

Eg Take \mathbb{H} = upper half plane, then $\text{hol}(x) = \text{PSL}_2(\mathbb{R})$.

What is $u_i: U_i \rightarrow \text{PSL}_2(\mathbb{R})$?

For $z = a+bi$, set $h_i(z) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in \text{GL}_2(\mathbb{R})^+$.

Then $h_i(z)$ stabilizes $i \in \mathbb{H}$,
but acts by mult by z/\bar{z} .

So if $t \in U_i \subset \mathbb{C}^*$, then $h_i(t) \in T_i \mathbb{H}$ as $t/\bar{t} = t^2$,

so $u_i(t) = \underbrace{h_i(\sqrt{t})}_{\text{PSL}_2 \text{ lifts ambiguity of } \sqrt{t}} \in \text{PSL}_2(\mathbb{R})$

PSL₂ lifts ambiguity of \sqrt{t} .

Recall: $X = \mathbb{HSD}$, then \exists real adj G st $G(\mathbb{R})^+ = \text{hol}(X)^+$.

For all $x \in X$, $\exists!$ $u_x: U_i \rightarrow K_x \subset G(\mathbb{R})^+$ st

$\forall t \in U_i$, $u_x(t) \in T_x X$ as $t \in U_i \subset \mathbb{C}^*$.

I.e., induced map $U_i \rightarrow \underbrace{\text{GL}(T_x X)}_{\text{a } \mathbb{C}\text{-vs}}$

is just $U_i \hookrightarrow \mathbb{C}^* \subset \text{GL}(T_x X)$.

So $u_x(\gamma) = \text{symmetry } s_x \text{ at } x$.

Let $V = \text{fd } \mathbb{R}\text{-vs.}$

Any hom $u: U_1 \rightarrow \text{GL}(V)$ induces grading

$$V_{\mathbb{C}} = \bigoplus_{k \in \mathbb{Z}} V_{\mathbb{C}}^{(k)}$$

with $\overline{V_{\mathbb{C}}^{(k)}} = V_{\mathbb{C}}^{(-k)}.$

But if V comes with a \mathbb{C} -structure, given by

$$\mathbb{C} \longrightarrow \text{End}_{\mathbb{R}}(V),$$

and $u = \text{this map restricted to } U_1$, then $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$

has an action of $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$ and

induced decomposition by idempotents

$$V_{\mathbb{C}} = V_{\mathbb{C}}^{(+)} \oplus V_{\mathbb{C}}^{(-)}$$

is the grading induced by u .

Moreover, natural map $V \hookrightarrow V_{\mathbb{C}} \twoheadrightarrow V_{\mathbb{C}}^{(+)}$

is a \mathbb{C} -linear isom.

\Rightarrow this gives restrictions on what $u_x: U_1 \rightarrow \text{GL}(\mathbb{R})^+$ can look like.

write $\mathfrak{g} = \text{Lie}(G(\mathbb{R})^+)$. And

$$\begin{array}{ccc}
 U_1 & \xrightarrow{ux} & G(\mathbb{R})^+ \xrightarrow{\text{Ad}} \text{Aut}(G(\mathbb{R})^+) \\
 & \searrow \text{Ad}(ux) & \downarrow \\
 & & \text{GL}(\mathfrak{g}).
 \end{array}$$

Proposition Grading on $\mathfrak{g}_{\mathbb{C}}$ induced by $\text{Ad}(ux)$ has the form

$$\mathfrak{g}_{\mathbb{C}} = \underbrace{\mathfrak{g}_{\mathbb{C}}^{(-1)}}_{T_{\mathbb{R}}X} \oplus \underbrace{\mathfrak{g}_{\mathbb{C}}^{(0)}}_{\text{Lie}(K_X)_{\mathbb{C}}} \oplus \underbrace{\mathfrak{g}_{\mathbb{C}}^{(1)}}_{T_{\mathbb{R}}X}.$$

Moreover, $\Theta := \text{Ad}(ux)(-1) \in \text{GL}(\mathfrak{g})$ decomposes

$$\mathfrak{g} = \underbrace{(\text{Lie } K_X)}_{\Theta = 1} \oplus \underbrace{T_{\mathbb{R}}X}_{\Theta = -1}.$$

as real-vs.

descends
from
 \mathbb{C} to \mathbb{R} .

Proof. start from $\mathcal{G}(\mathcal{K})^+ / \mathcal{K}_X \xrightarrow{\sim} X$.
 $g \mapsto gx$

Tangent space gives

$$\mathcal{G} / \text{Lie}(\mathcal{K}_X) \xrightarrow{\sim} T_x X \quad (\text{as } \mathbb{R}\text{-vs}).$$

Since $u_x: \mathcal{U}_1 \rightarrow \mathcal{K}_X$ lands in $\mathcal{Z}(\mathcal{K}_X)$,

$\text{Ad}(u_x): \mathcal{U}_1 \rightarrow \text{Aut}(\mathcal{K}_X)$ is trivial
 (since its conjugation).

So $\text{Ad}(u_x): \mathcal{U}_1 \rightarrow \text{GL}(\mathcal{G})$

induces trivial action on $\text{Lie}(\mathcal{K}_X) \subset \mathcal{G}$.

So we have induced action of \mathcal{U}_1 on $\mathcal{G} / \text{Lie}(\mathcal{K}_X)$ and

$$\mathcal{G} / \text{Lie}(\mathcal{K}_X) \xrightarrow{\sim} T_x X$$

is \mathcal{U}_1 -equivariant.

But recall $\mathcal{U}_1 \hookrightarrow T_x X$ via $\mathcal{U}_1 \subset \mathbb{C}^X$.

In particular (\mathcal{U}_1 -equiv)

$$0 \longrightarrow \text{Lie}(\mathcal{K}_X) \xrightarrow{\oplus=1} \mathcal{G} \xrightarrow{\oplus=-1} T_x X \longrightarrow 0$$

So we have splitting

$$\mathcal{G} = \text{Lie}(\mathcal{K}_X) \oplus T_x X.$$

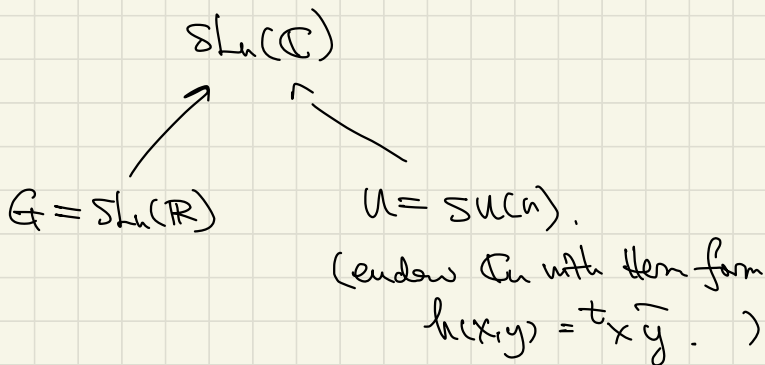
$$\oplus=1 \Rightarrow u_x(t)=1 \quad u_x(t)=t \quad \Leftarrow \quad \oplus=-1$$

Apply $-\otimes_{\mathbb{R}} \mathbb{C}$ to get the claim.

□

Cartan involution (Definition). — What is Θ ?

Example: Inside $SL_n(\mathbb{C})$, we have 2 real Lie subgroups

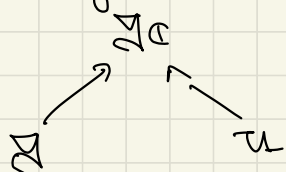


(1) $U = \text{compact}$

(2) "nice" relative position

$$G \cap U = SO(n)$$

On Lie algs we have



\exists Cartan decomposition

$$\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$$

where $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ inside $\mathfrak{g}_{\mathbb{C}}$.

\mathfrak{p} is NOT a Lie-subalg!

In fact,

$$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$$

$$[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$$

$$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$$

Remark: G, U are both simply connected, so

G -fd reps of $G \leftrightarrow$ reps of $\mathfrak{g} \leftrightarrow$ reps of $\mathfrak{g}_{\mathbb{C}}$

— U — of $U \leftrightarrow$ reps of $\mathfrak{k} \leftrightarrow$ reps of \mathfrak{u}

(So eg can show G -fd reps of G decompose by showing that for U : (arg argument.)

The Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$
induces "Cartan inv" $\Theta \in \text{GL}(\mathfrak{g})$ by

$$\Theta = \begin{cases} 1 & \text{on } \mathfrak{k} \\ -1 & \text{on } \mathfrak{p} \end{cases}.$$

NB: There's natural complex conj on

$$\mathfrak{u}_{\mathbb{C}} = \mathfrak{u} \otimes_{\mathbb{R}} \mathbb{C}$$

↑
=

under $\mathfrak{u}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$, which stabilizes $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ and $\mathfrak{i} \Theta$.

will be same Θ
in $\text{Ad}(u_X)(-1)$.