Digression on Cartan Involution

$$\begin{tikzcd}[column sep=small] & SL_n(\mathbb{C}) \arrow[dl, dash] \arrow[dr, dash] & \\ G = SL_n(\mathbb{R}) \arrow[dr, dash] & & SU(n) = \mathcal{U} \arrow[dl, dash] \\ & SO(n) & \end{tikzcd}
\qquad
\begin{tikzcd}[column sep=small] & \mathfrak{g}_{\mathbb{C}} \cong \mathfrak{u}_{\mathbb{C}} \arrow[dl, dash] \arrow[dr, dash] & \\ \mathfrak{g} \arrow[dr, dash] & & \mathfrak{u} \arrow[dl, dash] \\ & \mathfrak{k} = \mathfrak{so}(n) & \end{tikzcd}$$

We have real $\mathbb{R}$-vec. space decomposition:

$$\underbrace{\underset{\substack{\cup \\ \text{trace} \\ =0}}{\mathfrak{g}} = \underset{\substack{\cup \\ \text{skew} \\ \text{-symm.}}}{\mathfrak{k}} \oplus \underset{\substack{\cup \\ \text{symm.}}}{\mathfrak{p}}}_{\text{example of Cartan decomp.}} \quad , \quad \mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}$$

There is induced involution $\theta \in \mathrm{Aut}(\mathfrak{g})$ which is 1 on $\mathfrak{k}$ and -1 on $\mathfrak{p}$. This allows us to find compact form of real Lie grp.

There is global Cartan involution $\Theta \in \mathrm{Aut}(G)$ given by $\Theta(g) = {}^t g^{-1}$ whose fixed pts. are $SO(n)$. We get global Cartan decomposition $SO(n) \times \underset{\substack{\cup \\ \text{pos. def. symm.} \\ \text{matrices of trace } 0}}{P} \xrightarrow{\sim} SL_n(\mathbb{R})$ as smooth mflds.

Let $\mathfrak{g}$ be Lie alg. / field k. We have adjoint action $\mathrm{ad}: \mathfrak{g} \to \mathrm{End}_k(\mathfrak{g})$, $X \mapsto [X, \cdot]$.

$\rightsquigarrow$ symm. k-bilin. form (killing form) $B: \mathfrak{g} \times \mathfrak{g} \to k$, $(X,Y) \mapsto \mathrm{Tr}_k(\mathrm{ad}(X) \circ \mathrm{ad}(Y))$.

<u>Thm</u> (Cartan): $\mathfrak{g}$ is semisimple iff B is nondeg.

<u>Thm</u>: G semisimple real Lie grp. Then, G is compact iff center of G is finite and killing form on $\mathrm{Lie}(G)$ is neg. def.

<u>Cor</u>: G adjoint real Lie group. Then, G is compact iff killing form is neg. def.

<u>Def</u>: Real Lie alg. is <u>compact</u> if its killing form is neg. def.

<u>Def</u>: <u>Cartan involution</u> of real Lie alg. $\mathfrak{g}$ is $\theta \in \mathrm{Aut}(\mathfrak{g})$ s.t. $\theta^2 = \mathrm{id}_{\mathfrak{g}}$ and B_θ given by $B_\theta(X,Y) := -B(X, \theta(Y))$ is pos. def.

So, $\mathfrak{g}$ is compact iff $\mathrm{id}_{\mathfrak{g}}$ is Cartan involution.

Cartan involution $\theta \rightsquigarrow$ Cartan decomposition $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ w/ $\theta = 1$ on $\mathfrak{k}$ and $\theta = -1$ on $\mathfrak{p}$. What's the actual definition?

(1) $[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k}$
$[\mathfrak{k}, \mathfrak{p}] \subseteq \mathfrak{p}$
$[\mathfrak{p}, \mathfrak{p}] \subseteq \mathfrak{k}$

(2) B is ~~pos.~~ neg. def. on $\mathfrak{k}$
B is ~~neg.~~ pos. def. on $\mathfrak{p}$

This is equiv. to $\mathfrak{u} := \mathfrak{k} \oplus i\mathfrak{p}$ (in $\mathfrak{g}_{\mathbb{C}}$) being a compact Lie alg.

Thm (Cartan): Real Lie alg. admits Cartan involution iff it is semisimple.

Thm: Assume $\mathfrak{g}$ semisimple. (1) All Cartan involutions are conjugate by elts. of $\mathrm{Aut}(\mathfrak{g})^+$. Equiv., $\mathrm{Aut}(\mathfrak{g})^+$ acts transitively on Cartan decompositions.

(2) There is bij. correspondence between Cartan involutions and compact real forms of $\mathfrak{g}$

$$\theta \mapsto \mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p} \mapsto \mathfrak{u} = \mathfrak{k} \oplus i\mathfrak{p}.$$

(3) Compact real form is unique up to isom.

Back to Geometry

X HSD $\Rightarrow \exists$ real adjoint group G (in the sense of alg. gps.) s.t. $G(\mathbb{R})^+ = \overbrace{\mathrm{Hol}(X)^+}^{\text{(conn. component of identity of gp. of holomorphic automorphisms)}}$.

Given $x \in X$, stabilizer $K_x \in G(\mathbb{R})^+$ is compact and $\exists!\ u_x : \underbrace{U_1}_{\text{circle in } \mathbb{C}^\times} \to K_x \subseteq G(\mathbb{R})$ s.t. induced action of U_1 on $\mathrm{Aut}_{\mathbb{C}}(T_x X) = GL(T_x X)$ is the expected scalar action $v_x(t) =$ mult. by t.

$$\underbrace{U_1 \xrightarrow{u_x} G(\mathbb{R}) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(G(\mathbb{R})) \to}_{\mathrm{Ad}(u_x)} \mathrm{Aut}(\overset{\mathrm{Lie}\,G(\mathbb{R})}{\overset{\shortparallel}{\mathfrak{g}}})$$

induces grading $\mathfrak{g}_{\mathbb{C}} = \bigoplus_{k \in \mathbb{Z}} \mathfrak{g}_{\mathbb{C}}^{(k)}$. On $\mathfrak{g}_{\mathbb{C}}^{(k)}$ $t \in U_1$ acts by t^k.

This looks like $\mathfrak{g}_{\mathbb{C}}^{(-1)} \oplus \mathfrak{g}_{\mathbb{C}}^{(0)} \oplus \underbrace{\mathfrak{g}_{\mathbb{C}}^{(1)}}_{= T_x X}$, with $\mathfrak{g}_{\mathbb{C}}^{(0)} = (\mathfrak{k}_x)_{\mathbb{C}} = \mathrm{Lie}(K_x)_{\mathbb{C}}$. Moreover, $\mathfrak{g} = \mathfrak{k}_x \oplus T_x X$. $(*)$.

Prop: $(*)$ is Cartan decomp. and assoc. involution is $\theta = \mathrm{Ad}(u_x)(-1)$.

Thm: There is bij. between HSD's and pairs (G, X) where G is real adjoint gp. (w/ Lie alg. $\mathfrak{g}$) and $X \subseteq \mathrm{Hom}(U_1, G(\mathbb{R}))$ is a $G(\mathbb{R})^+$-conj. class of maps. These satisfy conditions...

(1) $(\forall u : U_1 \to G(\mathbb{R}) \in X)$ the grading on $\mathfrak{g}_{\mathbb{C}}$ induced by $\underbrace{U_1 \xrightarrow{u} G(\mathbb{R}) \xrightarrow{\mathrm{Ad}} \mathrm{Aut}(G(\mathbb{R})) \to}_{\mathrm{Ad}(u)} \mathrm{Aut}(\mathfrak{g})$ has form

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{g}_{\mathbb{C}}^{(-1)} \oplus \mathfrak{g}_{\mathbb{C}}^{(0)} \oplus \mathfrak{g}_{\mathbb{C}}^{(1)} \quad \text{(minuscule!)}.$$

(2) $\mathrm{Ad}(u)(-1) \in \mathrm{Aut}(\mathfrak{g})$ is Cartan involution,

[Recall HSD's don't have components which are of compact type...]

(3) There is no simple factor (over $\mathbb{R}$) of G into which $u(-1)$ has trivial projection.