Prismatization: $\mathrm{Sch}_{\mathbb{F}_p} \to \widehat{\mathrm{St}}_{\mathbb{Z}_p}$ (p-adic formal stack)
$X \mapsto X^{\Delta}$

$\widehat{\mathrm{St}}_{\mathbb{Z}_p} = \varprojlim_n \mathrm{St}_{\mathbb{Z}/p^n}$ (formal) inverse limit of stacks

$\mathrm{St}_{\mathbb{Z}_p} \to \widehat{\mathrm{St}}_{\mathbb{Z}_p}$ ("p-adic completion"), $X \mapsto \hat{X}$

$\mathrm{Crys}(X) := \mathrm{QCoh}(X^{\Delta})$

Taking "stacky" cohom. gives crystalline cohom.

"Zariski-locally quotient of affine scheme by flat gpoid."
"fpqc top." [work of Lurie]

__Ex 0__: $(\mathrm{Spec}\,\mathbb{F}_p)^{\Delta} = \mathrm{Spf}\,\mathbb{Z}_p \Rightarrow \mathrm{Crys}(\mathrm{Spec}\,\mathbb{F}_p) \simeq \mathrm{QCoh}(\mathrm{Spf}\,\mathbb{Z}_p)$

__Ex 1__: $(\mathbb{A}^1_{\mathbb{F}_p})^{\Delta} = \hat{\mathcal{R}}$, $\mathcal{R} = \mathrm{Cone}(\mathbb{G}_a^{\#} \xrightarrow{f} \mathbb{G}_a)$, $\mathbb{G}_a = \mathbb{G}_{a,\mathbb{Z}_p}$, $\mathbb{G}_a^{\#} \cong \mathrm{Spec}\,\mathbb{Z}_p[x, \frac{x^2}{2!}, \frac{x^3}{3!}, \dots]$
$\shortparallel$
PD hull of $0 \subseteq \mathbb{G}_a$ (closed subscheme)

__Remark__: $\exists!$ gp. scheme structure on $\mathbb{G}_a^{\#}$ s.t. f is homomorphism ($\mathbb{G}_a^{\#}$ is "smaller" or "sharper" à la Beilinson).

__Remark__: It is tempting to think of a $\mathbb{G}_a^{\#}$ as a formal nbhd (i.e., as a formal subscheme) of $\mathbb{G}_a$ but the map f is not inj. (among other problems). So, $\mathrm{Cone}(f)$ is the "stacky quotient" which we would want to take more classically but can't. $\mathcal{R}$ is a gp. stack (more precisely, a strictly comm. Picard stack).

$\mathrm{Crys}(\mathbb{A}^1_{\mathbb{F}_p})$ should involve nilpotent connections, and here we see we indeed get the same things.
This is what $\mathbb{G}_a^{\#}$-equivariance accomplishes.

(R, k, K) char. $(0,p)$ DVR
W_n scheme of length n Witt vectors

$\exists$ (explicit) isogeny $\mathcal{W}_n \to \mathcal{V}_n$ of smooth affine n-dim R-gp. schemes w/ special and generic fibers "very nice"

Special fiber: Artin-Schreier-Witt isogeny $\wp: W_{n,k} \to W_{n,k}$, $x \mapsto x^p - x$

Generic fiber: Kummer-type isogeny $\Theta: \mathbb{G}_{m,K}^n \to \mathbb{G}_{m,K}^n$, $(x_1, \dots, x_n) \mapsto (x_1^p, x_2^p x_1^{-1}, \dots, x_n^p x_{n-1}^{-1})$

s.t. __any__ p^n-cyclic fin. étale ext. of local flat R-alg.'s is obtained by base change from $\mathcal{W}_n \to \mathcal{V}_n$.

Indeterminants $\Lambda, \mathcal{U}, T \rightsquigarrow E_p(\mathcal{U}, \Lambda, T) \in \mathbb{Q}[\Lambda, \mathcal{U}][[T]]$ defined by

$$(1+\Lambda T)^{\mathcal{U}/\Lambda} \prod_{k \geq 1} (1 + \Lambda^{p^k} T^{p^k})^{1/p^k ((\mathcal{U}/\Lambda)^{p^k} - (\mathcal{U}/\Lambda)^{p^{k-1}})}$$

Artin-Hasse exponential

$$E_p(x) := \exp\left(x + \frac{x^p}{p} + \frac{x^{p^2}}{p^2} + \dots\right)$$

$$e^x = \prod_{n \geq 1} (1 - x^n)^{-\mu(n)/n} \in \mathbb{Q}[[x]]$$

$$E_p(x) = \prod_{(p,n)=1} (1 - x^n)^{-\mu(n)/n}$$

We will give three descriptions of prismatization, focusing on the affine case.

(1) Let Y be the PD hull of $X \hookrightarrow (\mathbb{A}^1_{\mathbb{F}_p})^I \hookrightarrow (\mathbb{A}^1_{\mathbb{Z}_p})^I$ ← (closed or loc. closed)

[If you want, embed X inside of an affine thing and then take a tubular nbhd.]

$(\mathbb{G}_a^\#)^I$ acts on $(\mathbb{A}^1_{\mathbb{Z}_p})^I$ and this lifts to Y. Consider $(Y/(\mathbb{G}_a^\#)^I)^\wedge$.

We think of identifying infinitesimally close pts. Embedding only necessary if X is not smooth.

This is a "coord.-based" approach.

(2) Let's axiomatize $X \mapsto X^{\Delta}$.

(a) Commutes w/ products (including infinite ones). ⟷ $\left(\prod_{i\in I} X_i\right)^{\Delta} \xrightarrow{\sim} \prod_{i \in I}(X_i^{\Delta})$ [property not data]

For free, $(\mathbb{A}^1_{\mathbb{F}_p})^{\Delta}$ must be comm. $\mathbb{F}_p$-alg. stack (since $\mathbb{A}^1_{\mathbb{F}_p}$ is comm. $\mathbb{F}_p$-alg. object).

(b) $(\mathbb{A}^1_{\mathbb{F}_p})^{\Delta} = \hat{\mathcal{R}}$.

$\mathcal{R}$ must be comm. ring stack and, moreover, comm. $\mathbb{F}_p$-alg. stack. The pt. is that $\mathbb{G}_a^\#$ is some kind of "ideal".

$\mathcal{R}$ is stack over $\mathbb{Z}_p$. [This is in some sense why crystalline cohom. even exists!]

Remark: Where do we get $\mathbb{F}_p$-alg. structure? We have $p \in \mathbb{Z}_p = \mathbb{G}_a(\mathbb{Z}_p)$, $\mathbb{G}_a^\#(\mathbb{Z}_p) \to \mathbb{G}_a(\mathbb{Z}_p)$ is inj., and all divided powers of p live in $\mathbb{Z}_p \Rightarrow p \in \mathbb{G}_a^\#(\mathbb{Z}_p)$.

(a) + (b) together (basically) determine restriction of prismatization to $\mathrm{Aff}_{\mathbb{F}_p}$. We then right Kan extend this (final object in some cat. of ext.'s). [One can give various recipes for this.]

(3) We want to think about things in terms of test objects - i.e., looking at functor-of-pts. of X^{Δ}. So, suppose $X = \mathrm{Spec}\, A$. Let $S = \mathrm{Spec}\, B$ be test object w/ B p-nilpotent (i.e., $p^n = 0$ in B for some n).

$X^{\Delta}(S) = \mathrm{Hom}_{\mathbb{F}_p}(A, \mathcal{R}(B))$. Classically, this is Hom in 2-cat. of $\mathbb{F}_p$-alg. grpoids.

As derived ring, $\mathcal{R}(B)$ only has π_0 and π_1.

Section 9 Math 2202

1. (Based on *Stewart 11.8 #6*) Consider the function $f(x, y) = e^{xy}$, and the constraint $x^3 + y^3 = 16$.

   (a) Use Lagrange multipliers to find the coordinates (x, y) of any points on the constraint where the function f could attain a maximum or minimum.

<u>Prop</u>: $\mathcal{R} = \widehat{\mathrm{Cone}}(W \xrightarrow{p} W)$, W = ring scheme of p-typical Witt vectors.

$$\begin{tikzcd} \mathbb{G}_a^{\#} \arrow[r] \arrow[d, dashed, "\exists!\cong"'] & \mathbb{G}_a \arrow[d, "\wr\|"] \\ W^{(F)} \arrow[r] & W/V(W) \end{tikzcd}$$

<u>Lemma</u>: $W/V(W)$ — Verschiebung

$W^{(F)} := \ker(F : W \to W)$ — Frobenius

Bottom Map: $W^{(F)} \to W \to W/V(W)$

$F : W \to W$ is flat and surj.

Remark: Lemma is easy using Joyal coords.

   (b) For each point you found in part (a), is the point a maximum, a minimum, both or neither? Explain your answer carefully. What are the minimum and maximum values of f on the constraint? Please explain your answers carefully.

SES : $0 \to W^{(F)} \to W \xrightarrow{F} W \to 0 \rightsquigarrow W \cong W/W^{(F)}$.

$$\begin{aligned} \mathrm{Cone}(W^{(F)} \to W/V(W)) &\cong \mathrm{Cone}(V(W) \to W/W^{(F)}) \\ &\cong \mathrm{Cone}(V(W) \xrightarrow{F} W) \\ &\cong \mathrm{Cone}(W \xrightarrow{FV=p} W) \end{aligned}$$

This construction of $\mathcal{R}$ is great because it allows us to naïvely take pts. (as ptwise cone).

   (c) The extreme value theorem which we discussed in class (See 11.7 in Stewart) guarantees that under the right circumstances, we are guaranteed to find absolute minima and maxima for a function f on a certain constraint. Explain why parts (a) and (b) don't violate the extreme value theorem.

The prismatic story of Bhatt-Scholze basically wants $(\cdot)^{\Delta} : \widehat{\mathrm{Sch}}_{\mathbb{Z}_p} \to \widehat{\mathrm{St}}_{\mathbb{Z}_p}$ by taking $\mathrm{Cone}(W \xrightarrow{3} W)$ w/ $\underbrace{2 \in W_{prim}/W^{\times}}_{=\Sigma \text{ [quotient stack]}}$ for $W_{prim} \subseteq W$ appropriate loc. closed subscheme. [Idea: W has coords. $x_0, x_1, \ldots$ and we are essentially enforcing $x_0 = 0$, $p = 0$, x_1 invertible (though not literally)]