$A \to S$ abelian scheme $\rightsquigarrow$ $\mathrm{Pic}_A = \mathrm{Pic}_{A/S} : \mathrm{Sch}_S \to \mathrm{Ab}$ via $\mathrm{Pic}_A(T) := \{$ line bundles L on A_T equipped w/ trivialization of pullback of L along $T \xrightarrow{e} A_T \}$.

Recall: $A \to S$ ab. scheme is smooth proper w/ geom. conn. fibers and grp. law.

Thm (Grothendieck, Raynaud): Pic_A is represented by smooth S-grp. scheme whose conn. components are proper $/S$ w/ geom. conn. fibers. $A \to S$ proj. $\Rightarrow$ same is true for conn. components of Pic_A.

Def: $A^\vee := \mathrm{Pic}_A^0$ (have to be careful about what this means for S not conn.).

$A \to S$ proj. $\Rightarrow A^\vee \to S$ proj.

Remark: Nat. map $A^\vee \hookrightarrow \mathrm{Pic}_A$ is pt. of $\mathrm{Pic}_A(A^\vee)$, which is line bundle on $A \times A^\vee$ w/ trivialization along $A^\vee \xrightarrow{\{e\}\times \mathrm{id}} A \times A^\vee$. This line bundle is the Poincaré bundle $\mathcal{P}_A$. This is "universal" in a precise sense.

Prop: $\exists$ canon. isom. $A \xrightarrow{\sim} A^{\vee\vee}$. [This is a purely formal result.]

$f : A \to B$ map of ab. schemes $\rightsquigarrow f^\vee : B^\vee \to A^\vee$ via $f^\vee(L) := f^*L$.

Polarizations

Given L line bundle on A, want symm. map $\phi_L : A \to A^\vee$.

First attempt: Imitate construction $/\mathbb{C}$. $T \in \mathrm{Sch}_S$, $x \in A(T) \rightsquigarrow t_x : A_T \to A_T$.

Let $L_T := L_{A_T}$. $t_x^* L_T \otimes L_T^{-1}$ line bundle on A_T but not trivialized along 0-section. This is not a problem. Thinking of x as a map $T \to A_T$, pulling back along $e : T \to A_T$ is $e^* t_x^* L_T \otimes e^* L_T^{-1} \cong x^* L_T \otimes e^* L_T^{-1}$.

So, we can modify ϕ_L to get canon. trivialization (twist by appropriate thing).

Second attempt: We have three maps $A \times_S A \to A$: μ, pr_1, pr_2. L line bundle on A

$\rightsquigarrow \Lambda(L) := \mu^* L \otimes pr_1^* L^{-1} \otimes pr_2^* L^{-1}$.

Remark: Morphism $A \to A^\vee$ is elt. of $A^\vee(A) \subseteq \mathrm{Pic}_A(A) = \{$ line bundles on $A \times_S A$ trivialized along 0-section $S \times_S A \to A \times_S A \}$.

So, we can take ϕ_L to "be" $\Lambda(L)$. By Thm of the Square we have $\phi_{L_1 \otimes L_2} = \phi_{L_1} + \phi_{L_2}$.

<u>Def</u>: <u>Polarization</u> of $A \to S$ is map $\lambda : A \to A^\vee$ satisfying the following (equiv.) conditions.

(1) étale locally on S, $\lambda = \phi_L$ w/ L ample. [check étale covers or, equivalently, surj. étale maps]

(2) at every geom. pt. $s \to S$, the map $\lambda_s : A_s \to A_s^\vee$ has form $\lambda_s = \phi_L$ w/ L ample on A_s.

(3) pullback of $\mathcal{P}_A$ along $\mathrm{id}_A \times \lambda : A \to A \times_S A^\vee$ is ample.