Prop: $\deg(D) \geq 1 \Rightarrow H^1(T, \mathcal{L}(D)) = 0$.

Pf: $\dim H^0(T, \mathcal{L}(D)) - \dim H^1(T, \mathcal{L}(D)) \geq 1 \Rightarrow H^0(T, \mathcal{L}(D)) \neq 0$. Let $f \in H^0(T, \mathcal{L}(D))$ nonzero. This means

$f \in \mathcal{M}(T)$ w/ $D' := \operatorname{div}(f) + D \geq 0$. We have $\mathcal{L}(D) \cong \mathcal{L}(D')$ and $\deg(D) = \deg(D')$. So, can assume WLOG D is effective

Fix p at which D has nonzero multiplicity. $\mathcal{L}(p) \subseteq \mathcal{L}(D)$ and this is isom. away from pts. of D.

$\rightsquigarrow$ SES $\quad 0 \to \mathcal{L}(p) \to \mathcal{L}(D) \to \overbrace{Q_{D-p}}^{\text{skyscraper sheaf} \, = \, i_* \mathcal{O}_{D-p}} \to 0$. Looking at LES, we can assume WLOG $D = [p]$.

Use group law to assume WLOG $p = 1$ in $\mathbb{G}_m^{an}/\langle q \rangle = T$. Now we do explicit calculation for $H^1(T, \mathcal{L}(1))$.

$T = \mathcal{U}_0 \cup \mathcal{U}_1$, $\mathcal{U}_0 = \mathcal{U}(|\pi|, |\pi|^{-1})$, $\mathcal{U}_1 = \mathcal{U}(|\pi|^2, |\pi|)$. Note $1 \in \mathcal{U}_0$, $1 \notin \mathcal{U}_1$. Associated Čech complex is

$$0 \to \overbrace{\tfrac{1}{z-1}}^{\text{accounting for potential pole of order 1 at 1}} \mathcal{O}_T(\mathcal{U}_0) \oplus \mathcal{O}_T(\mathcal{U}_1) \xrightarrow{\partial} \mathcal{O}_T(\mathcal{U}_0 \cap \mathcal{U}_1) \to 0.$$ Just do the power series calculation.

D divisor w/ $\deg(D) \geq 1 \Rightarrow \dim H^0(T, \mathcal{L}(D)) = \deg(D)$. Let $e \in T$ be image of $1 \in \mathbb{G}_m^{an}$.

$L(n[e]) = H^0(T, \mathcal{L}(n[e])) = \{ f \in \mathcal{M}(T) : f \text{ has pole of order} \leq n \text{ at } e \text{ and no other poles} \}$.

$\dim L(n[e]) = n \quad \forall n \geq 1$.

$\operatorname{ord}_e(x) = -2$, $\operatorname{ord}_e(y) = -3$

$L([e]) = k1$

$L(2[e]) = k1 \oplus kx$

$L(3[e]) = k1 \oplus kx \oplus ky$

Prop: Consider $\overbrace{k[x,y]}^{\text{alg. generated}} \subseteq \overbrace{k(x,y)}^{\text{field of fractions}} \subseteq \mathcal{M}(T)$.

(1) $k[x,y] = \bigcup_{n \geq 1} L(n[e])$.

(2) $k(x,y) = \mathcal{M}(T)$.

$\underline{\text{Pf}}$: (1) $A_n := \{ f \in k[x,y] : \mathrm{ord}_e(f) \geq -n \} \subseteq L(n[e])$. These are equal by comparing dimensions.

Now use $k[x,y] = \bigcup_{n \geq 1} A_n$. (Note: $k = k$)

(2) Given $t \in T$ $(e \neq)$, $\exists$ nonzero $g \in k[x,y]$ s.t. $g(t) = 0$. Why? Evaluating x at t gives $x(t) \in k_t$ (k_t / k).

Take the Galois closure k'/k and $g := \prod_{\sigma \in \mathrm{Gal}(k'/k)} (x - \sigma(x(t)))$. The point of doing this is to control supports.

Given $f \in \mathcal{M}(T)$ nonzero, we can thus choose $g \in k[x,y]$ s.t. $\mathrm{div}(fg)$ is supported at e. Hence,

$fg \in \bigcup_{n \geq 1} L(n[e]) = k[x,y] \Rightarrow f \in k(x,y)$. $\square$

$1, x, y, x^2, xy, x^3, y^2 \in L(6[e])$ have a relation. $0 = y^2 + \lambda_1 x^3 + \lambda_2 xy + \lambda_3 x^2 + \lambda_4 y + \lambda_5 x + \lambda_6 =: p(x,y)$. (can assume coeff. is 1 by looking at pole behavior)

Consider the proj. var. $p(x,y) = 0$ giving $E \subseteq \mathbb{P}^2_k$.

$\underline{\text{Prop}}$: E is elliptic curve and $T \cong E^{an}$.

$\underline{\text{Pf}}$: Assume k is alg. closed. $\phi : T \to \mathbb{P}^{2,an}$, $t \mapsto (x(t), y(t))$. This is the map associated to line bundle $\mathcal{L}(3[e])$ on T. To check ϕ is isom. onto its image we need

(1) ϕ is injective. ("separates pts.")

(2) ϕ is injective on tangent vectors. ("separates tangent vectors")

Suppose $t_1, t_2, e \in T$ pairwise distinct. $\underbrace{L(3[e]-t_1-t_2)}_{\dim = 1} \subseteq \underbrace{L(3[e]-t_1)}_{\dim = 2} \subseteq \underbrace{L(3[e])}_{\dim = 3} \subseteq k[x,y]$.

$\Rightarrow \exists f \in k[x,y]$ s.t. $f(t_1) = 0$ but $f(t_2) \neq 0 \Rightarrow (x(t_1), y(t_1)) \neq (x(t_2), y(t_2))$.

GAGA $\Rightarrow \exists$ subvar. $V \subseteq \mathbb{P}^2_k$ s.t. ϕ induces isom. $T \cong V^{an}$.

$\phi: T \to \mathbb{P}^{2,an}$ factors through $E^{an} \subseteq \mathbb{P}^{2,an}$ by definition of $E \Rightarrow V \subseteq E$. But E is irred. of dim $=1$, so $V = E$ and $T \cong E^{an}$. E is smooth because its completed local rings are the completed local rings of T, which are regular.

We see that E is elliptic curve either by Weierstrass theory or by using GAGA to compare cohom. of line bundle $\mathcal{L}(D)$ on E w/ those on T. We find E satisfies Riemann-Roch for a genus 1 curve, so is a genus 1 curve. $\square$

<u>Prop</u>: (1) pullback by $pr: \mathbb{G}_m^{an} \to T$ induces isom. $\mathcal{M}(T) \cong \mathcal{M}(\mathbb{G}_m^{an})^{\langle q \rangle}$.

(2) analytification induces isom. $\{$rational functions on $E\} \xrightarrow{\sim} \mathcal{M}(T)$.

($|q|$ and 1 identified in T, because gluing along mult. by q)

$V_0 := \{x \in \mathbb{G}_m^{an} : |q|^{1/2} \leq |z(x)| \leq 1\}$
$V_1 := \{x \in \mathbb{G}_m^{an} : |q| \leq |z(x)| \leq |q|^{1/2}\}$

<u>Pf</u>: (1) We want to prove surjectivity of $\mathcal{M}(T) \to \mathcal{M}(\mathbb{G}_m^{an})^{\langle q \rangle}$.

$\mathcal{U}_i := pr(V_i) \cong V_i \Rightarrow T = \mathcal{U}_0 \cup \mathcal{U}_1$ is adm. affinoid cover. $f \in \mathcal{M}(\mathbb{G}_m^{an})^{\langle q \rangle}$ can be restricted to V_0 and V_1, determining meromorphic functions $f_0 := f|_{V_0}$ on $\mathcal{U}_0 \cong V_0$, $f_1 := f|_{V_1}$ on $\mathcal{U}_1 \cong V_1$. We can glue because of $\langle q \rangle$-invariance.

(2) This can be proven using standard facts about Weierstrass equations / functions, but we want to do something more general. Need to show meromorphic $f \in \mathcal{M}(T)$ is analytification of some rational function on E. Let $D := -\mathrm{div}(f)$.

$\mathrm{div}(f) + D \geq 0 \Rightarrow f \in H^0(T, \mathcal{L}(D))$. View D as divisor on E and get line bundle $\mathcal{L}(D)$ on E. Using GAGA, $H^0(E, \mathcal{L}(D)) \cong H^0(T, \mathcal{L}(D))$. $\square$

Assume k alg. closed. Let E/k be elliptic curve. We know that $E \to \mathrm{Pic}^0(E)$, $p \mapsto [p] - [0]$ is bijection. This is good for establishing grp. law on E. Let $D = \sum_{x \in E(k)} n_x [x]$ be degree 0 divisor. D is principal iff $\sum_{x \in E(k)} n_x x = 0$, using the grp. law. Here's another way to see this.

We have exact sequence: $1 \to k^\times \to \{\text{rational functions}\}^\times \xrightarrow{\text{div}} \mathrm{Div}(E) \to \mathbb{Z} \times E \to 0.$

$$D = \sum n_x[x] \mapsto (\deg(D), \sum n_x x)$$

By what we know, same holds w/ E replaced by T. Let's prove this directly for T.

Lemma: Recall coord. $z \in H^0(\mathbb{G}_m^{an}, \mathcal{O}^\times_{\mathbb{G}_m^{an}})$ ← (meromorphic function w/ no zeros or poles, so gives trivial divisor). The following is exact.

$$1 \to k^\times \times \langle z \rangle \to \mathcal{M}(\mathbb{G}_m^{an})^\times \xrightarrow{\text{div}} \mathrm{Div}(\mathbb{G}_m^{an}) \to 0$$

$\mathrm{Div}(\mathbb{G}_m^{an})$ is grp. of formal linear combinations $D = \sum_{x \in \mathbb{G}_m^{an}} n_x [x]$ s.t. $\forall$ affinoid open $\mathcal{U} \subseteq \mathbb{G}_m^{an}$ we have

$\{x \in \mathcal{U} : n_x \neq 0\}$ is finite.

Pf: One can verify the following:

(1) Any $f \in H^0(\mathbb{G}_m^{an}, \mathcal{O}_{\mathbb{G}_m^{an}})$ is given by global power series $f = \sum_{n \in \mathbb{Z}} a_n z^n$ s.t. $r^n |a_n| \to 0 \ \forall r \in (0, \infty)$.

(2) Any nonzero $f \in \mathcal{M}(\mathbb{G}_m^{an})$ has form $f = p/q$ w/ $p, q \in H^0(\mathbb{G}_m^{an}, \mathcal{O}_{\mathbb{G}_m^{an}})$ having no common zeros.

(3) Given $f \in \mathcal{M}(\mathbb{G}_m^{an})$, $\mathrm{div}(f) = 0$ iff $f = \lambda z^m$ for some $\lambda \in k^\times, m \in \mathbb{Z}$.

(4) Every divisor is divisor of meromorphic function. For this suppose $D = \sum_x n_x [x]$ and check

$$f(z) = \prod_{|x|>1} \left(1 - \frac{z}{x}\right)^{n_x} \prod_{|x| \le 1} \left(1 - \frac{x}{z}\right)^{n_x}$$ converges to $f \in \mathcal{M}(\mathbb{G}_m^{an})$ w/ $\mathrm{div}(f) = D$. ∎

Now, exactness for T!

Prop: There is exact seq. $1 \to k^\times \to \mathcal{M}(T)^\times \xrightarrow{\alpha} \mathrm{Div}(T) \xrightarrow{\beta} \mathbb{Z} \times T \to 0$.

$$D = \sum n_t [t] \mapsto \left(\deg(D), \prod t^{n_t}\right)$$

Pf: Surjectivity of β is clear. $\Gamma = \langle q \rangle \subseteq \mathrm{Aut}(\mathbb{G}_m^{an})$ acts on all terms in

$1 \to k^\times \times \langle z \rangle \to \mathcal{M}(\mathbb{G}_m^{an})^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(\mathbb{G}_m^{an}) \to 0$. Take Γ-cohom. to get

$1 \to k^\times \to \mathcal{M}(T)^\times \xrightarrow{\mathrm{div}} \mathrm{Div}(T) \to H^1(\Gamma, k^\times \times \langle z \rangle)$. Let A be ab. grp. (written multiplicatively) w/ Γ-action.

$H^*(\Gamma, A)$ is cohom. of $1 \to A \xrightarrow{\left(a \mapsto \frac{qa}{a}\right)} A \to 1$. $H^0(\Gamma, A) \cong A^\Gamma$, $H^1(\Gamma, A) \cong A_\Gamma$.

$\Rightarrow H^1(\Gamma, k^\times \times \langle z \rangle) =$ cokernel of $k^\times \times \langle z \rangle \to k^\times \times \langle z \rangle$, $\lambda z^n \mapsto \dfrac{\lambda q^n z^n}{\lambda z^n} = q^n$ – i.e., $(\lambda, z^n) \mapsto (q^n, 1)$.

$\Rightarrow H^1(\Gamma, k^\times \times \langle z \rangle) \cong k^\times / \langle q \rangle \times \langle z \rangle \cong T \times \mathbb{Z}$. ∎

We will make the behavior of α totally explicit!

NB: I missed the subsequent lecture – get ideas from Tobi and Xinyu.