Let $\mathsf{Ch}(\mathbb{Z})$ denote the category of cochain complexes of abelian groups, $\mathsf{Ch}(\mathbb{Z})^{\text{free}}$ and $\mathsf{Ch}(\mathbb{Z})^{\text{tf}}$ the full subcategories of objects whose entries are free and p-torsion-free, and $D(\mathbb{Z})$ the derived category. Since $\mathbb{Z}$ has projective dimension 1, this is given by

$$D(\mathbb{Z}) \simeq h\,\mathsf{Ch}(\mathbb{Z})^{\text{free}} \simeq \mathsf{Ch}(\mathbb{Z})^{\text{tf}}[\text{qis}^{-1}].$$

The category $h\,\mathsf{Ch}(\mathbb{Z})^{\text{free}}$ is the homotopy category of $\mathsf{Ch}(\mathbb{Z})^{\text{free}}$, whose objects are $X, Y \in \mathsf{Ch}(\mathbb{Z})^{\text{free}}$ and morphisms are cochain homotopy classes of maps in $[X, Y]$. Note that our construction of the derived category requires choosing free or p-torsion-free resolutions for general cochain complexes.

Remark 1. *Let's talk a bit about homotopy. The abelian category $\mathsf{Ch}(\mathbb{Z})$ comes equipped with a natural tensor product $\otimes$ giving it the structure of a symmetric monoidal category. There is an internal Hom functor $\underline{\text{Hom}} : \mathsf{Ch}(\mathbb{Z})^{\text{op}} \times \mathsf{Ch}(\mathbb{Z}) \to \mathsf{Ch}(\mathbb{Z})$ which of course satisfies*

$$\text{Hom}(Z \otimes X, Y) \cong \text{Hom}(Z, \underline{\text{Hom}}(X, Y))$$

functorial in $X, Y, Z \in \mathsf{Ch}(\mathbb{Z})$.[1] *Explicitly,*

$$\underline{\text{Hom}}^n(X, Y) = \prod_{i \in \mathbb{Z}} \text{Hom}(X^i, Y^{i+n})$$

with differential $df = d_Y \circ f - (-1)^n f \circ d_X$ for f homogeneous of degree n.[2] *One of the nice things about $\underline{\text{Hom}}$ is that it captures homotopy. There is a natural group isomorphism*

$$\text{Hom}(X, Y) \cong Z^0(\underline{\text{Hom}}(X, Y)) = \{f \in \underline{\text{Hom}}^0(X, Y) : df = 0\}$$

that identifies the nullhomotopies of a fixed $f \in \text{Hom}(X, Y)$ with

$$B^0(\underline{\text{Hom}}(X, Y)) = \{h \in \underline{\text{Hom}}^{-1}(X, Y) : dh = f\}.$$

It follows that $H^0(\underline{\text{Hom}}(X, Y)) \cong [X, Y]$ and taking higher cohomology captures higher homotopy. Also note that if $X, Y \in \mathsf{Mod}_{\mathbb{Z}}$ then $\underline{\text{Hom}}(X, Y)$ is just $\text{Hom}(X, Y)$ concentrated in degree 0.[3]

Definition 2. ***Classical** p**-completion** is the functor*

$$\widehat{\cdot} : \mathsf{Mod}_{\mathbb{Z}} \to \mathsf{Mod}_{\mathbb{Z}}, \qquad X \mapsto \varprojlim_{n \geq 1} X/p^n X.$$

*We say $X \in \mathsf{Mod}_{\mathbb{Z}}$ is **classically** p**-complete** if the natural map $X \to \widehat{X}$ is an isomorphism. On a somewhat related note, $X \in D(\mathbb{Z})$ is **derived** p**-complete** if $\text{Hom}_{D(\mathbb{Z})}(Y, X) = 0$ for every $Y \in D(\mathbb{Z})$ such that $p : Y \xrightarrow{\sim} Y$. Such objects span a full subcategory $D_p(\mathbb{Z}) \subseteq D(\mathbb{Z})$.*

Proposition 3. *The inclusion $D_p(\mathbb{Z}) \hookrightarrow D(\mathbb{Z})$ admits a left adjoint $\widehat{\cdot} : D(\mathbb{Z}) \to D_p(\mathbb{Z})$ called the **derived** p**-completion** given by choosing a representative in $\mathsf{Ch}(\mathbb{Z})$ and applying classical p-completion in each degree.*

[1] $\underline{\text{Hom}}$ is sometimes referred to as the **mapping class group**.

[2] Be warned that I might be confusing homological and cohomological conventions.

[3] Hence, $\underline{\text{Hom}}$ extends the inner Hom on $\mathsf{Mod}_{\mathbb{Z}}$ (which is represented by the naïve Hom that is automatically enriched over $\mathbb{Z}$). Note that $\underline{\text{Hom}}$ gives rise to $\underline{\text{Ext}}^i$ on $\mathsf{Ch}(\mathbb{Z})$. For $X, Y \in \mathsf{Mod}_{\mathbb{Z}}$ it seems reasonable that we would have $\underline{\text{Ext}}^i(X, Y)$ is just $\text{Ext}^i(X, Y)$ in degree 0. More generally I expect there to be some spectral sequence relating the two notions.

We extend derived notions to $\mathsf{Mod}_{\mathbb{Z}}$ by thinking of abelian groups as complexes concentrated in degree 0. Given $X \in \mathsf{Mod}_{\mathbb{Z}}$, the classical p-completion of X represents the derived p-completion of X and so we may identify the two.

Proposition 4. *Let $X \in \mathsf{Mod}_{\mathbb{Z}}$.*

(a) *X is derived p-complete if and only if $\mathrm{Hom}(\mathbb{Z}[p^{-1}], X)$ and $\mathrm{Ext}^1(\mathbb{Z}[p^{-1}], X)$ are both contractible. This holds if and only if every short exact sequence*

$$0 \longrightarrow X \longrightarrow M \longrightarrow \mathbb{Z}[p^{-1}] \longrightarrow 0$$

admits a unique splitting.

(b) *X is classically p-complete if and only if it is p-adically separated and derived p-complete.*

(c) *X is **pro-free** (i.e., the p-completion of a free abelian group) if and only if it is derived p-complete and p-torsion-free.*

Remark 5. *I'm not entirely sure about the content of the above proposition. I know that $\mathrm{Ext}^1(\mathbb{Z}[p^{-1}], X)$ classifies extensions of $\mathbb{Z}[p^{-1}]$ by X, with the zero element corresponding to the trivial extension. Any splitting is by definition an extension isomorphic to the trivial extension as a short exact sequence. So it makes sense that if Ext^1 vanishes then there is a unique splitting (up to ismorphism). But what if there is only a weak equivalence to 0?*

Complexes of pro-free abelian groups span a full subcategory $\mathsf{Ch}(\mathbb{Z})^{\text{pro-free}} \subseteq \mathsf{Ch}(\mathbb{Z})$. The following result shows that this subcategory lets us get at $D_p(\mathbb{Z})$.

Theorem 6. *The functor $\mathsf{Ch}(\mathbb{Z})^{\text{pro-free}} \to D(\mathbb{Z})$ obtained by passing to (formal) qis classes has essential image $D_p(\mathbb{Z})$ and induces an equivalence $h\,\mathsf{Ch}(\mathbb{Z})^{\text{pro-free}} \xrightarrow{\sim} D_p(\mathbb{Z})$.*

Our goal now is to start tying in fixed points. Our first stop is deriving η_p.

Proposition 7. *There is an essentially unique functor $L\eta_p : D(\mathbb{Z}) \to D(\mathbb{Z})$ such that*

$$\begin{tikzcd} \mathsf{Ch}(\mathbb{Z})^{\mathrm{tf}} \arrow[r, "\eta_p"] \arrow[d] & \mathsf{Ch}(\mathbb{Z})^{\mathrm{tf}} \arrow[d] \\ D(\mathbb{Z}) \arrow[r, dotted, "\exists!\, L\eta_p"'] & D(\mathbb{Z}) \end{tikzcd}$$

commutes up to natural isomorphism.[4]

Definition 8. *Let $\mathcal{C}$ be a category and $T : \mathcal{C} \to \mathcal{C}$ an endofunctor. The **fixed point** category $\mathcal{C}^T$ of $\mathcal{C}$ with respect to T is the category whose objects are pairs (X, φ) with $X \in \mathcal{C}$ and $\varphi \in \mathrm{Isom}_{\mathcal{C}}(X, TX)$. The data of a morphism $f : (X, \varphi) \to (X', \varphi')$ is $f \in \mathrm{Hom}_{\mathcal{C}}(X, X')$ such that*

[4] In particular, there is no nontrivial homotopical coherence introduced at this stage in the game.

$$\begin{tikzcd} X \arrow[r, "f"] \arrow[d, "\varphi"'] & X' \arrow[d, "\varphi'"] \\ TX \arrow[r, "Tf"'] & TX' \end{tikzcd}$$

commutes.

Basically by definition, we immediately see that there is an equivalence $\mathsf{DC}_{\mathrm{sat}} \simeq (\mathsf{Ch}(\mathbb{Z})^{\text{pro-free}})^{\eta_p}$. In fact, more is true.

Theorem 9. $\mathsf{DC}_{\mathrm{sat}} \simeq (\mathsf{Ch}(\mathbb{Z})^{\text{pro-free}})^{\eta_p}$ *restricts to an equivalence* $\mathsf{DC}_{\mathrm{str}} \xrightarrow{\sim} D_p(\mathbb{Z})^{L\eta_p}$.