#### Virtual intersection theories

Zachary Gardner

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Good examples of admissible categories are  $St_k$  as well as the categories  $Sch_k$  of LFT k-schemes and  $QSch_k$  its full subcategory of qs schemes (viewing both as 2-categories in the trivial way).

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Note that the context of more narrow theories we also have access to more general *proper* pushforwards and *flat* pullbacks.

<sup>&</sup>lt;sup>1</sup>Here,  $X \times Y$  is the product formed in V and so is naturally fibered over Spec k.  $\circ \circ \circ$ zachary.gardner@bc.edu 4/20

• Refined Gysin pullback: For a Cartesian square

$$X' \xrightarrow{f'} Y'$$

$$g' \downarrow \qquad \qquad \downarrow g$$

$$X \xrightarrow{f} Y$$

with f a regular immersion of constant codimension c, we have a graded map

$$f^!: H_*(Y') \to H_{*-c}(X').$$

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• Exterior product: For  $X, Y \in V$ , we have a bilinear graded map

$$\times: H_*(X) \otimes_{\mathbb{Z}} H_*(Y) \to H_*(X \times Y)$$

which is commutative and associative with a distinguished element  $1 \in H_0(\operatorname{Spec} k)$  as unit.<sup>1</sup>

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- Given  $X, Y \in V$ , there are obvious maps  $i: X \to X \sqcup Y$  and  $j: Y \to X \sqcup Y$  and the induced map

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The other conditions are more mysterious or at least require more geometric context. But before that, some examples!

Given  $X \in V$ , endow  $G_0(X)$  with a grading by considering

$$G_0(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\beta^{\pm 1}] \cong \bigoplus_{n \in \mathbb{Z}} G_0(X) \cdot \beta^n$$

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$$f^*: [E] \cdot \beta^n \mapsto [f^*E] \cdot \beta^{n+e}.$$

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$$\times : ([E] \cdot \beta^n) \otimes ([E'] \cdot \beta^m) \mapsto [E \boxtimes E'] \cdot \beta^{n+m}.$$

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This is commutative and associative with a distinguished element  $1 \in G_0(\operatorname{Spec} k) \cong \mathbb{Z}$  as unit. Note that, by definition,  $E \boxtimes E'$  is given by the tensor product  $\operatorname{pr}_X^{-1} E \otimes_{\mathcal{O}_{X \times Y}} \operatorname{pr}_Y^{-1} E'$  for  $\operatorname{pr}_X : X \times Y \to X$  and  $\operatorname{pr}_Y : X \times Y \to Y$  the projection maps.

## Algebraic Cycles

<sup>&</sup>lt;sup>2</sup> "Nice" means separated, Noetherian, finite dimensional pand excellent. ■ ► ■ ✓ ९०

#### Algebraic Cycles

Our second and arguably most important example comes from Chow theory. For simplicity, let Sch denote the category of "nice" schemes.<sup>2</sup> We will try to do things carefully to clarify some things and make our lives easier in the future. In particular, we will be careful about matters of (co-)dimension as well as rational equivalence.

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• For any (general) scheme X, we let Z(X) denote the free abelian group on the set of closed integral subschemes of X, whose elements are called (algebraic) cycles on X.

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- For any (general) scheme X, we let Z(X) denote the free abelian group on the set of closed integral subschemes of X, whose elements are called (algebraic) cycles on X.
- The closed integral subschemes of X correspond bijectively with the (topological, field-valued) points of X. This is given by looking at generic points  $\zeta$  and we have

$$Z(X) \cong \bigoplus_{\zeta \in X} \mathbb{Z}.$$

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We define the sets  $X^{(p)}$  and  $X_{(p)}$  so that  $Z^p(X) \cong \bigoplus_{x \in X^{(p)}} \mathbb{Z}$  and  $Z_p(X) \cong \bigoplus_{x \in X_{(p)}} \mathbb{Z}$  – these are useful for bookkeeping purposes.

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For general X, we have the sheaf  $\mathcal{O}_X^{\text{reg}}$  of regular elements (which can be defined at the level of stalks). To this we associate  $\mathcal{K}_X := \mathcal{O}_X[(\mathcal{O}_X^{\text{reg}})^{-1}]$  and  $\mathcal{D}\text{iv}_X := \mathcal{K}_X^\times/\mathcal{O}_X^\times$ . These give rise to the additive group of *Cartier divisors*  $\text{Div}(X) := H^0(X, \mathcal{D}\text{iv}_X)$ .

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• There is a unique Zariski sheaf morphism  $\operatorname{div}: \mathcal{D}\mathrm{iv}_X \to \mathcal{Z}_X^1$  satisfying  $\operatorname{div}(f) = [\mathcal{O}_U/(f)]$  for  $U \subseteq X$  open and  $f \in \mathcal{O}_X^{\mathsf{reg}}(U)$ .

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Define  $R(X):=\bigoplus_{\zeta\in X}k(\zeta)^{\times}$ , which can be graded by (co-)dimension as appropriate. The amalgamation of  $\operatorname{div}:K(Z)^{\times}\to Z^1(Z)$  for  $Z\subseteq X$  closed integral induces  $\operatorname{div}:R(X)\to Z(X)$  and we set  $\operatorname{CH}(X):=\operatorname{coker}(\operatorname{div}:R(X)\to Z(X))$ . Being rationally equivalent to zero is captured by the image of  $\operatorname{div}$ .

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Note that there is another equivalent approach to rational equivalence that works for varieties over a field. Heuristically, two cycles  $\alpha, \beta$  are rationally equivalent if there is a family of cycles  $\{\zeta_t\}_{t\in\mathbb{P}^1}$  with  $\zeta_0=\alpha$  and  $\zeta_\infty=\beta$ . This can be made more precise using the language of flat families.

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which must be interpreted in terms of support and length

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Both of these constructions are invariant under rational equivalence and so descend to maps of Chow groups.

<sup>&</sup>lt;sup>3</sup>Defining the exterior product over a more general base should be possible using some kind of fibral procedure.

We ultimately want to grade by dimension and this affects pullbacks. If  $f: X \to Y$  is smooth of constant relative dimension e then we have

$$f^*: \mathsf{CH}_*(Y) \to \mathsf{CH}_{*+e}(X), \qquad [\eta] \mapsto [f^{-1}(\eta)]$$

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Working over *k*, the *exterior product* 

$$\times : \mathsf{CH}_*(X) \otimes_{\mathbb{Z}} \mathsf{CH}_*(Y) \to \mathsf{CH}_*(X \times Y)$$

is given by the recipe  $[Z_1] \otimes [Z_2] \mapsto [Z_1 \times Z_2]$  if k is algebraically closed and for general k in terms of the irreducible components of  $Z_1 \times Z_2$ .<sup>3</sup>

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Working over k, the exterior product

$$\times : \mathsf{CH}_*(X) \otimes_{\mathbb{Z}} \mathsf{CH}_*(Y) \to \mathsf{CH}_*(X \times Y)$$

is given by the recipe  $[Z_1] \otimes [Z_2] \mapsto [Z_1 \times Z_2]$  if k is algebraically closed and for general k in terms of the irreducible components of  $Z_1 \times Z_2$ .

The only thing left to describe for Chow theory is the refined Gysin pullback. This is a bit subtle and so we will come back to it later.

<sup>&</sup>lt;sup>3</sup>Defining the exterior product over a more general base should be possible using some kind of fibral procedure.

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#### Theorem

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- **③** Given  $X \in \operatorname{Sm}_k$  and  $V, W \subseteq X$  closed integral,  $[V \times_k X] \cdot [X \times_k W] = [V \times_k W]$  working with cycles on  $X \times_k X$ .

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- **1** Projection formula: Let  $f: X \to Y$  be proper with  $X, Y \in Sm_k$ ,  $\alpha \in CH^*(X)$ , and  $\beta \in CH^*(Y)$ . Then,  $f_*(\alpha \cdot f^*(\beta)) = f_*(\alpha) \cdot \beta$ .

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- **1** Homotopy invariance: Let  $p: V \to X$  be a smooth vector bundle. Then,  $p^*: \operatorname{CH}^*(X) \xrightarrow{\sim} \operatorname{CH}^*(V)$ .

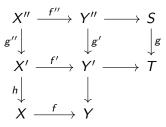
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Returning to our intersection theory  $H_*$  for V, what are the remaining properties we want to impose?

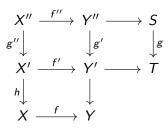
Returning to our intersection theory  $H_*$  for V, what are the remaining properties we want to impose? First, we have the base change condition which says that if

$$\begin{array}{ccc}
X & \xrightarrow{g} & Y \\
\downarrow q & & \downarrow p \\
Z & \xrightarrow{f} & W
\end{array}$$

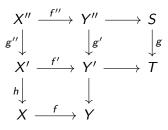
is a Cartesian square with f projective and p smooth then  $p^* \circ f_* = g_* \circ q^*$ .



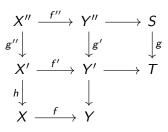
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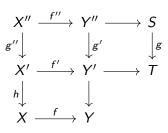


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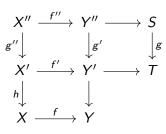
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- If g' is smooth (so that g'' is automatically smooth) then we demand  $f^! \circ g'^* = g''^* \circ f^!$ .
- If g is a regular immersion then we demand  $f^! \circ g^! = g^! \circ f^!$ .

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I believe this construction is closely linked to Grothendieck-Riemann-Roch but have not looked into the details. More on this later...

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The latter construction of  $\eta$  is linked to other constructions. For L a line bundle on X, the zero section  $0:X\to L$  defines the *first Chern class* homomorphism

$$c_1(L) := 0! \circ 0_* : H_*(X) \to H_{*-1}(X),$$

which you should check is well defined in the sense that 0 is a projective regular immersion.

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Let's round out our discussion of properties of  $H_*$ .

$$H_*(Z) \xrightarrow{i_*} H_*(X) \xrightarrow{j^*} H_*(U) \longrightarrow 0$$

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- Projective bundle formula: Let  $E \in \text{Vect}_r(X)$  with associated projective bundle  $p : \mathbb{P}(E) \to X$ . We demand that the induced map

$$\bigoplus_{i=0}^{r-1} H_{*-r+1+i}(X) \to H_*(\mathbb{P}(E)), \qquad (\xi_i) \mapsto \sum_i c_1(\mathcal{O}_{\mathbb{P}(E)}(1))^i \cdot (p^*\xi_i)$$

is an isomorphism.

<sup>4</sup>This should remind you of the theory of quasi-smooth-morphisms. ≥ → √ ≥ → ○ ○

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• Let  $f: X \to Y$  be an lci morphism of constant relative dimension d and choose a factorization  $f = h \circ g$  with  $g: X \to Z$  a regular closed immersion and  $h: Z \to Y$  smooth.

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