Prop: Equiv. of cat.'s $\{$elliptic curves $/\mathbb{C}\} \xrightarrow{\sim}$ pairs $\mathcal{U} \subseteq V$ w/ V vec. space of dim. $1/\mathbb{C}$ and $\mathcal{U}$ free rank 2 $\mathbb{Z}$-mod. and isom. $\mathcal{U} \otimes_{\mathbb{Z}} \mathbb{R} \xrightarrow{\sim} V\}$

$\mathcal{U} \subseteq V \mapsto V/\mathcal{U}$.

Remark: $V/\mathcal{U}$ is alg. by working w/ Weierstrass $\wp$-function. We can also use deep results on uniformization and algebraicity.

$H_1(E;\mathbb{Z}) = \mathcal{U}$, $H_1(E;\mathbb{R}) = \mathcal{U} \otimes_{\mathbb{Z}} \mathbb{R} = V = \mathrm{Lie}(E)$. $H_1(E;\mathbb{C}) = V \otimes_{\mathbb{R}} \mathbb{C}$ has two complex structures. $\mathbb{C}\times\mathbb{C}$ ~~These yields~~ idempotents

and thus yields a decomposition $H_1(E;\mathbb{C}) = \underbrace{F^0 H_1(E;\mathbb{C})}_{=V^{0,-1}} \oplus \underbrace{\overline{F^0 H^1(E;\mathbb{C})}}_{=V^{-1,0}}$

$$V \otimes_{\mathbb{R}} \mathbb{C} = \underbrace{V^{0,-1}}_{\substack{\text{Complex}\\\text{structures}\\\text{agree}}} \oplus \underbrace{V^{-1,0}}_{\substack{\text{Complex structures}\\\text{differ by a sign}}}$$

Remark: Dual to canon. Hodge decomposition of $H^1(E;\mathbb{C})$.

Varying ~~the~~ complex structure on fixed top. space coming from Hodge fil.

$V \hookrightarrow V \otimes_{\mathbb{R}} \mathbb{C} \twoheadrightarrow (V \otimes_{\mathbb{R}} \mathbb{C})/V^{0,-1}$ is isom. $\Rightarrow E = \mathcal{U}\backslash V = H_1(E;\mathbb{Z}) \backslash \underbrace{(H_1(E;\mathbb{C})/F^0 H_1(E;\mathbb{C}))}_{H_1(E;\mathbb{R})}$.

Prop: $\exists$ bij. $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R}) \longleftrightarrow \{$elliptic curves $E/\mathbb{C}$ w/ choice of $H_1(E;\mathbb{Z}) \cong \mathbb{Z}^2\}$

Pf: $F \in \mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$ is line $F \subseteq \mathbb{C}^2$ s.t. $F \neq \overline{F}$ – i.e., $\mathbb{C}^2 = F \oplus \overline{F}$. Hence, $\mathbb{R}^2 \hookrightarrow \mathbb{C}^2 \twoheadrightarrow \mathbb{C}^2/F$ is isom. and the nat. map $\mathbb{Z}^2 \to \mathbb{C}^2/F$ is an inj. Consider $E = \mathbb{Z}^2 \backslash \underbrace{(\mathbb{C}^2/F)}_{\mathbb{R}^2}$. For other direction pass to Hodge filtration. $\square$

Cor: $\{$elliptic curves $/\mathbb{C}\} \longleftrightarrow GL_2(\mathbb{Z}) \backslash (\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})) \longleftrightarrow GL_2(\mathbb{Z}) \backslash (\mathbb{C} \setminus \mathbb{R}) \longleftrightarrow SL_2(\mathbb{Z}) \backslash \mathcal{H}$. $(\tau \in \mathcal{H} \mapsto \mathbb{C}/(\mathbb{Z}\tau + \mathbb{Z}))$

Remark: Thinking of pts. of $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R})$ as Hodge structures on $\mathbb{R}^2$ we get a map $\mathbb{P}^1(\mathbb{C}) \setminus \mathbb{P}^1(\mathbb{R}) \to \mathrm{Hom}(\mathbb{S}, GL_{2,\mathbb{R}})$.

$\circlearrowleft$ transitive action of $GL_2(\mathbb{R})$ $\quad$ $\circlearrowleft$ conjugation action

Image is single $GL_2(\mathbb{R})$-conj. class $X \subseteq \mathrm{Hom}(\mathbb{S}, GL_{2,\mathbb{R}})$ and $\{$elliptic curves $/\mathbb{C}\} \Leftrightarrow GL_2(\mathbb{Z}) \backslash X$.

Remark: Definition of Shimura datum ensures X has nat. complex mfld structure and not just real mfld structure.

Special pts. of X related to CM.