DRINFELD MODULAR VARIETIES: HOMEWORK ON SMOOTHNESS

(1) For any map of commutative rings $B \to B'$ and any $S \in \mathrm{Alg}_B$, show that there are canonical isomorphisms of $S' = B' \otimes_B S$-modules:
$$B' \otimes_B \Omega^1_{S/B} \simeq S' \otimes_S \Omega^1_{S/B} \simeq \Omega^1_{S'/B'}.$$

(2) Suppose that $S \in \mathrm{Alg}_B$ and $T \in \mathrm{Alg}_S$. Exhibit a canonical exact sequence of T-modules
$$T \otimes_S \Omega^1_{S/B} \to \Omega^1_{T/B} \to \Omega^1_{T/S} \to 0.$$

(3) Show that any localization of a smooth B-algebra is formally smooth over B.
*Such a B-algebra is called **essentially smooth**.*

(4) Suppose that $B \to B'$ is a faithfully flat map, and that $S \in \mathrm{Alg}_B$. Show that S is smooth over B if and only if $S' = B' \otimes_B S$ is smooth over B'.
Hint: One direction is easy (and true for any base change). For the other, use the criterion in terms of surjections $P \to S$, the splitting of the fundamental short exact sequence, and the projectivity of the module of differentials. This argument appeared in disguise already in Lecture 22.

(5) Suppose that $B = k$ is a field. Show that the following are equivalent:
(a) S is a product of finite separable extensions of k.
(b) S is smooth over k of relative dimension 0.
(c) S is a finite dimensional k-vector space and formally smooth over k.
(d) S is finitely generated over k and $\Omega^1_{S/k} = 0$.
Hint: All these assertions can be equivalently checked after base changing to an algebraic closure of k (see the previous problem!). So you can assume that k is algebraically closed, and in particular infinite.
For (a)⇒(b), note more generally that a product of smooth algebras of relative dimension n is also smooth of relative dimension 0.
For (c)⇒(d), note that $\mathrm{Hom}(\Omega^1_{S/k}, k)$ is finite if and only if $\Omega^1_{S/k} = 0$.
For (d)⇒(a), note that all the maximal ideals of S must localize to (0) (Lemma from lecture 23). You will need the Nullstellensatz.

(6) Let S be a smooth k-algebra with maximal ideal $\mathfrak{m}$ such that $\mathfrak{m}S_{\mathfrak{m}}$ is nilpotent. Show that $\mathfrak{m}S_{\mathfrak{m}} = (0)$ and that $S_{\mathfrak{m}}$ is a field.
Hint: Show that $S_{\mathfrak{m}}$ is a finite dimensional k-vector space and use the previous problem.
Remark: With a bit more work, the same argument shows that for any prime $P \leq S$, if PS_P is nilpotent, then S_P is a field. This implies that smooth k-algebras cannot contain non-zero nilpotent elements.

(7) Let S be a smooth k-algebra, and suppose that $\mathfrak{m} \leq S$ is a maximal ideal such that $\mathfrak{m}S_{\mathfrak{m}} = (a)$ is principal. Show that one of the following is true:
(a) $S_{\mathfrak{m}}$ is a PID.
(b) $S_{\mathfrak{m}}$ is a field.
Hint: If $S_{\mathfrak{m}}$ is not an integral domain, show that $a^n = 0$ for some n, and use the previous problem.

(8) Suppose that k is a perfect field. Let S be a finitely generated k-algebra with a maximal ideal $\mathfrak{m}$ and residue field $L = S/\mathfrak{m}$. Let
$$\mathfrak{m}_L \leq L \otimes_k S$$
be the maximal ideal corresponding to the surjection
$$L \otimes_k S \xrightarrow{a\otimes s \mapsto a\pi(s)} L$$

where $\pi : S \to L$ is the quotient map. Show that $S/\mathfrak{m}^2 \simeq (L \otimes_k S)/\mathfrak{m}_L^2$, and hence that $\mathfrak{m}/\mathfrak{m}^2 \simeq \mathfrak{m}_L/\mathfrak{m}_L^2$.

Hint: The key is to show that $S/\mathfrak{m}^2$ is canonically an L-algebra. This uses the fact that L is a finite separable extension of k.

Remark: This completes the proof of the 'Consequence' from Lecture 23 under the additional perfectness hypothesis. The full proof needs a little dimension theory and will be skipped.

(9) Suppose that $S \in \mathrm{Alg}_k$ is a finitely generated k-algebra such that $\Omega^1_{S/k}$ is finite projective over S of rank n. Show that there exist $f_1, \dots, f_m \in S$ such that $(f_1, \dots, f_m) = S$ is the unit ideal and such that, for each i, $S[f_i^{-1}]$ is the quotient of a standard smooth S-algebra of relative dimension n.

Hint: See Lecture 22.

(10) Suppose that $S \in \mathrm{Alg}_k$ is a finitely generated k-algebra that is a Dedekind domain, and is such that $\Omega^1_{S/k}$ is projective of rank 1. Show that S is smooth.

Hint: Use the previous problem. This completes the proof of the proposition in Lecture 23.