Last time: Construction of Mumford's moduli space (w/ linear rigidifications)

Level Structure

Let $A \to S$ be ab. scheme of rel. dim. g. Fix $m \in \mathbb{Z}^{>0}$ w/ $m \in \mathcal{O}_S^\times$. Then, $[m]: A \to A$ is finite étale $/S$ and $\exists$ étale cover $S' \to S$ s.t. $A[m]_{S'} \cong (\underline{\mathbb{Z}/m\mathbb{Z}})^{2g}_{S'}$. [Can in fact take $S' = A[m]$.]

Def: (Full) level m structure on $A \to S$ is isom. of S-grp. schemes $\alpha : (\underline{\mathbb{Z}/m\mathbb{Z}})^{2g}_{S} \xrightarrow{\sim} A[m]_{S'}$.

$\lambda: A \to A^\vee$ polarization of degree d^2 prime to $m \Rightarrow \lambda: A[m] \xrightarrow{\sim} A^\vee[m]$ so we can view Weil pairing

$e_m : A[m] \times A^\vee[m] \to \mu_m$ as perfect alternating pairing $e_m : A[m] \times A[m] \to \mu_m$.

Def: symplectic level m structure on A is:

(1) isom. $\underline{\mathbb{Z}/m\mathbb{Z}} \cong \mu_m$.

(2) level m structure $\alpha : \underline{\mathbb{Z}/m\mathbb{Z}}^{2g} \cong A[m]$ identifying Weil pair on $A[m]$ w/ $\underline{\mathbb{Z}/m\mathbb{Z}}^{2g} \times \underline{\mathbb{Z}/m\mathbb{Z}}^{2g} \to \underline{\mathbb{Z}/m\mathbb{Z}}$, $(x,y) \mapsto {}^t x \begin{bmatrix} & I_g \\ -I_g & \end{bmatrix} y$.

Fix (g,d,m) w/ $\gcd(d,m) = 1$. $\mathcal{H}_{g,d,m} : Sch_{\mathbb{Z}[1/m]} \to Set$

$\mathcal{H}_{g,d,m}(S) := \{$ isom. classes of (A,λ,ϕ,α) w/ $(A,\lambda,\phi) \in \mathcal{H}_{g,d}(S)$ and α level m structure on it $\}$.

Thm: $\mathcal{H}_{g,d,m}$ is represented by quasi-proj. $\mathbb{Z}[1/m]$-scheme.

Pf: Consider the universal object

$$\begin{tikzcd} A \arrow[r, hook] \arrow[dr] & \mathbb{P}^{6^g d - 1} \times \mathcal{H}_{g,d} \arrow[d, "pr_2"] \\ & \mathcal{H}_{g,d} \end{tikzcd}$$

(This is all base-changed to $\mathbb{Z}[1/m]$.)

$H := \underbrace{A[m] \times_{\mathcal{H}_{g,d}} \cdots \times_{\mathcal{H}_{g,d}} A[m]}_{m^{2g} \text{ times}}$.

$H(S) = \{$ isom. classes of (A,λ,ϕ,α) w/ $(A,\lambda,\phi) \in \mathcal{H}_{g,d}(S)$ and $\alpha = (\alpha_1, \dots, \alpha_{2g})$ is a tuple of $\alpha_i : S \to A[m] \}$.

S conn. $\Rightarrow$ each tuple determines $(\mathbb{Z}/m\mathbb{Z})^{2g} = (\underline{\mathbb{Z}/m\mathbb{Z}})^{2g}(S) \to A[m](S)$. For $S = \coprod S_i$ in terms of conn. components,

$$\begin{array}{c} (\underline{\mathbb{Z}/m\mathbb{Z}})^{2g}(S) = \coprod (\underline{\mathbb{Z}/m\mathbb{Z}})^{2g}(S_i) \\ \downarrow \\ \coprod A[m](S_i) \\ \cap \\ A[m](S) \end{array}$$

Over H we have universal (A,λ,ϕ,α) w/ $\alpha : (\underline{\mathbb{Z}/m\mathbb{Z}})^{2g} \to A[m]$ arbitrary homomorphism. But $\ker\alpha \to H$ is finite étale so has loc. constant degree. Take $\mathcal{H}_{g,d,m}$ to be union of components of H on which degree is 1. (In other words, pass to where universal homomorphism α is isom. and discard the rest.)

For quasi-projectivity, $\underbrace{\mathcal{H}_{g,d,m} \to \mathcal{H}_{g,d}}_{\substack{\text{finite} \\ \Rightarrow \text{affine} \\ \Rightarrow \text{quasi-proj.}}} \underset{\substack{\text{quasi-} \\ \text{proj.}}}{\longrightarrow} \operatorname{Spec} \mathbb{Z}[1/m]$.

<u>Remark</u>: m not inv. over the base $\Rightarrow A[m] \to S$ <u>not</u> étale $\Rightarrow \ker \alpha \to H$ <u>not</u> loc. constant degree. $\square$

We can still look at $\mathcal{H}_{g,d,m} \to \mathcal{H}_{g,d}$, which won't be finite or surj. [ordinary locus]

<u>Quotients</u>

Recall $PGL_{6^g d^g} \curvearrowright \mathcal{H}_{g,d,m}$ by changing linear rigidification. Want to construct $PGL_{6^g d_g} \backslash \mathcal{H}_{g,d,m}$.

Fix $S \in Sch$. Let $G \to S$ be faithfully flat FT grp. scheme. Let $X \in Sch_S$ w/ action by G.

<u>Categorical quotient</u> of X by G is
$$\begin{tikzcd} X \arrow[r, "q"] \arrow[dr] & Y \arrow[d] \\ & S \end{tikzcd}$$
s.t. q is G-equivariant for trivial G-action on Y. (and initial

When does this exist?

<u>Example</u>: $X = \operatorname{Spec} B$, $S = \operatorname{Spec} A$, G constant $\rightsquigarrow G\backslash X \cong \operatorname{Spec} B^G$. [May not be FT!]

<u>Thm</u> (Hilbert): G reductive $\Rightarrow$ this quotient is FT. (Noether showed this for G finite)

<u>Thm</u> (SGA...): G finite grp. $\curvearrowright (X \to S)$ and every pt. has orbit contained in G-stable open affine

$\Rightarrow G\backslash X$ exists.

<u>Example</u>: Fix field k. View $X = \mathbb{A}^{n^2}_k$ in terms of $n \times n$ matrices. $PGL_{n^2} \curvearrowright X$ via conjugation.

$X \xrightarrow{q} \mathbb{A}^n_k$ exists and is $q: X \to \mathbb{A}^n_k$ w/ $q(A) :=$ coeffs. of char. polyn. of A. Natural map

$PGL_{n^2}(k) \backslash X(k) \to (PGL_{n^2} \backslash X)(k)$ is <u>not</u> injective.

Suppose $X \to Y$ is faithfully ~~flat~~ flat FT and $G \to Y$ faithfully flat FT grp. scheme acting on $X \to Y$.

$X \to Y$ is <u>G-torsor</u> (for fppf topology) if $G \underset{Y}{\times} X \to X \underset{Y}{\times} X$ sheaf map is isom.

<u>Example</u>: ~~Gx~~ $X = G$ is trivial G-torsor.

<u>Remark</u>: fppf locally any G-torsor $X \to Y$ is trivial. In fact, $X \to Y$ is an fppf cover over which

$X_X = X \underset{Y}{\times} X \cong G \underset{Y}{\times} X = G_X$. (so local triviality condition comes for free)

<u>Def</u>: G-torsor for Grothendieck top. (*) is G-torsor for fppf topology s.t. $\exists$ (*)-cover $Y' \to Y$ s.t.

$X_{Y'} \to Y'$ is trivial $G_{Y'}$-torsor. [Doing things for fppf topology is somehow the weakest possible geometric condition.]