Review and derived Artin stacks

Zachary Gardner

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- ∞-over-category dAlg_A of derived (commutative) A-algebras

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For A discrete we have $\mathrm{Anim}(\mathrm{Mod}_A) \simeq \mathcal{D}(A)_{\geq 0}$.

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- $M \in \mathcal{D}(A)$ is **strong** if the natural morphism

  $$\pi_0(M) \otimes_{\pi_0(A)} \pi_i(A) \to \pi_i(M)$$

  of $\pi_0(A)$-modules is an isomorphism for every $i \in \mathbb{Z}$

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$\mathcal{F} \in \mathrm{Pre}(\mathcal{T})$ is a **sheaf** if and only if

$$\mathcal{F}(U) \xrightarrow{\sim} \mathrm{hlim}\left(\prod_{\alpha\in\Lambda} \mathcal{F}(U_\alpha) \rightrightarrows \prod_{\alpha,\beta\in\Lambda} \mathcal{F}(U_\alpha \times_U U_\beta) \cdots\right)$$

for every $\mathcal{U} = \{U_\alpha \to U\}_{\alpha\in\Lambda}$ in $\widetilde{\tau}$. The simplicial object inside the limit is the **Čech nerve** of $\mathcal{U}$.

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Here, $\Omega^1_{B/A} \in \mathrm{Mod}_B$ is the module of Kähler differentials characterized by

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natural in $M \in \mathrm{Mod}_B$. We "left derive" this to get the **cotangent complex** $\mathbb{L}_{B/A} \in \mathcal{D}(B)$, which satisfies a similar universal property.

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Given $A \in \mathsf{CRing}$ and $M \in \mathsf{Mod}_A$, we have the split square-zero extension $A \oplus M \in \mathsf{CAlg}_A$ with

$$\begin{aligned} \mathrm{Der}_{\mathbb{Z}}(A, M) &\cong \mathrm{Hom}_{\mathsf{CRing}\,/A}(A, A \oplus M) \\ &= \mathrm{fib}_{\mathrm{id}_A}(\mathrm{Hom}_{\mathsf{CRing}}(A, A \oplus M), \mathrm{Hom}_{\mathsf{CRing}}(A, A)) \end{aligned}$$

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Given $A \to B \to C$ in CRing, there is an exact sequence

$$C \otimes_B \Omega^1_{B/A} \to \Omega^1_{C/A} \to \Omega^1_{C/B} \to 0$$

in Mod_C. This lets us make sense of the *relative* $\Omega^1_{B/A}$ in terms of the *absolute* Ω^1_A and Ω^1_B.

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$$B \otimes_A \mathbb{L}_A \to \mathbb{L}_B \to \mathbb{L}_{B/A}$$

in $\mathcal{D}(B)$. Note that $\pi_0(\mathbb{L}_A) \simeq \Omega^1_{\pi_0(A)}$ and $\mathbb{L}_{B/A} \simeq 0$ if and only if $B \otimes_A \mathbb{L}_A \xrightarrow{\sim} \mathbb{L}_B$.

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Fixing $A \in \mathrm{dRing}$, we say $P \in \mathcal{D}(A)_{\geq 0}$ is **projective** if $\mathrm{Hom}_{\mathcal{D}(A)_{\geq 0}}(P, -) : \mathcal{D}(A)_{\geq 0} \to \mathrm{Anim}$ commutes with geometric realization (of simplicial objects). This captures the usual notion when P is discrete.

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is surjective on π_0. For future reference, we say B is **homotopically finitely presented** if B is compact in dAlg_A – i.e., $\text{Hom}_{\text{dAlg}_A}(B, -)$ commutes with filtered homotopy colimits.

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- **étale** if it is formally étale and homotopically finitely presented.

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Smoothness is a more subtle notion. Let us content ourselves for now by saying that B is **smooth** if it is formally infinitesimally smooth, homotopically finitely presented, and if $M \in \mathcal{D}(B)_{\geq 0}$ with $\pi_0(M) = 0$ then $[\mathbb{L}_{B/A}, M] = 0$.

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One last notion to cover. We say $A \to B$ in dRing is a **(Zariski) open immersion** if it is flat, homotopically finitely presented, and epic (i.e., $B \otimes_A B \xrightarrow{\sim} B$).

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This gives us the small étale ∞-site $(\operatorname{Spec} A)_{\text{ét}}$.

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Theorem

Affine derived schemes satisfy étale descent – i.e., the Yoneda embedding $\mathrm{dAff} \hookrightarrow \mathrm{Pre}(\mathrm{dAff})$ *identifies* dAff *with a full subcategory of* dStk.

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- each U_α is affine;
- $U_\alpha \to U \to X$ is an open immersion;
- $V := \coprod_{\alpha\in\Lambda} U_\alpha \to U$ satisfies $\mathrm{hcolim}_n\, \check{C}(V/U)_n \xrightarrow{\sim} U$.

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We immediately get that dAff is a full subcategory of dSch. Any derived stack X has an underlying **classical stack** X_{cl}, basically characterized by $(\operatorname{Spec} A)_{\mathrm{cl}} \simeq \operatorname{Spec} \pi_0(A)$.

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We say $X \to Y$ in dSch is **smooth (resp., flat, étale)** if there exist affine Zariski coverings $\{\operatorname{Spec} B_i \to X\}$ and $\{\operatorname{Spec} A_{j_i} \to Y\}$ and commutative squares

$$\begin{tikzcd} \operatorname{Spec} B_i \arrow[r] \arrow[d] & X \arrow[d] \\ \operatorname{Spec} A_{j_i} \arrow[r] & Y \end{tikzcd}$$

such that each $\operatorname{Spec} B_i \to \operatorname{Spec} A_{j_i}$ is smooth (resp., flat, étale).

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- **Deligne-Mumford (DM) Stacks**: Good for working with stack quotients of schemes whose automorphism groups are finite groups (analogous to orbifolds)

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- $\mathcal{X}$ is n-**Artin** if there exists $X \in \mathrm{dSch}_{/\mathcal{X}}$ such that $X \to \mathcal{X}$ is $(n-1)$-smooth and epic.

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We take *derived algebraic stack* to be synonymous with derived Artin stack. Note that one can give equivalent descriptions of derived Artin and DM stacks in terms of representability of the diagonal and derived quotient stacks.