Section 2 Math 2202
Vector Addition, Dot and Cross Product

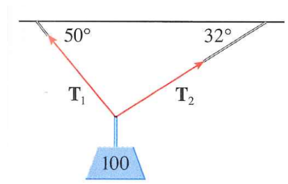
Comments for Facilitator:

- Do Quiz (10 min) - It's on basics of dot product and lines in R^3.
- Go over quiz (10 min). Poll first on which one they most want to discuss (We can use as a proxy for what students have most difficulty with.) NOTE: You don't need to go over the Think about it... question.

  Things to emphasize The coordinate planes and their equations. Students sometimes write "the x-plane" to mean $x = 0$ and then sometimes getting confused with that.

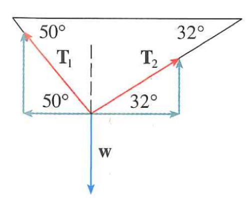
  Also, you can use the question about the line intersecting the plane to help them understand how you check something like that.
- Problem 1: Do a forces example similar to 9.2 Example 7. You can type the problem statement into this document so they don't have to spend time copying it down. I did not cover forces in lectures, so just explain that this is a nice application of vector addition and that if they don't understand the physics behind it, that is ok.
- NEW MATERIAL: Scalar Triple Product (10 min) Define the scalar triple product (see Stewart 659) and give intuition as to why it measures the *volume* of the parallelopiped. Then ask them what it would mean for $a \cdot (b \times c)$ to equal zero. Get them to see this is a test for coplanarity.
- Problem #2: This is a nice problem to make use of the scalar triple product.

1. **Resultant Forces** (*Stewart 9.2 Example 7*) A 100-lb weight hangs from two wires as shown below



Find tensions (forces) $\mathbf{T}_1$ and $\mathbf{T}_2$ in both wires and the magnitudes of the tensions.

Solution: We first express $\mathbf{T}_1$ and $\mathbf{T}_2$ in terms of their horizontal and vertical components.



We see that

$$\mathbf{T}_1 = -|\mathbf{T}_1| \cos 50° \mathbf{i} + |\mathbf{T}_1| \sin 50° \mathbf{j}$$
$$\mathbf{T}_2 = |\mathbf{T}_2| \cos 32° \mathbf{i} + |\mathbf{T}_2| \sin 32° \mathbf{j}$$

The resultant $\mathbf{T}_1 + \mathbf{T}_2$ of the tensions counterbalances the weight $\mathbf{w}$ and so we must have

$$\mathbf{T}_1 + \mathbf{T}_2 = -\mathbf{w} = 100\mathbf{j}.$$

2. **Scalar Triple Product**

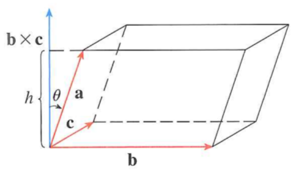


FIGURE 7

The product $\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})$ is called the **scalar triple product** of the vectors **a**, **b**, and **c**. Its geometric significance can be seen by considering the parallelepiped determined by the vectors **a**, **b**, and **c**. (See Figure 7.) The area of the base parallelogram is $A = |\mathbf{b} \times \mathbf{c}|$. If θ is the angle between the vectors **a** and $\mathbf{b} \times \mathbf{c}$, then the height h of the parallelepiped is $h = |\mathbf{a}||\cos\theta|$. (We must use $|\cos\theta|$ instead of $\cos\theta$ in case $\theta > \pi/2$.) Thus the volume of the parallelepiped is

$$V = Ah = |\mathbf{b} \times \mathbf{c}||\mathbf{a}||\cos\theta| = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Therefore we have proved the following:

> The volume of the parallelepiped determined by the vectors **a**, **b**, and **c** is the magnitude of their scalar triple product:
>
> $$V = |\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c})|$$

Instead of thinking of the parallelepiped as having its base parallelogram determined by **b** and **c**, we can think of it with base parallelogram determined by **a** and **b**. In this way, we see that

$$\mathbf{a} \cdot (\mathbf{b} \times \mathbf{c}) = \mathbf{c} \cdot (\mathbf{a} \times \mathbf{b})$$

3. Consider the four points in $\mathbf{R}^3$, $K(1,2,3)$, $L(1,3,6)$, $M(3,8,6)$ and $N(3,7,3)$.

(a) Show that the vectors $\overrightarrow{KL}$, $\overrightarrow{KM}$ and $\overrightarrow{KN}$ are coplanar. Explain why this means that K, L, M and N all lie in the same plane.

Solution: Recall that these three vectors are coplanar if their scalar triple product $\overrightarrow{KL} \cdot \left(\overrightarrow{KM} \times \overrightarrow{KN}\right) = 0$. (This is because all three vectors are perpendicular to $\overrightarrow{KM} \times \overrightarrow{KN}$.) Since

$$\overrightarrow{KL} = \langle 1-1, 3-2, 6-3\rangle = \langle 0,1,3\rangle, \qquad \overrightarrow{KM} = \langle 2,6,3\rangle, \qquad \text{and} \qquad \overrightarrow{KN} = \langle 2,5,0\rangle,$$

the scalar triple product is

$$\overrightarrow{KL} \cdot \left(\overrightarrow{KM} \times \overrightarrow{KN}\right) = \langle 0,1,3\rangle \cdot (\langle 2,6,3\rangle \times \langle 2,5,0\rangle) = \langle 0,1,3\rangle \cdot \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 2 & 6 & 3 \\ 2 & 5 & 0 \end{vmatrix} = \langle 0,1,3\rangle \cdot \langle -15, 6, -2\rangle = 0.$$

Thus these three vectors $\overrightarrow{KL}$, $\overrightarrow{KM}$ and $\overrightarrow{KN}$ are coplanar.
The four points K, L, M and N are thus also coplanar. One way to see this is to recognize that from K we can get to any of the other three points by adding a vector ($\overrightarrow{KL}$, $\overrightarrow{KM}$ or $\overrightarrow{KN}$) from a single plane. Another approach is to write down the equation of the plane they all lie in ($-15x + 6y - 2z = -9$) and verify that each point satisfies this equation.

(b) From part (a) we know that K, L, M and N are the vertices of a quadrilateral. Explain how you can tell that this quadrilateral is actually a parallelogram.

Solution: One simple way to check is to show that opposite sides are the same vectors: $\overrightarrow{KL} = \overrightarrow{NM}$ and $\overrightarrow{KN} = \overrightarrow{LM}$.

(c) (*Stewart 9.4 #22*) Find the area of the parallelogram with vertices K, L, M and N.

Solution: This area is the length of the vector $\overrightarrow{KL} \times \overrightarrow{KN}$. We find this without much comment:

$$\overrightarrow{KL} \times \overrightarrow{KN} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 0 & 1 & 3 \\ 2 & 5 & 0 \end{vmatrix} = \langle -15, 6, -2\rangle,$$

and so the area of the parallelogram is $|\overrightarrow{KL} \times \overrightarrow{KN}| = |\langle -15, 6, -2\rangle| = \sqrt{(-15)^2 + 6^2 + (-2)^2} = \sqrt{265} \approx 16.28$.

(d) What is the area of the triangle with vertices K, L, and M? How about the triangle with vertices L, M, N?

Solution: The area of each triangle is one half the area of the parallelogram, or $\frac{1}{2}\sqrt{265} \approx 8.14$.

(e) Find an equation for the plane in which K, L, M and N all lie.

(f) (*To think about...*) How many other points N' (different from N) are there such that K, L, M and N' form a parallelogram.

Solution: There are a total of three choices for N, so there are two other points N' and our original N. One way to see this is to pair opposite points in the parallelogram; there is one such "opposite point" for each of the points K, L and M. Here is a sketch with all three possibilities illustrated: