Deformation to the normal bundle

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Convention: Everything devived



End

Deformation theory

Deformation spaces

Deformation to the normal bundle

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Deformation theory

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Normal bundles

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Cotangent complex

16 QCCX) is almost connected

(A + spec A - x - 3 n st R(A)Cn) is

connective

Definition

Deformation theory

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Let $f: X \to Y$ be a morphism of stacks. Then f admits a cotangent complex if there is an almost connective $L_{X/Y} \in \mathrm{QCoh}(X)$ such that, for each $\eta: \mathrm{Spec}\ C \to X$ and each connective C-module M, we have an equivalence of spaces (Natural in N)

$$\operatorname{Mod}_{\mathcal{C}}(\eta^*L_{X/Y},M)\simeq\operatorname{Fib}(X(\mathcal{C}\oplus M)\to X(\mathcal{C})\times_{Y(\mathcal{C})}Y(\mathcal{C}\oplus M))$$

If such $L_{X/Y}$ exists, we define the *normal sheaf* as $N_{X/Y} := L_{X/Y}[-1]$.

Examples

Relation to deformation theory

Remark

Deformation theory

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Let $A \to B$ be a surjection in Alg.

- For $M \in (\operatorname{Mod}_B)_{\geq 0}$, have $\operatorname{Mod}_B(N_{B/A}, M) \simeq \operatorname{Alg}_{A/B}(B, B \oplus M[1])$. These are small extension of B by M.
- If $A \to B$ is discrete, then a small extension $A \to B \to B$ by a discrete B-module I is exactly a square-zero extension of $A \rightarrow B$ with ideal I.
- Upshot: $N_{Z/X} = L_{Z/X}[-1]$ classifies small / square-zero extension.



Relation to deformation theory

Example

Deformation theory

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Let $X = \operatorname{Spec} A$ be discrete, and let $x : \operatorname{Spec} k \to X$ be a point, with ideal $I = \mathfrak{m}_x$.

- Sections of $\mathbb{V}(N_{X/k}) \to \operatorname{Spec} k$ are classified by $\operatorname{Mod}_k(N_{X/k}, k) \simeq \operatorname{Mod}_k(\pi_0(N_{X/k}), k) \simeq \pi_0 N_{X/k}^{\vee}$.
- We have $\operatorname{Mod}_k(N_{X/k}, k) \simeq \operatorname{Alg}_{A/k}(, k, k \oplus k[1])$ is also classified by square-zero extensions of k over A. Since k is field, these are extensions of the form $A \to k[\epsilon] \to k$.
- Hence $\pi_0 N_{X/k}^{\vee}$ is the Zariski tangent space $(\mathfrak{m}_x/\mathfrak{m}_x^2)^{\vee}$.

SO NXM Controls def = Speck-ox, i.e. infinitesimal defourations.

Cotangent complexes for algebraic stacks

Proposition

Deformation theory

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Let $f: X \to Y$ be n-algebraic. Then $L_{X/Y} \in \mathrm{QCoh}(X)$ exists.

Using the universal property of the cotangent complex, one can reduce to Y is affine, say $Y = \operatorname{Spec} A$.

Now X is n-algebraic, hence we can take an (n-1)-smooth epimorphism $g:U\to X$, where U is a scheme. Let $\eta:\operatorname{Spec} A\to X$ be given. We want to construct $\eta^* L_{X/Y}$ with the desired universal property.

Since $QCoh(X) = \lim_{Spec A \to X} Mod_A$, where the limit is taken over all smooth maps Spec $A \to X$, we can assume that there is a factorization of

$$\eta$$
 through some $f: \operatorname{Spec} A \to U$. exists since exists Then we let $\eta^* L_{X/Y}$ be the fiber $U:_{Y}$ schemes $f:_{Y}$ be the fiber $U:_{Y}$ schemes $f:_{Y}$ for $f:_{Y}$

Examples

Deformation theory

Normal cones and normal bundles

Let $f: Z \to X$ be a closed immersion of classical schemes, with ideal I.

- The normal cone is $\operatorname{Spec}_{Z}(\bigoplus_{n} I^{n}/I^{n+1})$.
- The normal bundle is $\mathbb{V}_Z(I/I^2) = \operatorname{Spec}_Z(\operatorname{Sym} I/I^2)$.
- We always have a closed immersion from the normal cone into the normal bundle, which is an isomorphism if *f* is regular.
- It holds $I/I^2 \simeq \pi_1(L_{Z/X}) \simeq \pi_0(N_{Z/X})$.

But if Z-ox is questionth, Z, X discrete, then these two agree

Normal cones and normal bundles

Definition

Let X be a stack, and $M \in \mathrm{QCoh}(X)$. Write $\mathbb{V}_X(M) = \mathbb{V}(M)$ for the stack over X, defined on points $f: T \to X$ as $\text{pave } G_{n} \text{-extin}$ $\mathbb{V}(M)(T) = \mathrm{Map}(f^*M, \mathcal{O}_T)$

Definition

For $f: X \to Y$ a morphism between algebraic stacks, the *normal bundle* is defined as the stack $\mathbb{V}_X(N_{X/Y})$ over X.

Examples

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Weil restriction

Deformation theory

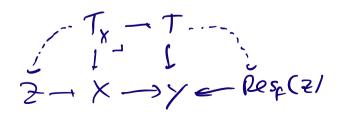
Let $f: X \to Y$ be an affine morphism of stacks. Then the pullback functor

$$f^*: \operatorname{St}_Y \to \operatorname{St}_X$$

has a right adjoint, written Res_f .

Definition

For $Z \to X$, we call $\operatorname{Res}_f(Z)$ the Weil restriction of Z along f.





The deformation space

Let $X \to Y$ be a morphism of stacks.

Definition

Deformation theory

The deformation space $\mathfrak{D}_{X/Y}$ of f is the Weil restriction of

$$X \times B\mathbb{G}_m \to Y \times B\mathbb{G}_m$$

along the zero section $Y \times B\mathbb{G}_m \to Y \times [\mathbb{A}^1/\mathbb{G}_m]$.



Examples

Definition

Deformation theory

A virtual Cartier divisor over $X \rightarrow Y$ is a commutative diagram

 $\begin{array}{ccc}
D & \longrightarrow & T \\
\downarrow & & \downarrow \\
X & \longrightarrow & Y
\end{array}$

DOT L'I

Examples

End

in St, such that $D \to T$ is a virtual Cartier divisor.

Lemma

The map $B\mathbb{G}_m \to [\mathbb{A}^1/\mathbb{G}_m]$ classifies virtual Cartier divisors.

Suppose $f: X \to Y$ has a cotangent complex. Put $\mathfrak{N}_{X/Y} := [N_{X/Y}/\mathbb{G}_m]$.

Fundamental Lemma

Deformation theory

or any T over Y, we have
$$\begin{cases} N_{X/Y} & \text{or } St_Y(T, N_{X/Y}) \simeq St_Y(\mathbb{V}_T(\mathcal{O}_T[1]), X) \cong \begin{pmatrix} \chi & \gamma \\ \chi & \gamma \end{pmatrix} \\ \text{lefty chech on fibers over Sty(T,X). So let fit-3X be given} \end{cases}$$

St_x(T, N_{X/Y}) = Map(f^xL_{x/Y}, O_TC(3) =
$$\int t \rightarrow x$$
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Corollary

Deformation theory

We have a Cartesian diagram D_{XY} is defined as: D_{XY} is also Weil restr: $N_{X/Y} \longrightarrow D_{X/Y} \longrightarrow D_{X/Y}$ $V_{X} = V_{X} = V_$

let + > y be giver. Then:

Remark

We also have a \mathbb{G}_m -equivariant version of the fundamental lemma. This gives us a Cartesian square

 $\mathfrak{N}_{X/Y} \to \mathfrak{D}_{X/Y}$ is then the universal virtual Cartier divisor over $X \to Y$.

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In terms of blow-ups

Proposition

Deformation theory

Let $Z \to X$ be a closed immersion. Then

$$D_{Z/X} \simeq \mathsf{Bl}_{Z imes \{0\}}(X imes \mathbb{A}^1) \setminus \mathsf{Bl}_{Z imes \{0\}}(X imes \{0\})$$

construction of
$$f$$
:

 $Tx \rightarrow T$

$$Tx \rightarrow T$$

$$Tx$$



Naturality of $D_{(-)/(-)}$

Proposition

Deformation theory

The functor

$$\operatorname{\mathsf{St}}_{[\mathbb{A}^1/\mathbb{G}_m]} o \operatorname{\mathsf{Ar}}(\operatorname{\mathsf{St}})$$

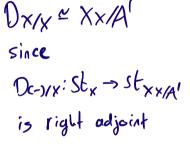
$$T \mapsto (T \times_{[\mathbb{A}^1/\mathbb{G}_m]} B\mathbb{G}_m \to T)$$

has a right adjoint, which sends $X \to Y$ to $\mathfrak{D}_{X/Y}$.

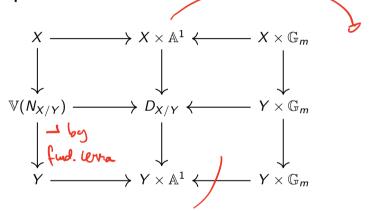
The deformation diagram

Let $f: X \to Y$ be a morphism of algebraic stacks. Then we have a

diagram of Cartesian squares



Deformation theory



P.B. For T-17×B4:



Quasi-smooth morphisms

Definition

Let $f: X \to Y$ be a morphism of algebraic stack.

- Recall that $A \to B$ in $\mathcal{A}\lg$ is locally of finite presentation if $\mathcal{A}\lg_A(B,-)$ commutes with filtered colimit.
- Now f is locally of finite presentation if for all Spec $B \to \operatorname{Spec} A$, smooth over f, $A \to B$ is locally of finite presentation.
- A module $M \in \mathrm{QCoh}(X)$ is of Tor-amplitude [n, m] if for all discrete $E \in \mathrm{QCoh}(M)$ it holds that $\pi_i(M \otimes E) = 0$ for i outside [n, m]
- Now f is *quasi-smooth* if it is locally of finite presentation and $L_{X/Y}$ is of Tor-amplitude $[-\infty, 1]$.



Examples

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Quasi-smooth morphisms

We say $A \to B$ is *finitely presented* if B can be obtained from A by a finite number of cell attachments. Now the following are equivalent:

- $A \rightarrow B$ is locally of finite presentation

 The personal preserves

 Fig. 1. Preserves
- B is a retract of a finitely presented A-algebra
- $\pi_0 A \to \pi_0 B$ is finitely presented, and $L_{B/A}$ is perfect (=compact).

Example

Deformation theory

The map $k[\epsilon] \to k$ is locally of finite presentation but not finitely represented. Can see this by cat. complex, which is K(t) where $K \to K(t)$ have $K \to K(t)$ is

Example

A closed immersion $Z \to X$ of derived schemes is quasi-smooth if and only if, Zarisksi locally on X, it is of the form $\operatorname{Spec} A/(f_1,\ldots,f_n) \to \operatorname{Spec} A$.

Deformation space in the quasi-smooth case

Proposition

Deformation theory

Suppose $f: X \to Y$ is quasi-smooth.

- $0 \bullet The structure map <math>D_{X/Y} \to Y \times \mathbb{A}^1$ is quasi-smooth
- ${\mathfrak O} \bullet {\it The map } X imes {\mathbb A}^1 o D_{X/Y} {\it is quasi-smooth.}$
 - Hence all maps in the deformation diagram are quasi-smooth.

Let's do
$$2 \iff \emptyset$$

(a) $| cal, so assure 2 \implies cour. to$
 $| A \rightarrow B = A/(f...f_n). Then$
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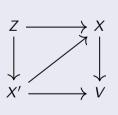
Rees algebras

Recall that we defined the extended Rees algebra of a closed immersion $Z \to X$ as the \mathbb{Z} -graded $\mathcal{O}_X[t^{-1}]$ -algebra $R_{Z/X}^{\mathrm{ext}}$ such that

$$D_{Z/X} = \operatorname{Spec} R_{Z/X}^{\operatorname{ext}}$$

Lemma

Suppose we have a commutative diagram



in St. Then naturality of $\mathfrak{D}_{(-)/(-)}$ gives

$$D_{Z/X} \simeq D_{X'/X} \times_{D_{X'/V}} D_{Z/V}$$

Examples

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End

Deformation theory

Example 1

Let $Z \to X$ correspond to $A \to B = A/(f_1, \ldots, f_n)$. Then

Examples

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Example 2

Deformation theory

Let $Z \to X$ correspond to Spec $k \to \operatorname{Spec} k[\epsilon]$. Then

$$R_{Z/X}^{\text{ext}} \simeq \pi_0 \left(\frac{k[\epsilon, t^{-1}, v]}{(vt^{-1} - \epsilon, \epsilon v)} \right) \qquad \mathbb{V}(N_{Z/X}) \simeq \mathbb{V}(k \oplus k[1])$$

$$P_{uf} = \sum_{j=1}^{N} \mathbb{V}_{j} \qquad \text{fines} \qquad D_{z/X} \simeq D_{x/X} \times D_{z/Y}$$

$$= \sum_{j=1}^{N} \mathbb{V}_{j} \qquad \text{fines} \qquad D_{z/X} \simeq D_{x/X} \times D_{z/Y}$$

$$S_{i} = \sum_{j=1}^{N} \mathbb{V}_{i} \qquad \text{fines} \qquad D_{z/X} \simeq \mathbb{V}(k \oplus k[1])$$

$$S_{i} = \sum_{j=1}^{N} \mathbb{V}_{i} \qquad D_{z/X} \qquad D_{z/X} \simeq \mathbb{V}(k \oplus k[1])$$

$$S_{i} = \sum_{j=1}^{N} \mathbb{V}_{i} \qquad \mathbb{V}_{$$

Example 3

Deformation theory

Let $Z \to X$ correspond to Spec $k \to \operatorname{Spec} k[x,y]/(xy^2,yx^2)$. Put $I = (x,y) \subset k[x,y]$. Then

$$R_{Z/X}^{\mathrm{ext}} \simeq rac{k[x,y][It,t^{-1}]}{(xy^2t,yx^2t)}$$

$$\mathbb{V}(N_{Z/X}) \simeq \mathbb{V}(k/(0,0)[u,v])$$

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References

Deformation theory



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