$(SL_n(\mathbb{R}), SO_n(\mathbb{R}))$ and $(GL_n(\mathbb{R})^+, SO_n(\mathbb{R})\mathbb{R}^+)$ induce same symm. space.

<u>Example</u>: $G = Sp_{2g}(\mathbb{R})$ = automorphisms of $\mathbb{R}^{2g}$ w/ symplectic form $\psi(x,y) = {}^t x \left[\begin{array}{c|c} & I_g \\ \hline -I_g & \end{array}\right] y$.

Acts on $\mathcal{H}_g^+ = \{ Z = X + iY \in \mathrm{Sym}_g(\mathbb{R}) : Y \text{ pos. def.} \}$ by $\begin{bmatrix} A & B \\ C & D \end{bmatrix} Z = (AZ+B)(CZ+D)^{-1}$. Let

$k = \left\{ \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \in G \right\}$ be stabilizer of $iI_g \in \mathcal{H}_g^+$. Let $\sigma \in \mathrm{Aut}(G)$ be $\sigma(g) = \left[\begin{array}{c|c} & I \\ \hline -I & \end{array}\right] g \left[\begin{array}{c|c} & I \\ \hline -I & \end{array}\right]^{-1}$.

Then, $k = G^\sigma \Rightarrow (G,k)$ Riemannian symm. pair and $G/k \cong \mathcal{H}_g^+$ symm. space.

<u>Example</u>: $p,q > 0$. Equip $\mathbb{R}^{p+q}$ w/ bilinear form $b(x,y) = {}^t x \left(\begin{array}{c|c} I_p & \\ \hline & -I_q \end{array}\right) y$. Let $G = SO(p,q) \subseteq \underset{\text{''}}{O(p,q)}$ where $\mathrm{Aut}(\mathbb{R}^{p+q}, b) = O(p,q)$.

$k = SO(p) \times SO(q) \underbrace{\subseteq}_{(\text{index } 2)} \{$ elts. of $SO(p,q)$ that preserve the decomposition $\mathbb{R}^{p+q} \cong \underbrace{\mathbb{R}^p}_{(\text{pos. def.})} \oplus \underbrace{\mathbb{R}^q}_{(\text{neg. def.})} \}$. (we've basically singled out this decomposition...)

Thing on the right is G^σ for $\sigma \in \mathrm{Aut}(G)$ conjugating by elt. $\begin{pmatrix} 1 \text{ on } \mathbb{R}^p \times \{0\} \subseteq \mathbb{R}^{p+q} \\ -1 \text{ on } \{0\} \times \mathbb{R}^q \subseteq \mathbb{R}^{p+q} \end{pmatrix} \in O(p,q)$.

Can recover $\mathbb{R}^q$ as orthogonal complement of $\mathbb{R}^p$ in $\mathbb{R}^{p+q}$. So, $G/k = \{$ oriented pos. def. subspaces of $\mathbb{R}^{p+q}$ of dim $p \}$.

<u>Problem</u>: This has 2 conn. components! We know this because of the spinor norm $SO(p,q) \to \{\pm 1\}$, which is surj.

Looking at spin double cover (which is only double cover in the sense of alg. gps.!) $\Rightarrow G/k$ union of two symm. spaces.

<u>Example</u>: Let $G = U(p,q)$, $k = U(p) \times U(q)$. Then, $\{G/k\}$ is disjoint union of symm. spaces (unless $p = q$, where we can swap things). This parameterizes orthog. decompositions $\mathbb{C}^{p+q} = \underbrace{H^+}_{\text{pos. def.}} \oplus \underbrace{H^-}_{\text{neg. def.}}$.

<u>Example</u>: $\mathbb{R}^n$, $\mathbb{R}^n/\mathbb{Z}^n$, S^n, $\mathbb{P}^n(\mathbb{C})$. [These are not the kind of examples we want!]

<u>Thm</u>: Every simply conn. symm. space M decomposes as $M = \underbrace{M^0}_{\text{Euclidean type}} \times \underbrace{M^-}_{\text{non-compact type}} \times \underbrace{M^+}_{\text{compact-type}}$.

Symm. space M is <u>Euclidean type</u> if all sectional curvatures vanish at every pt. of M. (e.g., $\mathbb{R}^n$ and $\mathbb{R}^n/\mathbb{Z}^n$).

" " <u>non-compact type</u> if " sectional curvatures are ≤ 0 but not all 0 (all interesting examples above).

" " <u>compact type</u> " " ≥ 0 but not all 0 (e.g., S^n, $\mathbb{P}^n(\mathbb{C})$).

We are interested in symm. spaces of non-compact type arising from reductive gps.

Hermitian Symm. Spaces

Def: Hermitian metric on complex mfld X is smoothly varying (not holomorphically varying) collection $\{h_x\}_{x\in X}$ of pos. def. Hermitian forms $h_x : T_xX \times T_xX \to \mathbb{C}$.

$g = \mathrm{Re}\, h$ gives Riemannian metric on X and $g(iv, iw) = g(v,w)$. Can go the other way.

Def: Hermitian symm. domain (HSD) is Hermitian symm. space of non-compact type.

Example: Siegel half-space!

Example: $U(p,q)/U(p)\times U(q)$. Complex structure comes from viewing this as open subset of $\{p\text{-dim subspaces of } \mathbb{C}^{p+q}\}$.

Non-Example: • $SL_n(\mathbb{R})/SO_n(\mathbb{R})$ (no nat. complex structure unless $n=2$)
• $\mathbb{C}^n, \mathbb{C}^n/\mathbb{Z}^{2n}$ (Euclidean type)
• $\mathbb{P}^n(\mathbb{C})$ (compact type)

Example: $SO(p,q)/SO(p)\times SO(q) \cong \{$oriented p-dim pos. def. subspaces of $\mathbb{R}^{p+q}$ w/ bilinear form $\left[\begin{array}{c|c} I_p & \\ \hline & -I_q\end{array}\right]\}$ is always ~~Hermitian~~ symm. space, but only Hermitian symm. space if p or q is 2 (in which case it is HSD). [Actually get disj. union of two HSD's.] Consider

$SO(2,q)/SO(2)\times SO(q) \cong \{$oriented pos. def. planes in $\mathbb{R}^{2+q}\}$. We get $\mathbb{C}$-structure by extending $\mathbb{C}$-bilinearly to $\mathbb{C}^{2+q}$.

Let $H \subseteq \mathbb{R}^{2+q}$ be oriented pos. def. plane. Let $\{e_1, e_2\}$ be oriented ON basis. $L := \mathbb{C}(e_1 + ie_2) \subseteq \mathbb{C}^{2+q}$.

Exercise: $H \mapsto L \rightsquigarrow$ isom. of smooth mflds $\{$oriented pos. def. planes in $\mathbb{R}^{2+q}\} \xrightarrow{\sim} \{$isotropic lines $L = \mathbb{C}v \subseteq \mathbb{C}^{2+q}$ s.t. $\langle v, \bar{v}\rangle > 0\}$.

RHS has complex structure!

Remark: When $p = 2 = q$ we get product of two upper-half-planes. Either choice defines the same complex structure.

Prop: $X \subseteq \mathbb{C}^n$ conn. bdd. open set. $\mathrm{Hol}(X) :=$ holomorphic auto. grp. of X. Assume $\mathrm{Hol}(X) \curvearrowright X$ transitively and $\exists$ symm. $s_x \in \mathrm{Hol}(X)\ \forall x \in X$. Then, $\exists$ Hermitian metric making it into HSD. [In fact our construction is canonical!]

Remark: This is called Bergman metric and arises from Bergman kernel.

Pf: $H(X) := \{L^2$ holomorphic functions on $X\}$. This is a Hilbert space! $x \in X \rightsquigarrow ev_x : H(X) \to \mathbb{C}$ cont. lin. functional.

This is represented by some $g_x \in H(X)$ - i.e., $ev_x(f) = \int_X f\,\overline{g_x}$.

Bergman kernel $k(x,y) := \overline{g_x(y)}$. Let $\{\varphi_i\}_{i\in I}$ be Hilbert space basis for $H(X)$.

$$k(x,y) = \sum_{i\in I} \varphi_i(x)\overline{\varphi_i(y)}.$$

Properties: (1) holomorphic in $x \in X$. (3) invariant under $\mathrm{Hol}(X)$.

(2) $k(x,y) = \overline{k(y,x)}$. (4) $\forall f \in H(X)$: $f(x) = \int_X k(x,y) f(y)\, dy$. ["reproducing kernel"]

Let $z_1, \dots, z_n$ be usual coords. on $X \subseteq \mathbb{C}^n$. $k_{i,j}(x) := \dfrac{\partial^2}{\partial z_i \partial \bar{z}_j} \log k(x,x)$. We get Hermitian metric

$h_x : T_xX \times T_xX \to \mathbb{C}$, $(v,w) \mapsto \sum_{i,j} k_{i,j}(x)\, dz_i(v)\, d\bar{z}_j(w)$. Making the identification $T_xX = \mathbb{C}^n$,

$\square$

$h_x(v,w) = {}^t\bar{w}\,(k_{ij}(x))\,v$.

Converse is also true. [This is far from obvious.]

Prop: Every HSD is biholomorphic to open subset of $\mathbb{C}^n$.

Remark: This requires knowing that isometries are basically the same thing as holomorphic automorphisms.

Example: $H_g \xrightarrow{\sim} \{X \in \mathrm{Sym}^g(\mathbb{C}) : I - {}^t\bar{X}X \text{ is pos. def.}\} \hookrightarrow \mathbb{C}^{g(g+1)/2}$ via $x \mapsto (x_{ij})_{j\geq i}$.
(open w/ bdd image)

Example: $U(n,1)/U(n)\times U(1) \cong$ open unit ball in $\mathbb{C}^n$. [isom. is not entirely obvious...]

Prop: (X,h) HSD.

(1) X is simply conn.

(2) Every elt. of $\mathrm{Hol}(X)$ preserves h.

(3) $\mathrm{Hol}(X) = \mathrm{Iso}(X,g)$ for $g := \mathrm{Re}\, h$.

So, we can just view HSD's as open bdd. conn. subsets of $\mathbb{C}^n$ s.t. every pt. admits holomorphic symm.