Section 7 Math 2202

Directional Derivatives, Gradients and Local Extrema

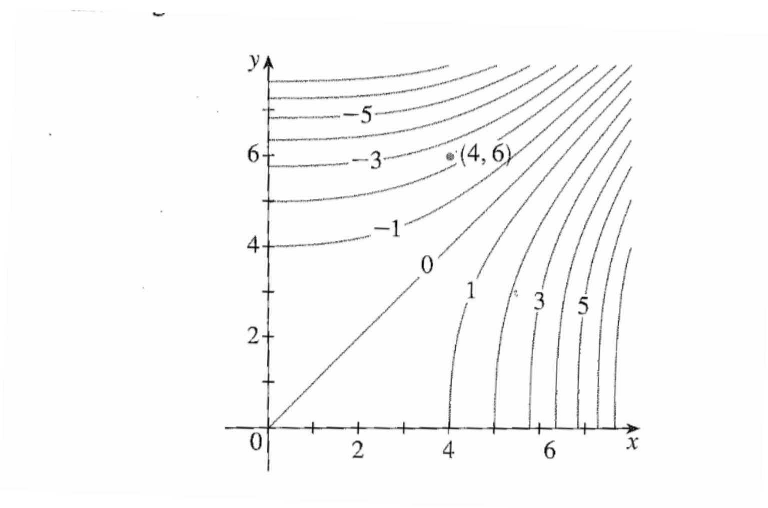
1. *Stewart 11.6 #36, modified*

   Consider a function $f(x, y)$ whose level curves are shown below.

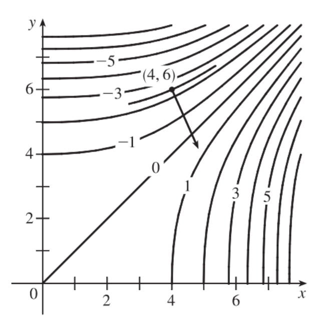
   (a) In what direction is the gradient vector $\nabla f(4, 6)$? Sketch a vector in that direction at $(4, 6)$ and explain how you chose the direction.

   (b) Approximate the length of $\nabla f(4, 6)$. Again, explain your reasoning.

   Hint: remember that directional derivative at (x_0, y_0) in the direction $\mathbf{v}$ is the rate of change of f at (x_0, y_0) with respect to distance (that is, distance between the inputs) in the direction of $\mathbf{v}$.



36. If we place the initial point of the gradient vector $\nabla f(4, 6)$ at $(4, 6)$, the vector is perpendicular to the level curve of f that includes $(4, 6)$, so we sketch a portion of the level curve through $(4, 6)$ (using the nearby level curves as a guideline) and draw a line perpendicular to the curve at $(4, 6)$. The gradient vector is parallel to this line, pointing in the direction of increasing function values, and with length equal to the maximum value of the directional derivative of f at $(4, 6)$. We can estimate this length by finding the average rate of change in the direction of the gradient. The line intersects the contour lines corresponding to -2 and -3 with an estimated distance of 0.5 units. Thus the rate of change is approximately $\dfrac{-2-(-3)}{0.5} = 2$, and we sketch the gradient vector with length 2.



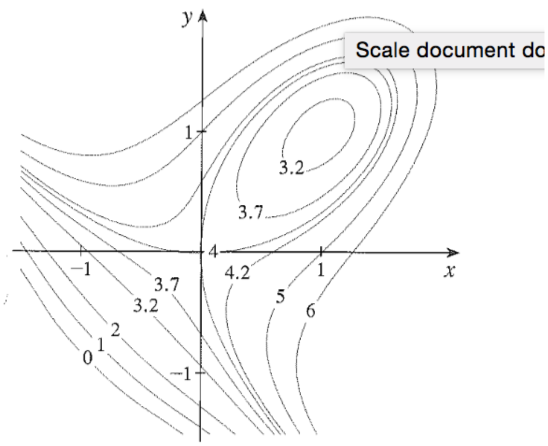
2. (*Stewart 11.7 #3*)

Let's consider

$$f(x, y) = 4 + x^3 + y^3 - 3xy$$

Use the level curves in the figures drawn below to predict the location of the critical points of f and whether f has a saddle point or a local maximum or minimum at each critical point. Explain your reasoning. Then use the Second Derivative Test to confirm your predictions.

3. $f(x, y) = 4 + x^3 + y^3 - 3xy$



Solution: From the picture, we guess that $(1, 1)$ is a local minimum as there is a nesting of closed level curves with z-values that increase as we move away from $(1, 1)$ in all directions. We also guess that $(0, 0)$ is a saddle point, since that point is on the level curve with $z = 4$ and there are level curves with z-values greater than it (to the bottom right of $(0, 0)$) and with z-values less than it (to the top right of $f(0, 0)$). Let's see if our computations reach the same conclusions. The first derivatives are $f_x(x, y) = 3x^2 - 3y$ and $f_y(x, y) = 3y^2 - 3x$, so the critical points satisfy both $y = x^2$ and $x = y^2$. Plugging the second into the first, we get $y = y^4$, so either $y = 0$ or $y^3 = 1$, which means either $y = 0$ or $y = 1$. Since $x = y^2$, we get the points $(0, 0)$ and $(1, 1)$ as our two critical points, as predicted.

To classify these critical points, we compute the second derivatives:

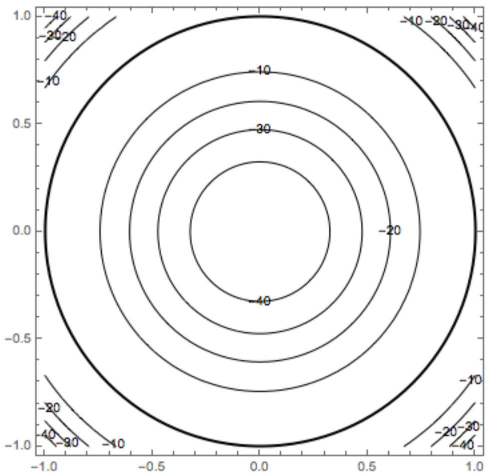
$$f_{xx}(x, y) = 6x, \qquad f_{xy}(x, y) = -3 \qquad \text{and} \qquad f_{yy}(x, y) = 6y$$

Thus the discriminant is $D = f_{xx}f_{yy} - f_{xy}^2 = 36xy - 9$. At the origin, this discriminant is $D(0, 0) = -9 < 0$, so it is indeed a saddle point. At the point $(1, 1)$, on the other hand, the discriminant is $D(1, 1) = 27 > 0$ and $f_{xx}(1, 1) = 6 > 0$, so the second derivative test confirms that there is a local minimum at this point.

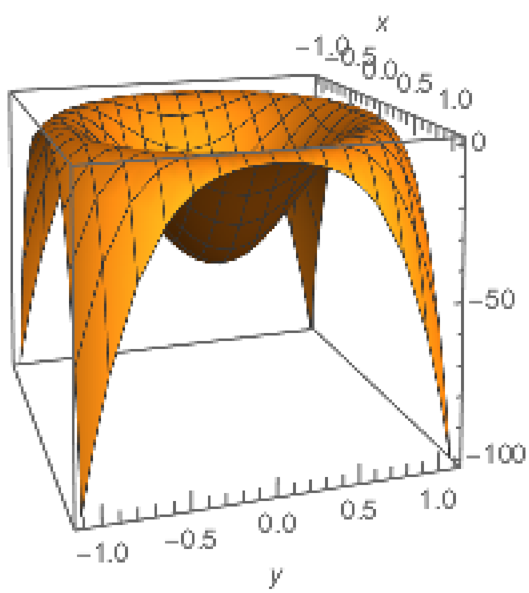
3. Consider a function $f(x, y)$ which has a local minimum at $(0, 0)$ and a whole circle of local maxima at $x^2 + y^2 = 1$. (In other words, $f(x, y)$ is the same value on the circle and this is the highest value of f in a neighborhood of this circle.)

   Sketch a possible contour plot of this function.

   Solution: Here is one possible contour plot.

   Here there is a local min with $z = -50$ at the origin and a whole circle of local maxima with $z = 0$ at $x^2 + y^2 = 1$ (shown in bold). This is the contours of the function $f(x, y) = -50((x^2 + y^2) - 1)^2$. Here is a graph on the same window:

   (Extra: can you come up with a possible formula for a function with these properties? Hint: Think of a radial function, involving $x^2 + y^2$.)