Tate Curve

$\rightsquigarrow$ automorphism of $\mathbb{G}_m^{an}$

Fix $q \in k$ w/ $0<|q|<1$. Consider $\mathbb{G}_m^{an}$. Given $R \in \mathrm{CAlg}_k$, q acts on $\mathbb{G}_m(R) = R^\times$ induced by $z \mapsto q^{-1}z$ on $k[z,z^{-1}]$.

As a set, $T = \mathbb{G}_m^{an}/\langle q \rangle$ and we want a rigid structure. $pr : \mathbb{G}_m^{an} \to T$ quotient map. (1) We define admissible opens and covers for T by pulling back by pr.

(2) Structure sheaf: given adm. open $\mathcal{U} \subseteq T$, $pr^{-1}(\mathcal{U})$ is q-stable adm. open in $\mathbb{G}_m^{an}$. Define $\mathcal{O}_T(\mathcal{U}) := \mathcal{O}_{\mathbb{G}_m^{an}}(pr^{-1}(\mathcal{U}))^{\langle q \rangle}$.

To see that $(T, \mathcal{O}_T)$ is rigid space, note that if $W \subseteq \mathbb{G}_m^{an}$ adm. open w/ $q^n W \cap W = \emptyset$ $\forall n>0$, then $\mathcal{U} = pr(W) \subseteq T$ satisfies $pr^{-1}(\mathcal{U}) = \bigsqcup_{n\in\mathbb{Z}} q^n W$ and so $\mathcal{O}_T(\mathcal{U}) = \mathcal{O}_{\mathbb{G}_m^{an}}(pr^{-1}(\mathcal{U}))^{\langle q \rangle} = \mathcal{O}_{\mathbb{G}_m^{an}}(W)$. What remains to show is that T can be covered by $\mathcal{U}$ of this form. By our choice of coords. we have distinguished $z \in H^0(\mathbb{G}_m^{an}, \mathcal{O}_{\mathbb{G}_m^{an}})$. Choose $r_1, r_2 \in \mathbb{R}$ s.t. $|q| r_2 < r_1 \leq r_2$. $W(r_1, r_2) := \{ x \in \mathbb{G}_m^{an} : r_1 \leq |z(x)| \leq r_2 \}$ satisfies $q^n W(r_1,r_2) \cap W(r_1,r_2) = \emptyset$ $\forall n \neq 0$.

$\mathcal{U}(r_1,r_2) := pr(W(r_1,r_2)) \subseteq T$ is affinoid. Consider adm. affinoid cov. $T = \mathcal{U}_0 \cup \mathcal{U}_1$ w/ $\mathcal{U}_0 := \mathcal{U}(|q|^{1/3}, |q|^{-1/3})$, $\mathcal{U}_1 := \mathcal{U}(|q|^{2/3}, |q|^{1/3})$.

We have two annuli, each sharing one "edge", which we then want to glue along the other "edge" after multiplying by q.

$\mathcal{U}_0 \cap \mathcal{U}_1 = \underbrace{\mathcal{U}(|q|^{1/3}, |q|^{1/3})}_{=:V^+} \sqcup \underbrace{\mathcal{U}(|q|^{-1/3}, |q|^{-1/3})}_{=:V^-}$. Get T by gluing $\mathcal{U}_0, \mathcal{U}_1$ along $\mathcal{U}_0 \cap \mathcal{U}_1$.

<u>Claim</u>: T is proper.

<u>Pf</u>: Enlarge $\{\mathcal{U}_0, \mathcal{U}_1\}$ to $\{\mathcal{U}_0', \mathcal{U}_1'\}$ w/ $\underbrace{\mathcal{U}_0' := \mathcal{U}(|q|^{2/5}, |q|^{-2/5})}_{\supset \mathcal{U}_0}$, $\underbrace{\mathcal{U}_1' := \mathcal{U}(|q|^{4/5}, |q|^{2/7})}_{\supset \mathcal{U}_1}$. □

(This helps us see that principal divisors over T are in fact finite sums.)

<u>Claim</u>: T is separated.

<u>Pf</u>: We need to check that $\exists$ adm. affinoid cov. $\{X_i\}$ s.t. $i \neq j$ w/ $X_i \cap X_j \neq \emptyset \Rightarrow X_i \cap X_j$ affinoid and $\mathcal{O}(X_i)\hat{\otimes}_k \mathcal{O}(X_j) \twoheadrightarrow \mathcal{O}(X_i \cap X_j)$. □

Explicitly check this for $\{\mathcal{U}_0, \mathcal{U}_1\}$.

$\mathcal{M}$:= sheaf of meromorphic functions on T.

Claim : $\mathcal{M}(T)$ is a field.

Pf: $\mathcal{O}(\mathcal{U}_0), \mathcal{O}(\mathcal{U}_1)$ are PIDs so $\underset{F_0}{\underset{\shortparallel}{\mathcal{M}(\mathcal{U}_0)}}, \underset{F_1}{\underset{\shortparallel}{\mathcal{M}(\mathcal{U}_1)}}$ are fields. $\mathcal{U}_0 \cap \mathcal{U}_1 = V^+ \sqcup V^-$ union of annuli

$\Rightarrow \mathcal{M}(\mathcal{U}_0 \cap \mathcal{U}_1) = \underset{F^+}{\underset{\shortparallel}{\mathcal{M}(V^+)}} \oplus \underset{F^-}{\underset{\shortparallel}{\mathcal{M}(V^-)}}$ sum of fields. $\mathcal{M}(T) = \ker(\mathcal{M}(\mathcal{U}_0) \oplus \mathcal{M}(\mathcal{U}_1) \to \mathcal{M}(\mathcal{U}_0 \cap \mathcal{U}_1))$ is a field, basically just because the intersection of two fields is a field. □

Goal: Prove that $\mathcal{M}(T)$ is field of rational functions on elliptic ~~curve~~ curve E/k and $T = E^{an}$.

Prop: (1) $\mathcal{F} \in \mathrm{Coh}(T) \Rightarrow H^i(T, \mathcal{F}) = 0 \ \forall\, i \geq 2$. ; (2) $\dim H^0(T, \mathcal{O}_T) = 1 = \dim H^1(T, \mathcal{O}_T)$.

Pf: (1) With $\mathcal{U}_0, \mathcal{U}_1$ as before, Tate's thm says $\underset{=}{\overset{\mathcal{U}}{}}\{\mathcal{U}_0, \mathcal{U}_1\}$ is locally acyclic for $\mathcal{F}$. (this is obscuring some technicalities) Just get two-term complex. ■

~~Pf~~ (2) We can take a finite ext. of $\hat{k}$, so can assume $\exists\, \pi \in k$ s.t. $\pi^3 = q$. So, (because cohom. plays nice w/ flat base change)

$\mathcal{U}_0 = \mathcal{U}(|\pi|, |\pi|^{-1})$, $\mathcal{U}_1 = \mathcal{U}(|\pi|^2, |\pi|)$, $V^+ = \mathcal{U}(|\pi|, |\pi|)$, $V^- = \mathcal{U}(|\pi|^{-1}, |\pi|^{-1}) \cong \mathcal{U}(|\pi|^2, |\pi|^2)$.

$\mathcal{O}(\mathcal{U}_0) = \mathrm{Sp}\, k\langle \pi z, \frac{\pi}{z} \rangle = \mathrm{Sp}\, k\langle S, T\rangle / (ST - \pi^2) = \{ f = \sum_{i \geq 0} a_i (\pi z)^i + \sum_{i > 0} b_i (\frac{\pi}{z})^i : a_i, b_i \to 0 \}$.

$\mathcal{O}(\mathcal{U}_1) = \mathrm{Sp}\, k\langle \frac{z}{\pi}, \frac{\pi^2}{z} \rangle = \mathrm{Sp}\, k\langle \pi^2 z, \frac{1}{\pi z} \rangle = \{ g = \sum_{i \geq 0} c_i (\frac{z}{\pi})^i + \sum_{i > 0} d_i (\frac{\pi^2}{z})^i : c_i, d_i \to 0 \}$.

$\mathcal{O}(V^+) = \mathrm{Sp}\, k\langle \frac{z}{\pi}, \frac{\pi}{z} \rangle$, $\mathcal{O}(V^-) = \mathrm{Sp}\, k\langle \pi z, \frac{1}{\pi z} \rangle$. We compute $H^*(T, \mathcal{O}_T)$ using the complex

$0 \to \mathcal{O}(\mathcal{U}_0) \oplus \mathcal{O}(\mathcal{U}_1) \xrightarrow{d} \mathcal{O}(V^+) \oplus \mathcal{O}(V^-) \to 0$ for d given by taking differences. What are the restriction maps?

$\mathcal{O}(\mathcal{U}_0) \to \mathcal{O}(V^+)$, $f \mapsto \sum_{i \geq 0} a_i \pi^{2i} (\frac{z}{\pi})^i + \sum_{i > 0} b_i (\frac{\pi}{z})^i$. (blah blah blah) We get relations

$f|_{V^+} = g|_{V^+} \Rightarrow \begin{array}{l} a_i \pi^{2i} = c_i \\ b_i = \pi^i d_i \end{array}$

$f|_{V^-} = g|_{V^-} \Rightarrow \begin{array}{l} a_i = \pi^i c_i \\ \pi^{2i} b_i = d_i \end{array}$

$\Rightarrow f, g$ constant hence $H^0(T, \mathcal{O}_T) \cong k$.

<u>Def</u>: <u>Divisor</u> on adm. open $\mathcal{U} \subseteq T$ is formal finite sum $D = \sum_{x \in \mathcal{U}} n_x [x]$ w/ $n_x \in \mathbb{Z}$. D is <u>effective</u> ($D \geq 0$) if

each $n_x \geq 0$. Define $\deg(D) := \sum_{x \in \mathcal{U}} n_x [k_x : k]$. $f \in \mathcal{O}_T(\mathcal{U}) \rightsquigarrow$ effective divisor $\operatorname{div}(f) := \sum_{x \in \mathcal{U}} \operatorname{ord}_x(f)\,[x]$.

Extend to meromorphic $\mathcal{M}(\mathcal{U})$ by taking differences. Divisor D on T $\rightsquigarrow$ line bundle $\mathcal{L}(D)$ w/

$\mathcal{L}(D)(\mathcal{U}) = \{ f \in \mathcal{M}(\mathcal{U}) : \operatorname{div}(f) + D|_{\mathcal{U}} \geq 0 \}$. $D \geq 0 \Rightarrow \underbrace{\mathcal{L}(-D) \subseteq \mathcal{O}_T}_{\text{coherent sheaf of ideals}} \subseteq \mathcal{L}(D)$. This defines analytic

subspace of T. Underlying set is support of D w/ $\overbrace{\text{rigid structure}}^{}$ $\bigsqcup_{x \in T} \operatorname{Sp}(\overbrace{\mathcal{O}_{T,x}/\mathfrak{m}_x^{n_x}}^{k_x\text{-alg. of finite dim}})$.

$\bigoplus_{x \in T} \mathcal{O}_{T,x}/\mathfrak{m}_x^{n_x}$

$i : D \to T$ inclusion $\rightsquigarrow$ SES $0 \to \mathcal{L}(-D) \to \mathcal{O}_T \to i_* \mathcal{O}_D \to 0$. We also have SES

$\bigoplus_{x \in T} \mathfrak{m}_x^{-n_x}/\mathcal{O}_{T,x}$

$0 \to \mathcal{O}_T \to \mathcal{L}(D) \to Q_D \to 0$. $\mathcal{O}_{T,x}$ is DVR w/ non-canonical isom. $i_* \mathcal{O}_D \cong Q_D$.

Prop (Riemann-Roch for T): D divisor on $T \Rightarrow \dim H^0(T, \mathcal{L}(D)) - \dim H^1(T, \mathcal{L}(D)) = \deg(D)$.

Pf: First suppose $D \geq 0$. SES $0 \to \mathcal{O}_T \to \mathcal{L}(D) \to i_* \mathcal{O}_D \to 0$. Pick $x \in T$ and consider $\mathcal{F} = \widetilde{\mathcal{O}_{T,x}/m_x^{n_x}}$. Pick loc. acyclic affinoid cov. $\mathfrak{U} = \{\mathcal{U}_0, \mathcal{U}_1\}$ of T, so that $x \in \mathcal{U}_0$ and $x \notin \mathcal{U}_1$. $H^*(T, \mathcal{F})$ is the cohom. of the complex

$$0 \to \underbrace{\mathcal{F}(\mathcal{U}_0)}_{= \mathcal{O}_{T,x}/m_x^{n_x}} \oplus \underbrace{\mathcal{F}(\mathcal{U}_1)}_{=0} \xrightarrow{\delta} \underbrace{\mathcal{F}(\mathcal{U}_0 \cap \mathcal{U}_1)}_{=0} \to 0$$ because we're just working w/ skyscraper sheaves.

$H^1(T, i_* \mathcal{O}_D) = 0$, $H^0(T, i_* \mathcal{O}_D) = \bigoplus_{x \in T} \mathcal{O}_{T,x}/m_x^{n_x} \Rightarrow \dim H^0(T, i_* \mathcal{O}_D) = \sum_{x \in T} n_x \underbrace{\dim(\mathcal{O}_{T,x}/m_x)}_{= [k_x : k]} = \deg(D)$.

Now look at SES above and consider the LES.

For arbitrary D, take $E \geq 0$ s.t. $E - D \geq 0$. Now just use SES $0 \to \mathcal{L}(D) \to \mathcal{L}(E) \to \underbrace{Q_{E-D}}_{\cong i_* \mathcal{O}_{E-D}} \to 0$. $\square$

Prop: Let $f \in H^0(T, \mathcal{M}_T)$ nonzero. Then, $\mathrm{div}(f)$ has degree 0.

Pf: Consider annulus $A \subseteq \mathbb{G}_m^{an}$ given by $|q|r \leq |z| \leq r$ for some $r > 0$. Quotient map $pr: \mathbb{G}_m^{an} \to T$ is surj. and can pull back to $f \in \mathcal{M}(\mathbb{G}_m^{an})^{\langle q \rangle}$. Recall: g meromorphic on closed disk $D = \mathrm{Sp}\, k\langle z \rangle$ w/ no zeros or poles on $\partial D = \mathrm{Sp}\, k\langle z, z^{-1} \rangle$ then $g|_{\partial D} = z^m c(1+h)$ for $m \in \mathbb{Z}$ order of g along ∂D $(\mathrm{ord}_{\partial D}(g))$, $c \in k^\times$, $h \in k\langle z, z^{-1} \rangle^{\circ\circ}$.

We proved $\deg(\mathrm{div}(g)) = \mathrm{ord}_{\partial D}(g)$. Same applies to restriction of g to smaller disk $|z| \leq |q|$ and shows

$\deg(\text{divisor of } g \text{ in annulus } |q| \leq |z| \leq 1) = \mathrm{ord}_{|z|=1}(g) - \mathrm{ord}_{|z|=|q|}(g)$. Apply (w/ some caveats) to $f|_A$. $\square$