Let K be nonarch. field w/ value $\overset{\text{dense}}{\wedge}$ grp. $\Gamma := |K^\times| \le \mathbb{R}^{>0}$. Let $\mathcal{O}$ be ring of integers, $\pi \in \mathcal{O}$ top. nil.

Logarithm w/ base $|\pi|$ lets us identify $\mathbb{R}^{>0}$ w/ $\mathbb{R}$ and induces val. map $v : K \to \mathbb{R} \cup \{\infty\}$, $\pi \mapsto 1$. Induced subgrp. is defined to be $\log \Gamma$. Given $r \in \log\Gamma$ we fix an elt., which we write $\pi^r \in K$, satisfying $|\pi^r| = |\pi|^r$.

Facts: Maximal ideal $\mathfrak{m} \trianglelefteq \mathcal{O}$ is gen. by $\{\pi^\varepsilon\}_{\varepsilon>0}$ and satisfies $\mathfrak{m}^2 = \mathfrak{m}$.

Def: $\mathrm{Mod}_{\mathcal{O}^a}$ = almost $\mathrm{Mod}_{\mathcal{O}}$ = quotient cat. of $\mathrm{Mod}_{\mathcal{O}}$ by almost zero modules (i.e., $M \in \mathrm{Mod}_{\mathcal{O}}$ s.t. $\mathfrak{m}M = 0$).

There is almostification functor $M \mapsto M^a$, $\mathrm{Mod}_{\mathcal{O}} \to \mathrm{Mod}_{\mathcal{O}^a}$. [When we take $\varepsilon > 0$ it will be implicit that $\varepsilon \in \log\Gamma$.]

Def: Let $M, N \in \mathrm{Mod}_{\mathcal{O}}$ and $\varepsilon > 0$ in $\log\Gamma$. $M \approx_\varepsilon N$ iff $\exists\, f_\varepsilon : M \to N$, $g_\varepsilon : N \to M$ s.t. $f_\varepsilon g_\varepsilon = \pi^\varepsilon = g_\varepsilon f_\varepsilon$.

$M \approx N$ iff $M \approx_\varepsilon N$ $\forall \varepsilon > 0$. [Note that $M \approx_\varepsilon N$ and $N \approx_\delta L \Rightarrow M \approx_{\varepsilon+\delta} L$.]

Facts:
- M is almost zero iff $M \approx 0$
- $M, N \in \mathrm{Mod}_{\mathcal{O}}$ w/ $M^a \cong N^a$ in $\mathrm{Mod}_{\mathcal{O}^a} \Rightarrow M \approx N$ (converse need not hold...)

Def: $M \in \mathrm{Mod}_{\mathcal{O}}$.
M is almost fin. gen. if $\forall \varepsilon > 0 : \exists\, N_\varepsilon \in \mathrm{Mod}_{\mathcal{O}}$ fin. gen. s.t. $M \approx_\varepsilon N_\varepsilon$.
M is almost fin. pres. " " fin. pres. " "

These notions depend only on M^a.

Examples: (1) Let $r \in \mathbb{R}^{\ge 0}$ and define $I_r := \bigcup_{\varepsilon \in \log\Gamma,\, \varepsilon > r} \pi^\varepsilon \mathcal{O}$, an ideal of $\mathcal{O}$. Every nonprincipal ideal of $\mathcal{O}$ looks like this, hence $I \subseteq \mathcal{O}$ nonzero ideal $\Rightarrow I \approx \mathcal{O} \Rightarrow I$ almost fin. pres.

(2) $\{\gamma_i\}_{i\ge 0} \subseteq \mathbb{R}^{\ge 0}$ s.t. $\lim_{i\to\infty} \gamma_i = 0 \Rightarrow \mathcal{O}/I_{\gamma_1} \oplus \mathcal{O}/I_{\gamma_2} \oplus \cdots$ is almost fin. pres.

Thm: $M \in \mathrm{Mod}_{\mathcal{O}}$ almost fin. gen. $\Rightarrow \exists!$ $\overset{r \in \mathbb{Z}^{\ge 0} \text{ and}}{\wedge}$ $\gamma_1 \ge \gamma_2 \ge \cdots \ge 0$ of real numbers w/ $\lim_{i\to\infty} \gamma_i = 0$ s.t. $M \approx \mathcal{O}^r \oplus \mathcal{O}/I_{\gamma_1} \oplus \mathcal{O}/I_{\gamma_2} \oplus \cdots$