Grothendieck Riemann Roch

Theorem (excess intersection theorem)

Let the following be a commutative square of derived schemes:

$$\begin{tikzcd} Z' \arrow[r, "i'"] \arrow[d, "g"'] \arrow[dr, phantom, "\checkmark"] & X' \arrow[d, "f"] \\ Z \arrow[r, "i"'] & X \end{tikzcd}$$

an excessive square

1) i, i' are quasi-smooth closed immersions, of v. codim d, d', resp.

2) $\square$ is Cartesian on underlying classical schemes ($Z'_{cl} \to (Z \times_X X')_{cl}$ is invertible)

3) $\underline{g^* N_{Z/X} \to N_{Z'/X'}}$ is surjective on π_0.

Let $\underline{\mathcal{E}}$ be the fiber of $g^* N_{Z/X} \to N_{Z'/X'}$, called the excess sheaf

We have

$$f^* i_*(x) = i'_*(g^*(x) \cdot \lambda_{-1}(\mathcal{E}))$$

$$\forall\, x \in K_0(Z)$$

$$f^* i_*^{\gamma}(x) = (i')_*^{\gamma}(g^*(x) \cdot c^{d-d'}(\mathcal{E}))$$

$\forall\, x \in K_0(\mathbb{Z})_{\mathbb{Q}}$

$$\lambda_{-1}(\mathcal{E}) = \sum_i (-1)^i [\Lambda^i(\mathcal{E})].$$

$c^{d-d'}(\mathcal{E})$ top chern class.

-GRR-

Let X derived scheme s.t. $\exists$ sheaf of spectra K s.t.

$$K_0(X) \simeq \pi_0 \Gamma(X, K).$$

$$Rk : K_0(X) \longrightarrow H^0(X_{zar}, \mathbb{Z}).$$

γ-filtration

$$\gamma^k(x) = \lambda^k(x+k-1)$$

$x \in K_0(X) \quad k \in \mathbb{Z}$

$\mathrm{Fil}^*_\gamma K_0(X)$

$$\mathrm{Fil}^1_\gamma K_0(X) = \ker(Rk)$$

$\mathrm{Fil}^k_\gamma K_0(X)$ gen by

$\gamma^{k_0}(x_0), \dots, \gamma^{k_n}(x_n)$

$x_i \in \mathrm{Fil}^1_\gamma K_0(X)$

$k_0 + \dots + k_n \geq k$.

Let $i : Z \hookrightarrow X$ be quasi-smooth closed immersion.

$$i_* : K_0(Z) \to K_0(X)$$

— $i^\gamma_\alpha : \mathrm{Gr}^*_\gamma K_0(Z)_\mathbb{Q} \to \mathrm{Gr}^{*+d}_\gamma K_0(X)_\mathbb{Q}$

Theorem

Let $i : \underline{Z} \hookrightarrow \underline{X}$ qcqs ds. v codim d.

— $i_* : K_0(Z)_\mathbb{Q} \to K_0(X)_\mathbb{Q}$

$$\underline{\mathrm{Fil}^{k}_\gamma K_0(Z)_\mathbb{Q} \to \mathrm{Fil}^{k+d}_\gamma K_0(X)_\mathbb{Q}} \quad \forall k \in \mathbb{Z}$$

Proof

Lemma

The image of i_* is contained in $\mathrm{Fil}^1_\gamma K_0(X)_{\mathbb{Q}}$

Mome λ-rings

If A is λ-ring, suppose $N \in A$, $\lambda^k(N) = 0$ for all $k > d$ $\exists d$.

$\Rightarrow \exists!$ elements $\lambda^p(N, x) \in A$ $\forall x \in A, p \geq 1$
$\gamma^p(N, x) \in A$ $\forall x \in A, p \geq 1$

With

$$\underline{\lambda^p(N, x)} \cdot \lambda_{-1}(N) = \lambda^p(x \cdot \lambda_{-1}(N))$$

$\gamma^p(N, x)$

$$\gamma^{k+d}(N, x) - (-1)^{k+d-1} (k+d-1)! \cdot x$$

$\in \mathrm{Fil}^{k+1}_\gamma(A)$.

Lemma

i

For any $x \in K_0(Z)$, any $\rho \geq 1$, we have the equality

$$i_*(\gamma^\rho(N_{Z/X}, x)) = \gamma^\rho(i_*(x))$$

in $K_0(X)_{\mathbb{Q}}$

— i. Blow up square

$$\begin{tikzcd} E \arrow[r, "i_E"] \arrow[d, "g"'] & \tilde{X} \arrow[d, "f"] \\ Z \arrow[r, "i"] & X \end{tikzcd}$$

$\mathcal{L}$ i_E, $\mathcal{E}$

locally free $\mathcal{O}_Z$-module N' s.t. $[N_{Z/X}]$
$= [N'] + 2$ in $K_0(Z)$

$$\lambda_{-1}(\mathcal{E}) \equiv 0 \ (\mathrm{mod}\ 1 - \mathcal{L})$$

in $K_0(Z)$.

i_* sends $\mathrm{Fil}^k_\gamma K_0(Z)_\mathbb{Q}$ to

$\mathrm{Fil}^{k+d}_\gamma K_0(X)_\mathbb{Q} \quad \forall k \in \mathbb{Z}$

If $x \in K_0(Z)_\mathbb{Q}$ $\quad \underline{\mathrm{Fil}^k_\gamma K_0(Z)_\mathbb{Q}}$ $\exists k \in \mathbb{Z}$.

$$a\,\gamma^{k_0}(\alpha_0)\cdots\gamma^{k_n}(\alpha_n) \qquad \alpha_i \in \mathrm{Fil}^1_\gamma(K_0(Z))$$

$a \in \mathbb{Q}$.

R sub-$\mathbb{Q}$-λ-algebra of $K_0(Z)_\mathbb{Q}$

$[N_{Z/X}]$ $\quad [\alpha_i]$

$$i(\mathrm{Fil}^k_\gamma(R)) \subseteq \mathrm{Fil}^{k+d}_\gamma K_0(X)_\mathbb{Q} \ldots$$

$$b_s = (-1)^{s-1}(s-1)!$$

$$\gamma^{k+d}(N_{Z/X}, x) - b_{k+d}\cdot x \in \mathrm{Fil}^{k+1}_\gamma(R)$$

$$i_*(\gamma^{k+d}(N_{Z/X}, x) - b_{k+d}\cdot x) \in \mathrm{Fil}^{k+d+1}_\gamma(K_0(X)_\mathbb{Q})$$

$\gamma^{n+1}(i_*(x)\, b_{n+1} \cdot i_*(x)) \in Fil_\gamma^{n+1+r}(K_0(X))_{\mathbb{Q}}$

$i_*(y) \in Fil_\gamma^1(K_0(X)_{\mathbb{Q}})$

$\gamma^{n+1}(i_*(x))$

$\Rightarrow i_*(x) \in Fil_\gamma^{n+d}(K_0(X)_{\mathbb{Q}})$.

Chern character map

$ch: K_0(X) \longrightarrow Gr_\gamma K_0(X)_{\mathbb{Q}}$.

A $\mathbb{N}$-graded

A^0 is binomial ring ($\mathbb{Z}$)

$\lambda^n(x) = \binom{x}{n}$

$\hat{A} = \prod_{i \geq 0} A^i \longrightarrow A^0 \quad \varepsilon$

$\ker(\varepsilon) = \cancel{1+}\hat{A}^+ \quad \hat{A}^+ \qquad 1 + \hat{A}^+$

Chern Ring

$\mathrm{Chern}_{A^0}(A)$ $\qquad A^0 \times (1+\hat{A}^+)$

$[n, x]$

$n \in A^0 \quad x = 1 + \sum_{i \geq 1} x^i \quad x^i \in A^i$

If k $\quad \Lambda$ aug k-λ-alg

$Gr_\gamma \Lambda \qquad c^i(x) \in Gr_\gamma^i \Lambda$

class of $\gamma^i(x - \varepsilon(x)) \in \mathrm{Fil}_\gamma^i(\Lambda)$

$\tilde{c}(x) = [\varepsilon(x), 1 + \sum_{i>0} c^i(x)] \ \forall x$

Completed chern char.

$$\mathrm{ch} : \mathrm{Chern}_{A^0}(A) \to \widehat{A}_{\mathbb{Q}} \qquad A_{\mathbb{Q}} = A \otimes \mathbb{Q}$$

- additive
- $1 \mapsto 1$
- $\mathrm{ch}[1, 1 + x'] = \exp(x') = \sum_{n \geq 0} (x')^n / n!$

$$f \in \mathbb{Q}[[t]] \qquad f(t) = \frac{t}{1 - \exp(-t)}$$

$$T_f : 1 + \widehat{A}^+ \to 1 + \widehat{A}_{\mathbb{Q}}^+$$

$K_0(X)$. $H^0(X_{\text{zan}}, \mathbb{Z})$

$\underline{ch}: K_0(X) \xrightarrow{\sim}$ chern $n(Gr_\gamma K_0(X))$

$\xrightarrow{ch} Gr_\gamma K_0(X)_{\mathbb{Q}}$

Theorem

X qcqs ds.

i r codim d

$: Z \hookrightarrow X$

$$\begin{tikzcd} K_0(Z) \arrow[r, "i_*"] \arrow[d, "ch"'] & K_0(X) \arrow[d, "ch"] \\ Gr_\gamma(K_0(Z)_{\mathbb{Q}}) \arrow[r, "\underline{i_*^\gamma(-\cdot Td(-N^\vee_{Z/X}))}"'] & Gr_\gamma(K_0(X)_{\mathbb{Q}}) \end{tikzcd}$$

$ch(i_*(\alpha)) = i_*^\gamma(ch(\alpha) \cdot Td(-N^\vee_{Z/X}))$
