What could go wrong w/ gluing reductions?

Assume k alg. closed and fix $\alpha, \beta \in k$ w/ $0 < |\alpha| < |\beta| < 1$. $D := \{ z \in k : |z| \le 1 \}$ w/ admissible affinoid cover $X_1 \cup X_2$

$X_1 := \{ z \in D : |z| \le |\beta| \}$, $X_2 := \{ z \in D : |\alpha| \le |z| \le 1 \}$ (genuine overlap) $X_1 \cap X_2 = \{ z \in D : |\alpha| \le |z| \le |\beta| \}$.

X_1 closed disk $\Rightarrow \overline{X}_1 = \mathbb{A}^1_{\bar{k}}$; $X_2, X_1 \cap X_2$ annuli $\Rightarrow$ reductions are $\mathbb{A}^1_{\bar{k}} \cup \mathbb{A}^1_{\bar{k}}$ (intersecting axes) ["interiors" get collapsed to a pt.]

<u>Exercise</u>: Supposed gluing data for $X_1, X_2, X_1 \cap X_2$ does not reduce to gluing data. $\overline{X_1 \cap X_2} \to \overline{X}_1, \overline{X}_2$ not open immersions.

Another weird thing: take $k = \mathbb{Q}_p$ w/ $|p| = 1/p$, $X := \{ z \in \hat{k}^{alg} : |z| \le |p|^{1/2} \}$. Rational subdomain w/ $\left|\frac{z^2}{p}\right| \le 1$.

$X = Sp(A)$ for $A = k\langle z, T \rangle / (T - z^2/p) = \{ f = \sum_i a_i z^i \in k[[z]] : \lim_{i \to \infty} |a_i| |p|^{i/2} = 0 \}$.

$A^\circ = \{ f : |a_i| |p|^{i/2} \le 1 \}$, $A^{\circ\circ} = \{ f : |a_i| |p|^{i/2} < 1 \}$.

$cz^{2i+1} \in A^\circ \Rightarrow |c| |p|^{(2i+1)/2} \le 1 \Leftrightarrow |cp^i| \le |p|^{-1/2} \Leftrightarrow |cp^i| < |p|^{-1/2} \Leftrightarrow cz^{2i+1} \in A^{\circ\circ}$.

So, $A^\circ \to \overline{A} = \bar{k}[z^2/p]$ kills odd powers of z. $\overline{X} = \mathbb{A}^1$ w/ red : $X \to \overline{X}$, $z \mapsto \overline{\left(\frac{z^2}{p}\right)} \in \bar{k}$. Funny business occurs because

$|\cdot|_{sp}$ on A takes values not in $|k|$.

<u>Prop</u>: Suppose $X = Sp(A)$ w/ A reduced and spectral seminorm on X takes values in $|k|$. Then, for every open affine subscheme $\mathcal{U} \subseteq \overline{X}$, $red^{-1}(\mathcal{U}) \subseteq X$ is affinoid reduced w/ spectral seminorm on $red^{-1}(\mathcal{U})$ taking values in $|k|$. Moreover,

$\overline{red^{-1}(\mathcal{U})} = \mathcal{U}$.

<u>Def</u>: X reduced rigid space, $\mathcal{U} = \{X_i\}_{i \in I}$ adm. cov. $\mathcal{U}$ is <u>pure affinoid cover</u> if

(1) Each X_i is affinoid.

(2) Spectral seminorm on X_i takes values in $|k|$.

(3) $\forall i \in I$ $\exists$ only finitely many $j \in I$ s.t. $X_i \cap X_j \neq \emptyset$.

(4) $X_i \cap X_j \neq \emptyset \Rightarrow$ nat. map $\mathcal{O}(X_i)^\circ \hat{\otimes}_{k^\circ} \mathcal{O}(X_j)^\circ \to \mathcal{O}(X_i \cap X_j)^\circ$ is surj.

(5) $X_i \cap X_j \neq \emptyset \Rightarrow \exists$ open affine $\mathcal{U}_{ij} \subseteq \overline{X}_i$ s.t. $red_i : X_i \to \overline{X}_i$ satisfies $X_i \cap X_j = red_i^{-1}(\mathcal{U}_{ij})$.

<u>Remark</u>: This is exactly what we need to glue reductions.

$\rightsquigarrow$ $\bar{k}$-scheme $\overline{(X, \mathcal{U})} = \cup \overline{X}_i$.

$\overline{(X, \mathcal{U})} \xleftarrow{red} X$ surj. on closed pts.

<u>Example</u>: Fix $\pi \in k$ w/ $0 < |\pi| < 1$. Let $X = \mathrm{Sp}\, k\langle z\rangle$. X_1 given by $|z| \le |\pi|$; X_2 given by $|\pi| \le |z| \le 1$.

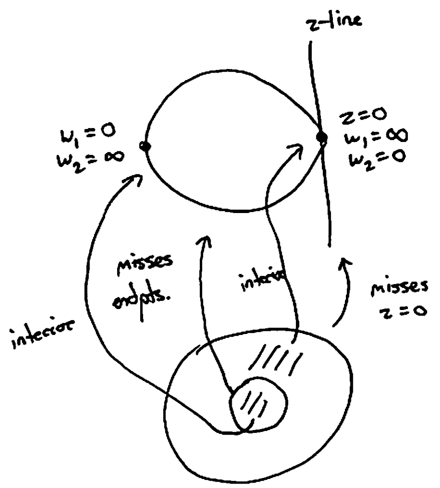
It's a fact that $\mathfrak{U} = \{X_1, X_2\}$ is pure affinoid.

$X_1 = \mathrm{Sp}(A_1)$ w/ $A_1 = k\langle z, z/\pi\rangle = k\langle z, w\rangle/(w - z/\pi)$. $\bar{A}_1 = \bar{k}[w_1] \Rightarrow \bar{X}_1 = \mathbb{A}^1_{w_1}$.

$X_2 = \mathrm{Sp}(A_2)$ w/ $A_2 = k\langle z, \pi/z\rangle = k\langle z, w_2\rangle/(zw_2 - \pi)$. $\bar{A}_2 = \bar{k}[z, w_2]/(zw_2) \Rightarrow \bar{X}_2 = \mathbb{A}^1_z \cup \mathbb{A}^1_{w_2}$.

$\Rightarrow X_1 \cap X_2 = \mathrm{Sp}\, k\langle w_1, w_2\rangle/(w_1w_2 - 1) \Rightarrow \overline{X_1 \cap X_2} \cong \mathbb{G}_{m,\bar{k}}$.

$\Rightarrow \overline{(X, \mathfrak{U})} = \mathbb{A}^1_z \cup \mathbb{P}^1_{w_1 = w_2^{-1}}$ joined at point $z=0$ in $\mathbb{A}^1$ and $w_1 = \infty, w_2 = 0$ in $\mathbb{P}^1$.



This depends on the choice of π!

<u>Ex</u>: $\pi \in k$ pseudo-uniformizer. Compute the reduction map for $X = \mathbb{P}^1$ and pure affinoid cover
$X_1 = \{|z| \le |\pi|\}$
$X_2 = \{|\pi| \le |z| \le 1\}$
$X_3 = \{|z| \ge 1\}$

<u>Ex</u>: Suppose valuation on k is discrete – e.g., $k = \mathbb{Q}_p$. $\mathcal{X}$ flat proj. scheme / k°. $X = \mathcal{X} \times_{\mathrm{Spec}\, k^\circ} \mathrm{Spec}\, k$ generic fiber and $X_s = \mathcal{X} \times_{\mathrm{Spec}\, k^\circ} \mathrm{Spec}\, \bar{k}$ special fiber. Naïve reduction $X^{an} = \{\text{closed pts. of } X\} \xrightarrow{\text{red}} X_s$. $x \in X$ closed pt. $\rightsquigarrow \mathrm{Spec}\, k_x \to X$. By valuative criterion of properness this extends to $\mathrm{Spec}\, k_x^\circ \to \mathcal{X}$. Reduce this to $\mathrm{Spec}\, \bar{k}_x \to X_s$.

This defines $\mathrm{red}: X^{an} \to X_s$ as desired.

<u>Claim</u>: This comes from pure affinoid cover $\mathfrak{U}$ of $\mathcal{X}$.

$$\begin{tikzcd} & X^{an} \arrow[dl, "\text{red}"'] \arrow[dr, "\text{red}"] \arrow[d, phantom, "\circlearrowleft"] & \\ \overline{(X,\mathfrak{U})} & \cong & X_s \end{tikzcd}$$

Fix $X \to \mathbb{P}^n_{k^\circ}$ w/ homogeneous coords. $z_0, \ldots, z_n$. Let $X_i \subseteq X$ given by $z_i \neq 0$ (so work in $\mathbb{P}^n_{k^\circ}$ and intersect w/ X).

$X_i \subseteq X$ generic fiber. Each X_i has coord. functions $\frac{z_0}{z_i}, \ldots, \frac{z_i}{z_i}, \ldots, \frac{z_n}{z_i}$. Define open affinoid $\mathcal{U}_i \subseteq X_i^{an}$ by

$\left|\frac{z_0}{z_i}\right|, \ldots, \left|\frac{z_n}{z_i}\right| \leq 1$. $X^{an} = \mathcal{U}_0 \cup \cdots \cup \mathcal{U}_n$. $\mathfrak{U} := \{\mathcal{U}_0, \ldots, \mathcal{U}_n\}$ is the desired pure affinoid cover.

<u>Remark</u>: The projectivity really does matter here.

<u>Separated and Proper Morphisms</u>

<u>Def</u>: $f: Z \to X$ map of rigid spaces is

- <u>closed immersion</u> if $\exists$ closed analytic subset $Y \subseteq X$ (defined by ideal sheaf) and factorization $Z \xrightarrow{\sim} Y \hookrightarrow X$, with composite f;
- <u>open immersion</u> if $\exists$ adm. open $\mathcal{U} \subseteq X$ s.t. $Z \xrightarrow{\sim} \mathcal{U} \hookrightarrow X$, with composite f;
- <u>locally closed immersion</u> if ... (usual definition – same for <u>separatedness</u>)

<u>Def</u>: Let $X = Sp(A) \supseteq Y = Sp(B)$ affinoid domains. Y <u>lies in the interior of</u> X ($Y \subset\subset X$) if $\exists\, k\langle z_1, \ldots, z_n\rangle \twoheadrightarrow A$ s.t.

$\exists \rho < 1$ w/ $Y \subseteq \{x \in X : |z_i(x)| \leq \rho \;\forall i\}$. So, there is closed immersion of X into closed polydisk w/ Y

contained in strictly smaller polydisk. Without coords., red : $X \to \bar{X}$ collapses Y to a pt.

<u>Def</u>: Rigid space X is <u>proper</u> if $\exists$ finite admissible affinoid cover $X = X_1 \cup \cdots \cup X_n$ and another finite admissible affinoid cover

$X = X_1' \cup \cdots \cup X_n'$ s.t. $X_i' \subset\subset X_i$.

For the relative notion, let $\underset{Sp(A)}{X} \to \underset{Sp(B)}{Y}$ be map of rigid spaces and $X' \subseteq X$ affinoid subdomain. $X' \subset\subset_Y X$ if $\exists \rho < 1$ and

$$\begin{tikzcd} B\langle z_1, \ldots, z_n\rangle \arrow[dr, two heads] & \\ B \arrow[u] \arrow[r] & A \end{tikzcd}$$

s.t. $X' \subseteq \{x \in X : |z_i(x)| < \rho \;\forall i\}$.

Def: $f: X \to Y$ map of rigid spaces is proper if $\exists$ adm. affinoid cov. $\{Y_i\}$ of Y s.t. every $f^{-1}(Y_i)$ admits finite adm. affinoid cov.'s $\{X_{ij}\}_j$ and $\{X'_{ij}\}_j$ w/ $X'_{i,j} \subset\subset_{Y_i} X_{i,j}$ $\forall i,j$.

Example: Closed immersion is proper.

Remark: X proper $\Rightarrow$ any $X \to Y$ is proper.

Cor: Image of proper morphism is closed analytic subset.

Pf: $f: X \to Y$ proper $\Rightarrow$ image defined by (coherent!) ideal sheaf $I := \ker(\mathcal{O}_Y \to f_* \mathcal{O}_X)$. $\square$

Thm: (1) X proper, $\mathcal{F}$ coherent sheaf $/X$ $\Rightarrow$ $\dim_k H^i(X, \mathcal{F}) < \infty$ $\forall i$.

(2) We have pushforward and higher direct images for proper morphisms.

Thm (Rigid GAGA): $\exists$ equiv. of cat's $\{\text{coh. sheaves} / \mathbb{P}^n\} \xrightarrow[(\cdot)^{an}]{\sim} \{\text{coh. sheaves} / \mathbb{P}^{n,an}\}$ respecting cohomology.

This plays nice w/ coh. ideal sheaves.

Consequence: Showing that rigid space is analytification of variety, it suffices to embed into proj. space.

Def: X reduced proper separated rigid space. Coh. line bundle L is gen. by global sections if $\exists$ $\underbrace{f_0, \dots, f_n}_{\text{(sections)}} \in H^0(X, L)$ and adm. cov. $\mathfrak{U}$ of X by open affinoids s.t.

(1) $\forall\, \mathcal{U} = Sp(A) \in \mathfrak{U}$: $L|_{\mathcal{U}} \cong \mathcal{O}_{\mathcal{U}}$;

(2) this induces $H^0(\mathcal{U}, L|_{\mathcal{U}}) \cong A$ w/ $\overbrace{(f_0, \dots, f_n)}^{\text{(functions)}} = A$.

We get induced (well-defined) morphism $X \xrightarrow{\phi} \mathbb{P}^n$.

Prop: ϕ inj. and separates tangent vectors $\Rightarrow$ ϕ is closed imm.

Df: A analytic reduction of rigid space X is scheme Z over $\tilde{k}$ of FT w/ $\mathrm{red}: X \to Z$ s.t. $\exists$ pure affinoid cover $\mathfrak{U}$ of X and

$$\begin{tikzcd} & X \arrow[dl, "\mathrm{red}"'] \arrow[dr, "\mathrm{red}"] \arrow[d, phantom, "\circlearrowleft"] & \\ \overline{(X,\mathfrak{U})} \arrow[rr, "\cong"] & & Z \end{tikzcd}$$

Thm: k discretely valued field, X irred. smooth proj. curve $/k$. There is a bijection between

(1) analytic reductions of X^{an};

(2) flat proj. k°-schemes $\mathcal{X}$ w/ generic fiber X and reduced special fiber.

Pf: Start w/ flat proj. model $\mathcal{X}$ as in (2). Fix proj. emb. $\mathcal{X} \to \mathbb{P}^n_{k^{\circ}}$ w/ homogeneous coords. $z_0, \dots, z_n$.

Consider adm. affinoid cover $X^{an} = X_0^{an} \cup \dots \cup X_n^{an}$ w/ X_i^{an} defined as before. Now start w/ analytic reduction $Z = \overline{(X,\mathfrak{U})}$.

Pick L very ample line bundle on X, so that $X = \mathrm{Proj}\left(\bigoplus_{n \geq 0} H^0(X, L^{\otimes n})\right)$. [enough just to take something ample]

Basic idea: Choose L (depending on Z) s.t. rigid line bundle L^{an} on X^{an} has natural subsheaf of k°-submodules $L^{an,o} \subseteq L^{an}$. Define $\mathcal{X} := \mathrm{Proj}\left(\bigoplus_{n \geq 0} H^0(X, (L^{an,o})^{\otimes n})\right)$.

Write $Z = Z_1 \cup \dots \cup Z_s$ union of irred. components. Pick closed pt. q_i in smooth locus of Z_i $\forall i$. Pick lifts $p_i \in X^{an}$ s.t. $\mathrm{red}(p_i) = q_i$. Pick $\mathfrak{U}_i \in \mathfrak{U}$ containing p_i $\Rightarrow$ $\overline{\mathfrak{U}_i}$ open affine nbhd of q_i. This could contain bits of other irred. components. Pick Zariski open affine $Z_i' \subseteq \overline{\mathfrak{U}_i} \cap Z_i$ small enough s.t. $\exists\, g_i \in \mathcal{O}_Z(Z_i')$ w/ $\mathrm{div}(g_i) = q_i$. (g_i has single simple zero) Let $\mathfrak{U}_i' = \mathrm{red}^{-1}(Z_i')$ open affinoid in $\mathfrak{U}_i$ s.t. $\overline{\mathfrak{U}_i'} = Z_i'$. Pick any

$$f_i \in \mathcal{O}_{X^{an}}(\mathfrak{U}_i') \;\mapsto\; g_i \in \overline{\mathcal{O}_{X^{an}}(\mathfrak{U}_i')} = \mathcal{O}_Z(Z_i')$$

Now check f_i generates maximal ideal of $\mathcal{O}_X(\mathfrak{U}_i')$.

So, for each Z_i we have $q_i \in Z_i$ and lift to $p_i \in \mathcal{U}_i'$ w/ $\mathcal{U}_i' \subseteq \mathcal{U}_i$ affinoid. Refine $\mathfrak{U}$ to pure affinoid cov. s.t.
(actually want disjoint from $\{p_1, \dots, p_s\}$)
$\mathcal{U}_1', \dots, \mathcal{U}_s' \in \mathfrak{U}$ and all other $\mathcal{U} \in \mathfrak{U}$ have $\overline{\mathcal{U}}$ disjoint from $\{q_1, \dots, q_s\}$. Define analytic line bundle $L \subseteq \mathcal{M}_{X^{an}}$ by

$$L|_{\mathcal{U}_i'} = \frac{1}{f_i} \mathcal{O}_{\mathcal{U}_i'}, \quad L|_{\mathcal{U}} = \mathcal{O}_{\mathcal{U}} \text{ for all other } \mathcal{U} \in \mathfrak{U}.$$

Observation: L has global section w/ divisor $p_1 + \dots + p_s$. (just use constant function $1 \in \mathcal{M}_{X^{an}}(X^{an})$)

GAGA $\Rightarrow$ this is analytification of line bundle/X, namely $L = \mathcal{O}_X(D)^{an}$ of positive degree! Riemann-Roch

$\Rightarrow$ some power $\mathcal{O}_X(mD)$ very ample $\Rightarrow X = \operatorname{Proj} \bigoplus_{n \geq 0} A_n$ w/ $A_n := H^0(X, \mathcal{O}_X(nmD)) = H^0(X^{an}, L^{\otimes mn})$.

Define subsheaf $L^\circ \subseteq L$ of k°-modules by $L^\circ|_{\mathcal{U}_i'} := \frac{1}{f_i} \mathcal{O}^\circ_{\mathcal{U}_i'}$, $L^\circ|_{\mathcal{U}} := \mathcal{O}^\circ_{\mathcal{U}}$ otherwise. Define $\mathfrak{X}$ by Proj. $\square$

<u>Remark</u>: What precisely is the nature of the bijection here? Does this depend on a choice of projective embedding?