Last time: Shimura datum is pair (G, X) w/ G reductive grp. $/\mathbb{Q}$ and $X \subseteq \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$ a $G(\mathbb{R})$-conj. class satisfying appropriate conditions. $K \le G(\mathbb{A}_f)$ compact open $\rightsquigarrow Sh_K(G,X) = G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K$. [$\mathbb{A}_f = \mathbb{A}_f$ = finite adèles]

Lemma: (1) $G(\mathbb{Q}) \subseteq G(\mathbb{R})$ is dense $\Rightarrow G(\mathbb{Q}) \curvearrowright \pi_0(X)$ transitively

(2) $G(\mathbb{Q})^+ := G(\mathbb{Q}) \cap G(\mathbb{R})^+$ has finite index in $G(\mathbb{Q})$.

(Need two conditions, one involving simple connectivity) [Strong Approximation]

(3) $G(\mathbb{Q}) \subseteq G(\mathbb{A}_f)$ need not be dense but if $K \le G(\mathbb{A}_f)$ compact open then $G(\mathbb{Q}) \backslash G(\mathbb{A}_f)/K$ is finite.

(4) Every conn. component of $X \supset G(\mathbb{R})$ has stabilizer $G(\mathbb{R})^+$.

Cor: (1) $X^+ \subseteq X$ conn. component $\Rightarrow G(\mathbb{Q})^+ \backslash X^+ \times G(\mathbb{A}_f)/K \xrightarrow{\sim} G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K$.

(2) $g \in G(\mathbb{A}_f) \rightsquigarrow \Gamma_g \backslash X^+ \hookrightarrow G(\mathbb{Q})^+ \backslash X^+ \times G(\mathbb{A}_f)/K$, $h \mapsto [h, g]$, $\overbrace{\Gamma_g := G(\mathbb{Q})^+ \cap gKg^{-1}}^{\text{congruence subgrp., hence arithmetic}}$.

(3) $Sh_K(G,X) = \coprod_g \Gamma_g \backslash X^+$, where $g \in G(\mathbb{A}_f)$ runs over set of reps. for finite set $G(\mathbb{Q})^+ \backslash G(\mathbb{A}_f)/K$.

K "small enough" (every Γ_g torsion-free) $\Rightarrow \Gamma_g \backslash X^+$ is (by Baily-Borel) the complex pts. of smooth quasi-proj. var. $/\mathbb{C}$.

What's this bit about?

Def: (1) $\gamma \in G(\mathbb{Q})$ is neat if $\forall$ reps. $(/\mathbb{Q})$ $G \to GL(V)$ the eigenvals. of γ generate torsion-free subgrp. of $\overline{\mathbb{Q}}^\times$.

(enough to check on just one faithful rep.)

(2) $\Gamma \le G(\mathbb{Q})$ is neat if all of its elts. are. (so Γ is torsion-free)

Def: (1) Suppose $g \in G(\mathbb{A}_f)$ w/ p-component $g_p \in G(\mathbb{Q}_p)$. Let $\Gamma_p \le \overline{\mathbb{Q}}_p^\times$ be subgrp. gen. by eigenvals. of $g \in GL(V \otimes \mathbb{Q}_p)$. Given emb. $\overline{\mathbb{Q}} \hookrightarrow \overline{\mathbb{Q}}_p$, form $(\overline{\mathbb{Q}}_p^\times \cap \Gamma_p)_{\text{tors}} \subseteq \overline{\mathbb{Q}}^\times$ (exercise: independent of choice of emb.).

$g \in G(\mathbb{A}_f)$ is neat if $\bigcap_p (\overline{\mathbb{Q}}^\times \cap \Gamma_p)_{\text{tors}} = \{1\}$ (so trivial intersection).

(2) $K \le G(\mathbb{A}_f)$ is neat if all of its elts. are.

Prop: (every) Compact open subgrp. of $G(\mathbb{A}_f)$ contains neat compact open w/ finite index. [In practice we don't need to shrink much...]

(2) $K \le G(\mathbb{A}_f)$ neat $\Rightarrow \Gamma_g = G(\mathbb{Q}) \cap gKg^{-1}$ neat $\forall g \in G(\mathbb{A}_f)$.

Conclusion: $K \le G(\mathbb{A}_f)$ neat $\Rightarrow Sh_K(G,X)$ is (complex pts. of) smooth quasi-proj. var.

"Small enough" basically just means neat.

More Modular Curves

[Hint: Noether-Skolem]

Exercise: All $\mathbb{R}$-alg. emb.'s $\mathbb{C} \hookrightarrow M_2(\mathbb{R})$ are $GL_2(\mathbb{R})$-conj.

Corresponding subset $X \subseteq \mathrm{Hom}(\mathbb{S}, GL_{2,\mathbb{R}})$ is $GL_2(\mathbb{R})$-conj. class $(X \cong \mathcal{H}^+ \sqcup \mathcal{H}^-)$.

We can view X as (set of) all $\mathbb{C}$-structures on $\mathbb{R}^2$. $h \in X \rightsquigarrow \mathbb{C}$-structure in which "mult. by i" is $h(i)$.

Prop: $K \leq GL_2(\hat{\mathbb{Z}})$ compact open $\Rightarrow GL_2(\mathbb{Q}) \backslash X \times GL_2(\mathbb{A}_f)/K \cong \{$isom. classes of elliptic curves $\underset{(*)}{E/\mathbb{C}}$ w/ K-orbit in $\mathrm{Iso}_{\hat{\mathbb{Z}}}(\underbrace{\hat{T}E}_{\text{(adèlic Tate module)}}, \hat{\mathbb{Z}}^2)$. $\}$

$$\hat{T}E = \varprojlim_{N \in \mathbb{Z}} E[N] \cong \prod_{\substack{p \in \mathbb{Z} \\ \text{prime}}} T_p E.$$

NB: $E = \mathbb{C}/L \Rightarrow \hat{T}E = \hat{L} = L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$.

(This is subtle, since K is far from being neat here.)

NB: $K = GL_2(\hat{\mathbb{Z}}) \Rightarrow$ only one K-orbit in $\mathrm{Iso}_{\hat{\mathbb{Z}}}(\hat{T}E, \hat{\mathbb{Z}}^2) \Rightarrow$ RHS is $\{$isom. classes of elliptic curves$\}$.

Pf: Given $(h,g) \in X \times GL_2(\mathbb{A}_f)$, we associate elliptic curve $E_{(h,g)} = \mathbb{R}^2/g \cdot \mathbb{Z}^2$ w/ $\mathbb{C}$-structure on $\mathbb{R}^2$ determined by $h \in X$. $g \cdot \mathbb{Z}^2 = g \cdot \hat{\mathbb{Z}}^2 \cap \mathbb{Q}^2$. In particular, $\hat{T}E_{(h,g)} = g \cdot \hat{\mathbb{Z}}^2$. Endow $E_{(h,g)}$ w/ K-orbit of isoms. containing $\hat{T}E_{(h,g)} = g \cdot \hat{\mathbb{Z}}^2 \underset{(\text{mult. by } g^{-1})}{\xrightarrow{\sim}} \hat{\mathbb{Z}}^2$. This gives $X \times GL_2(\mathbb{A}_f)/K \to (*)$.

$\gamma \in GL_2(\mathbb{Q}) \Rightarrow E_{(h,g)} = \mathbb{R}^2/g\mathbb{Z}^2 \underset{(\text{mult. by } \gamma)}{\longrightarrow} \mathbb{R}^2/\gamma g \mathbb{Z}^2 = E_{(\gamma h, \gamma g)}$ respecting $\mathbb{C}$-structure.

$\square$

Other direction is easier...

Remark: We need K here (though it may seem incidental) because we get a "nice" double quotient.

$\mathbb{A}_f^\times = \mathbb{Q}^\times \hat{\mathbb{Z}}^\times \Rightarrow GL_2(\mathbb{A}_f) = GL_2(\mathbb{Q})\, GL_2(\hat{\mathbb{Z}})$.