K-theory and G-theory of projective bundles and derived blow-ups (plus miscellany)

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Overview

The Noetherian assumption

Convention: everything derived, $(\mathcal{S}\mathrm{p}, \otimes)$ is the symmetric monoidal category of spectra.

Definition

A ring $A \in \mathrm{s}\mathcal{R}\mathrm{ing}$ is *Noetherian* is $\pi_0 A$ is Noetherian and each $\pi_n A$ is finitely generated ($=$ finitely presented) over $\pi_0 A$.

Definition

An algebraic stack X is *Noetherian* if it is qcqs and if for any smooth map $\operatorname{Spec} A \to X$, the ring A is Noetherian.

Throughout, we assume all algebraic stacks to be Noetherian, hence all rings to be Noetherian.

Perfect modules

Definition

Let $A \in \mathrm{s}\mathcal{R}\mathrm{ing}$.

- The *category of finitely presented modules* $\mathcal{M}\mathrm{od}_A^{fp}$ is the smallest stable subcategory of $\mathcal{M}\mathrm{od}_A$ which contains A.
- The *category of perfect modules* is the closure of $\mathcal{M}\mathrm{od}_A^{fp}$ under extensions in $\mathcal{M}\mathrm{od}_A$.

Lemma

$M \in \mathcal{M}\mathrm{od}_A$ is finitely presented if and only if it is obtained from 0 by a finite number of cell attachments.

Perfect modules

Lemma

$M \in \mathcal{M}\mathrm{od}_A$ is perfect iff it is compact iff it is dualizable .

Coherent modules

Definition

$M \in \mathcal{M}\mathrm{od}_A$ is *coherent* if $\pi_n M$ if finitely presented over $\pi_0 A$ for all n, and M has bounded homotopy. Notation: $\mathcal{C}\mathrm{oh}(A)$.[a]

[a] Lurie does not demand the boundedness assumption (e.g. in SAG). We want this due to the Eilenberg-Mazur swindle.

Coherent modules

Definition

$M \in \mathcal{M}\mathrm{od}_A$ is *coherent* if $\pi_n M$ if finitely presented over $\pi_0 A$ for all n, and M has bounded homotopy. Notation: $\mathcal{C}\mathrm{oh}(A)$.

Lemma

If $R \in \mathcal{C}\mathrm{oh}(A)$, then $\mathcal{P}\mathrm{erf}(A) \subset \mathcal{C}\mathrm{oh}(A)$.

Global versions & K-theory (once more)

Write $\mathcal{A}\mathrm{rt}$ for the category of algebraic stack. Fix $X \in \mathcal{A}\mathrm{rt}$.

- $\mathcal{M} \in \mathrm{Q}\mathcal{C}\mathrm{oh}(X)$ is *coherent* or *perfect* if it is so smooth-locally.
- Notation: $\mathcal{C}\mathrm{oh}(X)$ and $\mathcal{P}\mathrm{erf}(X)$.
- If X has bounded structure sheaf, then $\mathcal{P}\mathrm{erf}(X) \subset \mathcal{C}\mathrm{oh}(X)$.
- The *K-theory space* of X is $K(X) := K(\mathcal{P}\mathrm{erf}(X))$ (resp. the *spectrum* is $K^B(X) := K^B(\mathcal{P}\mathrm{erf}(X))$).
- The *G-theory space* of X is $G(X) := K(\mathcal{C}\mathrm{oh}(X))$ (resp. the *spectrum* is $G^B(X) := K^B(\mathcal{C}\mathrm{oh}(X))$.

Recall, $K^B(\mathcal{C})$ is roughly (equivalent to the spectrum defined) as follows:

- Define $\mathcal{C} \subset F\mathcal{C}$ such that $K(F\mathcal{C}) = 0$, and put $\Sigma\mathcal{C} := F\mathcal{C}/\mathcal{C}$.
- Then $\mathcal{C} \to F\mathcal{C} \to \Sigma\mathcal{C}$ is (strict?) exact, so $K_{n+1}(\Sigma\mathcal{C}) = K_n(\mathcal{C})$.
- Put $K^B(\mathcal{C}) := \mathrm{colim}_n\, \Omega^n K(\Sigma^n\mathcal{C})$.
- Note $\pi_n\Omega^m K(\Sigma^m\mathcal{C}) = \pi_{n+m}K(\Sigma^m\mathcal{C}) = \pi_n K(\mathcal{C})$.

Cup product

Lemma

A biexact functor $\mathcal{C} \times \mathcal{D} \to \mathcal{E}$ *induces* $K^B(\mathcal{C}) \otimes K^B(\mathcal{D}) \to K^B(\mathcal{E})$, *which induces maps* $K_n(\mathcal{C}) \times K_m(\mathcal{D}) \to K_{n+m}(\mathcal{E})$.

Now

$$\mathcal{P}\mathrm{erf}(X) \times \mathcal{P}\mathrm{erf}(X) \xrightarrow{(-)\otimes(-)} \mathcal{P}\mathrm{erf}(X)$$

is biexact, which gives us a map

$$\cup : K^B(X) \otimes K^B(X) \to K^B(X)$$

called the *cup product*. This makes $K^B(X)$ into an $\mathbb{E}_\infty$-ring spectrum. (Reason: use naturality in multilinear functors and symmetric monoidal structure on $\mathcal{P}\mathrm{erf}(X)$?)

Pullback & Gysin map

For $f : X \to Y$ in $\mathcal{A}\mathrm{rt}$, the exact, symmetric monoidal functor $f^* : \mathcal{P}\mathrm{erf}(Y) \to \mathcal{P}\mathrm{erf}(X)$ induces a map of $\mathbb{E}_\infty$-ring spectra

$$f^* : K^B(Y) \to K^B(X)$$

Definition

If $f_* : \mathrm{Q}\mathcal{C}\mathrm{oh}(X) \to \mathrm{Q}\mathcal{C}\mathrm{oh}(Y)$ preserves perfect complexes, then we have the *Gysin map*

$$f_* : K^B(X) \to K^B(Y)$$

Remark

In [K21], certain technical conditions are given to ensure the Gysin map exists and interacts nicely with the cup product. I will highlight one.

Finite cohomological dimension

Definition

Let $f : X \to Y$ in $\mathcal{A}\mathrm{rt}$.

- f is of *finite cohomological dimension* (fcd) if there is $n \geq 0$ such that $f_*(\mathrm{Q}\mathcal{C}\mathrm{oh}(X)_{\geq 0}) \subset \mathrm{Q}\mathcal{C}\mathrm{oh}(Y)_{\geq -n}$.
- f is *universally of fcd* if for all qcqs Y' over Y, the base change $X' \to Y'$ is of fcd

Now consider a cartesian square

$$\begin{tikzcd} X' \arrow[r, "g_2"] \arrow[d, "f'"] & X \arrow[d, "f"] \\ Y' \arrow[r, "g_1"] & Y \end{tikzcd}$$

This gives a natural map

$$\varphi : g_1^* f_* \to f'_* g_2^*$$

If f is universally of fcd, it satisfies base-change, i.e., φ is an equivalence.

Finite cohomological dimension

Proposition

If $f : X \to Y$ is universally of fcd, then $f_* : \mathrm{Q}\mathcal{C}\mathrm{oh}(X) \to \mathrm{Q}\mathcal{C}\mathrm{oh}(Y)$ preserves perfect complexes iff it does so smooth-locally.

Projection formula

Lemma

If $f : X \to Y$ is universally of fcd, then it satisfies the projection formula, stating that

$$f_*(M) \otimes N \to f_*(M \otimes f^* N)$$

is an equivalence, for all $M \in \mathrm{Q}\mathcal{C}\mathrm{oh}(X), N \in \mathrm{Q}\mathcal{C}\mathrm{oh}(Y)$.

Proposition

If $f : X \to Y$ is universally of fcd such that f_* preserves perfect complexes, then

$$f_*(m) \cup y \simeq f_*(m \cup f^*(y))$$

for all $m \in K^B(X), y \in K^B(Y)$.

Projection formula

Proposition

If $f : X \to Y$ is universally of fcd such that f_* preserves perfect complexes, then

$$f_*(m) \cup y \simeq f_*(m \cup f^*(y))$$

for all $m \in K^B(X), y \in K^B(Y)$.

Absolute perfection

Definition

Let $X \in \mathcal{A}\mathrm{rt}$.

- X is *perfect* if the canonical map $\mathsf{Ind}(\mathcal{P}\mathrm{erf}(X)) \to \mathrm{Q}\mathcal{C}\mathrm{oh}(X)$ is an equivalence.
- For $Z \subset |X|$ closed, write $\mathrm{Q}\mathcal{C}\mathrm{oh}(X \text{ on } Z)$ for the full subcategory of $F \in \mathrm{Q}\mathcal{C}\mathrm{oh}(X)$ supported on Z. Similarly for $\mathcal{P}\mathrm{erf}(X \text{ on } Z)$.
- Now X is *absolutely perfect* if

$$\mathsf{Ind}(\mathcal{P}\mathrm{erf}(X \text{ on } Z)) \xrightarrow{\simeq} \mathrm{Q}\mathcal{C}\mathrm{oh}(X \text{ on } Z)$$

for al cocompact closed $Z \subset |X|$.

Note: if X is perfect then $\mathcal{P}\mathrm{erf}(X) = \mathrm{Q}\mathcal{C}\mathrm{oh}(X)^{\omega}$.

Localization

Put $K^B(X \text{ on } Z) := K^B(\mathcal{P}\mathrm{erf}(X \text{ on } Z))$.

Proposition

If X is absolutely perfect, then for every cocompact $Z \subset |X|$, we have an exact triangle

$$K^B(X \ on\ Z) \to K^B(X) \xrightarrow{j^*} K^B(X \setminus Z)$$

The G-spectrum is the G-space

Proposition

The canonical map $G(X) \to G^B(X)$ is an equivalence.

Roughly:

- The *theorem of the heart* says that if $\mathcal{C}$ has bounded t-structure, then $K(\mathcal{C}) \simeq K(\mathcal{C}^\heartsuit)$.
- An abelian category is *noetherian* if all objects are noetherian.
- If $\mathcal{C}$ has bounded t-structure and the heart is noetherian, then $K(\mathcal{C}) \simeq K^B(\mathcal{C})$.
- Since $\mathcal{C}\mathrm{oh}(X)$ has bounded t-structure and $\mathcal{C}\mathrm{oh}(X)^\heartsuit$ is noetherian, the claim follows.

Cap product

Observe that

$$\mathcal{P}\mathrm{erf}(X) \times \mathcal{C}\mathrm{oh}(X) \xrightarrow{(-)\otimes(-)} \mathrm{Q}\mathcal{C}\mathrm{oh}(X)$$

lands in $\mathcal{C}\mathrm{oh}(X)$. Indeed, for $\operatorname{Spec} A \to X$,

$$\mathcal{M}\mathrm{od}_A^{fp} \times \mathcal{C}\mathrm{oh}(A) \xrightarrow{(-)\otimes(-)} \mathcal{M}\mathrm{od}_A$$

lands in $\mathcal{C}\mathrm{oh}(A)$ since $A \otimes M = M$. Now use that $\mathcal{C}\mathrm{oh}(A)$ is stable under retracts.

Definition

The functor $\mathcal{P}\mathrm{erf}(X) \times \mathcal{C}\mathrm{oh}(X) \xrightarrow{(-)\otimes(-)} \mathcal{C}\mathrm{oh}(X)$ induces the *cap product*

$$\cap : K^B(X) \otimes G(X) \to G(X)$$

making $G(X)$ a $K^B(X)$-module.

Gysin map

Suppose that $f : X \to Y$ is of finite Tor-amplitude n. Then f^* restricts to a functor $\mathrm{Q}\mathcal{C}\mathrm{oh}(Y)_{\leq 0} \to \mathrm{Q}\mathcal{C}\mathrm{oh}(X)_{\leq n}$, and therefore gives a functor

$$f^* : \mathcal{C}\mathrm{oh}(Y) \to \mathcal{C}\mathrm{oh}(X)$$

Definition

For f of finite Tor-amplitude, pulling back induces the *Gysin map*

$$f^* : G(Y) \to G(X)$$

Projection formula

Suppose $f_* : \mathrm{QCoh}(X) \to \mathrm{QCoh}(Y)$ preserves coherent complexes. Then we have a *direct image map*

$$f_* : G(X) \to G(Y)$$

If moreover f is universally of fcd, then

$$y \cap f_*(x) \simeq f_*(f^*(y) \cap x)$$

for all $x \in G(X), y \in K^B(Y)$. Moreover, base-change holds against maps of finite Tor amplitude.

Localization

Since $\mathcal{C}\mathrm{oh}(X)$ has bounded t-structure, the theorem of the heart says that

$$G(X) \simeq K(\mathcal{C}\mathrm{oh}(X)^{\heartsuit}) \simeq K(\mathcal{C}\mathrm{oh}(X_{\mathrm{cl}})^{\heartsuit}) \simeq G(X_{\mathrm{cl}})$$

Lemma

Let $i : Z \to X$ be a closed immersion with open complement $j : U \to X$. Then we have an exact triangle

$$G(Z) \xrightarrow{i_*} G(X) \xrightarrow{j^*} G(U)$$

As before, we have an exact sequence

$$\mathcal{C}\mathrm{oh}(X \text{ on } Z) \to \mathcal{C}\mathrm{oh}(X) \xrightarrow{j^*} \mathcal{C}\mathrm{oh}(U)$$

Dévissage for closed immersions

Lemma

Let $\mathcal{A} \subset \mathcal{B}$ be an inclusion of abelian categories, such that $\mathcal{A}$ is closed under subobjects and quotients, and each $B \in \mathcal{B}$ has a filtration

$$0 = B_r \subset B_{r-1} \subset \cdots \subset B_1 \subset B_0 = B$$

such that all B_i/B_{i-1} lie in $\mathcal{A}$. Then $K(\mathcal{B}) \simeq K(\mathcal{A})$.

By proper pushforward, we have $\mathcal{C}\mathrm{oh}(Z) \to \mathcal{C}\mathrm{oh}(X \text{ on } Z)$. We will show this induces an equivalence on K-theory.

Nil invariance

Corollary

Let $Z \to X$ be a surjective closed immersion. Then

$$i_* : G(Z) \to G(X)$$

is an equivalence.

Étale excision

Let $j : U \to X$ be an open immersion with closed complement $Z \subset |X|$. Let $X' \to X$ be étale ($\Rightarrow$ finite Tor-amplitude) such that $f^{-1}(Z)_{\mathrm{red}} \cong Z_{\mathrm{red}}$. Then the following induced square is cartesian

$$\begin{tikzcd} G(X) \arrow[r] \arrow[d, "f^*"] & G(U) \arrow[d, "f^*"] \\ G(X') \arrow[r] & G(f^{-1}U) \end{tikzcd}$$

Quasi-smoothness and virtual codimension

Let $f : X \to Y$ in $\mathcal{A}\mathrm{rt}$.

- f is *quasi-smooth* if it is locally of finite presentation and $L_{X/Y}$ has Tor-amplitude $[-\infty, 1]$.
- If f is a closed immersion of schemes, then it is quasi-smooth iff Zariski-locally on the target it is of the form $V(f_1, \ldots, f_n) \to Y$ for sections f_i on Y.
- If f is a closed immersion of algebraic stack, then it is quasi smooth iff it has a smooth atlas of schemes which is a quasi-smooth closed immersion.
- The *virtual codimension* of a quasi-smooth closed immersion is the number of sections being cut out.
- Equivalently, $N_{X/Y} := L_{X/Y}[-1]$ is smooth-locally of finite presentation with rank the virtual codimension.

Derived blow-ups

Let $Z \to X$ be a closed immersion in $\mathcal{A}\mathrm{rt}$. A *virtual Cartier divisor* is a quasi-smooth closed immersion $D \to T$ of virtual codimenion 1.

Definition

The *blow-up* of X in Z is the space

$$\mathrm{Bl}_Z X(T) := \left\{ \begin{tikzcd} D \arrow[r, "i_D"] \arrow[d, "g"] & T \arrow[d] \\ Z \arrow[r] & X \end{tikzcd} \;\middle|\; \begin{array}{l} \bullet\ i_D \text{ is a virtual Cartier divisor} \\ \bullet\ D_{\mathrm{cl}} \cong (T \times_X Z)_{\mathrm{cl}} \\ \bullet\ g^* N_{Z/X} \to N_{D/T} \text{ surjective} \end{array} \right\}$$

Proposition

The stack $\mathrm{Bl}_Z X$ is algebraic. If Z, X are schemes, then so is $\mathrm{Bl}_Z X$.

Projective bundles

Definition

Let $X \in \mathcal{A}\mathrm{rt}$ and $\mathcal{E} \in \mathrm{Q}\mathcal{C}\mathrm{oh}(X)$ locally free of finite rank. Then the *projective bundle* of $\mathcal{E}$ is the stack $\pi : \mathbb{P}(\mathcal{E}) \to X$ such that

$$\mathbb{P}(\mathcal{E})(f : T \to X) := \{(\mathcal{L}, u) \mid \mathcal{L} \in \mathcal{P}\mathrm{ic}(T), u : f^*(\mathcal{E}) \twoheadrightarrow \mathcal{L}\}$$

Since line bundles on X are defined smooth-locally, the data $(\mathcal{L}, u)$ glue into an invertible sheaf $\mathcal{O}(1)$ on $\mathbb{P}(\mathcal{E})$ and a surjection $\pi^*(\mathcal{E}) \to \mathcal{O}(1)$.

Universal virtual Cartier divisor

The identity map $\mathrm{Bl}_Z X \to \mathrm{Bl}_Z X$ corresponds to the square

$$\begin{tikzcd} \mathbb{P}_Z(N_{Z/X}) \arrow[r, "i_D"] \arrow[d, "g"] & \mathrm{Bl}_Z X \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

which is the universal square such that

- i_D is a virtual Cartier divisor
- It is cartesian on $(-)_{\mathrm{cl}}$
- $g^* N_{Z/X} \to N_{\mathbb{P}_Z(N_{Z/X})/\,\mathrm{Bl}_Z X}$ surjective

Semi-orthogonal decompositions

Definition

Let $\mathcal{C}$ be a stable category with full stable subcategory $\mathcal{D}$.

- The category of *left orthogonals to* $\mathcal{D}$ is the full subcategory

$$^{\perp}\mathcal{D} := \{x \in \mathcal{C} \mid \forall d \in \mathcal{D} : \mathcal{C}(x, d) \simeq *\}$$

Definition

Let $\mathcal{C}$ be stable. A *semi-orthogonal decomposition* of $\mathcal{C}$ is a sequence $\mathcal{C}(0), \dots, \mathcal{C}(-n)$ of full stable subcategories such that

- For all integers $i > j$ it holds $\mathcal{C}(i) \subset {}^{\perp}\mathcal{C}(j)$;
- $\mathcal{C}$ is generated by $\mathcal{C}(0), \dots, \mathcal{C}(-n)$ under finite limits and finite colimits.

Lemma

Let $\mathcal{C}$ be stable, with semi-orthogonal decompostion $(\mathcal{C}(0), \dots, \mathcal{C}(-n))$. For $0 \leq m \leq n$, define $\mathcal{C}_{\leq -m} := \mathrm{span}(\mathcal{C}(-m) \cup \cdots \cup \mathcal{C}(-n))$ and put $\mathcal{C}_{\leq -n-1} := \{0\}$. Then there are split short exact sequences

$$\mathcal{C}(-m) \to \mathcal{C}_{\leq -m} \to \mathcal{C}_{\leq -m-1}$$

for each $0 \leq m \leq n$.

Lemma ('Generalized additivity theorem')

Let $\mathcal{C}$ be stable, with semi-orthogonal decompostion $(\mathcal{C}(0), \dots, \mathcal{C}(-n))$. For E an additive invariant (= exact on split exact sequences), it holds

$$E(\mathcal{C}) \simeq \bigoplus_{0 \leq m \leq n} E(\mathcal{C}(-m))$$

Semi-orthogonal decomposition on $\mathrm{Q}\mathcal{C}\mathrm{oh}(\mathbb{P}(\mathcal{E}))$

Let $\mathcal{E}$ be locally free of rank $n+1$, and consider $\pi : \mathbb{P}(\mathcal{E}) \to X$.

Lemma

For each $0 \leq k \leq n$ we have a fully faithful functor

$$\mathrm{Q}\mathcal{C}\mathrm{oh}(X) \to \mathrm{Q}\mathcal{C}\mathrm{oh}(\mathbb{P}(\mathcal{E})) : \mathcal{F} \mapsto \pi^* \mathcal{F} \otimes \mathcal{O}(-k)$$

Definition

For any $-k$, let $\mathcal{C}(-k)$ be the essential image of the functor $\mathcal{F} \mapsto \pi^* \mathcal{F} \otimes \mathcal{O}(-k)$.

Semi-orthogonal decomposition on $\mathrm{Q}\mathcal{C}\mathrm{oh}(\mathbb{P}(\mathcal{E})), \mathcal{P}\mathrm{erf}(\mathbb{P}(\mathcal{E})), \mathcal{C}\mathrm{oh}(\mathbb{P}(\mathcal{E}))$

Proposition

The categories $\mathcal{C}(0), \ldots, \mathcal{C}(-n)$ form a semi-orthogonal decomposition of $\mathrm{Q}\mathcal{C}\mathrm{oh}(\mathbb{P}(\mathcal{E}))$. These restrict to $\mathcal{P}\mathrm{erf}(\mathbb{P}(\mathcal{E})), \mathcal{C}\mathrm{oh}(\mathbb{P}(\mathcal{E}))$.

Projective bundle formulae

Theorem

Let $\mathcal{E}$ be a locally free complex of rank $n+1$ on X. Then

$$K^B(\mathbb{P}(\mathcal{E})) \simeq \bigoplus_{0\le k\le n} K^B(X)$$

$$G(\mathbb{P}(\mathcal{E})) \simeq \bigoplus_{0\le k\le n} K^B(X)$$

Blow-up formulas

Let $Z \to X$ be a quasi-smooth closed immersion of virtual codimension n, write $\pi : \mathrm{Bl}_Z X \to X$ and $p : \mathbb{P}_Z(N_{Z/X}) \to Z$.

- $\pi^* : \mathrm{Q}\mathcal{C}\mathrm{oh}(X) \to \mathrm{Q}\mathcal{C}\mathrm{oh}(\mathrm{Bl}_Z X)$ is fully faithful. Write image as $\mathcal{D}(0)$.
- For all $1 \leq k \leq n-1$, the composition

$$\mathrm{Q}\mathcal{C}\mathrm{oh}(Z) \xrightarrow{p^*(-)\otimes\mathcal{O}(-k)} \mathrm{Q}\mathcal{C}\mathrm{oh}(\mathbb{P}(N_{Z/X})) \xrightarrow{i_*} \mathrm{Q}\mathcal{C}\mathrm{oh}(\mathrm{Bl}_Z X)$$

  is fully faithful. Write image as $\mathcal{D}(-k)$
- Now $\mathcal{D}(0), \ldots, \mathcal{D}(-n+1)$ forms a semi-orthogonal decomposition on $\mathrm{Q}\mathcal{C}\mathrm{oh}(\mathrm{Bl}_Z X)$.
- This restricts to perfect and coherent complexes.
- We thus have

$$K^B(\mathrm{Bl}_Z X) \simeq K^B(X) \oplus \bigoplus_{1\leq k\leq n-1} K^B(Z)$$

$$G(\mathrm{Bl}_Z X) \simeq G(X) \oplus \bigoplus_{1\leq k\leq n-1} G(Z)$$

Vector bundles

Let $\mathcal{E}$ be a locally free sheaf of finite rank on $X \in \mathcal{A}\mathrm{rt}$.

- The canonical map $h : \mathcal{E} \to \mathcal{E} \oplus \mathcal{O}_X$ induces a surjection $h^\vee : (\mathcal{E} \oplus \mathcal{O}_X)^\vee \to \mathcal{E}^\vee$.
- We thus have a closed immersion $j : \mathbb{P}(\mathcal{E}^\vee) \to \mathbb{P}((\mathcal{E} \oplus \mathcal{O}_X)^\vee)$.
- Let $\mathbb{V}(\mathcal{E}^\vee)$ be the vector bundle of sections of $\mathcal{E}$, i.e.

$$\mathbb{V}(\mathcal{E}^\vee)(f : T \to X) := \{v : f^*\mathcal{E}^\vee \to \mathcal{O}_T\}$$

- We have an obvious map $i : \mathbb{V}(\mathcal{E}^\vee) \to \mathbb{P}((\mathcal{E} \oplus \mathcal{O}_X)^\vee)$.
- The map i is the open complement of j.

Homotopy invariance

Proposition

For $\mathcal{E}$ locally free of finite rank on $X \in \mathcal{A}\mathrm{rt}$, the map

$$\pi^* : G(X) \to G(\mathbb{V}(\mathcal{E}))$$

induced by $\pi : \mathbb{V}(\mathcal{E}) \to X$, is invertible.

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Thank you!