Assignment 5

due midnight (Eastern Time), Monday, October 26, 2020

Prove all statements in the questions below. You must TeX your solutions and submit your writeup as a pdf at www.gradescope.com.
You can submit it any time until the deadline (try not to spend your Friday evening working on this!).

Question 1. (Folland 2.6.55 a and b) Let $E = [0,1] \times [0,1]$. Investigate the existence and equality of

$$\int_E f\ dm^2, \quad \int_0^1 \int_0^1 f(x,y)\ dxdy \quad \text{and} \int_0^1 \int_0^1 f(x,y)\ dydx$$

for the following f:

1. $f(x,y) = (x^2 - y^2)(x^2 + y^2)^{-2}$
2. $f(x,y) = (1 - xy)^{-a}$, $(a > 0)$.

Question 2. Let ν be a signed measure on $(X, \mathcal{M})$.

1. $L^1(\nu) = L^1(|\nu|)$.
2. If $f \in L^1(\nu)$, then $|\int f\ d\nu| \leq \int |f|\ d|\nu|$.
3. If $E \in \mathcal{M}$, $|\nu|(E) = \sup\{|\int_E f d\nu| : |f| \leq 1\}$.

Question 3. (Folland 3.2.13) Let $X = [0,1]$, $\mathcal{M} = \mathcal{B}_{[0,1]}$, $m =$ Lebesgue measure, and $\mu =$ counting measure on $\mathcal{M}$.

1. $m \ll \mu$ but $dm \neq f d\mu$ for any f.
2. μ has no Lebesgue decomposition with respect to m.

Question 4. (Folland 3.2.16) Suppose that μ, ν are measures on $(X, \mathcal{M})$ with $\nu \ll \mu$, and let $\lambda = \mu + \nu$. If $f = d\nu/d\lambda$, then $0 \leq f < 1$ μ-a.e. and $d\nu/d\mu = f/(1-f)$.

Question 5. (Folland 3.2.17) Let $(X, \mathcal{M}, \mu)$ be a σ-finite measure space, $\mathcal{N}$ a sub-σ-algebra of $\mathcal{M}$, and $\nu = \mu|_{\mathcal{N}}$. If $f \in L^1(\mu)$, there exists $g \in L^1(\nu)$ (thus g is $\mathcal{N}$-measurable) such that $\int_E f\ d\mu = \int_E g\ d\nu$ for all $E \in \mathcal{N}$; if g' is another such function, then $g = g'$ ν-a.e.
(In probability theory, g, is called the conditional expectation of f on $\mathcal{N}$.)