BLM Basics

Zachary Gardner

Introduction

Let X be an algebraic variety defined over a perfect field k of characteristic p. Let $W\Omega^\bullet_X$ denote the classical de Rham-Witt complex of X, which when $X = \operatorname{Spec} R$ is characterized up to unique isomorphism as the initial object of the category of so-called R-framed V-pro-complexes. Here are two key results.

Theorem

Suppose X is smooth. Then, there exists a canonical map $W\Omega^\bullet_X \to \Omega^\bullet_X$ inducing a qis $W\Omega^\bullet_X / pW\Omega^\bullet_X \to \Omega^\bullet_{X/k}$.

Theorem

Let $\mathfrak{X}$ be a smooth formal scheme over $\operatorname{Spf}(W(k))$ with special fiber $X := \operatorname{Spec} k \times_{\operatorname{Spf}(W(k))} \mathfrak{X}$. Suppose that the Frobenius map $\varphi_X : X \to X$ extends to a map of formal schemes $\varphi_{\mathfrak{X}} : \mathfrak{X} \to \mathfrak{X}$. Then, there exists a natural qis $\Omega^\bullet_{\mathfrak{X}/W(k)} \to W\Omega^\bullet_X$ which depends on the choice of $\varphi_{\mathfrak{X}}$ but is independent of this choice on the level of derived categories.

Introduction

Our goal is to construct a complex $\mathcal{W}\Omega^\bullet_X$, called the saturated de Rham-Witt complex, which agrees with $W\Omega^\bullet_X$ when X is smooth and more naturally satisfies the above results. The key comes from extrapolating properties of $\Omega^\bullet_{R/k}$ when $X = \operatorname{Spec} R$ is smooth. The construction of $\mathcal{W}\Omega^\bullet_R$ will proceed in two stages, the first stage providing us with a relevant Verschiebung map V through a saturation process and the second stage roughly forcing completeness with respect to V. In the interest of time, we will skip right to the algebra.

Dieudonné Complexes

Definition

Let DC denote the category of **Dieudonné complexes**, which are triples (M, d, F) with (M, d) a cochain complex of abelian groups and $F : M \to M$ the **Frobenius** map, a homomorphism of graded abelian groups satisfying

$$dF(x) = pF(dx)$$

for every $x \in M$. In practice, we will often omit d and F when they are clear from context. A morphism $f : (M, d, F) \to (M', d', F')$ of Dieudonné complexes is a map of cochain complexes $f : (M, d) \to (M', d')$ such that $F' \circ f = f \circ F$. Note that DC has a natural symmetric monoidal structure given by tensor product.

Remark

The function F is not a map of complexes from $(M/pM, d)$ to itself. It does, however, give a map of complexes from $(M/pM, 0)$ to $(M/pM, d)$. This becomes part of the initial inspiration for considering Dieudonné complexes when paired with the Cartier isomorphism (to be described later).

Décalage

Any p-torsion-free complex (M, d) may be regarded as a subcomplex of the localization $M[p^{-1}]$. Given such a complex, define the **décalage** $\eta_p M \subseteq M[p^{-1}]$ via

$$(\eta_p M)^n := \{x \in p^n M^n : dx \in p^{n+1} M^{n+1}\}.$$

Given $(M, d, F) \in \mathrm{DC}$ a p-torsion-free complex, we have a well-defined cochain map $\alpha_F : M \to \eta_p M$ defined on nth components by $x \mapsto p^n F(x)$. Conversely, given (M, d) simply a p-torsion-free complex and $\alpha \in \mathrm{Hom}(M, \eta_p M)$, we may define a Frobenius map $F : M \to M$ on nth components via $x \mapsto p^{-n}\alpha(x)$. These two constructions are inverse to one another.

Saturated Dieudonné Complexes

Definition

$(M, d, F) \in \mathrm{DC}$ is **saturated** if M is p-torsion-free and for every $n \in \mathbb{Z}$ the map F induces a group isomorphism $M^n \xrightarrow{\sim} \{x \in M^n : dx \in pM^{n+1}\}$ (equivalently, α_F is an isomorphism). We obtain a full subcategory $\mathrm{DC}_{\mathrm{sat}} \subseteq \mathrm{DC}$. Since $F : M \to M$ is necessarily injective with image containing pM, we obtain a unique **Verschiebung** map $V : M \to M$ satisfying $F(Vx) = px$ for every $x \in M$.

Proposition

Given $M \in \mathrm{DC}_{\mathrm{sat}}$, the Verschiebung map $V : M \to M$ is injective and satisfies

- $F \circ V = V \circ F = p \cdot \mathrm{id}$;
- $F \circ d \circ V = d$;
- $p \cdot (d \circ V) = V \circ d$.

Saturated Dieudonné Complexes

Definition

A morphism $f \in \mathrm{Hom}_{\mathrm{DC}}(M, N)$ **exhibits N as a saturation of M** if N is saturated and, for every $K \in \mathrm{DC}_{\mathrm{sat}}$, f induces a bijection $\mathrm{Hom}_{\mathrm{DC}}(N, K) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{DC}}(M, K)$. If it exists, the data of (N, f) is unique up to unique isomorphism. We call it the **saturation** of M and write $\mathrm{Sat}(M)$.

Proposition

Saturations exist.

Saturated Dieudonné Complexes

Proof.

Let $(M, d, F) \in \mathrm{DC}$. Consider the subcomplex $M[p^\infty] \subseteq M$ defined by

$$M[p^\infty] := \{x \in M : p^n x = 0 \text{ for } n \gg 0\}.$$

Replacing M by $M/M[p^\infty]$ if necessary, we may assume WLOG that M is p-torsion-free. The obstruction to M being saturated is measured by $\alpha_F : M \to \eta_p M$ failing to be an isomorphism. So, we just force things:

$$\mathrm{Sat}(M) := \mathrm{colim}(M \xrightarrow{\alpha_F} \eta_p M \xrightarrow{\eta_p(\alpha_F)} \eta_p(\eta_p M) \xrightarrow{\eta_p(\eta_p(\alpha_F))} \cdots).$$

Then, $\mathrm{Sat}(M)$ is saturated since η_p commutes with filtered colimits. Moreover, the tautological map $M \to \mathrm{Sat}(M)$ exhibits $\mathrm{Sat}(M)$ as a saturation of M. $\square$

Cartier Complexes

Given $M \in \mathrm{DC}$ and $x \in M$, we have $d(Fx) = pF(dx)$ and so the image of Fx in M/pM is a cycle. Hence, we obtain a graded map $M \to H^\bullet(M/pM)$ which necessarily factors through M/pM.

Definition

$M \in \mathrm{DC}_{\mathrm{sat}}$ is of **Cartier type** if M is p-torsion-free and F induces a graded isomorphism $M/pM \xrightarrow{\sim} H^\bullet(M/pM)$. We will often abbreviate this by simply saying that M is **Cartier**.

As it turns out, the Cartier type condition is the right condition to place on M so that we can control the behavior of the saturated de Rham-Witt complex (which is not yet defined). The first inkling of this is the following.

Theorem (Cartier Criterion)

Let M be Cartier. Then, the canonical map $M \to \mathrm{Sat}(M)$ induces a qis

$$M/pM \to \mathrm{Sat}(M)/p\mathrm{Sat}(M).$$

Cartier Complexes

How do we prove this result? The key is décalage. Given M a p-torsion-free complex, there is a map $\overline{\gamma} : \eta_p M \to H^\bullet(M/pM)$ of graded abelian groups defined on nth components by $x \mapsto [p^{-n}x]$. This factors uniquely as

$$\eta_p M \longrightarrow (\eta_p M)/p(\eta_p M) \xrightarrow{\gamma} H^\bullet(M/pM)$$

As we saw in BMS I, γ is a qis when $H^\bullet(M/pM)$ is equipped with the cochain complex structure coming from the Bockstein operator $\beta : H^\bullet(M/pM) \to H^{\bullet+1}(M/pM)$ induced by the short exact sequence

$$0 \longrightarrow M/pM \xrightarrow{p} M/p^2M \longrightarrow M/pM \longrightarrow 0$$

From this we conclude that, given $f : M \to N$ a map of p-torsion-free complexes that is a qis mod p, the induced map $(\eta_p M)/p(\eta_p M) \to (\eta_p N)/p(\eta_p N)$ is a qis.

Cartier Complexes

Using the presentation

$$\mathrm{Sat}(M) = \mathrm{colim}(M \xrightarrow{\alpha_F} \eta_p M \xrightarrow{\eta_p(\alpha_F)} \eta_p(\eta_p M) \xrightarrow{\eta_p(\eta_p(\alpha_F))} \cdots),$$

to show that the natural map $M/pM \to \mathrm{Sat}(M)/p\mathrm{Sat}(M)$ is a qis it suffices to show that

$$(\eta_p^k M)/p(\eta_p^k M) \to (\eta_p^{k+1} M)/p(\eta_p^{k+1} M)$$

is a qis for each $k \geq 0$. But by what just showed it suffices to check the case $k = 0$, which follows since

$$\begin{tikzcd} M/pM \arrow[r, "\alpha_F"] \arrow[d, "F"', "\cong"] & (\eta_p M)/p(\eta_p M) \arrow[dl, "\mathrm{qis}"', "\gamma"] \\ H^\bullet(M/pM) & \end{tikzcd}$$

commutes and so α_F is a qis.

Completion

Given $M \in \mathrm{DC}_{\mathrm{sat}}$ and $r \in \mathbb{Z}^{\geq 0}$, define

$$\mathcal{W}_r(M) := M/(\mathrm{im}(V^r) + \mathrm{im}(dV^r)),$$

which is a quotient complex of M. This comes equipped with natural restriction maps

$$\mathrm{Res} : \mathcal{W}_{r+1}(M) \to \mathcal{W}_r(M).$$

Using this we define the **completion** of M to be

$$\mathcal{W}(M) := \varprojlim_{r \geq 0} \mathcal{W}_r(M).$$

We may uniquely complete each diagram

$$\begin{tikzcd} M \arrow[r, "F"] \arrow[d] & M \arrow[d] \\ \mathcal{W}_r(M) \arrow[r, dotted, "\exists!\, F"'] & \mathcal{W}_{r-1}(M) \end{tikzcd}$$

and together these induce the Frobenius $F : \mathcal{W}(M) \to \mathcal{W}(M)$.

Completion

Similarly, we obtain the Verschiebung $V : \mathcal{W}(M) \to \mathcal{W}(M)$ from unique maps $V : \mathcal{W}_r(M) \to \mathcal{W}_{r+1}(M)$. We now enumerate a number of properties of the completion $\mathcal{W}(M)$.

- $\mathcal{W}(M)$ is naturally an object of DC.
- The association $M \mapsto \mathcal{W}(M)$ is functorial in M.
- The canonical map $\rho_M : M \to \mathcal{W}(M)$ of Dieudonné complexes is functorial in M.
- $\mathcal{W}(M)$ is p-adically complete. Hence, if M is strict then M itself is p-adically complete. It follows that M is ℓ-torsion-free for $\ell \neq p$ and thus that M is torsion-free since being p-torsion-free is part of being saturated.

Strict Dieudonné Complexes

Definition

$M \in \mathrm{DC}_{\mathrm{sat}}$ is **strict** if ρ_M is an isomorphism. We obtain a full subcategory $\mathrm{DC}_{\mathrm{str}} \subseteq \mathrm{DC}_{\mathrm{sat}}$.

Example

Regard $M \in \mathrm{Mod}_{\mathbb{Z}}$ as a complex concentrated in degree zero. Then, any endomorphism $F : M \to M$ endows M with the structure of a Dieudonné complex. Using this, we deduce that M is saturated if and only if M is p-torsion-free and F is an automorphism. Assuming M is saturated, M is strict if and only if it is p-adically complete.

It isn't hard to see that $\mathcal{W}(M)$ is saturated given $M \in \mathrm{DC}_{\mathrm{sat}}$. In fact, though, more is true.

Theorem

Let $M \in \mathrm{DC}_{\mathrm{sat}}$*. Then,* $\mathcal{W}(M) \in \mathrm{DC}_{\mathrm{str}}$*.*

Strict Dieudonné Complexes

Completion also satisfies a useful universal property.

Proposition

Let $M, N \in \mathrm{DC}_{\mathrm{sat}}$ with N strict. Then, ρ_M induces a natural bijection

$$\mathrm{Hom}_{\mathrm{DC}}(\mathcal{W}(M), N) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{DC}}(M, N).$$

Where the theorem comes from is, given $M \in \mathrm{DC}_{\mathrm{sat}}$ and $r \in \mathbb{Z}^{\geq 0}$, $F^r : M \to M$ induces an isomorphism of graded abelian groups $\mathcal{W}_r(M) \xrightarrow{\sim} H^\bullet(M/p^r M)$ and so the associated quotient map $M/p^r M \to \mathcal{W}_r(M)$ is a qis. From this, we conclude that, given $f \in \mathrm{Hom}_{\mathrm{DC}_{\mathrm{sat}}}(M, N)$, the induced map $M/p^r M \to N/p^r N$ is a qis if and only if the induced map $\mathcal{W}_r(M) \to \mathcal{W}_r(N)$ is an isomorphism. Moreover, we need only check at the $r = 1$ level to get equivalence for every r.

Getting Somewhere

Proposition

Let $M \in \mathrm{DC}_{\mathrm{sat}}$.

1. *Given* $r \in \mathbb{Z}^{\geq 0}$, ρ_M *induces a qis* $M/p^r M \to \mathcal{W}(M)/p^r\mathcal{W}(M)$.
2. ρ_M *exhibits* $\mathcal{W}(M)$ *as the* p*-completion of* M *in* $D(\mathbb{Z})$.

Applying the Cartier Criterion, we conclude the following.

Corollary

Let M *be Cartier with each* M^n p*-adically complete. Then, the canonical map* $M \to \mathcal{W}\mathrm{Sat}(M)$ *is a qis.*

With the module-theoretic stuff figured out, let's introduce some multiplicative structure into the mix.

Dieudonné Algebras

Definition

Let DA denote the category of **Dieudonné algebras**, which are triples (A, d, F) with (A, d) a cdga[a] and **Frobenius** $F : M \to M$ a homomorphism of graded abelian groups such that

- $A^n = 0$ for $n > 0$;
- given $x \in A^0$, $Fx \equiv x^p \pmod{p}$;
- given $x \in A$, $dF(x) = pF(dx)$.

Morphisms in DA are cochain maps with the expected compatibilities.

[a] Recall that this means that (A, d) is a cochain complex with graded ring structure such that multiplication on A is graded-commutative and d satisfies the Leibniz rule. Additionally, we require that if $x \in A^n$ is homogeneous of odd degree then $x^2 = 0$ in A^{2n}.

Roughly speaking, we think of Dieudonné algebras as Dieudonné complexes equipped with a ring structure compatible with Frobenius and the differential.

Dieudonné Algebra Constructions

Definition

$A \in \mathrm{DA}$ is **saturated** if it saturated as a Dieudonné complex. In this case, $\mathcal{W}(A) \in \mathrm{DC}_{\mathrm{str}}$ is naturally a Dieudonné algebra and the tautological map $\rho_A : A \to \mathcal{W}(A)$ is a morphism in DA. We say that A is **strict** if ρ_A is an isomorphism in DA, which holds if and only if A is strict regarded as an object of DC. From this we obtain full subcategories $\mathrm{DA}_{\mathrm{str}} \subseteq \mathrm{DA}_{\mathrm{sat}} \subseteq \mathrm{DA}$.

As with Dieudonné complexes, saturations and completions of Dieudonné algebras make sense and exist (and have the expected universal properties). In fact, they are obtained by placing suitable algebra structures on the relevant constructions for Dieudonné complexes.

Witt Vectors

Example

Let $R \in \mathrm{CAlg}_{\mathbb{F}_p}$ and regard $W(R)$ as a cdga concentrated in degree zero. Then, $W(R)$ equipped with its Witt vector Frobenius F is naturally a Dieudonné algebra which is saturated if and only if R is perfect. In the saturated case, $W(R)$ is automatically strict. Recall that, in the case that R is perfect, $W(R)$ is p-adically complete with canonical isomorphism $W(R)/pW(R) \xrightarrow{\sim} R$ and satisfies the universal property that, given p-adically complete $A \in \mathrm{CRing}$ equipped with a ring map $R \to A/pA$, we may uniquely complete the diagram

$$\begin{tikzcd} W(R) \arrow[r, "\exists!"] \arrow[d, two heads] & A \arrow[d, two heads] \\ R \arrow[r] & A/pA \end{tikzcd}$$

Closely related to the Witt vector story is the following. Fix p-torsion-free $R \in \mathrm{CRing}$. We will see shortly that $\Omega^\bullet_R$ is naturally a Dieudonné algebra (under an additional constraint). Before that, we briefly discuss the notion of δ-rings.

δ-Rings

Definition

Let CRing_δ denote the category of **(commutative) δ-rings**, which are pairs (A, δ) with $A \in \mathrm{CRing}$ and $\delta : A \to A$ a p-**derivation** satisfying

- $\delta(xy) = x^p\delta(y) + \delta(x)y^p + p\delta(x)\delta(y)$;
- $\delta(x+y) = \delta(x) + \delta(y) - \sum_{0<i<p} \frac{(p-1)!}{i!(p-i)!} x^i y^{p-i}$;
- $\delta(1) = 0$

for all $x, y \in A$. A morphism $f : (A, \delta_A) \to (B, \delta_B)$ in CRing_δ is a ring map $f : A \to B$ such that $f \circ \delta_A = \delta_B \circ f$.

Roughly speaking, the point of the second property above is to capture the identity

$$p(\delta(x+y) - \delta(x) - \delta(y)) = x^p + y^p - (x+y)^p$$

even when there is p-torsion. The category CRing_δ is nicely behaved (in ways that we won't make precise) and is the right place to study lifts of Frobenius.

Frobenius Lifts

Given $(A, \delta) \in \mathrm{CRing}_\delta$, the map

$$\varphi : R \to R, \qquad x \mapsto x^p + p\delta(x)$$

is a mod p lift of Frobenius. Conversely, given p-torsion-free $A \in \mathrm{CRing}$ with $\varphi : A \to A$ a mod p lift of Frobenius, the map

$$\delta : A \to A, \qquad x \mapsto \frac{\varphi(x) - x^p}{p}$$

is a p-derivation on A and so $(A, \delta) \in \mathrm{CRing}_\delta$.

Returning to $\Omega_R^\bullet$, suppose that we have chosen $\varphi : R \to R$ a mod p lift of Frobenius (which need not exist in general). Then, as above we have an associated p-derivation $\delta_\varphi : R \to R$.

Proposition

There exists a unique ring homomorphism $F : \Omega_R^\bullet \to \Omega_R^\bullet$ such that

- $F = \varphi$ *on* $R = \Omega_R^0$*;*
- *given* $x \in R$*,*

$$F(dx) = \underbrace{x^{p-1}dx + d\delta_\varphi(x)}_{=:\rho_\varphi(x)}.$$

Moreover, the triple $(\Omega_R^\bullet, d, F)$ *is a Dieudonné algebra.*

Proof.

Uniqueness is clear since $\Omega_R^\bullet$ is generated by elements of the form x and dx for $x \in R$. The main idea behind proving existence is to obtain F using the universal property of $\Omega_R^\bullet$. The first step is to verify that $\rho_\varphi : R \to \Omega_R^1$ is a group homomorphism and a φ-linear derivation in the sense that

$$\rho_\varphi(xy) = \varphi(y)\rho_\varphi(x) + \varphi(x)\rho_\varphi(y)$$

for all $x, y \in R$. The universal property of $\Omega_R^\bullet$ then gives that there is a unique φ-semilinear map $F : \Omega_R^1 \to \Omega_R^1$ such that $F \circ d = \rho_\varphi$. This extends uniquely to a ring homomorphism $F : \Omega_R^\bullet \to \Omega_R^\bullet$ satisfying the desired properties and one checks $(\Omega_R^\bullet, d, F) \in \mathrm{DA}$. $\square$

de Rham Complexes as Dieudonné Algebras

Unless otherwise stated, we will hereafter assume that $\Omega_R^\bullet$ is equipped with the Dieudonné algebra structure described by the proposition. This structure possesses a useful universal property.

Proposition

Given p-torsion-free $A \in \mathrm{DA}$, the restriction map

$$\mathrm{Hom}_{\mathrm{DA}}(\Omega_R^\bullet, A) \to \mathrm{Hom}_{\mathrm{CRing}}(R, A^0)$$

is injective with image $\{f \in \mathrm{Hom}_{\mathrm{CRing}}(R, A^0) : f \circ \varphi = F_A \circ f \ \ on \ R\}$.

Our aim now is to introduce the Cartier isomorphism in order to tackle our original goal. We start by expanding upon the above construction.

The Completed de Rham Complex

Fix $R \in \mathrm{CRing}$. The **completed de Rham complex** of R is

$$\widehat{\Omega}_R^\bullet := \varprojlim_n \Omega_R^\bullet / p^n \Omega_R^\bullet \cong \varprojlim_n \Omega_{R/p^nR}^\bullet,$$

which naturally satisfies $\widehat{\Omega}_R^\bullet / p^n \widehat{\Omega}_R^\bullet \cong \Omega_{R/p^nR}^\bullet$ and is p-adically complete. The completed de Rham complex admits a unique multiplication such that the tautological map $\Omega_R^\bullet \to \widehat{\Omega}_R^\bullet$ is a homomorphism of dga's. Return now to the assumption that R is p-torsion-free with φ a choice of mod p lift of Frobenius (which we encapsulate by saying that (R, φ) or simply R is **good**). Then, the morphism $F : \Omega_R^\bullet \to \Omega_R^\bullet$ induces $F : \widehat{\Omega}_R^\bullet \to \widehat{\Omega}_R^\bullet$ endowing $\widehat{\Omega}_R^\bullet$ with the structure of a Dieudonné algebra that has a universal property similar to the one for $\Omega_R^\bullet$.

Proposition

Given p-torsion-free and p-adically complete $A \in \mathrm{DA}$, the restriction map

$$\mathrm{Hom}_{\mathrm{DA}}(\widehat{\Omega}_R^\bullet, A) \to \mathrm{Hom}_{\mathrm{CRing}}(R, A^0)$$

is injective with image $\{f \in \mathrm{Hom}_{\mathrm{CRing}}(R, A^0) : f \circ \varphi = F_A \circ f \text{ on } R\}$.

The Cartier Map

Proposition

Let $A \in \mathrm{CAlg}_{\mathbb{F}_p}$. Then, there exists a unique homomorphism of graded algebras $\mathrm{Cart} = \mathrm{Cart}_A : \Omega_A^\bullet \to H^\bullet(\Omega_A^\bullet)$, called the **Cartier map**, *such that*

$$\mathrm{Cart}(x) = [x^p], \qquad \mathrm{Cart}(dy) = [y^{p-1}dy]$$

for all $x, y \in A$.

This map is necessarily given by

$$x_0 dx_1 \wedge \cdots \wedge dx_n \mapsto [x_0^p (x_1^{p-1} dx_1) \wedge \cdots \wedge (x_n^{p-1} dx_n)]$$

for $x_0, x_1, \ldots, x_n \in R$.

Theorem (Cartier Isomorphism)

Let $k \in \mathrm{CAlg}_{\mathbb{F}_p}$ be perfect and $A \in \mathrm{CAlg}_k$ smooth. Then, $\mathrm{Cart} : \Omega_A^\bullet \to H^\bullet(\Omega_A^\bullet)$ is an isomorphism.[a]

[a] This is a nontrivial result that requires some work done later in the BLM paper.

Cartier Pays Off

What does this have to do with our previous setting? The map $F : \widehat{\Omega}^\bullet_R \to \widehat{\Omega}^\bullet_R$ induces the composition

$$\Omega^\bullet_{R/pR} \xrightarrow{\sim} \widehat{\Omega}^\bullet_R/p\widehat{\Omega}^\bullet_R \xrightarrow{F} H^\bullet(\widehat{\Omega}^\bullet_R/p\widehat{\Omega}^\bullet_R) \xrightarrow{\sim} H^\bullet(\Omega^\bullet_{R/pR})$$

One can check that this agrees with the Cartier map and so we see that the composition is independent of the choice of φ.

Corollary

Let $R \in \mathrm{CRing}$ be good and suppose there exists $k \in \mathrm{CAlg}_{\mathbb{F}_p}$ perfect such that R/pR is a smooth k-algebra. Then, $\widehat{\Omega}^\bullet_R$ is Cartier and the canonical map $\widehat{\Omega}^\bullet_R \to \mathcal{W}\mathrm{Sat}(\widehat{\Omega}^\bullet_R)$ is a qis.

The only non-obvious part of the above result is that $\widehat{\Omega}^\bullet_R$ is p-torsion-free, which follows from the fact that each $\widehat{\Omega}^i_R$ is a projective module of finite rank over the completion $\widehat{R} = \varprojlim_n R/p^nR$.

The Saturated de Rham-Witt Complex

Definition

Let $A \in \mathrm{DA}_{\mathrm{str}}$ and $R \in \mathrm{CAlg}_{\mathbb{F}_p}$. We say that $f \in \mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{F}_p}}(R, A^0/VA^0)$ **exhibits A as a saturated de Rham-Witt complex of** R if, given $B \in \mathrm{DA}_{\mathrm{str}}$,

$$\mathrm{Hom}_{\mathrm{DA}}(A, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{F}_p}}(R, B^0/VB^0).$$

The pair (A, f) is unique up to unique isomorphism if it exists, so we call A the **saturated de Rham-Witt complex** of R and write $\mathcal{W}\Omega_R^\bullet$.

Proposition

Saturated de Rham-Witt complexes exist!

We construct $\mathcal{W}\Omega_R^\bullet$ by first replacing R by R^{red} if necessary (which changes nothing) and then taking $\mathcal{W}\mathrm{Sat}(\Omega^\bullet_{W(R)})$.

Comparison with a Smooth Lift

Returning to the completed de Rham complex, we have the following result.

Proposition

Let $R \in \mathrm{CRing}$ *be good and* $B \in \mathrm{DA}_{\mathrm{str}}$*. Then, there is a natural bijection*

$$\mathrm{Hom}_{\mathrm{DA}}(\widehat{\Omega}^{\bullet}_{R}, B) \xrightarrow{\sim} \mathrm{Hom}_{\mathrm{CAlg}_{\mathbb{F}_p}}(R, B^0/VB^0).$$

This ties into what we just did assuming we can "bridge characteristics."

Theorem

Let $R \in \mathrm{CRing}$ *be good and suppose there exists* $k \in \mathrm{CAlg}_{\mathbb{F}_p}$ *perfect such that* R/pR *is a smooth* k*-algebra. Then, there is a canonical morphism* $\mu : \widehat{\Omega}^{\bullet}_{R} \to \mathcal{W}\Omega^{\bullet}_{R/pR}$ *of Dieudonné algebras which is a qis.*

Comparison with the de Rham Complex

Given $R \in \mathrm{CAlg}_{\mathbb{F}_p}$, consider

$$\mathcal{W}_1\Omega_R^\bullet := \mathcal{W}\Omega_R^\bullet/(\mathrm{im}(V) + \mathrm{im}(dV))$$

which comes equipped with a tautological ring map $e : R \to \mathcal{W}\Omega_R^0$ extending uniquely to a map $\nu : \Omega_R^\bullet \to \mathcal{W}_1\Omega_R^\bullet$ of dga's.

Theorem

Suppose there exists $k \in \mathrm{CAlg}_{\mathbb{F}_p}$ perfect such that R is a smooth k-algebra. Then, ν is an isomorphism.

Under the above setup, the composition

$$\mathcal{W}\Omega_R^\bullet \longrightarrow \mathcal{W}_1\Omega_R^\bullet \xrightarrow{\nu^{-1}} \Omega_R^\bullet$$

is a map of cdga's which induces a qis $\mathcal{W}\Omega_R^\bullet/p\mathcal{W}\Omega_R^\bullet \to \Omega_R^\bullet$.