Finite Flat Group Schemes

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References:

- "There is no abelian scheme over $\mathbb{Z}$" by Gyujin Oh = [Oh]
- "A course on finite flat group schemes and p-divisible groups" by Jakob Stix = [Stix]
- Hard Arithmetic blog post "p-divisible groups, formal groups, and the Serre-Tate theorem" by Alex Youcis = [Youcis]

1 Introduction

Let S be a scheme. An S-**group scheme** is a group object in Sch_S – i.e., an S-scheme G together with morphisms $m : G \times_S G \to G$, $i : G \to G$, and $e : S \to G$ such that the following diagrams commute:

(Associativity)

$$\begin{tikzcd}
(G \times_S G) \times_S G \arrow[rr, "\cong"] \arrow[dr, "m \times \mathrm{id}_G"'] & & G \times_S (G \times_S G) \arrow[dl, "\mathrm{id}_G \times m"] \\
& G \times_S G \arrow[d, "m"] & \\
& G &
\end{tikzcd}$$

(Identity)

$$\begin{tikzcd}
G \times_S S \arrow[r, "\mathrm{id}_G \times e"] \arrow[dr, "\mathrm{pr}_1"'] & G \times_S G \arrow[d, "m"] & S \times_S G \arrow[l, "e \times \mathrm{id}_G"'] \arrow[dl, "\mathrm{pr}_2"] \\
& G &
\end{tikzcd}$$

(Inverses)

$$\begin{tikzcd}
G \arrow[r, "\Delta"] \arrow[d] & G \times_S G \arrow[dd, bend right, "\mathrm{id}_G \times i"'] \arrow[dd, bend left, "i \times \mathrm{id}_G"] \\
S \arrow[d, "e"'] & \\
G & G \times_S G \arrow[l, "m"]
\end{tikzcd}$$

In the above, $\Delta : G \to G \times_S G$ denotes the (canonical) diagonal morphism and pr_i denotes a (canonical) projection morphism. By Yoneda's Lemma, this is the same data as a group structure on the functor-of-points $h^G := \mathrm{Hom}_{\mathsf{Sch}_S}(\cdot, G)$ of G – i.e., for every S-scheme T a group structure on $G(T)$ that is functorial in T (note that this is entirely determined by affine T). Equivalently, there is a factorization

$$\begin{tikzcd} \mathsf{Sch}_S^{\mathrm{op}} \arrow[rr, "h^G"] \arrow[dr, dashed] & & \mathsf{Set} \\ & \mathsf{Grp} \arrow[ur] & \end{tikzcd}$$

where the solid unmarked arrow is forgetful. We say that G is **commutative** if there is a factorization

$$\begin{tikzcd} \mathsf{Sch}_S^{\mathrm{op}} \arrow[rr, "h^G"] \arrow[dr, dashed] & & \mathsf{Set} \\ & \mathsf{Ab} \arrow[ur] & \end{tikzcd}$$

which is equivalent to requiring that G is pointwise invariant under the action of conjugation. In the case that $S = \operatorname{Spec} R$ and $G = \operatorname{Spec} A$, A naturally has the structure of a **Hopf algebra** over R – i.e., A is an R-algebra with compatible R-coalgebra structure arising from the group scheme structure of G. Hopf algebras arise often in representation theory and algebraic topology, and will play an important structural role in the sequel. The multiplication on A is usually denoted by $\mu : A \otimes_R A \to A$ and the unit by $\eta : R \to A$. The comultiplication on A is usually denoted by $\Delta : A \to A \otimes_R A$, the counit by $\epsilon : A \to R$, and the antipode by $S : A \to A$. Exhibiting all of this information is clearly equivalent to exhibiting an affine group scheme over R.

Remark 1.1. *Much of the structure of A is encoded by its* ***augmentation ideal*** $I := \ker(\epsilon : A \to R)$. *Indeed, the short exact sequence*

$$0 \longrightarrow I \longrightarrow A \longrightarrow R \longrightarrow 0$$

of R-modules is canonically split by $\eta : R \to A$ and so $A \cong I \oplus R$. It follows that $\Omega^1_{A/R} \cong I/I^2 \otimes_R A$ with universal R-derivation $(\pi \circ \mathrm{id}_A) \circ \Delta : A \to I/I^2 \otimes_R A$ for π the composite $A \to I \twoheadrightarrow I/I^2$.

Example 1.2. *The following are important examples of S-group schemes. We assume for simplicity that $S = \operatorname{Spec} R$ is affine.*[1]

(1) *The* ***additive group scheme*** $\mathbb{G}_{a,S} = \mathbb{G}_{a,R} = \mathbb{G}_a$ *has functor-of-points*

$$\mathbb{G}_a(T) := \mathcal{O}_T(T) = \Gamma(T, \mathcal{O}_T)$$

and is represented by $\operatorname{Spec} R[t]$. *The underlying Hopf algebra structure is given by*

$$\begin{aligned} \Delta : R[t] \to R[t] \otimes_R R[t], &\qquad t \mapsto t \otimes 1 + 1 \otimes t \\ \epsilon : R[t] \to R, &\qquad t \mapsto 0 \\ S : R[t] \to R[t], &\qquad t \mapsto -t \end{aligned}$$

[1]Most of the constructions here can be generalized to work for any S using the relative spectrum $\underline{\mathrm{Spec}}_S = \underline{\mathrm{Spec}}$.

(2) *The **multiplicative group scheme** $\mathbb{G}_{m,S} = \mathbb{G}_{m,R} = \mathbb{G}_m$ has functor-of-points*

$$\mathbb{G}_m(T) := \mathcal{O}_T(T)^\times = \Gamma(T, \mathcal{O}_T^\times)$$

and is represented by $\operatorname{Spec} R[t^{\pm 1}]$. The underlying Hopf algebra structure is given by

$$\begin{aligned} \Delta : R[t^{\pm 1}] \to R[t^{\pm 1}] \otimes_R R[t^{\pm 1}], &\qquad t \mapsto t \otimes t \\ \epsilon : R[t^{\pm 1}] \to R, &\qquad t \mapsto 1 \\ S : R[t^{\pm 1}] \to R[t^{\pm 1}], &\qquad t \mapsto t^{-1} \end{aligned}$$

(3) *The **group scheme of nth roots of unity** $\mu_{n,S} = \mu_n$ has functor-of-points*

$$\mu_n(T) := \{f \in \mathbb{G}_m(T) : f^n = 1\}$$

and is represented by $\operatorname{Spec} R[t]/(t^n - 1)$. Its behavior depends heavily on the characteristic of R, as well as whether R contains the nth roots of unity. The underlying Hopf algebra structure is induced by that on $\mathbb{G}_m$.

(4) *Let $\Gamma \in \mathsf{Ab}$. The **diagonalizable group scheme** $D(\Gamma)$ associated to Γ has functor-of-points*

$$D(\Gamma)(T) := \operatorname{Hom}_{\mathsf{Ab}}(\Gamma, \Gamma(T, \mathcal{O}_T^\times))$$

and is represented by $\operatorname{Spec} R[\Gamma]$ for $R[\Gamma]$ the group algebra of Γ over R. Note that $D(\mathbb{Z}) \cong \mathbb{G}_m$ and $D(\mathbb{Z}/n) \cong \mu_n$. The underlying Hopf algebra structure is given by

$$\begin{aligned} \Delta : R[\Gamma] \to R[\Gamma] \otimes_R R[\Gamma], &\qquad g \mapsto g \otimes 1 + 1 \otimes g \\ \epsilon : R[\Gamma] \to R, &\qquad g \mapsto 0 \\ S : R[\Gamma] \to R[\Gamma], &\qquad g \mapsto -g \end{aligned}$$

This Hopf algebra is always commutative and cocommutative, with commutativity equivalent to Γ being abelian.

(5) *Let Γ be a finite group. The **constant group scheme** associated to Γ is $\underline{\Gamma} = \Gamma_R := \operatorname{Spec} R^\Gamma$, where R^Γ is the product of $|\Gamma|$ copies of R whose algebra structure comes from pointwise operations, characterized by the fact that*

$$\operatorname{Hom}_{\mathsf{Alg}_R}(R^\Gamma, T) \cong \Gamma$$

for $T \in \mathsf{Alg}_R$ connected (i.e., $0, 1$ are the only idempotents). More succinctly, for general $T \in \mathsf{Alg}_R$, $\underline{\Gamma}(T)$ records the decompositions of 1_T into mutually orthogonal idempotents (with group operation given by convolution). The underlying Hopf algebra structure is given by

$$\begin{aligned} \Delta : R^\Gamma \to R^\Gamma \otimes_R R^\Gamma, &\qquad f \mapsto ((x, y) \mapsto f(xy)) \\ \epsilon : R^\Gamma \to R, &\qquad f \mapsto f(1_\Gamma) \\ S : R^\Gamma \to R^\Gamma, &\qquad f \mapsto (x \mapsto f(x^{-1})) \end{aligned}$$

where he have identified R^Γ as a set with $\operatorname{Maps}(\Gamma, R)$.[2] *Note that the indicator functions $\mathbb{1}_g$ for $g \in \Gamma$ form an R-module basis for R^Γ under this identification. This Hopf algebra is always commutative and is cocommutative if and only if Γ is abelian.*

[2] We may then naturally identify $R^\Gamma \otimes_R R^\Gamma$ and $\operatorname{Maps}(\Gamma \times \Gamma, R)$ as sets.

A **morphism of S-group schemes**[3] is an S-morphism $\varphi : G \to H$ such that the diagram

$$\begin{tikzcd} G \times_S G \arrow[r, "\varphi\times\varphi"] \arrow[d, "m_G"] & H \times_S H \arrow[d, "m_H"] \\ G \arrow[r, "\varphi"] & H \end{tikzcd}$$

commutes. In this way, S-group schemes form their own (non-full) subcategory of Sch_S. As with usual group homomorphisms, we deduce that $\varphi \circ e_G = e_H$ and $\varphi \circ i_G = i_H \circ \varphi$. Similarly, G is commutative if and only if inversion $i : G \to G$ is a morphism of S-group schemes. Given $\varphi : G \to H$ a morphism of S-group schemes, $\ker\varphi$ is described as a space via

$$(\ker\varphi)(T) = \ker(\varphi(T) : G(T) \to H(T)),$$

where $T \in \mathsf{Sch}_S$. This space is represented by the S-scheme fiber product of

$$\begin{tikzcd} & S \arrow[d, "e_H"] \\ G \arrow[r, "\varphi"] & H \end{tikzcd}$$

which is a locally closed subscheme of G. It follows that $\ker\varphi$ is a well-defined S-group scheme.

Example 1.3. *Let $R \in \mathsf{CRing}$ have characteristic $p > 0$ and let $\varphi : \mathbb{G}_{a,R} \to \mathbb{G}_{a,R}$ be induced by $(\cdot)^p$ on $R[t]$. Then, $\alpha_{p,R} = \alpha_p := \ker\varphi$ is given by $\operatorname{Spec} R[t]/(t^p)$. The underlying Hopf algebra structure is induced by that on $\mathbb{G}_{a,R}$.*

The situation for cokernels is much more delicate since the naïve approach does not in general yield something representable. The classic example is $(\cdot)^n : \mathbb{G}_m \to \mathbb{G}_m$. Concretely, consider $(\cdot)^2 : \mathbb{G}_{m,\mathbb{Q}} \to \mathbb{G}_{m,\mathbb{Q}}$ and let C be its naïve cokernel. If C were representable then the map $\mathbb{Q} \hookrightarrow \mathbb{Q}(\sqrt{3})$ should induce $C(\mathbb{Q}) \hookrightarrow C(\mathbb{Q}(\sqrt{3}))$. This is not the case, however, since 3 represents a nontrivial class in $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ but not in $\mathbb{Q}(\sqrt{3})^\times/(\mathbb{Q}(\sqrt{3})^\times)^2$. This difficulty can be remedied somewhat by looking at finite flat group schemes.

2 Finite Flat Group Schemes

Given a locally Noetherian scheme S, $f : G \to S$ is finite flat if and only if $f_*\mathcal{O}_G$ is a finite locally free O_S-module.[4] It follows that G has a locally constant rank function that is constant under suitable circumstances – e.g., if S is connected.[5] If G is in addition a commutative group scheme then its **Cartier dual** is described as a space via

$$G^D(T) := \operatorname{Hom}_{\mathsf{Sch}_T}(G_T, \mathbb{G}_{m,T})$$

for $T \in \mathsf{Sch}_S$. Under the assumption that $S = \operatorname{Spec} R$ and $G = \operatorname{Spec} A$, we claim that G^D is represented by $\operatorname{Spec} A^\vee$ for $A^\vee := \operatorname{Hom}_R(A, R)$ the R-linear dual of A. Under the identifications $R^\vee \cong R$ and $A^\vee \otimes_R A^\vee \cong (A \otimes_R A)^\vee$, $A^\vee$ has a Hopf algebra structure with $(\mu_{A^\vee}, \eta_{A^\vee}, \Delta_{A^\vee}, \epsilon_{A^\vee}, S_{A^\vee})$ dual to $(\Delta_A, \epsilon_A, \mu_A, \eta_A, S_A)$. This algebra is a finite flat Hopf R-algebra just like A, which is always

[3] Common alternative names include S-group homomorphism or simply S-homomorphism.

[4] The locally Noetherian condition ensures that the structure morphism for G is automatically locally of finite presentation. Note that many sources require finite flat group schemes to have constant rank.

[5] If $S = \operatorname{Spec} k$ for k a field then the rank is $\dim_k H^0(G, \mathcal{O}_G)$.

cocommutative and is commutative precisely when A is cocommutative or, equivalently, G is commutative. It is clear that A is canonically isomorphic to its double dual and thus, assuming representability, that taking the Cartier dual yields a contravariant involutary autoequivalence on the category of commutative finite flat affine group schemes over R.[6]

Let's return to the representability claim, which amounts to the existence of a natural isomorphism

$$\mathrm{Hom}_{\mathsf{Alg}_R}(A^\vee, T) \cong \mathrm{Hom}_{\mathsf{Alg}_T}(T[t^{\pm 1}], A \otimes_R T)$$

functorial in $T \in \mathsf{Alg}_R$. We could construct this isomorphism explicitly by hand but a slicker approach is to consider the R-algebra morphism $R[t^{\pm 1}] \to A^\vee \otimes_R A$ given by $t \mapsto \mathrm{id}_A$ under the identification $\mathrm{End}_R(A) \cong A^\vee \otimes_R A$, which yields a "perfect" pairing $G \times \operatorname{Spec} A^\vee \to \mathbb{G}_m$.[7]

Example 2.1.

(1) *Let $\Gamma \in \mathsf{Ab}$ be finite. Then, $D(\Gamma)$ and $\underline{\Gamma}$ are Cartier dual. As an exercise, try working out the pairing $D(\Gamma) \times \underline{\Gamma} \to \mathbb{G}_m$.*

(2) *Let $R \in \mathsf{CRing}$ with characteristic $p > 0$. Then, α_p is its own Cartier dual. In this case, the Cartier duality pairing $\alpha_p \times \alpha_p \to \mathbb{G}_m$ is induced by the truncated exponential*

$$R[t^{\pm 1}] \to R[x]/(x^p) \otimes_R R[y]/(y^p), \qquad t \mapsto \exp(\overline{x} \otimes \overline{y}) := \sum_{k=0}^{p-1} \frac{1}{k!}(\overline{x}^k \otimes \overline{y}^k).$$

Be warned that the related schemes α_{p^n} for $n \geq 2$ are not Cartier self-dual!

The following is a useful structural result for commutative finite flat group schemes. Note that this result must be somewhat deep since it is not known to hold in general in the non-commutative case.

Theorem 2.2 (Deligne)**.** *Let G be a commutative finite flat group scheme of constant rank n over a Noetherian ring R. Then, the "order kills the group" in the sense that there is a factorization*

$$\begin{tikzcd} G \arrow[r, "{[n]}"] \arrow[d, dotted, "\exists!"'] & G \\ \operatorname{Spec} R \arrow[ur, "e"'] & \end{tikzcd}$$

Of course, this theorem readily generalizes to commutative finite flat group schemes defined over any locally Noetherian base.

Proof. See [Oh, Theorem 1.2.1] for a local proof using Cartier duality and [Stix, Theorem 3.3.6] for a global proof using norms and traces. □

Corollary 2.3. *Let G be a commutative finite flat group scheme of constant rank n over a locally Noetherian base S. Suppose that n is invertible in S – i.e., $n \in \Gamma(S, \mathcal{O}_S^\times)$. Then, G/S is étale.*

[6]This may involve flatness in an essential way, to ensure that we do not "lose" any information in the process of dualizing.

[7]The isomorphism $\mathrm{End}_R(A) \cong A^\vee \otimes_R A$ can be checked locally, which is good since A is locally free.

In particular, if $S = \operatorname{Spec} k$ for k a field of characteristic 0 then G/k is étale (this is Cartier's theorem).

Proof. Since G/S is flat, we need only check $\Omega^1_{G/S} = 0$.[8] By the above theorem, we may factor $[n] : G \to G$ as a composition $G \to S \xrightarrow{e} G$. Thus, the induced map $[n]^* : \Omega^1_{G/S} \to \Omega^1_{G/S}$ factors through $\Omega^1_{S/S} = 0$ hence vanishes. At the same time, $[n]^*$ is multiplication by n on the level of stalks and so is an isomorphism. $\square$

An alternative proof of this fact comes from [Stix, Proposition 8.2.41], which asserts that, for $G = \operatorname{Spec} A$ a commutative finite flat group scheme over a field k, there is a k-vector space isomorphism $T_eG \cong \operatorname{Hom}(G^D, \mathbb{G}_a)$ for $\operatorname{Hom}(G^D, \mathbb{G}_a)$ equipped with the k-vector space structure induced by the identification $k \cong \underline{\operatorname{End}}(\mathbb{G}_a)(k)$. By passing to geometric fibers, we may assume that G as in the theorem is defined over an algebraically closed field k with $n \in k^\times$. Any $\varphi \in \operatorname{Hom}(G^D, \mathbb{G}_a)$ has image torsion of rank dividing n. At the same time, the endomorphism ring k of $\mathbb{G}_a$ has no nontrivial n-torsion by assumption.[9] Thus, $\varphi = 0$ and it follows that $T_eG = 0$ hence $\Omega^1_{G/k} = 0$.

We will see that most of the structure of finite flat group schemes comes from understanding those that are connected and those that are étale. Over a field k connectedness should be viewed as a local condition. Indeed, if we write $G = \operatorname{Spec} A$ then A is a finite k-module hence Artinian and so decomposes as a product of local Artinian k-algebras. It follows that G is the disjoint union of the spectra of these algebras.

If k is in addition perfect then we see that étale and reduced are the same (we must replace reduced by geometrically reduced for general k). Indeed, since k is perfect we have that G is reduced if and only if $G_{\overline{k}}$ is reduced. If $G_{\overline{k}}$ is reduced then we have some regular nonempty open $U \subseteq G_{\overline{k}}$ and we may use the group structure to move this open around and conclude that $G_{\overline{k}}$ is regular at all closed points hence regular. By flatness G is then smooth and thus étale since $G \to \operatorname{Spec} k$ is finite of relative dimension 0.

3 Quotients, Cokernels, and Exact Sequences

One setting in which finite flat group schemes are very nicely behaved is the étale case. For example, if $R \in \mathsf{CRing}$ is connected Noetherian then the category of finite commutative étale R-group schemes is equivalent to the category of finite abelian groups with continuous $\pi_1^{\text{ét}}(\operatorname{Spec} R, \overline{s})$-action (for $\overline{s}$ some geometric point of $\operatorname{Spec} R$), hence is abelian.

In the case that $(R, \mathfrak{m}, k)$ is Henselian local we can be a bit more explicit. To G an étale R-group scheme we may associate the G_k-module $G(R^{\text{unr}})$ for R^{unr} defined to be the integral closure of R in K^{unr} for $K := \operatorname{Frac}(R)$ (note that R^{unr} carries a natural action by G_k). We thus see, for example, that the category of finite étale $\mathbb{Z}_p$-group schemes is equivalent to the category of finite continuous $\widehat{\mathbb{Z}}$-modules. The underlying isomorpism $\pi_1^{\text{ét}}(\operatorname{Spec} R) \cong G_k$ arises from the equivalence of categories $\text{FÉt}_R \to \text{FÉt}_k$ given by the special fiber functor, whose quasi-inverse we may construct explicitly using Witt vectors. This is a generalization of the familiar statement that unramified extensions of $\mathbb{Q}_p$ correspond to extensions of $\mathbb{F}_p$.

[8] This uses the fact that the structure morphism for G/S is automatically locally of finite presentation. A reference for this is [Stacks, Tag 02GU].

[9] What precisely Stix means by endomorphism ring here is not entirely clear to me.

Remark 3.1. *That Henselian local rings appear here should not be surprising. Indeed, the fact that we have an equivalence of categories* $\mathrm{F\acute{E}t}_R \to \sim \mathrm{F\acute{E}t}_k$ *may be viewed as a geometric characterization of what it means for* $(R, \mathfrak{m}, k)$ *local to be Henselian.*

In more general settings it is natural to ask when quotients and cokernels exist. This merits a definition.

Definition 3.2. *Let* $S \in \mathsf{Sch}$ *and* $G, X \in \mathsf{Sch}_S$ *with* G *a group scheme. A **right group action** of* G *on* X *is an* S*-scheme morphism* $\varphi : X \times_S G \to X$ *inducing right group actions on* T*-points for every* $T \in \mathsf{Sch}_S$*. The action* φ *is **strictly free** if* $(\mathrm{id}_X, \varphi) : X \times_S G \to X \times_S X$ *is a closed immersion.*

Given a right group action φ *of* G *on* X*, a morphism* $f : X \to Y$ *is **constant on orbits** if* $f \circ \varphi = f \circ \mathrm{pr}_1$ *– i.e.,* $f(xg) = f(x)$ *for every* $T \in \mathsf{Sch}_S$*,* $x \in X(T)$*, and* $g \in G(T)$*. Such morphisms are important since they allow us to categorically describe the quotient* X/G*. Namely, the quotient* $q : X \to X/G$ *(if it exists) is the initial (hence unique up to unique isomorphism) object in the category of* S*-morphisms* $X \to Z$ *constant on orbits.*

Theorem 3.3 (Grothendieck)**.** *Let* $S \in \mathsf{Sch}$ *be locally Noetherian,* $X \in \mathsf{Sch}_S$ *finite type, and* $G \in \mathsf{Sch}_S$ *a finite flat group scheme of constant rank with a strictly free action on* X *such that every* G*-orbit of a closed point is contained in an affine open set.*[10] *Then,* $q : X \to X/G$ *exists and satisfies the following properties.*

(a) q *is finite flat with* $\deg q = \operatorname{rank} G$*.*

(b) *Given* $T \in \mathsf{Sch}_S$*, there is a natural injection* $X(T)/G(T) \hookrightarrow (X/G)(T)$*.*

(c) *Suppose* $S = \operatorname{Spec} R, G = \operatorname{Spec} A, X = \operatorname{Spec} B$*. Then,* $X/G = \operatorname{Spec} B_0$ *for* B_0 *the equalizer of the induced maps* $\widetilde{\mathrm{pr}}_1, \widetilde{\varphi} : B \to B \otimes_R A$*.*

The quotient $q : X \to X/G$ is built via a faithfully flat descent procedure, using the (scheme theoretic) equivalence relation $\mathcal{R} \subseteq X \times_S X$ induced by the group action.

Remark 3.4. *All of this is reminiscent of when a Lie group* G *acts on a smooth manifold* X*. To recall, let* φ *be a right group action of* G *on* X *that is smooth, free (i.e., the stabilizer of every point is trivial), and proper (i.e., the obvious map* $X \times G \to X \times X$ *is topologically proper). Then,* X/G *admits a unique smooth structure such that the canonical map* $X \to X/G$ *is a smooth submersion. Note, moreover, that the map* $X \to X/G$ *is a covering space projection if and only if* φ *is properly discontinuous (i.e., every point of* X *has an open neighborhood disjoint from its conjugate under any non-identity element of* G*).*

Corollary 3.5. *Let* R *be a Noetherian ring,* $G = \operatorname{Spec} A$ *an affine* R*-group scheme, and* $H \trianglelefteq G$ *a finite flat closed normal* R*-subgroup scheme. Then,* $G \to G/H$ *exists and is finite faithfully flat,* G/H *is affine, and* G/H *is finite (resp., finite flat) if* G *is finite (resp., finite flat).*

In this case, $H = \operatorname{Spec} A/I$ acts on $\operatorname{Spec} A$ via right multiplication and $\widetilde{\varphi}$ is given explicitly by the

[10] This occurs, for example, when $G = \operatorname{Spec} k$ for k an infinite field and X is a quasiprojective variety.

composition $(\mathrm{id}_A \otimes \pi) \circ \Delta_A$ for $\pi : A \twoheadrightarrow A/I$. The map $\widetilde{\mathrm{pr}}_1 : A \to A \otimes_R A/I$ is given explicitly by $x \mapsto x \otimes (1 \bmod I)$, hence B_0 as above is given by $\{x \in A : (\mathrm{id}_A \otimes \pi)(\Delta_A(x)) = x \otimes (1 \bmod I)\}$.

Corollary 3.6. *Given a field k, the category of commutative finite flat k-group schemes is abelian.*

Proof. Let $\mathcal{A}$ denote the category in the claim and consider $\varphi \in \mathrm{Hom}_{\mathcal{A}}(G, H)$. Then, $\ker \varphi$ as defined in the category of all k-group schemes fits into a Cartesian square

$$\begin{tikzcd} \ker\varphi \arrow[r] \arrow[d] & \operatorname{Spec} k \arrow[d, "e_H"] \\ G \arrow[r, "\varphi"'] & H \end{tikzcd}$$

and so, by stability of finiteness and flatness under base change, $\ker \varphi$ is finite flat since φ itself is finite flat. Alternatively, we could use the fact that $\ker \varphi$ is locally closed in G. Either way, we then have $\ker \varphi \in \mathcal{A}$ since commutativity is clear and so we may consider $\mathrm{coker}(\varphi) := G/\ker \varphi \in \mathcal{A}$ as defined above. This satisfies the universal property that a cokernel in $\mathcal{A}$ should have and hence we obtain a natural morphism $\mathrm{coim}(\varphi) \to \mathrm{im}(\varphi)$. One first checks that this morphism is bijective and then uses Cartier duality to show that bijective homomorphisms of commutative finite flat k-group schemes are isomorphisms. $\square$

4 Classification of Finite Flat Group Schemes

Given a group scheme G over a field k, there is a well-defined k-subgroup scheme G^0 of G whose underlying topological space is the connected component of the identity $e : \operatorname{Spec} k \to G$. This construction generalizes to work over any base, but it may not be so well-behaved. Over k, the scheme G^0 is normal in G, geometrically irreducible, and quasicompact with $G^0 \hookrightarrow G$ a flat closed immersion. Letting G be commutative finite flat and $G^{\text{ét}} := G/G^0$, we obtain the **connected-étale sequence**

$$0 \longrightarrow G^0 \longrightarrow G \longrightarrow G^{\text{ét}} \longrightarrow 0$$

which is exact. Intuitively, $G^{\text{ét}}$ is $\pi_0(G)$ and there is some mileage to be gained from this perspective. In addition, if k is perfect then the connected-étale sequence splits canonically. How this works is as follows. Letting $G = \operatorname{Spec} A$, A has a maximal étale subalgebra $A^{\text{ét}}$ (which exists since composita are well-behaved) and $G^{\text{ét}} = \operatorname{Spec} A/\operatorname{Spec} A^{\text{ét}}$ is the quotient in the category of commutative finite flat group schemes over k. In the case that k is perfect, the reduced locus G_{red} is $\operatorname{Spec} A_{\text{red}}$ for A_{red} the maximal reduced quotient of A and $G_{\text{red}} \hookrightarrow G$ provides a section of $G \to G^{\text{ét}}$ since the composition $G_{\text{red}} \hookrightarrow G \to G^{\text{ét}}$ is an isomorphism.

Example 4.1. *The following provide examples of the connected-étale sequence in action.*

(1) *Let $E/\overline{\mathbb{F}_p}$ be an (ordinary) elliptic curve. Then, $E[p^n] \cong \underline{\mathbb{Z}/p^n} \times \mu_{p^n}$.*

(2) *Let $E/\overline{\mathbb{F}_p}$ be a supersingular elliptic curve. Then, $E[p]$ is an extension of α_p by itself.*

Aside from the splitting, all of this works just as well over a Henselian local base.[11]

Theorem 4.2. *Let $(R, \mathfrak{m}, k)$ be a Henselian local ring and G a commutative finite flat R-group scheme.*

(a) *If $G = \operatorname{Spec} A$ is affine then G^0 is the spectrum of a Henselian local ring with residue field k.*

(b) *$G^0 \trianglelefteq G$ is flat and closed.*

(c) *The quotient $G^{\text{ét}} := G/G^0$ exists and is finite étale over R.*

(d) *Let H be a finite étale R-group scheme. Then, there is a factorization*

$$\begin{tikzcd} G \arrow[d] \arrow[dr, "\varphi"] & \\ G^{\text{ét}} \arrow[r, dotted, "\exists!"'] & H \end{tikzcd}$$

given any $\varphi \in \operatorname{Hom}(G, H)$.

(e) *$G \mapsto G^0$ and $G \mapsto G^{\text{ét}}$ define exact endofunctors on the category of commutative finite flat R-group schemes.*

The takeaway is that finite flat group schemes over an appropriate base are "built up" from connected and étale parts.

Corollary 4.3. *Let $(R, \mathfrak{m}, k)$ be a Henselian local ring.*

(a) *An extension of a connected commutative finite flat R-group scheme by a connected commutative finite flat R-group scheme is connected.*

(b) *An extension of an étale commutative finite flat R-group scheme by an étale commutative finite flat R-group scheme is étale.*

(c) *An extension of a connected commutative finite flat R-group scheme by an étale commutative finite flat R-group scheme is trivial – i.e., given by a product.*

Theorem 4.4. *Let k be a perfect field with characteristic $p > 0$ and $G = \operatorname{Spec} A$ a connected commutative finite flat k-group scheme. Then,*

$$A \cong k[x_1, \ldots, x_r]/(x_1^{p^{e_1}}, \ldots, x_r^{p^{e_r}})$$

for $r \in \mathbb{N}$ unique and $(e_1, \ldots, e_r) \in \mathbb{N}^r$ unique up to reordering. Moreover, A has p-power rank.

It follows from the above and faithfully flat descent that any connected commutative finite flat group scheme over a field of characteristic $p > 0$ has p-power rank. The above result also gives rise to a local version (via Nakayama's lemma).

Corollary 4.5. *Let $(R, \mathfrak{m}, k)$ be a complete DVR with k perfect of characteristic $p > 0$ and $G =$*

[11] Recall that strictly Henselian local rings are precisely the local rings of geometric points in the étale topology. Henselian local rings correspond to something slightly more general, namely the Nisnevich topology.

Spec A *a connected commutative finite flat R-group scheme. Then,*

$$A \cong R[\![x_1, \dots, x_r]\!]/(f_1, \dots, f_r),$$

where there exist $e_1, \dots, e_r \in \mathbb{N}$ *such that each* $f_i - x_i^{p^{e_i}} \in \mathfrak{m}R[x_1, \dots, x_r]$ *is a polynomial in* x_i *of degree* $< p^{e_i}$.

Over a Dedekind domain R with $K := \mathrm{Frac}(R)$, the generic fiber carries a lot of information in the sense that, given a fixed finite flat R-group scheme G, the generic fiber functor is an equivalence of categories from the category of closed flat R-subgroup schemes of G to the category of closed flat K-subgroup schemes of G_K. Thinking more locally, how much information about a finite flat group scheme over a DVR can we recover from its generic fiber? Surprisingly a lot under certain ramification conditions.

Theorem 4.6 (Raynaud)**.** *Let R be a DVR with mixed characteristic $(0,p)$ and $K := \mathrm{Frac}(R)$. Choose a uniformizer π with associated normalized valuation v (i.e., $v(\pi) = 1$) and ramification index $e := v(p)$. Suppose that $e < p - 1$.*

(a) *Let G_0 be a finite (flat) commutative K-group scheme killed by some power of p. Then, G_0 admits at most one* prolongation *over R – i.e., at most one finite flat commutative R-group scheme G such that $G_K \cong G_0$. In particular, any finite flat commutative R-group scheme is the unique prolongation of its generic fiber.*

(b) *The generic fiber functor from the category of finite flat commutative R-group schemes to the category of finite flat commutative K-group schemes is fully faithful with (essential) image stable under taking sub-objects and quotients.*

We say G_0 in the above theorem has **unique prolongation** (or **UP** for short).

Remark 4.7. *The condition $e < p - 1$ is necessary as can be seen by considering a suitable finite extension $K/\mathbb{Q}_p$ containing the pth roots of unity and comparing μ_p and $\underline{\mathbb{Z}/p}$ over K and $\mathcal{O}_K$.*

The proof of this result rests on the structure theory of so-called *Raynaud F-module schemes*. The major steps in the proof are roughly as follows.

(1) If G_0 is an extension whose sub and quotient have UP then G_0 has UP, hence we need only consider simple groups.

(2) Classify the simple groups and their prolongations.

(3) Check by hand that UP holds when $e < p - 1$.