Let X be smooth compact complex d-mfld. <u>Current</u> $T \in D^{p,q}(X)$ is $\mathbb{C}$-linear form $T : A^{d-p,d-q}(X) \to \mathbb{C}$

cont. for the Schwartz top. (top. dual of appropriate space of forms)

<u>Example</u>: (1) Let $\eta \in L^1(X) \otimes_{C^\infty(X)} A^{p,q}(X)$ be integrable diff. We get current $\eta(\omega) := \int_X \eta \wedge \omega$.

(2) $Z = \sum_\alpha n_\alpha Z_\alpha$ codim p cycle on X $\rightsquigarrow$ <u>Dirac current</u> $\delta_Z \in D^{p,p}(X)$ via

$\delta_Z(\omega) := \sum_\alpha n_\alpha \int_{Z_\alpha} \omega$. This converges because...

$T \in D^{p,q}(X) \rightsquigarrow \partial T(\omega) := (-1)^{p+q+1} T(\partial \omega)$

$\bar{\partial} T(\omega) := (-1)^{p+q+1} T(\bar{\partial}\omega)$

$$\begin{array}{ccc} D^{p,q}(X) & \xrightarrow{\partial} & D^{p+1,q}(X) \\ \uparrow & \circlearrowleft & \uparrow \\ A^{p,q}(X) & \xrightarrow[\partial]{} & A^{p+1,q}(X) \end{array}$$

<u>Prop 3.9</u> (Poincaré–Lelong): $\bar{L}$ hermitian line bundle $/X$, s meromorphic section of L. In $D^{1,1}(X)$ we have:

$$dd^c(-\log\|s\|^2) + \delta_{\mathrm{div}(s)} = c_1(\bar{L}).$$

Let X be $(*)$ and $p \in \mathbb{Z}^{\geq 0}$. <u>Green current</u> for $Z \in Z^p(X)$ is real current $g \in D^{p-1,p-1}(X(\mathbb{C}))$ s.t.

$F_\infty^* g \overset{(-1)^{p-1}}{=} g$ and $dd^c g + \delta_Z = \omega$ for some $\omega \in A^{p,p}(X(\mathbb{C}))$. This induces additive grp. $\hat{Z}^p(X)$.

Let $Y \subseteq X$ be closed irred. of codim $p-1$. Choose rational function $f \in K(Y)$. Define $\log|f|^2 \in D^{p-1,p-1}(X(\mathbb{C}))$ via:

$(\log|f|^2)(\omega) := \int_{Y(\mathbb{C})} \log|f|^2 \omega$. This can be viewed as rational section of trivial line bundle on Y.

Poincaré–Lelong $\Rightarrow dd^c(-\log|f|^2) + \delta_{\mathrm{div}(f)} = 0 \Rightarrow \widehat{\mathrm{div}}(f) := (\mathrm{div}(f), -\log|f|^2) \in \hat{Z}^p(X)$.

Also note: $u \in D^{p-2,p-1}(X(\mathbb{C}))$, $v \in D^{p-1,p-2}(X(\mathbb{C})) \Rightarrow dd^c(\partial u + \bar{\partial} v) = 0$.

Why care about these two relations?

X sm. proj. complex var., Y analytic subvar. of X not containing any irred. components of X

<u>Def</u>: Smooth form α on $X-Y$ is of <u>logarithmic type</u> along Y if $\exists$ proj. map $\pi : \tilde{X} \to X$ s.t.

- $E := \pi^{-1}(Y)$ is divisor w/ normal crossings;
- $\pi : \tilde{X} - E \to X - Y$ is smooth;
- α is direct image by π of form β on $\tilde{X} - E$ w/ property (*).

$E = \{(z_1, \ldots, z_k) : z_1 \cdots z_k = 0\}$ locally

(*) Locally, $\beta = \sum_{i=1}^{k} \alpha_i \log|z_i|^2 + \gamma$ w/

- z_i defining local equation for E
- α_i ∂- and $\bar{\partial}$-closed forms
- γ smooth form

<u>Observation</u>: α of logarithmic type along Y $\Rightarrow$ α loc. integrable on X $\Rightarrow$ $\alpha \rightsquigarrow$ current $[\alpha]$ = direct image of $[\beta]$ by π

<u>Lemma 2</u>: (i) Assuming components are nice, logarithmic type condition preserved by pullback. [29]

(ii) Assuming some component conditions, logarithmic type condition preserved by pushforward by projective morphisms.

<u>Thm 3</u>: $Y \subseteq X$ irred. subvar. $\Rightarrow$ smooth form g_Y on $X-Y$ of logarithmic type along Y s.t. $[g_Y]$ is Green current for Y.

<u>Lemma 3</u>: $\exists$ closed form $\alpha \in A^{d-1,d-1}(W)$ s.t. $\pi_*(\delta_E \wedge [\alpha]) = \delta_P$. [Notation from proof of thm]

How do we prove Thm 3?

(1) Suppose Y is a divisor on X. Then, $\exists$ line bundle L on X w/ herm. metric $\|\cdot\|$ and section s s.t.

$Y = \mathrm{div}(s)$. $g_Y := -\log\|s\|^2 \overset{\text{(Poincaré–Lelong)}}{\Rightarrow} dd^c[g_Y] + \delta_Y = [c_1(L, \|\cdot\|)]$. We see that g_Y has logarithmic type along Y by making Y divisor w/ normal crossings.

$X \to \operatorname{Spec} \mathbb{Z}$ regular, projective, flat (= <u>arithmetic scheme</u> in <u>my</u> language)

~~[illegible]~~ $X(\mathbb{C}) \circlearrowleft F_\infty$ cont. involution via complex conj.

<u>Thm 1</u>: We have exact seq's :

Work of Burgos Gil contextualizes these seq's ...

(1) $CH^{p-1,p}(X) \xrightarrow{\rho} H^{p-1,p-1}(X) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{(\zeta,\omega)} CH^p(X) \oplus Z^{p,p}(X) \xrightarrow{cl} H^{p,p}(X) \to 0$

(2) $CH^{p-1,p}(X) \xrightarrow{\rho} \tilde{A}^{p-1,p-1}(X) \xrightarrow{a} \widehat{CH}^p(X) \xrightarrow{\zeta} CH^p(X) \to 0$

$Z^p_{fin}(X) := \{ Z \in Z^p(X) : \overbrace{|Z|}^{\text{support}} \cap X_{\mathbb{Q}} = \emptyset \}$

(All auxiliary considerations used in rationalizing the picture.)

$CH^p_{fin}(X) := Z^p_{fin}(X) / \langle \operatorname{div} f \rangle$ for $f \in k(y)^\times$ w/ $y \in X^{(p-1)} - X_{\mathbb{Q}}$

<u>Fact</u>: Any $Z \in Z^p(X)$ decomposes uniquely as $Z_1 + Z_2$ w/ $Z_1 \in Z^p_{fin}(X)$ and $Z_2 \in Z^p(X_{\mathbb{Q}})$.

$\hat{Z}^p(X_{\mathbb{Q}}) := \{ (Z, g_Z) : Z \in Z^p(X_{\mathbb{Q}}),\ g_Z \text{ Green current for } Z \}$

<u>Thm 2</u>: $\exists$ pairing $\widehat{CH}^p(X) \otimes \widehat{CH}^q(X) \to \widehat{CH}^{p+q}(X)_{\mathbb{Q}}$ s.t.

(i) $\bigoplus_{p \geq 0} \widehat{CH}^p(X)_{\mathbb{Q}}$ is comm. graded unital $\mathbb{Q}$-alg. ;

(ii) (ζ, ω) : $\bigoplus_{p\geq 0} \widehat{CH}^p(X)_{\mathbb{Q}} \to \bigoplus_{p \geq 0} (CH^p(X) \oplus Z^{p,p}(X))_{\mathbb{Q}}$ is map of $\mathbb{Q}$-alg's.

The remarks (§2.3) after the proof are really valuable. If we can avoid ~~[illegible]~~ requiring that our cycles meet properly then we may not need to tensor up to $\mathbb{Q}$.

Thm 3 : Assume $Y/\operatorname{Spec}\mathbb{Z}$ is also regular, proj., flat. Let $f: Y \to X$ be any map of schemes.

$$\overbrace{\text{mult.}}^{f^*(\alpha\cdot\beta) = f^*(\alpha)\cdot f^*(\beta)}$$

(i) $\exists$ well-defined mult. pullback $f^* : \widehat{CH}^p(X) \to \widehat{CH}^p(Y)_{\mathbb{Q}}$. [In fact, this can be done without rationalizing.]

(ii) f proper, $f_{\mathbb{Q}} : Y_{\mathbb{Q}} \to X_{\mathbb{Q}}$ smooth, X, Y equidimensional $\Rightarrow$ $\exists$ well-defined pushforward

$$f_* : \widehat{CH}^p(Y) \to \widehat{CH}^{p-\delta}(X) \quad \text{for } \delta := \dim Y - \dim X.$$

(iii) We have projection formula $f_*(f^*(\alpha)\cdot\beta) = \alpha\cdot f_*(\beta) \in \widehat{CH}^{p+q-\delta}(X)_{\mathbb{Q}}$ $\forall\, \alpha \in \widehat{CH}^p(X)$, $\beta \in \widehat{CH}^q(Y)$.

(iv) Pullback and pushforward are functorial when defined.

[40]

Prop 1 : $\hat{c}_1 : \widehat{Pic}(X) \xrightarrow{\sim} \widehat{CH}^1(X)$, isom. class of $(L, \|\cdot\|) \mapsto$ equiv. class of $(\operatorname{div}(s), -[\log\|s\|^2])$.

s any choice of rational section of L

$\widehat{\deg}$ and Beilinson regulator business... Obviously we want to apply derived perspective to Faltings height.

Let X be arith. scheme and ω_0 kähler metric on $X(\mathbb{C})$ inv. under F_∞. $\overline{X} := (X, \omega_0)$ is Arakelov variety.

Hodge decomp. $\Rightarrow$ $A^{p,p}(X) = \mathcal{H}^{p,p}(X) \oplus \operatorname{im} d \oplus \operatorname{im} d^*$ $\rightsquigarrow$ notion of Arakelov Chow group. [43]

$\widehat{\mathrm{Pic}}(X) :=$ grp. of isom. classes of hermitian line bundles / X

$[\bar{L}] \in \widehat{\mathrm{Pic}}(X)$, s nonvanishing rational section $\rightsquigarrow$ $\widehat{\mathrm{div}}(s) := (\mathrm{div}(s), -\log\|s\|^2) \in \hat{Z}^1(X)$

$\hat{c}_1(\bar{L}) :=$ class of $\widehat{\mathrm{div}}(s)$ in $\widehat{CH}^1(X)$.

<u>Prop 4.4</u>: $\hat{c}_1 : \widehat{\mathrm{Pic}}(X) \xrightarrow{\sim} \widehat{CH}^1(X)$.

<u>Thm 4.5</u>: $\bigoplus_{p \geq 0} \widehat{CH}^p(X)_{\mathbb{Q}}$ has intersection pairing $(\cdot)$ making it a comm. graded $\mathbb{Q}$-alg.

$\zeta : \widehat{CH}^p(X) \to CH^p(X)$ forgetful, $\omega : \widehat{CH}^p(X) \to A^{p,p}(X)$, given by $(Z, g) \mapsto dd^c g + \delta_Z$.

<u>Facts</u>: $\zeta(x \cdot y) = \zeta(x)\zeta(y)$ and $\omega(x \cdot y) = \omega(x)\omega(y)$.

One difficulty that arises is defining products of currents.

Let $f : X \to \mathrm{Spec}\,\mathbb{Z}$ be proper flat. Note that image of integral closed subscheme is integral closed.

- $f(Z) = \mathrm{Spec}\,\mathbb{Z}$: Z is <u>horizontal</u>
- $f(Z) = \{p\}$: Z is <u>vertical</u>

Exact seq.: $\Gamma(X, \mathcal{O}_X^\times) \to C^\infty(X(\mathbb{C}), \mathbb{R}) \xrightarrow{\text{}} \widehat{\mathrm{Pic}}(X) \xrightarrow{\text{(forgetful)}} \mathrm{Pic}(X) \to 0$

$(f \mapsto \log|f|)$ $\quad$ $(\phi \mapsto$ triv. line bundle $\mathcal{O}_X$ w/ herm. metric for which $\log\|1\|^{-1} = \phi)$

How do we get (Faltings) height? X reg. proj. flat scheme $/\mathbb{Z}$, $\bar{L}$ hermitian line bundle $/X$

$Y \subseteq X$ integral closed $\rightsquigarrow$ $h_{\bar{L}}(Y) \in \mathbb{R}$

Let $A \in \mathrm{CRing}$ be int. dom. of Krull dim 1. $K := \mathrm{Frac}(A)$

$\rightsquigarrow \mathrm{ord}_A : K - \{0\} \to \mathbb{Z}$, $a/b \mapsto \ell_A(A/aA) - \ell_A(A/bA)$ (specializes to usual ord for DVR)

X analytic smooth mfld $/\mathbb{C}$ w/ sheaf $\mathcal{O}_{X,an}$. Let L be holomorphic line bundle $/X$. Metric on L is data of $\|\cdot\| : L(x) \to \mathbb{R}_+$ for fibers $L(x) = L_x/m_x$ s.t.

(i) $\|\lambda s\| = |\lambda| \cdot \|s\|$ $\forall \lambda \in \mathbb{C}$;
(ii) $\|s\| = 0$ iff $s = 0$;
(iii) $\mathcal{U} \subseteq X$ open and $s \in \Gamma(\mathcal{U}, L)$ vanishing nowhere $\Rightarrow$ $x \mapsto \|s(x)\|^2$ is C^∞. $\rightsquigarrow \bar{L} := (L, \|\cdot\|)$

$A^n(X) :=$ $\mathbb{C}$-vec. space of C^∞ deg n complex diff. forms $/X$ $\Rightarrow A^n(X) = \bigoplus_{p+q=n} A^{p,q}(X)$

$d = \partial + \bar{\partial}$, $d : A^n(X) \to A^{n+1}(X)$, $\partial : A^{p,q}(X) \to A^{p+1,q}(X)$, $\bar{\partial} : A^{p,q}(X) \to A^{p,q+1}(X)$.

$\partial^2 = \bar{\partial}^2 = d^2 = 0$. $d^c := \dfrac{\partial - \bar{\partial}}{4\pi i}$ $\rightsquigarrow dd^c = \dfrac{\bar{\partial}\partial}{2\pi i}$

Lemma 2.11: $\exists\, c_1(\bar{L}) \in A^{1,1}(X)$ s.t. $\forall$ nonvanishing $s \in \Gamma(\mathcal{U}, L)$: $\overbrace{c_1(\bar{L})}^{\text{(first Chern form)}}|_{\mathcal{U}} = -dd^c \log\|s\|^2$.

As before let X be reg. flat proj. scheme $/\mathbb{Z}$ (*): $X(\mathbb{C})$ is complex mfld.

Hermitian line bundle on X is $\bar{L} = (L, \|\cdot\|)$ w/ L line bundle $/X$ and $\|\cdot\|$ metric on $L_{\mathbb{C}} := L|_{X(\mathbb{C})}$.

Assume $\|\cdot\|$ invariant under complex conj. $F_\infty : X(\mathbb{C}) \to X(\mathbb{C})$. Define $c_1(\bar{L}) := c_1(\bar{L}_{\mathbb{C}}) \in A^{1,1}(X(\mathbb{C}))$.