Algebraic K theory

Dirk van Bree

15 feb. 2022

K_0 of a ring

- Recall: if R is a (discrete) ring, define the group

$$K_0(R) = \bigoplus_{\text{f.g. proj. modules } P} \mathbb{Z}P$$

  modulo $[P] = [P'] + [P'']$ for every s.e.s. $0 \to P' \to P \to P'' \to 0$.
- Universal property: the map $\chi(P) = [P]$ is the universal map satisfying $\chi(P) = \chi(P') + \chi(P'')$ for a s.e.s as above.
- Can also define $K_0(R)$ using finite complexes of f.g. proj. modules using triangles. Then $-[C] = [\Sigma C]$.

K_0 of a category

- For C a stable ∞-category,

$$K_0(C) = \{P \mid P \text{ is a compact object in } C\}/\sim$$

  such that $[P] = [P'] + [P'']$ for every cofiber sequence $P' \to P \to P''$.
- This recovers $K_0(R)$ if $C = \mathrm{Mod}_R$.
- In particular, we can now take the K_0 of a (derived) scheme.

Why compact objects?

The *Eilenberg-Mazur swindle*: if $N = M \oplus M \oplus M \ldots$, then $M \oplus N = N$, so $M = 0$ in K_0. Hence, we want to avoid infinite sums.

A localisation sequence

- If $f \in R$, then we have a *localisation sequence*

$$K_0(\mathrm{Mod}_{R,Z(f)}) \to K_0(R) \to K_0(R_f) \to 0.$$

  Here $\mathrm{Mod}_{R,Z(f)}$ are those R-modules supported on $Z(f)$.
- Can we turn this into a long exact sequence?
- Central idea: find a natural space $K(R)$ such that $\pi_0(K(R)) = K_0(R)$. Then define $K_i(R) = \pi_i(K(R))$.

Motivation for the construction

From now on, fix a stable ∞-category C. Notation: if $X \to Y$ is a morphism, denote by Y/X its cofiber.

Lemma

If $X \to Y \to Z$, then $[Z] = [X] + [Y/X] + [Z/Y]$.

Proof. We get cofiber sequences

$$X \to Z \to Z/X \quad \text{and} \quad Y/X \to Z/X \to Z/Y$$

telling us that $[Z] = [X] + [Z/X]$ and $[Z/X] = [Y/X] + [Z/Y]$. Combine these.

Proof 2. Instead use the sequences

$$X \to Y \to Y/X \quad \text{and} \quad Y \to Z \to Z/Y.$$

Thinking "homotopically", we have found two paths given the same data.

Gaps

Denote by $\mathrm{Gap}_n(C)$ the set of equivalence classes of n-fold compositions $X_1 \to \ldots \to X_n$ together with a choice of cofibers X_i/X_j, arranged in a nice diagram

$$
\begin{tikzcd}
0 \arrow[r] & X_1 \arrow[r] \arrow[d] & X_2 \arrow[r] \arrow[d] & X_3 \arrow[r] \arrow[d] & \ldots \arrow[r] & X_n \arrow[d] \\
 & 0 \arrow[r] & X_2/X_1 \arrow[r] \arrow[d] & X_3/X_1 \arrow[r] \arrow[d] & \ldots \arrow[r] & X_n/X_1 \arrow[d] \\
 & & 0 & \ddots & \vdots \arrow[d] & \vdots \arrow[d] \\
 & & & & 0 \arrow[r] & X_{n-1}/X_n \arrow[d] \\
 & & & & & 0
\end{tikzcd}
$$

Building a simplicial set

Thus, $\mathrm{Gap}_0(C) \cong *$, $\mathrm{Gap}_1(C) \cong C$, $\mathrm{Gap}_2(C)$ are the cofiber sequences in C and $\mathrm{Gap}_3(C)$ are the compositions $X \to Y \to Z$ with a choice of cofibers.

- There are three maps $\mathrm{Gap}_2(C) \to \mathrm{Gap}_1(C)$ by sending a cofiber sequence to three of its objects.
- There are four maps $\mathrm{Gap}_3(C) \to \mathrm{Gap}_2(C)$ by extracting from $X \to Y \to Z$ the three cofiber sequences we did above.
- Idea: in both cases, we want the "alternating sum" of the maps to be zero. So we are building some kind of simplicial set!

The K-theory space

- Homework: actually turn $\mathrm{Gap}_n(C)$ into a simplical set. I did the first few face maps already.
- But $\mathrm{Gap}_n(C)$ is not really a set...
- Instead of considering objects/diagrams up to equivalence, we should remember the equivalences and instead consider $\mathrm{Gap}_n(C)$ as a space. Hence we get a simplicial space $\mathrm{Gap}_\bullet(C)$.

Proposition

The following functors $(\mathrm{Set}_\Delta)_\Delta \to \mathrm{Set}_\Delta$ coincide:

- The left adjoint of the constant functor, called *geometric realisation*.
- The homotopy colimit of $S_{n,\bullet}$.
- The diagonal $d(S_{\bullet,\bullet})_n = S_{n,n}$.

Denote the resuling space by $|\mathrm{Gap}_\bullet(C)|$ and define $K(C)$ to be the loop space of this space. Then $K(C)$ is a *connective spectrum*, i.e., some kind of ∞-group object.

The fundamental groupoid

Is $\pi_0(K(C))$ actually $K_0(C)$? Equivalent: is $\pi_1(|\operatorname{Gap}_\bullet(C)|)$ actually $K_0(C)$? Let's pretend that $|\operatorname{Gap}_\bullet(C)|$ is a Kan complex.

- We only have to look at $\operatorname{Gap}_n(C)_n$ for $n = 0, 1, 2$.
- It is easy (but tedious) to see that we get the "equivalence classes" description from before.
- So the fundamental group consists of equivalence classes $[X]$ of objects C with $[X'] + [X''] = [X]$ for every cofiber sequence.
- Formally, we need to replace $|\operatorname{Gap}_n(C)|$ by a Kan complex, we can verify using an explicit model (the Ex^∞ functor) that this gives the same answer.

Other constructions

- One might remark that we used only a very small part of the structure of stable ∞-categories. This construction applies to a more general class of categories known as *Waldhausen categories*.
- For an "exact category", Quillen also provides a construction, called the *Q-construction*. It is based on using "roofs", instead of "cofiber sequences".
- For the K-theory of a ring R: let $\mathrm{GL}(R)$ be the direct limit of $\mathrm{GL}_n(R)$. Then

$$K_1(R) = \mathrm{GL}(R)/[\mathrm{GL}(R), \mathrm{GL}(R)] = H_1(B\,\mathrm{GL}(R), \mathbb{Z})$$

  But we want homotopy, not homology! We have $\pi_1(B\,\mathrm{GL}(R)) = \mathrm{GL}(R)$. There exists a space $B\,\mathrm{GL}^+$ such that any map $B\,\mathrm{GL} \to Y$ which kills $[\mathrm{GL}, \mathrm{GL}]$ on π_1 factors through $B\,\mathrm{GL}^+$. Then $B\,\mathrm{GL}(R)^+ \times K_0(R) = K(R)$.

Computational results

- By a result of Quillen:

$$K_i(\mathbb{F}_q) = \begin{cases} \mathbb{Z}/(q^n - 1) \text{ if } i = 2n - 1 \\ 0 \text{ otherwise} \end{cases}$$

- As far as I know, there is no easy description for $K_i(\mathbb{C})$. We know something though:
- We know $K_0(\mathbb{C}) = \mathbb{Z}$, $K_{2n}(\mathbb{C})$ is a $\mathbb{Q}$-vector space for $n > 0$, and $K_{2n+1}(\mathbb{C})$ is the direct sum of $\mathbb{Q}/\mathbb{Z}$ and a $\mathbb{Q}$-vector space.
- In general, we know that $K_1(F) = F^*$ for a field F. Indeed, if I mod out $\mathbb{Q}/\mathbb{Z} \cong \{\zeta \mid \exists n\, \zeta^n = 1\} \subseteq \mathbb{C}^*$ we get a $\mathbb{Q}$-vector space, so our descriptions are compatible.

Long exact sequences

We would like to construct long exact sequences of K-theory groups, or equivalently, fiber sequences of spectra

$$K(C) \to K(C') \to K(C'').$$

For example, we want this for the sequence

$$\mathrm{Mod}_{R,Z(f)} \to \mathrm{Mod}_R \to \mathrm{Mod}_{R_f}$$

of stable infinity categories.
Central idea: this sequence should be a cofiber sequence of stable ∞-categories!

Big and small categories

I said that we only want to apply K-theory to compact objects.

- If C is a presentable stable ∞-category, then we can look only at its compact objects.
- If C is a small stable infinity category, then $\mathrm{Ind}(C)$ is a presentable stable ∞-category.
- This gives an equivalence of categories between *compactly-generated* presentable stable infinity categories C whose morphisms preserve colimits and compact objects and *idempotent-complete* small compact stable infinity categories.

Idempotent completeness

This is a notion we will not need: any stable ∞-category C admits an *idempotent completion* C'; then we have $K(C) = K(C')$ (and most interesting categories are idempotent complete anyway).

Exact sequences of big categories

Let $C \to D$ be a colimit preserving exact functor of presentable stable ∞-categories.

- There is a universal functor $D \to D/C$ such that the composition $C \to D \to D/C$ is zero. (The *Verdier quotient*.)
- Explicitely, we localise at all $X \to Y$ whose cofiber is in the image of C.
- We say that the sequence $C \to D \to D/C$ is *exact* if in addition $C \to D$ is fully faithful. Any sequence equivalent to such a sequence is also exact.
- We call such a sequence *strict* if C is the universal stable infinity category such that $C \to D \to D/C$ is zero.
- We call such a sequence *split* if it is strict and there is a right adjoint $D/C \to D$ which is fully faithful.

Via the correspondence above, we can also define these notions for small stable ∞-categories.

K-theory and exact sequences

Let $A \to B \to C$ be an exact sequence of stable infinity categories.

Fact: Waldhausen Additivity theorem

If the sequence is split, then $A \to B$ and the right adjoint $r : C \to B$ induce an equivalence $K(A) \oplus K(C) \to K(B)$.

Fact: localisation theorem

If the sequence is *strict exact*, then the induced sequence $K(A) \to K(B) \to K(C)$ is a cofiber sequence.

It is not true for arbitrary exact sequences. For this, you need *nonconnective K-theory*.

Properties of K-theory

- There is a natural map $C^{\cong} \to K(C)$ sending objects of C to their K-theory classes. ($C^{\cong}$ is simply C by forgetting noninvertible maps, the largest Kan complex in C.)
- K-theory, as a functor from small stable idempotent-complete ∞-categories, preserves filtered colimits.
- Idea of proof: prove that all steps in the construction preserve filtered colimits.

Universal property of K-theory

Definition: additive invariants

Let $\mathcal{E}$ be a presentable stable ∞-category. An *$\mathcal{E}$-valued additive invariant* is a functor E from idempotent-complete small stable ∞-categories to $\mathcal{E}$ such that F preserves filtered colimits and such that $E(A) \oplus E(C) \to E(B)$ is an equivalence for all split exact sequences.

So, K-theory is a spectrum-valued additive invariant.

Theorem: universal property of K-theory

If E is any spectrum-valued additive invariant with a natural map $C^{\cong} \to E(C)$, then this map factors through some $K \to E$.

Universal property of K-theory

- Note: an object of C is an exact functor $\mathrm{Sp}^\omega \to C$. (Where Sp^ω is the category of finite spectra.)
- Hence, the universal property says that

$$\mathrm{Map}(K, E) = \mathrm{Map}((-)^{\cong}, E) = \mathrm{Map}(\mathrm{Map}(\mathrm{Sp}^\omega, -), E) = E(\mathrm{Sp}^\omega).$$

- Idea: the category $\mathrm{Map}^{\mathrm{add}}(\mathrm{Sp})$ of additive invariants is a subcategory of $\mathrm{Map}(\mathrm{Cat}_{\text{st, small}}, \mathrm{Sp})$. Then K should be the "additification" of the functor corepresented by Sp^ω!

A more ambitious universal property

- **Theorem.** There is a category M_{add} such that

$$\text{Map}^{\text{add}}(\mathcal{E}) = \text{Map}^{L}(M_{\text{add}}, \mathcal{E}).$$

  Here Map^{add} means the additive invariants and Map^{L} means colimit preserving functors.

- In particular, there is a universal additive invariant U_{add} in $\text{Map}^{\text{add}}(M_{\text{add}})$.

-

$$\begin{tikzcd} \text{Cat}_{\text{st, small}} \arrow[rr, "U_{\text{add}}"] \arrow[dr, "E"'] & & M_{\text{add}} \arrow[dl, dashed, "\exists!"] \\ & \mathcal{E} & \end{tikzcd}$$

A more ambitious universal property

- **Theorem.** K-theory induces a functor $M_{\mathrm{add}} \to \mathrm{Sp}$. This functor is corepresented by $U_{\mathrm{add}}(\mathrm{Sp}^{\omega})$.
- This implies the universal property. If E is in $\mathrm{Map}^{\mathrm{add}}(\mathrm{Sp})$ with corresponding $E' : M_{\mathrm{add}} \to \mathrm{Sp}$ then

$$\begin{aligned}\mathrm{Map}(K, E) = \mathrm{Map}(\mathrm{Map}(U_{\mathrm{add}}(\mathrm{Sp}^{\omega}), -), E') &= E'(U_{\mathrm{add}}(\mathrm{Sp}^{\omega})) \\ &= E(\mathrm{Sp}^{\omega}).\end{aligned}$$

- Concretely, the corepresentation statement means that $K(A) = \mathrm{Map}(U_{\mathrm{add}}(\mathrm{Sp}^{\omega}), U_{\mathrm{add}}(A))$.
- To prove this we will actually look at $\mathrm{Map}(-, U_{\mathrm{add}}(A))$. The claim is that $\mathrm{Map}(U_{\mathrm{add}}(B), U_{\mathrm{add}}(A))$ is *relative K-theory* $K_A(B) = K(\mathrm{Fun}^{\mathrm{ex}}(B, A))$. Note that indeed $K_A(\mathrm{Sp}^{\omega}) = A$.
- Technical detail: B is compact in the above discussion.

M_{add} and U_{add}

Sketch of the construction.

- Consider the Yoneda embedding $\mathrm{Cat}_{\text{st, small, compact}}$ into its presheaf ∞-category.
- Localise some maps in the presheaf category to force additivity. (A "sheaf-like" situation.)
- Stabilise the resulting category to create a stable ∞-category.
- The map U_{add} is what remains of the Yoneda embedding.

Confusing: we can now see K_A as an element of M_{add}!

Proof of universal property

We are done if we know that

- K_A is a local object with respect to the localisation.
- $U_{\mathrm{add}}(A) = K_A$ in M_{add}

because then

$$\begin{aligned}\mathrm{Map}_{M_{\mathrm{add}}}(U_{\mathrm{add}}(B), U_{\mathrm{add}}(A)) &= \mathrm{Map}_{M_{\mathrm{add}}}(U_{\mathrm{add}}(B), K_A) \\ &= \mathrm{Map}_{\mathrm{Presheaves}}(\mathrm{Map}(-, B), K_A) \\ &= K_A(B)\end{aligned}$$

The fact that K_A is local is the Waldhausen fibration theorem. The other one needs a proof.

Proof of universal property

We need to mix the Waldhausen and the M_{add} constructions.

- Note that $\mathrm{Gap}_n(A)$ is not just an ∞-groupoid, but a stable ∞-category as well.
- Denote by $S_\bullet A$ the resulting simplicial stable ∞-category.
- Apply U_{add} levelwise. Then as a simplicial object of M_{add}, we have

$$\begin{aligned} U_{\text{add}}(S_\bullet A) &= \mathrm{Map}^{\text{ex}}(-, S_\bullet A) \\ &= S_\bullet \mathrm{Map}^{\text{ex}}(-, A) \\ &= S_\bullet \mathrm{Fun}^{\text{ex}}(-, A)_{\cong} \end{aligned}$$

- If you geometrically realise the RHS, you get $\Sigma K_A(-)$.

Proof of universal property

To relate $U_{\mathrm{add}}(S_\bullet A)$ and $U_{\mathrm{add}}(A)$ consider the sequence

$$A_\bullet \to PS_\bullet A \to S_\bullet A$$

Here $A_\bullet$ is a constant simplicial category and $PS_n A = S_{n+1} A$. The map $A_\bullet \to PS_\bullet A$ is given by $A \mapsto (0 \to A \to \ldots \to A)$ and $PS_\bullet A \to S_\bullet A$ is given by the zeroeth face map.

- This sequence is *levelwise split*. A splitting is given by the zeroeth degeneracy map.
- Hence, after applying U_{add} and geometrically realising, we get a cofiber sequence

$$U_{\mathrm{add}}(A) = |U_{\mathrm{add}}(A_\bullet)| \to |U_{\mathrm{add}}(PS_\bullet A)| \to |U_{\mathrm{add}}(S_\bullet(A))| = \Sigma K_A$$

- General fact: $|PS_\bullet A| = S_0 A$ is contractible! So we have that $U_{\mathrm{add}}(A) = \Omega\Sigma K_A = K_A$. We are done!

Nonconnective K-theory

Let A be a small stable ∞-category. I will construct a “K-theoretic cone” and a “K-theoretic suspension” of A.

- Consider $F_\kappa A = (\mathrm{Ind}_\omega(A))^\kappa$ for $\kappa > \omega$ regular. This means the κ-compact objects of the Ind-completion. Then $K(F_\kappa A) = *$ (the swindle again).
- Define $\Sigma_\kappa A = F_\kappa A / A$.
- Define $\mathbf{K}(A) = \mathrm{colim}_n \Omega^n K(\Sigma_\kappa^n A)$. This is *nonconnective K-theory*.
- This is interesting even for ordinary rings: negative K-groups exist!

The universal property of nonconnective K-theory

- We can perform an analogous construction where we consider functors that turn *arbitrary* exact sequences to cofiber sequences.
- Call the result M_{loc} and U_{loc}. Then we have a similar theorem saying that

$$\mathrm{Map}(U_{\mathrm{loc}}(\mathrm{Sp}^{\omega}), U_{\mathrm{loc}}(A)) = \mathbf{K}(A)$$

  which is the *relative nonconnective K-theory*.
- There is also a relative version: we have

$$\mathrm{Map}(U_{\mathrm{loc}}(B), U_{\mathrm{loc}}(A)) = \mathbf{K}(B^{\mathrm{op}} \hat{\otimes} A)$$

  if B is "smooth and proper".

References

- Blumberg, Gepner, Tabuada *A universal characterisation of higher algebraic K-theory*.
- Moi, *Universality and localisation for K-theory*
- Lurie, *Higher K-theory of ∞-categories* (Lecture notes)
- Waldhausen, *Algebraic K-theory of spaces*