Remark: Homomorphism $f : X_1 \to X_2$ is induced by $\mathbb{C}$-linear $f : V_1 \to V_2$ s.t. $f(\mathcal{U}_1) \subseteq \mathcal{U}_2$. Passing to
$\qquad\qquad X_1 = V_1/\mathcal{U}_1, \quad X_2 = V_2/\mathcal{U}_2$

$\bar{\mathbb{C}}$-duals gives $\hat{f} : \hat{V}_2 \to \hat{V}_1$ s.t. $\hat{f}(\hat{\mathcal{U}}_2) \subseteq \hat{\mathcal{U}}_1$. This descends to $\hat{f} : \hat{X}_2 \to \hat{X}_1$.

Def: $f : X \to \hat{X}$ is symm. if $\hat{f} = f$. All such maps constitute $\mathrm{Hom}_{sym}(X, \hat{X})$.

Prop: $X = V/\mathcal{U}$ complex torus $\Rightarrow NS(X) \xrightarrow{\sim} \mathrm{Hom}_{sym}(X, \hat{X})$.

Pf: Recall $NS(X) = \{$Hermitian $H : V \times V \to \mathbb{C}$ s.t. $\mathrm{Im} H : \mathcal{U} \times \mathcal{U} \to \mathbb{Z}\}$.

$H \in NS(X) \rightsquigarrow V \to \mathrm{Hom}_{\bar{\mathbb{C}}}(V, \mathbb{C}) = \hat{V}$, $x \mapsto H(x, \cdot)$. Integrality condition on $H \Rightarrow \mathcal{U}$ maps to $\hat{\mathcal{U}} \rightsquigarrow X \to \hat{X}$.

One checks this is symm., which follows from $\overline{H(x,y)} = H(y,x)$.

Q: Which elts. of $\mathrm{Hom}_{sym}(X, \hat{X})$ correspond to polarizations? What is the composition $\mathrm{Pic}(X) \to NS(X) \xrightarrow{\sim} \mathrm{Hom}_{sym}(X, \hat{X})$?

$L \in \mathrm{Pic}(X) \rightsquigarrow \phi_L(x) = (t_x^* L) \otimes L^{-1} \in \mathrm{Pic}(X)$ (for $x \in X$ and t_x left translation map). We get homomorphism

$\phi_L : X \to \mathrm{Pic}^0(X) = \hat{X}$. $L \mapsto \phi_L$ is the desired composition. So, one answer is that a polarization of X

is a morphism $X \to \hat{X}$ of form ϕ_L w/ L ample.

Remark: This is not good! L is not uniquely determined. Over $\mathbb{Q}$ we want the morphism $X \to \hat{X}$ to be defined over

$\mathbb{Q}$ (not some ext. field). Things are even worse for familes of ab. var's.

Poincaré Bundle

$x \in \hat{X} = \mathrm{Pic}^0(X)$ is a line bundle $\mathcal{P}_x \in \mathrm{Pic}^0(X)$. $\mathcal{P}_x$ depends holomorphically on x as follows.

Prop: $\exists!$ line bundle $\mathcal{P} \in \mathrm{Pic}(X \times \hat{X})$ Poincaré bundle s.t.

(1) $\forall x \in \hat{X}$ restriction of $\mathcal{P}$ to $X = X \times \{*\} \subseteq X \times \hat{X}$ is the line bundle $\mathcal{P}_x$ that x "is".

(2) Restriction of $\mathcal{P}$ to $\{0\} \times \hat{X} \subseteq X \times \hat{X}$ is trivial (and we know about the trivialization).

Pf: Write down Appell-Humbert data. $X \times \hat{X} = (V \times \hat{V})/(\mathcal{U} \times \hat{\mathcal{U}})$. We need Hermitian form

$H : (V \times \hat{V}) \times (V \times \hat{V}) \to \mathbb{C}$ and semi-char. $\alpha : \mathcal{U} \times \hat{\mathcal{U}} \to S^1$. $H((v, \hat{v}), (w, \hat{w})) = \overline{\hat{w}(v)} + \hat{v}(w)$.

$\alpha(u, \hat{u}) = e^{i\pi \, \mathrm{Im}\, \hat{u}(u)}$. $\square$

Any map $f \in \mathrm{Hom}(X, \hat{X}) \rightsquigarrow$ line bundle $\mathcal{P}_f :=$ $\left(\text{pullback Poincaré bundle } \mathcal{P} \text{ via } X \xrightarrow[\mathrm{id}_X \times f]{} X \times \hat{X} \right) \in \mathrm{Pic}(X)$.

OTOH, $L \in \mathrm{Pic}(X) \rightsquigarrow \phi_L \in \mathrm{Hom}_{\mathrm{sym}}(X, \hat{X})$. These constructions are <u>not</u> inverse to one another.

<u>Prop</u>: (1) $f \in \mathrm{Hom}_{\mathrm{sym}}(X, \hat{X}) \Rightarrow \phi_{\mathcal{P}_f} = 2f$. (Probably $f + \hat{f}$ for general f.)

(2) $L \in \mathrm{Pic}(X) \Rightarrow \mathcal{P}_{\phi_L} \otimes L^{\otimes -2} \in \mathrm{Pic}^0(X)$.

(Best definition!)

<u>Cor</u>: $f \in \mathrm{Hom}_{\mathrm{sym}}(X, \hat{X}) \cong NS(X)$ is polarization iff $\mathcal{P}_f \in \mathrm{Pic}(X)$ is ample.

(In fact, as we will need later, $\mathcal{P}_f^{\otimes 3}$ is very ample!)

(by Prop (2)).

<u>Pf</u>: f is polarization iff $f = \phi_L$ for ample $L \in \mathrm{Pic}(X)$. But, L is ample iff $L^{\otimes 2}$ is ample iff $\mathcal{P}_f$ ample. $\square$