Basics of Perfectoid Rings

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Our goal is to compare and contrast various algebro-geometric perspectives on perfectoid rings. Recall that we call a (commutative unital) ring S perfectoid if

(1) there exists $\pi \in S$ such that $\pi^p \mid p$ and S is π-adically complete;[1]

(2) S/p is semi-perfect – i.e., the Frobenius map $\varphi : S/p \to S/p$ is surjective; and

(3) $\ker(\theta : \mathbb{A}_{\text{inf}}(S) \to S)$ is principal.

An element π as above (which is typically not unique) is often called a **pseudo-uniformizer**.

Lemma 0.1. *Suppose S satisfies condition* (1) *above. TFAE:*

(i) *Every element of $S/\pi p$ is a pth power.*

(ii) *Every element of S/p is a pth power.*

(iii) *Every element of S/π^p is a pth power.*

(iv) *$F : W_{r+1}(S) \to W_r(S)$ is surjective for every $r \geq 1$.*

(v) *θ_r is surjective for every $r \geq 1$.*

Moreover, if any of the above conditions hold then there exist $u, v \in S^\times$ such that $u\pi$ and vp admit systems of p-power roots in S.

Another way of viewing the moreover condition above is that θ is surjective (by (v)) and there exist $u, v \in S^\times$ and $\alpha, \beta \in S^\flat$ such that $\theta([\alpha]) = u\pi$ and $\theta([\beta]) = vp$. For concreteness, recall that we have a factorization

$$\begin{tikzcd} S^\flat \arrow[r, "{[\cdot]}"] \arrow[dr, "(\cdot)^\#"'] & \mathbb{A}_{\text{inf}}(S) \arrow[d, "\theta"] \\ & S \end{tikzcd}$$

where $(\cdot)^\#$ is the "un-tilt" or "sharp map" that takes in $(\overline{x}_0, \overline{x}_1, \ldots) \in S^\flat \subseteq \prod_{n\geq 0} S/p$ and outputs the limit of $x_n^{p^n}$ as $n \to \infty$ for any choice of lifts $x_n \in S$.

Checking whether $\ker\theta$ is principal a priori seems somewhat difficult to do. Inspiration comes from the following observation.

[1] Part of the reason we want $\pi^p \mid p$ is that then, given any $x, z \in S$, $(x + \pi z)^p - x^p \in \pi^p S$.

Exercise 0.2. *Let $A \in \mathsf{CRing}$ and $\xi = \xi_0 + \xi_1 t + \cdots \in A[\![t]\!]$ a **distinguished** element – i.e., A is ξ_0-adically complete and $\xi_1 \in A^\times$. Show that $A[\![t]\!]/\xi \cong A$ canonically (we think of this as an evaluation procedure).*

In the above, the map $A[\![t]\!]/\xi \xrightarrow{\sim} A$ should be viewed as roughly analogous to $\mathbb{A}_{\mathrm{inf}}(S)/\ker\theta \xrightarrow{\sim} S$. Borrowing the above terminology, we call an element $\xi = (\xi_0, \xi_1, \ldots) \in \mathbb{A}_{\mathrm{inf}}(S)$ **distinguished** if $S^\flat$ is ξ_0-adically complete and $\xi_1 \in (S^\flat)^\times$. The following theorem demonstrates that under appropriate conditions distinguished elements correspond precisely to principal generators of $\ker\theta$.

Theorem 0.3. *Suppose S satisfies condition (1) above and that $\varphi : S/\pi \to S/\pi^p$ is surjective.*

(a) *Suppose $\ker\theta$ is principal. Then, φ is an isomorphism and any generator of $\ker\theta$ is an NZD.*

(b) *Conversely, suppose φ is an isomorphism and π is an NZD. Then, $\ker\theta$ is principal (and so S is perfectoid).*

Proof. By pre-multiplying π be an element of $S^\times$ if necessary and using the last part of Lemma 0.1, we may assume without loss of generality that there exists $\pi^\flat \in S^\flat$ such that $\theta([\pi^\flat]) = \pi$. Assuming that $\ker\theta$ is principal, we would first like to understand what generators of $\ker\theta$ look like. To that end, choose $x \in \mathbb{A}_{\mathrm{inf}}(S)$ such that $\theta(-x) = p/\pi^p$ and consider $\xi := p + [\pi^\flat]^p x \in \ker\theta$.[2] If now $\xi' = (\xi'_0, \xi'_1, \ldots) \in \mathbb{A}_{\mathrm{inf}}(S)$ is a principal generator of $\ker\theta$ then $\xi = \xi' a$ for some $a \in \mathbb{A}_{\mathrm{inf}}(S)$. Comparing Witt vector expansions, we have

$$((\pi^\flat)^p x_0, 1 + (\pi^\flat)^{p^2} x_1, \ldots) = (\xi'_0 a_0, (\xi'_0)^p a_1 + \xi'_1 a_0^p, \ldots).$$

We wish to show that $a_0 \in (S^\flat)^\times$ and hence that $a \in \mathbb{A}_{\mathrm{inf}}(S)^\times$.[3] To do this, it suffices to show that the image of a_0 in S/π under projection is a unit, remembering that $S^\flat \cong \varprojlim_\varphi S/\pi$.[4] From the above we get $\xi'_1 a_0^p = 1 + (\pi^\flat)^{p^2} x_1 - (\xi'_0)^p a_1 \in S^\flat$. Using the commutative diagram

$$\begin{array}{ccc} S^\flat & \xrightarrow{(\cdot)^\#} & S \\ {\scriptstyle\cong}\downarrow & & \downarrow \\ \varprojlim_\varphi S/\pi & \longrightarrow & S/\pi \end{array}$$

we see that the image under projection of $\pi^\flat$ is trivial. Meanwhile, the fact that $\xi' \in \ker\theta$ shows that the image under projection of ξ'_0 is also trivial. Hence, $\xi'_1 a_0^p \equiv 1 \pmod{\pi S}$ and we conclude that $a \in \mathbb{A}_{\mathrm{inf}}(S)^\times$.

(a) Since $\ker\theta$ is principal, the above argument shows that ξ as above generates $\ker\theta$. We thus obtain an isomorphism $\overline{\theta} : \mathbb{A}_{\mathrm{inf}}(S)/\xi \xrightarrow{\sim} S$ induced by θ fitting into a commutative diagram

[2] The symbol p here refers to the p-adic expansion of the Witt vector $V(1)$, which satisfies $\theta(p) = p \in S$ since θ is a ring homomorphism. This notation emphasizes that ξ is "almost p" in a sense that can be made precise.

[3] Remember that $S^\flat$ is perfect with characteristic p and so this is easy to verify directly.

[4] Let A be a characteristic p ring with Frobenius φ and $x = (x_0, x_1, \ldots) \in \varprojlim_\varphi A$ such that x_0 has inverse $y_0 \in A$. It is then easily seen that x is a unit in $\varprojlim_\varphi A$ with inverse $(y_0, x_1^{p-1} y_0, x_2^{p^2-1} y_0, \ldots)$.

$$
\begin{tikzcd}
\mathbb{A}_{\inf}(S)/\xi \arrow[r, "\overline{\theta}"] \arrow[d] & S \arrow[d] \\
\mathbb{A}_{\inf}(S)/(\xi, [\pi^\flat]^p) \arrow[r] & S/\pi^p
\end{tikzcd}
$$

where the bottom horizontal arrow is the composition

$$\mathbb{A}_{\inf}(S)/(\xi, [\pi^\flat]^p) = W(S^\flat)/(p, [\pi^\flat]^p) \cong S^\flat/(\pi^\flat)^p \twoheadrightarrow S/\pi^p$$

induced by the projection

$$S^\flat \cong \varprojlim_{\varphi} S/\pi^p \twoheadrightarrow S/\pi^p.$$

It follows that the map $S^\flat/(\pi^\flat)^p \to S/\pi^p$ is an isomorphism. The map $\varphi : S/\pi \to S/\pi^p$ fits into a commutative diagram

$$
\begin{tikzcd}
S^\flat/\pi^\flat \arrow[r, "\sim"] \arrow[d] & S^\flat/(\pi^\flat)^p \arrow[d, "\cong"] \\
S/\pi \arrow[r, "\varphi"'] & S/\pi^p
\end{tikzcd}
$$

and so is injective.[5] We conclude that φ is an isomorphism since it is surjective by assumption. To see that ξ (and hence any principal generator of $\ker\theta$) is an NZD, let $b \in \mathbb{A}_{\inf}(S)$ such that $\xi b = 0$. Given $r \geq 1$ odd, we know that $\xi = p + [\pi^\flat]^p x$ divides $p^r + [\pi^\flat]^{pr} x^r$ and so $(p^r + [\pi^\flat]^{pr} x^r)b = 0$. Hence, $p^r b \in [\pi^\flat]^{pr} \mathbb{A}_{\inf}(S)$ and so, writing $b = (b_0, b_1, \dots)$, we conclude $b_i^{p^r} \in (\pi^\flat)^{rp^{r+i+1}} S^\flat$ for every $i \geq 0$.[6] Since $S^\flat$ is perfect we get $b_i \in (\pi^\flat)^{rp^{i+1}} S^\flat$ and so, since we may take r arbitrarily large and $S^\flat$ is $\pi^\flat$-adically complete and separated, $b_i = 0$ for every $i \geq 0$ hence $b = 0$.

(b) By assumption we have $\varphi : S/\pi \xrightarrow{\sim} S/\pi^p$, which induces isomorphisms $S/\pi^{1/p^n} \cong S/\pi^{1/p^{n-1}}$ for every $n \geq 0$. We claim first that $\ker(S^\flat \twoheadrightarrow S/\pi)$ is generated by $\pi^\flat$. To see this, let $y \in \ker(S^\flat \twoheadrightarrow S/\pi)$ and write $\pi^\flat = (\pi, \pi^{1/p}, \pi^{1/p^2}, \dots) \in \varprojlim_{(\cdot)^p} S$. We may write

$$y = (y^{(0)}, y^{(1)}, \dots) \in \varprojlim_{(\cdot)^p} S \text{ with } y^{(0)} \in \pi S.$$

The isomorphism $S/\pi \cong S/\pi^{1/p}$ forces $\pi^{1/p} \mid y^{(1)}$ and, inductively,

$$S/\pi^{1/p^n} \cong S/\pi^{1/p^{n-1}} \text{ forces } \pi^{1/p^n} \mid y^{(n)} \text{ for every } n \geq 0.$$

Hence, $\pi^\flat \mid y$ in $\varprojlim_{(\cdot)^p} S$ and so $\pi^\flat$ generates $\ker(S^\flat \twoheadrightarrow S/\pi)$. As above we thus have a commutative diagram

$$
\begin{tikzcd}
S^\flat/\pi^\flat \arrow[r, "\sim"] \arrow[d, "\cong"'] & S^\flat/(\pi^\flat)^p \arrow[d] \\
S/\pi \arrow[r, "\sim", "\varphi"'] & S/\pi^p
\end{tikzcd}
$$

which in turn forces $S^\flat/(\pi^\flat)^p \xrightarrow{\sim} S/\pi^p$ and gives a commutative diagram[7]

[5] The upper horizontal arrow in this diagram is always an isomorphism since $S^\flat$ is perfect.

[6] Recall that, given $f \in S^\flat$ and $z = (z_0, z_1, \dots) \in \mathbb{A}_{\inf}(S)$, $[f]z = (fz_0, f^p z_1, f^{p^2} z_2, \dots)$.

[7] This observation tells us that the bottom horizontal arrow is an isomorphism.

$$\begin{array}{ccc}
\mathbb{A}_{\inf}(S) & \xrightarrow{\quad\theta\quad} & S \\
\downarrow & & \downarrow \\
\mathbb{A}_{\inf}(S)/(\xi, [\pi^\flat]^p) & \xrightarrow{\sim} & S/\pi^p
\end{array}$$

Given $z \in \ker\theta$, we therefore have $y_0, z_0' \in \mathbb{A}_{\inf}(S)$ such that $z = \xi y_0 + [\pi^\flat]^p z_0'$. Then,

$$\pi^p\theta(z_0') = \theta([\pi^\flat]^p z_0') = 0 \implies \theta(z_0') = 0$$

since π is an NZD and so we can apply the same procedure to z_0'. We may thus inductively write $z = \xi(y_0 + [\pi^\flat]^p y_1 + [\pi^\flat]^{p^2} y_2 + \cdots)$. □

Remark 0.4. *Here is another way to view the above proof. In particular, we get a more natural perspective on condition* (3) *in the definition of perfectoid. ...*

Where is the geometry in all of this?

Definition 0.5. *A* ***complete Tate ring*** *is a complete topological ring*[8] *R for which there exists an open subring R_0 such that $R = R_0[1/\pi]$ and the topology on R_0 is π-adic for some $\pi \in R_0$.*[9]

The subring R_0 is not considered to be part of the data of R and is not unique. In practice, complete Tate rings are often constructed by first defining R_0 and then inverting an appropriate element π. Before looking at some examples, let's introduce a bit more terminology that will help us describe such objects.

Definition 0.6.

- *A subset $X \subseteq R$ is* ***bounded*** *if for every $n \geq 1$ there exists $N \geq 1$ such that $X \cdot \pi^N R_0 \subseteq \pi^n R_0$. Equivalently, there exists $N \geq 1$ such that $X \subseteq \pi^{-N} R_0$. For convenience, we call any choice of such N a* ***bounding exponent****.*
- *Let R° denote the set of* ***power-bounded*** *elements $x \in R$ satisfying that $\{x^k : k \geq 0\} \subseteq R$ is bounded. We say R is* ***uniform*** *if R° is itself bounded – i.e., there is a uniform bounding exponent for all elements of R°.*
- *Let $R^{\circ\circ} \subseteq R^\circ$ denote the collection of* ***topologically nilpotent*** *elements $x \in R$ satisfying that $x^k \to 0$.*
- *A* ***ring of integral elements*** *is an open integrally closed subring $R^+ \subseteq R^\circ$.*[10] *From this perspective, R° is a maximal ring of integral elements.*

Remark 0.7.

[8] By definition, a complete topological ring is a Hausdorff topological ring such that every Cauchy net converges.

[9] This latter condition means that the π-adic topology on R_0 (which makes it into a complete Hausdorff space) is equivalent to the subspace topology inherited from R. Note that only sequences and not general nets should be needed to assess completeness since the π-adic topology is first-countable. This should translate over to the whole of R since all we do is invert p.

[10] Note that R° is integrally closed in R. It is not necessarily true, however, that R° is open in R. Hence, a ring of integral elements is not the same thing as an open integrally closed subring of R that is contained in R°.

- *The set R° forms a subring of R since if $x, y \in R^\circ$ with bounding exponents M, N then any power of xy or $x+y$ has bounding exponent $M+N$.*
- *If $x \in R$ with $x^k \in R^\circ$ then $x \in R^\circ$.*[11]
- *The set $R^{\circ\circ}$ forms an ideal of R°.*
- *$R^{\circ\circ} \subseteq R^+$ and, more generally, if $x \in R$ with $x^k \in R^+$ then $x \in R^+$.*[12]
- *The above notions are closely related to those of Tate and affinoid k-algebras, defined over a nonarchimedean field k. In more detail, the data of an* ***affinoid k-algebra*** *is as follows. First we have R a* ***Tate k-algebra*** *– i.e., a topological k-algebra for which there exists a subring $R_0 \subseteq R$ such that $\{aR_0 : a \in k^\times\}$ is a basis of open neighborhoods of 0 in R.*[13] *Second we have $R^+ \subseteq R^\circ$ an open integrally closed subring. Associated to this is the space*

$$X = \mathrm{Spa}(R, R^+) := \{|\cdot| \text{ a continuous valuation on } R : |f| \leq 1 \text{ for every } f \in R^+\}/\sim$$

The role of R^+ is that it imposes necessary finiteness conditions on the "points" of X while still allowing X to have "enough" points. More precisely, if we assume R is complete (which can be done without loss of generality and mirrors our situation of interest) then

(a) *$X = \emptyset \implies R = 0$;*

(b) *if $f \in R$ such that $|f(x)| \neq 0$ for every $x \in X$ then f is invertible; and*

(c) *if $f \in R$ such that $|f(x)| \leq 1$ for every $x \in X$ then $f \in R^+$.*

The finiteness conditions ensure that X behaves like an affine scheme and has a structure sheaf which is, well, a sheaf.

Example 0.8.

(1) *Take $(R, R_0, \pi) = (\mathbb{Q}_p, \mathbb{Z}_p, p)$. In this case, R is uniform with $R^\circ = \mathbb{Z}_p$ and $R^{\circ\circ} = p\mathbb{Z}_p$. We see that $\mathbb{Z}_p$ is* ***the*** *ring of integral elements of $\mathbb{Q}_p$.*

(2) *Take $(R, R_0, \pi) = (\mathbb{Z}_p[\![t]\!][1/p], \mathbb{Z}_p[\![t]\!], p)$. In this case, R is uniform with $R^\circ = \mathbb{Z}_p[\![t]\!]$ and $R^{\circ\circ} = p\mathbb{Z}_p[\![t]\!]$. Note that $\mathbb{Z}_p[\![t]\!][1/p] \subseteq \mathbb{Q}_p[\![t]\!]$ is a proper subring since $\mathbb{Q}_p[\![t]\!]$ allows arbitrarily high powers of p in the denominator.*

(3) *Given A a ring with nonarchimedean valuation $|\cdot|$, define $A\langle t\rangle := \left\{\sum_{i\geq 0} a_i t^i \in A[\![t]\!] : |a_i| \to 0\right\}$, which is the ring of formal power series converging on the unit disc in A.*[14] *Then we may take $(R, R_0, \pi) = (\mathbb{Q}_p\langle t\rangle, \mathbb{Z}_p\langle t\rangle, p)$. This example is much like the previous two.*

(4) *For a non-uniform example, let $R_0 := \left\{\sum_{i\geq 0} a_i \in \mathbb{Z}_p[\![t]\!] : v_p(a_i) \geq \sqrt{i}\right\}$ and $R := R_0[1/p]$ (which contains $\mathbb{Q}_p[t]$ as a subring). Then, $p\mathbb{Z}_p[t] \subseteq R^\circ$ but is unbounded since it contains the unbounded set $\{pt, pt^2, pt^3, \ldots\}$.*

[11] This is not the same as R/R° being reduced since R° may not be an ideal of R and so the quotient may not even make sense.

[12] This is a general fact about normal ring extensions $A \subseteq B$. Indeed, if $x \in B$ with $x^k \in A$ then x is a root of $t^k - x^k \in A[t]$.

[13] As above, note that R_0 is not considered part of the data. The notion of boundedness is slightly different in this context: $X \subseteq R$ is bounded if $X \subseteq aR_0$ for some $a \in k^\times$.

[14] The valuation $|\cdot|$ extends to $A[\![t]\!]$ (hence any subring) by defining $|f - g| := \sup_{i\geq 0}\{|a_i - b_i|\}$ for a_i, b_i the ith coefficient of f, g.

We are now in a position to provide a more geometric perspective on perfectoid rings.

Definition 0.9. *A complete Tate ring R is **Fontaine perfectoid** if*

(1) *there exists a topologically nilpotent unit $\pi \in R$ such that $\pi^p \mid p$ in R°;*

(2) *the Frobenius map $\varphi : R^\circ/\pi \to R^\circ/\pi^p$ is surjective; and*

(3) *R is uniform.*

The π in the above definition may not be the same as the π in the definition of a complete Tate ring, though they are often the same in practice. We will take them to be the same for part (a) of the below theorem.

Theorem 0.10. *Let R be a complete Tate ring with R^+ a ring of integral elements.*

(a) *Suppose R is Fontaine perfectoid. Then, R^+ is perfectoid.*

(b) *Suppose R^+ is perfectoid and bounded. Then, R is Fontaine perfectoid.*

Proof.

(a) We first show that R° is perfectoid. The subring R° is bounded by assumption and thus π-adically complete.[15] Since by assumption $\varphi : R^\circ/\pi \to R^\circ/\pi^p$ is surjective and π is a unit hence an NZD, it suffices by Theorem 0.3 to show that φ is injective. To that end, let $x \in R^\circ$ such that $x^p = \pi y^p$ for some $y \in R^\circ$ – i.e., x represents an element of $\ker \varphi$. Since π is a unit, we may consider $z := x/\pi \in R$. Then,

$$z^p = y \in R^\circ \implies z \in R^\circ \implies x = \pi z \in \pi R^\circ$$

and so $\ker \varphi$ is trivial.

Now we show that R^+ is perfectoid. By definition, R^+ is open in R° and thus is complete.[16] As before it suffices to show that $\varphi : R^+/\pi \to R^+/\pi^p$ is an isomorphism. To check injectivity, either argue as above or use the commutative diagram

$$\begin{tikzcd} R^+/\pi \arrow[r, "\varphi"] \arrow[d, hook] & R^+/\pi^p \arrow[d, hook] \\ R^\circ/\pi \arrow[r, "\sim", "\varphi"'] & R^\circ/\pi^p \end{tikzcd}$$

To check surjectivity, it suffices to show that $\varphi : R^+/p \to R^+/p$ is surjective. To that end, let $x \in R^+$. Every element of $R^\circ/p\pi$ is a pth power and so we may write $x = y^p + p\pi z$ for some $y, z \in R^\circ$. Then,

$$z' := \pi z \in R^{\circ\circ} \subseteq R^+ \implies y^p = x - pz' \in R^+ \implies y \in R^+$$

and so φ sends y mod pR^+ to x mod pR^+.

[15] Limits of π-adic Cauchy sequences exist in R_0 since R_0 is a π-adically topologized subspace of the complete space R. Any such limit then lies in R° by boundedness, with bounding exponent any uniform bounding exponent of R°.

[16] The main idea here is that open sets in a nonarchimedean setting are closed, as can be seen by working locally with balls.

(b) To begin, R is uniform since R^+ is bounded and $\pi R^\circ \subseteq R^+$ (for π as in the definition of complete Tate ring). We seek $\pi \in R$ such that

(1) π is a topologically nilpotent unit satisfying $\pi^p \mid p$ and

(2) $\varphi : R^\circ/\pi \to R^\circ/\pi^p$ is surjective.

Skipping a few details, here we go.

(1) The tricky part is ensuring that π is a unit in R. Start by picking $\pi_0 \in R$ any topologically nilpotent unit, which is automatically an element of R^+. Choose a distinguished generator ξ of $\ker(\theta : \mathbb{A}_{\mathrm{inf}}(R^+) \twoheadrightarrow R^+)$.

<u>Fact</u>: We may use ξ to construct $\pi^\flat \in (R^+)^\flat$ and $u \in (R^+)^\times$ such that $\theta([\pi^\flat]) = u\pi_0$.[17]

Finally, taking $\pi := \theta([(\pi^\flat)^{1/p^n}])$ for $n \gg 1$ does the trick.

(2) We show $\varphi : R^\circ/p \to R^\circ/p$ is surjective hence $\varphi : R^\circ/\pi \to R^\circ/\pi^p$ is a fortiori surjective. Changing π by a unit if necessary, we may assume without loss of generality that π has a pth root $\pi^{1/p} \in R^+$. Given $x \in R^\circ$, we may write $\pi x \in R^+$ as $\pi x = y^p + p\pi z$ for some $y, z \in R^+$. Then, $y' := y/\pi^{1/p} \in R$ lies in R° since $(y')^p = x - pz \in R^\circ$. Hence, the equation $x = (y')^p + pz$ gives that φ sends y' mod pR° to x mod pR°. $\square$

Example 0.11. *Recall the perfectoid ring $\mathbb{Z}_p^{\mathrm{cycl}}$ defined to be the p-adic completion of $\mathbb{Z}_p[\zeta_p^{1/p^\infty}]$. If we let $R := \mathbb{Q}_p^{\mathrm{cycl}} = \mathbb{Q}_p(\zeta_p^{1/p^\infty})$ then $R^\circ = \mathbb{Z}_p^{\mathrm{cycl}}$ is bounded [Why?] and so $\mathbb{Q}_p^{\mathrm{cycl}}$ is Fontaine perfectoid by the above theorem. In the same vein, $\mathbb{Q}_p^{\mathrm{cycl}}\langle t^{1/p^\infty}\rangle$ is Fontaine perfectoid with bounded maximal ring of integral elements $\mathbb{Z}_p^{\mathrm{cycl}}\langle t^{1/p^\infty}\rangle$ that is perfectoid.*[18]

The following theorem makes even more clear the relationship between the notions of perfectoid and Fontaine perfectoid.

Theorem 0.12. *Let R_0 be a perfectoid ring with $\pi \in R_0$ an NZD satisfying condition (1) of the definition of perfectoid. Equip $R := R_0[1/\pi]$ with the topology induced by the π-adic topology on R_0. Then, R is a complete Tate ring which is Fontaine perfectoid and satisfies $\pi R^\circ \subseteq R_0$.*

The proof uses almost mathematics and so requires some setup. We will return to this. For now, let us first define a notion which has surprisingly not yet come up.

Definition 0.13.

- *A **(nonarchimedean) valuation** on a ring A is a multiplicative map $|\cdot| : A \to \Gamma \cup \{0\}$ with Γ a multiplicative totally ordered abelian group such that $|a| = 0 \iff a = 0$, $|1| = 1$, and $|a+b| \le \max\{|a|, |b|\}$ for all $a, b \in A$. The pair $(A, |\cdot|)$ is called a **valued ring**.*

[17] Note how this differs from the moreover part of Lemma 0.1.

[18] Given A a ring with nonarchimedean valuation $|\cdot|$, $A\left\langle t^{1/p^\infty}\right\rangle$ is defined to be the colimit of $A\langle t\rangle \subseteq A\left\langle t^{1/p}\right\rangle \subseteq A\left\langle t^{1/p^2}\right\rangle \subseteq \cdots$. Elements of this ring can in some sense be viewed as convergent sums $\sum_{j\in\mathbb{N}^{\mathbb{N}}} a_j t^{c_j}$ with $a_j \in A$ and $c_j \in \mathbb{Q}_p$.

- *If A is a topological ring then $|\cdot|$ is* ***continuous*** *if the ray $\{a \in A : |a| < \gamma\}$ is open for every $\gamma \in \Gamma$. If A is just a ring then these rays induce a (minimal) topology on A making $|\cdot|$ continuous. If no topology is specified then $(A, |\cdot|)$ should be assumed to have this topology.*

- *A complete valued field $(K, |\cdot|)$ is* ***perfectoid*** *if*

  (1) *the local valuation ring $\mathcal{O} = \mathcal{O}_{|\cdot|} := \{x \in K : |x| \leq 1\}$ has residue characteristic $p > 0$ – i.e., $\mathcal{O}/\mathfrak{m}$ is an $\mathbb{F}_p$-algebra for $\mathfrak{m} \subseteq \mathcal{O}$ the unique maximal ideal;*

  (2) *the Frobenius map $\varphi : \mathcal{O}/p \to \mathcal{O}/p$ is surjective; and*

  (3) *$|\cdot|$ is non-discrete of rank 1 – equivalently, the image of $|\cdot|$ may be viewed as a non-cyclic subgroup of $\mathbb{R}^{>0}$.*

Claim 0.14. *Let $(K, |\cdot|)$ be a perfectoid field. Then, K is Fontaine perfectoid with $R^\circ = \mathcal{O}$ and $R^{\circ\circ} = \mathfrak{m}$. Moreover, $\mathfrak{m}^2 = \mathfrak{m}$ and the image of $|\cdot|$ is p-divisible.*