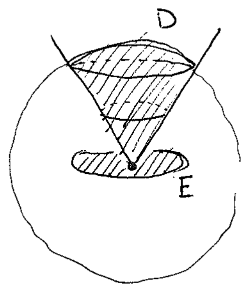
Section 10 Math 2202
Volume between Surfaces and
Spherical Coordinates Introduction

1. **Volume Between Surfaces**

   Set up an integral which represents the volume of the solid bounded by $z = \sqrt{x^2+y^2}$ and $x^2+y^2+z^2=4$.

   You may find the guidelines below helpful.

$$\iint_E \left(\sqrt{4-x^2-y^2} - \sqrt{x^2+y^2}\right) dA$$



$E = \{(x,y) \mid x^2+y^2=2\}$

to find the shadow E, we look for the intersection of the two surfaces

$$4-x^2-y^2 = x^2+y^2$$
$$4 = 2(x^2+y^2)$$
$$2 = x^2+y^2$$

Volume Via Double Integration Some general guidelines:
Before setting up $\int\int_D f(x,y)\,dA$, you need to

- Identify the region D over which you will integrate- this is the projection ('shadow') of the solid in the xy-plane.
- Identify the function $f(x,y)$ which you will integrate - this should be the function whose graph is the surface under which the solid lies.

To do this, it is helpful to

- Draw a 'good enough' picture of what's happening in $\mathbf{R}^3$ with the bounding surfaces to see what the solid looks like. Find intersections of the surfaces.
- Draw a good picture of D in the xy-plane, making sure you have all the intersections and bounds correct. You can reality check with the equations - find the points or curves at which the surfaces intersect and think about the projection of those points or curves in the xy-plane.
- Consider if you need two or more integrals, for example if the volume of the solid is best described as a difference of volumes.

Then set up the integral(s) and then decide the best way to integrate it (switching orders, doing a change of coordinates).

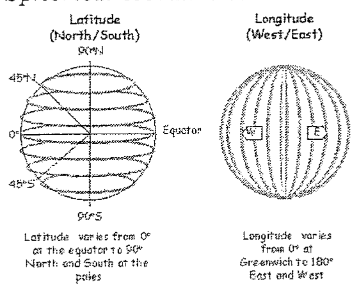
The usual way we represent a point in $\mathbf{R}^3$ is as (x, y, z). These are called **rectangular/Cartesian coordinates**. There are two other commonly used ways to think about a point.

Spherical Coordinates for $\mathbf{R}^3$

A point P in $\mathbf{R}^3$ is represented as (ρ, ϕ, θ) where

- ρ is the distance to the origin O, which is $|OP|$
- ϕ is the angle OP makes with the positive z-axis
- θ is the angle the projection of OP in the xy-plane makes with the positive x-axis (the same angle as in cylindrical coordinates)

Spherical coordinates takes its intuition from latitude ($\pi/2 - \phi$) and longitude (θ).



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- Spherical $\to$ Rectangular: $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$

- Rectangular $\to$ Spherical:

2. Let's figure out why this is the relationship between rectangular and spherical coordinates, and how to go back the other way.

3. (a) Write the point $(5, \pi/4, \pi/3) = (\rho, \theta, \phi)$ in rectangular coordinates.

$\sin\phi = \sqrt{3}/2$

$\cos\phi = 1/2$

$\sin\theta = \sqrt{2}/2$

$\cos\theta = \sqrt{2}/2$

$\Rightarrow$ $x = \dfrac{5\sqrt{6}}{4}$ $\quad z = 5/2$

$y = \dfrac{5\sqrt{6}}{4}$

(b) Write the point $(0, 2\sqrt{3}, -2) = (x, y, z)$ in spherical coordinates. (Hint: it might help to plot and visualize it.)

$\rho = \sqrt{x^2+y^2+z^2} = \sqrt{12+4} = 4$

$\theta = \arctan(+\infty) = \pi/2$

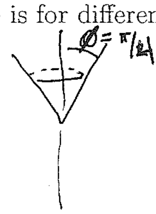
$\phi = \arccos(-2/4) = 2\pi/3$

(c) Identify the surface described by each equation and write the equation in rectangular coordinates if it is in spherical or spherical if it is in rectangular:

- In rectangular coordinates, $x^2 + y^2 + z^2 = 4$
- In spherical coordinates $\rho = 4$

- sphere of radius 2:
  $x^2+y^2+z^2=4 \Leftrightarrow \rho = 2$
- Sphere of radius 4:
  $x^2+y^2+z^2=16 \Leftrightarrow \rho = 4$

(d) What is the equation of the half cone $z = \sqrt{x^2 + y^2}$ in in spherical coordinates? (Hint: Think about what the angle ϕ is for different points on the cone.)



the eq. for the half cone is
$\phi = \pi/4$

See Chapter 9.7 for more on these coordinates.

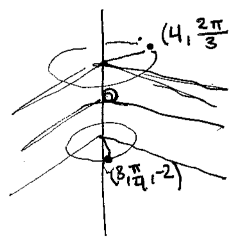
Cylindrical Coordinates for $\mathbf{R}^3$ (We'll cover this in homework and class next week.)
A point P in $\mathbf{R}^3$ is represented as (r, θ, z) where r and θ are the polar coordinates of the projection of P into the xy-plane and z is the usual z-coordinate. (For a refresher on polar coordinates, see pages A6-A11 in the text (Stewart's *Calculus: Concepts and Contexts, Multivariable (4th ed)*).)

- Cylindrical $\to$ Rectangular: $x = r\cos\theta$, $y = r\sin\theta$, $z = z$
- Rectangular $\to$ Cylinder: $r^2 = x^2 + y^2, \tan\theta = \frac{y}{x}, z = z$

  Here r is always the positive square root and which θ is determined by considering where the point is located in space.

4. (a) Plot the point whose cylindrical coordinates are given. Then find the Cartesian coordinates of the same point.

    i. $(r, \theta, z) = (4, \frac{2\pi}{3}, 5)$
    ii. $(r, \theta, z) = (3, \frac{\pi}{4}, -2)$



$(4, \frac{2\pi}{3}, 5) \Rightarrow x = -2, \ y = 2\sqrt{3}, \ z = 5$

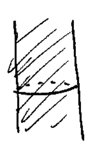
$(3, \frac{\pi}{4}, -2) \Rightarrow x = \frac{3}{2}\sqrt{2}, \ y = \frac{3}{2}\sqrt{2}, \ z = -2$

(b) Plot the point whose rectangular coordinates are given. Then find the polar coordinates of the same point where $r > 0$ and $0 \leq \theta \leq 2\pi$.

    i. $(x, y) = (2, -2, 2)$ $\quad (r, \theta, z) = (2\sqrt{2}, \frac{7\pi}{4}, 2)$
    ii. $(x, y) = (-1, \sqrt{3}, 0)$ $\quad (r, \theta, z) = (2, \frac{2\pi}{3}, 0)$

(c) Sketch the surface or space in $\mathbf{R}^3$ that is described by

    i. In rectangular coordinates: $x^2 + y^2 = 9$
    ii. In cylindrical coordinates: $r \leq 5$

i.  cylinder shell of radius 3

ii.  solid cylinder of radius 5