Remark: This is primarily work of Mumford, w/ input from Grothendieck.

$id_A \times \lambda : A \to A \times A^\vee$

Mumford Moduli over Fields [Last time: Mumford vanishing thm]

Suppose (A,λ) polarized ab. var. $/F$. $g = \dim A$, $d^2 = \deg(\lambda : A \to A^\vee)$. Let $M = (id_A \times \lambda)^* \mathcal{P}_A^{\otimes 3}$. This is very ample line bundle on A w/ $\dim H^0(A,M) = 6^g d$. In fact, $\dim H^0(A, M^{\otimes k}) = d(6k)^g \ \forall k \geq 1$.

Def: Linear rigidification of (A,λ) is isom. of F-schemes $\phi : \mathbb{P}(H^0(A,M)) \xrightarrow{\sim} \mathbb{P}_F^{6^g d - 1}$.

Convention: $\mathbb{P}(V)$ classifies hyperplanes $H \subseteq V$ w/ $\mathcal{O}_{\mathbb{P}(V)}(1)$ line bundle whose fiber at H is V/H.

M very ample $\Rightarrow \exists$ canon. proj. emb. $A \to \mathbb{P}(H^0(A,M)) \cong \mathbb{P}_F^{6^g d - 1}$, $x \mapsto$ hyperplane $\ker(H^0(A,M) \to M_x)$ s.t. $M \cong \mathcal{O}_A(1)$ (= pullback of $\mathcal{O}_{\mathbb{P}}(1)$). [This is canon. up to scale factor because A is proper.]

Essential Takeaway: Triple (A,λ,ϕ) is completely determined by closed subvar. $A \subseteq \mathbb{P}_F^{6^g d-1}$ and chosen pt. $e \in A$.

(1) Ab. scheme structure on A is determined by $e \in A$.

(2) Polarization $\lambda : A \to A^\vee$ determined by inclusion $A \subseteq \mathbb{P}^{6^g d - 1}$ since the inclusion determines $\mathcal{O}_A(1) \cong M$, which determines $\phi_{\mathcal{O}_A(1)} : A \to A^\vee$ which is $\phi_M = \phi_{(id_A \times \lambda)^* \mathcal{P}_A^{\otimes 3}} = 3\phi_{(id_A \times \lambda)^* \mathcal{P}_A} = 6\lambda$. This gives λ since $\mathrm{Hom}(A, A^\vee)$ is $\mathbb{Z}$-torsion-free.

(3) V vec. space $\rightsquigarrow$ canon. isom. $V \xrightarrow{\sim} H^0(\mathbb{P}(V), \mathcal{O}_{\mathbb{P}(V)}(1))$, $v \mapsto$ section whose fiber at hyperplane $H \subseteq V$ is image of x in $V/H \cong (\mathcal{O}_{\mathbb{P}(V)}(1))_H$. So, we have canon. isom. $F^{6^g d - 1} \xrightarrow{\sim} H^0(\mathbb{P}_F^{6^g d - 1}, \mathcal{O}(1)) \to H^0(A, \mathcal{O}_A(1)) \xrightarrow{\sim} H^0(A,M)$.

Last step only canon. up to scaling (whole composition is isom.) $\rightsquigarrow \phi$ we started w/ (canonically).

We want to construct moduli space $\mathcal{H}_{g,d}$ parametrizing triples (A,λ,ϕ).

Idea: First construct Hilbert scheme classifying $\underline{\underline{\text{all}}}$ subvar.'s of $\mathbb{P}^{6^g d - 1}$ w/ chosen pt. This will be infinite disjoint union of proj. schemes. We then want to realize $\mathcal{H}_{g,d}$ as locally closed subscheme of this Hilbert scheme. $PGL_{6^g d - 1}$ acts on $\mathbb{P}^{6^g d - 1}$ and on $\mathcal{H}_{g,d}$. The final step is to form the quotient $A_{g,d} = PGL_{6^g d - 1} \backslash \mathcal{H}_{g,d}$. [This is hard! Mumford shows the strongest possible quotient result by showing we get a torsor.]

Remark: Artin has a method to obtain $A_{g,d}$ as an alg. space (much weaker than getting a scheme) by looking at formal deformations of appropriate complete local rings.

Mumford Moduli over Schemes

Thm (Grothendieck): $\pi : X \to Y$ proper flat morphism of schemes, $\mathcal{F} \in \mathrm{Vect}(X)$, $H^1(X_y, \mathcal{F}_y) = 0 \ \forall y \in Y$.

(1) $\pi_* \mathcal{F}$ is vector bundle / Y (not just a coherent sheaf).

(2) Formation of $\pi_* \mathcal{F}$ commutes w/ base change : $\forall$ Cartesian

$$\begin{tikzcd} X' \arrow[r, "\pi'"] \arrow[d, "g"'] & Y' \arrow[d, "f"] \\ X \arrow[r, "\pi"'] & Y \end{tikzcd} \quad : \quad f^* \pi_* \mathcal{F} \cong \pi'_* g^* \mathcal{F}.$$

Remark: Is (2) not just proper base change (+ Grauert's Thm)? This comes up when working w/ moduli of curves over a fixed base.

Cor: $\pi : A \to S$ conn. ab. scheme, L ample line bundle / A $\Rightarrow$ $\phi_L : A \to A^\vee$ is isogeny (of some degree d^2) and $(\pi_* L)^{\otimes z}$ is very ample $\forall z \geq 3$.

$\pi_* L$ is vector bundle / S of rank d whose formation commutes w/ base change and

$S \in \mathrm{Sch}$, $\mathcal{M} \in \mathrm{Vect}(S) \rightsquigarrow \mathbb{P}(\mathcal{M}) \to S$. Polarized ab. scheme

$$\begin{tikzcd} A \arrow[r, "\lambda"] \arrow[dr, "\pi"'] & A^\vee \arrow[d] \\ & S \end{tikzcd}$$

$\leftarrow$ (deg $= d^2$) $\rightsquigarrow$ line bundle $\mathcal{M} = (id_A \times \lambda)^* \mathcal{P}_A^{\otimes 3}$ on A.

$\pi_* \mathcal{M}$ line bundle / S of ~~bon~~ rank $6^g d$.

[I think this should be Mumford bundle.]

Def: Linear rigidification of (A, λ) is $\phi : \mathbb{P}_S^{6^g d - 1} \to \mathbb{P}(\pi_* \mathcal{M})$. Given $S \in \mathrm{Sch}$,

$\mathcal{H}_{g,d}(S) := \{$ isom. classes (A, λ, ϕ) s.t. $\pi : A \to S$ ab. scheme of dim. g, $\lambda : A \to A^\vee$ polarization of degree d^2, ϕ linear rigidification $\}$

$\rightsquigarrow$ functor $\mathcal{H}_{g,d} : \mathrm{Sch} \to \mathrm{Set}$.

Thm (Mumford): $\mathcal{H}_{g,d}$ representable by (smooth) quasiproj. $\mathbb{Z}$-schemes. ($\leftarrow$ over $\mathbb{Z}[1/d]$)