Derived Schemes

Let SCR be the ∞-cat. of simplicial comm. rings (perhaps best viewed in terms of animation).

We let Anim be the ∞-cat. of anima, more traditionally referred to as spaces or ∞-gpds.

Following the last talk, we define the ∞-cat. of affine derived schemes to be $\mathrm{DAff} := \mathrm{SCR}^{op}$.

Grothendieck tells us that the cat. Sch of schemes can be understood in terms of fpqc descent, and we aim to generalize this to the derived setting. First, though, we make a few remarks about ∞-topoi.

∞-Topoi

One important insight in alg. geometry is that it is often helpful to work not w/ bona fide open sets (i.e., a genuine topology in the classical sense) but w/ open coverings. This is the perspective of a pretopology. More generally, one considers Grothendieck topologies built from things called sieves (and pretopologies generate Grothendieck topologies in an obvious way). (Site is cat. equipped w/ Grothendieck top.) This lets us make sense of a notion of sheaves. These form a full subcat. of the cat. of presheaves, w/ there being a left exact localization functor from presheaves to sheaves called sheafification. This gives an extrinsic characterization of a kind of cat. called a topos. We can give an intrinsic description in terms of the so-called Giraud axioms.

In the ∞-categorical setting we can generalize all three descriptions. Given an ∞-cat. $\mathcal{J}$, one now considers the presheaf cat. $\mathcal{P}_\infty(\mathcal{J}) := \mathrm{Fun}(\mathcal{J}^{op}, \mathrm{Anim})$. ["$\infty$" is for emphasis here – will probably drop in the future...]

(finiteness condition...)

Thm: An ∞-cat. $\mathcal{X}$ satisfies generalized Giraud axioms iff $\mathcal{X}$ is accessible and there is adjunction

$L: \mathcal{P}_\infty(\mathcal{C}) \rightleftarrows \mathcal{X} : i$ w/ L left exact, i fully faithful, $\mathcal{C}$ small ∞-cat.

Intuitively, L encodes sheafification as above but this is not quite right.

$X \in \mathrm{Top}$, $\mathcal{B}$ basis for X. Assume $\mathcal{B}$ <u>intersection closed</u> – i.e., given $\mathcal{U} \subseteq X$ open, $\exists$ covering by $\mathcal{U}_i \in \mathcal{B}$ s.t. $\mathcal{U}_{i_0} \cap \cdots \cap \mathcal{U}_{i_n} \in \mathcal{B}$ for all finite subsets $\{i_0, \dots, i_n\} \subseteq I$.

Given $\mathcal{F} \in \mathcal{P}(X)$, TFAE:
(1) $\mathcal{F}$ is a sheaf.
(2) given $\mathcal{U} \subseteq X$ open w/ $\mathcal{U}_i \in \mathcal{B}$ covering $\mathcal{U}$, $\mathcal{F}(\mathcal{U}) \xrightarrow{\sim} \lim\left(\prod_i \mathcal{F}(\mathcal{U}_i) \rightrightarrows \prod_{ij} \mathcal{F}(\mathcal{U}_i \cap \mathcal{U}_j) \right)$.

With some additional constraints/assumptions this is equiv. to Mayer–Vietoris – type condition.

<u>Def</u>: Let $\mathcal{J}$ be ∞-cat.

- <u>Sieve</u> on $X \in \mathcal{J}$ is full subcat. $T \subseteq \mathcal{J}_{/X}$ s.t. $\left[\begin{tikzcd} Y' \arrow[r] \arrow[dr] & Y \arrow[d] \\ & X \end{tikzcd}\right]$, $\begin{matrix} Y' \\ \downarrow \\ Y \end{matrix} \in \mathcal{J}_{/X}$ w/ $Y \in T \Rightarrow Y' \in T$.
- Let $f \in \mathrm{Hom}_{\mathcal{J}}(X, Y)$ and T sieve on Y. We have <u>pullback</u> $f^*T \subseteq \mathcal{J}_{/X}$ which is full subcat. spanned by $\mathcal{U} \to X$ s.t. $(\mathcal{U} \to X \to Y) \in T$.

<u>Notice</u>: Sieve approach does <u>not</u> specify any notion of "openness."

(<u>Remark</u>: For both, it should be possible to "double-count" some $X \in \mathcal{J}$.)

- <u>Grothendieck top.</u> τ on $\mathcal{J}$ is collection of sieves T (<u>covering sieves</u>) for each $X \in \mathcal{J}$ s.t.
  - trivial sieve $\mathcal{C}_{/X}$ is cov.
  - pullbacks of cov. sieves are cov. sieves
  - Let T be cov. sieve on X and S <u>any</u> sieve on X s.t. f^*S is cov. sieve $\forall f \in T$. Then, S is cov. sieve.

<u>Def</u>: <u>Pretop.</u> $\tilde{\tau}$ on $\mathcal{J}$ (assuming $\mathcal{J}$ has pullbacks) is assignment $\{\mathcal{U}_\alpha \to X\}_\alpha$ of <u>coverings</u> for each $X \in \mathcal{J}$ s.t.

- isom. $Y \xrightarrow{\sim} X$ yields cov. $\{Y \xrightarrow{\sim} X\}$.
- $\{\mathcal{U}_\alpha \to X\}_\alpha$ cov. and $\{V_{\alpha\beta} \to \mathcal{U}_\alpha\}_\beta$ is cov. $\forall \alpha \Rightarrow \{V_{\alpha\beta} \to X\}_{\alpha,\beta}$ is cov.
- $\{\mathcal{U}_\alpha \to X\}_\alpha$ cov. $\Rightarrow \{\mathcal{U}_\alpha \times_X Y \to Y\}_\alpha$ is cov. $\forall\, Y \to X$.

Pretop. $\tilde{\tau}$ generates top. s.t. sieve T on $X \in \mathcal{J}$ is cov. sieve iff it contains $\{\mathcal{U}_\alpha \to X\}_\alpha$ from $\tilde{\tau}$.

Intuitively, we see what we can build from cov.'s in $\tilde{\tau}$.

Let $\mathcal{T}$ be ∞-site, so an ∞-cat. equipped w/ Grothendieck top. (can assume $\mathcal{T}$ has fiber products) Each $X \in \mathcal{T}$ induces functor-of-pts.

$h_X \in \mathcal{P}_\infty(\mathcal{T})$. To $X \in \mathcal{T}$ and $\mathcal{U} \to h_X$ a map of presheaves we may associate $T(\mathcal{U})$ the full subcat. of $\mathcal{T}_{/X}$ spanned by $Y \to X$ s.t. $h_Y \to h_X$ factors through $\mathcal{U}$. This $T(\mathcal{U})$ is a sieve and so it makes sense to ask if $T(\mathcal{U})$ is a covering sieve. We say $\mathcal{F} \in \mathcal{P}_\infty(\mathcal{T})$ is a <u>sheaf</u> if

$$\mathrm{Hom}_{\mathcal{P}_\infty(\mathcal{T})}(h_X, \mathcal{F}) \to \mathrm{Hom}_{\mathcal{P}_\infty(\mathcal{T})}(\mathcal{U}, \mathcal{F}) \text{ is equiv. } \forall\, \mathcal{U} \to h_X \text{ s.t. } T(\mathcal{U}) \text{ is a covering sieve.}$$

We obtain $\mathrm{Shv}_\infty(\mathcal{T}) \subseteq \mathcal{P}_\infty(\mathcal{T})$ full subcat. This has a "computational" equivalent.

<u>Prop</u>: Let $\mathcal{T}$ be ∞-site, $\tilde{\tau}$ pretop. generating the Grothendieck top. on $\mathcal{T}$, $\mathcal{F} \in \mathcal{P}_\infty(\mathcal{T})$. Then,

$$\mathcal{F} \text{ is a sheaf iff } \mathcal{F}(X) \xrightarrow{\sim} \lim\Big(\prod_\alpha \mathcal{F}(\mathcal{U}_\alpha) \rightrightarrows \prod_{\alpha,\beta} \mathcal{F}(\mathcal{U}_\alpha \times_X \mathcal{U}_\beta) \substack{\rightarrow\\\rightarrow\\\rightarrow} \cdots\Big) \ \forall \{\mathcal{U}_\alpha \to \mathcal{U}\}_{\alpha \in \Lambda} \text{ in } \tilde{\tau}.$$

Given $\mathcal{T}$ an ∞-site, $\mathrm{Shv}_\infty(\mathcal{T})$ is an ∞-topos. However, not all ∞-topoi arise in this way. (Also, not all ∞-topoi are <u>hypercomplete</u>.)

To any ∞-cat. $\mathcal{C}$ (okay, satisfying mild assumptions) we may associate $\mathrm{Shv}_{\mathcal{C}}(\mathcal{T})$ defined as above but w/ $\mathcal{P}_{\mathcal{C}}(\mathcal{T}) := \mathrm{Fun}(\mathcal{T}^{op}, \mathcal{C})$.

Note that $\mathrm{Shv}_\infty(\mathcal{T})$ and $\mathrm{Shv}_{\mathrm{Set}}(\mathcal{T})$ are generally very different. How does all of this compare w/ stacks?

Given ∞-topos $\mathcal{X}$ and ∞-cat. $\mathcal{C}$, a <u>$\mathcal{C}$-valued stack</u> on $\mathcal{X}$ is precisely an object of $\mathrm{Fun}_{\mathrm{cont}}(\mathcal{X}^{op}, \mathcal{C})$.

[$\mathcal{C}$ not specified means $\mathcal{C}$ is Anim.] This gives $\mathrm{Stk}_{\mathcal{C}}(\mathcal{X})$ and $\mathrm{Stk}(\mathcal{X}) := \mathrm{Stk}_{\mathrm{Anim}}(\mathcal{X})$.

<u>Prop</u>: $\mathcal{T}$ ∞-site, $\mathcal{C}$ complete ∞-cat. $\Rightarrow$ composition of Yoneda and sheafification $\rightsquigarrow$ $\mathrm{Stk}_{\mathcal{C}}(\mathrm{Shv}_\infty(\mathcal{T})) \xrightarrow{\sim} \mathrm{Shv}_{\mathcal{C}}(\mathcal{T})$.

<u>Remark</u>: Fibered perspective on stacks is also equivalent...

2. Derived schemes. Any scheme S represents a presheaf

$$X \mapsto \mathrm{Maps}(X, S)$$

on the category of schemes, which satisfies fpqc descent by a theorem of Grothendieck. The fact that every scheme admits an affine Zariski cover implies that the inclusion of affine schemes into arbitary schemes induces an equivalence at the level of Zariski or fpqc sheaves. Therefore there is a fully faithful embedding of the category of schemes into the category of sheaves on the affine fpqc site. On the other hand, if we identify its essential image, we can take this as our *definition* of scheme. This is the philosophy we will take in our definition of derived scheme.

Start here

2.1. The *fpqc pretopology* on $(\mathrm{SCRing})^{\mathrm{op}}$ is defined as follows.

Definition 2.2. *A family of homomorphisms* $(R \to R_\alpha)_{\alpha\in\Lambda}$ *is* fpqc covering *if the following conditions hold:*

(i) The set Λ *is finite.* ← Need this for technical reasons.

(ii) For each $\alpha \in \Lambda$, *the homomorphism* $R \to R_\alpha$ *is flat (i.e. the underlying* R*-module of* R_α *is flat).*

(iii) The induced homomorphism $R \to \prod_\alpha R_\alpha$ *is faithfully flat.*

Remark: Classically, fpqc top. is finer than Zariski, fppf, étale, and smooth top's.

Recall that a connective R-module M is *flat* if for any discrete R-module N, the tensor product $R \otimes_M N$ is discrete. It is *faithfully flat* if it is flat, and a connective R-module N is zero iff $M \otimes_R N$ is zero.

Should be same as $\pi_0 M$ flat as $\pi_0 R$-module and $\pi_i M \simeq \pi_i R \otimes_{\pi_0 R} \pi_0 M$ as $\pi_0 R$-modules.

Definition 2.3.

(i) *A* derived prestack *is a presheaf of spaces on* ~~$(\mathrm{SCRing})^{\mathrm{op}}$~~ DAff.

Maybe should say fpqc stack... → (ii) *A* derived stack *is an fpqc sheaf of spaces on* ~~$(\mathrm{SCRing})^{\mathrm{op}}$~~ DAff, *i.e. a derived prestack which satisfies fpqc descent.* DStk

Let us recall the descent condition in this setting. Let $(R \to R_\alpha)_\alpha$ be an fpqc covering family, and write $\tilde{R} = \prod_\alpha R_\alpha$. Let $\check{C}(R/\tilde{R})_\bullet$ denote the Čech nerve of $R \to \tilde{R}$, a cosimplicial object given degree-wise by the $(n+1)$-fold tensor product

$$\check{C}(R/\tilde{R})^n = \tilde{R} \otimes_R \cdots \otimes_R \tilde{R}.$$

$A \to B \rightsquigarrow A \to B \rightrightarrows B \otimes_A B \ \substack{\to\\\to\\\to} \ B \otimes_A B \otimes_A B \cdots$

Now, a derived prestack $\mathcal{X}$ satisfies fpqc descent if for all such fpqc covering families, the canonical morphism

$$\mathcal{X}(R) \xrightarrow{\sim} \varprojlim_{n\in\Delta} \mathcal{X}(\check{C}(R/\tilde{R})^n) \tag{2.1}$$

~~is invertible.~~

2.4. Given a simplicial commutative ring R, we let Spec(R) denote the derived prestack represented by R.

Proposition 2.5. *For any simplicial commutative ring* R, *the presheaf* Spec(R) *is an fpqc sheaf. In particular, the fpqc topology is subcanonical.*

This follows from the fact that, for any fpqc covering morphism $A \to B$ in SCRing, the canonical morphism $A \to \varprojlim_{n\in\Delta} \check{C}(A/B)^n$ is invertible. This can be shown using the associated Bousfield–Kan spectral sequence, which degenerates on the second page. Alternatively it follows immediately from some general machinery developed by Lurie, see [2, Thm. D.6.3.5]. ← SAG

Definition 2.6. *An* affine derived scheme *is a derived stack which is isomorphic to* Spec(R) *for some simplicial commutative ring* R.

In other words, this is the essential image of DAff in DStk.

~~We let $\mathrm{DSch}^{\mathrm{aff}}$ denote the ∞-category of derived affine schemes, which is equivalent to $(\mathrm{SCRing})^{\mathrm{op}}$ by construction.~~

2.7. In order to give the definition of derived scheme, we need to define the notion of *open immersion* between derived stacks.

We begin with the following preliminary definition:

Definition 2.8. *A homomorphism of simplicial commutative rings* $\mathrm{R} \to \mathrm{R}'$ *is* locally of finite presentation *if it exhibits* R' *as a compact object of* $\mathrm{SCRing}_{\mathrm{R}}$, *i.e. if the functor*

$$\mathrm{A} \mapsto \mathrm{Maps}_{\mathrm{SCRing}_{\mathrm{R}}}(\mathrm{R}', \mathrm{A})$$

commutes with filtered colimits.

Q: How does this compare w/ "finite pres." and "almost of finite pres."?

Now let $j : \mathcal{U} \to \mathcal{X}$ be a morphism of derived stacks. First suppose that $\mathcal{X} = \mathrm{Spec}(\mathrm{R})$ and $\mathcal{U} = \mathrm{Spec}(\mathrm{A})$ are both affine. In this case we say that j is an *open immersion* if the corresponding homomorphism $\mathrm{R} \to \mathrm{A}$ is locally of finite presentation, flat, and an epimorphism, i.e. the co-diagonal homomorphism $\mathrm{A} \otimes_{\mathrm{R}} \mathrm{A} \to \mathrm{A}$ is invertible.

Isn't this truncated somehow?

Next suppose that $\mathcal{U}$ is possibly non-affine. Then we say that j is an open immersion if it is a monomorphism, and there exists a family $(\mathcal{U}_\alpha \to \mathcal{U})_\alpha$ which induces an effective epimorphism[1] $\coprod_\alpha \mathcal{U}_\alpha \to \mathcal{U}$, such that each $\mathcal{U}_\alpha$ is affine, and each composite $\mathcal{U}_\alpha \to \mathcal{X}$ is an open immersion of affine derived schemes.

$$\begin{tikzcd} & \mathcal{U}_\alpha \arrow[dl] \arrow[d, hook] \\ \mathcal{U} \arrow[r, hook] & \mathcal{X} \end{tikzcd}$$

Finally, we define j to be an open immersion in the general case if, for any affine derived scheme $\mathrm{Spec}(\mathrm{R})$ and any morphism $\mathrm{Spec}(\mathrm{R}) \to \mathcal{X}$, the base change $\mathcal{U} \times_{\mathcal{X}} \mathrm{Spec}(\mathrm{R}) \to \mathrm{Spec}(\mathrm{R})$ is an open immersion in the above sense.

2.9. We are now ready to give the definition of derived scheme.

Definition 2.10.

(i) *A* Zariski cover *of a derived stack* $\mathcal{X}$ *is a family* $(j_\alpha : \mathcal{U}_\alpha \hookrightarrow \mathcal{X})_\alpha$ *where each* j_α *is an open immersion, and the induced morphism*

$$\coprod_\alpha \mathcal{U}_\alpha \to \mathcal{X}$$

is an effective epimorphism.

(ii) *An* affine Zariski cover *of a derived stack* $\mathcal{X}$ *is a Zariski cover* $(\mathcal{U}_\alpha \hookrightarrow \mathcal{X})_\alpha$ *where each* $\mathcal{U}_\alpha$ *is an affine derived scheme.*

sometimes called an atlas

(iii) *A derived stack* $\mathcal{X}$ *is* schematic *if it admits an affine Zariski cover. A* derived scheme *is a schematic derived stack.*

Effectivity equiv. to epi. of sheaves on π_0

We get $\mathrm{DSch} \subseteq \mathrm{DStk}$.

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[1]Recall that a morphism of sheaves $\mathcal{X} \to \mathcal{Y}$ is an *effective epimorphism* if the canonical morphism of sheaves

$$\varinjlim_{n \in \Delta^{\mathrm{op}}} \check{\mathrm{C}}(\mathcal{X}/\mathcal{Y})_n \to \mathcal{Y}$$

is invertible. Here $\check{\mathrm{C}}(\mathcal{X}/\mathcal{Y})_\bullet$ is the Čech nerve, a simplicial object with $\check{\mathrm{C}}(\mathcal{X}/\mathcal{Y})_n = \mathcal{X} \times_{\mathcal{Y}} \cdots \times_{\mathcal{Y}} \mathcal{X}$ (the $(n+1)$-fold fibred product).

__Classical scheme__ admits affine Zariski cover by __classical affine schemes__ (underlying simplicial ring is discrete). Such X is then discrete as a presheaf in the sense that it is Set-valued.

Any derived prestack X has restriction X_{cl} to the classical site. If $X = \operatorname{Spec} R$ then this is $X_{cl} = \operatorname{Spec} \pi_0 R$.

$$\begin{tikzcd} \mathrm{Aff} \arrow[r, hook] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowright"] & \mathrm{DAff} \arrow[d, hook] \\ \mathrm{Sch} \arrow[r, hook] & \mathrm{DSch} \end{tikzcd}$$

$\pi_0 : SCR \rightleftarrows CRing$

$\leftarrow (\cdot)_{cl}$ is adjoint to the inclusion

__Fact__: Let $X \in DSch$. Then, X is affine iff $X_{cl} \in Sch$ is affine.