Assignment 9

due by midnight, Friday, December 4, 2020

Prove all statements in the questions below. You must TeX your solutions and submit your writeup as a pdf at www.gradescope.com.
You can submit it any time until the deadline.

Question 1. (Folland 6.1.2) Prove Theorem 6.8.

Question 2. (Folland 6.1.3) If $1 \leq p < r \leq \infty$, $L^p \cap L^r$ is a Banach space with norm $||f|| = ||f||_p + ||f||_r$, and if $p < q < r$, the inclusion map $L^p \cap L^r \to L^q$ is continuous.

Question 3. (Folland 6.1.7) If $f \in L^p \cap L^\infty$ for some $p < \infty$, so that $f \in L^q$ for all $q > p$, then $||f||_\infty = \lim_{q \to \infty} ||f||_q$.

Question 4. (Folland 6.3.33) Given $1 < p < \infty$, let $Tf(x) = x^{-1/p} \int_0^x f(t)dt$. It $p^{-1} + q^{-1} = 1$, then T is a bounded linear map from $L^q((0, \infty))$ to $C_0((0, \infty))$.

Question 5. (Folland 7.1.2) Let μ be a Radon measure on X (a LCH space).

1. Let N be the union of all open $U \subset X$ such that $\mu(U) = 0$. Then N is open and $\mu(N) = 0$. (The complement of N is called the support of μ.)

2. $x \in \text{supp}(\mu)$ iff $\int f \, d\mu > 0$ for every $f \in C_c(X, [0, 1])$ such that $f(x) > 0$.

Question 6. (Folland 7.2.8) Suppose that μ is a Radon measure on X. If $\phi \in L^1(\mu)$ and $\phi \geq 0$, then $\nu(E) = \int_E \phi \, d\mu$ is a Radon measure. (Use Corollary 3.6.)