Connected Shimura Data

Given G semisimple alg. gp. (over field), $G^{ad} := G/Z(G)$ ~~[illegible]~~ is adjoint gp.

Def: <u>connected Shimura datum</u> is pair (G, D) consisting of semisimple gp. G over $\mathbb{Q}$ and $G^{ad}(\mathbb{R})^+$-conj. class.

$D \subseteq \mathrm{Hom}(U_1, G^{ad}(\mathbb{R}))$ s.t. $\forall u \in D$:

(1) only the characters $t \mapsto t^{-1}$, $t \mapsto 1$, $t \mapsto t$ appear in the action $Ad(u) \to GL(\mathfrak{g}_{\mathbb{R}}) \to GL(\mathfrak{g}_{\mathbb{C}})$. $\qquad \mathfrak{g} = \mathrm{Lie}\, G$ is $\mathbb{Q}$-Lie alg.

(2) $Ad(u)$ is Cartan involution of $\mathfrak{g}_{\mathbb{R}}$. [instead of, say, over $\mathbb{R}$ (for emphasis)]

(3) G^{ad} has no simple factor H (defined over $\mathbb{Q}$) s.t. $H(\mathbb{R})$ is compact.

(1)+(2)+(3) $\Rightarrow$ D is HSD.

Remark: Recall where the $\mathbb{C}$-structure on D comes from. $x = u_x \in D \rightsquigarrow G^{ad}(\mathbb{R})^+/K_x \xrightarrow{\sim} D$ as real mfld's.

Cartan involution $\theta = Ad(u)(-1) \in GL(\mathfrak{g}_{\mathbb{R}}) \rightsquigarrow$ splitting $\mathfrak{g}_{\mathbb{R}} = \underbrace{\mathfrak{k}_x}_{\theta = +1} \oplus \underbrace{T_x D}_{\theta = -1}$.

$\Rightarrow \mathfrak{g}_{\mathbb{R}}/\mathfrak{k}_x \xrightarrow{\sim} T_x D$.

(1) $\Rightarrow$ $T_x D$ stable under $Ad(u)$. Hence, $\exists!$ $\mathbb{C}$-structure on $T_x D$ s.t. action of U_1 is via the inclusion $U_1 \hookrightarrow \mathbb{C}^\times$. So, we get almost complex mfld which is in fact complex mfld (have to check some integrability condition).

Remark: In fact, it's much more than integrability. We work w/ filtered vector bundles, ultimately using Griffiths transversality.

Example: $G = SL_2$ (over $\mathbb{Q}$), $D \subseteq \mathrm{Hom}(U_1, PSL_2(\mathbb{R}))$ is $PSL_2(\mathbb{R})$-conj. class containing $u(t) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in PSL_2(\mathbb{R})$ where $t = (a+bi)^2$.

More natural to look at $h: \mathbb{C}^\times \to GL_2(\mathbb{R})$, $a+bi \mapsto \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$ so that

$$\begin{array}{ccccc} & \mathbb{S}(\mathbb{R}) = \mathbb{C}^\times & \xrightarrow{h} & GL_2(\mathbb{R}) \\ & \downarrow & & \downarrow \\ t & U_1 & \xrightarrow{h} & SL_2(\mathbb{R}) \\ \downarrow & \downarrow & & \\ t^2 & U_1 & \xrightarrow{h} & PSL_2(\mathbb{R}) \end{array}$$

more on this later...

Def: G alg. grp. / $\mathbb{Q}$. Subgrps. $\Gamma_1, \Gamma_2 \subseteq G(\mathbb{Q})$ are <u>commensurable</u> if $\Gamma_1 \cap \Gamma_2$ has finite index in Γ_1 and Γ_2. $\Gamma \leq G(\mathbb{Q})$ is <u>arithmetic</u> if $\exists$ faithful rep. $G \hookrightarrow GL_n$ s.t. Γ is commensurable w/ $G(\mathbb{Q}) \cap GL_n(\mathbb{Z})$. [This is indep. of choice of faithful rep.]. $\Gamma \leq G(\mathbb{Q})$ is <u>congruence subgrp.</u> if $\exists N$ s.t. $G(\mathbb{Q}) \cap \{g \in GL_n(\mathbb{Z}) : g \equiv I_n \bmod N\}$ has finite index in Γ.

Thm (Baily-Borel): (G,D) conn. Shimura datum. Suppose $\Gamma \subseteq G^{ad}(\mathbb{Q})^+ := G^{ad}(\mathbb{Q}) \cap G^{ad}(\mathbb{R})^+$ is torsion-free arithmetic subgp. $D(\Gamma) := \Gamma \backslash D$ has canon. structure of open subvar. of proj. var. $D(\Gamma)^*$ over $\mathbb{C}$.

Remark: $D(\Gamma)^*$ is Baily-Borel-(Satake) compactification.

Modular Curves

$(G,D) = (SL_2, \mathcal{H}^+)$. Let $\Gamma(N) := \{ g \in SL_2(\mathbb{Z}) : g \equiv I_2 \bmod N \}$. Form $\Gamma(N) \backslash \mathcal{H}^+$. We also have moduli space $Y(N)$ (over $\mathbb{Q}$) parametrizing elliptic curves $E \to \underset{\mathbb{Q}\text{-scheme}}{\underbrace{S}}$ together w/ full level N structure. $(\underline{\mathbb{Z}/N})^2 \cong E[n]$, $\overset{(1,0)}{\underset{''}{P}}, \overset{(0,1)}{\underset{''}{Q}}$ determined by sections $P, Q \in E[N](S)$.

$\Gamma(N) \backslash \mathcal{H}^+ \to Y(N)(\mathbb{C})$, $\tau \mapsto (E_\tau, P_\tau, Q_\tau)$ w/ $E_\tau = \mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$, $P_\tau = \tau/N \in E_\tau[N]$, $Q_\tau = 1/N$.

This is isom. onto $\underline{\text{one}}$ conn. component of $Y(N)(\mathbb{C})$ (which has multiple conn. components!).

Elliptic curve $E \to \operatorname{Spec} \mathbb{C}$ has Weil pairing $e_N : E[N] \times E[N] \to \mu_n$ which is nondeg. This induces

$e_N : Y(N)(\mathbb{C}) \to \mu_N$, $(E,P,Q) \mapsto e_N(P,Q) \rightsquigarrow$ bij. $\pi_0(Y(N)(\mathbb{C})) \xrightarrow{\sim} \mu_N^*(\mathbb{C}) =: \{$primitive Nth roots of unity in $\mathbb{C}\}$.

Compute: $e_N(P_\tau, Q_\tau) = e^{2\pi i/N}$ (or its inverse), which is indep. of τ.

Upshot: $Y(N)$ is conn. but not when base changed to $\mathbb{C}$. It has $\varphi(n)$ conn. components indexed by $\mu_N^*(\mathbb{C})$.

Each component has complex pts. $\Gamma(N) \backslash \mathcal{H}^+$.

Let now $\mathbb{Q}(\zeta_N) := \mathbb{Q}[X]/(\Phi_N(X))$. Define (as in Katz-Mazur) $Y(N)_{\mathbb{Q}[\zeta_N]}$ (not base change!) to be like $Y(N)$ $\underline{\text{but}}$ we also require $e_N(P,Q) = \zeta_N$. Choice of $\zeta \in \mu_N^*(\mathbb{C}) \rightsquigarrow \mathbb{Q}(\zeta_N) \hookrightarrow \mathbb{C}$ via $\zeta_N \mapsto \zeta$.

$Y(N)_{\mathbb{Q}[\zeta_N]} \times_{\mathbb{Q}[\zeta_N]} \operatorname{Spec} \mathbb{C} \subseteq Y(N)_{\mathbb{C}}$ as $\underline{\text{one}}$ conn. component.