Arithmetic Intersection Theory II

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Goal

Define the $*$-product of Green currents properly (using Green forms), and prove its commutativity and associativity.

Notations

For a smooth manifold X (all manifolds in this talk are without boundary) and an integer $r \geq 0$ we let

- $\mathrm{A}^r(X)$ be the topological vector space over $\mathbb{C}$ of complex-valued (smooth differential) forms of degree r;
- $\mathrm{A}_c^r(X) \subset \mathrm{A}^r(X)$ be the subspace of r-forms with compact support;
- $\mathscr{D}_r(X) := \mathrm{A}_c^r(X)'$ be the topological dual.

Definition

A **current** is an element in $\mathscr{D}_r(X)$ for some r; the number r is called the **dimension** of the current.

If X is a complex manifold, then we have a decomposition

$$\mathrm{A}^r(X) = \bigoplus_{p+q=r} \mathrm{A}^{p,q}(X); \tag{1}$$

elements of the RHS are called (p,q)-forms. Set

$$\mathrm{A}_c^{p,q}(X) := \mathrm{A}^{p,q} \cap \mathrm{A}_c^r(X) \quad (p+q=r).$$

The decomposition of forms induces a decomposition of currents

$$\mathscr{D}_r(X) = \bigoplus_{p+q=r} \mathscr{D}_{p,q}(X), \tag{2}$$

where $\mathscr{D}_{p,q}(X) := \mathrm{A}_c^{p,q}(X)'$.

We put

$$\mathscr{D}^r(X) := \mathscr{D}_{2d-r}(X)$$

respectively

$$\mathscr{D}^{p,q}(X) := \mathscr{D}_{d-p,d-q}(X),$$

where $d := \dim X$ is the *complex* dimension of X.
The *sup*script r or (p, q) is called the **degree** of the current.

If we don't want to specify the degree, we only write $\mathrm{A}(X)$ and so on.

Current of integration

Recall that the current of integration or Dirac current δ_X over a complex manifold X is given by

$$\delta_X(\alpha) := \int_X \alpha, \quad \alpha \in \mathrm{A}_c^{2\dim X}(X).$$

Wedge

Let X be a complex manifold. We have a pairing

$$\wedge : \mathscr{D}_{p,q}(X) \times \mathrm{A}^{r,s}(X) \to \mathscr{D}_{p-r,q-s}(X)$$

given by the formula

$$(T, \alpha) \mapsto (\beta \mapsto T(\alpha \wedge \beta)) =: T \wedge \alpha.$$

In particular, if X is a complex manifold, we have a map

$$\mathrm{A}^{p,q}(X) \to \mathscr{D}^{p,q}(X), \quad \alpha \mapsto \delta_X \wedge \alpha.$$

Remark

One can show that this is a continuous injective $\mathbb{C}$-linear with *dense image*. However, the embedding is not canonical as it depends on the chosen volume form on X.

Currents associated to (smooth) forms

Using the notation from the last slice, we will write

$$[\alpha] := \delta_X \wedge \alpha,$$

i.e.

$$[\alpha](\beta) = \int_X \alpha \wedge \beta$$

for $\beta \in A_c(X)$.

Definition

By an L^1-form on a complex manifold X, we mean that in any coordinate patch, α has coefficients which are L^1-functions (locally integrable functions).

Remark

For an L^1-form α, we can still makes sense of $[\alpha]$.
The main difference is that the equality of currents

$$[d\alpha] = d[\alpha],$$

which is true for smooth forms, no longer holds for general L^1-forms.

If α is only L^1, the symbol $[d\alpha]$ means the weak derivative of α.

Poincaré-Lelong: statement

Theorem (Poincaré-Lelong formula)

Let X be a connected complex projective manifold and $(L, ||\ ||)$ be a holomorphic hermitian line bundle on X. For any nonzero meromorphic section s of L, we have an equality

$$dd^c[-\log ||s||^2] + \delta_{\mathrm{div}(s)} = [dd^c(-\log ||s||^2)]$$

in $\mathscr{D}^{1,1}(X)$.

Recall $dd^c := \frac{i}{2\pi}\partial\overline{\partial}$.

We first make some comments on the last theorem.

1. The term $-\log ||s||^2$ is an L^1-function on X, it has poles exactly at the support $|\operatorname{div}(s)|$.

2. By GAGA principle, the analytic divisor $\operatorname{div}(s)$ can be defined as the complex analytification of the algebraic divisor of the meromorphic section s.

(cont.)

3. On the open set $U := X \setminus |\operatorname{div}(s)|$, we have

$$dd^c(-\log ||s||^2) = c_1(L, ||\ ||)$$

as *smooth* forms, where $c_1(L, ||\ ||)$ is the first Chern form attached to the hermitian line bundle $(L, ||\ ||)$.

4. Without referring to the theorem, the first Chern form $c_1(L, ||\ ||)$ is D-closed ($D \in \{d, \partial, \overline{\partial}\}$) by the more standard construction, and it induces the first Chern class

$$c_1(L) \in H^{1,1}(X; \mathbb{R}) := H^{1,1}(X) \cap H^2(X; \mathbb{R})$$

of the holomorphic line bundle L.

Since s is represented by a family of meromorphic functions (satisfying the cocycle condition), and meromorphic functions are locally fraction of holomorphic functions, we obtain from Poincaré-Lelong's formula the following

Corollary (Poincaré-Lelong equation)

If f is a nonzero meromophic function on X, then

$$dd^c[-\log|f|^2] + \delta_{\mathrm{div}(f)} = 0.$$

See [Huybrechts, §2.3] for the notions of analytic divisors, meromophic sections etc.

Poincaré-Lelong: proof

Want: the identity

$$\int_X \log ||s||^2 \partial\overline{\partial}\alpha + 2\pi i \int_{\mathrm{div}(s)} \alpha = \int_X (\partial\overline{\partial} \log ||s||^2) \wedge \alpha$$

for $\alpha \in \mathrm{A}_c^{d-1,d-1}(X)$, $d := \dim X$.

We will make sense of integration over a possibly singular complex analytic space in Theorem 2.

Proof.

Assume $|\operatorname{div}(s)|$ is a simple normal crossing divisor.

Then there is a locally finite coordinate open cover $\{U_\lambda\}_\lambda$ of X such that we can choose the coordinates $(z_1, \ldots, z_k)$ (depend on λ) on U_λ and a local frame e_λ of L over U_λ satisfying

$$s|_{U_\lambda} = z_1 \cdots z_k e_\lambda. \quad (k \leq d)$$

Since the statement is local, using a partition of unity subordinate to $\{U_\lambda\}_\lambda$, we are reduced to show the following

Lemma

Let $(z_1, \ldots, z_d)$ be the coordinates of $\mathbb{C}^d$. Then for every $i = 1, \ldots, d$ we have

$$\int_{\mathbb{C}^d} \log |z_i|^2 \partial\overline{\partial}\alpha + 2\pi i \int_{\{z_i=0\}} \alpha = 0, \quad \alpha \in \mathrm{A}_c^{d-1,d-1}(\mathbb{C}^d).$$

Proof.

Let $\alpha \in \mathrm{A}_c^{d-1,d-1}(\mathbb{C}^d)$. Observe for degree reasons,

$$\log |z_1|^2 \partial\overline{\partial}\alpha = d(\log |z_1|^2 \overline{\partial}\alpha) - \frac{dz_1}{z_1} \wedge \overline{\partial}\alpha$$

and

$$d\left(\frac{dz_1}{z_1} \wedge \alpha\right) = -\frac{dz_1}{z_1} \wedge \overline{\partial}\alpha.$$

Proof.

(cont.) Then we find for every $\epsilon > 0$,

$$\int_{|z_1|\geq\epsilon} \log|z_1|^2 \partial\overline{\partial}\alpha = \int_{|z_1|\geq\epsilon} d(\log|z_1|^2\overline{\partial}\alpha) - \int_{|z_1|\geq\epsilon} \frac{dz_1}{z_1} \wedge \overline{\partial}\alpha$$
$$= -\int_{|z_1|=\epsilon} \log|z_1|^2 \partial\alpha - \int_{|z_1|=\epsilon} \frac{dz_1}{z_1} \wedge \alpha,$$

where in the last equality we used Stokes' theorem for both integrals. One then shows that on taking the limit $\epsilon \to 0$, the first integral is zero, whereas the other is exactly $2\pi i \int_{z_1=0} \alpha$ (see [Moriwaki, Claim 8 on p.38]). □

Proof for Poincaré-Lelong.

In the general case where $|\operatorname{div}(s)|$ is not necessarily a normal crossing, we apply *resolution of singularities* $\pi : \tilde{X} \to X$ along $|\operatorname{div}(s)| \subset X$ such that $\pi^{-1}(|\operatorname{div}(s)|)$ is a normal crossing.

We then "pull everything back" to $\tilde{X}$, in which we have the Poincaré-Lelong formula by the last lemma, and conclude by using the fact that the pullback does not change the value of the integrals with the log terms and observe that

$$\int_{\pi^* \operatorname{div}(s)} \pi^* \alpha = \int_{\operatorname{div}(s)} \alpha.$$

This settles the proof of Poincaré-Lelong. □

Remark

It is possible to prove Poincaré-Lelong's formula without using resolution of singularities, however, this requires more analysis.

Green currents

Definition

Let $Y \subset X$ be an irreducible analytic subvariety of codimension p of a d-dimensional complex manifold X. A **Green current** g_Y for Y is an element of $\mathscr{D}^{p-1,p-1}(X)$ such that

$$dd^c g_Y + \delta_Y = [\omega_Y],$$

where $\omega_Y \in \mathrm{A}^{d-p,d-p}(X)$ is a *smooth* form. We extend this definition to general analytic cycles by additivity.

Example

The Poincaré-Lelong formula (in case X is projective) says that $[-\log ||s||^2]$ is a Green current for the Weil divisor $\mathrm{div}(s)$, the smooth form being the first Chern form.

On the other hand, the Poincaré-Lelong equation implies that if two subvarieties Y and Y' are (algebraically) rationally equivalent, then they have the same Green current.

Resolution of singularities

Theorem

Let Y be a complex analytic space. There exist a complex manifold $\tilde{Y}$ and a morphism $\pi : \tilde{Y} \to Y$ with the following properties.

1. *π is proper.*
2. *π is isomorphic over Y^0, i.e. $\pi|_{\pi^{-1}(Y^0)} \cong Y^0$ as complex analytic spaces.*
3. *$E := \pi^{-1}(Y \setminus Y^0)$ is a simple normal crossing divisor, that is, the local equations of E are of the form $z_1 \ldots z_r = 0$ for some $r \leq \dim(\tilde{Y})$.*

Here $Y^0 \subset Y$ denotes the regular locus of Y.

Proof.

See [Włodarczyk, Theorem 2.0.1]. □

Integration over complex analytic space

Let $Y \subset X$ be an analytic subvariety of a complex manifold X. For a compactly supported form $\alpha \in \mathrm{A}_c(X)$ we define

$$\int_Y \alpha \overset{\text{def}}{=} \int_{Y^0} \alpha|_{Y^0} = \int_{\pi^{-1}(Y^0)} \pi^*(\alpha|_{Y^0}) = \int_{\tilde{Y}} \pi^* \alpha.$$

The middle equality is by Property (2) of the last Theorem 2, and the last equality is by Property (3) combined with Sard's theorem, and finally the last integral is finite because the support

$$\mathrm{supp}(\pi^*\alpha) \subset \pi^{-1}(\mathrm{supp}(\alpha)) \subset \tilde{Y}$$

is compact by properness of π.

Remark

The above construction shows that the value $\int_Y \alpha = \int_{Y^0} \alpha|_{Y^0}$ does not depend on the particular choice of the resolution $\pi : \tilde{Y} \to Y$. This will be useful for various discussions later.

We want to show that Green currents exist – at least for cycles on a *compact Kähler* manifold. This is because of the following deep results from Complex Geometry.

Theorem (Hodge decomposition)

Let X be a compact Kähler manifold. Then there are natural ***orthogonal*** *decompositions*

$$\mathrm{A}^{p,q}(X) = \partial \mathrm{A}^{p-1,q} \oplus H^{p,q}(X) \oplus \partial^* \mathrm{A}^{p+1,q}(X).$$

and

$$\mathrm{A}^{p,q}(X) = \overline{\partial} \mathrm{A}^{p,q-1} \oplus H^{p,q}(X) \oplus \overline{\partial}^* \mathrm{A}^{p,q+1}(X).$$

(The decomposition is independent of the choice of the Kähler metric.)

Here ∂^* (resp. $\overline{\partial}^*$) denotes the adjoint operator with respect to the L^2-product on $\mathrm{A}^{p,q}(X)$.

Corollary (Serre duality)

Let X be a compact Kähler manifold of (complex) dimension d. The natural pairing

$$\langle -, - \rangle : H^{p,q}(X) \times H^{d-p,d-q}(X) \to \mathbb{C}$$

is (well-defined and) nondegenerate.

We want to transfer the above results to the cohomology of currents. First note that we can define the differentials d, ∂, and $\overline{\partial}$ on the space of currents $\mathscr{D}(X)$ with suitable sign such the diagram

$$\begin{tikzcd} \mathrm{A}^r(X) \arrow[r, hook, "{[\,]}"] \arrow[d, "d"] & \mathscr{D}^r(X) \arrow[d, "d"] \\ \mathrm{A}^{r+1}(X) \arrow[r, hook, "{[\,]}"] & \mathscr{D}^{r+1}(X) \end{tikzcd} \tag{3}$$

commutes (similarly for ∂, $\overline{\partial}$ in place of d).
We thus obtain naturals maps of cohomology groups, e.g.

$$H^{p,q}(X) \to H^q(\mathscr{D}^{p,\bullet}, \overline{\partial}).$$

Theorem (Smoothing of cohomology)

The above maps of cohomology groups are isomorphism.

Corollary (dd^c-Lemma for currents)

Let X be a compact Kähler manifold. Consider a current $T \in \mathscr{D}^{p,q}(X)$. If T is d-exact, then it is also dd^c-exact.

Proof.

Let $T = \partial S + \overline{\partial} S$ for some current S. By smoothing of cohomology and Hodge decomposition, there are currents of suitable degree such that

$$S = h_1 + \partial x_1 + \partial^* y_1 = h_2 + \overline{\partial} x_2 + \overline{\partial}^* y_2,$$

which yields

$$\partial S = \partial\overline{\partial} x_2 + \partial\overline{\partial}^* y_2, \quad \overline{\partial} S = \overline{\partial}\partial x_1 + \overline{\partial}\partial^* y_1.$$

So

$$T = \overline{\partial}\partial x_1 + \partial\overline{\partial} x_2 + \partial\overline{\partial}^* y_2 + \overline{\partial}\partial^* y_1.$$

Proof.

(cont.) Since T is d-exact, $\partial T = \overline{\partial} T = 0$. Thus from the last display we find $\partial\overline{\partial}\partial^* y_1 = 0$ and $\overline{\partial}\partial\overline{\partial}^* y_2 = 0$. By Kähler identities (for a complete list, see [Huybrechts, Prop.3.1.12]), we have moreover $\partial^*\overline{\partial} = -\overline{\partial}\partial^*$. Therefore

$$0 = \langle \partial\overline{\partial}\partial^* y_1, \overline{\partial} y_1 \rangle = -\langle \overline{\partial}\partial^* y_1, \overline{\partial}\partial^* y_1 \rangle.$$

By Serre duality, the pairing $\langle -, - \rangle$ is non-degenerate. We conclude that $\overline{\partial}\partial^* y_1 = 0$. Similarly $\partial\overline{\partial}^* y_2 = 0$. The claim follows (recall $dd^c = \frac{i}{2\pi}\partial\overline{\partial}$ and $\overline{\partial}\partial = -\partial\overline{\partial}$). □

Corollary (Existence of Green currents)

Let Y be a codimension p analytic cycles on a compact Kähler manifold X. Then there is a Green current for Y.

Proof.

By Stokes' Theorem, we have

$$d\delta_Y = 0 \quad \text{(as currents).}$$

By smoothing of cohomology, there is a smooth form $\omega \in \mathrm{A}^{p,p}(X)$ and a current $S \in \mathscr{D}^{p-1,p-1}(X)$ such that

$$\delta_Y - \omega = dS.$$

So the claim follows from the dd^c-Lemma. □

Convention for Green currents

Next we discuss the uniqueness of Green currents.

Lemma

Let X be a complex manifold. Let T be a current on X such that $\partial\overline{\partial}T$ is a smooth current, then there exist currents α and β and a smooth form ω such that

$$T = [\omega] + \partial\alpha + \overline{\partial}\beta.$$

By a smooth current we mean a current associated to some smooth form.

Proof.

Using smoothing of cohomology, this is an elementary exercise using inductive arguments. See [GS90, 1.2.2]. □

Corollary (Uniqueness of Green currents)

Let X be a compact Kähler manifold and g_1, g_2 be Green currents for the same analytic cycle. Then there exist currents α and β and a smooth form ω such that

$$g_1 - g_2 = [\omega] + \partial\alpha + \overline{\partial}\beta.$$

Moreover, ω can be chosen to be a harmonic form.

Proof.

Apply the last lemma with $T := g_1 - g_2$. The harmonicity statement follows from the Hodge decomposition. □

An equivalence relation

Put

$$\tilde{\mathrm{A}}(X) := \mathrm{A}(X)/\mathrm{im}(\partial) + (\bar{\partial})$$

and

$$\tilde{\mathscr{D}}(X) := \mathscr{D}(X)/\mathrm{im}(\partial) + (\bar{\partial}).$$

Towards the *-product

Let Y, Z be two analytic cycles on a complex projective manifold/smooth variety X, let g_Y, g_Z be some choice of Green currents respectively. We wish to define a product $g_Y * g_Z$ such that

$$g_Y * g_Z = g_{Y.Z} \quad (\text{in } \tilde{\mathscr{D}}(X))$$

is the Green current associated to the intersection cycle (or even the cycle class of) $Y.Z$.

The *-product of Green currents

Definition

Using the previous notations, define

$$g_Y * g_Z := [g_Y] \wedge \delta_Z + \omega_Y \wedge g_Z.$$

The second wedge

The wedge in the second summand has been defined (after exchanging the position). We will make sense of the other wedge in the next slice.

Let X be a complex projective manifold and $Y \subset X$ and irreducible closed subvariety. Assume $i : Z \subset X$ is another closed irreducible subvariety such that $Z \not\subset Y$. Let $\pi : \tilde{Z} \to Z$ be a resolution and consider the composition $\nu := i\pi : \tilde{Z} \to X$. If η is a “logarithmic” L^1-form on X which is smooth away from Y, then we define a current

$$\eta \wedge \delta_Z := \delta_Z \wedge \eta := \nu_*[\nu^*\eta].$$

Explicitly,

$$\eta \wedge \delta_Z(\alpha) = \int_{\tilde{Z}} \nu^*\eta \wedge \nu^*\alpha, \quad \alpha \in \mathrm{A}_c(X).$$

The definition does not depend on the choice of resolution.

Definition

The form η is said to be **logarithmic or form of log type along** Y if there exists a proper map $\pi : M \to X$ and a smooth form ϕ on $M \setminus \pi^{-1}(Y)$ such that

1. M is a complex manifold and $\pi^{-1}(Y) =: E$ is a divisor with normal crossing on M and $\pi|_{M\setminus E} : M \setminus E \to X \setminus Y$ is a submersion.
2. $\pi_*[\phi] = [\eta]$ (more generally $(\pi|_{M\setminus E})_*\phi = \eta$).
3. If $U \subset M$ is any nonempty open subset with holomorphic coordinates $(z_1, \ldots, z_n)$ such that $E \cap U$ is given by the equation $z_1 \ldots z_k = 0$ for some $k \leq n$, then there exist ∂- and $\overline{\partial}$-closed smooth forms α_i on U, $i = 1, \ldots, k$, and a smooth form β on U such that
$$\phi|_U = \sum_1^k \alpha_i \log |z_i|^2 + \beta.$$

Green forms

Definition

Let $Y \subset X$ be an irreducible analytic subvariety of a complex (projective) manifold X. A **Green form (of log type) along** Y is a smooth form of log type along $|Y|$ whose associated current is a Green current for Y. We extend this definition to general analytic cycles by additivity.

Pushforward of currents

Let $f : X \to Y$ be a *proper* holomorphic map of complex manifolds. Then we have a well-defined pullback map of differential forms

$$f^* : \mathrm{A}_c^{p,q}(Y) \to \mathrm{A}_c^{p,q}(X).$$

Let $T \in \mathscr{D}_{p,q}(X)$ be a current. Consider the continuous $\mathbb{C}$-linear functional

$$\alpha \mapsto T(f^*\alpha), \quad \alpha \in \mathrm{A}_c^{p,q}(Y).$$

By Riesz's representation theorem, there is a unique current $f_*T \in \mathscr{D}_{p,q}(Y)$ such that

$$f_*T(\alpha) = T(f^*\alpha).$$

We thus obtain a map

$$f_* : \mathscr{D}_{p,q}(X) \to \mathscr{D}_{p,q}(Y), \quad T \mapsto f_*T,$$

called the **pushforward** of currents.

Pullback of currents

Let $f : X \to Y$ be a proper *submersion* of complex manifolds of relative dimension $r := \dim f^{-1}(y) = \dim X - \dim Y$. Let $\alpha \in \mathrm{A}_c^{p,q}(X)$ and consider the pushforward $f_*[\alpha] \in \mathscr{D}_{n-p,n-q}(Y)$ of the current $[\alpha] \in \mathscr{D}_{n-p,n-q}(X)$. Then there is a smooth form, denoted $f_*\alpha \in \mathrm{A}_c^{p-r,q-r}(Y)$, such that

$$f_*[\alpha] = [f_*\alpha] \quad \text{as currents.}$$

In fact, $f_*\alpha$ can be realized by **integration along fibers**:

$$f_*\alpha(y) =: \int_{x \in f^{-1}(y)} \alpha(x),$$

where the fiber $f^{-1}(y) \subset X$ has induced orientation.

We need to show

$$\int_X \alpha \wedge f^* u = \int_{y \in Y} \left(\int_{x \in f^{-1}(y)} \alpha(x) \right) \wedge u(y), \quad u \in \mathrm{A}_c^{n-p,n-q}(Y).$$

Using a partition of unity argument, one recognizes that the above can be proved using the classical *Fubini's theorem* (see [Demailly, I.2C] for the argument in the smooth case).
Therefore, we can define the **pullback** of currents by the formula

$$f^* T(\alpha) := T(f_* \alpha), \quad \alpha \in \mathrm{A}_c^{p,q}(X),\ T \in \mathscr{D}_{p-r,q-r}(Y),$$

thus obtaining a map

$$f^* : \mathscr{D}_{p-r,q-r}(Y) \to \mathscr{D}_{p,q}(X).$$

Lemma (Green current of inverse image)

Let X be a complex projective manifold, $Y \subset X$ be an irreducible closed subvariety and η be a log type form along Y on X. Let $f : X' \to X$ be a holomorphic map of complex projective manifolds. Suppose that $f^{-1}(Y)$ does not contain any irreducible component of X'. Then $f^*\eta$ is a log type form along $f^{-1}(Y)$.
Moreover, if η is a Green form along Y, then $[f^*\eta]$ is a Green current for $f^{-1}(Y)$, i.e.

$$dd^c[f^*\eta] + \delta_{f^{-1}(Y)} = [f^*\omega_Y].$$

Remark

There is a similar statement and formula for pullback of an analytic cycle (assuming there is no excess component in the pullback cycle), see [GS90, Theorem 2.1.4(ii)]).

Proof.

Consider a smooth proper map $\pi : M \to X$ and an L^1-form ϕ on M as in the definition of log form so that $\pi_*[\phi] = [\eta]$. Let Z' be the Zariski closure of the subvariety

$$(X' \setminus f^{-1}(|Y|)) \times_X (M \setminus \pi^{-1}(|Y|)) \subset X' \times_X M.$$

Let $M' \to X' \times_X M$ be an embedded resolution of singularities along Z'. Denote the obvious maps $M' \to X'$ and $M' \to M$ by π' and f' respectively so that we have a commutative square

$$\begin{tikzcd} M' \arrow[r, "f'"] \arrow[d, "\pi'"] & M \arrow[d, "\pi"] \\ X' \arrow[r, "f"] & X. \end{tikzcd}$$

Proof.

(cont.) Moreover, the resolution gives us a *Cartesian* diagram

$$\begin{tikzcd} M' \setminus (f\pi')^{-1}(Y) \arrow[r] \arrow[d] & M \setminus \pi^{-1}(Y) \arrow[d] \\ X' \setminus f^{-1}(Y) \arrow[r] & X \setminus Y \end{tikzcd}$$

so that π' is smooth proper map over $X' \setminus f^{-1}(Y)$. We then compute

$$f^*\eta = f^*\pi_*\phi = \pi'_* f'^* \phi \quad \text{in } \mathrm{A}(X' \setminus f^{-1}(Y))$$

Proof.

Next we use that $(f\pi')^{-1}(Y)$ is a divisor with simple normal crossing. Let E_i be the irreducible components of $\pi^{-1}(Y)$ with local equation $z_i = 0$. The pullback of each of the E_i by f' as a Cartier divisor is $\sum_j n_{ij}E'_j$, where the E'_j is an irreducible component of $(f\pi')^{-1}(Y)$ with local equation $z'_j = 0$. Therefore, locally on M', we may write

$$f'^*\phi = \sum_i f'^*\alpha_i \sum_j n_{ij} \log |z'_j|^2 + f'^*\beta$$

for some local smooth form β. From the last two displays we conclude that $f^*\eta$ is of log type form along $f^{-1}(Y)$.

The moreover part is clear from the fact that the operator dd^c commutes with pushforward and pullback of currents. □

Using similar techniques, one can show ([GS90, Lemma 1.3.3(ii)])

Lemma (Green current of image)

Let η be as in the last lemma. Let $f : X \to X'$ be a (proper) holomorphic map of complex projective manifolds. Assume that $f|_{X \setminus Y}$ is a submersion and that $f(Y)$ does not contain any irreducible component of X'. Then $f_*\eta$ is a log type form along $f(Y)$. Moreover, we have an equality of currents

$$f_*[\eta] = [f_*\eta].$$

Lemma

Let X be a complex projective manifold, Y be an analytic cycle on X and η_Y be a Green form on X along $|Y|$. Assume Y and Z intersect properly. Then we have an equality of currents

$$dd^c(\eta_Y \wedge \delta_Z) = \omega_Y \wedge \delta_Z - \delta_{Y.Z}.$$

It follows that

$$\begin{aligned} dd^c(\eta_Y * g_Z) + \delta_{Y.Z} &= dd^c(\eta_Y \wedge \delta_Z + \omega_Y \wedge g_Z) + \delta_{Y.Z} = \\ &= \omega_Y \wedge \delta_Z + (dd^c \omega_Y) \wedge g_Z + \omega_Y \wedge (-\delta_Z + \omega_Z) = [\omega_Y \wedge \omega_Z]. \end{aligned}$$

In other words, $\eta_Y * g_Z$ is a Green current for $Y.Z$.

Proof of the lemma.

Let $f := (\tilde{Z} \xrightarrow{\pi} Z \xrightarrow{i} X)$, where π is a resolution and i is the closed immersion. Let ω_Y be a smooth form attached to the Green current $[\eta_Y]$ for Y. We compute that

$$\begin{aligned} dd^c(\eta_Y \wedge \delta_Z) &= f_*(dd^c[f^*\eta_Y]) \\ &= f_*([f^*\omega_Y] - \delta_{f^*Y}) \quad \text{(lemma on inverse image)} \\ &= \omega_Y \wedge \delta_Z - \delta_{f_*f^*Y}. \end{aligned}$$

Using that the cycle map $\mathrm{cl} : \mathrm{CH}(|Y| \cap Z) \to H(|Y| \cap Z)$ is a ring homomorphism, where the RHS denotes the Borel-Moore homology of the underlying topological space of the subvariety $|Y| \cap Z$ ([Fulton, §19]), one can show that $\delta_{f_*f^*Y}$ and $\delta_{Y.Z}$ represent the same (cohomology class of the) current.
(And note that resolution does not change the value of the integral by birationality.) $\square$

We give a more precise version:

Theorem

Suppose two analytic cycles Y and Z on a complex projective manifold intersect properly (no excess component); let $S_1, \ldots, S_k$ be the irreducible components of $|Y| \cap |Z|$. Then

$$dd^c(\eta_Y * g_Z) + \sum_1^k \mu_i \delta_{S_i} = [\omega_Y \wedge \omega_Z],$$

where μ_i denote the Serre intersection multiplicities.

Proof.

See Theorem 4 from [Sou+, §II.3] for a more explicit proof. (The case where Y, Z do not intersect properly is treated in [GS90, 2.1.4].) □

Next we show that Green forms actually exist and discuss the uniqueness problem in defining the *-product.

Consider the following situation. Let X be a complex manifold and $Y \subset X$ a closed complex *submanifold* of codimension p. Let $\nu : \tilde{X} := \mathrm{Bl}_Y(X) \to X$ be the blow-up of X along Y, and $E := \nu^{-1}(Y)$ its exceptional divisor, so that we have a Cartesian diagram

$$\begin{array}{ccc} E & \xhookrightarrow{j} & \tilde{X} \\ \downarrow{\scriptstyle \nu_Y} & & \downarrow{\scriptstyle \nu} \\ Y & \xhookrightarrow{i} & X \end{array}$$

(compare with diagram (*) in [Fulton, §6.7]).

- Set $N := N_{Y/X} = T_X|_Y / T_Y$, the normal bundle to Y.
- Recall $E = \mathbb{P}(N)$.
- Set $Q := \nu_Y^* N / \mathcal{O}_N(-1)$.

We choose a smooth Hermitian metric $|| \, ||$ on the holomorphic line bundle $\mathcal{O}_{\tilde{X}}(E)$, and a holomorphic section s of $\mathcal{O}_{\tilde{X}}(E)$ such that $\mathrm{div}(s) = E$. Finally we let $\beta := c_1(\mathcal{O}_{\tilde{X}}(E), || \, ||)$ be the first Chern form so that we have an equality of currents

$$dd^c[\log ||s||^{-2}] + \delta_E = [\beta]$$

by Poincaré-Lelong's formula.

Let $\mathrm{cl}(T)$ denote the cohomology class of a closed current T on X.

Lemma (Key Lemma I, part 1)

Let $\alpha \in \mathrm{A}^{p-1,p-1}(\tilde{X})$ be a closed form such that the cohomology class $\mathrm{cl}(j^*\alpha)$ is the $(p-1)$-th Chern class $c_{p-1}(Q) \in H^{2p-2}(E;\mathbb{C})$. Then we have the following equalities

$$\nu_*(\alpha \wedge \delta_E) = \delta_Y \quad \text{in } \mathscr{D}^{p-1,p-1}(X), \tag{4}$$

and

$$\nu^* \mathrm{cl}(\delta_Y) = \mathrm{cl}(\alpha \wedge \beta) \quad \text{in } H^{2p}(\tilde{X};\mathbb{C}). \tag{5}$$

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[d, "\nu_Y"] & \tilde{X} \arrow[d, "\nu"] \\ Y \arrow[r, hook, "i"] & X \end{tikzcd}$$

Lemma (Key Lemma I, part 2)

Moreover, if there is a closed form $\omega \in \mathrm{A}^{p,p}(X)$ such that the cohomology classes of ω and of δ_Y coincide, and if there is a smooth form $\gamma \in \mathrm{A}^{p-1,p-1}(X)$ satisfying

$$dd^c[\gamma] = [\nu^*\omega] - [\alpha \wedge \beta],$$

then for the current $g := \nu_*([\log ||s||^{-2}] \wedge \alpha + \gamma)$ we have

$$dd^c g + \delta_Y = [\omega] \quad \text{in } \mathscr{D}^{p,p}(X), \tag{6}$$

that is, g is a Green form along Y (provided all the assumptions are satisfied).

Proof.

For the first equality, we compute that

$$\nu_*(\alpha \wedge \delta_E) = \nu_*(\alpha \wedge j_*[1]) = \nu_*(j_*(j^*\alpha \wedge [1])) = i_*(\nu_Y)_*j^*\alpha.$$

By properness of ν_Y, we find that $(\nu_Y)_*j^*\alpha \in \mathrm{A}^0(Y)$ is actually a smooth form given by

$$(\nu_Y)_*j^*\alpha(y) = \int_{\nu_Y^{-1}(y)} j^*\alpha = \int_{\mathbb{P}(N_y)} c_{p-1}(Q) = 1 \quad (y \in Y)$$

by the very definition of the Chern class of Q. So

$$\nu_*(\alpha \wedge \delta_E) = i_*[1] = \delta_Y.$$

Proof.

(cont.) For the second identity, we use the “Key Lemma” from [Fulton, 6.7(a)], which gives

$$\nu^* \operatorname{cl}(\delta_Y) = j_* c_{p-1}(Q).$$

The assumption $\operatorname{cl}(j^*\alpha) = c_{p-1}(Q)$, Poincaré-Lelong’s formula and the projection formula then imply

$$\nu^* \operatorname{cl}(\delta_Y) = j_*(j^* \operatorname{cl}(\alpha) \cup 1) = \operatorname{cl}(\alpha) \cup j_*[1] = \operatorname{cl}(\alpha) \cup \operatorname{cl}(\delta_E) = \operatorname{cl}(\alpha \wedge \beta).$$

(Recall that the cup product can be computed by the wedge product of forms.)

Proof.

(cont.) Finally, for the moreover part, we compute

$$\begin{aligned} dd^c g &= \nu_* dd^c([\log ||s||^{-2}] \wedge \alpha + [\gamma]) \\ &= \nu_*(dd^c[\log ||s||^{-2}] \wedge \alpha + dd^c[\gamma]) \\ &= \nu_*(([\beta - \delta_E] \wedge \alpha + [\nu^*\omega] - [\alpha \wedge \beta])) \\ &= -\delta_Y + \nu_*[\nu^*\omega]. \end{aligned}$$

Since ν is birational, we conclude that

$$dd^c g + \delta_Y = [\omega].$$

$\square$

Assumptions in Key Lemma I

If X is a compact Kähler manifold, then we know that the Green current for $Y \subset X$ exists. Hence on choosing $\omega := \omega_Y$ (closed smooth form attached to g_Y), we get

$$dd^c g_Y + \delta_Y = [\omega], \quad \text{or} \quad \mathrm{cl}(\delta_Y) = \mathrm{cl}(\omega).$$

This shows that $\pi^*\omega$ represents an d-exact current. On considering $d(\alpha \wedge \beta)$, we see that $[\nu^*\omega] - [\alpha \wedge \beta]$ represents a d-exact current. Knowing that any blow-up of a compact Kähler manifold is still compact Kähler, we may use dd^c-Lemma on $\tilde{X}$ to find a smooth form γ satisfying the equation in the moreover part of Key Lemma I.

It remains to study the assumption on α. We keep assuming that X is compact Kähler. Let α be a closed $(p-1,p-1)$-form on $\tilde{X}$. By Hodge theory, the pullback j^* of cohomology classes preserves the grading, hence the condition amounts to

$$c_{p-1}(Q) \in j^* H^{2p-2}(\tilde{X}; \mathbb{C}).$$

By the characterising properties of Chern classes, we have that

$$c_{p-1}(Q) = \sum_{i=0}^{p-1} \nu_Y^* c_{p-1-i}(N) \cup c_1(\mathcal{O}_N(1))^i \quad \text{in } H^{2p-2}(E; \mathbb{C}).$$

Thus we are led to ask when does the total Chern class $c(N)$ belong to the image of

$$i^* : H^\bullet(X; \mathbb{C}) \to H^\bullet(Y; \mathbb{C}). \tag{7}$$

$$\begin{tikzcd} E \arrow[r, hook, "j"] \arrow[d, "\nu_Y"] & \tilde{X} \arrow[d, "\nu"] \\ Y \arrow[r, hook, "i"] & X \end{tikzcd}$$

Consider the special case where $X = M \times M$ for some complex projective manifold M, and $Y = \Delta \subset X$ the diagonal. Then both X and Y are compact Kähler, and the map (7) by Künneth's formula.
By the discussion so far, we obtain a Green form g_Δ on $M \times M$ along Δ.

Let $\mathrm{pr}_1 : M \times Z \to M$ be the induced projection.

Theorem (Existence of Green form)

Let Z be any analytic cycle (of fixed codim.) on M. Then the current

$$g := \mathrm{pr}_{1,*}(g_\Delta \wedge \delta_{M\times Z})$$

is a smooth current associated to a Green form.

The product $g_\Delta \wedge \delta_{M\times Z}$ is well-defined because the subvariety $M \times |Z| \subset M \times M$ and the diagonal $\Delta \subset M \times M$ meet properly.

Proof.

We may assume that Z is a prime cycle. Let $\nu : \tilde{Z} \to Z$ be a resolution of singularities. Let

$$\mu := \mathrm{id}_M \times \nu : M \times \tilde{Z} \to M \times Z$$

and $p := (\mathrm{pr}_1 \circ \mu) : M \times \tilde{Z} \to M$ be the projection. By definition

$$g_\Delta \wedge \delta_{M \times Z} = \mu_*[\mu^*(g_\Delta|_{M \times Z})],$$

thus we can rewrite

$$g = p_*[\mu^*(g_\Delta|_{M \times Z})].$$

The last display implies that g is a current on M associated to an L^1-form, which is smooth on $M \setminus Z$.

Proof.

(cont.) By the lemma on Green current of inverse image, the form is a Green form of log type along Z. Let

$$[\omega_\Delta] := dd^c[g_\Delta] + \delta_\Delta$$

(ω_Δ is smooth). By the moreover part of the lemma loc.cit., we get

$$dd^c[\mu^*(g_\Delta|_{M\times Z})] = \mu^*(\omega_\Delta|_{M\times Z}) - \delta_{\mu^*(\omega_\Delta|_{M\times Z})}. \tag{8}$$

On the other hand, observe that we have an equality of cycles

$$p_*\mu^*\Delta = Z,$$

which gives rise to an equality of currents

$$p_*\delta_{\mu^*\Delta} = \delta_Z. \tag{9}$$

Proof.

(cont.) Combining (8) and (9) and using the properness of p, we deduce that

$$dd^c g + \delta_Z = p_*[\mu^*(\omega_\Delta|_{M\times Z})] = [p_*\mu^*(\omega_\Delta|_{M\times Z})],$$

as wanted. □

We aim to show the commutativity and associativity of the *-product. Along the way we will talk about the well-definedness problem.

Setup

Consider the following situation. Let X be a complex projective manifold. Let Y, Z, W be closed subvarieties of X of codimension p, q, r with $p > 0$ and $q > 0$. Assume that $Y \cap Z$, $Y \cap W$, $Z \cap W$ and $Y \cap Z \cap W$ respectively intersects properly.

The technical ground for the subsequent discussion is the following

Lemma (Key Lemma II)

Let g_Y, g_Z be Green forms along Y, Z respectively. Then we have an equality

$$g_Y \wedge \delta_{Z.W} + \omega_Y \wedge (g_Z \wedge \delta_W) = g_Z \wedge \delta_{Y.W} + \omega_Z \wedge (g_Y \wedge \delta_W) \quad \text{in } \widetilde{\mathscr{D}}^{n,n}(X),$$

where $n := p + q + r - 1$.

Proof.

See [GS90, 2.2.2 ff.]. Note that the irreducibility assumption in loc. cit. is not necessarily. $\square$

Corollary (Commutativity)

We use the previous setup. Then

$$g_Y * g_Z = g_Z * g_Y \quad \text{in } \widetilde{\mathscr{D}}^{p+q-1,p+q-1}(X).$$

Proof.

Take $W := X$. □

Corollary (Associativity)

Let X be a complex projective manifold. Let Z_i be codimension p_i cycles on X, $i = 1, 2, 3$. Let g_i be Green currents of them respectively. Assume that the cycles intersect properly as in the previous setup. Then

$$g_1 * (g_2 * g_3) = (g_1 * g_2) * g_3 \quad \textit{in } \widetilde{\mathscr{D}}(X).$$

Proof.

Let ω_i be some smooth forms attached to the Green currents g_i $(i = 1, 2, 3)$.
Using the commutativity, we compute on the one hand, (for simplicity we suppress the wedge in $g \wedge \delta$)

$$g_1 * (g_2 * g_3) = g_1 * (g_3 * g_2) = g_1\delta_{Z_3.Z_2} + \omega_1 \wedge (g_3 * g_2) = \\ = g_1\delta_{Z_3.Z_2} + \omega_1 \wedge g_3\delta_{Z_2} + (\omega_1 \wedge \omega_3) \wedge g_2.$$

On the other hand,

$$(g_1 * g_2) * g_3 = g_3 * (g_1 * g_2) = g_3\delta_{Z_1.Z_2} + \omega_3 \wedge g_1\delta_{Z_2} + (\omega_3 \wedge \omega_1) \wedge g_2.$$

We then apply Key Lemma II to see that the right most terms of the last two displays are the same (in $\widetilde{\mathscr{D}}(X)$). □

Dependence of *-product on Green forms

Corollary (of Key Lemma II)

*Let X be a complex projective manifold, Y, Z be cycles on X which intersect properly. Then the *-product in $\tilde{\mathscr{D}}^p(X)$ (for all p in question) is independent of the choices of the Green forms along Y, Z.*

Proof.

By commutativity it suffices to show, say, $g_Y * g_Z = g_Y * g_Z'$ in $\tilde{\mathscr{D}}(X)$ (all Green forms), where $g_Z - g_Z' = \partial u + \overline{\partial} v$ for some currents u, v. Using the closeness of ω_Y, we easily compute

$$g_Y * g_Z - g_Y * g_Z' = \omega_Y \wedge (g_Z - g_Z') = \partial(\omega_Y \wedge u) + \overline{\partial}(\omega_Y \wedge v).$$

$\square$

Lemma

Let $f : X \to X'$ be a holomorphic map between complex projective manifold, let Z_1, Z_2 be cycles on X of codim. p_1, p_2 respectively. Suppose Z_1, Z_2 intersect properly. Assume moreover $f^{-1}(|Z_1|), f^{-1}(|Z_2|)$ and $f^{-1}(|Z_1| \cap |Z_2|)$ have codim. p_1, p_2 and $p_1 + p_2$. Then we have equalities in $\widetilde{\mathscr{D}}(X)$:

$$f^*(g_1 * g_2) = f^* g_1 * f^* g_2$$

and

$$f^*(g_1 \wedge \delta_{Z_2}) = f^* g_1 \wedge \delta_{f^* Z_2}.$$

Proof.

For the first see [GS90, Lemma in 4.4.3]. The second follows from the first. □

Projection formula

There is also a projection formula for the *-product (holds in $\widetilde{\mathscr{D}}(X)$), see the proof of [GS90, Theorem 4.4.3(7)].

Outlook

There are ongoing works on "derived Green currents", in which the commutativity and associative, say, of the (derived) *-product holds naturally "up to homotopy".

More on this see later talks of the Seminar.

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Thank you!