Section 7 Math 2202

Directional Derivatives, Gradients and Local Extrema

Comments for Facilitator:

- Do Quiz (7 min)
- Go over quiz (13 min). You don't need to poll first on which one they most want to discuss - I imagine it will be (b). (NOTE: You don't need to go over the Exploratory quiz question.)

  Refresh the computation of directional derivative especially that we need to create a unit vector. Then focus on carefully articulating the meaning of directional derivatives here and again with the gradients that arise in discussing the T/F.

  I think the computations are relatively easy for students - it's interpreting them where the work is. That's the focus of today.
- Problems (20 min): Have them do the level curves one in small groups and talk about meaning. This should be about 10 minutes.

  Then have them try the max/min problem - again having them make educated guess about max, min and saddles with the contours and then confirming these with derivative computations. You can build off the intuition from the T/F on the quiz: for critical points, we look at where $\nabla f = \mathbf{0}$ because if we are at max, then the direction to travel to increase most would be zero - we're at the highest point (and analogously for min).
- Open Questions (5 min)

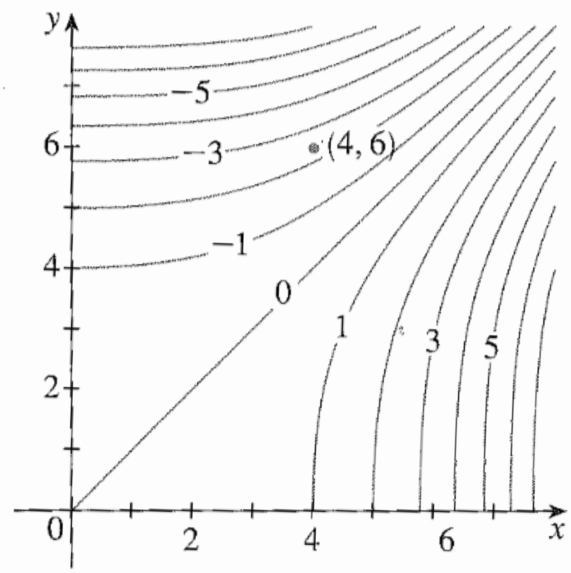
1. *Stewart 11.6 #36, modified*

   Consider a function $f(x, y)$ whose level curves are shown below.

   (a) In what direction is the gradient vector $\nabla f(4, 6)$? Sketch a vector in that direction at $(4, 6)$ and explain how you chose the direction.

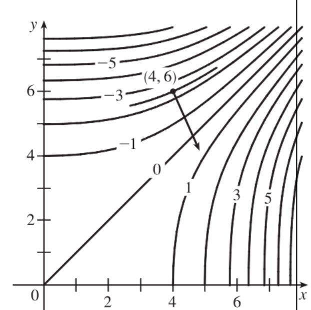
   (b) Approximate the length of $\nabla f(4, 6)$. Again, explain your reasoning.

   Hint: remember that directional derivative at (x_0, y_0) in the direction $\mathbf{v}$ is the rate of change of f at (x_0, y_0) with respect to distance (that is, distance between the inputs) in the direction of $\mathbf{v}$.



Comments for Facilitator: This is to get students remembering the meaning of the directional derivative (by connecting it to thinking about average rate of change of f wrt distance) and gradient, especially the idea that we're talking about change wrt distance between input points.
It also is to enforce the idea that gradients are perpendicular to level curves.

36. If we place the initial point of the gradient vector $\nabla f(4,6)$ at $(4,6)$, the vector is perpendicular to the level curve of f that includes $(4,6)$, so we sketch a portion of the level curve through $(4,6)$ (using the nearby level curves as a guideline) and draw a line perpendicular to the curve at $(4,6)$. The gradient vector is parallel to this line, pointing in the direction of increasing function values, and with length equal to the maximum value of the directional derivative of f at $(4,6)$. We can estimate this length by finding the average rate of change in the direction of the gradient. The line intersects the contour lines corresponding to -2 and -3 with an estimated distance of 0.5 units. Thus the rate of change is approximately $\dfrac{-2-(-3)}{0.5}=2$, and we sketch the gradient vector with length 2.



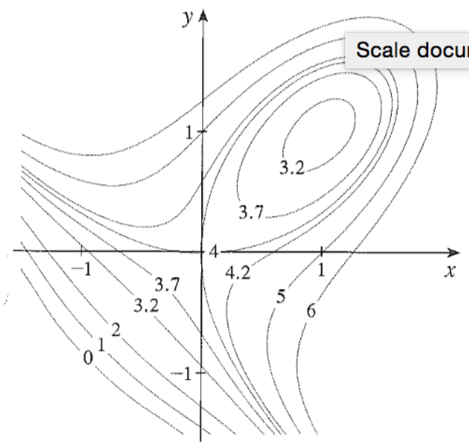
2. (*Stewart 11.7 #3*)

   Let's consider

$$f(x, y) = 4 + x^3 + y^3 - 3xy$$

   Use the level curves in the figures drawn below to predict the location of the critical points of f and whether f has a saddle point or a local maximum or minimum at each critical point. Explain your reasoning. Then use the Second Derivative Test to confirm your predictions.

   3. $f(x, y) = 4 + x^3 + y^3 - 3xy$



3. Consider a function $f(x, y)$ which has a local minimum at $(0, 0)$ and a whole circle of local maxima at $x^2 + y^2 = 1$. (In other words, $f(x, y)$ is the same value on the circle and this is the highest value of f in a neighborhood of this circle.)

   Sketch a possible contour plot of this function.

   (Extra: can you come up with a possible formula for a function with these properties? Hint: Think of a radial function, involving $x^2 + y^2$.)