Section 10 Math 2202
Volume between Surfaces and
Spherical Coordinates Introduction

1. **Volume Between Surfaces**

   Set up an integral which represents the volume of the solid bounded by $z = \sqrt{x^2 + y^2}$ and $x^2 + y^2 + z^2 = 4$.

   You may find the guidelines below helpful.

Volume Via Double Integration Some general guidelines:

Before setting up $\int \int_D f(x, y)\, dA$, you need to

- Identify the region D over which you will integrate- this is the projection ('shadow') of the solid in the xy-plane.

- Identify the function $f(x, y)$ which you will integrate - this should be the function whose graph is the surface under which the solid lies.

To do this, it is helpful to

- Draw a 'good enough' picture of what's happening in $\mathbf{R}^3$ with the bounding surfaces to see what the solid looks like. Find intersections of the surfaces.

- Draw a good picture of D in the xy-plane, making sure you have all the intersections and bounds correct. You can reality check with the equations - find the points or curves at which the surfaces intersect and think about the projection of those points or curves in the xy-plane.

- Consider if you need two or more integrals, for example if the volume of the solid is best described as a difference of volumes.

Then set up the integral(s) and then decide the best way to integrate it (switching orders, doing a change of coordinates).
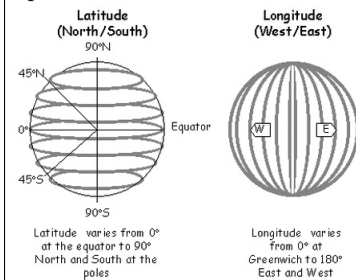
The usual way we represent a point in $\mathbf{R}^3$ is as (x, y, z). These are called **rectangular/Cartesian coordinates**. There are two other commonly used ways to think about a point.

Spherical Coordinates for $\mathbf{R}^3$

A point P in $\mathbf{R}^3$ is represented as (ρ, ϕ, θ) where

- ρ is the distance to the origin O, which is $|OP|$

- ϕ is the angle OP makes with the positive z-axis

- θ is the angle the projection of OP in the xy-plane makes with the positive x-axis (the same angle as in cylindrical coordinates)

Spherical coordinates takes its intuition from latitude $(\pi/2 - \phi)$ and longitude (θ).

Latitude (North/South)
90°N
45°N
0°
45°S
90°S
Latitude varies from 0° at the equator to 90° North and South at the poles

Longitude (West/East)
Equator
W E
Longitude varies from 0° at Greenwich to 180° East and West

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- Spherical $\to$ Rectangular: $x = \rho \sin\phi \cos\theta$, $y = \rho \sin\phi \sin\theta$, $z = \rho \cos\phi$

- Rectangular $\to$ Spherical:

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2. Let's figure out why this is the relationship between rectangular and spherical coordinates, and how to go back the other way.

3. (a) Write the point $(5, \pi/4, \pi/3) = (\rho, \theta, \phi)$ in rectangular coordinates.

(b) Write the point $(0, 2\sqrt{3}, -2) = (x, y, z)$ in spherical coordinates. (Hint: it might help to plot and visualize it.)

(c) Identify the surface described by each equation and write the equation in rectangular coordinates if it is in spherical or spherical if it is in rectangular:

   • In rectangular coordinates, $x^2 + y^2 + z^2 = 4$
   • In spherical coordinates $\rho = 4$

(d) What is the equation of the half cone $z = \sqrt{x^2 + y^2}$ in in spherical coordinates? (Hint: Think about what the angle ϕ is for different points on the cone.)

See Chapter 9.7 for more on these coordinates.

4. (a) Plot the point whose cylindrical coordinates are given. Then find the Cartesian coordinates of the same point.

     i. $(r, \theta, z) = (4, \frac{2\pi}{3}, 5)$
     ii. $(r, \theta, z) = (3, \frac{\pi}{4}, -2)$

  (b) Plot the point whose rectangular coordinates are given. Then find the polar coordinates of the same point where $r > 0$ and $0 \le \theta \le 2\pi$.

     i. $(x, y) = (2, -2, 2)$
     ii. $(x, y) = (-1, \sqrt{3}, 0)$

  (c) Sketch the surface or space in $\mathbf{R}^3$ that is described by

     i. In rectangular coordinates: $x^2 + y^2 = 9$
     ii. In cylindrical coordinates: $r \le 5$