Rolling in the Deep

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Remark. *By a locally compact space we mean a Hausdorff topological space X such that every point of X has a compact neighborhood (i.e., for every $x \in X$ there is some $K \subset\subset X$ containing an open neighborhood of x).*

1 Number Theory Review

Definition 1.1. *A **number field** is a field which is a finite dimensional $\mathbb{Q}$-vector space. A **global function field** is a field which is a finite dimensional $\mathbb{F}_p(t)$-vector space, for p a prime and t an indeterminant. A **global field** is either a number field or global function field.*

A general rule of thumb is that if a result holds for one type of global field then an analogous result holds for the other type. This is why number fields and global function fields are placed under the same umbrella term. Note that things are often easier to prove for global function fields than for number fields. In what follows, F denotes a field.

Definition 1.2. *A **(discrete) valuation** on F is a map $v : F \to \mathbb{Z} \cup \{\infty\}$ such that, for all $a, b \in F$,*

(a) $v(a) = \infty \iff a = 0$;

(b) $v(ab) = v(a) + v(b)$; *and*

(c) $v(a + b) \geq \min\{v(a), v(b)\}$.

Example 1.3.

(1) *Let $F = \mathbb{Q}$ and consider the map $v_2 : \mathbb{Q} \to \mathbb{Z} \cup \{\infty\}$ which takes a/b to the highest power of 2 that divides it when a/b is written in lowest terms – e.g., $v_2(1/2) = -1$ and $v_2(4/5) = 2$. Then, v_2 is a discrete valuation.*

(2) *More generally, let $\mathcal{O}$ be a Dedekind domain with fraction field F and $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$. The map*

$$v_{\mathfrak{p}} : F \to \mathbb{Z} \cup \{\infty\}, \qquad a \mapsto \sup\{n \in \mathbb{Z} : a \in \mathfrak{p}^n\}$$

is a discrete valuation.

Remark 1.4. *Getz and Hahn drop the word discrete and allow the codomain of a discrete valuation to be $\mathbb{R} \cup \{\infty\}$. It is important to remember that there is a more general notion of valuation where one has a surjective group homomorphism $v : F^\times \to \Gamma$ for Γ a totally ordered abelian group (called* *the value group) that can be extended to a map $v : F \to \Gamma \cup \{\infty\}$ by declaring $v(0) := \infty$. What makes a discrete valuation discrete is that its value group is discrete. We recover Getz and Hahn's notion of valuation by noting that all discrete subgroups of $(\mathbb{R}, +)$ look like $(\mathbb{Z}, +)$ up to scaling.*

Definition 1.5. *An **absolute value** on F is a map $|\cdot| : F \to \mathbb{R}^{\geq 0}$ such that, for all $a, b \in F$,*

(a) $|a| = 0 \iff a = 0$*;*

(b) $|ab| = |a||b|$*; and*

(c) $|a + b| \leq |a| + |b|$.

*Condition (c) is called the **triangle inequality** and admits a stronger version fittingly called the **strong triangle inequality** or **ultrametric inequality**:*

(c') $|a + b| \leq \max\{|a|, |b|\}$.

*Absolute values satisfying (c') are called **nonarchimedean** and the rest are called **archimedean**. Note that the pair $(F, |\cdot|)$ is often called a **valued field**.*

For bookkeeping reasons we do not consider the **trivial** absolute value which returns 1 on $K^\times$ to be an absolute value. Every absolute value on F induces a metric on F.[1] We really only care about the induced topology and so consider two absolute values on F to be equivalent if they induce the same metric topology. A **place** of F is then the class of an absolute value on F under this equivalence relation. We denote the collection of places of F by Ω_F.[2] A place is said to be **(non-)archimedean** if it is represented by a (non-)archimedean absolute value. Clearly, no place is both archimedean and nonarchimedean.

Example 1.6.

(1) *Let v be a discrete valuation on F and $c \in (0, 1)$. The map $|\cdot|_{v,c} : F \to \mathbb{R}^{\geq 0}$ defined by $a \mapsto c^{v(a)}$ is a nonarchimedean absolute value.*

(2) *The standard absolute value on $\mathbb{Q}$ is an archimedean absolute value. In what follows, unless otherwise stated, $|\cdot|$ will refer to this absolute value or its extension to any subfield of $\mathbb{C}$.*

(3) *The map $|\cdot|_\infty : \mathbb{F}_p(t) \to \mathbb{R}^{\geq 0}$ defined by $|f/g|_\infty := p^{\deg f - \deg g}$ is a nonarchimedean absolute value.*

Carrying out the procedure in (a) above under the setup of Example 1.3(2) with $c = |\mathcal{O}/\mathfrak{p}|^{-1}$ produces the **normalized** absolute value $|\cdot|_{v_\mathfrak{p}} = |\cdot|_\mathfrak{p}$. This is very important because of the following result, which shows that the places of number fields are well understood.

Theorem 1.7 (Ostrowski)**.** *Let F be a number field and $\omega \in \Omega_F$.*

(a) *Suppose ω is nonarchimedean. Then, there is $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_F)$ such that ω is represented by $|\cdot|_\mathfrak{p}$.*

(b) *Suppose ω is archimedean. Then, there is an embedding $\sigma : F \hookrightarrow \mathbb{C}$ fixing $\mathbb{Q}$ pointwise such*

[1] Note that the trivial absolute value induces the discrete topology and so is often rather unseemly to work with.
[2] Another common choice of notation is M_F.

that ω is represented by

$$|\cdot|_\sigma := \begin{cases} |\cdot| \circ \sigma, & \sigma(F) \subseteq \mathbb{R}, \\ |\cdot|^2 \circ \sigma, & \textit{otherwise.} \end{cases}$$

*The embedding σ is called **real** in the first case and **complex** in the second case, and $|\cdot|_\sigma$ is considered to be normalized.*

What about absolute values on global function fields? It turns out that a similar result holds. It is a fact that an absolute value on a field of positive characteristic must be nonarchimedean. If F is a global function field it follows that all the places of F are nonarchimedean. To figure out the rest of the picture, let's do an example and then generalize from there.

Example 1.8. *Consider the case $F = \mathbb{F}_p(t)$. Using the above procedure, for each $\mathfrak{p} \in \operatorname{Spec} \mathbb{F}_p[t]$ we get a place represented by the normalized absolute value $|\cdot|_\mathfrak{p}$ on $\mathbb{F}_p(t)$. It turns out that this accounts for every place on $\mathbb{F}_p(t)$ save one, namely the place represented by the absolute value $|\cdot|_\infty$ in Example 1.6(3).*

For F a more general global function field, F can be written as $\mathbb{F}_q(t)[s]/(m(s))$ for q a power of p and $m(s) \in \mathbb{F}_q[t, s]$ separable.[3] The places of F are then either induced by prime ideals of $\operatorname{Spec}(\mathbb{F}_q[t, s]/(m(s)))$ or are represented by absolute values on F extending $|\cdot|_\infty$. This prompts a definition.

Definition 1.9. *Let F be a global field and $\omega \in \Omega_F$.*

- *Suppose F is a number field. Then, ω is **finite** if it is nonarchimedean and **infinite** if it is archimedean.*
- *Suppose F is a global function field. Then, ω is **finite** it is induced by a prime ideal of the above form and **infinite** if it is induced by $|\cdot|_\infty$.*

Ostrowski's Theorem and the above discussion tell us that every place of a global field is either finite or infinite.

Warning: Though the above terminology agrees with Getz and Hann, some sources use the words "finite" and "nonarchimedean" as well as "infinite" and "archimedean" interchangeably when describing places.

Remark 1.10. *For F a global function field the apparent asymmetry between the finite and infinite places of F is a farce. This is best seen by considering the case $F = \mathbb{F}_p[t]$. The absolute value $|\cdot|_\infty$ corresponds to the prime ideal $(t^{-1}) \subseteq \mathbb{F}_p[t^{-1}]$. Geometrically, one should think about choosing different closed points on the curve $\mathbb{P}^1_{\mathbb{F}_p}$. This kind of thinking applies to more general global function fields F, thinking of F as the function field of some smooth proper curve X over $\mathbb{F}_p$ and places of F as closed points of X. Algebraically, every place of F arises from some prime ideal of some valuation subring $A \subseteq F$. This gives us a notion of normalized absolute values representing infinite places of F.*

Question: Is it possible to think of the infinite places of a number field geometrically as points on some generalized scheme?

[3] A priori, F is of the form $\mathbb{F}_q(t)[s]/(m(s))$ for $m(s) \in \mathbb{F}_q(t)[s]$ separable. Clearing denominators lets us assume $m(s) \in \mathbb{F}_q[t][s] = \mathbb{F}_q[t, s]$.

For the sake of convenience, a place is often denoted by a representative absolute value. Similarly, a place induced by a discrete valuation is often denoted by that discrete valuation. With this in mind, we will follow Getz and Hahn and use the symbol v to refer to a place from here on out. We use the notation $v \mid \infty$ to suggest that a place v is infinite and $v \nmid \infty$ to suggest that it is finite.[4]

Definition 1.11. *Let F be a global field and $v \in \Omega_F$. Let $|\cdot|_v$ denote the normalized absolute value representing v. Define F_v to be the metric completion of F with respect to the metric induced by $|\cdot|_v$.[5] The absolute value $|\cdot|_v$ extends to an absolute value on F_v which is traditionally also denoted $|\cdot|_v$ and is (non-)archimedean if v is (non-)archimedean. The pair $(F_v, |\cdot|_v)$ is then a valued field which is a non-discrete locally compact topological field. Such fields are called **local fields** and are, up to isomorphism of topological rings, of the form $\mathbb{R}$, $\mathbb{C}$, a finite extension of $\mathbb{F}_q((t))$, or a finite extension of $\mathbb{Q}_p$ for p a prime and q a power of p (so, in other words, all local fields arise from completing global fields). In the case that v is nonarchimedean, the set[6]*

$$\mathcal{O}_{F_v} := \{x \in F_v : |x|_v \leq 1\}$$

*is a local valuation ring with unique principal maximal ideal $\mathfrak{m}_v := \{x \in F_v : |x|_v < 1\}$. Any generator of this ideal is called a **uniformizer** or **uniformizing parameter** for F_v and is often denoted by π_v or ϖ_v (the v may be dropped if it is clear from context). The associated **residue field** $k_v := \mathcal{O}_{F_v}/\mathfrak{m}_v$ often comes up a lot in practice. Note that the pair $(\mathcal{O}_{F_v}, \mathfrak{m}_v)$ remains unchanged if $|\cdot|_v$ is replaced with an equivalent absolute value. Crucially, $\mathcal{O}_{F_v}$ is open in F_v and is compact when equipped with the subspace topology. For the sake of convenience, we define $\mathcal{O}_{F_v} := F_v$ for v archimedean.*

Example 1.12.

(1) $\mathbb{Q}_{|\cdot|} = \mathbb{R}$.

(2) $\mathbb{Q}(i)_{|\cdot|} = \mathbb{C}$.

(3) *Let v_p be the discrete valuation on $\mathbb{Q}$ arising from the prime p. Then, $\mathbb{Q}_{v_p} = \mathbb{Q}_p$ (you can take this as a definition if that suits your fancy).*

(4) *Let v_t be the discrete valuation on $\mathbb{F}_p(t)$ arising from the prime t. Then, $\mathbb{F}_p(t)_{v_t} = \mathbb{F}_p((t))$.*

Exercise 1.13. *Determine $\mathbb{F}_p(t)_{|\cdot|_\infty}$.*

Remark 1.14. *Thinking of all of this in terms of topology is in some ways the wrong approach, at least in the nonarchimedean case, in part because it obscures the underlying geometry. What really matters are the underlying filtrations. Given a Dedekind domain $\mathcal{O}$ and $\mathfrak{p} \in \mathrm{Spec}(\mathcal{O})$, there is an associated multiplicative $\mathfrak{p}$-adic filtration $\mathfrak{p} \supseteq \mathfrak{p}^2 \supseteq \cdots$ and $\mathfrak{p}$-adic completion*

$$\widehat{\mathcal{O}}_{\mathfrak{p}} := \varprojlim \mathcal{O}/\mathfrak{p}^n.$$

*This ring is a discrete valuation ring with maximal ideal $\mathfrak{p}\widehat{\mathcal{O}}_{\mathfrak{p}}$ and is naturally a topological ring under the so-called **Krull topology** in which $\{\mathfrak{p}^n\widehat{\mathcal{O}}_{\mathfrak{p}}\}_{n \geq 0}$ is a local basis at 0 (and gives a local basis*

[4]Another common notation for finite places is $v < \infty$.

[5]Since equivalent absolute values induced the same metric topology on F, the universal property of the completion guarantees that we could have chosen any absolute value representing v.

[6]Getz and Hahn call it the **ring of integers** of F_v.

everywhere by translating). We get a field by taking the fraction field of $\widehat{\mathcal{O}}_{\mathfrak{p}}$. The affine scheme $\operatorname{Spec}(\widehat{\mathcal{O}}_{\mathfrak{p}})$ *should be thought of as an infinitesimal neighborhood of the point $\mathfrak{p}$ on the curve* $\operatorname{Spec}(\mathcal{O})$.

All of this applies to our setting by taking $\mathcal{O}$, for example, to be $\mathcal{O}_F$ for F a number field. Let $v \in \Omega_F$ be finite corresponding to $\mathfrak{p} \in \operatorname{Spec}(\mathcal{O}_F)$. One can deduce the following.

(a) *There is a natural identification of $\widehat{\mathcal{O}}_{F,\mathfrak{p}} := (\widehat{\mathcal{O}_F})_{\mathfrak{p}}$ and $\mathcal{O}_{F_v}$ as rings.*

(b) *There is a natural identification of the fraction field of $\widehat{\mathcal{O}}_{F,\mathfrak{p}}$ and F_v as fields.*

(c) *The various topologies on all rings in question agree with one another.*

Note that a similar analysis carries through for the units $\mathcal{O}^\times$.

Exercise 1.15. *Let F be a global function field and $v \in \Omega_F$ infinite. Recall that v corresponds to $\mathfrak{p} \in \operatorname{Spec}(A)$ for $A \subseteq F$ a suitable valuation subring, so that the normalized absolute value $|\cdot|_v$ is given by*

$$|x|_v = |A/\mathfrak{p}|^{-v_{\mathfrak{p}}(x)}.$$

Show that

$$|x|_v = |N_{F_v/\mathbb{F}_p((t))}(x)|_\infty^{1/[F_v:\mathbb{F}_p((t))]}.$$

The following result is extremely useful and illustrates why we normalize our absolute values.[7]

Theorem 1.16 (Product Formula)**.** *Let $x \in F^\times$. Then, there are only finitely many $v \in \Omega_F$ such that $|x|_v \neq 1$ and*

$$\prod_{v \in \Omega_F} |x|_v = 1.$$

Exercise 1.17. *Use Ostrowski's Theorem to verify the Product Formula directly when $F = \mathbb{Q}$.*

2 Adèles

Throughout this section, F denotes a fixed global field. Unless otherwise stated, all products are taken over Ω_F. Before defining the adèles, we first consider a useful general construction.

Let A be a countable index set, $S_0 \subseteq A$ finite, and $\{X_\alpha\}_{\alpha \in A}$ a collection of locally compact topological spaces. Given $\alpha \in A - S_0$, let $K_\alpha \subseteq X_\alpha$ be compact open. The **restricted direct product** associated to this data is

$$X = \prod_{\alpha \in A}{}' X_\alpha := \left\{ (x_\alpha) \in \prod_{\alpha \in A} X_\alpha : x_\alpha \in K_\alpha \text{ for every } \alpha \in A - S_0 \right\},$$

where the apostrophe indicates that there is a restriction. X carries a topology whose open sets look like

$$U \times \prod_{\alpha \in A - S_0} K_\alpha$$

[7] The geometric significance of this result will become clear in the next section once we discuss adèles.

for $U \subseteq \prod_{\alpha \in S} X_\alpha$ open and $S \subseteq A$ finite containing S_0, where each K_α is endowed with the subspace topology and everything else is endowed with the appropriate product topology.[8] Equipped with this topology, X is then necessarily locally compact.[9] If all of the X_α are groups and the K_α are subgroups then X carries a natural group structure making it into a topological group. The same remains true with every instance of "group" replaced by "ring."

Remark 2.1. *The above construction seems to depend on the choice of S_0 but it turns out that the restricted product topology remains unchanged if we replace S_0 by a larger finite subset of A. So, the above picture does not change if we let S_0 vary over all finite subsets of A. Viewed categorically, the restricted product X is naturally a colimit (in fact, a direct limit) in the category of topological spaces. Note that we could also work in the more restrictive category of locally compact topological spaces and make everything "play nice" with suitable algebraic structures (e.g., group and ring structures).*

Let's port all of this over to our number theoretic setting.

Definition 2.2. *The set of **adèles** of F is the restricted direct product*

$$\mathbb{A}_F := \prod_{v \in \Omega_F}{}' F_v = \left\{ (x_v) \in \prod_v F_v : x \in \mathcal{O}_{F_v} \text{ for cofinitely many } v \in \Omega_F \right\}$$

endowed with the restricted direct product topology described above.

By the above, $\mathbb{A}_F$ is naturally a locally compact topological ring. We care because $\mathbb{A}_F$ therefore inherits a Haar measure and so we can do harmonic analysis. Moving forward, it is useful to introduce some specialized notation for discussing adèles.

Definition 2.3. *Let $S \subseteq \Omega_F$ be finite.*

- *Define $\mathbb{A}_F^S := \prod_{v \notin S}{}' F_v$ and $\mathbb{A}_{F,S} = F_S := \prod_{v \in S} F_v$. Endowing $\mathbb{A}_F^S$ with the subspace topology and F_S with the product topology yields a natural isomorphism $\mathbb{A}_F \cong F_S \times \mathbb{A}_F^S$ of topological rings. Note that $\mathbb{A}_F^S$ and F_S are both naturally open subrings of $\mathbb{A}_F$.*
- *Given $S' \subseteq S$, define $F_S^{S'} := \prod_{v \in S - S'} F_v$.*
- *Assuming S contains the infinite places of F, define $\widehat{\mathcal{O}}_F^S := \prod_{v \notin S} \mathcal{O}_{F_v}$.*

Using this notation, the open subsets of $\mathbb{A}_F$ look like $U_S \times \widehat{\mathcal{O}}_F^S$ for $U_S \subseteq F_S$ open and every point of $\mathbb{A}_F$ has a locally compact open neighborhood of the form $F_S \times \widehat{\mathcal{O}}_F^S$.[10]

Remark 2.4. *The notation for $\widehat{\mathcal{O}}_F^S$ is similar to that for profinite completion and this is no coincidence. Namely, given $S \subseteq \Omega_F$ finite containing the infinite places, there is a natural isomorphism of topological rings between $\widehat{\mathcal{O}}_F^S$ and the profinite completion of the ring $\mathcal{O}_F^S$ of S-integers of F.*

[8] In general, for A infinite, this restricted product topology does not agree with the subspace topology induced from the product topology. We will see an example of this in a little bit.

[9] This is a direct consequence of Tychonoff's theorem.

[10] Note that the induced topology here coincides with the product topology.

This is perhaps best seen in the case $F = \mathbb{Q}$ and $S = \emptyset$, where we have

$$\widehat{\mathbb{Z}} := \varprojlim \mathbb{Z}/n \cong \prod_p \mathbb{Z}_p.$$

Geometrically, for F a number field, we often think of $\operatorname{Spec}(\mathcal{O}_F^S)$ as $\operatorname{Spec}(\mathcal{O}_F)$ with some of the points removed. This kind of geometric reasoning can be carried over to $\mathbb{A}_F$.

Remark 2.5. *The term adèle is short for "additive idèle," with idèle itself being short for "ideal element."*[11] *The group $\mathbb{I}_F = I_F$ of* ***idèles*** *of F can be described algebraically as $\mathbb{A}_F^\times$. Note, however, that it carries its own restricted product topology which differs from the subspace topology arising from $\mathbb{A}_F$.*

To see this in action, let's look at $\mathbb{A}_\mathbb{Q}$ and $I_\mathbb{Q}$. A sequence with terms $x^n = (x_v^n) \in \mathbb{A}_\mathbb{Q}$ converges to $x \in \mathbb{A}_\mathbb{Q}$ if x_v^n is contained in any given open neighborhood of x for $n \gg 0$. A similar notion of convergence holds for $I_\mathbb{Q}$. In $\mathbb{A}_\mathbb{Q}$, an open neighborhood of 0 looks like $U \times \prod_p p^{n_p}\mathbb{Z}_p$ for $U \subseteq \mathbb{R}$ an open neighborhood of 0 and $n_p \in \mathbb{Z}$ nonnegative for cofinitely many p.[12] *Translating gives that an open neighborhood of 1 in $\mathbb{A}_\mathbb{Q}$ looks like $(1+U) \times \prod_p (1+p^{n_p}\mathbb{Z}_p)$. Meanwhile, an open neighborhood of 1 in $I_\mathbb{Q}$ looks like $V \times \prod_p p^{m_p}\mathbb{Z}_p^\times$ for $V \subseteq \mathbb{R}^\times$ an open neighborhood of 1 and $m_p \in \mathbb{Z}^{\geq 0}$ with value 0 for cofinitely many p. With all of this in mind, define a sequence (x^n) in $\mathbb{A}_\mathbb{Q}$ as follows. Enumerate the positive primes of $\mathbb{Z}$ in increasing order as $p_1, p_2, \ldots$ and define*

$$x_v^n := \begin{cases} p_n, & v = p_n, \\ 1, & v \neq p_n. \end{cases}$$

Then, (x^n) converges to 1 in $\mathbb{A}_\mathbb{Q}$ but not in $I_\mathbb{Q}$ since it never enters the neighborhood $V \times \prod_p \mathbb{Z}_p^\times$ for $V \subseteq \mathbb{R}^\times$ a fixed open ball centered at 1.

Exercise 2.6. *Let F be a number field.*

(a) *Show that there is a canonical isomorphism $(\widehat{\mathbb{Z}} \times \mathbb{R}) \otimes_\mathbb{Z} F \xrightarrow{\sim} \mathbb{A}_F$ of topological rings.*

(b) *Show that there is a canonical isomorphism $\widehat{\mathbb{Z}}^\times \times \mathbb{Q}^\times \times \mathbb{R}^{>0} \xrightarrow{\sim} I_\mathbb{Q}$ of topological groups.*

(c) *How is I_F related to the ideal class group of F?*

One of the major perks of adèles is that they make it smoother to do local-global arguments.[13] This would not be possible without the following result.

Proposition 2.7. *The image of the diagonal embedding $F \hookrightarrow \mathbb{A}_F$ is closed and discrete.*[14]

We often identify F with its image, the ring of **principal adèles**, under this diagonal embedding.

Proof. It is a general fact that discrete subgroups of Hausdorff topological groups are closed. Hence, it suffices to show that F is discrete in $\mathbb{A}_F$ and thus that 0 is an isolated point of F in $\mathbb{A}_F$. Given

[11] Some people amusingly reverse the history and say that idèle is short for "invertible adèle."

[12] This shows that the restricted product topology on $\mathbb{A}_\mathbb{Q}$ is not the subspace topology arising from the relevant product topology.

[13] If you aren't convinced or aware of the power of the local-global principle then you should look up Hensel's Lemma and Hasse's work on quadratic forms.

[14] It is also true that the quotient $\mathbb{A}_F/F$ is compact, though we won't prove this.

a place $v \mid \infty$, let U_v denote the open unit ball $\{x \in F_v : |x|_v < 1\}$. We claim that the open set $U := \prod_{v|\infty} U_v \times \prod_{v\nmid\infty} \mathcal{O}_{F_v}$ is an isolated neighborhood of 0. To see this, let $y \in U \cap F$. Then,

$$\prod_{v\in\Omega_F} |y|_v = \prod_{v|\infty} |y|_v \prod_{v\nmid\infty} |y|_v < 1$$

and so the Product Formula gives $y = 0$. $\square$

The following exercise illustrates one way in which many statements about global fields can be reduced to checking the simplest case – i.e., the global fields $\mathbb{Q}$ and $\mathbb{F}_p(t)$.

Exercise 2.8. *Let L/F be a finite separable extension of global fields. Show that there is a natural isomorphism $\mathbb{A}_L \xrightarrow{\sim} \mathbb{A}_F \otimes_F L$ of both topological rings and F-vector spaces such that the diagram*

$$\begin{tikzcd} L \arrow[r, "\sim"] \arrow[d, hook] & F \otimes_F L \arrow[d, hook] \\ \mathbb{A}_L \arrow[r, "\sim"] & \mathbb{A}_F \otimes_F L \end{tikzcd}$$

commutes.[15]

By construction, F is dense in F_v for every $v \in \Omega_F$. This fact admits a vast generalization.

Theorem 2.9 (Strong Approximation)**.** *Let $S \subseteq \Omega_F$ be finite and nonempty. Then, the image of the embedding $F \hookrightarrow \mathbb{A}_F^S$ is dense.*

There is a related notion of strong approximation for algebraic groups that Getz and Hahn discuss at some length. For a discussion of some of the applications of strong approximation in practice, see this MathOverflow post.

3 Applications for Algebraic Groups

Let R be a topological ring and X an affine R-scheme of finite type – i.e., $\mathcal{O}(X) \cong R[t_1, \ldots, t_n]/I$ with $I \subseteq R[t_1, \ldots, t_n]$ an ideal. We will most often care about the case that X is an algebraic group such as $\mathbb{G}_a^n$ for some $n > 0$ and R is a number theoretic topological ring such as $\mathbb{A}_F$ for F a global field. One might reasonably hope that there is some way of topologizing the R-points $X(R)$ that is reflective of the topology on R and has certain nice properties. What sort of properties are desirable?

(a) If we replace X by an isomorphic scheme then the topology on $X(R)$ should be essentially unchanged – i.e., if there is an isomorphism $X \to Y$ of R-schemes then the natural map $X(R) \to Y(R)$ should be a homeomorphism. This boils down to the topology on $X(R)$ being functorial – i.e., a morphism $X \to Y$ of finite type R-schemes gives rise to a natural map $X(R) \to Y(R)$ that is continuous. Stated in fancy language, there should be a factorization

$$\begin{tikzcd} \mathsf{Alg}_R \arrow[r, dotted, "\exists"] \arrow[dr, "X"'] & \mathsf{Top} \arrow[d] \\ & \mathsf{Set} \end{tikzcd}$$

[15] If it helps, $\mathbb{A}_F \otimes_F L$ can be identified with $\mathbb{A}_F^{[L:F]}$ as a topological F-vector space.

where the unmarked vertical arrow is the forgetful functor.

(b) Consider the simplest case $X = \mathbb{G}_a$. There is a natural set bijection $X(R) \cong R$ that should be compatible with the topology on $X(R)$ and R.

(c) Consider now the case $X = \operatorname{Spec}(R[t_1, \ldots, t_n])$. There is a natural R-algebra isomorphism

$$R[t_1, \ldots, t_n] \cong R[t_1] \otimes_R \cdots \otimes_R R[t_n]$$

and thus a natural isomorphism of R-schemes $X \cong \mathbb{G}_a^n$. By (b) we understand the topology on $\mathbb{G}_a$ and this should be compatible with the topology on $\mathbb{G}_a^n$ hence X by (a). With some hindsight, what we want is for the topology we place on finite type affine R-schemes to be compatible with fiber products – i.e., given morphisms $X \to Y$ and $Z \to Y$ of finite type affine R-schemes, the natural bijection $(X \times_Y Z)(R) \cong X(R) \times_{Y(R)} Z(R)$ should be a homeomorphism.

(d) Finally, consider the case $X = \operatorname{Spec}(R[t_1, \ldots, t_n]/I)$. There is a natural surjection

$$R[t_1, \ldots, t_n] \twoheadrightarrow R[t_1, \ldots, t_n]/I$$

and thus a natural closed immersion $X \hookrightarrow \mathbb{G}_a^n$.[16] The corresponding map $X(R) \hookrightarrow \mathbb{G}_a^n(R)$ should be a topological embedding. With some hindsight, what we want is for all closed immersions of finite type R-schemes to give rise to topological embeddings on the level of R-points.

The above suggests a recipe for topologizing $X(R)$. Choose an isomorphism

$$A := \mathcal{O}(X) \cong R[t_1, \ldots, t_n]/I \tag{1}$$

The set $\{\alpha \in R^n : f(\alpha) = 0 \text{ for every } f \in I\}$ comes equipped with the subspace topology arising from the product topology on R^n. Identifying this set with $X(R)$ endows $X(R)$ with a topology. The choice of isomorphism in (1) does not matter since the induced topology on $X(R)$ agrees with the subspace topology arising from the canonical injection

$$X(R) = \operatorname{Hom}_{\mathsf{Alg}_R}(A, R) \hookrightarrow R^A = \operatorname{Mor}_{\mathsf{Set}}(A, R).$$

Checking that conditions (a)-(d) hold is then a routine matter that we leave to the reader.[17]

Let's apply this. Let F be a global field and X an affine algebraic group over F. We wish to endow $X(\mathbb{A}_F)$ with a well-behaved topology such that $X(\mathbb{A}_F)$ is a locally compact topological group. To that end, we will prove the following theorem.

Theorem 3.1. *Choose a faithful representation $X \hookrightarrow \mathrm{GL}_n$ and identify X with its image under this embedding. To each $v \nmid \infty$, associate the compact open subgroup $K_v := X(F_v) \cap \mathrm{GL}_n(\mathcal{O}_{F_v})$. Then, there is an isomorphism of topological groups*

$$\prod_v{}' X(F_v) \to X(\mathbb{A}_F), \tag{2}$$

[16] Recall that a closed immersion $X \hookrightarrow Y$ of R-schemes corresponds to a surjective ring homomorphism $\mathcal{O}(Y) \twoheadrightarrow \mathcal{O}(X)$. It is a fact worth checking that the map $X(C) \to Y(C)$ is then an injection for every R-algebra C.

[17] It is not hard to see that conditions (a)-(d) *uniquely* specify a topology on $X(R)$, at least up to equivalence of sets.

where the restricted product is taken with respect to the subgroups K_v and $X(R)$ is given the subspace topology induced by $\mathrm{GL}_n(R)$ for R a topological ring.[18]

Proof. Our previous discussion shows that the topology on each $X(R)$ for R a topological ring is independent of the choice of faithful representation of X. With this in mind, let's construct the map in (2). Consider first what happens in the case $X = \mathrm{GL}_n$. Write $\mathrm{GL}_n = \mathrm{Spec}(A)$. In what follows, all direct limits are taken over $S \subseteq \Omega_F$ finite containing the infinite places.[19] We have

$$\begin{aligned}
\prod_v{}' \mathrm{GL}_n(F_v) &\cong \varinjlim \prod_{v\in S} \mathrm{GL}_n(F_v) \times \prod_{v\notin S} \mathrm{GL}_n(\mathcal{O}_{F_v}) \\
&= \varinjlim \prod_{v\in S} \mathrm{Hom}(A, F_v) \times \prod_{v\notin S} \mathrm{Hom}(A, \mathcal{O}_{F_v}) \\
&\cong \varinjlim \mathrm{Hom}\left(A, \prod_{v\in S} F_v \times \prod_{v\notin S} \mathcal{O}_{F_v}\right) \\
&= \varinjlim \mathrm{GL}_n(F_S \times \widehat{\mathcal{O}}_F^S)
\end{aligned}$$

an equivalence of groups. Since $\mathbb{A}_F = \varinjlim F_S \times \widehat{\mathcal{O}}_F^S$, we have a compatible family of maps $\mathrm{GL}_n(F_S \times \widehat{\mathcal{O}}_F^S) \to \mathrm{GL}_n(\mathbb{A}_F)$ and thus a map of the form (2) by the universal property of the colimit. This map is a group isomorphism since $\mathrm{GL}_n : \mathsf{Alg}_F \to \mathsf{Grp}$ commutes with filtered colimits.[20]

What about more general X? The map in (2) is defined to be the map so that

$$\begin{tikzcd}
\prod_v{}' X(F_v) \arrow[r] \arrow[d, hook] & X(\mathbb{A}_F) \arrow[d, hook] \\
\prod_v{}' \mathrm{GL}_n(F_v) \arrow[r] & \mathrm{GL}_n(\mathbb{A}_F)
\end{tikzcd}$$

is a commutative diagram in the category of groups, where the restricted product in the lower left corner is taken with respect to $\mathrm{GL}_n(\mathcal{O}_{F_v})$ for $v \nmid \infty$. If the lower horizontal map is an isomorphism of topological groups then so is (2) since the vertical maps amount to restriction. Hence, to prove the theorem it suffices to consider the case $X = \mathrm{GL}_n$ and $K_v = \mathrm{GL}_n(\mathcal{O}_{F_v})$. The topology associated to GL_n can be understood by way of the closed immersion $\mathrm{GL}_n \hookrightarrow M_n \times \mathbb{G}_a$ of $\mathbb{Z}$-schemes (hence also F-schemes) given on R-points by

$$\mathrm{GL}_n(R) \hookrightarrow M_n(R) \times \mathbb{G}_a(R), \qquad g \mapsto (g, \det g^{-1}). \tag{3}$$

Since $M_n \times \mathbb{G}_a^n \cong \mathbb{G}_a^{n^2+1}$, if R is a topological ring then the topology that $\mathrm{GL}_n(R)$ inherits from this closed immersion is the same as the subspace topology arising from $M_n(R) \times \mathbb{G}_a(R)$. Note that this need not be the same as the standard subspace topology $\mathrm{GL}_n(R) \subseteq M_n(R)$ (as we will see in a minute). For $R = F_v$ no difficulties arise. Hence, the topology on $\mathrm{GL}_n(F_v)$ can as usual

[18] Remember that being an isomorphism of topological groups is stronger than being a homeomorphism which is also a group isomorphism since the group operations must be continuous.

[19] Wherever S appears, this assumptions on S will be implicit unless otherwise stated.

[20] Getz and Hahn equivalently obtain this map by way of the isomorphism $\mathbb{A}_F \cong F_S \times \mathbb{A}_F^S$.

be entirely understood in terms of a convenient local basis of I_n. For v nonarchimedean this looks like $\{I_n + \mathfrak{m}_v^k M_n(\mathcal{O}_{F_v})\}_{k\geq 0}$ and for v archimedean this looks like

$$\{I_n + \epsilon_v M_n(F_v) : \epsilon_v \in \mathbb{R} \text{ such that } 0 < \epsilon_v \leq r_v\}$$

for $r_v \in \mathbb{R}^{>0}$ sufficiently small. For $R = \mathbb{A}_F^S$ things are more complicated. A local basis of $(I_n, 1)$ in $M_n(\mathbb{A}_F^S) \times \mathbb{A}_F^S$ looks like

$$\{(I_n + mM_n(\widehat{\mathcal{O}}_F^S)) \times (1 + m\widehat{\mathcal{O}}_F^S) : m \subseteq \mathcal{O}_F^S \text{ a proper ideal}\},$$

where $mM_n(\widehat{\mathcal{O}}_F^S) := \prod_{v\notin S} \mathfrak{m}_v^{v(m)} M_n(\mathcal{O}_{F_v})$ and we have implicitly used the previously mentioned isomorphism of $\widehat{\mathcal{O}}_F^S$ with the profinite completion of $\mathcal{O}_F^S$. Using (3), we may thus identify

$$\left\{ \prod_{\substack{v\notin S\\ v(m)\neq 0}} (I_n + \mathfrak{m}_v^{v(m)} M_n(\mathcal{O}_{F_v})) \times \prod_{\substack{v\notin S\\ v(m)=0}} \mathrm{GL}_n(\mathcal{O}_{F_v}) : m \subseteq \mathcal{O}_F^S \text{ a proper ideal} \right\}$$

with a local basis of I_n in $\mathrm{GL}_n(\mathbb{A}_F^S)$.[21] Working similarly with the archimedean places of F and using the isomorphism $\mathbb{A}_F \cong F_S \times \mathbb{A}_F^S$ of topological rings, we conclude that the map (2) pushes forward basic open sets to basic open sets and so defines a homeomorphism and thus an isomorphism of topological groups (because of the nature of the specific group structures in question). □

An immediate corollary of this theorem is that $X(\mathbb{A}_F)$ is locally compact since $\mathbb{G}_a^{n^2+1}(\mathbb{A}_F)$ hence $\mathrm{GL}_n(\mathbb{A}_F)$ is locally compact.

Remark 3.2. *The above can be made a bit more clear by examining the relationship between $\mathcal{O}_F^S$ and $\mathcal{O}_{F_v}$ for $v \notin S$. By definition, $\mathcal{O}_F^S = \{x \in F : |x|_v \leq 1\}$ and so we may identify $\mathcal{O}_F^S$ as a subring of each $\mathcal{O}_{F_v}$. Under this identification, given $m \subseteq \mathcal{O}_F^S$ a proper ideal, we have $m \subseteq \mathfrak{m}_v^{v(m)}$ and $m = \prod_{v\notin S}(\mathfrak{m}_v^{v(m)} \cap F)$ as a product of ideals.*

[21]Note that this is not the same as the subspace topology arising from $\mathrm{GL}_n(\mathbb{A}_F^S) \subseteq M_n(\mathbb{A}_F^S)$ because of the mediating role played by the ideal m.