Recall: A CM pair (E, Φ) determines a Shimura datum $(T, \{h_\Phi\})$. Let $T_E = \mathrm{Res}_{E/\mathbb{Q}} \mathbb{G}_m$. Φ determines isom. of $\mathbb{R}$-alg.'s $E \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \mathbb{C}^d$ ($[E:\mathbb{Q}] = 2d$), $x \otimes 1 \mapsto (\varphi(x))_{\varphi \in \Phi}$. Hence, we get morphism of $\mathbb{R}$-alg.'s $\mathbb{C} \overset{\text{diag}}{\hookrightarrow} \mathbb{C}^d \cong E \otimes_{\mathbb{Q}} \mathbb{R}$ and thus morphism of real alg. grps. $h_\Phi : \mathbb{S} \to (T_E)_{\mathbb{R}}$, whose image lies in slightly smaller torus

$$\begin{tikzcd} T \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & T_{\mathbb{C}} \arrow[d, "N_{E/F}"] \\ \mathbb{G}_m \arrow[r, hook] & T_F \end{tikzcd}$$

$F \subseteq E$ maximal tot. real subfield. That is, (or maybe e.g.)

$T(\mathbb{Q}) = \{x \in E^\times : x\bar{x} \in \mathbb{Q}^\times\}$ so $h_\Phi : \mathbb{S} \to T_{\mathbb{R}}$.

Fix 1-dim E-vec. space V and $\mathbb{Q}$-symplectic form $\psi : V \times V \to \mathbb{Q}$ s.t. $\psi(\alpha x, y) = \psi(x, \bar{\alpha} y)$ $\forall x, y \in V$, $\alpha \in E$.

Remark: Such ψ always exists. Identify $V \cong E$ and take $F \subseteq E$ max. tot. real subfield. Write $E = F(\zeta)$ w/ $\zeta^2 \in F$. $\bar{\zeta} = -\zeta$ and we can take $\psi(x,y) = \mathrm{Tr}_{E/\mathbb{Q}}(\zeta x \bar{y})$. $\blacksquare$

Then, $T \to GL_E(V)$. Given $\alpha \in T(\mathbb{Q})$ and $x, y \in V$, $\psi(\alpha x, \alpha y) = \psi(x, \overbrace{\alpha\bar{\alpha}}^{\text{rational }\#} y) = (\alpha\bar{\alpha})\psi(x,y)$. So, $T \hookrightarrow GSp(V, \psi)$. Induces map of Shimura data $(T, \{h_\Phi\}) \to (GSp(V), X)$. (Siegel Shimura variety)

Now, fix $\mathcal{O}_E$-lattice $L \subseteq V$ s.t. $\psi(L, L) \subseteq \mathbb{Z}$. (integral condition)

Prop: $\forall K \leq T(\mathbb{A}_f)$ compact open, suff. small (i.e., neat), and stabilizing $\hat{L} \subseteq \hat{V}$ (adelic), (don't need suff. small if we only want bijection – comes in for moduli interpretation)

$T(\mathbb{Q}) \backslash \{h_\Phi\} \times T(\mathbb{A}_f)/K$ is in bij. w/ tuples $(A, i, \lambda, [\eta])$ where:

[Note: Last time we discussed Rosati involution] (?)

- A ab. var. $/\mathbb{C}$
- $i : \mathcal{O}_E \to \mathrm{End}(A)$
- $\lambda : A \to A^\vee$ polarization
- $[\eta] \subseteq \mathrm{Iso}_{\hat{\mathbb{Z}}}(\hat{T}A, \hat{L})$ is K-orbit

s.t. (1) A has CM type Φ. $\longleftarrow$ Focus more on this!!! [reflex fields]

(2) $\forall \alpha \in \mathcal{O}_E$:
$$\begin{tikzcd} A \arrow[r, "\lambda"] \arrow[d, "\alpha"'] \arrow[dr, phantom, "\circlearrowright"] & A^\vee \arrow[d, "\bar{\alpha}^\vee"] \\ A \arrow[r, "\lambda"'] & A^\vee \end{tikzcd}$$

(3) $\eta : \hat{T}A \xrightarrow{\sim} \hat{L}$ is $\mathcal{O}_E$-linear and
$$\begin{tikzcd} \hat{T}A \times \hat{T}A \arrow[r, "\text{Weil}"] \arrow[d, "\eta \times \eta"'] \arrow[dr, phantom, "\circlearrowright"] & \hat{\mathbb{Z}}(1) \arrow[d, "\text{some isom...}"] \\ \hat{L} \times \hat{L} \arrow[r, "\psi"'] & \hat{\mathbb{Z}} \end{tikzcd}$$

What is natural field of definition of the moduli problem? Choose alg. closure $\overline{\mathbb{Q}} \subseteq \mathbb{C}$.

$\mathrm{Hom}(E, \mathbb{C}) = \mathrm{Hom}(E, \overline{\mathbb{Q}})$. $\mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$ acts on CM types in $\mathrm{Hom}(E, \mathbb{C})$.

Def: Given CM pair (E, Φ), let $G_\Phi := \{\sigma \in \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) : \Phi^\sigma = \Phi\}$. Reflex field of (E, Φ) is $E_\Phi := \overline{\mathbb{Q}}^{G_\Phi}$.

This has distinguished emb. into $\mathbb{C}$, by construction.

[Most people take this to be the definition...]

Prop: $E_\Phi/\mathbb{Q}$ is finite ext. Moreover,

(1) E_Φ is gen. as $\mathbb{Q}$-alg. by image of $\mathbb{Q}$-linear map $E \to \mathbb{C}$, $\alpha \mapsto \sum_{\varphi \in \Phi} \varphi(\alpha)$.

(2) Given $\alpha \in E$, define $p_\alpha(x) := \prod_{\varphi \in \Phi} (x - \varphi(\alpha)) \in \overline{\mathbb{Q}}[x]$. E_Φ is gen. by all coeffs. of all p_α as $\alpha \in E$ varies.

In particular, $p_\alpha(x) \in E_\Phi[x]$.

(unique up to unique isom. suitably defined)

(3) $\exists$ E_Φ-vec. space V_Φ and $E \to \mathrm{End}_{E_\Phi}(V_\Phi)$ s.t., $\forall \alpha \in E$, char. poly. of $\alpha \in \mathrm{End}_{E_\Phi}(V_\Phi)$ is $p_\alpha(x)$.

Note: As $E \otimes_{\mathbb{Q}} \mathbb{C}$-module, $V_\Phi \otimes_{E_\Phi} \mathbb{C} \cong \bigoplus_{\varphi \in \Phi} \mathbb{C}(\varphi)$. So, RHS descends to $E \otimes_{\mathbb{Q}} E_\Phi$.

Note: All three of (1)-(3) can be used to characterize E_Φ.

Fun fact: E_Φ is CM! (something like this is true: biquadratic imag. field is CM but not CM type of some field)

Prop: Fix $K \leq T(\mathbb{A}_f)$ compact open as before. Consider the functor $Y_K : \mathrm{Sch}_{E_\Phi} \to \mathrm{Set}$ defined by taking $Y_K(S)$ as before, so we get tuple $(A, i, \lambda, [\eta])$, w/ extra conditions that $A \to S$ is abelian scheme and $i : \mathcal{O}_E \to \mathrm{End}(A)$ satisfying: every $\alpha \in \mathcal{O}_E$ acts on $\mathcal{O}_S$-module $\mathrm{Lie}(A)$ w/ char. poly. $\prod_{\varphi \in \Phi} (x - \varphi(\alpha)) \in E_\Phi[x] \hookrightarrow \mathcal{O}_S[x]$. This is rep. by finite étale E_Φ-scheme and $Y_K(\mathbb{C}) = T(\mathbb{Q}) \backslash T(\mathbb{A}_f)/K$.

Q: What do we do when not given a CM type?