$X = Sp(A)$. $(A \neq 0)$

Def: $R \subseteq X$ is rational domain if $\exists f_0, f_1, \ldots, f_s \in A$ s.t. $(f_0, \ldots, f_s) = A$ and $R = \{x \in X : |f_i(x)| \leq |f_0(x)| \ \forall i\}$.

We call this latter set $X(\frac{f_1}{f_0}, \ldots, \frac{f_s}{f_0})$.

Prop: (1) Rational domains are clopen for canon. top.; (2) Finite intersections of rat. domains are rat. domains.

Pf: Pick $x \in X(\frac{f_1}{f_0}, \ldots, \frac{f_s}{f_0})$. Note $f_0(x) \neq 0$. Chose open nbhd $X(g) \ni x$ on which $|f_0| = |f_0(x)| = \varepsilon$. Then,

$X(g) \cap X(f_1; \varepsilon) \cap \cdots \cap X(f_s; \varepsilon) \subseteq R$ is open nbhd of x in $X(\frac{f_{01}}{f_0}, \ldots, \frac{f_s}{f_0})$. This proves (1). For (2),

check $X(\frac{f_i}{g f_0}) \cap X(\frac{g_j}{g_0}) = X(\frac{f_i g_j}{f_0 g_0})$. $\square$

Given $R = X(\frac{f_1}{f_0}, \ldots, \frac{f_s}{f_0})$, define affinoid alg.

$B = \mathcal{O}(R) = A\langle \frac{f_1}{f_0}, \ldots, \frac{f_s}{f_0} \rangle := A\langle z_1, \ldots, z_s \rangle / (f_1 - f_0 z_1, \ldots, f_s - f_0 z_s)$. Natural map $\phi : A \to B$ induces

$\phi : Sp(B) \to X$. Choices don't matter because...

Prop: (1) ϕ is a homeomorphism onto R; (2) $\psi : A \to C$ map of affinoid alg.'s s.t. $Sp(C) \xrightarrow{\psi} Sp(A)$ factors through R

$$\Rightarrow \begin{tikzcd} A \arrow[r, "\phi"] \arrow[d, "\psi"'] \arrow[dr, phantom, "\circlearrowleft"] & B \arrow[dl, dashed, "\exists!"] \\ C & \end{tikzcd}$$ inducing the above factorization.

Pf: (1) We show first $\hat{\phi}(Sp(B)) \subseteq R$. Pick $y \in Sp(B)$. Let $x = \phi^{-1}(y) \in Sp(A)$. We have
$$\begin{tikzcd} A \arrow[r] \arrow[d, "\phi"'] \arrow[dr, phantom, "\circlearrowleft"] & k_x \arrow[d] \\ B \arrow[r] & k_y \end{tikzcd}.$$

The images of $z_1, \ldots, z_s$ in k_y are power bounded $\Rightarrow |z_i(y)| \leq 1 \ \forall i$. But,

$f_i(y) - z_i(y) f_0(y) = 0$ in k_y $\Rightarrow |f_i(y)| = |z_i(y) f_0(y)| \leq |f_0(y)|$. So, in k_x we have

$|f_i(x)| \leq |f_0(x)| \ \forall i \Rightarrow x \in R$. Start now w/ $x \in R$. We have $|f_i(x)| \leq |f_0(x)| \ \forall i$, hence $\frac{f_i(x)}{f_0(x)} \in k_x^\circ$.

Choose $A\langle z_1, \ldots, z_s \rangle \to k_x$, $z_i \mapsto \frac{f_i(x)}{f_0(x)}$.
$$\begin{tikzcd} A \arrow[d] \arrow[dr, bend left] & \\ A\langle z_1, \ldots, z_s \rangle \arrow[r] \arrow[d] & k_x \\ B \arrow[ur, bend right] & \end{tikzcd}$$
kernel of $B \to k_x$ is pt. of $Sp(B)$.

(2) Pick ψ as in statement. Given $y \in Sp(C)$ let $x = \psi^{-1}(y) \in R \subseteq Sp(A)$.

$$\begin{tikzcd} A \arrow[r] \arrow[d, "\psi"'] \arrow[dr, phantom, "\circlearrowright"] & k_x \arrow[d] \\ C \arrow[r] & k_y \end{tikzcd}$$

We have $f_0(x) \neq 0$ and $\left|\frac{f_i(x)}{f_0(x)}\right| \leq 1$. Let $g_0, \dots, g_s \in C$ be images of $f_0, \dots, f_s \in A$. Then, $g_0(y) \neq 0$ and $\left|\frac{g_i(y)}{g_0(y)}\right| \leq 1$. This holds $\forall y \in Sp(C)$, so g_0 is nowhere vanishing hence a unit. Also, $\left|\frac{g_i}{g_0}\right|_{sp} \leq 1 \Rightarrow \frac{g_i}{g_0} \in C^\circ$. So, we get unique $A\langle z_1, \dots, z_s\rangle \to C$, $z_i \mapsto \frac{g_i}{g_0}$ (i.e., we get desired factor-ization) as desired. $\square$

<u>Def</u>: $\mathcal{U} \subseteq X = Sp(A)$ is <u>affinoid subdomain</u> if $\exists$ affinoid alg. $B = \mathcal{O}(\mathcal{U})$ and map $A \to B$ s.t.

(1) image of $Sp(B) \to X$ is contained in $\mathcal{U}$

(2) for every map $A \to C$ of affinoid alg's s.t. $Sp(C) \to Sp(A)$ factors through $\mathcal{U}$ $\Rightarrow$

$$\begin{tikzcd} A \arrow[r] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[dl, dashed, "\exists!"] \\ C & \end{tikzcd}$$

inducing factorization

<u>Example</u>: Rational subdomains are affinoids by the above.

<u>Remark</u>: Very weak top. has admissibles the rational subdomains. Slightly finer refinements provided by finite unions of rational subdomains and affinoid subdomains.

<u>Prop</u>: Let $\mathcal{U} \subseteq X$ affinoid subdomain and $A \xrightarrow{i} \mathcal{O}(\mathcal{U}) = B$.

(1) Induced $Sp(B) \to \mathcal{U}$ is bijective.

(2) $y \in Sp(B)$ and $x = i^{-1}(y) \in \mathcal{U} \Rightarrow$ induced $A/x^n \to B/y^n$ is isom. $\forall n$.

<u>Pf</u>: Start w/ $x \in \mathcal{U} \rightsquigarrow$

$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d] \arrow[dr, phantom, "\circlearrowright"] & B \arrow[d] \\ A/x^n \arrow[r] & B/i(x^n)B \end{tikzcd} \rightsquigarrow \begin{tikzcd} X \supseteq \mathcal{U} & Sp(B) \arrow[l] \\ Sp(A/x^n) \arrow[u] \arrow[ur, dashed, "\exists!\alpha"] & Sp(B/i(x^n)B) \arrow[l] \arrow[u] \end{tikzcd}$$

α having the property that it makes the upper triangle commute.

<u>Claim</u>: The lower triangle commutes (as well!).

key comes in applying univ. prop. of i to
$$\begin{tikzcd} \mathcal{U} & Sp(B) \arrow[l, "i"'] \\ & Sp(B/i(x^n)B) \arrow[ul] \arrow[u, dashed, "\exists!"'] \end{tikzcd}$$

In other words, we have two ways to complete the same diagram so they must be the same.

So, now we have comm. diagram
$$\begin{tikzcd} A \arrow[r, "i"] \arrow[d, two heads] & B \arrow[d] \arrow[dl, "\alpha"'] \\ A/x^n \arrow[r] & B/i(x^n)B \end{tikzcd}$$

Purely formally we conclude that bottom arrow is surj.

Easy to show bottom arrow is inj. Taking $n=1$ gives $i: A/x \xrightarrow{\sim} B/i(x)B$. So, $i(x)B \in Sp(B)$ and we get the desired inverse $\mathcal{U} \to Sp(B)$. □

Prop: $\mathcal{U} \subseteq Sp(A)$ affinoid subdomain, which we identify set-theoretically w/ $Sp(B)$ by the above. Let $V \subseteq Sp(B)$ be affinoid subdomain. Then, V is affinoid subdomain of $Sp(A)$.

Remark: This is purely formal. The same statement is true if we replace "affinoid" by "rational". This is not purely formal!

Prop: $\phi: \underset{Sp(B)}{Y} \to \underset{Sp(A)}{X}$ morphism of affinoid spaces. $\mathcal{U} \subseteq X$ affinoid $\Rightarrow \phi^{-1}(\mathcal{U}) \subseteq Y$ affinoid. $\mathcal{U} = X\left(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0}\right)$

$$\Rightarrow \phi^{-1}(\mathcal{U}) = Y\left(\frac{\phi^* f_1}{\phi^* f_0}, \dots, \frac{\phi^* f_n}{\phi^* f_0}\right).$$

Cor: $\mathcal{U}, V \subseteq X = Sp(A)$ affinoid subdomains $\Rightarrow \mathcal{U} \cap V$ is as well and $\mathcal{O}(\mathcal{U} \cap V) = \mathcal{O}(\mathcal{U}) \hat{\otimes}_A \mathcal{O}(V)$.

[intersection is just pullback of an appropriate inclusion map]

Warning: Unions of affinoids need not be affinoid.

Thm: Every affinoid subdomain $\mathcal{U} \subseteq X$ is finite union of rational subdomains (Gerritzen–Grauert) hence open for canon. top. Moreover, the bijection $\mathcal{U} \leftrightarrow Sp(\mathcal{O}(\mathcal{U}))$ is a homeomorphism.

The Weak Topology

Fix $X = Sp(A)$.

Def: Very weak top. on X is given by:

- admissible opens: rational subdomains;
- admissible cover: (of $\mathcal{U} \subseteq X$ rational domain) is naive cover $\mathcal{U} = \bigcup_i \mathcal{U}_i$ w/ $\mathcal{U}_i$ rational subdomain

  s.t. $\mathcal{U}$ is actually cov. by fin. many of the $\mathcal{U}_i$.

Weak top. on X is given by: ← (default for the book we are using)

- admissible opens: finite unions of rational subdomains
- admissible cover: mimic the above...

Presheaf $\mathcal{O}_X$ for very weak top. given by $\mathcal{O}_X(X(\frac{f_1}{f_0}, \dots, \frac{f_n}{f_0})) := A\langle z_1, \dots, z_n\rangle / (f_i - f_0 z_i)$.

$M \in Mod_A^{fg} \rightsquigarrow$ presheaf $\tilde{M} : \mathcal{U} \mapsto M \hat{\otimes}_A \mathcal{O}_X(\mathcal{U})$ (completion not necessary because M is fin. gen.)

Thm (Tate): $M \in Mod_A^{fg}$, $\mathfrak{U}$ (very weak) admissible cover of X.

(1) $\check{H}^0(\mathfrak{U}, \tilde{M}) = M$.

(2) $\check{H}^i(\mathfrak{U}, \tilde{M}) = 0 \ \forall i > 0$.

Cor: $\tilde{M}$ is a sheaf (for the very weak top.).

Pf: Given $\mathcal{U}$ admissible open and $\mathfrak{U}$ admissible cover, we need $\check{H}^0(\mathfrak{U}, \tilde{M}|_{\mathcal{U}}) = \tilde{M}(\mathcal{U})$. Writing $\mathcal{U} = Sp(B)$,

$\tilde{M}|_{\mathcal{U}} = \widetilde{M \otimes_A B}$, $\tilde{M}(\mathcal{U}) = M \otimes_A B$. Now use Tate's thm. $\square$

<u>Lemma 1</u>: Fix $f \in A$. Consider adm. cov. $X = \underbrace{\{x \in X : |f(x)| \le 1\}}_{=: X_1} \cup \underbrace{\{x \in X : |f(x)| \ge 1\}}_{=: X_2}$. The seq.

$$0 \to \widetilde{M}(X) \to \widetilde{M}(X_1) \oplus \widetilde{M}(X_2) \to \widetilde{M}(X_1 \cap X_2) \to 0 \quad (*)$$

is exact (in fact, split in Mod_A).

<u>Shorthand</u>:

$X_1 = \{|f| \le 1\}$
$X_2 = \{|f| \ge 1\}$

<u>Pf</u>: Suppose $M = A$, $\widetilde{M} = \mathcal{O}_X$. $(*)$ becomes $0 \to A \to \underbrace{A\langle f \rangle}_{A\langle T\rangle/(T-f)} \oplus \underbrace{A\langle f^{-1}\rangle}_{A\langle S\rangle/(1-Sf)} \to \underbrace{A\langle f, f^{-1}\rangle}_{A\langle S,T\rangle/(T-f,\,1-Sf)} \to 0$.

$$\begin{tikzcd}
 & & 0 & 0 & \\
0 \arrow[r] & A \arrow[r] & A\langle f\rangle \oplus A\langle f^{-1}\rangle \arrow[r] \arrow[u] & A\langle f, f^{-1}\rangle \arrow[r] \arrow[u] & 0 \\
0 \arrow[r] & A \arrow[r] \arrow[u, equal] & A\langle T\rangle \oplus A\langle S\rangle \arrow[r] \arrow[u] & A\langle T, T^{-1}\rangle \arrow[r] \arrow[u] & 0 \\
 & & (T-f)A\langle T\rangle \oplus (1-Sf)A\langle S\rangle \arrow[r] \arrow[u] & (T-f)A\langle T, T^{-1}\rangle \arrow[u] & \\
 & & 0 \arrow[u] & 0 \arrow[u] &
\end{tikzcd}$$

$(a(T), b(S)) \mapsto a(T) - b(T^{-1})$

All columns are exact.

Middle row is split by isolating "positive" and "non-positive" terms.

Splitting is inherited by top row, showing exactness and splitness simultaneously. $\square$

<u>Remark</u>: Still need to handle general M, but this is easy.

<u>Def</u>: $\mathfrak{U}$ adm. cov. of adm. open $\mathcal{U} \subseteq X$. We say $\widetilde{M}$ is <u>acyclic</u> for $\mathfrak{U}$ if
- $\check{H}^0(\mathfrak{U}, \widetilde{M}|_{\mathcal{U}}) = \widetilde{M}(\mathcal{U})$;
- $\check{H}^i(\mathfrak{U}, \widetilde{M}|_{\mathcal{U}}) = 0 \ \forall i \ge 1$.

<u>Observations</u>:

(1) $\mathfrak{U} = \{\mathcal{U}\} \Rightarrow \widetilde{M}$ acyclic for $\mathfrak{U}$.

(2) $\widetilde{M}$ acyclic for $\mathfrak{U}$ and $V \subseteq \bigcup_{\mathcal{U} \in \mathfrak{U}} \mathcal{U} \Rightarrow \widetilde{M}$ acyclic for $\mathfrak{U} \cup \{V\}$.

<u>Lemma 2</u>: X_1, X_2 as before, $\mathfrak{U} = \{\mathcal{U}_i\}$ adm. cov. of X s.t.
- $\widetilde{M}|_{X_1}$ acyclic for $\mathfrak{U} \cap X_1$ $\quad (:= \{\mathcal{U}_i \cap X_1\})$;
- $\widetilde{M}|_{X_2}$ acyclic for $\mathfrak{U} \cap X_2$;
- $\widetilde{M}|_{X_1 \cap X_2}$ acyclic for $\mathfrak{U} \cap (X_1 \cap X_2)$ }.

Then, $\widetilde{M}$ is acyclic for $\mathfrak{U}$.

Pf: Use previous lemma to get SES of Čech complexes hence LES in Čech cohom. Now explicitly describe what "remains." ☐

Lemma 3: $\tilde{M}$ is acyclic for the cover $\{X_1, X_2\}$.

Pf: This is trivial consequence of previous lemma + the observations. ☐

Now take $f_1, f_2 \in A$ s.t. $(f_1, f_2) = A$. Consider adm. cov. $X = \underbrace{\mathcal{U}_1}_{\{x\in X:\ |f_1(x)| \geq |f_2(x)|\}} \cup \underbrace{\mathcal{U}_2}_{\{x \in X:\ |f_1(x)| \leq |f_2(x)|\}}$

Lemma 4: $\tilde{M}$ is acyclic for $\mathcal{U} := \{\mathcal{U}_1, \mathcal{U}_2\}$.

(So, the larger of the two is always "big" in some sense.)

Pf: By assumption, choose $a_1, a_2 \in A$ s.t. $a_1 f_1 + a_2 f_2 = 1$ in A. Given $x \in X$,

$$1 = |a_1(x) f_1(x) + a_2(x) f_2(x)| \leq \max\{\|a_1\|_{sp} |f_1(x)|, \|a_2\|_{sp} |f_2(x)|\} \leq C \max\{|f_1(x)|, |f_2(x)|\}$$ for some constant $C > 0$. Choose $\varepsilon \in |k^\times|$ s.t. $0 < \varepsilon < \max\{|f_1(x)|, |f_2(x)|\}\ \forall x \in X$.

Consider adm. cov. $X = \underset{\{|f_2| \leq \varepsilon\}}{X_1} \cup \underset{\{|f_2| \geq \varepsilon\}}{X_2}$. On X_1, $|f_1| > \varepsilon$. Easy to see $\tilde{M}|_{X_1}$ acyclic for $\mathcal{U} \cap X_1$ and

$\tilde{M}|_{X_1 \cap X_2}$ acyclic for $\mathcal{U} \cap (X_1 \cap X_2)$. On X_2, $|f_2| > \varepsilon \Rightarrow f_2 \in \mathcal{O}_X(X_2)^\times$. By previous lemma,

$\tilde{M}|_{X_2}$ is acyclic for $\{\mathcal{U}_1 \cap X_2, \mathcal{U}_2 \cap X_2\}$ since $\mathcal{U}_1 \cap X_2 = \{x \in X_2 : \left|\frac{f_1(x)}{f_2(x)}\right| \geq 1\}$ and $\mathcal{U}_2 \cap X_2 = \{\left|\frac{f_1}{f_2}\right| \leq 1\}$.

(This is of a form we already know how to handle.) ☐

Now extend to any $f_1, \ldots, f_n \in A$ s.t. $(f_1, \ldots, f_n) = A$ (by induction!). Final step is to show any adm. cov. of X can be suitably refined and then doing some combinatorics.