Uniqueness of canonical models

Milne's Intro to Shvarks [SVI] (§15)
Deligne's Travaux de Shimura [TdS] (§5)

(*) $M_k(G,X)_{\mathbb{C}} \overset{\varphi}{\xrightarrow{\sim}} Sh_k(G,X) \overset{\varphi'}{\xleftarrow{\sim}} M_{k'}(G,X)$ ← descend this to $E(G,X)$

Slogan 1: Var. V/k is uniquely (up to isom.) determined by $V_{\bar{k}}$ and $\mathrm{Aut}(\bar{k}/k) \curvearrowright V(\bar{k})$.

(Somehow we only really understand Shimura varieties by way of their special pts.)

Slogan 2 (Deligne): Canonical models "work" because they have a lot of special pts.

Let (G,X) be Shimura datum w/ reflex field $E(G,X)$. We have shown that X admits a special pt. x_0.

Lemma: Fix $x \in X$. Then, $\{[x,a] : a \in G(\mathbb{A}_f)\}$ is dense in $Sh_k(G,X)$. [The significance is that we can understand behavior at every pt. in terms of behavior at a chosen pt.]

We have $E(G,X) \subset E(x_0)$. Hope is that (1) $(\varphi')^{-1}\circ\varphi$ descends to $E(x_0)$ and (2) $\bigcap_{x_0 \text{ special}} E(x_0) = E(G,X)$.

Let's sketch (2).

Thm (TdS §5.1): $F/E(G,X)$ finite degree field ext. We can find $(T,h) \hookrightarrow (G,X)$ w/ $E(T,h)$ lin. disjoint from F.

The proof is very similar to the one used to show we have a special pt. in the first part.

We make use of a version of Hilbert's Irreducibility Thm [TdS, §5.1.3]. Original version can be seen as saying $p(t,x) \in \mathbb{Q}[t,x]$ irred. $\Rightarrow \exists t_0 \in \mathbb{Q}$ s.t. $p(t_0,x) \in \mathbb{Q}[x]$ is irred. (proof of this is not a purely algebraic matter).

Milne handles the situation for (*) where allow ourselves to vary k. If we look at conjugate compact opens then we can compare using a right translation map (which is actually a map of $\mathbb{C}$-varieties by a result of Borel).

Deligne's treatment somehow combines all of the above ideas but mixes them in an interesting way.