Deformation to the normal bundle

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Overview

Cotangent complex

Definition

Let $f : X \to Y$ be a morphism of stacks. Then f *admits a cotangent complex* if there is an almost connective $L_{X/Y} \in \mathrm{QCoh}(X)$ such that, for each $\eta : \mathrm{Spec}\, C \to X$ and each connective C-module M, we have an equivalence of spaces

$$\mathcal{M}\mathrm{od}_C(\eta^* L_{X/Y}, M) \simeq \mathrm{Fib}(X(C \oplus M) \to X(C) \times_{Y(C)} Y(C \oplus M))$$

If such $L_{X/Y}$ exists, we define the *normal sheaf* as $N_{X/Y} := L_{X/Y}[-1]$.

Relation to deformation theory

Remark

Let $A \to B$ be a surjection in $\mathcal{A}\mathrm{lg}$.

- For $M \in (\mathcal{M}\mathrm{od}_B)_{\geq 0}$, have $\mathcal{M}\mathrm{od}_B(N_{B/A}, M) \simeq \mathcal{A}\mathrm{lg}_{A/B}(B, B \oplus M[1])$. These are *small extension of B by M*.
- If $A \to B$ is discrete, then a small extension $A \to \widetilde{B} \to B$ by a discrete B-module I is exactly a square-zero extension of $A \to B$ with ideal I.
- Upshot: $N_{Z/X} = L_{Z/X}[-1]$ classifies small / square-zero extension.

Relation to deformation theory

Example

Let $X = \operatorname{Spec} A$ be discrete, and let $x : \operatorname{Spec} k \to X$ be a point, with ideal $I = \mathfrak{m}_x$.

- Sections of $\mathbb{V}(N_{X/k}) \to \operatorname{Spec} k$ are classified by $\mathcal{M}\mathrm{od}_k(N_{X/k}, k) \simeq \mathcal{M}\mathrm{od}_k(\pi_0(N_{X/k}), k) \simeq \pi_0 N_{X/k}^\vee$.
- We have $\mathcal{M}\mathrm{od}_k(N_{X/k}, k) \simeq \mathcal{A}\mathrm{lg}_{A/k}(, k, k \oplus k[1])$ is also classified by square-zero extensions of k over A. Since k is field, these are extensions of the form $A \to k[\epsilon] \to k$.
- Hence $\pi_0 N_{X/k}^\vee$ is the Zariski tangent space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee$.

Cotangent complexes for algebraic stacks

Proposition

Let $f : X \to Y$ be n-algebraic. Then $L_{X/Y} \in \mathrm{QCoh}(X)$ exists.

Using the universal property of the cotangent complex, one can reduce to Y is affine, say $Y = \operatorname{Spec} A$.

Now X is n-algebraic, hence we can take an $(n-1)$-smooth epimorphism $g : U \to X$, where U is a scheme. Let $\eta : \operatorname{Spec} A \to X$ be given. We want to construct $\eta^* L_{X/Y}$ with the desired universal property.

Since $\mathrm{QCoh}(X) = \lim_{\operatorname{Spec} A \to X} \mathcal{M}\mathrm{od}_A$, where the limit is taken over all smooth maps $\operatorname{Spec} A \to X$, we can assume that there is a factorization of η through some $f : \operatorname{Spec} A \to U$.

Then we let $\eta^* L_{X/Y}$ be the fiber

$$\eta^* L_{X/Y} \to f^* L_{U/Y} \to f^* L_{U/X}$$

Normal cones and normal bundles

Let $f : Z \to X$ be a closed immersion of classical schemes, with ideal I.

- The normal cone is $\mathrm{Spec}_Z(\bigoplus_n I^n/I^{n+1})$.
- The normal bundle is $\mathbb{V}_Z(I/I^2) = \mathrm{Spec}_Z(\mathrm{Sym}\, I/I^2)$.
- We always have a closed immersion from the normal cone into the normal bundle, which is an isomorphism if f is regular.
- It holds $I/I^2 \simeq \pi_1(L_{Z/X}) \simeq \pi_0(N_{Z/X})$.

Normal cones and normal bundles

Definition

Let X be a stack, and $M \in \mathrm{Q}\mathcal{C}\mathrm{oh}(X)$. Write $\mathbb{V}_X(M) = \mathbb{V}(M)$ for the stack over X, defined on points $f : T \to X$ as

$$\mathbb{V}(M)(T) = \mathrm{Map}(f^* M, \mathcal{O}_T)$$

Definition

For $f : X \to Y$ a morphism between algebraic stacks, the *normal bundle* is defined as the stack $\mathbb{V}_X(N_{X/Y})$ over X.

Weil restriction

Let $f : X \to Y$ be an affine morphism of stacks. Then the pullback functor

$$f^* : \mathcal{S}\mathrm{t}_Y \to \mathcal{S}\mathrm{t}_X$$

has a right adjoint, written Res_f.

Definition

For $Z \to X$, we call $\mathrm{Res}_f(Z)$ the *Weil restriction* of Z along f.

The deformation space

Let $X \to Y$ be a morphism of stacks.

Definition

The *deformation space* $\mathcal{D}_{X/Y}$ of f is the Weil restriction of

$$X \times B\mathbb{G}_m \to Y \times B\mathbb{G}_m$$

along the zero section $Y \times B\mathbb{G}_m \to Y \times [\mathbb{A}^1/\mathbb{G}_m]$.

Virtual Cartier divisors

Definition

A *virtual Cartier divisor over* $X \to Y$ is a commutative diagram

$$\begin{tikzcd} D \arrow[r] \arrow[d] & T \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

in $\mathcal{S}\mathrm{t}$, such that $D \to T$ is a virtual Cartier divisor.

Lemma

The map $B\mathbb{G}_m \to [\mathbb{A}^1/\mathbb{G}_m]$ *classifies virtual Cartier divisors.*

Virtual Cartier divisors

Suppose $f : X \to Y$ has a cotangent complex. Put $\mathcal{N}_{X/Y} \coloneqq [N_{X/Y}/\mathbb{G}_m]$.

Fundamental Lemma

For any T over Y, we have

$$\mathcal{S}\mathrm{t}_Y(T, N_{X/Y}) \simeq \mathcal{S}\mathrm{t}_Y(\mathbb{V}_T(\mathcal{O}_T[1]), X)$$

Virtual Cartier divisors

Corollary

We have a Cartesian diagram

$$\begin{tikzcd} N_{X/Y} \arrow[r] \arrow[d] & D_{X/Y} \arrow[d] \\ Y \arrow[r] & Y \times \mathbb{A}^1 \end{tikzcd}$$

Virtual Cartier divisors

Remark

We also have a $\mathbb{G}_m$-equivariant version of the fundamental lemma. This gives us a Cartesian square

$$\begin{tikzcd}
\mathcal{N}_{X/Y} \arrow[r] \arrow[d] & \mathcal{D}_{X/Y} \arrow[d] \\
Y \times B\mathbb{G}_m \arrow[r] & Y \times [\mathbb{A}^1/\mathbb{G}_m]
\end{tikzcd}$$

$\mathcal{N}_{X/Y} \to \mathcal{D}_{X/Y}$ is then the universal virtual Cartier divisor over $X \to Y$.

In terms of blow-ups

Proposition

Let $Z \to X$ be a closed immersion. Then

$$D_{Z/X} \simeq \mathrm{Bl}_{Z\times\{0\}}(X \times \mathbb{A}^1) \setminus \mathrm{Bl}_{Z\times\{0\}}(X \times \{0\})$$

Naturality of $D_{(-)/(-)}$

Proposition

The functor

$$\begin{aligned} \mathcal{S}\mathrm{t}_{[\mathbb{A}^1/\mathbb{G}_m]} &\to \mathrm{Ar}(\mathcal{S}\mathrm{t}) \\ T &\mapsto (T \times_{[\mathbb{A}^1/\mathbb{G}_m]} B\mathbb{G}_m \to T) \end{aligned}$$

has a right adjoint, which sends $X \to Y$ to $\mathcal{D}_{X/Y}$.

The deformation diagram

Let $f : X \to Y$ be a morphism of algebraic stacks. Then we have a diagram of Cartesian squares

$$\begin{tikzcd}
X \arrow[r] \arrow[d] & X \times \mathbb{A}^1 \arrow[d] & X \times \mathbb{G}_m \arrow[l] \arrow[d] \\
\mathbb{V}(N_{X/Y}) \arrow[r] \arrow[d] & D_{X/Y} \arrow[d] & Y \times \mathbb{G}_m \arrow[l] \arrow[d] \\
Y \arrow[r] & Y \times \mathbb{A}^1 & Y \times \mathbb{G}_m \arrow[l]
\end{tikzcd}$$

Quasi-smooth morphisms

Definition

Let $f : X \to Y$ be a morphism of algebraic stack.

- Recall that $A \to B$ in $\mathcal{A}\mathrm{lg}$ is *locally of finite presentation* if $\mathcal{A}\mathrm{lg}_A(B, -)$ commutes with filtered colimit.
- Now f is *locally of finite presentation* if for all $\operatorname{Spec} B \to \operatorname{Spec} A$, smooth over f, $A \to B$ is locally of finite presentation.
- A module $M \in \mathrm{Q}\mathcal{C}\mathrm{oh}(X)$ is *of Tor-amplitude* $[n, m]$ if for all discrete $E \in \mathrm{Q}\mathcal{C}\mathrm{oh}(M)$ it holds that $\pi_i(M \otimes E) = 0$ for i outside $[n, m]$
- Now f is *quasi-smooth* if it is locally of finite presentation and $L_{X/Y}$ is of Tor-amplitude $[-\infty, 1]$.

Quasi-smooth morphisms

We say $A \to B$ is *finitely presented* if B can be obtained from A by a finite number of cell attachments. Now the following are equivalent:

- $A \to B$ is locally of finite presentation
- B is a retract of a finitely presented A-algebra
- $\pi_0 A \to \pi_0 B$ is finitely presented, and $L_{B/A}$ is perfect (=compact).

Example

The map $k[\epsilon] \to k$ is locally of finite presentation but not finitely presented.

Example

A closed immersion $Z \to X$ of derived schemes is quasi-smooth if and only if, Zariski locally on X, it is of the form $\operatorname{Spec} A/(f_1, \ldots, f_n) \to \operatorname{Spec} A$.

Deformation space in the quasi-smooth case

Proposition

Suppose $f : X \to Y$ is quasi-smooth.

- *The structure map $D_{X/Y} \to Y \times \mathbb{A}^1$ is quasi-smooth*
- *The map $X \times \mathbb{A}^1 \to D_{X/Y}$ is quasi-smooth.*
- *Hence all maps in the deformation diagram are quasi-smooth.*

Rees algebras

Recall that we defined the extended Rees algebra of a closed immersion $Z \to X$ as the $\mathbb{Z}$-graded $\mathcal{O}_X[t^{-1}]$-algebra $R^{\text{ext}}_{Z/X}$ such that

$$D_{Z/X} = \operatorname{Spec} R^{\text{ext}}_{Z/X}$$

Lemma

Suppose we have a commutative diagram

$$\begin{tikzcd} Z \arrow[r] \arrow[d] & X \arrow[d] \\ X' \arrow[r] \arrow[ur] & V \end{tikzcd}$$

in $\mathcal{S}\mathrm{t}$. Then naturality of $\mathcal{D}_{(-)/(-)}$ gives

$$D_{Z/X} \simeq D_{X'/X} \times_{D_{X'/V}} D_{Z/V}$$

Example 1

Let $Z \to X$ correspond to $A \to B = A/(f_1, \dots, f_n)$. Then

$$R^{\text{ext}}_{B/A} = \frac{A[t^{-1}, v_1, \dots, v_n]}{(v_1 t^{-1} - f_1, \dots, v_n t^{-1} - f_n)} \qquad N_{B/A} = B^{\oplus n}$$

Example 2

Let $Z \to X$ correspond to $\operatorname{Spec} k \to \operatorname{Spec} k[\epsilon]$. Then

$$R^{\mathrm{ext}}_{Z/X} \simeq \pi_0\left(\frac{k[\epsilon, t^{-1}, v]}{(vt^{-1} - \epsilon, \epsilon v)}\right) \qquad \mathbb{V}(N_{Z/X}) \simeq \mathbb{V}(k \oplus k[1])$$

Example 3

Let $Z \to X$ correspond to $\operatorname{Spec} k \to \operatorname{Spec} k[x, y]/(xy^2, yx^2)$. Put $I = (x, y) \subset k[x, y]$. Then

$$R^{\mathrm{ext}}_{Z/X} \simeq \frac{k[x, y][It, t^{-1}]}{(xy^2 t, yx^2 t)} \qquad \mathbb{V}(N_{Z/X}) \simeq \mathbb{V}(k/(0, 0)[u, v])$$

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Thank you!