Eg Take $\mathcal{H}$ = upper half plane, then $\mathrm{Hol}(X) = PSL_2(\mathbb{R})$.

What is $u_i : U_1 \longrightarrow PSL_2(\mathbb{R})$ ?

For $z = a+bi$, set $h_i(z) = \begin{pmatrix} a & b \\ -b & a \end{pmatrix} \in GL_2(\mathbb{R})^+$.

Then $h_i(z)$ stabilizes $i \in \mathcal{H}$,

but acts by mult by $z/\bar{z}$.

So if $t \in U_1 \subset \mathbb{C}^\times$, then $h_i(t) \curvearrowright T_i\mathcal{H}$ as $t/\bar{t} = t^2$,

so $u_i(t) = \underbrace{h_i(\sqrt{t}) \in PSL_2(\mathbb{R})}_{PSL_2 \text{ kills ambiguity of } \sqrt{\ }}$.

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Recall: $X$ = HSD, then $\exists$ real adj $G$ st $G(\mathbb{R})^+ = \mathrm{Hol}(X)^+$.

For all $x \in X$, $\exists!\ u_x : U(1) \longrightarrow K_x \subset G(\mathbb{R})^+$ st

$\forall t \in U_1$, $u_x(t) \curvearrowright T_xX$ as $t \in U_1 \subset \mathbb{C}^\times$.

Ie, induced map $U_1 \longrightarrow \underbrace{GL(T_xX)}_{\text{a } \mathbb{C}\text{-VS}}$

is just $U_1 \hookrightarrow \mathbb{C}^\times \subset GL(T_xX)$.

So $u_x(-1)$ = symmetry $s_x$ at $x$.

Let $V$ = fd $\mathbb{R}$-vs.

Any hom $u : U_1 \longrightarrow GL(V)$ induces grading

$$V_{\mathbb{C}} = \bigoplus_{k \in \mathbb{Z}} V_{\mathbb{C}}^{(k)}$$

with $\overline{V_{\mathbb{C}}^{(k)}} = V_{\mathbb{C}}^{(-k)}$.

But if $V$ comes with a $\mathbb{C}$-structure given by

$$\mathbb{C} \longrightarrow \mathrm{End}_{\mathbb{R}}(V),$$

and $u$ = this map restricted to $U_1$, then $V_{\mathbb{C}} = V \otimes_{\mathbb{R}} \mathbb{C}$ has an action of $\mathbb{C} \otimes_{\mathbb{R}} \mathbb{C} = \mathbb{C} \oplus \mathbb{C}$ and induced decomposition by idempotents

$$V_{\mathbb{C}} = V_{\mathbb{C}}^{(-1)} \oplus V_{\mathbb{C}}^{(1)}$$

is the grading induced by $u$.

Moreover, natural map $V \hookrightarrow V_{\mathbb{C}} \twoheadrightarrow V_{\mathbb{C}}^{(1)}$ is a $\mathbb{C}$-linear isom.

$\Longrightarrow$ this gives restrictions in what $u_x : U_1 \longrightarrow G(\mathbb{R})^+$ can look like.

Write $\mathfrak{g} = \mathrm{Lie}(G(\mathbb{R})^+)$. And

$$\begin{tikzcd} U_1 \arrow[r, "u_x"] \arrow[drr, "\mathrm{Ad}(u_x)"'] & G(\mathbb{R})^+ \arrow[r, "\mathrm{Ad}"] & \mathrm{Aut}(G(\mathbb{R})^+) \arrow[d] \\ & & GL(\mathfrak{g}). \end{tikzcd}$$

<u>Proposition</u> Grading on $\mathfrak{g}_{\mathbb{C}}$ induced by $\mathrm{Ad}(u_x)$ has the form

$$\mathfrak{g}_{\mathbb{C}} = \underset{T_xX}{\mathfrak{g}_{\mathbb{C}}^{(-1)}} \oplus \underset{\mathrm{Lie}(K_x)_{\mathbb{C}}}{\mathfrak{g}_{\mathbb{C}}^{(0)}} \oplus \underset{T_xX}{\mathfrak{g}_{\mathbb{C}}^{(1)}}.$$

descends from $\mathbb{C}$ to $\mathbb{R}$.

Moreover, $\Theta := \mathrm{Ad}(u_x)(-1) \in GL(\mathfrak{g})$ decomposes

$$\mathfrak{g} = \underbrace{(\mathrm{Lie}\, K_x)}_{\Theta = 1} \oplus \underbrace{T_xX}_{\Theta = -1}.$$

as real-vs.

Proof. start from $GL(\mathbb{R})^+/K_x \xrightarrow{\sim} X$.
$$g \longmapsto gx$$

Tangent space gives
$$\mathfrak{g}/\mathrm{Lie}(K_x) \xrightarrow{\sim} T_xX \quad (\text{as } \mathbb{R}\text{-vs}).$$

Since $u_x : U_1 \longrightarrow K_x$ lands in $Z(K_x)$,
$$\mathrm{Ad}(u_x) : U_1 \longrightarrow \mathrm{Aut}(K_x) \quad \text{is trivial}$$
(since its conjugation).

So $\mathrm{Ad}(u_x) : U_1 \longrightarrow GL(\mathfrak{g})$
induces trivial action on $\mathrm{Lie}(K_x) \subset \mathfrak{g}$.

So we have induced action of $U_1$ on $\mathfrak{g}/\mathrm{Lie}(K_x)$ and
$$\mathfrak{g}/\mathrm{Lie}(K_x) \xrightarrow{\sim} T_xX$$
is $U_1$-equivariant.

But recall $U_1 \circlearrowright T_xX$ via $U_1 \subset \mathbb{C}^\times$.

In particular ($U_1$-equiv)
$$0 \longrightarrow \underset{\oplus = 1}{\mathrm{Lie}(K_x)} \longrightarrow \mathfrak{g} \longrightarrow \underset{\oplus = -1}{T_xX} \longrightarrow 0$$

So we have splitting
$$\mathfrak{g} = \underbrace{\mathrm{Lie}(K_x)}_{\oplus = 1 \Rightarrow u_x(t) = 1} \oplus \underbrace{T_xX}_{u_x(t) = t} .$$
$\Longleftarrow \oplus = -1$

Apply $- \otimes_{\mathbb{R}} \mathbb{C}$ to get the claim. $\square$

Cartan involution (Digression). —— What is $\Theta$?

Example: Inside $SL_n(\mathbb{C})$, we have 2 real Lie subgroups

$$\begin{tikzcd} & SL_n(\mathbb{C}) & \\ G = SL_n(\mathbb{R}) \arrow[ur] & & U = SU(n). \arrow[ul] \end{tikzcd}$$

(endow $\mathbb{C}^n$ with Herm form $h(x,y) = {}^t x \bar{y}$.)

(1) $U$ = compact

(2) "nice" relative position $G \cap U = SO(n)$

On Lie algs we have

$$\begin{tikzcd} & \mathfrak{g}_{\mathbb{C}} & \\ \mathfrak{g} \arrow[ur] & & \mathfrak{u} \arrow[ul] \end{tikzcd}$$

$\exists$ Cartan decomposition $\underbrace{\mathfrak{g}}_{\text{trace } 0} = \underbrace{\mathfrak{k}}_{\text{skew-symm}} \oplus \underbrace{\mathfrak{p}}_{\text{symm}}$

where $\mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$ inside $\mathfrak{g}_{\mathbb{C}}$.

$\mathfrak{p}$ is NOT a Lie-subalg! In fact,

$[\mathfrak{k}, \mathfrak{k}] \subset \mathfrak{k}$

$[\mathfrak{k}, \mathfrak{p}] \subset \mathfrak{p}$

$[\mathfrak{p}, \mathfrak{p}] \subset \mathfrak{k}$

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Remark: $G, U$ are both simply connected, so

$\mathbb{C}$-fd reps of $G \longleftrightarrow$ reps of $\mathfrak{g} \longleftrightarrow$ reps of $\mathfrak{g}_{\mathbb{C}}$

$\quad$ — " — of $U \longleftrightarrow$ reps of $\mathfrak{u} \longleftrightarrow$ reps of $\mathfrak{u}_{\mathbb{C}}$

(So eg can show $\mathbb{C}$-fd reps of $G$ decompose by showing that for $U$: (avg argument.)).

The Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$
induces "Cartan inv" $\Theta \in GL(\mathfrak{g})$ by

$$\Theta = \begin{cases} 1 & \text{on } \mathfrak{k} \\ -1 & \text{on } \mathfrak{p}. \end{cases}$$

NB: There's natural complex conj on

$$\mathfrak{u}_{\mathbb{C}} = \mathfrak{u} \otimes_{\mathbb{R}} \underset{\overline{\cdot}}{\mathbb{C}}$$

under $\mathfrak{u}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}$, which stabilizes $\mathfrak{g} \subset \mathfrak{g}_{\mathbb{C}}$ and is $\Theta$.

will be same $\Theta$ in $\mathrm{Ad}(u_X)(-1)$.