§ Deformation theory

• Recall, for $A \to B$, the cot. complex $L_{B/A}$ has the univ. property

$$\mathrm{Mod}_C(L_{B/A} \otimes_B C, M) \simeq \mathrm{Alg}_{A/C}(B, C \oplus M)$$

($\forall\, B \to C$ in SCR, $M \in \mathrm{Mod}_C^{cn}$).

$$\Rightarrow \quad \mathrm{Mod}_B(L_{B/A}, M[1]) \simeq \mathrm{Alg}_{A/B}(B, B \oplus M[1])$$

$$\simeq \{A\text{-alg. sections of } B \oplus M[1] \to B\}$$

classical:

$$\pi_0 \mathrm{Mod}_B(L_{B/A}, M[1]) \simeq \pi_1 \mathrm{Mod}_B(L_{B/A}, M) = \mathrm{Ext}^1(L_{B/A}, M)$$

$$= \{\text{sq-0 extensions } M \to \tilde{B} \to B \text{ of } A\text{-algs}\}/\simeq$$

$(s \in \mathrm{Alg}_{A/B}(B, B \oplus M[1]) \rightsquigarrow$

$$\begin{tikzcd} M \arrow[r] \arrow[d] & B \times_{B \oplus M[1]} B \arrow[r] \arrow[d] & B \arrow[d, "0"] \\ 0 \arrow[r] & B \arrow[r, "s"] & B \oplus M[1] \end{tikzcd}$$

Def A small (infinitesimal) extension of an A-algebra B by $M \in \mathrm{Mod}_B^{cn}$ is:

$\tilde{B} \in \mathrm{Alg}_B$ plus:

$$\begin{tikzcd} \tilde{B} \arrow[r] \arrow[d] & B \arrow[d, "s"] \\ B \arrow[r, "0"] & B \oplus M[1] \end{tikzcd} \quad s \in \mathrm{Alg}_{A/B}(B, B \oplus M[1])$$

Rem: $\{\text{inf. ext. of } B \text{ by } M\} \simeq \mathrm{Alg}_{A/B}(B, B \oplus M[1])$

$\simeq \mathrm{Map}_B(L_{B/A}, M[1])$

//

Def/prop. If B is k-truncated, $I \subset \pi_k(B)$ is a $\pi_0(B)$-module (st $I^2=0$ if $k=0$)

then $\exists!$ k-truncated ring B/I st

$$\mathrm{Alg}(B/I, C) = \{f: B \to C \mid \pi_k(f)(I) = 0\}$$

($\forall$ k-truncated C)

we say B is the square-zero ext. of B/I by $I[k]$.

Rem A sq-0 ext $B/I \to B$ by $I[k]$ is an infinitesimal extension

//

prop/def

classical

$$\begin{tikzcd} X \arrow[d, "\text{flat}: f"'] & \\ Y \arrow[r, "j"', "\text{sq-0}"] & Y' \\ J & \end{tikzcd}$$

then $o(f) \in \mathrm{Ext}^2_{\mathcal{O}_X}(L_{X/Y}, f^*J)$

st $o(f) = 0$ iff

$\exists$ deformation

$$\begin{tikzcd} X \arrow[r, dashed, "\text{sq-0}"] \arrow[d] & X' \arrow[d, dashed, "\text{flat}"] \\ Y \arrow[r] & Y' \end{tikzcd}$$

For $M \in QC(X)^{cn}$, define

$$D_{i,X}(M) := \mathrm{Spec}_X(\mathcal{O}_X \oplus M[i])$$

then
$$\mathrm{Ext}^i(L_X, M) = \pi_0\, \mathrm{QCoh}(X)(L_X, M[i])$$
$$= [D_{i,X}(M), X]_*$$

$$\begin{tikzcd} X \arrow[d] \\ Y \arrow[r, "j"] & Y' \end{tikzcd} \qquad J \to \tilde{B} \to B \to B \oplus J[1]$$

$$Y' \xleftarrow{j} Y \leftarrow D_{1,Y}(J)$$

so j induces $c_j \in [D_{1,Y}(J), Y] \cong \mathrm{Ext}^1_{\mathcal{O}_Y}(L_Y, J)$

$$(L_Y \to J[1]) \rightsquigarrow (L_Y[1] \xrightarrow{\theta} J[2])$$

$$\begin{tikzcd}
\mathrm{Spec}(\mathcal{O}_X \oplus f^* J[2]) \arrow[r, "\theta^*"] \arrow[dd, equal] & \mathrm{Spec}(\mathcal{O}_X \oplus f^* L_Y[1]) \arrow[d, "(f^*L_Y \to L_X \to L_{X/Y} \to f^*L_Y[1])"] \\
& \mathrm{Spec}(\mathcal{O}_X \oplus L_{X/Y}) \arrow[d] \\
D_{2,X}(f^* J) \arrow[r] & X
\end{tikzcd}$$

$n+1$

$\wedge$

$$\mathrm{Ext}^2_{\mathcal{O}_X}(L_{X/Y}, f^*J)$$

(Formally) smooth, étale

Def. $A \to B$ is formally smooth if
it's $\cdot$ $L_{B/A}$ is a fin. gen. proj. B-module, i.e. $L_{B/A}$ is direct summand of some $B^{\oplus n}$.

- is formally étale if $L_{B/A} = 0$
- is lfp if B is compact in Alg_A, i.e. $\mathrm{Alg}_A(B, -)$ preserves filt. colims.
- is smooth / étale if formally so + lfp.

//

Now $f: X \to Y$ of der. schemes is étale/smooth/flat/lfp $= P$

if $\exists$ affine covers $\{U_\alpha \to X\}$, $\{V_\beta \to Y\}$ +

$$\begin{array}{ccc} U_\alpha & \overset{P}{\dashrightarrow} & V_\beta \\ \forall \downarrow & & \downarrow \exists \\ X & \longrightarrow & Y \end{array}$$

$\#$

Quasi-smoothness

Def A closed immersion $Z \to X$ is q.smooth if locally on X, it's of the form

$$\begin{array}{ccc} Z & \longrightarrow & X \\ \downarrow & & \downarrow \\ \{0\} & \longrightarrow & \mathbb{A}^n \end{array} \qquad (n = \text{vir. codim}_X Z)$$

- $X \to Y$ is q. smooth if locally on X $\exists$ factorization

$$X \xhookrightarrow{q.sm} Y' \xrightarrow{sm} Y$$

(Khan & Rydh '19)

Prop
- A closed immersion $Z \to X$ is q.smooth (of virt. codim $= n$) iff lfp & $L_{Z/X}[-1]$ is loc free of rank n
- A map $X \to Y$ is q.smooth iff lfp + $L_{X/Y}$ is of Tor-amplitude ≤ 1

(M has tor amplitude $[a,b]$ if $\forall$ discrete Σ $\pi_i(M \otimes \Sigma) = 0$ $\forall i \notin [a,b]$)

§ Virtual Cartier Divisors

Throughout, fix $Z \hookrightarrow X$: closed immersion in dSch

Def. · A virt. cart div. is a q.smooth closed immersion $D \hookrightarrow S$ of virtual codim 1

· For $S \to X$, a vCD on $S/(X,Z)$ is a square

$$\begin{tikzcd} D \arrow[r] \arrow[d, "g"'] & S \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$

·) $D \to S$ is a vCD

·) $g^*(\mathcal{L}_{Z/X}(-1)) \to \mathcal{L}_{D/S}(-1)$ is surj. on π_0

·) $D_{cl} \cong (Z \times_X S)_{cl}$

Let $\quad vDiv_Z X : Sch_{/X}^{op} \to \mathcal{S}$

$$(S \to X) \mapsto \{vCD \text{ on } S/(X,Z)\}$$

thm (KhR'19) If $Z \to X$ is q. smooth

then $\mathrm{vDiv}_Z X$ is representable (by scheme)

& • vDiv is stable under BC

• If Z, X classical, then $\mathrm{vDiv}_Z X = Bl_Z^{cl} X$

•
$$\begin{tikzcd} \mathbb{P}_Z(\mathcal{L}_{Z/X}(-1)) \arrow[r] \arrow[d] & Bl_Z X \arrow[d] \\ Z \arrow[r] & X \end{tikzcd}$$
is the universal vCD

---

§ Graded Algebras & Projective spectra

Def Let Γ: comm. monoid,

$$\mathrm{Poly}^\Gamma := \left\{ \begin{array}{l} \text{fin. gen, } \Gamma\text{-graded poly} \\ \text{rings / } \underset{\text{over } \mathbb{Z}}{w/} \text{ graded hom}^s \end{array} \right\}.$$

$$\mathrm{Alg}^\Gamma := \mathcal{P}_\Sigma(\mathrm{Poly}^\Gamma) = (\mathrm{Alg}^\Gamma)^{\Delta^{op}}$$

$$= \left\{ \begin{array}{l} \text{fin. prod. preserving} \\ \text{pre sheaves } (\mathrm{Poly}^\Gamma)^{op} \to \mathcal{S} \end{array} \right\}$$

$= \infty$- cat of Γ-graded animated rings

$\mathrm{Alg}^\Gamma \ni B: \mathrm{Poly}^\Gamma \longrightarrow \mathcal{S}$

$$\mathbb{Z}[x(d)] \longmapsto Bd$$

$$\begin{pmatrix} \mathbb{Z}[x(d+e)] \to \mathbb{Z}[x(d), x(e)] \\ x(d+e) \mapsto x(d)\cdot x(e) \end{pmatrix} \mapsto \begin{pmatrix} Bd \times Be \xrightarrow{\times} Bd{+}e \\ \ \end{pmatrix}$$

$R \in \mathrm{Alg}$: $\mathrm{Alg}^\Gamma_R := \cancel{(\mathrm{Alg}^\Gamma)_{R/}}$

Can globalize this: X: scheme

$\mathrm{Alg}^\Gamma(X) = \{\Gamma\text{-graded}, \text{quasi-coherent}$
$\mathcal{O}_X\text{-alg}\}$

$= \lim\limits_{\mathrm{Spec}\, A \to X} \mathrm{Alg}^\Gamma_A$

<u>Rem</u> $P_\Sigma(\mathcal{C})$ is the ∞-cat of a model cat on

$\mathbb{C} := \{\mathcal{C}^{op} \to \mathrm{Set} \text{ pres. fin lims}\}$

$\mathcal{C}^{\Delta^{op}}$ Lurie HTT

Thm $\mathrm{Alg}^{\mathbb{Z}}(X) \simeq \mathrm{Aff}^{\mathbb{G}_m}(X)$ $\quad$ $\mathrm{Pols} \to \mathrm{Pols}^{\mathbb{Z}} \to S$, $\quad \mathbb{Z}(x) \mapsto \mathbb{Z}[x(0)]$

For $B \in \mathrm{Alg}^{\mathbb{N}}(X)$ $\quad B_+ := \bigoplus_{n>0} B_n$

Have $B_0 \to B \to B_0$ $\quad B_+ \to B \to B_0$

Have $\mathrm{Spec}(B_0) =: V(B_+) \hookrightarrow \mathrm{Spec}\, B$

$\mathrm{Proj}(B) := \mathcal{C}(\mathrm{Spec}\, B \setminus V(B_+)) / \mathbb{G}_m]$

Prop this is a scheme /X
(if B is gen. in deg 1 over B_0)

idea:

$$\mathrm{Proj}(B) = \bigcup_{f \in B_1} \mathrm{Spec}(B_{(f)})$$

$$B_{(f)} = (B_f)_0$$

Rees Algebra

Classically: let $A \xrightarrow{I} B$ surj. of rings

$$R := \bigoplus_{n \in \mathbb{Z}} I^n t^n \subseteq A[t^{\pm 1}]$$

Q: $\mathbb{Z}$-graded $A[t^{-1}]$-alg, t^{-1}-regular.

$$\begin{tikzcd} B \arrow[r, dashed, "\exists"] & Q/(t^{-1}) \\ A \arrow[u] \arrow[ur] & \end{tikzcd}$$

iff $I \subset (t^{-1})$

$$\begin{tikzcd} A = R_0 \arrow[r] & Q_0 \\ It = R_1 \arrow[u, "\times t^{-1}"] \arrow[r, dashed, "\exists"] & Q_1 \arrow[u, "\times t^{-1}"'] \end{tikzcd}$$

iff $(I) \subseteq (t^{-1})$

$$\rightsquigarrow \mathrm{Alg}^{\mathbb{Z}}_{A[t^{-1}]}(R, Q) \simeq \mathrm{Alg}_A(B, Q/(t^{-1}))$$

__Def./prop__ let $Z \to X$: closed imm. of derived schemes.

consider $\xi: B\mathbb{G}_{m,X} \to [\mathbb{A}^1_X / \mathbb{G}_{m,X}]$

induced by $0: X \to \mathbb{A}^1_X$

Have

$$\xi^* : St_{[\mathbb{A}^1_X/\mathbb{G}_{m,X}]} \longrightarrow St_{B\mathbb{G}_{m,X}} : \mathrm{Res}_\xi$$

$\exists$

(by Lurie, SAG, weil restrictions)

$[Z/\mathbb{G}_{m,X}] \to [X/\mathbb{G}_{m,X}] = B\mathbb{G}_{m,X}$

<u>Define</u>:

$$\begin{tikzcd} D_{Z/X} \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & \mathrm{Res}_\xi [Z/\mathbb{G}_{m,X}] \arrow[d] \\ \mathrm{Spec}(\mathcal{O}_X[t^{-1}]) = \mathbb{A}^1_X \arrow[r] & [\mathbb{A}^1_X/\mathbb{G}_{m,X}] \end{tikzcd}$$

<u>Thm</u> $D_{Z/X}$ is affine $/\mathbb{A}^1_X$.

$\rightsquigarrow$ $R^{ext}_{Z/X} \in \mathrm{Alg}^{\mathbb{Z}}(\mathbb{A}^1_X)$:

$$D_{Z/X} = \mathrm{Spec}\, R^{ext}_{Z/X}$$

$$[D_{Z/X}/\mathbb{G}_{m,X}] = \mathrm{Res}_\xi [Z/\mathbb{G}_{m,X}]$$

<u>recover universal property</u>.

Let $Q \in \mathrm{Alg}^{\mathbb{Z}}(\mathbb{A}^1_X)$. Then:

$T := \mathrm{spec}\, Q$ w/ $\mathbb{G}_{m,X}$-action

$$\mathrm{Alg}^{\mathbb{Z}}(\mathbb{A}^1_X)(R^{ext}_{Z/X}, Q) \simeq \mathrm{St}^{\mathbb{G}_{m,X}}_{/\mathbb{A}^1_X}(T, D_{Z/X})$$

$$\simeq \mathrm{St}_{[\mathbb{A}^1_X/\mathbb{G}_{m,X}]}([T/\mathbb{G}_{m,X}], \mathrm{Res}_{\varepsilon}([Z/\mathbb{G}_{m,X}]))$$

$$\simeq \mathrm{St}_{B\mathbb{G}_{m,X}}([(T \times_{\mathbb{A}^1_X} X)/\mathbb{G}_{m,X}], Z)$$

$$\simeq \mathrm{Alg}^{\mathbb{Z}}(X)(B, Q/(t^{-1}))$$

//

Cor For $A \to B$ in Alg, $(\pi_0 R^{ext}_{B/A})^{t^{-1}\text{-reg}}$

is the classical extended Rees

of $\pi_0 A \to \pi_0 B$

//

Ex $\overset{A}{\overset{\shortparallel}{k[\varepsilon]}} \to k = B$

$ut^{-1} \mapsto \varepsilon$

$$R^{ext}_{B/A} = \frac{K[v, t^{-1}]}{(v^2 t^{-1})} \leftarrow K[\varepsilon]$$

$$\left(R^{ext}_{B/A}\right)^{t^{-1}\text{-ver}} \quad \frac{K[v, t^{-1}]}{(v^2)} \cong A[\varepsilon t, t^{-1}]$$

$\underline{\text{Def}}$ $Bl_Z X = \text{Proj}(R_{Z/X})$

$\underline{\text{Thm}}$ $Z \to X$ is q. smooth:

$$Bl_Z X = \mathrm{vDiv}_Z X$$

$$\begin{tikzcd}
\underline{\mathbb{V}(L_{Z/X}[-1])} = N_{Z/X} \arrow[r] \arrow[d] & D_{Z/X} = \mathrm{Spec}(R_{Z/X}) \arrow[d] \\
X \arrow[r] & \mathbb{A}^1_X
\end{tikzcd}$$

$\mathbb{V}(\mathcal{E}) = \operatorname{Spec} \operatorname{Sym}(\mathcal{E})$

$Z \to X$ is classical. $\pi_0(L_{Z/X}(-1)) = I/I^2$

$(N_{Z/X})_{cl}$ = normal bundle

$N_{Z/X} = \operatorname{Spec}(R_{Z/X}/(t^{-1}))$

If $\pi_0 R_{Z/X}$ is t^{-1}-regular

$\pi_0 R_{Z/X}/(t^{-1}) = \bigoplus I^n/I^{n+1}$

$(N_{Z/X})_{cl}$: normal cone