neq 0$. Then, $\exists\, g \in A$ s.t. $g(x) = 0$ and $X(g) \subseteq \{y \in X : |f(y)| = |f(x)|\}$.

In particular, $\{x \in X : |f(x)| \,?\, \varepsilon\}$ w/ $? \in \{=, \ge, >, \le, <\}$ are all open hence clopen.

Pf: Consider $f(x) \in k_x = A/x$. Let $P(T) \in k[T]$ be its minimal polyn. and consider $g := P(f) \in A$.

$g(x) =$ image of $P(f)$ in $A/x = P(f(x)) = 0$. Let $\alpha_1, \dots, \alpha_n \in k^{alg}$ be roots of $P(T) = (T-\alpha_1)\cdots(T-\alpha_n)$.

$\forall i\ \exists\, k_x \hookrightarrow k^{alg}$ s.t. $f(x) \mapsto \alpha_i \Rightarrow \forall i\ |\alpha_i| = |f(x)| = \varepsilon$.

Claim: If $y \in X$ satisfies $|g(y)| < \varepsilon^n \Rightarrow |f(y)| = \varepsilon$.

Indeed, $|f(y)| \neq \varepsilon \Rightarrow \forall\, k_y \hookrightarrow k^{alg}: |f(y) - \alpha_i| = \max\{|f(y)|, |\alpha_i|\} \ge |\alpha_i| = \varepsilon$.

$\Rightarrow g(y) = P(f(y)) = (f(y)-\alpha_1)\cdots(f(y)-\alpha_n)$ has $|g(y)| \ge \varepsilon^n \Rightarrow\Leftarrow$

So, $\{y \in X : |g(y)| < \varepsilon^n\} \subseteq \{y \in X : |f(y)| = \varepsilon\}$. Lemma $\Rightarrow$ this contains $X(g')$ for some $g' \in A$ w/ $g'(x) = 0$. $\square$