<u>Recall</u>: Given $X = V/\mathcal{U}$ complex torus, $H^2(X;\mathbb{Z}) \cong \{\text{alternating forms } E : \mathcal{U}\times\mathcal{U}\to\mathbb{Z}\}$

$$\cup\!\text{I}$$

$$NS(X) \cong \operatorname{Im}(c_1 : Pic(X) \to H^2(X;\mathbb{Z})) = H^2(X;\mathbb{Z}) \cap H^{1,1}(X)$$

(V complex vec. space of dim g, $\mathcal{U}$ free $\mathbb{Z}$-mod. of rank $2g$ s.t. $\mathcal{U}\otimes_{\mathbb{Z}}\mathbb{R} \xrightarrow{\sim} V$ as $\mathbb{R}$-vec. spaces.)

(same as $E(zx,y) = E(x,\bar{z}y)$ for $z\in\mathbb{C}$)

$NS(X) \cong \{\text{alt. forms } E : \mathcal{U}\times\mathcal{U}\to\mathbb{Z} \text{ s.t. } \mathbb{R}\text{-lin. ext. } E : V\times V\to\mathbb{R} \text{ satisfies } E(ix,iy) = E(x,y)\}$.

$\cong \{\text{Hermitian forms } H : V\times V\to\mathbb{C} \text{ s.t. } \operatorname{Im}H \text{ is } \mathbb{Z}\text{-valued on } \mathcal{U}\}$

$\operatorname{Im}E \longleftrightarrow H$

$E \longmapsto H(x,y) := E(ix,y) + iE(x,y)$

Inside of this we have $\{\text{polarizations of } X\} \cong \{H \text{ as above that are pos. definite}\}$.

<u>Remark</u>: Polarization is alt. $E : \mathcal{U}\times\mathcal{U}\to\mathbb{Z}$ s.t. bilinear form $B(x,y) = E(ix,y)$ on V is symm. and pos. def.

<u>Remark</u>: X has dim 1 $\Rightarrow$ $\operatorname{Hom}(\Lambda^2_{\mathbb{Z}}\mathcal{U},\mathbb{Z})$ free of rank 1 over $\mathbb{Z}$. Hence, $NS(X)\cong\mathbb{Z}$ and so $\{\text{polarizations of } X\}\cong\mathbb{Z}^{>0}$.

So, polarizations not that interesting in dim 1. For greater dim this is very interesting (and in fact polarizations are rare).

Why do we care?

<u>Thm</u>: $L\in Pic(X)$ ample iff $c_1(L)\in NS(X)$ is a polarization.

Combining w/ Chow's Thm says X admits polarization iff (Riemann) X embeds into proj. space iff (Lefschetz) X is proj. alg. var.

<u>Line bundles on complex tori</u>

Given $X = V/\mathcal{U}$ complex torus, want to make $c_1 : Pic(X)\to NS(X)$ more explicit.

Given $X = V/\mathcal{U}$ complex torus, describe alt. form $c_1(L) = E : \mathcal{U}\times\mathcal{U}\to\mathbb{Z}$. Let $\pi : V\to X$ be quotient map.

<u>First approach</u>: Given $L\in Pic(X)$, describe alt. form $c_1(L) = E : \mathcal{U}\times\mathcal{U}\to\mathbb{Z}$. Let $\pi : V\to X$ be quotient map. $u\in\mathcal{U} \rightsquigarrow$ left translation $u : V\to V$.

Line bundles / V trivial so we can choose nowhere vanishing $s\in H^0(V,\pi^*L)$. $u\in\mathcal{U}\rightsquigarrow$ left translation $u: V\to V$.

$\pi\circ u = \pi \Rightarrow u^*s\in H^0(V,u^*\pi^*L) = H^0(V,\pi^*L)$. L line bundle $\Rightarrow u^*s = e_u s$ for e_u holo. nowhere vanishing function on V. We have cocycle condition $e_{u+u'}(z) = e_u(z+u')e_{u'}(z)$. $e_u(z) = e^{2\pi i f_u(z)}$.

$F(u_1,u_2) := f_{u_2}(z+u_1) + f_{u_1}(z) - f_{u_1+u_2}(z)\in\mathbb{Z}$ by cocycle condition. $E(u_1,u_2) := F(u_1,u_2) - F(u_2,u_1)$.

<u>Second approach</u>: <u>Thm</u> (Appell-Humbert): $Pic(X)\cong\{\text{pairs } (H,\alpha) \text{ w/ } H : V\times V\to\mathbb{C} \text{ Hermitian form s.t. } \operatorname{Im}H \text{ is } \mathbb{Z}\text{-valued}$ on $\mathcal{U}$ and $\alpha : \mathcal{U}\to S^1$ "almost a homomorphism" via $\alpha(u_1+u_2) = \underbrace{e^{i\pi(\operatorname{Im}H)(u_1,u_2)}}_{\in\{\pm1\}}\alpha(u_1)\alpha(u_2)\}$.

$c_1 : Pic(X)\to NS(X)$ is $c_1(H,\alpha) = H$.

We can explicitly describe the line bundle $L = L(H,\alpha)$. Let $p: V \to X$ be the proj.

$\{$hol. sections of L over open set $X_0 \subseteq X\} = \{$ hol. functions Θ on $p^{-1}(X_0) \subseteq V$ s.t. $\forall u \in \mathcal{U}, z \in V$

$$\Theta(z+u) = \alpha(u) e^{\pi H(z,v) + \frac{1}{2}\pi H(v,v)} \Theta(z) \}.$$

(*)

Remark: This may not make you happy but the point is that it's concrete. Global sections are classical Θ functions.

We can study (*) to produce invariant things (by taking infinite sums). For pos. def H we often get convergence and ampleness comes for free! This is what Riemann focused on a lot, as did Mumford.

Dual torus

Let $Pic^0(X) := \ker(c_1 : Pic(X) \to NS(X))$. By Appell-Humbert these are pairs (H,α) w/ $H = 0$ and $\alpha: \mathcal{U} \to S^1$ a char. $Pic^0(X) \cong Hom_{\mathbb{Z}}(\mathcal{U}, S^1)$. RHS is top. torus and in fact has complex structure.

$\hat{V} = Hom_{\overline{\mathbb{C}}}(V, \mathbb{C})$ set of conj.-lin. functionals on V. Lattice $\mathcal{U} \subseteq V$ determines a dual lattice w/ appropriate properties

$\hat{\mathcal{U}} := \{\lambda \in \hat{V} \mid \forall u \in \mathcal{U}: \operatorname{Im} \lambda(u) \in \mathbb{Z}\}$. [Exercise: This is a lattice.]

Def: Dual torus of X is $\hat{X} := \hat{V}/\hat{\mathcal{U}}$. [Note: $\hat{\hat{V}} = V, \hat{\hat{\mathcal{U}}} = \mathcal{U} \Rightarrow \hat{\hat{X}} = X$.]

Prop: There are grp. isomorphisms $\hat{X} \cong Hom(\mathcal{U}, S^1) = Pic^0(X)$.

Pf: $\hat{V} \to Hom(\mathcal{U}, S^1)$, $\lambda \mapsto e^{2\pi i \operatorname{Im}\lambda(\cdot)}$ has kernel $\hat{\mathcal{U}}$ and is surj. $\square$

Remark: Lurking in the background is the obvious grp. structure on Appell-Humbert data.