DRINFELD MODULAR VARIETIES: HOMEWORK 2

A **quasi-projective scheme** over a ring B is an open subscheme of a projective scheme over B. An **ample** line bundle on such a scheme is one obtained via pull-back of an ample line bundle from the projective scheme.

A quasi-projective scheme X over a field k is **connected** if, for any two open subschemes $V_1, V_2 \subset X$, we have $V_1 \cap V_2 \neq \infty$. It is **integral** if, for any affine open $\operatorname{Spec} A \subset X$, A is an integral domain.

(1) Suppose that X is integral and connected. Show that, for any two non-empty affine opens $\operatorname{Spec} A_1, \operatorname{Spec} A_2 \subset X$, there is a canonical isomorphism $\operatorname{Frac}(A_1) \simeq \operatorname{Frac}(A_2)$ of fraction fields.

Therefore, we can associate with X a field $K(X)$, which is called the field of **rational functions on** X. In other words, a rational function is simply a global section defined over a non-empty affine open.

(2) Suppose that $\mathcal{L}$ is a line bundle on a quasi-projective scheme X. For any section $s \in H^0(X, \mathcal{L})$, the **zero locus** $Z(s) \subset X$ is the sub-functor given for $T \in \mathrm{Alg}_B$ by

$$Z(s)(T) = \{x \in X(T) : \; s_x = 0 \in \mathcal{L}_x\}.$$

(a) Show that $Z(s)$ is a closed subscheme of X, whose ideal sheaf is isomorphic to $\mathcal{L}^\vee$.
(b) Show that the open complement $D(s) \subset X$ of $Z(s)$ is given by

$$D(s)(T) = \{x \in X(T) : \; T \cdot s_x = \mathcal{L}_x\}.$$

(c) Suppose that $\mathcal{L}$ is ample for the rest of the problem. Show that $D(s)$ is affine.
(d) For d sufficiently large, show that there are finitely many sections $s_1, \ldots, s_r \in H^0(X, \mathcal{L}^{\otimes d})$ such that $\{D(s_i)\}$ is an affine open covering for U.
Hint: For these two parts, reduce to the case where $X = \mathbb{P}^n_B$ and where $\mathcal{L} = \mathcal{O}(1)$.
(e) Show that there is a surjective map

$$\mathcal{O}_X^r \xrightarrow{(s_1,\ldots,s_r)} \mathcal{L}^{\otimes d}.$$

(3) Let X be a quasi-projective scheme over B. For any $f \in H^0(X, \mathcal{O}_X)$, and any quasi-coherent sheaf $\mathcal{F} \in \mathrm{QCoh}(X)$, show that we have a canonical isomorphism

$$H^0(D(f), \mathcal{F}) \simeq H^0(X, \mathcal{F})[f^{-1}].$$

Hint: Put X inside $\mathbb{P}^n_B$ and compute the left hand side using a Cech complex.

(4) In the above situation, conclude that the following are equivalent:
(a) $D(f)$ is affine.
(b) $D(f) \simeq \operatorname{Spec} H^0(X, \mathcal{O}_X)[f^{-1}]$.

for any $F : \mathrm{Alg}_B \to \mathrm{Set}$, there is a canonical map $F \to \operatorname{Spec} H^0(F, \mathcal{O}_F)$: For any $x \in F(S)$, we obtain a map of rings $H^0(F, \mathcal{O}_F) \to S$ given by evaluating global sections at x.

(5) Show that the following are equivalent for a quasi-projective scheme X over B:
(a) The map $X \to \operatorname{Spec} H^0(X, \mathcal{O}_X)$ is an open immersion.
(b) There exist $f_1, \ldots, f_r \in H^0(X, \mathcal{O}_X)$ such that $\{D(f_i)\}_{1 \le i \le r}$ is an affine open cover of X.

(6) Show that the following are equivalent for X as above:
(a) The map $X \to \operatorname{Spec} H^0(X, \mathcal{O}_X)$ is an isomorphism.
(b) X is affine.
(c) There exist $f_1, \ldots, f_r \in H^0(X, \mathcal{O}_X)$ such that $\{D(f_i)\}$ is an open affine cover for X, and such that $(f_1, \ldots, f_r) = H^0(X, \mathcal{O}_X)$ as ideals.

(7) (Serre's criterion for affineness) Let X be a quasi-projective scheme over B, and let $\mathcal{L}$ be an ample line bundle over X, so that the conclusions of 2(c),(d),(e) hold. Assume that $H^1(X, \mathcal{F}) = 0$ for all coherent sheaves $\mathcal{F} \in \mathrm{Coh}(X)$.
  (a) Tensor the surjection from 2(e) with $\mathcal{L}^{\vee,\otimes d}$ to obtain a surjection $(\mathcal{L}^{\vee,\otimes d})^r \to \mathcal{O}_X$.
  (b) Use the vanishing assumption to show that, for some $t \leq r$, there are non-zero homomorphisms
$$\varphi_1, \ldots, \varphi_t : \mathcal{L}^{\otimes d} \to \mathcal{O}_X$$
  such that $\varphi_1(s_1) + \ldots + \varphi_t(s_t) = 1$ for some non-zero sections $s_1, \ldots, s_t \in H^0(X, \mathcal{L}^{\otimes d})$
  (c) Set $f_i = \varphi_i(s_i)$ and show that $D(f_i) = D(s_i)$ is affine.
  (d) Conclude that X is affine.

Fix a ring R and an ideal $I \leq R$.[1] Consider the **blowup algebra**
$$\mathrm{Bl}_I(R) = \bigoplus_{n \geq 0} I^n t^n = \{\sum_n a_n t_n \in R[t] : \ a_n \in I^n\}.$$

For any R-module equipped with a descending filtration
$$\cdots \subset M^n \subset M^{n-1} \subset \cdots \subset M^1 \subset M^0 = M$$
set
$$\mathrm{Bl}(M) = \bigoplus_{n \geq 0} M^n t^n = \{\sum_n m_n t^n \in R[t] \otimes_R M : \ m_n \in M^n\}.$$

(8) (a) Show that $\mathrm{Bl}_I(R)$ is a graded finitely generated R-algebra and hence Noetherian.
  (b) Suppose that M is an R-module equipped with a descending filtration M^n such that $I \cdot M^n \subset M^{n+1}$ for all n. Show that $\mathrm{Bl}(M)$ has the natural structure of a graded $\mathrm{Bl}_I(R)$-module.
  (c) With the previous hypotheses, show that the following are equivalent:
    (i) $\mathrm{Bl}(M)$ is generated as a $\mathrm{Bl}_I(R)$-module by homogeneous elements of bounded degree.
    (ii) For all n sufficiently large, we have $I \cdot M^n = M^{n+1}$.

We will say that a descending filtration M^n on an R-module M is **I-adic** if:
- $I \cdot M^n \subset M^{n+1}$ for all n.
- $I \cdot M^n = M^{n+1}$ for all n sufficiently large.

(9) Suppose that M is a finitely generated R-module equipped with a descending filtration satisfying the first condition, so that $\mathrm{Bl}(M)$ is a graded $\mathrm{Bl}_I(R)$-module. Show that the following are equivalent:
  (a) $\mathrm{Bl}(M)$ is finitely generated over $\mathrm{Bl}_I(R)$.
  (b) The filtration is I-adic.

(10) (Artin-Rees) Suppose that M is a finitely generated R-module equipped with a descending I-adic filtration M^n. Then show that the filtration $N \cap M^n$ is I-adic for any R-submodule $N \subset M$. In particular, we have
$$N \cap I^n M \subset IN$$
for n sufficiently large.

(11) Let X be a smooth quasi-projective curve over a field k[2]. Exhibit a bijection between:
  (a) $K(X)$;
  (b) Morphisms $X \to \mathbb{P}^1_k$ of schemes over k.
  Hint: You only have to deal with the case where $X = \mathrm{Spec}\, A$ where A is a Dedekind domain. To go from (b) to (a), note that if you have $f : X \to \mathbb{P}^1_k$, then the pull-back of $t = T_1/T_0$ from U_0 defines a rational function on X.

[1] Recall that all our rings are Noetherian and commutative unless otherwise stated.
[2] This just means that you're looking at an open subscheme of a smooth projective curve

A map $f : F \to G$ of functors on Alg_B has **finite fibers** if, for all fields $k \in \mathrm{Alg}_B$, the map $F(k) \xrightarrow{f_k} G(k)$ has finite fibers. We will say that F has finite fibers if the map $F \to \operatorname{Spec} B$ does. We will say that F is **quasi-finite** if it is quasi-projective and has finite fibers.

(12) (a) Suppose that $F = \operatorname{Spec} A$ where A is a finitely generated B-module. Show that $F \to \operatorname{Spec} A$ has finite fibers.
  (b) Let X be a smooth projective curve, and let $Z \subset X$ be a proper closed subscheme. Show that the map $Z \to \operatorname{Spec} k$ has finite fibers.
  Hint: This amounts to the fact that any non-trivial quotient of a Dedekind domain A that is a finite dimensional k-algebra is a finite dimensional k-vector space.
  (c) Let X be a smooth projective curve, and let $f : X \to \mathbb{P}^1_k$ be a map associated with a non-zero rational function in $K(X)$. Show that f has finite fibers.

(13) Let Z be a quasi-finite scheme over k.
  (a) Show that Z admits an open cover by disjoint affine schemes of the form $\operatorname{Spec} A$ with A a local finite dimensional k-algebra.
  (b) Conclude that for any coherent sheaf $\mathcal{F} \in \mathrm{Coh}(Z)$, we have $H^i(Z, \mathcal{F}) = 0$ for $i > 0$.