In our setting, we have affine base $S = \operatorname{Spec} B$. Let H be affine grp. scheme $/B$.

<u>Def</u>: $H \curvearrowright Y$, $\pi: Y \to X$, is $\underline{(\text{fppf})\ H\text{-torsor}}$ if

(/X and relative to the base B)

(i) Y is quasi-proj. $/B$.

(ii) $H \times Y \xrightarrow{\sim} Y \times_X Y$, $(h,y) \mapsto (hy, y)$ [shear map]

(iii) For every affine open $\operatorname{Spec} A \subseteq X$, $\exists\, A' \in \mathrm{Alg}_A$ faithfully flat s.t. $\exists$ section

$$\begin{tikzcd} \operatorname{Spec} A' \arrow[r, "s"] \arrow[d] \arrow[dr] & Y \arrow[d, "\pi"] \\ \operatorname{Spec} A \arrow[r, hook] & X \end{tikzcd}$$

(X quasi-proj. $/B$)

<u>Remark</u>: (iii) $\Leftrightarrow$ $(\cdots)$.

[pullback the action (just use the original action on the first factor)]

$$\begin{tikzcd} H \curvearrowright Y_{x'} \arrow[r] \arrow[d] & Y \arrow[d] \\ \operatorname{Spec} A' \arrow[r, "x'"'] \arrow[ur, "s"] & X \end{tikzcd} \rightsquigarrow \begin{tikzcd} H \times \operatorname{Spec} A' \arrow[r, "\sim"] \arrow[d, "(\mathrm{id}_H, s)"'] & Y \times_X \operatorname{Spec} A' = Y_{x'} \arrow[d, "(\mathrm{id}_Y, s)"] \\ H \times Y \arrow[r, "\sim"] & Y \times_X Y \end{tikzcd}$$

(trivial H-torsor) (Fiber square)

Isom. on top of RHS is H-equivariant, and this characterizes (the data of) (iii). (ii) tells us the fibers are simply transitive H-sets, assuming they are nonempty. (iii) ensures the fibers are nonempty. ~~[Need faithful flatness so tensor product does not vanish when computing pullback]~~

Faithfully flat descent tells us: $\{\mathbb{G}_m\text{-torsors}\} \leftrightarrow \{\text{line bundles}\}$, $\{GL_n\text{-torsors}\} \leftrightarrow \{\text{rank } n \text{ vec. bundles}\}$.

(one thing is X is affine)

<u>Lemma</u>: Descent statement for torsors (yielding representability of Y in certain circumstances).

"local representability" (in some sense)

To descend just need to produce suitable descent datum. $\Rightarrow$ "global representability"

<u>Prop</u>: $\pi: Y \to X$ H-torsor of $\mathrm{Sch}_{/B}$ $\Rightarrow$ given $C \in \mathrm{Alg}_B$, $\exists$ nat. bij. $\underbrace{X(C) \xrightarrow{\sim} \{(j, \gamma) :}$ $\begin{tikzcd} Z \arrow[r, "\gamma"] \arrow[d, "j"'] & Y \\ \operatorname{Spec} C & \end{tikzcd}$ $\}$ $= (Y/H)(C)$

H-torsor, H-equivariant

[Equiv. of cat's] $\to$ equiv. of grpoids.

A priori we get a groupoid, but actually this is discrete (hence we essentially just get a set).

<u>Remark</u>: To check if two morphisms are the same, we can do so over faithfully flat cover.

<u>Thm 1.1.1</u>: S Noe. scheme, V vec. bundle of rank n. Given integer $0 < k < n$,

$Gr_S(k,V) : Sch_S \to Set$, $(T \xrightarrow{f} S) \mapsto \{$vector bundle quotients $V_T = f^* V \to Q$ of rank $k\}$

is rep. by proj. S-scheme.

$$\begin{tikzcd} F_s & F \arrow[l, squiggly] \\ X_s \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & X \arrow[d] \\ \operatorname{Spec} k(s) \arrow[r, "s"'] & S \end{tikzcd}$$

[Maybe "more geometric" to interpret as derived intersection.]

Given $X \to S$ proj. morphism, $\mathcal{O}_X(1)$ rel. ample, $s \in S$, <u>Hilbert polyn.</u> of F_s is $P_{F_s}(z) := \chi(X_s, F_s(z))$

where $F_s(z) := F_s \otimes \mathcal{O}_{X_s}(n)$. [Not sure about precisely what z means here.]

<u>Fact</u>: $P_{F_s}(z) \in \mathbb{Q}[z]$ and $P_{F_s}(z) = h^0(X_s, F_s(z))$ for $z \gg 0$.

Remarkably, Hilbert polyns. measure flatness of coherent sheaves!

<u>Thm 1.1.2</u>: $X \to S$ proj. morphism of Noe. schemes, $\mathcal{O}_X(1)$ rel. ample line bundle on X. Given $P \in \mathbb{Q}[z]$,

$Hilb^P(X/S) : Sch_S \to Set$, $(T \to S) \mapsto \{$subschemes $Z \subseteq X_T$ flat and fin. pres. $/T$ s.t. $Z_t \subseteq X \times_S k(t)$ has Hilbert polyn. T $\forall t \in T\}$ is rep. by proj. S-scheme.

<u>Thm 1.1.3</u>: $X \to S$ proj. morphism of Noe. schemes, $\mathcal{O}_X(1)$ rel. ample line bundle $/X$, $F \in Coh(X)$. Given $P \in \mathbb{Q}[z]$,

$Quot^P(F/X/S) : Sch_S \to Set$, $(T \to S) \mapsto \{$coh. quotients $F_T \to Q$ on X_T of fin. pres. s.t. $Q|_{X \times_S k(t)}$ on $X \times_S k(t)$ has Hilbert polyn. P $\forall t \in T\}$ is rep. by proj. S-scheme.

<u>Q</u>: How do these help us understand $\underbrace{\overline{\mathcal{M}}_g}_{\text{stable curves}}$ and $\underbrace{V^{ss}_{r,d}}_{\text{semi-stable vec. bundles}}$? [Invariant here are: genus, rank, degree]

<u>Thm A</u>: Moduli space $\overline{\mathcal{M}}_g$ of stable curves of genus $g \geq 2$ is smooth proper irred. Deligne-Mumford stack of dim $3g-3$ admitting proj. coarse moduli space.

$Z(s)$ vanishing locus of section, which is affine in our case so has no higher coh.

so $H^0(\mathcal{O}(1)) \rightsquigarrow$ SES $0 \to \mathcal{O}(-1) \xrightarrow{s} \mathcal{O} \to \mathcal{O}_{Z(s)} \to 0$

$H^0(K(l))$ depends on l, but $H^0(K(l) \otimes \mathcal{O}_{Z(s)})$ does not! This is great, and gives us a growth factor for comparing Euler characteristics of twists.

The fact that we are playing our coh. game over a curve means that we have to deal w/ possibly nonzero genus.

Global generation of twist of kernel $\Rightarrow$ we get subfunctor of a suitable Grassmannian

Thm (Flattening Stratification): We can stratify coherent sheaves over projective space w/ a factoring property based on flatness and controlled by Hilbert polyns.

Hilbert polyn. is actually polyn. because we can pick sections avoiding pts. ...

Q: How far is a general coherent sheaf from being a vector bundle? (depends on fiber-by-fiber criterion for flatness)
- Vector bundle locus over a projective scheme should be an open guy
[Mumford - Lectures on Curves on a Surface]

Next time: deformation theory

$C \to \tilde{C}$ square zero ext. $\Rightarrow$ $\mathrm{Bun}_n(C) \to \mathrm{Bun}_n(\tilde{C})$ is ess. surj. (depends fundamentally on the fact that we are working over a curve, so H^2 vanishes) $\Longleftrightarrow$ Bun_n is formally smooth (much nicer than working w/ some schematic presentation).

Theorem 4. $H^*(BU;\mathbb{Z}) \cong \mathbb{Z}[c_1, c_2, \dots]$ *with* $|c_k| = 2k$.

We define $c_k(E) := f_E^* c_k$ for $f_E : M \to BU_n \hookrightarrow BU$ (unique up to homotopy) representing E. One advantage of this approach is that it allows us to define characteristic classes for more general principal G-bundles.

Deformation Theory and Dimension [def. theory = lifting across nilpotent thickenings]

$R \in \mathrm{CAlg}_B$ smooth of rel. dim n (i.e., R is finitely pres., formally smooth, and $\Omega^1_{R/B}$ is proj. of rank n).

$$\begin{tikzcd} & C \arrow[r, "j"] & \bar{C} \\ & R \arrow[ul, "f"] \arrow[u, "\bar{f}"'] & \end{tikzcd}$$

$j: C \to \bar{C}$ square zero ext. in CAlg_B $(I = \ker j) \Rightarrow \mathrm{Lift}(\bar{f}; C \xrightarrow{j} \bar{C}) \underbrace{\simeq}_{\text{canon. for } \underline{\text{split}} \text{ square zero ext.}} \mathrm{Der}_B(R, I) \simeq \mathrm{Hom}_R(\Omega^1_{R/B}, I)$

Q: What can be said about the grpoid morphism $\mathrm{Bun}_n(C) \to \mathrm{Bun}_n(\bar{C})$?

When we pass from a module to its endomorphisms, automorphisms go to conjugation maps.

Keerthi gives a very nice explanation for why H^2 arises in deformation theory (specifically, when deforming sheaves). Choices are involved and we pass to Čech cohom. This probably has a better feel to it in the derived setting.

$0 \to \mathcal{E}_I \to GL_{n,X_C} \to GL_{n,X_{\bar{C}}} \to 0$ fppf sheaves of groups (not ab, but crucially $\mathcal{E}_I$ is)

$\rightsquigarrow \cdots \to H^1(GL_n(X_C)) \to H^1(GL_n(X_{\bar{C}})) \to H^2(\mathcal{E}_I) \to \cdots$ ← [Need nonab. cohom. theory.]

Of course, to get a well-defined (and make sense of this) LES like this we need gerbes (gerbes?), more or less by definition.

$S \in \mathrm{Sch}$ base scheme, $j: C \to \bar{C}$ square-zero ext. of (affine) S-alg.'s, G grp. scheme $/S$ (possibly w/ some conditions),

$X \in \mathrm{Sch}_S$

$k := \text{"}\ker\text{"}(G_{X_C} \to G_{X_{\bar{C}}}) \rightsquigarrow 0 \to k \to G_{X_C} \to G_{X_{\bar{C}}} \to 0$ SES of fppf sheaves of grps.

Nonab. cohom. theory and gerbes