Exam 2

Mon. Nov. 16, 2020

This is a closed book exam; you may not consult any references or notes. Upload your completed exam to gradescope immediately after completing the exam.

Question 1. Let μ and ν be finite measures on a measurable space$(X, \mathcal{M})$ with $\nu \ll \mu$. Denote by f the Radon-Nikodym derivative $\frac{d\nu}{d(\nu+\mu)}$ and denote by g the Radon-Nikodym derivative $\frac{d\nu}{d\mu}$. Show that for any set $E \in \mathcal{M}$,

$$\int_E g \; d\mu = \int_E (f + fg) \; d\mu.$$

Question 2. Let μ be counting measure on $\mathbb{N}$.

1. Prove that the space (of real-valued functions) $\ell^1 := L^1(\mathbb{N}, \mathcal{P}(\mathbb{N}), \mu)$ is a Banach space.

2. What link does the Radon-Nikodym theorem yield between ℓ^1 and the set of all finite signed measures $\lambda \ll \mu$ on $\mathbb{N}$?

Question 3. Prove that if $\mathcal{X}$ and $\mathcal{Y}$ are normed linear spaces and $T : \mathcal{X} \to \mathcal{Y}$ is a linear map, then T is continuous if and only if T is bounded.

Question 4. Let $(X, \mathcal{M}, \mu)$ be the real interval $[0, 1]$ with Lebesgue measure. Prove that for any continuous function $g : X \to \mathbb{R}$, the map $\Phi_g : L^1(\mu) \to \mathbb{R}$ defined by

$$\Phi_g(f) = \int fg \; d\mu$$

is is an element of the dual space $L^1(\mu)^*$ and satisfies $||\Phi_g||_{op} = ||g||_{sup}$.