Dieudonné Theory

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Let k be a perfect field of characteristic p. Let φ denote the Frobenius on k and σ the Frobenius on $W(k)$.

Definition 0.1. *The* ***Dieudonné ring*** *of* k *is the associative ring* $\mathscr{D}_k := W(k)[F,V]$ *with relations*

- $FV = p = VF$;
- $Fx = \sigma(x)F$ *for every* $x \in W(k)$; *and*
- $xV = V\sigma(x)$ *for every* $x \in W(k)$.

In analogy with Witt vector terminology we call F ***Frobenius*** *and* V ***Verschiebung****.*

If $k = \mathbb{F}_p$ then $\mathscr{D}_k \cong \mathbb{Z}_p[x,y]/(xy-p)$. Otherwise, $\mathscr{D}_k$ is non-commutative since φ is not the identity on k.

Definition 0.2. *A (left)* ***Dieudonné module*** *is a left* $\mathscr{D}_k$*-module – i.e., a left* $W(k)$*-module* D *equipped with* σ*-semilinear* $F : D \to D$ *and* σ^{-1}*-semilinear* $V : D \to D$ *such that* $FV = [p]_D = VF$.

We see that $\mathscr{D}_k$ is naturally a Dieudonné module over itself.

Theorem 0.3. *There exists an additive anti-equivalence of categories*

$$\{\text{finite flat commutative } k\text{-group schemes of } p\text{-power order}\}$$

$$\downarrow$$

$$\{\text{Dieudonné modules } D \text{ of finite } W(k)\text{-length } \ell_{W(k)}(D)\}$$[1]

given by $D \rightsquigarrow D(G)$ *satisfying the following properties.*

(a) *The order of* G *is* $p^{\ell_{W(k)}(D(G))}$.

(b) *Let* k'/k *be an extension of perfect fields. Then, there is a natural isomorphism*

$$W(k') \otimes_{W(k)} D(G) \cong D(G_{k'})$$

of (left) $\mathscr{D}_{k'}$*-modules. In particular, there is a natural isomorphism* $\sigma^*(D(G)) \cong D(G^{(p)})$ *of (left)* $W(k)$*-modules.*

[1] This is the same as the full subcategory of finitely generated Dieudonné modules. We simply include the length notion for ease of description.

(c) *Let $F_{G/k} : G \to G^{(p)}$ be the relative Frobenius. Then, the action of F on $D(G)$ is given by the σ-semilinear map $D(G) \to D(G)$ corresponding to the $W(k)$-linear map*

$$\sigma^*(D(G)) \cong D(G^{(p)}) \xrightarrow{D(F_{G/k})} D(G).$$

Moreover, G is connected if and only if F is nilpotent on $D(G)$.

(d) *The k-vector space $D(G)/FD(G)$ is canonically isomorphic to the k-linear dual $T_G^\vee := \mathrm{Hom}_k(T_eG, k)$ of $T_eG := \ker(G(k[\epsilon]/(\epsilon^2)) \to G(k))$. In particular, G is étale if and only if F is bijective on $D(G)$.*[2]

By taking inverse limits the previous theorem gives us a classification result for p-divisible groups over k.

Corollary 0.4. *There exists an anti-equivalence of categories*

$$\{p\text{-divisible groups over } k\}$$

$$\downarrow$$

$$\{\text{finite free } W(k)\text{-modules with } \sigma\text{-semilinear endomorphism } F \text{ such that } pD \subseteq F(D)\}$$[3]

given by

$$G = \{G_n\} \rightsquigarrow D(G) := \varprojlim D(G_n)$$

satisfying the following properties.

(a) *The height of G is the rank of $D(G)$ as a $W(k)$-module.*

(b) *The functor D is compatible with extension of perfect fields analogous to the previous theorem.*

(c) *The torsion-levels G_n of the p-divisible group G satisfy $D(G_n) \cong D(G)/p^n$ compatible with change in n. In particular, G is connected if and only if F is topologically nilpotent on $D(G)$ equipped with the p-adic topology.*

Recall that to an abelian scheme $\mathcal{A}/S$ and rational prime ℓ we may associate the ℓ-divisible group $\mathcal{A}(\ell) = \mathcal{A}[\ell^\infty] := \{\mathcal{A}[\ell^n]\}$ and ℓ-adic Tate module $T_\ell(\mathcal{A}) := \varprojlim \mathcal{A}[\ell^n]$. Assume that we have A, B abelian varieties over k and $\ell \neq p$ a rational prime, dropping for now the assumption that k is perfect.

Theorem 0.5 (Tate). *The natural map*

$$\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathrm{Hom}_k(A, B) \to \mathrm{Hom}_{\mathbb{Z}_\ell[\mathrm{Gal}(k)]}(T_\ell(A), T_\ell(B))$$

is injective. Moreover, it is an isomorphism when k is finite.

The isomorphism result in the above is highly nontrivial (What's the sketch?). Injectivity is an easier result whose proof we sketch here. One passes to the category of abelian varieties over k up to isogeny, which is nicely behaved and can be concretely realized as having objects abelian varieties over k and morphisms elements of $\mathrm{Hom}_k^0(X, Y) := \mathbb{Q} \otimes_{\mathbb{Z}} \mathrm{Hom}_k(X, Y)$.

[2] This last statement is not so surprising since, given a flat scheme X over a perfect base S of characteristic p, X/S is étale if and only if the associated relative Frobenius $F_{X/S} : X \to X^{(p)}$ is an isomorphism.

[3] The condition $pD \subseteq F(D)$ ensures that there is a well-defined natural analogue of V on D. Intuitively, we obtain freeness from the fact that any torsion does not grow fast enough and so dies with the inverse limit.

(1) Reduce to the case $A = B$ by considering $\mathrm{End}_k(A \times_k B)$.

(2) Using the Poincaré Complete Reducibility Theorem, reduce to the case that A is simple (i.e., has no nontrivial abelian subvarieties).

(3) Show that the degree function $\deg : \mathrm{End}_k(A) \to \mathbb{Z}$ extends uniquely to a homogeneous polynomial function $\deg : \mathrm{End}^0(A) \to \mathbb{Q}$ of degree $2 \dim A$ (polynomial in the sense of being polynomial in a basis when restricted to any finite dimensional subspace).

(4) Show that $\mathrm{End}_k(A)$ is a finitely generated abelian group.

(5) Explicitly show that $\mathbb{Z}_\ell \otimes_{\mathbb{Z}} \mathrm{Hom}_k(A, B) \to \mathrm{Hom}_{\mathbb{Z}_\ell[\mathrm{Gal}(k)]}(T_\ell(A), T_\ell(B))$ is injective. This is where the condition $\ell \neq p$ enters the picture (though I'm not exactly sure of its role).

As stated at the beginning, our goal is to get analogues of such statements when $\ell = p$. The first order of business is to rewrite

$$\mathrm{Hom}_{\mathbb{Z}_\ell[\mathrm{Gal}(k)]}(T_\ell(A), T_\ell(B)) \cong \mathrm{Hom}_k(A(\ell), B(\ell)),$$

with the RHS denoting morphisms of ℓ-divisible groups over k (this rests crucially on the condition $\ell \neq p$ since then $A(\ell), B(\ell)$ are étale ℓ-divisible groups. In the case $\ell = p$ we expect

$$\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Hom}_k(A, B) \hookrightarrow \mathrm{Hom}_k(A(p), B(p)).$$

This can be checked over perfect extensions of k and so we may assume without loss of generality that k is perfect. The result then follows from the following consequence of Dieudonné theory.

Theorem 0.6. *Let A, B be abelian varieties over a perfect field k of characteristic p. Then, there is a natural injective homomorphism*

$$\mathbb{Z}_p \otimes_{\mathbb{Z}} \mathrm{Hom}_k(A, B) \to \mathrm{Hom}_{\mathscr{D}_k}(D(B(p)), D(A(p)))$$

that is an isomorphism if k is finite.

The method of proof is similar to that for the previous theorem of Tate (How exactly?). However, there are some important differences to keep in mind for the two settings.

- We cannot ignore the $\mathscr{D}_k$-module structure in the case that k is separably closed, in contrast with the $\mathrm{Gal}(k)$-action being trivial in this case.
- The RHS of the above does not have a natural $W(k)$-module structure when $k \neq \mathbb{F}_p$ since $W(k)$ is not central in $\mathscr{D}_k$ and so does not act through $\mathscr{D}_k$-linear endomorphisms on a general $\mathscr{D}_k$-module.

Remark 0.7. *There is a connection to crystalline cohomology. Let K be a p-adic local field with residue field k and $\mathcal{A}/\mathcal{O}_K$ an abelian scheme. Then, it is a theorem of Mazur-Messing-Oda (I think?) that $H^1_{crys}(\mathcal{A}_k/W(k)) \cong D(\mathcal{A}_k(p))$. There is more that can be said about this, e.g. by using isocrystals.*

Our goal now is to classify finite flat commutative k-group schemes of p-power order and p-divisible groups over $W(k)$ in terms of the Dieudonné module $D(G_k)$ of the special fiber together with some "lifting data." Our motivation comes from the following result.

Theorem 0.8 (Fontaine)**.**

(a) *Let G_k be a p-divisible group over k with $p > 2$. Then, any lift G of G_k to $W(k)$ yields a $W(k)$-submodule $L \subseteq D(G_k)$ of "logarithms" such that $L/p \xrightarrow{\sim} D(G_k)/FD(G_k)$. The same result holds true if $p = 2$ under the additional assumption that G_k is connected.*

(b) *Let G_k be a finite flat commutative k-group scheme of p-power order (assumed connected if $p = 2$). Then, any lift G of G_k to $W(k)$ yields a $W(k)$-submodule $L \subseteq D(G_k)$ such that $L/p \xrightarrow{\sim} D(G_k)/FD(G_k)$ and $V|_L : L \hookrightarrow D(G_k)$.*[4]

Remark 0.9. *Why is this an appropriate use of the term logarithm?*

Definition 0.10. *A **Honda system** over $W(k)$ is a pair (M, L) with M a finite free $W(k)$-module and L a $W(k)$-submodule equipped with a σ-semilinear endomorphism $F : M \to M$ such that*

- *$pM \subseteq F(M)$ and*
- *$L/p \xrightarrow{\sim} M/F(M)$.*

*We say (M, L) is **connected** if F is topologically nilpotent. Analogously, a **finite Honda system** is a pair (M, L) with M a left $\mathscr{D}_k$-module of finite $W(k)$-length and L a $W(k)$-submodule such that*

- *$V|_L : L \hookrightarrow M$ and*
- *$L/p \xrightarrow{\sim} M/F(M)$.*

*We say (M, L) is **connected** if F is nilpotent.*

It is a true but non-obvious fact that if (M, L) is a finite Honda system over $W(k)$ then $(M/p^n, L/p^n)$ is a finite Honda system over $W(k)$ for every $n \geq 1$. With this in mind, we have the following restatement and refinement of Fontaine's earlier theorem.

Theorem 0.11 (Fontaine)**.**

(a) *The assignment $G \rightsquigarrow (D(G_k), L(G))$ defines an anti-equivalence from the category of p-divisible groups over $W(k)$ to the category of Honda systems over $W(k)$, where we must add the word "connected" to both sides if $p = 2$.*

(b) *The assignment $G \rightsquigarrow (D(G_k), L(G))$ defines an anti-equivalence from the category of finite flat commutative $W(k)$-group schemes of p-power order to the category of finite Honda systems over $W(k)$, where we must add the word "connected" to both sides if $p = 2$.*

(c) *Both anti-equivalences are compatible with each other and with extension of perfect residue fields.*

[4] The latter condition is immediate for a finite free $W(k)$-module since $F \circ V = p$.