Virtual intersection theories

Zachary Gardner

zachary.gardner@bc.edu

Our goal is to describe a general context for intersection theories and virtual fundamental classes, mostly following the axiomatic framework of Kiem and Park. We start with a mostly technical consideration. Let St_k denote the (2-)category of qs LFT algebraic stacks over k, for k a field of characteristic zero (which need not be algebraically closed). We fix an *admissible* full (2-)subcategory $V \subseteq \mathrm{St}_k$, which is by definition required to satisfy the following properties.

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Good examples of admissible categories are St_k as well as the categories Sch_k of LFT k-schemes and QSch_k its full subcategory of qs schemes (viewing both as 2-categories in the trivial way).

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Note that the context of more narrow theories we also have access to more general *proper* pushforwards and *flat* pullbacks.

[1] Here, $X \times Y$ is the product formed in V and so is naturally fibered over $\operatorname{Spec} k$.

- *Refined Gysin pullback*: For a Cartesian square

$$\begin{tikzcd} X' \arrow[r, "f'"] \arrow[d, "g'"'] & Y' \arrow[d, "g"] \\ X \arrow[r, "f"'] & Y \end{tikzcd}$$

  with f a regular immersion of constant codimension c, we have a graded map

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- *Exterior product*: For $X, Y \in V$, we have a bilinear graded map

$$\times : H_*(X) \otimes_{\mathbb{Z}} H_*(Y) \to H_*(X \times Y)$$

  which is commutative and associative with a distinguished element $1 \in H_0(\operatorname{Spec} k)$ as unit.[1]

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- Given $X, Y \in V$, there are obvious maps $i : X \to X \sqcup Y$ and $j : Y \to X \sqcup Y$ and the induced map

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The other conditions are more mysterious or at least require more geometric context. But before that, some examples!

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Given $X \in V$, endow $G_0(X)$ with a grading by considering

$$G_0(X) \otimes_{\mathbb{Z}} \mathbb{Z}[\beta^{\pm 1}] \cong \bigoplus_{n \in \mathbb{Z}} G_0(X) \cdot \beta^n$$

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  This is commutative and associative with a distinguished element $1 \in G_0(\operatorname{Spec} k) \cong \mathbb{Z}$ as unit. Note that, by definition, $E \boxtimes E'$ is given by the tensor product $\mathrm{pr}_X^{-1} E \otimes_{\mathcal{O}_{X \times Y}} \mathrm{pr}_Y^{-1} E'$ for $\mathrm{pr}_X : X \times Y \to X$ and $\mathrm{pr}_Y : X \times Y \to Y$ the projection maps.

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Our second and arguably most important example comes from Chow theory. For simplicity, let Sch denote the category of "nice" schemes.[2] We will try to do things carefully to clarify some things and make our lives easier in the future. In particular, we will be careful about matters of (co-)dimension as well as rational equivalence.

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- The closed integral subschemes of X correspond bijectively with the (topological, field-valued) points of X. This is given by looking at generic points ζ and we have

$$Z(X) \cong \bigoplus_{\zeta \in X} \mathbb{Z}.$$

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We define the sets $X^{(p)}$ and $X_{(p)}$ so that $Z^p(X) \cong \bigoplus_{x \in X^{(p)}} \mathbb{Z}$ and $Z_p(X) \cong \bigoplus_{x \in X_{(p)}} \mathbb{Z}$ – these are useful for bookkeeping purposes.

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For general X, we have the sheaf $\mathcal{O}_X^{\mathrm{reg}}$ of regular elements (which can be defined at the level of stalks). To this we associate $\mathcal{K}_X := \mathcal{O}_X[(\mathcal{O}_X^{\mathrm{reg}})^{-1}]$ and $\mathcal{D}\mathrm{iv}_X := \mathcal{K}_X^\times / \mathcal{O}_X^\times$. These give rise to the additive group of *Cartier divisors* $\mathrm{Div}(X) := H^0(X, \mathcal{D}\mathrm{iv}_X)$.

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Define $R(X) := \bigoplus_{\zeta \in X} k(\zeta)^\times$, which can be graded by (co-)dimension as appropriate. The amalgamation of $\mathrm{div} : K(Z)^\times \to Z^1(Z)$ for $Z \subseteq X$ closed integral induces $\mathrm{div} : R(X) \to Z(X)$ and we set $\mathrm{CH}(X) := \mathrm{coker}(\mathrm{div} : R(X) \to Z(X))$. Being *rationally equivalent to zero* is captured by the image of div.

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Note that there is another equivalent approach to rational equivalence that works for varieties over a field. Heuristically, two cycles α, β are rationally equivalent if there is a family of cycles $\{\zeta_t\}_{t \in \mathbb{P}^1}$ with $\zeta_0 = \alpha$ and $\zeta_\infty = \beta$. This can be made more precise using the language of flat families.

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Both of these constructions are invariant under rational equivalence and so descend to maps of Chow groups.

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We ultimately want to grade by dimension and this affects pullbacks. If $f : X \to Y$ is smooth of constant relative dimension e then we have

$$f^* : \mathrm{CH}_*(Y) \to \mathrm{CH}_{*+e}(X), \qquad [\eta] \mapsto [f^{-1}(\eta)]$$

and this construction is functorial.

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The only thing left to describe for Chow theory is the refined Gysin pullback. This is a bit subtle and so we will come back to it later.

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4. *Projection formula: Let* $f : X \to Y$ *be proper with* $X, Y \in \mathrm{Sm}_k$, $\alpha \in \mathrm{CH}^*(X)$, *and* $\beta \in \mathrm{CH}^*(Y)$. *Then,* $f_*(\alpha \cdot f^*(\beta)) = f_*(\alpha) \cdot \beta$.

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5. *Homotopy invariance: Let $p : V \to X$ be a smooth vector bundle. Then, $p^* : \mathrm{CH}^*(X) \xrightarrow{\sim} \mathrm{CH}^*(V)$.*

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Returning to our intersection theory H_* for V, what are the remaining properties we want to impose? First, we have the base change condition which says that if

$$\begin{tikzcd} X \arrow[r, "g"] \arrow[d, "q"'] & Y \arrow[d, "p"] \\ Z \arrow[r, "f"'] & W \end{tikzcd}$$

is a Cartesian square with f projective and p smooth then $p^* \circ f_* = g_* \circ q^*$.

Consider the following behemoth commutative diagram

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X'' \arrow[r, "f''"] \arrow[d, "g''"'] & Y'' \arrow[r] \arrow[d, "g'"] & S \arrow[d, "g"] \\
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- If g is a regular immersion then we demand $f^! \circ g^! = g^! \circ f^!$.

Suppose that f' is also a regular immersion, and let $E := h^* N_{X/Y}/N_{X'/Y'}$ be the *excess normal bundle*.

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I believe this construction is closely linked to Grothendieck-Riemann-Roch but have not looked into the details. More on this later...

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The latter construction of η is linked to other constructions. For L a line bundle on X, the zero section $0 : X \to L$ defines the *first Chern class* homomorphism

$$c_1(L) := 0^! \circ 0_* : H_*(X) \to H_{*-1}(X),$$

which you should check is well defined in the sense that 0 is a projective regular immersion.

Given $i : D = s^{-1}(0) \hookrightarrow X$ an effective Cartier divisor (for s some nonzero section of a line bundle on X), we have the *divisor intersection product*

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Let's round out our discussion of properties of H_*.

- *Excision sequence*: Given a closed immersion $i : Z \hookrightarrow X$ with complement $j : U \hookrightarrow X$, we demand that the sequence

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$$\bigoplus_{i=0}^{r-1} H_{*-r+1+i}(X) \to H_*(\mathbb{P}(E)), \qquad (\xi_i) \mapsto \sum_i c_1(\mathcal{O}_{\mathbb{P}(E)}(1))^i \cdot (p^*\xi_i)$$

is an isomorphism.

[4] This should remind you of the theory of quasi-smooth morphisms.

- Let $f : X \to Y$ be an lci morphism of constant relative dimension d and choose a factorization $f = h \circ g$ with $g : X \to Z$ a regular closed immersion and $h : Z \to Y$ smooth.

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