Section 9 Math 2202
Optimization

1. (Based on *Stewart 11.8 #6*) Consider the function $f(x, y) = e^{xy}$, and the constraint $x^3 + y^3 = 16$.

   (a) Use Lagrange multipliers to find the coordinates (x, y) of any points on the constraint where the function f could attain a maximum or minimum.

   Solution: We wish to find the extreme values of the function $f(x, y) = e^{xy}$ subject to the constraint $g(x, y) = x^3 + y^3 = 16$. Lagrange multipliers tells us that $\nabla f = \lambda \nabla g$ or $\langle ye^{xy}, xe^{xy} \rangle = \langle 3\lambda x^2, 3\lambda y^2 \rangle$, and so we get the system

   $$\begin{aligned} ye^{xy} &= 3\lambda x^2 \\ xe^{xy} &= 3\lambda y^2 \\ x^3 + y^3 &= 16. \end{aligned}$$

   Note that if either x or y is zero, then $x = y = 0$, which contradicts $x^3 + y^3 = 16$, so we can assume that $x \neq 0$ and $y \neq 0$. Then

   $$\lambda = \frac{ye^{xy}}{3x^2} = \frac{xe^{xy}}{3y^2},$$

   from which we get $x^3 = y^3$ and so $x = y$. Since $x^3 + y^3 = 16$, we get $2x^3 = 16$ or $x = y = 2$. Thus the point $(2, 2)$ is a point on the constraint where the function f could attain a maximum or minimum.

   (b) For each point you found in part (a), is the point a maximum, a minimum, both or neither? Explain your answer carefully. What are the minimum and maximum values of f on the constraint? Please explain your answers carefully.

   Solution: The maximum value is $f(2, 2) = e^4$.

   We can see that there is no minimum value, since we can choose points satisfying the constraint $x^3 + y^3 = 16$ that make $f(x, y) = e^{xy}$ arbitrarily close to 0 (but never equal 0). More specifically, if x is given, then $y = \sqrt[3]{16 - x^3}$. If x grows positive without bound ($x \to +\infty$), then y grows *negative* without bound ($y \to -\infty$) and so e^{xy} approaches zero ($xy \to -\infty$ and so $e^{xy} \to 0$). So there is *no* minimum value of f.

   (c) The extreme value theorem which we discussed in class (See 11.7 in Stewart) guarantees that under the right circumstances, we are guaranteed to find absolute minima and maxima for a function f on a certain constraint. Explain why parts (a) and (b) don't violate the extreme value theorem.

   Solution: The extreme value theorem states that if a real-valued function f is continuous in a closed and bounded region, then f must attain its maximum and minimum value, each at least once.

   But $x^3 + y^3 = 16$ is not a bounded region. The curve it describes cannot be contained in a circle. So it's possible f does not attain a max or min value.

   In this case, we don't attain a min value since the constraint curve crosses level curves with ever-decreasing values (going toward 0) as $x \to \infty$ and $x \to -\infty$.

   Note: the level curves are always positive valued, because $e^{xy} > 0$.

Solutions

Compute the value of the integrals for the definite integrals. For indefinite integrals, find the antiderivative. You may need to refresh your integration techniques such as substitution, integration by parts, using trigonometric identities, as well as your basic antiderivatives.

1. $\int \frac{1}{x^{2/3}}\,dx$

   Solution:

$$\begin{aligned}\int \frac{1}{x^{2/3}}\,dx &= \int x^{-2/3}\,dx \\ &= \frac{1}{(-2/3)+1}x^{(-(2/3)+1)} + C \\ &= \frac{1}{(1/3)}x^{1/3} + C \\ &= 3x^{1/3} + C\end{aligned}$$

   Check:

$$\frac{d}{dx}(3x^{1/3} + C) = 3(1/3)x^{(1/3)-1} = x^{-2/3}$$

2. $\int \sin x\,dx$

$$\int \sin x\,dx = -\cos x + C$$

   Because:

$$\frac{d}{dx}(-\cos x + C) = -(-\sin x) = \sin x$$

3. $\int \tan x \, dx$

   Note that $\tan x = \frac{\sin x}{\cos x}$, let $u = \cos x$, so $du = -\sin x \; dx$ and $-du = \sin x \; dx$

4. $\int_e^{e^4} \frac{3}{x \ln x} \, dx$

   Let $u = \ln x$. The bounds $x = e$ and $x = e^4$ translate to $u = 1$ and $u = 4$, respectively. Since $du = 1/x dx$, the integral translates to

   $$\int_1^4 \frac{3}{u} \, du.$$

   This gives $3 \ln u|_1^4 = 3 \ln 4$.

5. $\int_0^1 x e^x \, dx$

   Integration by parts. Let $u = x$ and $dv = e^x$. Then $du = dx$ and $v = e^x$. Then integration by parts says:

   $$\int x e^x = x e^x - \int e^x \, dx = x e^x - e^x.$$

   Now plug in the bounds and subtract, to get 1.

6. $\quad (\sin x + \cos x)^2 \, dx$

   First expand the integrand to get $\sin^2 x + \cos^2 x + 2 \sin x \cos x$. This becomes $1 + 2 \sin x \cos x$. Then, you can either proceed by the substitution $u = \sin x$, or you could use the trigonometric identity $2 \sin x \cos x = \sin 2x$.

   So our integral is $\int 1 + \sin 2x \, dx$. This integral is $x - \frac{1}{2} \cos 2x + C$. (Here we substituted $u = 2x$.)

7. $\int \frac{x}{\sqrt{1-x^2}} \, dx$

   The trick here is to substitute $u = 1 - x^2$. Then $du = -2x dx$, so our integral becomes

   $$-1/2 \int \frac{1}{\sqrt{u}} \, du = -1/2 \int u^{-1/2} \, du.$$

   Now apply the power rule to get $-u^{1/2} = -(1 - x^2)^{1/2} = -\sqrt{1 - x^2} + C$.

8. $\int \frac{1}{\sqrt{1-x^2}}\, dx$

   Here we must use a *trigonometric substitution*. Let $\sin\theta = x$, i.e. $\theta = \arcsin x$ Then $\cos\theta d\theta = dx$, while $1 - x^2 = 1 - \sin^2\theta = \cos^2\theta$.

   Therefore, our integral transforms into

$$\int \frac{\cos\theta}{\cos\theta}\, d\theta.$$

   This integral is clearly $\theta + C$, which is $\arcsin x + C$.