$X \in Sch$ has a structure sheaf $\mathcal{O}_X$.

$\rho : Y \to X$ map of schemes

$$\begin{array}{ccc} Shv(Y) & \underset{\rho^*}{\overset{\rho_*}{\rightleftarrows}} & Shv(X) \\ \Psi & & \Psi \\ \mathcal{G} & & \mathcal{F} \end{array}$$

$\rho_*\mathcal{G} \in Shv(X)$, $\rho^*\mathcal{F} \in Shv(Y)$

$V \subseteq X$ open

$(\rho_*\mathcal{G})(V) := \mathcal{G}(\rho^{-1}(V))$

$U \subseteq Y$ open

$W' \subseteq \rho(U) \supseteq W$

$(\rho^*_{pre}\mathcal{F})(U) := \underset{W \supseteq \rho(U)}{\varinjlim} \mathcal{F}(W)$

$\mathcal{F}(W')$ $\mathcal{F}(W)$

$\rho^*\mathcal{F} := (\rho^*_{pre}\mathcal{F})^{sh}$

$Op(X) = Ouv(X)$

$\mathcal{P}(X) = Fun(Op(X)^{op}, Set) \underset{sh}{\rightleftarrows} Shv(X)$

$$\begin{array}{ccc} \mathcal{P}(Y) & \underset{\rho^*_{pre}}{\overset{\rho_*}{\rightleftarrows}} & \mathcal{P}(X) \\ \uparrow \downarrow sh & & \uparrow \downarrow sh \\ Shv(Y) & \underset{\rho^*}{\overset{\rho_*}{\rightleftarrows}} & Shv(X) \end{array}$$

RAPL : right adjoints preserve limits

Limits commute w/ limits

$\rho_* \dashv \rho^*_{pre}$

$Hom_{\mathcal{P}(X)}(\rho_*\mathcal{G}, \mathcal{F}) \cong Hom_{\mathcal{P}(Y)}(\mathcal{G}, \rho^*_{pre}\mathcal{F})$

$(\cdot)^{sh} \circ \rho_* = \rho_* \circ (\cdot)^{sh}$

$$\begin{array}{ccc} \widetilde{N} & & \widetilde{M} \\ | & & | \\ Y & \xrightarrow{\rho} & X \\ \| & & \| \\ Spec B & & Spec A \end{array}$$

$M \in Mod_A$, $N \in Mod_B$

$\widetilde{M}(\operatorname{Spec} A) = M$, $\widetilde{M}(\operatorname{Spec} A_f) = M_f$

$\rho_* \widetilde{N}$, $\rho^* \widetilde{M}$

$\widetilde{M} \in \mathrm{Mod}_{\mathcal{O}_{\operatorname{Spec} A}} \subseteq \mathrm{Shv}(\operatorname{Spec} A)$

$$(\rho^*_{pre} \widetilde{M})(\operatorname{Spec} B) = \underset{W \supseteq \rho(\operatorname{Spec} B)}{\underrightarrow{\mathrm{colim}}} \widetilde{M}(W) \quad ?$$

$$(\rho_* \widetilde{N})(\operatorname{Spec} A) = \widetilde{N}(\rho^{-1}(\operatorname{Spec} A)) = \widetilde{N}(\operatorname{Spec} B) = N$$

$$\begin{array}{ccc} \rho^*\widetilde{M} = ? & & \widetilde{M} \\ | & & | \\ \operatorname{Spec} B & \xrightarrow{\rho} & \operatorname{Spec} A \\ | & & \\ \widetilde{N} & & \end{array}$$

$$\widetilde{B \otimes_A M} \cong \rho^* \widetilde{M}$$

$$\rho_* \widetilde{N} \cong \widetilde{N_{[A]}}$$

$X \in \mathrm{Sch} \rightsquigarrow \mathrm{QCoh}(X)$ quasicoherent sheaves

$\mathrm{Fun}(\mathrm{AffSch}^{op}_{/X}, \mathrm{Ab})$

$$\begin{tikzcd} \operatorname{Spec} B \arrow[r, "g"] \arrow[dr, "f \circ g"'] & \operatorname{Spec} A \arrow[d, "f"] \\ & X \end{tikzcd} \rightsquigarrow \mathcal{F}_f \in \mathrm{Mod}_A$$

$$\mathcal{F}_{f\circ g} \in \mathrm{Mod}_B \xleftarrow[\alpha_{f,g}]{\sim} B \otimes_A \mathcal{F}_f$$

$\operatorname{Spec} A$

$f \downarrow \quad \longmapsto \quad \mathcal{F}_f \in \mathrm{Mod}_A$

$$\begin{tikzcd} \operatorname{Spec} A \arrow[d, "f"'] \\ X \end{tikzcd} \longmapsto \mathcal{F}_f \in \mathrm{Mod}_A$$

$$\begin{tikzcd} \operatorname{Spec} B \arrow[d, "f\circ g"'] \\ X \end{tikzcd} \longmapsto \tilde{\mathcal{F}}_{f\circ g} \in \mathrm{Mod}_B$$

$$\begin{tikzcd} \operatorname{Spec} B \arrow[d, "f\circ g"'] \\ X \end{tikzcd} \longrightarrow \begin{tikzcd} \operatorname{Spec} A \arrow[d, "f"] \\ X \end{tikzcd} \longmapsto \mathcal{F}_f \to \mathcal{F}_{f\circ g}$$

$$\mathcal{F}_f \xrightarrow{1\otimes \mathrm{id}} B \underset{A}{\otimes} \mathcal{F}_f \xrightarrow{\alpha_{f,g}} \mathcal{F}_{f\circ g}$$

$$\mathcal{O}_X : \begin{tikzcd} \operatorname{Spec} A \arrow[d, "f"'] \\ X \end{tikzcd} \longmapsto A$$

$M \in \mathrm{Mod}_A$, $N \in \mathrm{Mod}_B$

$$\begin{tikzcd} \tilde{N} \arrow[d, dash] & \tilde{M} \arrow[d, dash] \\ \operatorname{Spec} B \arrow[r, "\rho"'] & \operatorname{Spec} A \end{tikzcd}$$

$$\mathrm{Hom}_B\left(B \underset{A}{\otimes} M, N\right) \cong \mathrm{Hom}_A\left(M, N_{[A]}\right)$$

$$\mathrm{Hom}_{QCoh(\operatorname{Spec} B)}\left(\rho^* \tilde{M}, \tilde{N}\right) \cong \mathrm{Hom}_{QCoh(\operatorname{Spec} A)}\left(\tilde{M}, \rho_* \tilde{N}\right)$$

$$G = G(*) \qquad \begin{tikzcd} G \arrow[d, dash] & F \arrow[d, dash] \\ * \arrow[r, "\rho"] & X \end{tikzcd} \qquad\qquad \begin{tikzcd} G \arrow[d, dash] & F \arrow[d, dash] \\ Y \arrow[r, "\pi"] & * \end{tikzcd} \qquad F = F(*)$$

$G = G(*)$

$$(p_* G)(\mathcal{U}) = G(p^{-1}(\mathcal{U})) = \begin{cases} \emptyset, & p(*) \notin \mathcal{U} \\ G, & p(*) \in \mathcal{U} \end{cases}$$ skyscraper sheaf

$$(p^* F)(\emptyset) = \emptyset$$

$$(p^*_{pre} F)(*) = \varinjlim_{W \ni p(*)} F(W) =: F_{p(*)}$$

$\parallel$

$(p^* F)(*)$

$$\mathrm{Hom}_*(p^* F, G) \to \mathrm{Hom}_X(F, p_* G)$$

$F_{p(*)} \to G$ $\quad \{F(\mathcal{U}) \to G\}$ compatible $\mathcal{U} \ni p(*)$

$$(\pi_* G)(*) = G(\pi^{-1}(*)) = G(Y)$$

$$(\pi_* G)(\emptyset) = \emptyset$$

$$(\pi^*_{pre} F)(V) = \varinjlim_{W \supseteq \pi(V)} F(W) = \begin{cases} \emptyset, & V = \emptyset \\ F = F(*), & V \neq \emptyset \end{cases}$$

$$\mathrm{Hom}_Y(\pi^* F, G) \to \mathrm{Hom}_*(F, \pi_* G)$$

$\pi^* F = \underline{F}$ locally constant

$\underbrace{\{F \to G(V)\} \text{ compatible} \qquad F \to G(Y)}_{\text{same by sheaf property for } G}$

Claim: $\mathcal{F} \to \mathcal{G}$ map of sheaves
"$\Updownarrow$"
$\{\mathcal{F}_x \to \mathcal{G}_x\}_{x \in X}$.

Pf: $\alpha : \mathrm{Hom}_{Shv(X)}(\mathcal{F}, \mathcal{G}) \to \{\mathrm{Hom}_{Set}(\mathcal{F}_x, \mathcal{G}_x)\}_{x \in X}$

$$\varphi \longmapsto \{\varphi_x\}_{x \in X}$$

φ surj. $\Rightarrow$ φ_x surj.

Want: $(F^{sh})_x \cong F_x$.

$F \in \mathcal{P}(X) \to F^{sh} = ?$

$$F \in \mathcal{P}(X) \Rightarrow F^{sh} = ?$$

$$(F^{sh})_x = \varinjlim_{x \in U} F^{sh}(U)$$

$$F_x = \varinjlim_{x \in U} F(U)$$

φ not injective $\Rightarrow$ $\exists U \subseteq X$ open s.t. $\varphi(U)$ not inj.

$\Rightarrow \exists x, y \in F(U)$ w/ $x \neq y$ s.t. $\varphi(U)(x) = \varphi(U)(y) \in G(U)$.

$$\varphi_x = \varinjlim_{x \in U} \left(\varphi(U) : F(U) \to G(U) \right)$$

$$\{ \underset{\in F_x \times F_x}{(a,b)} : \varphi_x(a) = \varphi_x(b) \} = eq\left(F_x \times F_x \underset{\varphi_x \circ pr_2}{\overset{\varphi_x \circ pr_1}{\rightrightarrows}} G_x \right)$$

$$\cong \varinjlim_{x \in U} eq\left(F(U) \times F(U) \underset{\varphi(U) \circ pr_2}{\overset{\varphi(U) \circ pr_1}{\rightrightarrows}} G(U) \right)$$