Key operations in K-theory and G-theory

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Overview

1. Review
2. K_0 and G_0 classically
3. Additive K-theory
4. λ-rings
5. End

Derived stacks

Convention: everything derived

- $\mathrm{s}\mathcal{R}\mathrm{ing} = \mathcal{P}_\Sigma(\mathcal{P}\mathrm{oly})$, and $\mathcal{A}\mathrm{ff} := \mathrm{s}\mathcal{R}\mathrm{ing}^{\mathrm{op}}$
- $\mathcal{S}\mathrm{t} \subset \mathrm{Fun}(\mathcal{A}\mathrm{ff}^{\mathrm{op}}, \mathcal{S})$ spanned by functors satisfying étale descent
- A stack is a *scheme* if it is Zariski locally an affine scheme
- A stack X is *n-algebraic* if there is a $(n-1)$-smooth and epic morphism $U \to X$, where U is a scheme

Perfect, coherent, and locally free complexes

Let X be an algebraic stack, and $F \in \mathrm{QCoh}(X) = \lim_{\mathrm{Spec}\, R \to X} \mathrm{Mod}_R$

- F is *perfect* if $F(A)$ is compact for all $\mathrm{Spec}\, A \to X$
- (for X a scheme) F is *locally free of finite rank* if there is a Zariski cover $\{U_\alpha \to X\}_\alpha$ and equivalences $F_{|U_\alpha} \simeq \mathcal{O}_{U_\alpha}^{n_\alpha}$
- F is *coherent* if $F(A) \in \mathrm{Mod}_A$ is, for all $\mathrm{Spec}\, A \to X$, meaning that $F(A)$ has bounded homotopy, and each $\pi_n F(A)$ is finitely generated over $\pi_0(A)$.

$\mathrm{Perf}(X) \subset \mathrm{QCoh}(X)$ is the stable subcategory of perfect complexes

Abstract K-theory

Let $\mathcal{C}$ be a stable category

- We defined the *K-theory space* $K(\mathcal{C})$ via the $S_\bullet$-construction
- $K(\mathcal{C})$ is a like a space, where the points are objects of $\mathcal{C}$ and the paths are induced by exact sequences

$$C_0 \to C_1 \to C_2$$

  giving $C_1 \sim C_0 + C_2$. Higher homotopties by 'staircases'.
- Then $K_n(\mathcal{C}) := \pi_n(K(\mathcal{C}))$

K_0 of exact categories

- An *exact 1-category* is a 1-category $\mathcal{E}$ with specified 'exact sequences'
$$M' \to M \to M''$$
satisfying certain stability conditions (Ex: abelian cats)
- For $\mathcal{E}$ and exact category, the Grothendieck group $K_0(\mathcal{E})$ is the abelian group freely generated by objects of $\mathcal{E}$ modulo
$$[M] = [M'] \oplus [M'']$$
for each exact sequence $M' \to M \to M''$
- We have $K_0(\mathrm{h}\mathcal{C}) = K_0(\mathcal{C})$ for $\mathcal{C}$ stable.

Definition

- For a scheme X, we defined

$$K(X) := K(\mathcal{P}\mathrm{erf}(X))$$
$$G(X) := K(\mathcal{C}\mathrm{oh}(X))$$

- For X classical, we get
  - $G_0(X)$ is the Grothendieck group of the abelian 1-category of the classical $\mathcal{C}\mathrm{oh}(X)$
  - $K_0(X)$ is the Grothendieck group of the exact 1-category $\mathrm{Q}\mathcal{C}\mathrm{oh}^{lf}(X)$ of the locally free sheaves on X.

Cartan map

- If $A \in \mathrm{s}\mathcal{R}\mathrm{ing}$ has bounded homotopy, then $\mathcal{P}\mathrm{erf}(A) \subset \mathcal{C}\mathrm{oh}(A)$
- If a Noetherian, algebraic stack X has bounded $\mathcal{O}_X$, then $\mathcal{P}\mathrm{erf}(X) \subset \mathcal{C}\mathrm{oh}(X)$, giving us

$$\mathrm{C} : K(X) \to G(X)$$

  called the *Cartan map*
- If X is moreover regular, then C is an equivalence
- If X is a classical scheme, then $\mathcal{O}_X$ is always bounded, hence the Cartan map always exists.
- For classical X, the Cartan map

$$K_0(X) \to G_0(X)$$

  comes about by the classical $\mathrm{Q}\mathcal{C}\mathrm{oh}^{lf}(X) \subset \mathcal{C}\mathrm{oh}(X)$

Dévissage

Theorem

Let $\mathcal{A} \subset \mathcal{B}$ be an exact abelian sub-category of an abelian 1-category, closed under subobjects and quotients. Suppose for each $B \in \mathcal{B}$ there is a filtration

$$0 = B_r \subset B_{r-1} \subset \cdots \subset B_0 = B$$

such that each B_i/B_{i+1} lies in $\mathcal{A}$. Then

$$K(\mathcal{A}) \simeq K(\mathcal{B})$$

Nil-invariance

Proposition

For X a Noetherian classical scheme, it holds $G_0(X) \cong G_0(X_{\mathrm{red}})$

$K_0^{\oplus}$ of additive categories

- A category $\mathcal{C}$ is additive if $M \cup N \simeq M \times N$ for all $M, N \in \mathcal{C}$
- For $\mathcal{C}$ additive, we define $K_0^{\oplus}(\mathcal{C})$ as the abelian group freely generated by objects of $\mathcal{C}$, modulo the relations

$$[M \oplus N] = [M] + [N]$$

Let $R \in \mathrm{s}\mathcal{R}\mathrm{ing}$. Recall $M \in \mathcal{M}\mathrm{od}_R$ is finitely generated projective if it is a direct summand of some $R^{\oplus n}$, equivalently, if it is finitely generated and locally free. Then put

$$K_0^{\oplus}(R) := K_0^{\oplus}(\mathcal{M}\mathrm{od}_R^{\mathrm{proj}})$$

$K_0(R)$: projectives vs perfects

Theorem

Let $R \in \mathrm{s}\mathfrak{R}\mathrm{ing}$. Then $K_0^{\oplus}(R) \cong K_0(R)$.

Derived nil-invariance

Theorem

Let $R \in \mathrm{s}\mathcal{R}\mathrm{ing}$. *Then* $K_0(R) \cong K_0(\pi_0(R))$.

Nonabelian derived functors

- Write $\mathrm{s}\mathcal{R}\mathrm{ing}\mathcal{M}\mathrm{od}^{\mathrm{cn}}$ for the category of pairs (R, M), where $R \in \mathrm{s}\mathcal{R}\mathrm{ing}$ and $M \in \mathcal{M}\mathrm{od}_R^{\mathrm{cn}}$
- Let $\mathcal{C}$ be the full subcategory of $\mathrm{s}\mathcal{R}\mathrm{ing}\mathcal{M}\mathrm{od}^{\mathrm{cn}}$ spanned by (R, M) where R is finitely generated polynomial and M finitely generated free
- It holds $\mathcal{P}_\Sigma(\mathcal{C}) \simeq \mathrm{s}\mathcal{R}\mathrm{ing}\mathcal{M}\mathrm{od}^{\mathrm{cn}}$
- Consequently

$$\mathrm{Fun}_\Sigma(\mathrm{s}\mathcal{R}\mathrm{ing}\mathcal{M}\mathrm{od}^{\mathrm{cn}}, \mathcal{E}) \simeq \mathrm{Fun}(\mathcal{C}, \mathcal{E})$$

  for any cocomplete $\mathcal{E}$, where $\mathrm{Fun}_\Sigma(-,-)$ means sifted-colimit preserving

Derived symmetric and exterior powers

Definition

Taking the nonabelian derived functor of the classical symmetric nth powers gives us the derived functor

$$\mathrm{s}\mathcal{R}\mathrm{ing}\mathcal{M}\mathrm{od}^{\mathrm{cn}} \to \mathrm{s}\mathcal{R}\mathrm{ing}\mathcal{M}\mathrm{od}^{\mathrm{cn}} : (R, M) \mapsto (R, \mathrm{Sym}^n_A(M))$$

We do the same of the classical exterior powers, giving us

$$\mathrm{s}\mathcal{R}\mathrm{ing}\mathcal{M}\mathrm{od}^{\mathrm{cn}} \to \mathrm{s}\mathcal{R}\mathrm{ing}\mathcal{M}\mathrm{od}^{\mathrm{cn}} : (R, M) \mapsto (R, \bigwedge\nolimits^n_A(M))$$

Lemma

The derived nth symmetric powers assemble into a ring $\mathrm{Sym}_A(M)$. This construction is left adjoint to the forgetful functor $\mathcal{A}\mathrm{lg}_A \to \mathcal{M}\mathrm{od}_A$

λ-rings

Definition

Let K be a discrete ring. A pre-λ-ring structure on K is a family of operations $\lambda^n : K \to K$, $n \geq 0$, such that

$$\lambda^n(x+y) = \sum_{i=0}^{n} \lambda^i(x)\lambda^{n-i}(y)$$

and $\lambda^0 \equiv 1, \lambda^1 \equiv \mathrm{id}$. K with such structure is a λ-ring if moreover

- $\lambda^n(xy)$ can be expressed in terms of $\lambda^1(x), \lambda^1(y), \ldots, \lambda^n(x), \lambda^n(y)$ through a fixed polynomial P_n
- $\lambda^m(\lambda^n(z))$ can be expressed in terms of $\lambda^1(z), \ldots, \lambda^{mn}(z)$ in terms of a fixed polynomial $P_{m,n}$

λ-rings

We have an adjunction

$$V : \lambda\mathfrak{R}\mathrm{ing} \rightleftarrows \mathfrak{R}\mathrm{ing} : \Lambda$$

here, $\Lambda(R) = 1 + tR[[t]]$. In fact, a pre-λ-ring is a λ-ring if

$$R \to \Lambda(R)$$

is a morphism of pre-λ-rings.
In doubly fact, this adjunction is comonadic, meaning that $\lambda\mathfrak{R}\mathrm{ing}$ is the category of coalgebras over

$$\Lambda : \mathfrak{R}\mathrm{ing} \to \mathfrak{R}\mathrm{ing}$$

The λ-ring structure on $K_0(R)$

We use the model $K_0(R) = K_0^{\oplus}(\mathcal{M}\mathrm{od}_R^{\mathrm{proj}})$.

- For $M \in \mathcal{M}\mathrm{od}_R$ locally free of finite rank m, $\bigwedge^n(M)$ is locally free of finite rank $\binom{m}{n}$
- We can thus define

$$\lambda^n : K_0(R) \to K_0(R) : [M] \mapsto \left[\bigwedge\nolimits_R^n(M)\right]$$

Proposition

These operations make $K_0(R)$ into a λ-ring.

Next time

- Coniveau filtration
- γ filtration
- Gysin map
- Localization
- Excision

References

Adeel Kahn (2018)
The Grothendieck–Riemann–Roch theorem (lecture notes)

Adeel Kahn (2021)
K-theory and G-theory of algebraic stacks

Jacob Lurie
Spectral Algebraic Geometry

Aaron Landesman
Some basics of algebraic K-theory (lecture notes)

Thank you!