DRINFELD MODULAR VARIETIES: HOMEWORK 1

(1) Let $U = \mathbb{A}^2_R - \{(0,0)\}$ be the open subscheme of $\mathbb{A}^2_R$ defined in Lecture 3. Find an R-algebra S and a point $(t_1, t_2) \in U(S)$ such that neither t_1 nor t_2 belongs to $S^\times$.

A **global section** α over a functor $F : \mathrm{Alg}_R \to \mathrm{Set}$ is an assignment that associates with every $S \in \mathrm{Alg}_R$ and $x \in F(S)$ an element $\alpha_x \in S_x$ such that for any map $\varphi : S \to T$, we have $\varphi(\alpha_x) = \alpha_{F(\varphi)(x)}$. The set of global sections is denoted by $\Gamma(F, \mathcal{O})$.

(2) (a) Show that $\Gamma(F, \mathcal{O})$ naturally has a structure of a commutative algebra over R.
  (b) If $F = \operatorname{Spec} S$ show that $\Gamma(\operatorname{Spec} S, \mathcal{O}) = S$.
  (c) Let U be as in the previous problem. Show that $\Gamma(U, \mathcal{O}) = R[x_1, x_2] = \Gamma(\mathbb{A}^2_R, \mathcal{O})$.*Hint: This ring of global sections is naturally contained in both $R[x_1, x_2][x_1^{-1}]$ and $R[x_1, x_2][x_2^{-1}]$.*

(3) Fix $S \in \mathrm{Alg}_R$. Show that the two notions of an open subfunctor of $\operatorname{Spec} S$ given in Lecture 3 agree. More precisely, prove the following: If $F \subset \operatorname{Spec} S$ is an open subscheme (defined as a 'union' of basic open subschemes), then for any map of rings $\varphi : S \to T$, the subfunctor $F_\varphi \subset \operatorname{Spec} T$ given by

$$F_\varphi(U) = \{\psi : T \to U : \ \psi \circ \varphi \in F(U) \subset (\operatorname{Spec} S)(U)\}$$

is again an open subscheme.

A **graded ring**[1] is a ring R equipped with a decomposition $R = \bigoplus_{d=0}^\infty R_d$ into a direct sum of abelian subgroups satisfying:

- $R_0 \subset R$ is a subring.
- $R_i R_j \subset R_{i+j}$ for all $i, j \geq 0$.[2]

In particular, each R_d is naturally an R_0-module. An element in R_d for some $d \geq 0$ is called a **homogeneous element of degree** d. Every element of R is a sum of homogeneous elements of R in a unique way.

Example 1. The polynomial ring $R = R_0[T]$ with $R_d = R_0 \cdot T^d$. More generally, $R = R_0[T_0, \dots, T_n]$ is a graded ring with $R_d = \bigoplus_{i_0 + \dots + i_n = d} R_0 \cdot T_0^{i_0} \cdots T_n^{i_n}$.

(4) Let R be a graded ring. Show that the following are equivalent for an ideal $I \leq R$:
  (a) $I = \bigoplus_{d=0}^\infty I_d$ with $I_d = R_d \cap I$.
  (b) R/I is a graded ring and the quotient map $R \to R/I$ is a graded homomorphism of rings.
  (c) For every $j \in I$, the homogeneous components of j are also in I.
  (d) I is generated by homogeneous elements of R.

A homomorphism $f : R \to S$ of graded rings is a **graded homomorphism** or a **homomorphism of graded rings** if it satisfies $f(R_d) \leq S_d$ for all $d \geq 1$.

A graded ring as above is **generated in degree** 1 if, for every $d \geq 1$, R_d is generated as an R_0-module by products of elements in R_1. Such a ring is **finitely generated** if R_1 is a finitely generated R_0-module.

(5) Show that the following are equivalent:
  (a) R is finitely generated in degree 1.
  (b) There exists a surjective R_0-algebra map of graded rings

$$R_0[T_0, \dots, T_n] \to R.$$

  (c) There exists a homogeneous ideal $I \leq R[T_0, \dots, T_n]$ and a graded isomorphism of R_0-algebras

$$R_0[T_0, \dots, T_n] \xrightarrow{\simeq} R.$$

[1]Technically, we're defining a $\mathbb{Z}$-graded ring concentrated in non-negative degrees.

[2]Here, for any two subsets $X, Y \subset R$, $X \cdot Y$ is the subset consisting of the products of elements of X with those of Y.

(6) Consider the elliptic curve X over an algebraically closed field k of characteristic $\neq 2$ given by the graded ring $k[T_0, T_1, T_2]/(T_2^2T_0 - T_1^3 - T_1T_0^2)$. Show that for any point $x \in X(k)$ the scheme $X - \{x\}$ is affine.

For the rest of this problem set, we will consider the question of *descent*: when does a module over an R-algebra S arise via base change (or tensor product) from an R-module? We will apply this to R-algebras of the form $\prod_{i=1}^r R[f_i^{-1}]$, which will enable us to show that the notion of a quasi-coherent sheaf is *Zariski local* (in fact, the argument shows that it is quite a bit more local than that).

Suppose that we have an R-algebra S and an S-module N. Then we obtain two $S \otimes_R S$-modules, $S \otimes_R N = N_1$ and $N \otimes_R S = N_2$: An element $s_1 \otimes s_2$ of $S \otimes_R S$ acts on $S \otimes_R N$ by

$$(s_1 \otimes s_2)(s \otimes n) = s_1 s \otimes s_2 n,$$

and on $N \otimes_R S$ by

$$(s_1 \otimes s_2)(n \otimes s) = s_1 n \otimes s_2 s.$$

For any map $\varphi : S \to T$ of rings write $\varphi^* N$ for $T \otimes_{S,\varphi} N$; then $N \otimes_R S = j_1^* N$ and $S \otimes_R N = j_2^* N$, where $j_1 : S \xrightarrow{s \mapsto s \otimes 1}$ and $j_2 : S \xrightarrow{s \mapsto 1 \otimes s}$.

If $\varphi_1, \varphi_2 : S \to T$ are two maps of R-algebras, then we can combine them to a single map

$$\varphi = \varphi_1 \otimes \varphi_2 : S \otimes_R S \to T$$

such that $\varphi \circ j_i = \varphi_i$ for $i = 1, 2$.

(7) Show that giving an isomorphism $\alpha : N_1 \xrightarrow{\simeq} N_2$ of $S \otimes_R S$-modules is equivalent to giving, for every pair of maps[3] $\varphi_1, \varphi_2 : S \to T$, an isomorphism

$$\alpha(varphi_1, \varphi_2) : \varphi_1^* N \xrightarrow{\simeq} \varphi_2^* N$$

such that for any map $\psi : T \to U$ we have

$$\alpha_{\varphi_1 \circ \psi, \varphi_2 \circ \psi} = 1 \otimes_{T,\psi} \alpha_{\varphi_1, \varphi_2} : U \otimes_{T,\psi} \varphi_1^* N \xrightarrow{\simeq} U \otimes_{T,\psi} \varphi_2^* N.$$

A **descent datum** (over R) for N is an isomorphism $\alpha : N_1 \xrightarrow{\simeq} N_2$ of $S \otimes_R S$-modules such that for any *triple* of maps

$$\varphi_1, \varphi_2, \varphi_3 : S \to T$$

we have $\alpha(\varphi_1, \varphi_3) = \alpha(\varphi_2, \varphi_3) \circ \alpha(\varphi_1, \varphi_2)$.

The category $\mathrm{Des}(S/R)$ will be the category of pairs (N, α) where N is an S-module and α is a descent datum for N over S: the maps are maps of S-modules that commute with the descent data.

(8) Show that there is a canonical functor

$$\mathrm{Mod}_R \to \mathrm{Des}(S/R)$$

arising from base-change $M \mapsto S \otimes_R M$.

(9) If R' is another R-algebra, show that there is a canonical functor

$$\mathrm{Des}(S/R) \to \mathrm{Des}(R' \otimes_R S/R')$$

arising from base-change $N \mapsto R' \otimes_R N$.

We say that S has **effective descent over** R if the functor

$$\mathrm{Mod}_R \to \mathrm{Des}(S/R)$$

is an equivalence of categories.

There is a direct criterion for this. For any pair (N, α) in $\mathrm{Des}(S/R)$, we obtain a map of R-modules

$$f_\alpha : N \xrightarrow{n \mapsto 1 \otimes n - \alpha(n \otimes 1)} S \otimes_R N.$$

Set $M = \ker f_\alpha$: this is an R-submodule of N, and so there is a natural map

$$g_{N,\alpha} : S \otimes_R M \xrightarrow{s \otimes n \mapsto sn} N.$$

[3] All maps here will be maps of commutative R-algebras

(10) Show that the following are equivalent:
 (a) S has effective descent over R.
 (b) For every pair (N, α) in $\mathrm{Des}(S/R)$, the map $g_{N,\alpha}$ is an isomorphism of S-modules.

It turns out that there are large categories of R-algebras that enjoy effective descent.

The simplest are of the following type: We will say that S has a **section over** R if there exists a map of R-algebras $\pi : S \to R$. For instance, polynomial algebras over R admits several sections over R by evaluating the variables at elements of R.

Choose a map of R-algebras $\pi : S \to R$; this gives us an R-algebra endomorphism

$$i_\pi : S \xrightarrow{\pi} R \to S,$$

where the second map is simply the R-algebra structure on S.

(11) Suppose that $(N, \alpha) \in \mathrm{Des}(S/R)$. Use the isomorphism

$$\alpha(i_\pi, \mathrm{id}_S) : S \otimes_{S,t_\pi} N \xrightarrow{\simeq} S \otimes_S N$$

to show that $g_{N,\alpha}$ is an isomorphism and conclude that S has effective descent over R.

Let's say that an R-algebra S is **faithfully flat** if a complex

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

of R-modules is a short exact sequence *if and only if* the base change

$$0 \to S \otimes_R M_1 \to S \otimes_R M_2 \to S \otimes_R M_3 \to 0$$

is a short exact sequence.

(12) Show that any R-algebra that is free as an R-module is faithfully flat over R; in particular, any polynomial algebra over R is faithfully flat over R.

(13) (Theorem of faithfully flat descent) Suppose that S is faithfully flat, and suppose that $(N, \alpha) \in \mathrm{Des}(S/R)$. Using the map $R \to S$, we obtain a pair $(N', \alpha') \in \mathrm{Des}(S \otimes_R S/S)$.
 (a) Show that $g_{(N,\alpha)}$ is an isomorphism if and only if $g_{(N',\alpha')}$ is an isomorphism.
 (b) Now use the fact the previous problem and the fact that $S \otimes_R S$ admits a section over S to conclude that S has effective descent over R.

(14) Let $f_1, \ldots, f_n \in R\backslash\{0\}$ be elements such that $(f_1, \ldots, f_n) = R$.
 (a) Show that the algebra $S = \prod_{i=1}^n R[f_i^{-1}]$ is faithfully flat over R.
 (b) Show that giving an object in $\mathrm{Des}(S/R)$ is equivalent to giving the following data:
  (i) An $R[f_i^{-1}]$-module M_i for every i.
  (ii) For every pair i, j, an isomorphism of $R[(f_i f_j)^{-1}]$-modules

$$\alpha(i,j) : M_i[f_j^{-1}] \xrightarrow{\simeq} M_j[f_i^{-1}]$$

  such that for any triple i, j, k, we have

$$\alpha(i,k)[f_j^{-1}] = \alpha(j,k)[f_i^{-1}] \circ \alpha(i,j)[f_k^{-1}]$$

  as isomorphisms from $M_i[(f_k f_j)^{-1}]$ to $M_k[(f_i f_j)^{-1}]$.
 (c) Conclude that giving an R-module M is equivalent to giving the data in the previous assertion.