AIT Recap

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Intro

In the past two talks we surveyed many of the foundational notions and constructions in arithmetic intersection theory. Our goal for today is to review the key content and start orienting ourselves toward derived horizons. At the center of it all is the notion of arithmetic Chow groups, which we want to behave like algebraic Chow groups while also capturing key analytic phenomena. The current version of the theory works with arithmetic cycles, which pair algebraic cycles with Green currents. The essential ingredient is an intersection product for these arithmetic cycles, which requires the notion of $*$-product for Green currents.

Forms and Currents

Fix X a complex manifold. Given $r \geq 0$, we have $A^r(X)$ the space of $\mathbb{C}$-valued smooth r-forms on X with decomposition

$$A^r(X) = \bigoplus_{p+q=r} A^{p,q}(X).$$

Both of these constructions have compactly supported variants. We define $D_r(X)$ and $D_{p,q}(X)$ to be the continuous $\mathbb{C}$-linear duals of $A^r_c(X)$ and $A^{p,q}_c(X)$, respectively, yielding

$$D_r(X) = \bigoplus_{p+q=r} D_{p,q}(X).$$

Letting n be the dimension of X, define $D^r(X) := D_{2n-r}(X)$ and $D^{p,q}(X) := D_{n-p,n-q}(X)$ (with expected decomposition).

Dirac Currents

The space X has an associated **Dirac current** $\delta_X \in D_{2n}(X) = D^0(X)$ sending $\alpha \in A_c^{2n}(X)$ to $\int_X \alpha$. We can in fact integrate along any analytic subvariety $Y \subseteq X$ to get δ_Y, by passing to the nonsingular locus and possibly using a resolution of singularities as a computational aid (the choice does not matter; note that we must restrict down from X to Y). We have an evaluation pairing

$$\wedge : D_{p,q}(X) \times A^{r,s}(X) \to D_{p-r,q-s}(X), \ (T, \alpha) \mapsto (T \wedge \alpha : \beta \mapsto T(\alpha \wedge \beta)),$$

which is well-defined since β and hence $\alpha \wedge \beta$ is compactly supported. We have no reason to prefer this ordering of the inputs and so we also define an evaluation pairing by the same formula but with the inputs swapped.

Remark

This notation is somewhat unfortunate since swapping terms on a traditional wedge product introduces a negative sign. We will, however, follow convention. Some extra care will be needed later to avoid confusion.

The Bracket

This gives rise to the map

$$[\cdot] : A^{p,q}(X) \to D^{p,q}(X), \qquad \alpha \mapsto \delta_X \wedge \alpha,$$

which by definition satisfies $[\alpha](\beta) := \int_X \alpha \wedge \beta$ and gives $[\cdot] : A^r(X) \hookrightarrow D^r(X)$ a $\mathbb{C}$-linear embedding. This is injective with dense image but **not** surjective.

Remark

In order to make sense of $[\cdot]$ we need only be able to integrate and so $[\cdot]$ clearly makes sense for forms which are locally L^1. This observation will be very important when working with Green currents.

Differential Operators

Letting $D \in \{d, \partial, \overline{\partial}\}$ be a suitable differential operator, we can uniquely extend D to $D^\bullet(X)$ so that

$$\begin{tikzcd} A^r(X) \arrow[r, hook, "{[\cdot]}"] \arrow[d, "D"'] & D^r(X) \arrow[d, "D"] \\ A^{r+1}(X) \arrow[r, hook, "{[\cdot]}"'] & D^{r+1}(X) \end{tikzcd}$$

commutes for every $r \geq 0$. The recipe is $DT(\alpha) := \pm T(D\alpha)$ with sign depending on α. We will have use for the operator

$$d^c := \frac{i}{4\pi}(\overline{\partial} - \partial),$$

which satisfies

$$dd^c = -d^c d = \frac{i}{2\pi}\partial\overline{\partial}$$

noting that $d = \partial + \overline{\partial}$.

Green Currents

Given $Y \subseteq X$ an irreducible analytic subvariety of codimension p, a **Green current** for Y is a current g_Y such that

$$dd^c g_Y + \delta_Y = [\omega_Y]$$

for ω_Y smooth. Note that this forces $g_Y \in D^{p-1,p-1}(X)$ and $\omega_Y \in A^{n-p,n-p}(X)$. Why should we care about Green currents?

- They form a suitable class for intersection theory.
- There are enough of them.
- They are sufficiently computable and we understand reasonably well what they look like.

Smoothing of Cohomology

Let's elaborate on these points. One of the keys is that the cohomology of currents is computed by the cohomology of forms. We are interested in the complex de Rham cohomology $H^\bullet_{\mathrm{dR}}(X) := H^\bullet_{\mathrm{dR}}(X;\mathbb{R}) \otimes_{\mathbb{R}} \mathbb{C}$ obtained from $(A^\bullet(X), d)$ and the Dolbeault cohomology $H^{p,\bullet}_{\overline{\partial}}(X)$ obtained from $(A^{p,\bullet}(X), \overline{\partial})$.

Theorem (Smoothing of Cohomology)

The natural maps

$$H^\bullet_{\mathrm{dR}}(X) \to H(D^\bullet(X), d), \qquad H^{p,\bullet}_{\overline{\partial}}(X) \to H(D^{p,\bullet}(X), \overline{\partial})$$

(arising from the underlying complexes) are functorial isomorphisms.

The significance of this result is of course that we can transfer cohomological tools and methods (e.g., cup products) from the setting of forms to the setting of currents. In the derived setting we hope for some analogue.

Uniqueness

Theorem

Let X be a compact Kähler manifold and g_1, g_2 Green currents for the same analytic cycle on X. Then, there exist currents α, β and a smooth form ω such that $g_1 - g_2 = [\omega] + \partial\alpha + \overline{\partial}\beta$.

Where does this uniqueness result come from? The trick is that $T := g_1 - g_2$ is such that $\partial\overline{\partial}T$ is smooth, which allows us to inductively apply Smoothing of Cohomology.

Remark

What about the compactness and Kähler conditions? Smoothing of Cohomology does not seem to require such conditions, so if we are just bootstrapping then something is fishy...

Existence

What about existence? Using a mix of Hodge decomposition, Kähler identities, and Serre duality, we see that d-exactness implies dd^c-exactness. From this we see that analytic cycles on compact Kähler manifolds admit compatible Green currents, using Stokes's theorem. This is all wonderfully abstract. We want something a little more hands-on. The previous slide suggests that we work with the quotient complexes

$$\widetilde{A}^\bullet(X) := A^\bullet(X)/(\mathrm{im}\,\partial + \mathrm{im}\,\overline{\partial}), \qquad \widetilde{D}^\bullet(X) := D^\bullet(X)/(\mathrm{im}\,\partial + \mathrm{im}\,\overline{\partial})$$

in order to avoid worrying about various auxiliary choices.

Remark

If we build a homotopy coherent theory then we can probably avoid the "brutal" effect of taking quotients here, allowing us to in some sense work more directly with $A^\bullet(X)$ and $D^\bullet(X)$. The computations we perform relative to $\widetilde{A}^\bullet(X)$ and $\widetilde{D}^\bullet(X)$ should be more thoroughly "refined" so that we understand how they "lift."

Green Currents Redux

On our way to unpacking the other points, let's consider the problem of playing two Green currents off of each other to produce a third. Let Y, Z be analytic cycles on X with Green current setup

$$dd^c g_Y + \delta_Y = [\omega_Y], \qquad dd^c g_Z + \delta_Z = [\omega_Z].$$

Assume that Y, Z meet properly so that the intersection product $[Y].[Z]$ is represented by the naïve set-theoretic intersection $Y \cap Z$. A natural aim is to produce a current $g_Y * g_Z$ so that

$$dd^c(g_Y * g_Z) + \delta_{Y \cap Z} = [\omega_Y \wedge \omega_Z],$$

which is to say we expect a Green current with some control on the "shape." The key is that we can write down a suitable "formal expression" which we can then check actually has the right analytic properties. From there, we want to understand how to generalize to more complicated intersections. All of this makes it necessary to understand how to represent Green currents by forms. With this in mind, we introduce the following definition.

Log Type Forms

Let X be a quasi-projective complex manifold and $Y \subseteq X$ an analytic subvariety not containing any irreducible components of X. Let η be a form on X which is smooth on $X \setminus Y$. We say that η is of **log type** along Y if there exists a proper map $\pi : M \to X$ and smooth form ϕ on $M \setminus \pi^{-1}(Y)$ such that

- M is a complex manifold, $E := \pi^{-1}(Y)$ is a normal crossings divisor on M, and $\pi|_{M \setminus E} : M \setminus E \to X \setminus Y$ is a submersion;
- $(\pi|_{M\setminus E})_* \phi = \eta$; and
- given $U \subseteq M$ nonempty open with holomorphic coordinates $(z_1, \ldots, z_n)$ such that $E \cap U$ looks like $z_1 \cdots z_k = 0$ for some $k \leq n$, then there exist a smooth form β on U and ∂- and $\overline{\partial}$-closed forms α_i $(1 \leq i \leq k)$ on U such that

$$\phi|_U = \sum_{i=1}^{k} \alpha_i \log |z_i|^2 + \beta.$$

Pushforward and Pullback

The unknown ingredient in the previous definition is a pushforward for currents, which actually fits into a good theory of both pushforwards and pullbacks. Given $f : X \to Y$ a proper holomorphic map of complex manifolds, we have $f_* : D_{p,q}(X) \to D_{p,q}(Y)$ induced by $f^* : A_c^{p,q}(Y) \to A_c^{p,q}(X)$. Assuming moreover that X, Y have dimensions n, m and f has constant fiber dimension $r := n - m$, we have $f^* : D_{p-r,q-r}(Y) \to D_{p,q}(X)$. Note that the relationship of f_* and f^* with the bigrading changes if we use upper instead of lower indexing. Given $T \in D(X)$ and $D \in \{d, \partial, \overline{\partial}\}$, the following hold.

- $D(f_* T) = f_*(DT)$.
- We have the projection formula $f_*(T \wedge f^*\alpha) = f_* T \wedge \alpha$ for $\alpha \in A^\bullet(X)$.
- Suppose that $g : Y \to Z$ is another proper holomorphic map of complex manifolds. Then, $(g \circ f)_* T = g_* f_* T$.
- Given $Z \subseteq X$ an analytic cycle, we have $f_* \delta_Z = \delta_{f_* Z}$.

Pushforward and Pullback

What about properties of pullbacks? Suppose moreover that f is a surjective submersion. Then, the following hold.

- $D(f^*T) = f^*(DT)$.
- $f^*(T \wedge \alpha) = f^*T \wedge f^*\alpha$ for $\alpha \in A^\bullet(Y)$.
- Suppose that $g : Y \to Z$ is another proper surjective submersion of complex manifolds. Then, $(f \circ g)^*T = g^*f^*T$.
- Given $Z \subseteq Y$ an analytic cycle, we have $f^*\delta_Z = \delta_{f^*Z}$.

Poincaré-Lelong

Log type forms arise for us because they play especially nice with dd^c and thus lead us to Green currents. Under suitable conditions which ensure everything is well-defined, the pushforward and pullback of log type forms are themselves log type. In turn, the pushforward and pullback of Green currents represented by log type forms are themselves Green currents represented by log type forms. Understanding how dd^c interacts with log type forms ultimately boils down to the Poincaré-Lelong formula, which tells us how to construct Green currents in codimension 1. The bootstrapping technique for dealing with higher codimension is resolution of singularities, which is also often used to prove Poincaré-Lelong itself.

Poincaré-Lelong

Theorem (Poincaré-Lelong)

Let X be a complex manifold and $Y \subseteq X$ a divisor given as $\mathrm{div}(s)$ for s a meromorphic section of some holomorphic line bundle L on X. Choose $\|\cdot\|$ some smooth Hermitian metric on L and let $c_1(L, \|\cdot\|)$ be the associated first Chern form. Then, $-\log \|s\|^2$ is L^1 on X and we have

$$dd^c[-\log \|s\|^2] + \delta_{\mathrm{div}(s)} = [c_1(L, \|\cdot\|)].$$

Unpacking definitions, we have the residue calculation $dd^c[\log \|s\|^2] - [dd^c \log \|s\|^2] = \delta_{\mathrm{div}(s)}$ or, in other words,

$$\int_X \log \|s\|^2 \, dd^c\eta - \int_X dd^c(\log \|s\|^2) \wedge \eta = \int_{\mathrm{div}(s)} \eta$$

for every $\eta \in A_c^{n-1,n-1}(X)$ with $n := \dim X$. Poincaré-Lelong is useful in part because it is somewhat explicit.

An Example

Example

Take $X = \mathbb{C}^2$ with coordinates z_1, z_2 and $Y = \{0\}$. The form

$$g_0 := \frac{1}{2\pi} \log(|z_1|^2 + |z_2|^2) \frac{(z_1 dz_1 - z_2 dz_2)(\overline{z}_1 d\overline{z}_1 - \overline{z}_2 d\overline{z}_2)}{(|z_1|^2 + |z_2|^2)^2}$$

satisfies $dd^c[g_0] = \delta_0$.

Note that there is a theory of canonical Green currents associated to Schubert cells in Grassmannians but this theory is not as explicit as one might hope.

Green Forms

By definition, a **Green form of log type** is a real form of log type inducing a Green current under $[\cdot]$. Under such conditions I believe it is the case that every Green current for Y is represented by a Green form of log type.

Remark

It's a little unclear how the realness condition should be implemented in the derived setting. I'm not sure how much this matters...

Theorem

Let X be a quasi-projective complex manifold and $Y \subseteq X$ irreducible closed. Then, there exists a Green form of log type for Y.

$*$-Products

What, then, can we say about $*$-products? Let $Y, Z \subseteq X$ be irreducible closed subvarieties of pure codimension with Z not contained in Y. Let η be a form of log type along Y. Define

$$\eta\delta_Z := h_*[h^*\eta] =: \delta_Z\eta,$$

where $h := i \circ \pi$ for $i : Z \hookrightarrow X$ the associated closed immersion and $\pi : \widetilde{Z} \to Z$ a choice of resolution of singularities (which exists!). This is a current on X independent of the choice of π (Why?). These notions extend to suitable pairs of analytic cycles by imposing linearity. Let once again Y, Z be analytic cycles on X with Green current setup

$$dd^c g_Y + \delta_Y = [\omega_Y], \qquad dd^c g_Z + \delta_Z = [\omega_Z].$$

Ignoring how Y and Z interact, define the $*$-product of g_Y and g_Z to be

$$g_Y * g_Z := g_Y \delta_Z + \omega_Y \wedge g_Z,$$

which requires a choice of representing Green form for g_Y.

Commutativity

The $*$-product is supposed to be commutative and associative, viewed in the proper context. Let's think about this heuristically. Switching the order and choosing a representing Green form for g_Z, we have

$$g_Z * g_Y = g_Z \delta_Y + \omega_Z \wedge g_Y.$$

By assumption we have $[\omega_Y] = dd^c g_Y + \delta_Y$ and so it is reasonable to expect

$$\omega_Y \wedge g_Z = dd^c(g_Y) \wedge g_Z + \delta_Y g_Z$$

as currents. Similarly, we expect

$$\omega_Z \wedge g_Y = dd^c(g_Z) \wedge g_Y + \delta_Z g_Y,$$

which gives the commutator

$$g_Y * g_Z - g_Z * g_Y = dd^c(g_Y) \wedge g_Z - dd^c(g_Z) \wedge g_Y.$$

There is in turn some reason to expect $g_Y \wedge g_Z = g_Z \wedge g_Y$ and that we can relate the expression $dd^c(g_Y \wedge g_Z)$.

Closure

Again heuristically, we expect

$$\begin{aligned} dd^c(g_Y * g_Z) &= dd^c(g_Y\delta_Z + \omega_Y \wedge g_Z) \\ &= dd^c(g_Y)\delta_Z + \omega_Y \wedge dd^c(g_Z) \\ &= ([\omega_Y] - \delta_Y)\delta_Z + \omega_Y \wedge ([\omega_Z] - \delta_Z) \\ &= \omega_Y\delta_Z - \delta_Y\delta_Z + \omega_Y \wedge [\omega_Z] - \omega_Y \wedge \delta_Z \\ &= -\delta_Y\delta_Z + [\omega_Y \wedge \omega_Z] \\ &= -\delta_{Y.Z} + [\omega_Y \wedge \omega_Z], \end{aligned}$$

where $\delta_Y\delta_Z$ is at present undefined and $Y.Z$ should be understood as the intersection product $[Y].[Z]$, which is in general only well-defined up to rational equivalence.

Remark

Expected here is the implicit equation $\omega_Y\delta_Z = \omega_Y \wedge \delta_Z$.

Associativity

Working heuristically, we have

$$\begin{aligned}(g_Y * g_Z) * g_W &= (g_Y * g_Z)\delta_W + (\omega_Y \wedge \omega_Z) \wedge g_W \\ &= (g_Y\delta_Z + \omega_Y \wedge g_Z)\delta_W + (\omega_Y \wedge \omega_Z) \wedge g_W \\ &= (g_Y\delta_Z)\delta_W + (\omega \wedge g_Z)\delta_W + (\omega_Y \wedge \omega_Z) \wedge g_W\end{aligned}$$

and

$$\begin{aligned}g_Y * (g_Z * g_W) &= g_Y\delta_{Z.W} + \omega_Y \wedge (g_Z * g_W) \\ &= g_Y\delta_{Z.W} + \omega_Y \wedge (g_Z\delta_W + \omega_Z \wedge g_W) \\ &= g_Y\delta_{Z.W} + \omega_Y \wedge (g_Z\delta_W) + \omega_Y \wedge (\omega_Z \wedge g_W).\end{aligned}$$

We expect to be able to match terms.

The Transverse Case

Through involved analysis, it is possible to make some headway in the case of transverse intersection.

Theorem

Let Y, Z, W be a transverse triple of irreducible closed subvarieties of X of codimension p, q, r with $p, q > 0$. Then,

$$g_Y \delta_{Z \cdot W} + \omega_Y \wedge g_Z \delta_W = g_Z \delta_{Y . W} + \omega_Z \wedge g_Y \delta_W$$

in $\widetilde{D}^{n,n}(X)$ for $n := p + q + r - 1$.

Remark

Ideally, we would like to have more refined information about the difference between $g_Y \delta_{Z \cdot W} + \omega_Y \wedge g_Z \delta_W$ and $g_Z \delta_{Y . W} + \omega_Z \wedge g_Y \delta_W$ living in $\operatorname{im} \partial + \operatorname{im} \overline{\partial}$.

The Transverse Case

Since $r = 0$ is allowed, taking $W = X$ tells us that the $*$-product is commutative at least in the context of $\widetilde{D}^\bullet(X)$. Note that we also get independence from the choice of representing Green form. This follows from taking g_Y, g'_Y to both represent the same Green current and considering the equation

$$g_Y \delta_{Y.Z} + \omega_Y \wedge g'_Y \delta_Z = g'_Y \delta_{Y.Z} + \omega_Y \wedge g_Y \delta_Z.$$

The Transverse Case

Associativity for a transverse triple seems a bit suspect but should hold. Let g_i be associated to ω_i and Z_i, for $i \in \{1, 2, 3\}$. We have

$$\begin{aligned} g_1 * (g_2 * g_3) &= g_1 * (g_3 * g_2) \\ &= g_1 \delta_{Z_3 . Z_2} + \omega_1 \wedge (g_3 * g_2) \\ &= g_1 \delta_{Z_3 . Z_2} + \omega_1 \wedge (g_3 \delta_{Z_2} + \omega_3 \wedge g_2) \\ &= g_1 \delta_{Z_3 . Z_2} + \omega_1 \wedge g_3 \delta_{Z_2} + \omega_1 \wedge (\omega_3 \wedge g_2) \\ &= g_1 \delta_{Z_3 . Z_2} + \omega_1 \wedge g_3 \delta_{Z_2} + (\omega_1 \wedge \omega_3) \wedge g_2 \end{aligned}$$

and

$$\begin{aligned} (g_1 * g_2) * g_3 &= g_3 * (g_1 * g_2) \\ &= g_3 \delta_{Z_1 . Z_2} + \omega_3 \wedge g_1 \delta_{Z_2} + (\omega_3 \wedge \omega_1) \wedge g_2 . \end{aligned}$$

Am I crazy or is there a glaring sign error to deal with here?

The Non-Transverse Case

We have access to more elaborate formulas for all of these things when working with non-transverse intersections, though the fully general case is not known. What are we to do with all of this? Let's start off with a first approximation of a derived theory.

Conjecture

- *There is a "robust" theory of derived currents on derived complex manifolds, being closely tied to a derived theory of differential forms and admitting nicely behaved homotopical notions of pushforward and pullback.*
- *There is a suitably "stable" class of derived Green currents, generalizing classical Green currents, which admits a derived $*$-product that is canonically commutative and associative up to coherent homotopy.*
- *This derived $*$-product generalizes the classical $*$-product, thereby "geometrizing" many ad hoc analytic manipulations.*

Some Questions

- When is the derived $*$-product of classical Green currents itself a classical Green current?
- What happens if we take the derived $*$-product of a classical Green current with itself?
- Can we recover the analysis used to compute classical $*$-products, in the transverse case or more generally?
- How do log type forms fit into the picture and can we generalize their associated blowup arguments using derived blowups?
- Does the derived setting suggest new classes of "canonical" (possibly classical) Green forms?[1]

[1] Log type forms are stable under the manipulations that go into computing classical $*$-products. Messing with heat flux produces Green forms which are not so stable. Can we get around this by framing things in a derived context?