$\underline{Q}$: Why do both perspectives agree?

<u>Complex Multiplication</u>

Why is our comparison so difficult? Let $E/\mathbb{C}$ be elliptic curve and choose some presentation $E(\mathbb{C}) \cong \mathbb{C}/L$. Let

$\sigma \in \mathrm{Aut}(\mathbb{C}) \rightsquigarrow E^\sigma$ via
$$\begin{tikzcd} E^\sigma \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & E \arrow[d] \\ \operatorname{Spec}\mathbb{C} \arrow[r, "\sigma"'] & \operatorname{Spec}\mathbb{C} \end{tikzcd}$$
Want $E^\sigma(\mathbb{C}) = \mathbb{C}/L'$. What is L', in terms of σ?

Amazingly, we can get around this fundamental obstacle in the CM setting. Recall the Hilbert class field...

Suppose E elliptic curve $/\mathbb{C}$, $k \subseteq \mathbb{C}$ quad. imaginary, $\mathcal{O}_k \hookrightarrow \mathrm{End}(E)$, $E(\mathbb{C}) \cong \mathbb{C}/L$. Necessarily, L must be stable under mult. action of $\mathcal{O}_k$. $\mathfrak{a} \subseteq k$ fractional ideal $\rightsquigarrow$ new lattice $\mathfrak{a}L := \mathcal{O}_k$-submodule of $\mathbb{C}$ gen. by products of pairs coming from $\mathfrak{a} \times L$.

[H = Hilbert class field of k]

$\underline{\text{Thm}}$: In the above setting, suppose
$$\begin{array}{ccccc} \sigma & \longmapsto & \sigma|_H & \longleftarrow & \mathfrak{a} \\ \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} & & \rotatebox{90}{$\in$} \\ \mathrm{Aut}(\mathbb{C}/k) & \twoheadrightarrow & \mathrm{Gal}(H/k) & \cong & \mathrm{Cl}(k) \end{array}$$
Then, $E^\sigma(\mathbb{C}) \cong \mathbb{C}/\mathfrak{a}L$.

$\underline{\text{Cor}}$: $\sigma \in \mathrm{Aut}(\mathbb{C}/H) \Rightarrow E^\sigma \cong E$.

Hence, $j(E) = j(E^\sigma) = j(E)^\sigma \Rightarrow j(E) \in H \Rightarrow E$ admits model over H. [Kronecker was interested in this because, in fact, $H = k(j(E))$.]

Now work adelically to deal w/ torsion thus level structure.

[isom. from above]
$$\begin{tikzcd} \mathrm{Aut}(\mathbb{C}/k) \arrow[r, two heads] & \mathrm{Gal}(k^{ab}/k) \arrow[d] & k^\times \backslash \mathbb{A}_{k,f}^\times \arrow[l, "\overset{\text{act}}{\sim}"'] \arrow[d] \\ & \mathrm{Gal}(H/k) & k^\times \backslash \mathbb{A}_{k,f}^\times / \hat{\mathcal{O}}_k^\times \cong \mathrm{Cl}(k) \arrow[l, "\sim"'] \end{tikzcd}$$

Now suppose $E/\mathbb{C}$ elliptic curve, $k \subseteq \mathbb{C}$ quad. imaginary, $k \hookrightarrow \mathrm{End}(E) \otimes \mathbb{Q}$, $E(\mathbb{C}) \cong \mathbb{C}/L$. Then, k acts on $H_1(E(\mathbb{C}), \mathbb{Q}) \cong L \otimes_{\mathbb{Z}} \mathbb{Q}$ 1-dim k-vec. space. Also makes $L \otimes_{\mathbb{Z}} \mathbb{A}_f$ into free rank 1 module over $\mathbb{A}_{k,f} = k \otimes_{\mathbb{Z}} \mathbb{A}_f$.

$$L \otimes_{\mathbb{Z}} \mathbb{A}_f \supseteq \hat{L} = L \otimes_{\mathbb{Z}} \hat{\mathbb{Z}}$$

Given $s \in \mathbb{A}_{k,f}^{\times} \rightsquigarrow s\hat{L} \subseteq L \otimes \mathbb{A}_f \rightsquigarrow sL := s\hat{L} \cap (L \otimes \mathbb{Q})$ giving new $\mathbb{Z}$-lattice in $L \otimes \mathbb{R} \subseteq \mathbb{C}$.

Adèlic version of what we did before.

<u>Thm</u> (Main Thm of CM): In the above setting, suppose

$$\begin{array}{ccccc} \mathrm{Aut}(\mathbb{C}/k) & \twoheadrightarrow & \mathrm{Gal}(k^{ab}/k) & \xleftarrow{\text{act}} & k^{\times} \backslash \mathbb{A}_{k,f}^{\times} \\ \Psi & & \Psi & & \Psi \\ \sigma & \longmapsto & \sigma|_{k^{ab}} & \longleftarrow & s \end{array}$$

Then, $E^{\sigma}(\mathbb{C}) \cong \mathbb{C}/sL$. Moreover, the grp. isom. $\sigma : E(\mathbb{C}) \xrightarrow{\sim} E^{\sigma}(\mathbb{C})$ restricts to isom. of torsion subgrps. hence of adèlic Tate modules. We get comm. diagram

$$\begin{tikzcd} \hat{T}E \arrow[r, "\sigma", "\sim"'] \arrow[d, equal] \arrow[dr, phantom, "\circlearrowright"] & \hat{T}E^{\sigma} \arrow[d, equal] \\ \hat{L} \arrow[r, "s"'] & s\hat{L} \end{tikzcd}$$

Let's put this into action for 0-dim Shimura varieties. $k \subseteq \mathbb{C}$ quad. imaginary, $T = \mathrm{Res}_{k/\mathbb{Q}} \mathbb{G}_m$. So, $\mathbb{C} \cong k \otimes_{\mathbb{Q}} \mathbb{R}$ restricts to $h : \mathbb{C}^{\times} \xrightarrow{\sim} (k \otimes_{\mathbb{Q}} \mathbb{R})^{\times} = T(\mathbb{R})$. Then, $(T, \{h\})$ is 0-dim Shimura datum.

Consider $Sh(T, \{h\}) = T(\mathbb{Q}) \backslash \{h\} \times T(\mathbb{A}_f) \cong T(\mathbb{Q}) \backslash T(\mathbb{A}_f)$. V 1-dim k-vec. space $\Rightarrow V_{\mathbb{R}} = V \otimes_{\mathbb{Q}} \mathbb{R}$ is 1-dim vec. space / $k \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{C}$. Choose $\mathbb{Z}$-lattice $L \subseteq V$ stable under $\mathcal{O}_k$. Then,

$T(\mathbb{Q}) \backslash T(\mathbb{A}_f) \cong \{$ elliptic curve $E/\mathbb{C}$ w/ $\mathcal{O}_k \hookrightarrow \mathrm{End}(E)$ and $\mathcal{O}_k$-linear isom. $\eta : \hat{T}E \xrightarrow{\sim} \hat{L}\}$, $g \mapsto (E_g, \eta_g)$,

$E_g(\mathbb{C}) = V_{\mathbb{R}}/gL$, $\eta_g : \hat{T}E_g = g\hat{L} \xrightarrow{g^{-1}} \hat{L}$.

How $\sigma \in \mathrm{Aut}(\mathbb{C}/k)$ act on such a pair? This is basically the Main Thm of CM!

$(E_g)^{\sigma} = E_{sg}$. What is η_g^{σ}?

$$\begin{tikzcd} \hat{T}E_g \arrow[r, "\eta_g"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\circlearrowright"] & \hat{L} \\ \hat{T}E_g^{\sigma} \arrow[ur, "\eta_g^{\sigma}"'] & \end{tikzcd}$$

One sees that $(\eta_g)^{\sigma} = \eta_{sg}$!