[HAG2, Remark 1.3.3.2]

Let $\mathcal{P}$ be auxiliary class of morphisms. Stack is geometric if it is quotient of (disjoint union of) representable stack X by grpoid. object X_1 acting on X s.t. X_1 is representable and $X_1 \to X$ lies in $\mathcal{P}$.

Before all of that, let's survey the (classical) stack essentials. We will follow Olsson.

Let $\mathcal{C}$ be cat. *Cat. over $\mathcal{C}$* is cat. F w/ functor $p : F \to \mathcal{C}$. We encode this by pair (F, p).

$\phi \in \mathrm{Hom}_F(u,v)$ is *Cartesian* if, given $\psi \in \mathrm{Hom}_F(w,v)$ and factorization

$$\begin{tikzcd} p(w) \arrow[r, "h"] \arrow[rr, bend right, "p(\psi)"'] & p(u) \arrow[r, "p(\phi)"] & p(v) \end{tikzcd}$$

$\exists!\ \lambda \in \mathrm{Hom}_F(w,u)$ s.t. $\phi \circ \lambda = \psi$ and $p(\lambda) = h$.

$$\begin{tikzcd} w \arrow[r, dashed, "\exists!\lambda"'] \arrow[d, mapsto] \arrow[rr, bend left, "\psi"] & u \arrow[r, "\phi"] \arrow[d, mapsto] & v \arrow[d, mapsto] \\ p(w) \arrow[r, "h"'] & p(u) \arrow[r, "p(\phi)"'] & p(v) \end{tikzcd}$$

[I believe this is somewhat related to retraction...]

The name comes from thinking of this as a pullback.

View this as the *fiber* of p above $\mathcal{U}$.

Let (F,p) over $\mathcal{C}$ and $\mathcal{U} \in \mathcal{C}$. $F(\mathcal{U})$ is the cat. w/ objects $u \in F$ s.t. $p(u) = \mathcal{U}$ and morphisms $f \in \mathrm{Hom}_F(u',u)$ s.t. $p(f) = \mathrm{id}_{\mathcal{U}}$. We're now able to discuss fibered cat.'s.

We say (F,p) over $\mathcal{C}$ is *fibered* if $\forall f \in \mathrm{Hom}_{\mathcal{C}}(\mathcal{U},\mathcal{V})$ and $v \in F(\mathcal{V})$ $\exists$ Cartesian $\phi \in \mathrm{Hom}_F(u,v)$ s.t. $p(\phi) = f$.

$(F, p_F) \to (G, p_G)$ is the data of $g \in \mathrm{Fun}(F,G)$ s.t. $p_G \circ g = p_F$ and g sends Cartesian to Cartesian.

If g, g' are two such then *base-preserving* $\alpha : g \to g'$ is nat. trans. s.t. $\forall u \in F$: $\alpha_u \in \mathrm{Hom}_G(g(u), g'(u))$ projects to identity morphism in $\mathcal{C}$.

$$\begin{tikzcd} g(u) \arrow[rr, "\alpha_u"] \arrow[dr, "p_G"'] & \circlearrowright & g'(u) \arrow[dl, "p_G"] \\ & p_F(u) & \end{tikzcd}$$

$\rightsquigarrow$ $\mathrm{HOM}_{\mathcal{C}}(F,G)$ cat. w/ objects morphisms of fibered cat.'s $F \to G$ / $\mathcal{C}$ and morphisms base-preserving nat. trans.'s.

$g \in HOM_{\mathcal{C}}(F,G)$ is *equivalence* if $\exists\, h \in HOM_{\mathcal{C}}(G,F)$ and base-preserving isom.'s

$h \circ g \simeq id_F$ and $g \circ h \simeq id_G$. This holds:

$\Leftrightarrow \forall\, U \in \mathcal{C}$: $g_U : F(U) \to G(U)$ is equiv.

Cat. fibered in grpoids over $\mathcal{C}$ is fibered $(F,p)/\mathcal{C}$ s.t. $\forall\, U \in \mathcal{C}$: $F(U)$ is grpoid.

Prop: F, F' cat.'s fibered in grpoids $/\mathcal{C} \Rightarrow HOM_{\mathcal{C}}(F,F')$ is grpoid.

Let $\mathcal{C}$ be cat. (w/ finite fiber products). *Grpoid in $\mathcal{C}$* is just grpoid object in $\mathcal{C}$, like a "decoupled" grp. object. This is data of $(X_0, X_1, s, t, \varepsilon, i, m)$. What can we do w/ it? [79] Olsson describes a cat. $\{X_0(U)/X_1(U)\}$ w/ objects $u \in X_0(U)$ and morphisms $\xi \in X_1(U)$ s.t. $s(\xi) = u$ and $t(\xi) = u'$. We can write this as $u \xrightarrow{\xi} u'$ or just $u \to u'$. $Hom_{\{X_0(U)/X_1(U)\}}(u,u') = \{\xi \in X_1(U) : s(\xi) = u, t(\xi) = u'\}$

$$u'' \xrightarrow{\eta} u' \xrightarrow{\xi} u \rightsquigarrow \xi \circ \eta := \text{image under } m \text{ of } (\xi, \eta) \in X_1(U) \underset{X_0(U),s,t}{\times} X_1(U).$$

Fact: $\{X_0(U)/X_1(U)\}$ is grpoid.

Remark: In the above we have chosen $U \in \mathcal{C}$. $X_0(U), X_1(U)$ is just notation indicating dependence on U.

We wish to define a fibered cat. $p : \{X_0/X_1\} \to \mathcal{C}$. The objects are (U, u) w/ $U \in \mathcal{C}$ and $u \in \{X_0(U)/X_1(U)\}$. $(V, v) \to (U, u)$ is data of (f, α) w/ $f \in Hom_{\mathcal{C}}(V, U)$ and $\alpha : v \to f^* u$ isom. in $\{X_0(V)/X_1(V)\}$. $f^* : \{X_0(U)/X_1(U)\} \to \{X_0(V)/X_1(V)\}$ here is induced by

$f^* : X_0(U) \to X_0(V)$ and $f^* : X_1(U) \to X_1(V)$.

$\mathcal{C}$ cat. (w/ finite fiber products), $(F,p)/\mathcal{C}$ fibered in grpoids, $X \in \mathcal{C}$

⇝ $p_{/X} : F_{/X} \to (\mathcal{C}/X)$ fibered in grpoids.

$F_{/X}$ has objects (y,f) w/ $y \in F$, $f \in \mathrm{Hom}_{\mathcal{C}}(p(y), X)$ and compatible morphisms. $p_{/X}$ is forgetful.

Remark: $f : Y \to X$ has fiber $F_{/X}(f : Y \to X) \underset{(\text{canon.})}{\simeq} F(Y)$, a grpoid.

Let $X \in \mathcal{C}$ and $x, x' \in F(X)$. We get presheaf $\underline{\mathrm{Isom}}(x,x') : (\mathcal{C}/X)^{op} \to \mathrm{Set}$ via...

Given $f \in \mathrm{Hom}_{\mathcal{C}}(Y,X)$, choose pullbacks f^*x, f^*x' and define $\underline{\mathrm{Isom}}(x,x')(f) := \mathrm{Isom}_{F(Y)}(f^*x, f^*x')$.

$\underline{\mathrm{Isom}}(x,x) =: \underline{\mathrm{Aut}}_x$. $\underline{\mathrm{Isom}}(x,x')$ is independent of choice of pullbacks up to canon. isom.

Torsors and principal homogeneous spaces

pair $(\mathcal{P}, \rho)$

$\rho : \mu \times \mathcal{P} \to \mathcal{P}$

$\mathcal{C}$ site, μ sheaf of grps. on $\mathcal{C}$. $\underline{\mu\text{-torsor}}$ on $\mathcal{C}$ is sheaf $\mathcal{P}$ on $\mathcal{C}$ w/ left action ρ of μ on $\mathcal{P}$ s.t.

(T1) $\forall X \in \mathcal{C}\ \exists$ cov. $\{X_i \to X\}$ s.t. $\mathcal{P}(X_i) \neq \emptyset\ \forall i$;

(T2) shear map $\mu \times \mathcal{P} \to \mathcal{P} \times \mathcal{P}$ is isom. $\Leftrightarrow$ ($\mathcal{P}(X) \neq \emptyset \Rightarrow \mu(X) \curvearrowright \mathcal{P}(X)$ simply transitive).

$(\mathcal{P}, \rho)$ is trivial if $\mathcal{P}$ has global section. This section yields isom. $\mu \xrightarrow{\sim} \mathcal{P}$.

Remark: Notion of μ-torsor depends only on the underlying topos.

Fix now $X \in \mathrm{Sch}$ and equip Sch/X w/ fppf top. Assume μ representable by flat loc. fin. pres. X-grp. scheme G.

Principal G-bundle over X is (π, ρ) w/ $\pi : P \to X$ flat, LFP, surj. and $\rho : G \times_X P \to P$ satisfying expected axioms.

Facts: (1) Yoneda ⇝ $\{$principal G-bundles $/X\} \hookrightarrow \{\mu\text{-torsors} /X\}$. ← (Need to prove here that suitable sections exist.)

(2) $G \to X$ affine $\Rightarrow$ Yoneda ⇝ equiv. of cat.'s.

$\mathcal{C}$ site, $p: F \to \mathcal{C}$ fibered in gpoids. (F,p) is stack if $\forall X \in \mathcal{C}$ and cov. $\{X_i \to X\}_{i \in I}$ the functor
(arises from choice of pullback functors $F(X) \to F(X_i)$)

$F(X) \to \underbrace{F(\{X_i \to X\})}_{\text{descent cat.}}$ is equiv. of cat.'s. Let's unpack this last functor. We care about the iterated fiber products $X_{i_1} \times_X \cdots \times_X X_{i_n}$. We write down isom.'s over double intersections compatible over triple intersections. $F(\{X_i \to X\})$ consists of $E_i \in F(X_i)$ together w/ $\sigma_{ij} \in \underbrace{\mathrm{Isom}_{F(X_i \times_X X_j)}(pr_1^* E_i, pr_2^* E_j)}$ s.t. compatibility holds relative to all i, j, k.

$$\begin{tikzcd} & X_i \times_X X_j \arrow[dl, "pr_1"'] \arrow[dr, "pr_2"] & \\ X_i & & X_j \end{tikzcd}$$

Lemma 4.2.7: Effectivity can be characterized by gluing (assuming relevant coproducts exist).

$(F(X) \xrightarrow{\sim} F(f))$

$\Updownarrow$

Claim: $(F,p)/\mathcal{C}$ fibered as above, $f: X \to Y$ in $\mathcal{C}$. If f admits section then f is effective descent morphism.

Prop: Let $p: F \to \mathcal{C}$ be cat. fibered in gpoids.

(i) $\forall X \in \mathcal{C}$, $x, y \in F(X)$: Isom(x,y) presheaf on $\mathcal{C}/X$ is a sheaf (prestack condition)

(ii) $\forall$ cov. $\{X_i \to X\}$ of $X \in \mathcal{C}$: descent data w.r.t. this cov. is effective.

Then, (F,p) is stack iff (i)+(ii) holds.

Fact: Any cat. fibered in gpoids admits a stackification.

We will be especially interested in stacks fibered over Sch/S for the étale top.

stack = étale stack

$f : \mathcal{X} \to \mathcal{Y}$ morphism of stacks is representable if $\forall\, \mathcal{U} \in Sch$ and $y : \mathcal{U} \to \mathcal{Y}$ the fiber product $\mathcal{X} \times_{\mathcal{Y}, y} \mathcal{U}$ is alg. space.

Lemma : $f : \mathcal{X} \to \mathcal{Y}$ representable $\Rightarrow \forall$ alg. space V and $y : V \to \mathcal{Y}$ the fiber product $V \times_{\mathcal{Y}} \mathcal{X}$ is alg. space

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Contrast the above notion w/ the following. Let $S \in Sch$ and $f \in Hom_{Shv(S_{ét})}(F, G)$.

(i) f is representable by schemes if $\forall\, T \in Sch/S$ and $T \to G$ the fiber product $F \times_G T$ is scheme.

(ii) Let P be stable morphism property and f representable by schemes. Then, f has P if $\forall\, T \in Sch/S$ the morphism $pr_2 : F \times_G T \to T$ has P.

Stack $\mathcal{X}/S$ is Artin if
(i) $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is representable.
(ii) $\exists$ smooth surj. $\pi : X \to \mathcal{X}$ w/ $X \in Sch$.

Lemma : Let $\mathcal{X}$ /S be ~~st~~ stack/S. Then, $\Delta : \mathcal{X} \to \mathcal{X} \times_S \mathcal{X}$ is rep. iff $\forall\, \mathcal{U} \in Sch/S$ and $u_1, u_2 \in \mathcal{X}(\mathcal{U})$ the sheaf $\underline{Isom}(u_1, u_2)$ on $Sch/\mathcal{U}$ is alg. space. [This lets us not think about diagonals...]

Let X be alg. space and G/S smooth gp. scheme acting on S. Define $[X/G]$ to be the stack w/ objects $(T, \mathcal{P}, \pi)$ s.t.

(i) $T \in Sch/S$;
(ii) $\mathcal{P}$ is G_T-torsor on big étale site of T ;
(iii) $\pi : \mathcal{P} \to X_T$ is G_T-equivariant morphism of sheaves on Sch/T.

Morphism ~~of~~ $(T', \mathcal{P}', \pi') \to (T, \mathcal{P}, \pi)$ is (f, f^b) w/ $f \in Hom_{Sch/S}(T', T)$ and $f^b : \mathcal{P}' \to f^*\mathcal{P}$ isom. of $G_{T'}$-torsors on (Sch/T'). s.t.

$$\begin{tikzcd}
\mathcal{P}' \arrow[rr, "f^b"] \arrow[dr, "\pi'"'] & \arrow[d, phantom, "\circlearrowleft"] & f^*\mathcal{P} \arrow[dl, "f^*\pi"] \\
& X_{T'} &
\end{tikzcd}$$

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Fact : $[X/G]$ is Artin stack. [171]

$BG := [S/G]$ for $G \curvearrowright S$ smooth trivially.

$X \in Sch$, μ sheaf of ab. grps. on $X_{ét}$ $\Rightarrow H^1_{ét}(X,\mu) \cong \{\mu\text{-torsors on } X\}/_{\cong}$ (grp. isom.)

__Example__: $H^1_{ét}(X, \mathbb{G}_m) \cong Pic(X)$.

__Remark__: Need to be careful since $H^1_{ét}(X,\mu)$ and $H^1(X_{ét},\mu)$ are different in general (I think...).

We will see $H^2_{ét}(X,\mu) \leftrightarrow \{\overbrace{\mu\text{-gerbes}}^{\text{(loosely, twists of } B\mu)} \text{ on } X\}$.

Let $\mathcal{C}$ be site and μ sheaf of ab. grps. $/\mathcal{C}$. Let $p: F \to \mathcal{C}$ be stack $/\mathcal{C}$. To $x \in F$ we have $X := p(x)$ and sheaf $\underline{Aut}_x$ over $\mathcal{C}/X$. __μ-gerbe__ $/\mathcal{C}$ is data of stack $/\mathcal{C}$ $p: F \to \mathcal{C}$ w/ isom. of sheaves of grps.

$\iota_x : \mu|_{\mathcal{C}/p(x)} \to \underline{Aut}_x$ $\forall x \in F$ s.t.

(G1) $\forall Y \in \mathcal{C}$ $\exists$ cov. $\{Y_i \to Y\}$ s.t. $F(Y_i) \neq \emptyset$ $\forall i$;

(G2) $\forall y, y' \in F(Y)$ over $Y \in \mathcal{C}$ $\exists$ cov. $\{f_i : Y_i \to Y\}$ s.t. $f_i^* y \cong f_i^* y'$ in $F(Y_i)$ $\forall i$;

(G3) $\forall Y \in \mathcal{C}$, isom. $\sigma : y \to y'$ in $F(Y)$:

$$\begin{tikzcd} & \mu \arrow[dl, "\iota_y"'] \arrow[dr, "\iota_{y'}"] \arrow[d, phantom, "\circlearrowright"] & \\ \underline{Aut}_y \arrow[rr, "\sigma"'] & & \underline{Aut}_{y'} \end{tikzcd}$$

$\mathcal{X}/S$ alg. stack. $\underline{\mathcal{X}\text{-space}}$ is (T,t) w/ T alg. space $/S$ and $t : T \to \mathcal{X}$.

k field of char. $p>0$, $R := k[\varepsilon,t]/(\varepsilon^2)$ $\not k$

$\mathbb{Z}/(p) \curvearrowright R$ over $k[\varepsilon]/(\varepsilon^2)$ via $t \mapsto t+\varepsilon$

$R^{\mathbb{Z}/(p)} \otimes_{k[\varepsilon]/(\varepsilon^2),\, \varepsilon\mapsto 0} k \to (R \otimes_{k[\varepsilon]/(\varepsilon^2)} k)^{\mathbb{Z}/(p)}$ is $\underline{\text{not}}$ inj. or surj.

Indicates some kind of infinitesimal phenomenon.

$X \xrightarrow{f} Y \xrightarrow{g} Z$ w/ gf formally unram. ⇒ f formally unram.

$0 \twoheadrightarrow \Omega^1_{X/Y} \Rightarrow \Omega^1_{X/Y} = 0$

$\Omega^1_{X/\cancel{Y}Z} = 0$

$f^*\Omega^1_{Y/Z} \to \underset{\overset{\shortparallel}{0}}{\Omega^1_{X/Z}} \to \Omega^1_{X/Y} \to 0$

(4.4.13) ⇒ $\mathcal{M}_g$ is stack for étale top.

Fix $g \in \mathbb{Z}^{>1}$ ⇝ $\mathcal{M}_g$ cat. fibered over Sch w/ objects (S,f) for $S \in \mathrm{Sch}$ and $f \in \mathrm{Hom}_{\mathrm{Sch}}(C,S)$ s.t.

f is proper smooth w/ geom. fibers each conn. genus g curve. $\underline{\text{Thm}}$: $\mathcal{M}_g$ is DM stack! (8.4.5)

↖ (One of the most "geometric" things in the notes...)

Note that there is morphism of fibered cat.'s $\mathcal{M}_g \to \mathcal{P}ol$, $(C \to S) \mapsto (C \to S, \Omega^1_{C/S})$. What's the latter cat.?

Object of $\mathcal{P}ol$ is (f, L) w/ $f : X \to Y$ proper flat in Sch and

L rel. ample inv. sheaf $/ X$.

Work w/ relevant stack examples ↘

(vii) ↗

equiv. relation ↓

$X \times_{\underbrace{X/R}_{\text{sheafified quotient}}} X \cong R$

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1. (16 points)

(a) Write an equation for the line L through the point $(3,-7,1)$ and parallel to $\vec{v} = \langle 2,5,0\rangle$.

(b) Does the line $\mathcal{I}$ given by

$$x = 6s-3, y = 4s, z = s-1$$

intersect line L? If so, at what point? If not, why not?

(c) Write an equation for the plane P through the point $(0,1,3)$ and perpendicular to $\vec{w} = \langle 1,4,2\rangle$.

(d) Consider the curve of intersection of $z = x^2$ and the plane P. Find a parametrization of this curve.