Consider $\mathcal{H} = \{ \mathbb{R}\text{-alg. maps } \mathbb{C} \to M_2(\mathbb{R}) \} \subseteq \mathrm{Hom}(\mathbb{S}, GL_{2,\mathbb{R}})$. Any $h \in \mathcal{H}$ determines $h : \mathbb{C}^\times \times \mathbb{C}^\times \to GL_2(\mathbb{C})$.

This is conjugate to $h(z,w) = \begin{pmatrix} z & 0 \\ 0 & w \end{pmatrix}$. So, Hodge cochar. of any $h \in \mathcal{H}$ is conjugate to $\mu(z) = \begin{pmatrix} z & 0 \\ 0 & 1 \end{pmatrix}$.

<u>Example</u>: Let $F$ be tot. real field and $B$ is quaternion alg. $/F$. Write $\mathrm{Hom}(F, \mathbb{R}) = S_0 \sqcup S_1$, where $v : F \to \mathbb{R}$

<u>Remark</u>: Best place to learn about quaternion alg.'s is in some book on 3-mflds.

Benson Farb – Brauer groups

$$B \underset{F,v}{\otimes} \mathbb{R} \cong \begin{cases} M_2(\mathbb{R}), & v \in S_0 \\ \mathbb{H}, & v \in S_1 \end{cases}$$ [has to do with split or non-split places]

[would be restriction of scalars if $B$ were comm.]

Let $G = B^\times$ viewed as alg. grp. $/\mathbb{Q}$. $G(R) = (B \underset{\mathbb{Q}}{\otimes} R)^\times$ ($R$ some $\mathbb{Q}$-alg.)

So, $G(\mathbb{R}) = (B \underset{\mathbb{Q}}{\otimes} \mathbb{R})^\times \cong \prod_{S_0} GL_2(\mathbb{R}) \times \prod_{S_1} \mathbb{H}^\times$. Let $X = \{ \mathbb{R}\text{-alg. maps } h : \mathbb{C} \to \prod_{S_0} M_2(\mathbb{R}) \} \cong \prod_{S_0} \mathcal{H}$.

Then, $X \subseteq \mathrm{Hom}(\mathbb{C}^\times, G(\mathbb{R})) = \prod_{S_0} GL_2(\mathbb{R}) \times \prod_{S_1} \mathbb{H}^\times$, w/ $h \mapsto (h, 1)$.

Hodge cochar. of any $h \in X \subseteq \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$ is $G(\mathbb{C})$-conj. to $\mu(z) = \Big( \underbrace{\begin{pmatrix} z & \\ & 1 \end{pmatrix}}_{S_0 \text{ copies}}, \underbrace{\begin{pmatrix} 1 & \\ & 1 \end{pmatrix}}_{S_1 \text{ copies}} \Big)$.

<u>Q</u>: What is field of definition of this conj. class?

$$\sigma \in \mathrm{Aut}(\mathbb{C}) \rightsquigarrow \mu^\sigma(z) = \underbrace{\sigma}_{\text{action of } \sigma \text{ on } G(\mathbb{C}) = (B \otimes_{\mathbb{Q}} \mathbb{C})^\times}(\mu(\overbrace{\sigma^{-1} z}^{\text{usual action on } \mathbb{C}}))$$

Action of $\sigma$ on $B \underset{\mathbb{Q}}{\otimes} \mathbb{C} \cong \prod_{v \in S_0} B \underset{F,v}{\otimes} \mathbb{C} \times \prod_{v \in S_1} B \underset{F,v}{\otimes} \mathbb{C}$ permutes the factors in the product.

So, $\mu^\sigma = \mu$ w/ factors permuted. Hence, $\mu^\sigma$ $G(\mathbb{C})$-conj. to $\mu$ iff $\sigma$ preserves decomposition $\mathrm{Hom}(F, \mathbb{R}) = S_0 \sqcup S_1$

iff $\sigma(S_0) = S_0$. So, $H := \{ \sigma \in \mathrm{Aut}(\mathbb{C}) : \sigma(S_0) = S_0 \} \Rightarrow E(G, X) = \mathbb{C}^H$.

**Prop**: $(G,X) \to (G',X')$ morphism of Shimura data $\Rightarrow E(G',X') \subseteq E(G,X)$.

**Pf**: Natural map $\mathcal{X}(\mathbb{C}) = G(\mathbb{C}) \backslash \mathrm{Hom}(\mathbb{G}_{m,\mathbb{C}}, G_{\mathbb{C}})$ is $\mathrm{Aut}(\mathbb{C})$-equivariant. So, fixed property passes to the image.

$$\begin{array}{c} \mathcal{X}(\mathbb{C}) = G(\mathbb{C}) \backslash \mathrm{Hom}(\mathbb{G}_{m,\mathbb{C}}, G_{\mathbb{C}}) \\ \downarrow \\ \mathcal{X}'(\mathbb{C}) = G'(\mathbb{C}) \backslash \mathrm{Hom}(\mathbb{G}_{m,\mathbb{C}}, G'_{\mathbb{C}}) \end{array}$$

$\square$

## Canonical Models

Let $(G,X)$ be Shimura datum, $K \leq G(\mathbb{A}_f)$ compact open.

**Def**: A point $(h: \mathbb{S} \to G_{\mathbb{R}}) \in X$ is special if $\exists$ torus $T \subseteq G$ (over $\mathbb{Q}$) s.t. $h$ factors ~~as~~ through (image of) $T_{\mathbb{R}} \subseteq G_{\mathbb{R}}$.

A point $[h,g] \in G(\mathbb{Q}) \backslash X \times G(\mathbb{A}_f)/K$ is special if $h$ is special. [This is related to map from 0-dim Shimura variety, which ~~and~~ carries some Galois action.]

**Remark**: $h \in X$ special $\Rightarrow h: \mathbb{S} \to G(\mathbb{R})$ factors through $T(\mathbb{R}) \subseteq G(\mathbb{R})$ for some torus $T \subseteq G$. Given any $\gamma \in G(\mathbb{Q})$, conjugate ~~this~~ map $(\gamma h)(z) = \gamma h(z) \gamma^{-1}$ factors through $\gamma T \gamma^{-1} \Rightarrow \gamma h$ special [so "specialness" is well-defined].

**Example**: Let $V$ be 2-dim $\mathbb{Q}$-vec. space, $G = GL(V)$, $\mathcal{H} = \{\mathbb{R}\text{-alg. maps } \mathbb{C} \to \mathrm{End}(V_{\mathbb{R}})\} \subseteq \mathrm{Hom}(\mathbb{S}, G_{\mathbb{R}})$.

**Q**: What are the special pts. of $\mathcal{H}$?

List all tori $T \subseteq G$. Embed $E \subseteq \mathrm{End}(V)$ and take $T = E^\times \subseteq G$ where $E \cong \begin{cases} \mathbb{Q} \\ \mathbb{Q} \times \mathbb{Q} \\ \text{real quad. field} \\ \text{imag. quad. field} \end{cases}$

(**Remark**: Look at image of $E$ in $\mathrm{End}(V)$ and take its centralizer.)

$\Rightarrow T(\mathbb{R}) \cong \begin{cases} \mathbb{R}^\times \\ \mathbb{R}^\times \times \mathbb{R}^\times \\ \mathbb{R}^\times \times \mathbb{R}^\times \\ \mathbb{C} \end{cases}$ $\leftarrow$ (only this one allowed for map from Deligne's torus)

So, special pt. $h \in \mathcal{H}$ must factor as $h: \mathbb{C}^\times \to (E \otimes_{\mathbb{Q}} \mathbb{R})^\times \subseteq GL(V_{\mathbb{R}})$ for some quad. imag. $E \subseteq \mathrm{End}(V)$.

$\Rightarrow$ Action of $\mathbb{C}$ on $V_{\mathbb{R}}$ induced by $h$ commutes w/ action of $E$ on $V$. That is, $V$ has CM in the sense that, given any $\mathbb{Z}$-lattice $L \subseteq V$, the elliptic curve $V_{\mathbb{R}}/L$ has action of $\mathcal{O} = \{\alpha \in E : \alpha L \subseteq L\} \subseteq E$.

[$V_{\mathbb{R}}/L$ has CM by $\mathcal{O}$.]