Def: Suppose (G,X), (G',X') are Shimura data. A <u>morphism</u> $f:(G,X)\to(G',X')$ is morphism of alg. grps. $G\to G'$ s.t. induced $\mathrm{Hom}(\mathbb{S},G_{\mathbb{R}})\xrightarrow{f}\mathrm{Hom}(\mathbb{S},G'_{\mathbb{R}})$ satisfies $f(X)\subseteq X'$.

Example: $E=\mathbb{Q}(\sqrt{-D})$ imaginary quad., (V,H) Hermitian space $/E$ of signature (p,q). Define symplectic form $\psi(x,y):=\mathrm{Tr}_{E/\mathbb{Q}}\left(\frac{1}{\sqrt{-D}}H(x,y)\right)$. Recall $X:=\{E\otimes_{\mathbb{Q}}\mathbb{R}\text{-stable}\perp\text{ decomp.'s } V_{\mathbb{R}}=W_0\oplus W_1\}$ is realized inside of $\mathrm{Hom}(\mathbb{S},\underbrace{GU(V_{\mathbb{R}})}_{\text{(unitary similitudes)}})$ as follows: given $V_{\mathbb{R}}=W_0\oplus W_1\in X$ and $z\in\mathbb{C}\cong E\otimes_{\mathbb{Q}}\mathbb{R}$ we have $h(z)\in\mathrm{End}(V_{\mathbb{R}})$ via z on W_0 and $\bar z$ on W_1. Inclusion $GU(V)\hookrightarrow GSp(V)\rightsquigarrow$ morphism of Shimura data $(GU(V),X)\to(GSp(V),\mathcal{H})$.

Example: Define $h:\mathbb{C}^\times\to U(V_{\mathbb{R}})$, $z\mapsto\begin{cases} z/\bar z, & \text{on } W_0,\\ 1, & \text{on } W_1.\end{cases}$ This realizes $X\subseteq\mathrm{Hom}(\mathbb{S},U(V_{\mathbb{R}}))$ as $U(V_{\mathbb{R}})$-conj. class. In fact, $(U(V),X)$ is Shimura datum. Inclusion $U(V)\hookrightarrow GU(V)$ does <u>not</u> determine morphism of Shimura data.

$$\begin{array}{ccc} X & \hookrightarrow & \mathrm{Hom}(\mathbb{S},U(V_{\mathbb{R}})) \\ \| & & \downarrow \\ X & \hookrightarrow & \mathrm{Hom}(\mathbb{S},GU(V_{\mathbb{R}})) \end{array}$$

does <u>not</u> commute!

Let $K\subseteq U(\hat V)$, $K'\subseteq GU(\hat V)$ be compact opens s.t. $K\subseteq K'$ there is morphism of complex mflds

$$U(V)\backslash X\times U(\hat V)/K\to GU(V)\backslash X\times GU(\hat V)/K'.$$

Remark: Deligne says this is <u>not</u> nice map from POV of canon. models over # fields. [complex geom. is fine but not the number theory]

<u>0-dim Shimura Varieties</u>

T torus $/\mathbb{Q}$. Assume T has no $\mathbb{Q}$-rational subtorus w/ compact $\mathbb{R}$-pts. [This is needed for consistency but is somehow unnecessary.] Any $T(\mathbb{R})$-conj. class in $\mathrm{Hom}(\mathbb{S},T_{\mathbb{R}})$ is single pt. $\{h_0\}$ and $\forall\, K\subseteq T(\mathbb{A}_f)$ compact open: $Sh_K(T,\{h_0\})=T(\mathbb{Q})\backslash T(\mathbb{A}_f)/K$ is finite set. For <u>some</u> $h:\mathbb{S}\to T_{\mathbb{R}}$ this has a moduli interpretation.

Def: <u>CM field</u> is fin. ext. $E/\mathbb{Q}$ s.t. the following equiv. conditions (not obvious) hold.

(1) E is totally imaginary quadratic ext. of totally real field.

(2) $\exists\, c\in\mathrm{Aut}(E/\mathbb{Q})$ of order 2 s.t. $\forall\, i:E\hookrightarrow\mathbb{C}$: $i(c(x))=\overline{i(x)}$ $\forall x\in E$.

. E^c is maximal tot. real subfield of E.

Def: <u>CM type</u> of CM field E is subset $\Phi \subseteq \mathrm{Hom}(E,\mathbb{C})$ s.t. $\Phi \sqcup \overline{\Phi} = \mathrm{Hom}(E,\mathbb{C})$ (choose minimal set of rep.'s of conj. pairs)

Example: Quadratic imaginary field is CM and CM type is choice of complex embedding.

Example: $\mathbb{Q}(\mu_n)$ is CM field for $n>2$.

Example: Suppose $E \subseteq \mathbb{C}$ is finite Galois ext. of $\mathbb{Q}$ s.t. $E \not\subseteq \mathbb{R}$ and complex conj. of E lies in center of $\mathrm{Gal}(E/\mathbb{Q})$.

Then, E is CM.

Prop: A ab. var. $/\mathbb{C}$ of dim d. Suppose E is # field and $E \hookrightarrow \overbrace{\mathrm{End}^\circ(A)}^{\text{(isogeny endomorphisms)}} \cong \mathrm{End}(A) \otimes_{\mathbb{Z}} \mathbb{Q}$. Then, $[E:\mathbb{Q}] \leq 2d$.

Equality holds $\Rightarrow$ E is CM and $\exists!$ CM type $\underset{=\{\varphi_1,\ldots,\varphi_d\}}{\Phi} \subseteq \mathrm{Hom}(E,\mathbb{C})$ s.t. $\mathrm{Lie}(A) \cong \mathbb{C}^d$ w/ $\alpha \in E$ acting as

$\begin{pmatrix} \varphi_1(\alpha) & & 0 \\ & \ddots & \\ 0 & & \varphi_d(\alpha) \end{pmatrix}$. That is, $\mathrm{Lie}(A) \cong \bigoplus_{\varphi \in \Phi} \mathbb{C}(\varphi)$ as $E \otimes_{\mathbb{Q}} \mathbb{C}$-modules.

Remark: We say A has CM by (E,Φ).

Pf: $\mathrm{End}^\circ(A) \hookrightarrow \mathrm{End}_{\mathbb{Q}}(H_1(A,\mathbb{Q})) \cong M_{2d}(\mathbb{Q})$. So, $E \hookrightarrow \mathrm{End}^\circ(A) \Rightarrow \mathbb{Q}^{2d}$ is vec. space over $E \Rightarrow [E:\mathbb{Q}] \leq 2d$.

Equality holds $\Rightarrow H_1(A,\mathbb{Q})$ is 1-dim E-vec. space. So, $H_1(A,\mathbb{C}) \cong E \otimes_{\mathbb{Q}} \mathbb{C} \overset{(*)}{\cong} \bigoplus_{\varphi \in \mathrm{Hom}(E,\mathbb{C})} \mathbb{C}(\varphi)$. Using Hodge theory, write

$A = V/L$ w/ V d-dim $\mathbb{C}$-vec. space and $L \subseteq V$ a $\mathbb{Z}$-lattice of rank $2d$. $H_1(A,\mathbb{R}) = L \otimes_{\mathbb{Z}} \mathbb{R} = V$ has $\mathbb{C}$-structure.

So, $\mathbb{C} \otimes \mathbb{C} \cong \mathbb{C} \oplus \mathbb{C}$ acts on $H_1(A,\mathbb{C}) \rightsquigarrow H_1(A,\mathbb{C}) = \mathrm{Lie}(A) \oplus \overline{\mathrm{Lie}(A)}$.

$(*) \Rightarrow \mathrm{Lie}(A) \cong \bigoplus_{\varphi \in \Phi} \mathbb{C}(\varphi)$ w/ $\Phi \sqcup \overline{\Phi} = \mathrm{Hom}(E,\mathbb{C})$. In particular, E is totally imaginary.

<u>Hard Part</u>: E is CM field. [needs polarization!]