Last time: Unitary Shimura variety example

E quadratic imaginary field, (V,H) Hermitian space of signature (p,q).

View $V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$ as $E \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}$-module. Let $X := \{ \perp \text{ decompositions } V_{\mathbb{R}} = \overset{\text{pos.def.}}{\overbrace{W_0}} \oplus \overset{\text{neg.def.}}{\overbrace{W_1}} \}$

Real pts. of $GU(V)$ act on X. Write $E = \mathbb{Q}(\sqrt{-D})$ and define symplectic form $\psi: V \times V \to \mathbb{Q}$ by

$(x,y) \mapsto Tr_{E/\mathbb{Q}} \frac{1}{\sqrt{-D}} H(x,y)$ so that $GU(V) \subseteq GSp(V)$. Given $W_0 \oplus W_1 \in X$ define new complex structure

$h: \mathbb{C} \to End_{\mathbb{R}}(V_{\mathbb{R}})$, $z \mapsto \begin{cases} z \cdot & \text{on } W_0, \\ \bar{z} \cdot & \text{on } W_1. \end{cases}$

Such h restricts to $h: \mathbb{C}^\times \to GU(V_{\mathbb{R}})$ and satisfies $\psi(h(i)x,y)$ is pos. or neg. def. symm. bilin. form

$\Rightarrow h$ lies in Siegel space defined by (V,ψ). This realizes $X \subseteq Hom(\mathbb{S}, GU(V_{\mathbb{R}}))$ as $GU(V_{\mathbb{R}})$-conj. class,

and $(GU(V_{\mathbb{R}}), X)$ is Shimura datum.

Prop: Fix $\mathcal{O}_E$-stable lattice $L \subseteq V$ s.t. $\lambda(L,L) \subseteq \mathbb{Z}$. Assume $K \subseteq GU(\hat{V})$ stabilizes $\hat{L} \subseteq \hat{V}$.

Define $M_K: Sch_E \to Set$, $S \mapsto \{$isom. classes of $(A, i, \lambda, [\eta])$ s.t. $\cdots\}$:

- $A \to S$ abelian scheme
- $i: \mathcal{O}_E \to End(A)$
- $\lambda: A \to A^\vee$ polarization
- (Fix $s \in S$ geom. pt. for every conn. component.) $[\eta] = K \cdot \eta \subseteq Isom_{\hat{\mathbb{Z}}}(\hat{T}A_s, \hat{L})$ is K-orbit stable under $\pi_1^{ét}(S,s)$ and the following are satisfied:

[$S = Spec\, R \Rightarrow$ char. poly. has coeffs. in R]

(1) signature (p,q) condition: every $\alpha \in \mathcal{O}_E$ acts on $Lie(A)$ (vec. bundle$/S$) w/ characteristic poly. $(x-\alpha)^p(x-\bar{\alpha})^q$.
We can view this as living over E instead of just S ($\mathcal{O}_E$ rather than $\mathcal{O}_S$).

[NB: $S = Spec\, \mathbb{C} \Rightarrow$ this is equiv. to $Lie(A) \cong \mathbb{C}^{p+q}$ s.t. $\alpha \cdot = (\alpha \cdots \alpha\, \bar{\alpha} \cdots \bar{\alpha})$]

(2) $\alpha \in \mathcal{O}_E \Rightarrow$
$$\begin{array}{ccc} A & \xrightarrow{\lambda} & A^\vee \\ {\scriptstyle\alpha}\downarrow & \circlearrowleft & \downarrow{\scriptstyle\bar{\alpha}^\vee} \\ A & \xrightarrow[\lambda]{} & A^\vee \end{array}.$$

(3) $\eta: \hat{T}A_s \xrightarrow{\sim} \hat{L}$ is $\mathcal{O}_E$-linear and identifies Weil pairing $\hat{T}A_s \times \hat{T}A_s \to \hat{\mathbb{Z}}(1) \leadsto \lambda: \hat{L} \times \hat{L} \to \hat{\mathbb{Z}}$

for some $\hat{\mathbb{Z}} \cong \hat{\mathbb{Z}}(1)$ (which may depend on $\eta \in [\eta]$).

K suff. small $\Rightarrow$ M_K representable by quasi-proj. E-scheme w/ $M_K(\mathbb{C}) \cong Sh_K(GU(V), X)$.

<u>Remark</u>: Common to have something like $K = \ker(GU(\hat{L}) \to GU(\hat{L}/N))$ for $N \geq 3$.

<u>Pf</u>: First construct $Sh_K(GU(V), X) \to M_K(\mathbb{C})$. Given $[h, g] \in GU(V) \backslash X \times GU(\hat{V})/K$ form

$A = V_{\mathbb{R}}/gL$ w/ complex structure determined by h. This has $\mathcal{O}_E$-action. As in Siegel case we can use similitude $\nu: GU(V) \to \mathbb{G}_m$ to get $\nu(g) \in \mathbb{A}_f^\times = \mathbb{Q}^\times \hat{\mathbb{Z}}^\times$. $\nu(g) = \mathrm{rat}(g) k$ w/ $\mathrm{rat}(g) \in \mathbb{Q}^\times$, $k \in \hat{\mathbb{Z}}^\times$. Define $\mathbb{Z}$-valued symplectic form $\frac{1}{\mathrm{rat}(g)} \psi : gL \times gL \to \mathbb{Z}$. This defines $(\pm)$ a polarization on A. Level structure is $\eta: \hat{T}A \cong g\hat{L} \underset{\cong}{\xrightarrow{g^{-1}}} \hat{L}$. For representability, exploit $GU(V) \subseteq GSp(V)$. Assume $K = \ker(GU(\hat{L}) \to G\cancel{U}L(\hat{L}/N))$.

Set $K' := \ker(GSp(\hat{L}) \to GL(\hat{L}/N)) \Rightarrow K = K' \cap GU(\hat{L})$. We already have moduli space $M_{K'}$ parametrizing polarized ab. schemes w/ level-K'-structure on $\hat{T}A$. Look at universal $A \to M_{K'}$.

[This is just standard Hilbert scheme stuff.]

$\underline{\mathrm{End}}(A) : Sch_{M_{K'}} \to \mathrm{CRing}$, $T \mapsto \mathrm{End}(A_T)$ is represented by $M_{K'}$-scheme. $\underline{\mathrm{End}}(A) \to M_{K'}$. Pullback of A to $\underline{\mathrm{End}}(A)$ has universal endomorphism $f \in \mathrm{End}(A)$. Write $\mathcal{O}_E \cong \mathbb{Z}[x]/(p(x))$. Define $\tilde{M}_{K'} \subseteq \underline{\mathrm{End}}(A) \to M_{K'}$, the clopen subscheme defined by $p(f) = 0$. Over $\tilde{M}_{K'}$ the pullback $A \to \tilde{M}_{K'}$ comes w/ $\mathcal{O}_E \to \mathrm{End}(A)$, $x \mapsto f$.

Cut more to account for more conditions.