Approaches to AG

1. Classical (algebraic sets)
2. Semi-Modern (locally ringed spaces)
3. Modern (functor-of-points)

1. Classical

($k^n = \mathbb{A}^n_k$ affine n-space)

Pros:
- Easy to define
- Geometrically clear (some things)

Cons:
- Not expensive enough
- Geometrically poor

2. Semi-Modern

Pros:
- Borrows intuition from manifold theory
- Manifold theory (and generalizations) make sense in the language of ringed spaces
- Very general

- Bad categorically

Cons:

- Topology sucks!
- Can obscure the geometry
- Not very algebraic

3. Modern

Cons:

- Very abstract (category theory!)
- Harder to borrow intuition from manifold theory

Pros: We talked about it...

Classical

$\mathbb{A}^n_k = k^n$ affine n-space (simplest example of an affine variety)

$k[x_1, \dots, x_n]/(f_1, \dots, f_r)$

$I \trianglelefteq k[x_1, \dots, x_n]$ ideal

$\rightsquigarrow V(I) = \{\alpha \in k^n : f(\alpha) = 0 \ \forall f \in I\}$

$S \subseteq k^n$

$\rightsquigarrow I(S) = \{ f \in k[x_1, \dots, x_n] ; f(S) = 0 \}$

$\trianglelefteq k[x_1, \dots, x_n]$

<u>Fact</u>: $\{ V(I) \}_I$ is a topology on k^n.

$V(I) \cup V(J) = V(IJ)$

$V(I) \cap V(J) = V(I+J)$

<u>Semi-Modern</u>

Given $A \in \mathrm{CRing}$, let $\operatorname{Spec} A := \{ \mathfrak{p} \trianglelefteq A : \mathfrak{p} \text{ is prime} \}$.

$V(I) = \{ \mathfrak{p} \in \operatorname{Spec} A : I \subseteq \mathfrak{p} \}$

$V((0)) = \operatorname{Spec} A$

<u>Fact</u>: (prime) ideals of $A/I \longleftrightarrow$ (prime) ideals of A containing I

$V(I) \cong \operatorname{Spec} A/I$

$\operatorname{Spec} A \setminus V(I) = D(I)$ nonvanishing locus

if $I = fA$ then $D(f) \cong \operatorname{Spec} A_f$

$$B \xrightarrow{\varphi} A \rightsquigarrow \operatorname{Spec} A \xrightarrow{\varphi^* = \operatorname{Spec}\varphi} \operatorname{Spec} B$$

$$\mathfrak{p} \longmapsto \varphi^{-1}(\mathfrak{p})$$

<u>Fact</u>: φ^* is continuous for the Zariski topology.

<u>Modern</u>

<u>Q</u>: What if we take the above as our starting point?

<u>Def</u>: The category of <u>spaces</u> is $\mathrm{Space} := \mathrm{Fun}(\mathrm{CRing}, \mathrm{Set})$.

$X \in \mathrm{Space}$ $\qquad A \longmapsto X(A) \in \mathrm{Set}$

$$\begin{tikzcd} A \arrow[d] & \longmapsto & X(A) \arrow[d] \arrow[r] & Y(A) \arrow[d] \\ B & & X(B) \arrow[r] & Y(B) \end{tikzcd}$$

An <u>affine scheme</u> is a space $\operatorname{Spec} A \in \mathrm{Space}$ given by

$$(\operatorname{Spec} A)(B) = \operatorname{Hom}_{\mathrm{CRing}}(A, B).$$

$B \to C$, $A \to B$ $\rightsquigarrow$ $A \to B \to C$

These span a full subcategory AffSch $\subseteq$ Space.

$V(I) := \mathrm{Spec}(A/I)$

Examples: • $A \mapsto A^\times$

• $A \mapsto A$

• $A \mapsto M_n(A)$

• $A \mapsto GL_n(A)$

complete
cocomplete

$\downarrow$

Why are spaces nice? Because Set is nice!

Fact: All limits are built from products and equalizers.

In Set, $\quad \mathrm{eq}\left(X \underset{g}{\overset{f}{\rightrightarrows}} Y\right) = \{x \in X : f(x) = g(x)\}$,

In Ab, $\quad \mathrm{eq}\left(X \underset{g}{\overset{f}{\rightrightarrows}} Y\right) = \ker(f-g)$.

Space (or really any $\mathrm{Fun}(\mathcal{C}, \mathrm{Set})$) inherits (co-)limits from Set, by evaluation.

$$\underbrace{(X \times Y)}_{\text{in Space}}(A) = \underbrace{X(A) \times Y(A)}_{\text{in Set}} = \{(x,y) : x \in X(A), y \in Y(A)\}$$

<u>Yoneda's lemma</u>: $\mathcal{C}$ category

$$\mathcal{C} \to \mathrm{Fun}(\mathcal{C}^{op}, \mathrm{Set}) = \mathcal{P}(\mathcal{C})$$

Yoneda functor

$$X \mapsto \mathrm{Hom}_{\mathcal{C}}(\cdot, X) = h^X$$

is an embedding.

$$\mathrm{Hom}_{\mathcal{C}}(X, Y) \xrightarrow{\sim} \mathrm{Hom}_{\mathcal{P}(\mathcal{C})}(h^X, h^Y)$$

Injectivity ~ faithful

Surjectivity ~ full

$\mathrm{CRing} \hookrightarrow \mathcal{P}(\mathrm{CRing}) = \mathrm{Fun}(\mathrm{CRing}^{op}, \mathrm{Set})$

$\mathrm{CRing}^{op} \hookrightarrow \mathrm{Space}$

$\Vert$ (semi-modern)

AffSch