Math 2202 Exam I Rubric

4. (15 points) Consider the plane $\mathcal{P}$ in $\mathbf{R}^3$ with equation $3x + 2y - z + 12 = 0$.

(a) Write the equation of the plane parallel to $\mathcal{P}$ and containing the point $A = (0, 8, 0)$.

(We are shifting the plane.)

Approach 1: Note that getting a parallel plane is the same as changing the constant. So, $3x+2y-z+d=0$ and we need to solve for d.

$0 = 3(0)+2(8)-0+d = 16+d \Rightarrow d=-16.$

Approach 2: $\mathcal{P}$ has normal vector $\vec{n} = \langle 3,2,-1\rangle$. So, taking $\vec{v} = \langle x-0, y-8, z-0\rangle$,

$$0 = \vec{n}\cdot\vec{v}$$
$$= 3(x-0)+2(y-8)+(-1)(z-0)$$
$$= 3x+2y-z-16.$$

[1 pt.] Correct values of a,b,c
[1 pt.] Correct value of d
[3 pts.] Explanation

(b) Find the distance between the two planes

Given planes $\mathcal{P}_1 : ax+by+cz+d_1=0$ and $\mathcal{P}_2 : ax+by+cz+d_2=0$, the distance formula says

$$\operatorname{dist}(\mathcal{P}_1,\mathcal{P}_2) = \frac{|d_1-d_2|}{\sqrt{a^2+b^2+c^2}}.$$

[1 pt.] Correct distance (need not be simplified)
[1 pt.] Work solving for distance

For us, $a=3, b=2, c=-1, d_1=12, d_2=-16$.

[1 pt.] Correct identification of d_1 and d_2

$$\Rightarrow \operatorname{dist} = \frac{|12-(-16)|}{\sqrt{3^2+2^2+(-1)^2}} = \frac{28}{\sqrt{14}} = 2\sqrt{14}.$$

[2 pts.] Explanation

(c) Consider the set of all points that have the same distance from $\mathcal{P}$ and the plane you found in (a). Describe this set with an equation or in words.

The distance from a point (x_0, y_0, z_0) to a plane $ax+by+cz+d=0$ is

$$\frac{|ax_0+by_0+cz_0+d|}{\sqrt{a^2+b^2+c^2}}.$$ We can describe the desired set of points by

$$\{(x,y,z) \in \mathbb{R}^3 : |3x+2y-z+12| = |3x+2y-z-16|\}.$$

Geometrically, this is a plane parallel to both of the planes and equidistant to both (it "sits in the middle").

Intuitively, we get a plane because if this were not flat then where it curves it would be closer to one of the two planes. [5 pts.]
Words or equations

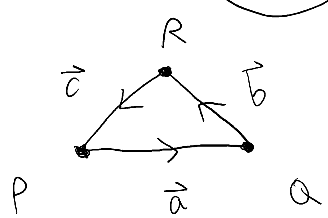
(a) If $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are both non-zero vectors and $\vec{\mathbf{u}} \cdot \vec{\mathbf{v}} = 0$, then $\vec{\mathbf{u}}$ and $\vec{\mathbf{v}}$ are parallel.

Circle One: **TRUE** **FALSE**

Let $\vec{u} = \vec{i}$ and $\vec{v} = \vec{j}$. Then, $\vec{i} \cdot \vec{j} = 0$ but $\vec{i}$ and $\vec{j}$ are not parallel.

[2 pts.] Correct answer
[3 pts.] Counter-example

(b) Let $\vec{\mathbf{a}} = \overrightarrow{\mathbf{PQ}}$, $\vec{\mathbf{b}} = \overrightarrow{\mathbf{QR}}$, and $\vec{\mathbf{c}} = \overrightarrow{\mathbf{RP}}$ with P, Q, R, three distinct points in $\mathbf{R}^3$. Then the vectors $\vec{\mathbf{a}} \times \vec{\mathbf{b}}, \vec{\mathbf{a}} \times \vec{\mathbf{c}}$ and $\vec{\mathbf{b}} \times \vec{\mathbf{c}}$ are all parallel to each other. **Circle One:** **TRUE** **FALSE**



$$\overrightarrow{QP} = \overrightarrow{QR} + \overrightarrow{RP} = \vec{b} + \vec{c}$$
$$\| $$
$$-\overrightarrow{PQ} = -\vec{a}$$

[2 pts.] Correct answer

$$\vec{a} \times \vec{b} = (-\vec{b} - \vec{c}) \times \vec{b} = -\vec{c} \times \vec{b} = \vec{b} \times \vec{c}$$
$$\vec{a} \times \vec{c} = (-\vec{b} - \vec{c}) \times \vec{c} = -\vec{b} \times \vec{c}$$

[3 pts.] Explanation (algebraic or geometric)

Geometrically, all three are parallel because they are all perpendicular to the same plane (the plane in which the triangle PQR lives).

(c) The equation $\langle x-2, y+4, z-8\rangle \cdot \langle 1,1,1\rangle = 0$ is the equation of a line parallel to $\langle 1,1,1\rangle$.

[2 pts.] Correct answer
[3 pts.] Explanation (any method works)

Circle One: **TRUE** **FALSE**

$$0 = \langle x-2, y+4, z-8\rangle \cdot \langle 1,1,1\rangle = x-2+y+4+z-8$$
$$= x+y+z-6$$

is the equation of a plane.

Geometrically, being normal to a vector is not enough data to determine a line. Also, the condition being enforced here is "being perpendicular to $\langle 1,1,1\rangle$" and not "being parallel to $\langle 1,1,1\rangle$." (I think it's not necessary to provide a counterexample here.)