We'll supplement Haines + Harris w/ Milne.

Motivation

$\mathcal{H} \subseteq \mathbb{C}$ upper half-plane w/ usual action by $SL_2(\mathbb{R})$. $\Gamma(m) \overset{\ker(SL_2(\mathbb{Z}) \to SL_2(\mathbb{Z}/m))}{=}$ full congruence subgp. of level m.

$Y(m)(\mathbb{C}) := \Gamma(m) \backslash \mathcal{H}$. This is complex mfld (w/ some care needed for $m \leq 2$). Fix primitive $\zeta_m \in \mu_m \subseteq \mathbb{C}$.

Bijection $Y(m)(\mathbb{C}) = \{$elliptic curves $E/\mathbb{C}$ w/ generators $P, Q \in E[m]$ s.t. $\overset{\leftarrow \text{ Weil pairing}}{e_m(P,Q)} = \zeta_m\}' = (*)$.

We get smooth quasiproj. alg. var. $Y(m)/\mathbb{Q}(\zeta_m)$ s.t. $Y(m)(\mathbb{C}) = (*)$.

Remark: This particular descent is canonical in terms of minimal fields of definition. We will gloss over the fact that we get equiv. mfld structures from soln. of moduli problem and gp. action quotient.

How did Shimura generalize this? Let F be degree d totally real # field. Label $\tau_1, \dots, \tau_d \in \mathrm{Hom}_{\mathbb{Q}}(F, \mathbb{R})$.

$F \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \mathbb{R}^{\times d}$, $x \otimes \alpha \mapsto (\tau_1(x)\alpha, \dots, \tau_d(x)\alpha)$. Let B be quaternion alg./F split at $\tau_1, \dots, \tau_r$ nonsplit at $\tau_{r+1}, \dots, \tau_d$.

B is fin. dim. F-alg. s.t. $B \otimes_{F,\tau_i} \mathbb{R} \cong \begin{cases} M_2(\mathbb{R}), & 1 \leq i \leq r \\ \mathbb{H}, & r < i \leq d \end{cases} \Rightarrow B \otimes_{\mathbb{Q}} \mathbb{R} \cong \prod_{i=1}^{d} B \otimes_{F,\tau_i} \mathbb{R} \cong M_2(\mathbb{R})^r \times \mathbb{H}^{d-r}$. In particular,

$B \subseteq M_2(\mathbb{R})^r \Rightarrow B^\times \subseteq GL_2(\mathbb{R})^r$. $\Gamma \subseteq B^\times \cap SL_2(\mathbb{R})^r$ congruence subgp. $\rightsquigarrow$ r-dim complex mfld $Y_\Gamma(\mathbb{C}) := \Gamma \backslash \mathcal{H}^r$.

Remark: For general setup this has no moduli interpretation. In the totally split case one can work w/ ab. var.'s.

Baily-Borel: $Y_\Gamma(\mathbb{C})$ is smooth quasiproj. alg. var. $/\mathbb{C}$.

Shimura: $\exists$ distinguished # field $E_\Gamma \subseteq \mathbb{C}$ and distinguished choice of smooth quasiproj. alg. var. Y_Γ recovering $Y_\Gamma(\mathbb{C})$.

Deligne: reformulated this to emphasize the sense in which Y_Γ is "canonical" and handle other quotients.

Goal: Count pts. over finite fields and work w/ zeta functions.

Hodge structures

Thm (Hodge): X smooth proj. var. $/\mathbb{C}$ (or more generally complex Kähler). $\exists$ canon. decomposition $\underset{\text{(only knows about X as top. space)}}{H^d(X, \mathbb{C})} \cong \bigoplus_{\substack{p+q=d \\ p,q \geq 0}} \underset{\text{(can tell us about complex structure)}}{H^{p,q}(X)}$.

$\overline{H^{p,q}(X)} \cong H^{q,p}(X)$ and $H^{p,q}(X) \cong H^q(X, \Omega^p_X)$. $\rightsquigarrow$ Hodge filtration $F^p H^d(X, \mathbb{C}) = \bigoplus_{p' \geq p} H^{p',q}(X)$ w/ $H^{p,q}(X) = F^p \cap \overline{F^q}$.

Thm (Deligne?, Grothendieck?): X smooth proj. var. $/k$.

$\Rightarrow \exists$ canon. filtration $F^p H^d_{dR}(X/k) \subseteq H^d_{dR}(X/k)$ s.t. $gr^p \cong H^q(X, \Omega^p_X)$. This agrees w/ Hodge's thm when $k = \mathbb{C}$.

Def: $V :=$ fin. dim. vec. space/$\mathbb{R}$. Hodge structure of weight ℓ on V is decomposition

$$V_{\mathbb{C}} = \bigoplus_{\substack{p+q=\ell \\ p,q \geq 0}} V^{p,q} \quad \text{s.t.} \quad \overline{V^{p,q}} = V^{q,p}.$$ We have filtration picture from before.

(minus signs à la Deligne)
(different conventions ...)

This bigrading is encoded by $h : \mathbb{C}^\times \times \mathbb{C}^\times \to GL(V_{\mathbb{C}})$ w/ $h(z_1, z_2) \cdot v = z_1^p z_2^q v$ for $v \in V^{p,q}$ s.t.
$h(z,z) = z^\ell \, \mathrm{id}_{V_{\mathbb{C}}}$ $\forall z \in \mathbb{C}^\times$ and $\overline{h(z_1, z_2)} = h(\overline{z_2}, \overline{z_1})$ $\forall z_1, z_2 \in \mathbb{C}^\times$. Consider now Deligne's torus

$\mathbb{S} := \mathrm{Res}_{\mathbb{C}/\mathbb{R}} \mathbb{G}_{m,\mathbb{R}}$. $\mathbb{S}(\mathbb{C}) \cong (\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C})^\times \cong \mathbb{C}^\times \times \mathbb{C}^\times$ (Exercise: $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} \xrightarrow{\sim} \mathbb{C} \times \mathbb{C}$, $z \otimes w \mapsto (zw, z\overline{w})$).

$$\begin{array}{ccccc} \mathbb{S}(\mathbb{C}) & = & \mathbb{C}^\times \times \mathbb{C}^\times & (z, \overline{z}) & \\ \cup\!| & & \cup\!| & \uparrow & \\ \mathbb{S}(\mathbb{R}) & = & \mathbb{C}^\times & z & \text{(fixed pts.)}. \end{array}$$

We have action on $\mathbb{C}^\times \times \mathbb{C}^\times$ that conjugates and swaps the entries.

(also copy of $\mathbb{G}_m(\mathbb{C}) = \mathbb{C}^\times$ emb. diagonally via $z \mapsto (z,z)$)

Second condition says $h : \mathbb{S}(\mathbb{C}) \to GL(V_{\mathbb{C}})$ descends to $h : \mathbb{S}(\mathbb{R}) \to GL(V)$.

Upshot: Weight ℓ Hodge structure on V is given by morphism $\mathbb{S} \to GL(V)$ of real alg. grps. s.t. restriction to $\mathbb{G}_{m,\mathbb{R}} \to GL(V)$ is $z \mapsto z^\ell$.