Assignment 1

due midnight (Eastern Time), Friday, Sept. 11, 2020

Prove all statements in the questions below. You must TeX your solutions and submit your writeup as a pdf at www.gradescope.com.
You can submit it any time until the deadline (try not to spend your Friday evening working on this!). (Use the course code listed in the syllabus to enroll in our class on gradescope.).

Question 1. (Folland 1.2.4) An algebra $\mathcal{A}$ is a σ-algebra iff $\mathcal{A}$ is closed under countable increasing unions (i.e., if $\{E_j\}_1^\infty \subset \mathcal{A}$ and $E_1 \subset E_2 \subset \dots$, then $\bigcup_1^\infty E_J \in \mathcal{A}$).

Question 2. (Folland 1.2.5) If $\mathcal{M}$ is the σ-algebra generated by $\mathcal{E}$, then $\mathcal{M}$ is the union of the σ-algebras generated by $\mathcal{F}$ as $\mathcal{F}$ ranges over all countable subsets of $\mathcal{E}$. (Hint: show that the latter object is a σ-algebra.)

Question 3. (Folland 1.3.9). If $(X, \mathcal{M}, \mu)$ is a measure space and $E, F \in \mathcal{M}$, then $\mu(E) + \mu(F) = \mu(E \cup F) + \mu(E \cap F)$.

Question 4. (Folland 1.3.11.) A finitely additive measure μ is a measure iff it is continuous from below as in Theorem 1.8c. If $\mu(X) < \infty$, μ is a measure iff it is continuous from above as in Theorem 1.8d.

Question 5. (Folland 1.3.12) Let $(X, \mathcal{M}, \mu)$ be a finite measure space.

a. If $E, F \in \mathcal{M}$ and $\mu(E \triangle F) = 0$, then $\mu(E) = \mu(F)$.

b. Say that $E \sim F$ if $\mu(E \triangle F) = 0$; then $\sim$ is an equivalence relation on $\mathcal{M}$.

c. For $E, F \in \mathcal{M}$, define $\rho(E, F) = \mu(E \triangle F)$. Then $\rho(E, G) \leq \rho(E, F) + \rho(F, G)$, and hence ρ defines a metric on the space $\mathcal{M}/\sim$ of equivalence classes.