An Overview of the Braverman-Kazhdan-Ngô Program

Zachary Gardner

May 5, 2021

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- some kind of (ρ-)Schwartz space $C_c^\infty(G(F)) \subseteq \mathscr{S}^\rho(G(F)) \subseteq C^\infty(G(F))$;
- some kind of (ρ-)Fourier transform $\mathbb{F}_\psi^\rho$ acting on $\mathscr{S}^\rho(G(F))$.

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In more detail, let $B \leq G$ be a Borel subgroup and $2\eta_B$ the sum of the associated positive roots. Define $\ell_\rho := \langle 2\eta_B, \lambda_\rho \rangle$, for λ_ρ the highest weight of ρ.

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- *The* $\mathbb{C}[q^{\pm s}]$*-module* $I(s, \pi, \rho)$ *spanned by* $\{Z(s, f, \phi) : f \in \mathscr{S}(G(F)), \phi \in \mathcal{C}(\pi)\}$ *is a principal fractional ideal of* $\mathbb{C}[q^{\pm s}]$ *with generator* $L(s, \pi, \rho)$.

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- *Suppose* π *is unramified, with zonal spherical function* Γ_π. *Then, there is a distinguished* **basic function** $\mathbb{L}^\rho \in \mathscr{S}(G(F))$ *such that* $Z(s, \mathbb{L}^\rho, \Gamma_\pi) = L(s, \pi, \rho)$.

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The basic function $\mathbb{L}^\rho$ is uniquely determined up to a choice of maximal compact special $K \leq G(F)$ by the additional requirement that $\mathbb{L}^\rho$ is bi-K-invariant.

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What about the functional equation? We expect that there is a (local) γ-factor $\gamma(s, \pi, \rho, \psi)$ which is a rational function in q^{-s} such that

$$Z(1 - s, \mathbb{F}_{\psi}^{\rho}(f), \phi^{\vee}) = \gamma(s, \pi, \rho, \psi) Z(s, f, \phi)$$

for every $s \in \mathbb{C}$, $f \in \mathscr{S}^{\rho}(G(F))$, and $\phi \in \mathcal{C}(\pi)$.

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$$\mathscr{S}^{\text{std}}(\mathrm{GL}_n(F)) = C_c^\infty(M_n(F)), \qquad \mathbb{F}_\psi^{\text{std}} = \mathbb{F}_\psi, \qquad \mathbb{L}^{\text{std}} = \mathbb{1}_{M_n(\mathcal{O}_F)},$$

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Letting $\Phi_\psi^{\mathrm{std}} := \psi(\mathrm{tr})\,|\det|_F^n$ and taking $f \in C_c^\infty(\mathrm{GL}_n(F))$ with $f^\vee(g) := f(g^{-1})$, we can rewrite this as

$$\mathbb{F}_\psi^{\mathrm{std}}(f) = |\det|_F^{-n}(\Phi_\psi^{\mathrm{std}} * f^\vee) = |\det|_F^{-\ell_{\mathrm{std}}-1}(\Phi_\psi^{\mathrm{std}} * f^\vee).$$

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for every $f \in C_c^\infty(G(F)) \subseteq \mathscr{S}^\rho(G(F))$. We also expect as in the standard case that $\mathbb{F}_\psi^\rho$ extends to $L^2(G(F), |\sigma|_F^{\ell_\rho + 1}\, dg)$, a useful analytic result in the local setting.

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- The endomorphism ring of the identity functor of the category of smooth representations of $G(F)$.
- The space of $G(F)$-conjugation-invariant essentially compactly supported distributions on Φ – i.e., those Φ such that $\Phi * C_c^\infty(G(F)) = C_c^\infty(G(F))$.

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Given Φ a distribution on $G(F)$ and $n \in \mathbb{Z}$, define

$$G(F)_n := \{g \in G(F) : |\sigma(g)| = q^{-n}\}$$

and $\Phi_n := \Phi \cdot \mathbb{1}_{G(F)_n}$.

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- for every irreducible admissible representation π of $G(F)$ with twist $\pi_s := \pi\, |\sigma|_F^s$, the Laurent series
$$f^{\Phi}(\pi_s) := \sum_{n \in \mathbb{Z}} f^{\Phi_n}(\pi_s)$$
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Such distributions are hand-crafted to give functional equations, as [Luo, Lemma 5.2.4] shows. While this analytic picture is nice, it is not so well suited to passing to the global setting. So, we seek a more geometric perspective.

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It turns out that, under some mild conditions on ρ, there is a more or less unique way to associate an M^ρ as desired to (G, ρ).

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Our first order of business is to find something that plays the same role as M_n does for GL_n. What properties do we want? In the standard setup, the open embedding $\mathrm{GL}_n \hookrightarrow M_n$ has dense image that is invariant under the appropriate action of $\mathrm{GL}_n \times \mathrm{GL}_n$. Moreover, M_n is a monoid with GL_n as unit group. With this in mind, for a pair (G, ρ) we seek a reductive monoid M^ρ with open embedding $G \hookrightarrow M^\rho$ realizing G as the unit group whose image is dense and invariant under an appropriate action of $G \times G$. The following definition makes this more precise.

Definition

A **reductive monoid** over a field k is a normal affine irreducible algebraic variety equipped with the structure of a monoid such that the unit group (i.e., the open subset of invertible elements) is a reductive algebraic group over k.

It turns out that, under some mild conditions on ρ, there is a more or less unique way to associate an M^ρ as desired to (G, ρ). Before we get into the details, though, we first mention what we can do with this.

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$$G(F) = \{g \in \mathrm{GL}_4(F) : {}^t g J_4 g = \lambda J_4 \text{ for some } \lambda \in F^\times\}$$

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One thing this example shows us is that the "obvious" choice for Γ^ρ may be too naïve. This deficiency can be accounted for by examining the singular locus of $\mathrm{MSp}_4(F)$, where Γ^ρ should not be constant but instead satisfy some kind of moderate growth condition.

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Reductive Monoids

Returning to our discussion of reductive monoids, the first item indicating we are on the right track is the following result.

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Let G be a reductive algebraic group over a field k. Then, the category of reductive monoids with unit group G is equivalent to the category of $G \times G$-affine spherical embeddings of G – i.e., those embeddings for which there is an open dense orbit of some Borel subgroup of $G \times G$.

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- Let M be a smooth reductive monoid with one-dimensional center. Then, $M \cong M_n$ for some n. Hence, our desired M^ρ will in general be singular and so perverse sheaves enter the fray (with basic functions arising as traces of suitable intersection complexes).

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$$\mathscr{S}^\rho(\mathbb{A}_F) := \varinjlim \bigotimes_{v \in S} \mathscr{S}^\rho(G(F_v))$$

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equipped with a Fourier transform (built from the local Fourier transforms) satisfying a Poisson summation formula. Here, the direct limit is taken over S *a finite set of places of* F *containing the archimedean places and the transition maps are given on pure tensors by*

$$\otimes_{v \in S} f_v \mapsto \otimes_{v \in S' - S} \mathbb{L}_v^\rho \otimes \otimes_{v \in S} f_v$$

with $S \subseteq S'$ *and* $f_v \in \mathscr{S}^\rho(G(F_v))$.