Section 1 Math 2202
The Next Dimension: Distance and Equations in R^3

Comments for Facilitator:

- Do first day introductions in whatever way you choose, making sure students share their name with the class. (5 min)
- Explain there will usually be a quiz at the start of section (graded on completion only) and your expectations (be on time for credit, do best effort). These start next week.
- Problems (20 min) The purpose is to (a) get students to start to work together to and justify their reasoning and (b) explore new ideas in R3, specifically distance, and equations for coordinate planes and spheres, and equations in R3 vs R2.

  Start with #1. Let students work a little 10 minutes, then recap at least through number 3. You don't need to get into the details for equations vs dimension as a section.

  Then

  Things to emphasize The coordinate planes and their equations. Students like to write "the x-plane" to mean $x = 0$ and then sometimes getting confused with that.

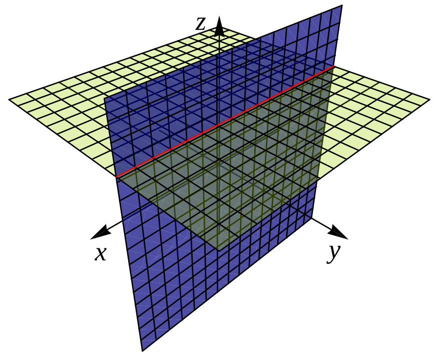
  Visualizing regions in 3D. This is fun stuff to talk about and really key to where we're going (in thinking about graphing $f(x, y) = z$, quadric surfaces, etc).

1. This problem is meant to help you explore what equations describe in the plane $\mathbf{R}^2$ compared to in 3-dimensional space $\mathbf{R}^3$.

   (a) What does the equation $x = 4$ represent in the xy-plane, $\mathbf{R}^2$? What does it represent in xyz-space, $\mathbf{R}^3$? Illustrate with sketches.

   (b) What does the equation $y = 3$ represent in $\mathbf{R}^3$? What does $z = 5$ represent? What does the pair of equations $\{y = 3, z = 5\}$ represent? In other words, describe the set of points (x, y, z) such that we have both $y = 3$ and $z = 5$. Illustrate with a sketch.

   Solution: In three-space, the equation $y = 3$ represents a vertical plane parallel to the xz-plane and passing through the point $(0, 3, 0)$. The equation $z = 5$ represents a horizontal plane, parallel to the xy-plane and at height 5. The points that satisfy both equations $y = 3$ and $z - 5$ are the points on the line that lies on both of the planes described above (that is, the intersection of the two lines). It's all the points of the form $(x, 3, 5)$.



(c) What does the set of three equations $\{x = 4, y = 5, z = 3\}$ represent in $\mathbf{R}^3$? In other words, describe the set of points (x, y, z) such that $x = 4$ and $y = 5$ and $z = 3$. Illustrate with a sketch.

Solution: There is only one such point. It's the point $(4, 5, 3)$. To give a point in $\mathbf{R}^3$ is the same as giving its x-coordinate, y-coordinate and z-coordinate.

(d) Points are considered to be "0-dimensional", lines are "1-dimensional", planes are "2-dimensional", and $\mathbf{R}^3$ is "3-dimensional". Describe anything you notice about the relationship between the number of equations in x, y and z and the dimension of the shape they represent in $\mathbf{R}^3$. Do the same thing for $\mathbf{R}^2$.

Solution: In three-space, when there is one equation, we get a shape of dimension two. When there are two equations, we get a shape of dimension one, when there are three equations, we get a shape of dimension zero. It seems like the dimension of the shape described by a set of equations is three minus the number of equations. Similarly, in two-space, it seems like the dimension of the shape described by a set of equations is two minus the number of equations. Loosely speaking, the *codimension* of an object in $\mathbf{R}^3$ is 3 minus the dimension. The codimension of an objection in $\mathbf{R}^2$ is 2 minus the dimension. Most of the time, the codimension is equal to the number of equations.

2. In this problem, you'll explore the set of points equidistant from two given points in $\mathbf{R}^3$. As you do, reflect back to what the set of points equidistant from two points in $\mathbf{R}^2$ is (Hint: high school geometry).

(a) Plot the points $(2, -1, 5)$ and $(2, 6, 5)$. Write an equation for the set of points equidistant from $(2, -1, 5)$ and $(2, 6, 5)$ and simplify it as much as possible. Describe the set as specifically as possible. (Hint: Start with a generic point (x, y, z). What would it mean for this point to be equidistant from the two given points? Write an equation to describe this.)

(b) Plot the points $(2, -1, 5)$ and $(3, 6, 0)$. Write an equation for the set of points equidistant from $(2, -1, 5)$ and $(3, 6, 0)$ and simplify it as much as possible. Describe the set as specifically as possible.

Solution:

Intuitively, we expect this set to be an object that "cuts halfway" between the two points. In $\mathbf{R}^2$, the set of points equidistant from two points is the perpendicular bisector, the line that bisects the line connecting the two points at a right angle. What happens in $\mathbf{R}^3$?

(a) We want to find all points (x, y, z) such that the distance from (x, y, z) to $(2, -1, 5)$ equals the distance from $(x, y, z) = (2, 6, 5)$. This translates to solving $\sqrt{(x-2)^2 + (y+1)^2 + (z-5)^2} = \sqrt{(x-2)^2 + (y-6)^2 + (z-5)^2}$. Squaring both sides, we have the equation $(x-2)^2 + (y+1)^2 + (z-5)^2 = (x-2)^2 + (y-6)^2 + (z-5)^2$ which simplifies to $2y + 1 = -12y + 36$. Solving this, we get $y = \frac{5}{2}$.

This is the equation for the plane parallel to the xz-plane and passing through the y-axis at $\frac{5}{2}$. Notice that this plane bisects the line connecting the two points and is perpendicular to that line.

(b) We use the same strategy as above. The equation we get is $2x + 14y - 10z = 15$. As we will confirm later, this describes a plane in $\mathbf{R}^3$ which perpendicularly bisects the line connecting the two points.

3. **What region is this in $\mathbf{R}^3$?** For each of the following, describe as fully as possible the region represented by the equation or set of equations.

Here are some things to address and tips for thinking and visualizing:

- Is it a familiar geometric object?
- What *dimension* is it? Is it a line or curve (one-dimensional/1D)? Is it two-dimensional like a plane? For example, if you think "cone", is it the surface of the cone (2D) or a solid cone (3D)?
- If the equation doesn't involve one of the variables (for example, there is no z in the equation), what does that mean about the possible values of z for points in the region? (A variable which does not have any restriction on it is sometimes called a *free variable* or *independent variable*.)
- How does the region relate to the coordinate planes or axes? Is is parallel or perpendicular to any of them?
- For some, it may be useful to think about whether the object intersects the coordinate planes. To test intersection with coordinate plane (eg: xy-plane) we use equation for that plane (eg: $z = 0$) and solve those equations together. We call such an intersection a *trace* of the region. However, we still have to consider the equation in $\mathbf{R}^3$ - what's the role of the other variable (eg: z)?

(a) $x = 3, y = -7$

(b) $y = 2x + 3$

(c) $y = 2x + 3, z = 5$

(d) $y = 3x + 1, z = -2, x = y$

(e) $x^2 = 3$

(f) $x^2 + y^2 + z^2 = 3$

(g) $y = x^2 + 3$

(h) $x^2 + z^2 = 3$

(i) $y^2 + 2z^2 = 3$

Comments for Facilitator:
Some key ideas to highlight:

- Even if they have no idea what it is, they can think about the dimension of the object by thinking about number of "free variables". Is x determined completely? Is y? Is z?
- What happens if a variable is not present in the equation (the idea that it is free and can take on any value, essentially meaning you can stretch the object in a 2D plane into that third dimension.
- Being specific about parallel and perpendicular.
- We'll get to planes next week, so you don't need to worry about getting through all of that. But the idea that two planes intersects in a line, usually, is something to highlight.
- Even students who zip through this can be challenged to describe more specifically, to come up with alternate equations for these regions. OR you can have them think about the problems on the back intersections of planes and cone.

4. (Exploratory) Another way to think about describing regions in $\mathbf{R}^2$, $\mathbf{R}^3$ and higher dimensions is in terms of a *parameteric equation.* For example, in $\mathbf{R}^2$, consider all points

$$\langle \cos t, \sin t \rangle$$

where t is any real value. What is this object? What about the following?

$$\langle 2\cos t, \sin t \rangle$$