Worksheet Solutions

Section 6 Math 2202

Functions of Two and Three Variables, Partial Derivatives and Chain Rule

1. Let's try to understand the function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$.

   (a) What is the domain of this function?

The expression $x^2+y^2+z^2$ makes sense for all $(x,y,z) \in \mathbb{R}^3$ and is always nonnegative, so the domain is $\mathbb{R}^3$.

   (b) We can't graph this in our three dimensions, because each point in the domain is already in three space, leaving no other dimension to plot the output $f(x, y, z)$.

   For a function of 3 variables, we can do the analog of level curves, called **level surfaces**. Find the level surfaces of $f(x, y, z)$ for $k = -2, -1, 0, 1, 2, 3$ and describe what each looks like as two dimensional surface. Then try sketching them all on the same xyz-axes.

k	Description
-2	$\emptyset$ (no solutions)
-1	$\emptyset$ (no solutions)
0	$\{(0,0,0)\}$
1	Upper hemisphere (radius $= 1$)
2	Upper hemisphere (radius $= \sqrt{2}$)
3	Upper hemisphere (radius $= \sqrt{3}$)

The shape given by $f(x,y,z)$ could be called a "hypercone" (it is 3D).

2. The function $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$ gives the distance to the origin of the input point (x, y, z).

   (a) Let's consider a particle moving according to $x(t) = t^2, y(t) = 3 \sin t, z(t) = t + 1$. Using chain rule, compute $\frac{df}{dt}$, the derivative of the composite function $f(x(t), y(t), z(t))$ with respect to t.

Let $p(t) = (x(t), y(t), z(t))$. The partial derivatives of f are given by:

$$f_x = \frac{x}{\sqrt{x^2+y^2+z^2}}, \quad f_y = \frac{y}{\sqrt{x^2+y^2+z^2}}, \quad f_z = \frac{z}{\sqrt{x^2+y^2+z^2}}$$

Note: $x'(t) = 2t \qquad y'(t) = 3\cos t \qquad z'(t) = 1$

$$\frac{\partial f}{\partial t}(t) = f_x(p(t))x'(t) + f_y(p(t))y'(t) + f_z(p(t))z'(t)$$

$$= \frac{1}{\sqrt{x(t)^2+y(t)^2+z(t)^2}} \left[x(t)x'(t) + y(t)y'(t) + z(t)z'(t) \right]$$

$$= \frac{1}{\sqrt{x(t)^2+y(t)^2+z(t)^2}}\left[t^2(2t) + 3\sin t(3\cos t) + (t+1)(1) \right]$$

$$= \frac{2t^3 + 9\sin t\cos t + t + 1}{\sqrt{t^4 + 9\sin^2 t + (t+1)^2}}$$

Note: An alternative strategy is to expicitly compute the composite function $f(x(t), y(t), z(t))$. However, this requires much more work.

(b) This derivative of the composite function $f(x(t), y(t), z(t))$ with respect to t can be interpreted as the rate of change of $f(x, y, z)$ with respect to t as the point (x, y, z) moves along the curve C described by those parametric equations.
In this case, recall that $f(x, y, z)$ is the distance to the origin of (x, y, z). Interpret $\frac{df}{dt}|_{t=2}$ in words.

Imagine $(x(t), y(t), z(t))$ as the position of a particle on C at time t. $\frac{df}{dt}\Big|_{t=2}$ is the instantaneous rate of change of the distance from the origin to this particle along C at the point $(x(2), y(2), z(2))$. The sign of this tells us if the particle is moving toward or away from the origin at time $t=2$ (or neither if the derivative is 0).

(c) Let's consider a particle moving according to $x(s) = 2\sin s, y(s) = 2\cos s, z(s) = 1$. Using chain rule, compute the derivative of $f(x(s), y(s), z(s))$ with respect to s. Does your answer make sense? Why or why not?

We have $x'(s) = 2\cos s$, $y'(s) = -2\sin s$, $z'(s) = 0$. As above, computing $\frac{df}{ds}$ requires us to compute $x(s)x'(s) + y(s)y'(s) + z(s)z'(s)$. This is

$2\sin s(2\cos s) + 2\cos s(-2\sin s) + 1(0) = 0$. This makes sense since we are moving along a circle centered at the origin.

Quiz Solutions

1. Let $f(x, y) = \sin(-xy) + y^2$.

(c) Find an equation for the tangent plane to the surface $z = f(x, y)$ at the point

$$P = (\frac{\pi}{4}, -1, \frac{\sqrt{2}}{2} + 1).$$

We first compute $f_y(x,y) = \cos(xy)(-x) + 2y = -x\cos(-xy) + 2y$

$$\Rightarrow f_y(\tfrac{\pi}{4}, -1) = -\tfrac{\pi}{4}\cos(-\tfrac{\pi}{4}(-1)) + 2(-1)$$

$$= -\tfrac{\pi}{4}(\tfrac{1}{\sqrt{2}}) - 2 = -\frac{\pi}{4\sqrt{2}} - 2.$$

Let $P = (x_0, y_0, z_0)$. The equation of the tangent plane looks like

$$z = z_0 + f_x(x_0, y_0)(x - x_0) + f_y(x_0, y_0)(y - y_0)$$

$$= \left(\frac{\sqrt{2}}{2} + 1\right) + \frac{1}{\sqrt{2}}\left(x - \frac{\pi}{4}\right) + \left(-\frac{\pi}{4\sqrt{2}} - 2\right)(y + 1).$$

2. (a) Which of the following functions describes a two dimensional surface lying in $\mathbf{R}^3$? (In other words, which has a graph which is a 2-D surface?)

(b) Which function, if any, describes a plane in $\mathbf{R}^3$?

(c) Which function, if any, describes the surface of a hemisphere in $\mathbf{R}^3$?

(d) Which function, if any, has level curves which are hyperbolas?

(e) Which function, if any, has level curves which are lines?

A. $f(x, y, z) = \sqrt{x^2 + y^2 + z^2}$

B. $g(x, y) = \sqrt{9 - x^2 - y^2}$

C. $h(x, y) = 2x - 3y + 7$

D. $k(x, y, z) = 2x - 3y + 6z + 7$

E. $m(x, y) = \sqrt{x^2 + y^2}$

F. $n(x, y) = 3x^2 - y^2$

Function	Shape	a	b	c	d	e
f	"Hypercone"	N	N	N	N	N
g	Upper hemisphere	Y	N	Y	N	N
h	Plane	Y	Y	N	N	Y
k	"Hyperplane"	N	N	N	N	N
m	Cone	Y	N	N	N	N
n	Hyperbolic paraboloid	Y	N	N	Y	N