Can view $\mathbb{G}_m^{an}/\langle q\rangle$ as quotient of $\mathbb{G}_m^{an} \subseteq \mathbb{P}^1$ by action of $\Gamma := \langle \overset{\gamma}{\overset{\shortparallel}{\begin{pmatrix} q & \\ & 1\end{pmatrix}}} \rangle \leq PGL_2(\underset{\wedge}{\overset{k}{\ }})$. The action of Γ has two fixed pts.

These are "limit pts." $\lim_{n\to\infty} \gamma^n \cdot 1 = \lim_{n\to\infty} q^n = 0$, $\lim_{n\to\infty} \gamma^{-n}\cdot 1 = \lim_{n\to\infty} q^{-n} = \infty$. (limits in metric topology)

Since k may not be alg. closed, we have distinction between $\mathbb{P}^1$ = analytification of alg. curve $\mathbb{P}^1$ over k and

$\mathbb{P}^1(k) = \{x \in \mathbb{P}^1 : k_x = k\} = \{\text{lines in } k^2\}$. $\mathbb{P}^1(k) \subseteq \mathbb{P}^1$. k locally compact $\Rightarrow \mathbb{P}^1(k)$ compact.

Fix $\Gamma \leq PGL_2(k)$.

<u>Def</u>: Given $w \in \mathbb{P}^1(k)$, $z \in \mathbb{P}^1(k)$ is <u>limit pt.</u> of (Γ, w) if $\exists$ seq. $\gamma_1, \gamma_2, \ldots \in \Gamma$ pairwise distinct s.t.

$z = \lim_{n\to\infty} \gamma_n w$. The set of all such z is $L_\Gamma(w)$. $L_\Gamma := \bigcup_{w \in \mathbb{P}^1(k)} L_\Gamma(w)$. $\Omega_\Gamma := \mathbb{P}^1 \setminus L_\Gamma$ "<u>ordinary pts.</u>".

(Ω_Γ contains at least one k-pt.) (automatic for k locally compact)

<u>Def</u>: Γ is <u>discontinuous</u> if $L_\Gamma \subsetneq \mathbb{P}^1(k)$ and $w \in \mathbb{P}^1(k) \Rightarrow$ closure $\overline{\Gamma\cdot w}$ is compact.

<u>Def</u>: $\Gamma \leq PGL_2(k)$ is <u>Schottky grp.</u> if it is fin. gen., torsion-free, discontinuous.

We want to make $\Gamma\backslash\Omega_\Gamma$ a rigid space (a Mumford curve). From now on we will assume Γ is Schottky.

<u>Def</u>: Suppose $\gamma \in GL_2(k)$ has eigenvals. $\lambda_1, \lambda_2 \in k^{alg}$.

- $|\lambda_1| \neq |\lambda_2| \Rightarrow \gamma$ <u>hyperbolic</u>
- $|\lambda_1| = |\lambda_2| \Rightarrow \gamma$ <u>elliptic</u>
- $\lambda_1 = \lambda_2 \Rightarrow \gamma$ <u>parabolic</u>

These descend to $PGL_2(k)$.

Given hyperbolic $\gamma \in GL_2(k)$, order eigenvals so that $|\lambda_1| < |\lambda_2|$. It follows min. poly. of γ is split. Hence, $\lambda_1, \lambda_2 \in k$ and

γ is $GL_2(k)$-conjugate to $\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix}$. For $PGL_2(k)$, $\begin{pmatrix} \lambda_1 & \\ & \lambda_2 \end{pmatrix} = \begin{pmatrix} q_\gamma & \\ & 1 \end{pmatrix}$ w/ $q_\gamma := \frac{\lambda_1}{\lambda_2}$ so $0 < |q_\gamma| < 1$.

Hence, γ has two limit pts. (the fixed pts.) $a^+, a^- \in \mathbb{P}^1(k)$ ordered so that $\lim_{n\to\infty} \gamma^n \cdot w = a^+$, $\lim_{n\to\infty} \gamma^{-n}\cdot w = a^-$ $\forall w \in \mathbb{P}^1(k)$.

$L_{\langle\gamma\rangle}(w) = \{a^+, a^-\}$ and $\langle\gamma\rangle$ is Schottky w/ $\Omega_{\langle\gamma\rangle} = \mathbb{P}^1 \setminus \{a^+, a^-\}$.

Prop: Γ Schottky grp. $\Rightarrow \gamma \in \Gamma$ w/ $\gamma \neq 1$ is hyperbolic. L_Γ is closed in $\mathbb{P}^1(k)$ and $L_\Gamma = L_\Gamma(w)$ for any $w \in \Omega_\Gamma \cap \mathbb{P}^1(k)$.

Genus 0: trivial grp.

Genus 1: just done - all examples are like what we saw

Thm (Ihara): Γ is free on $g \geq 0$ generators. (genus of Γ)

Fix rational disks ($a_i \in k$, $r_i \in |k^\times|$)

$B_1^\circ = \{z \in \mathbb{P}^1 : |z-a_1| < r_1\}$ $\quad\dots\quad$ $B_{2g}^\circ = \{z \in \mathbb{P}^1 : |z-a_{2g}| < r_{2g}\}$

$B_1 = \{z \in \mathbb{P}^1 : |z-a_1| \leq r_1\}$ $\quad\quad\quad$ $B_{2g} = \{z \in \mathbb{P}^1 : |z-a_{2g}| \leq r_{2g}\}$

s.t. $\infty \notin B_i$ and $B_i \cap B_j = \emptyset$ $\forall i \neq j$. For each $1 \leq i \leq g$ chose $\gamma_i \in PGL_2(k)$ s.t. $\gamma_i(\mathbb{P}^1 \setminus B_i) = B_{g+i}^\circ$.

Fact: $\Gamma = \langle \gamma_1, \dots, \gamma_g \rangle$ is Schottky of genus g.

Thm: Up to conjugacy, every Schottky grp. arises from above description. Consider the affinoid $F := \mathbb{P}^1 - \bigcup_{i=1}^{2g} B_i^\circ$.

The ordinary pts. are $\Omega_\Gamma = \bigcup_{\gamma \in \Gamma} \gamma(F)$ and F is a <u>fundamental domain</u> for Γ:

(1) $F \cap \gamma(F) = \emptyset$ for all but finitely many $\gamma \in \Gamma$.

(2) $F^\circ := \mathbb{P}^1 - \bigcup_{i=1}^{2g} B_i$ satisfies $F \cap \gamma(F^\circ) = \emptyset$ $\forall \gamma \in \Gamma$ w/ $\gamma \neq 1$.

NB: There aren't many translate intersections and they all occur "on the boundary."

This is highly non-obvious! We give Ω_Γ the rigid structure obtained by gluing the rigid structures on each $\gamma(F) \cong F$.

So, $\mathcal{U} \subseteq \Omega_\Gamma$ is admissible open iff $\mathcal{U} \cap \gamma(F) \subseteq \gamma(F)$ is admissible open $\forall \gamma \in \Gamma$. (same for adm. covers).

Let $pr: \Omega_\Gamma \to \Gamma \backslash \Omega_\Gamma$.

Prop: $\exists$ adm. cov. $F = \bigcup_{i=1}^{2g} \mathcal{U}_i$ s.t. $pr|_{\mathcal{U}_i}: \mathcal{U}_i \to \Gamma\backslash\Omega_\Gamma =: X_\Gamma$ identifies $\mathcal{U}_i$ w/ its image $V_i := pr(\mathcal{U}_i)$.

So, $X_\Gamma = pr(F)$ and bij. $pr: \mathcal{U}_i \xrightarrow{\sim} V_i$ makes each V_i a rigid space. Gluing gives rigid structure on X_Γ s.t. $pr: \Omega_\Gamma \to X_\Gamma$ is morphism of rigid spaces (loc. an isom., so "étale").

Thm (Mumford): X_Γ is analytification of smooth geom. conn. proj. genus g curve $/K$.

Pf: We sketch the key steps.

(NB: Same proof as for Tate curve. Twist the divisor to assume it is effective and then inductively work one pt. at a time.)

(1) Check properness directly from the construction.

(2) Prove a form of Riemann-Roch. Given a divisor D on $X = X_\Gamma$, $\dim H^0(X, \mathcal{L}(D)) - \dim H^1(X, \mathcal{L}(D)) = \deg(D) + (1-g')$

for $g' := \dim H^1(X, \mathcal{O}_X)$.

(analogous to Riemann's strategy for showing Riemann surfaces are proj.)

(3) Use Riemann-Roch to choose D s.t. $\mathcal{L}(D)$ is very ample.

(4) GAGA $\Rightarrow X = Y^{an}$ for some proj. var. Y whose completed local rings are same as those of X. So, Y is smooth curve.

GAGA $\Rightarrow Y$ has same coherent cohom. as X. So, g' = genus of Y. (NB: So far we have not used any explicit constructions.)

(5) Use construction of X_Γ to produce meromorphic 1-form whose divisor has degree $2g-2$. We get a natural 1-form on Y and then can use Riemann-Roch + Serre duality for Y to get genus of Y is g. Hence, $g' = g$! $\square$

Reductions of $\mathbb{P}^1$

Instead of choosing coords. on $\mathbb{P}^1$, let V be some 2-dimensional k-vector space and set $\mathbb{P}^1 = \mathbb{P}^1(V)$. Assume k alg. closed so $\mathbb{P}^1 = \{\text{lines in } V\}$. Let $M \subseteq V$ be a k°-lattice - i.e., M is free k°-submodule of V of rank 2 s.t. $M \otimes_{k^\circ} k \xrightarrow{\sim} V$.

(nonzero and quotient is torsion-free)

We get $\bar{k}$-vector space $\bar{M} = M \otimes_{k^\circ} \bar{k}$ and reduction $Red_M : \mathbb{P}^1 \to \mathbb{P}^1(\bar{M})$. Given line $\ell \subseteq V$, can check $\ell \cap M \subseteq M$ is k°-direct summand (i.e., we get a line in M). $Red_M(\ell) := (\ell \cap M) \otimes_{k^\circ} \bar{k} \subseteq \bar{M}$. This is actually the reduction assoc. to pure affinoid cover of $\mathbb{P}^1$. Choose k°-basis $e_1, e_2 \in M$. These form k-basis of V hence there is dual basis $e_1^*, e_2^* \in \mathrm{Hom}_k(V,k)$.

$z := e_2^*/e_1^*$ gives coord. on $\mathbb{P}^1$. Now take $\mathbb{P}^1 = \{|z| \le 1\} \cup \{|z| \ge 1\}$.

Remark: Everything is invariant under homothety - i.e., under replacing M by λM for $\lambda \in k^\times$. We want to work up to homothety.

Now take two lattices $M, N \subseteq V$. This determines $Red_M \times Red_N : \mathbb{P}^1 \to \mathbb{P}^1(\bar{M}) \times \mathbb{P}^1(\bar{N})$ which is not surj. (*)

After homothety can assume $\exists$ basis $e_1, e_2 \in N$, $e_1, \pi e_2 \in M$ for some $\pi \in k^\times$ w/ $0 < |\pi| < 1$ (some version of elementary divisor thm). We get pt. $(a,b) \in \mathbb{P}^1(\bar{M}) \times \mathbb{P}^1(\bar{N})$ via

$a :=$ image of πN under $M \to \bar{M}$ $=$ $\bar{k}$-span of πe_2 in $\bar{M}$
$b :=$ image of M under $N \to \bar{N}$ $=$ $\bar{k}$-span of e_1 in $\bar{N}$

(union of two $\mathbb{P}^1$'s crossing transversely)

Image of $(*)$ is $Z = (\{a\} \times \mathbb{P}^1(\bar{N})) \cup (\mathbb{P}^1(\bar{M}) \times \{b\})$ inside of $\mathbb{P}^1(\bar{M}) \times \mathbb{P}^1(\bar{N})$.

Reduction $\mathbb{P}^1 \to Z$ is assoc. to some pure affinoid cover of $\mathbb{P}^1$. Again let $z := e_2^* / e_1^*$, a coord. on $\mathbb{P}^1$.

Reductions
$\mathbb{P}^1(V) \to \mathbb{P}^1(\bar{N})$, $\{|z| \geq 1\} \cup \{|z| \leq 1\}$,
$\mathbb{P}^1(V) \to \mathbb{P}^1(\bar{M})$, $\{|z| \geq |\pi|\} \cup \{|z| \leq |\pi|\}$.

$$\begin{tikzcd} & \mathbb{P}^1 \arrow[d] & \\ & Z \arrow[dl] \arrow[dr] & \\ \mathbb{P}^1(\bar{M}) & & \mathbb{P}^1(\bar{N}) \end{tikzcd}$$

Common refinement $\mathbb{P}^1(V) = \{|z| \leq |\pi|\} \cup \{|\pi| \leq |z| \leq 1\} \cup \{|z| \geq 1\}$.

Slogan: Three pts. in $\mathbb{P}^1$ determine a line!

How so? Pick three distinct lines $x_1, x_2, x_3 \in \mathbb{P}^1$ and generators $y_1, y_2, y_3 \in V$. There is linear relation $\lambda_1 y_1 + \lambda_2 y_2 + \lambda_3 y_3 = 0$. Define $M_{(x_1,x_2,x_3)} := k^\circ$-span of $\{\lambda_1 y_1, \lambda_2 y_2, \lambda_3 y_3\} \subseteq V$, independent of choice of generators up to homothety.

Fact: $M = M_{(x_1,x_2,x_3)}$ is unique lattice s.t. x_1, x_2, x_3 have distinct images under $\mathrm{Red}_M : \mathbb{P}^1 \to \mathbb{P}^1(\bar{M})$.

Let $S \subseteq \mathbb{P}^1$ be finite. Every distinct triple $x = (x_1, x_2, x_3) \in S^3$ determines a lattice $M_x \subseteq V$ w/ reduction $\mathrm{Red}_x : \mathbb{P}^1 \to \mathbb{P}^1(\bar{M}_x)$. These combine to $R_S : \mathbb{P}^1 \to \prod_{\substack{x \in S^3 \\ \text{distinct}}} \mathbb{P}^1(\bar{M}_x)$, which is <u>not</u> surj. Let image be Z_S.

For $x, y \in S^3$ the image of $\mathbb{P}^1 \to \mathbb{P}^1(\bar{M}_x) \times \mathbb{P}^1(\bar{M}_y)$ is $(\{a_x\} \times \mathbb{P}^1(\bar{M}_y)) \cup (\mathbb{P}^1(\bar{M}_x) \times \{b_y\})$.

So $x \neq y \Rightarrow$ lattice M_x determines a pt. of $\mathbb{P}^1(\bar{M}_y)$.

$$C_x := \mathbb{P}^1(\overline{M}_x) \subseteq \prod_y \mathbb{P}^1(\overline{M}_y)$$

$$\parallel$$

$$\prod_{y \neq x} \underbrace{\{b_y\}}_{\text{depends on } x} \times \mathbb{P}^1(\overline{M}_x)$$

So, $Z_S = \bigcup_{x \in S^3} C_x$ finite union of $\mathbb{P}^1$'s crossing transversely.

<u>NB</u>: Only need to worry about (at worst) ordinary double pts.