(i) $\Leftrightarrow$ $\forall \operatorname{Spec} B \to X$

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$\exists \mathcal{U} \in \mathfrak{U}$ s.t. $\operatorname{Spec} B \times_X \mathcal{U} \neq \emptyset$

<u>Thm</u>: Let $X = \operatorname{Spec} A$ be nonempty and $\mathfrak{U} = \{(\mathcal{U}, i_{\mathcal{U}} : \mathcal{U} \hookrightarrow X)\}$ a collection of open embeddings. TFAE:

(i) $\mathfrak{U}$ is an open cov.

(X is quasicompact or qc.)

(ii) $\exists$ finite subcollection $\mathfrak{U}' \subseteq \mathfrak{U}$ s.t. $\mathfrak{U}'$ is a cov.

(iii) Let $x \in \operatorname{Hom}(\operatorname{Spec} k, X)$ for k a field. Then, $\exists \mathcal{U} \in \mathfrak{U}$ s.t. x factors through $i_{\mathcal{U}}$.

(iv) For each $\mathcal{U} \in \mathfrak{U}$, $\mathcal{U} = X \setminus Z_{\mathcal{U}}$ w/ $Z_{\mathcal{U}} = \operatorname{Spec} A/I_{\mathcal{U}}$.

Then, $\sum_{\mathcal{U} \in \mathfrak{U}} I_{\mathcal{U}} = A$.

<u>Pf</u>: Let $f \in \operatorname{Hom}_{\text{Space}}(\operatorname{Spec} B, X)$ and $\mathcal{U} \in \mathfrak{U}$. Then,

$$\operatorname{Spec} B \times_X \mathcal{U} \cong \operatorname{Spec} B \times_X (\operatorname{Spec} A \setminus \operatorname{Spec} A/I_{\mathcal{U}})$$
$$\cong \operatorname{Spec} B \setminus \operatorname{Spec} B/I_{\mathcal{U}}B$$

$$\operatorname{Hom}_{\text{Space}}(\operatorname{Spec} C, \operatorname{Spec} B \times_X \mathcal{U}) \cong \{\varphi \in \operatorname{Hom}_{\text{CRing}}(B, C) : \varphi(I_{\mathcal{U}}B)C = C\}.$$

$$(X \setminus Z)(T) \neq X(T) \setminus Z(T)$$

$$\cap|$$

$$X(T) \simeq \mathrm{Hom}_{Space}(\operatorname{Spec} T, X) \ni x$$

$$\begin{tikzcd} = \emptyset \arrow[r] \arrow[d] \arrow[dr, phantom, "\ulcorner"] & Z \arrow[d] \\ \operatorname{Spec} T \arrow[r, "x"'] & X \end{tikzcd}$$

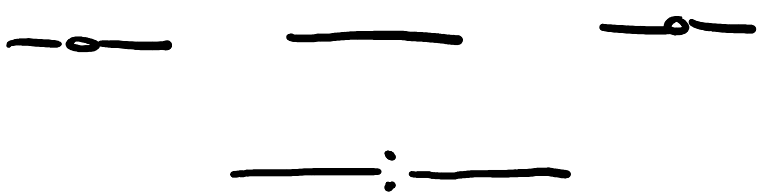
$X = \operatorname{Spec} A$

$Z = \operatorname{Spec} A/f$

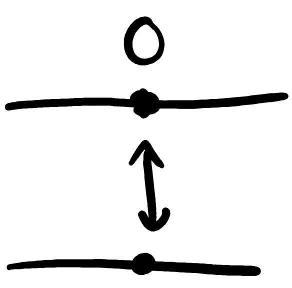
$U = \operatorname{Spec} A_f$

$$Z(T) = \mathrm{Hom}_{CRing}(A/f, T)$$

$$U(T) = \mathrm{Hom}_{CRing}(A_f, T)$$



$\mathbb{A}^1_{\mathbb{Z}} = \operatorname{Spec}\mathbb{Z}[t]$



$\mathbb{A}^1$

$\mathbb{A}^1$

$0 = \operatorname{Spec}\mathbb{Z}[t]/(t)$

$g \in \mathrm{Hom}_{Space}(Y, X)$. Saying this factors through i_U means

$$Y \overset{\exists}{\dashrightarrow} U$$

If $Y = \operatorname{Spec} C$ then g corresponds to

$$\begin{tikzcd} Y \arrow[r, dashed] \arrow[dr, "g"'] & \mathcal{U} \arrow[d, hook] \\ & X \end{tikzcd}$$

If $Y = \operatorname{Spec} C$ then g corresponds to $\psi \in \operatorname{Hom}_{CRing}(A, C)$ and the factoring condition says $\psi(I_{\mathcal{U}})C = C$.

$(i) \Rightarrow (iv)$: Let $B := A / \sum_{\mathcal{U} \in \mathfrak{U}} I_{\mathcal{U}}$. We claim $B = 0$, this is the same as $\operatorname{Spec} B = \emptyset$. Given $\mathcal{U} \in \mathfrak{U}$,

by construction $I_{\mathcal{U}} B = 0 \Rightarrow \operatorname{Spec} B / I_{\mathcal{U}} B \cong \operatorname{Spec} B$.
Hence, $\operatorname{Spec} B \times_X \mathcal{U} = \emptyset$. Thus, $\operatorname{Spec} B = \emptyset$.

$(iv) \Rightarrow (iii)$ Let $x : \operatorname{Spec} k \to X$ w/ k a field.

$x \sim \varphi : A \to k$ map of rings. We need $\varphi(I_{\mathcal{U}})k = k$

for some $\mathcal{U} \in \mathfrak{U}$. This is the same as $\varphi(I_{\mathcal{U}}) \neq 0$.

But, $\varphi(1_A) = 1_k$, We also need $\sum_{\mathcal{U} \in \mathfrak{U}} I_{\mathcal{U}} = A$.

$(iii) \Rightarrow (i)$: Let $f : \operatorname{Spec} B \to X$ w/ $\operatorname{Spec} B \neq \emptyset$. $B \neq 0$

means B has a maximal ideal $\mathfrak{m} \rightsquigarrow$ closed pt.

$\operatorname{Spec} B/\mathfrak{m} \hookrightarrow \operatorname{Spec} B$.

$\operatorname{Hom}(\operatorname{Spec} B/\mathfrak{m}, \operatorname{Spec} B \times \mathcal{U}) \cong \{ \varphi \in \operatorname{Hom}_{CRing}(B, B/\mathfrak{m}) :$

$$\mathrm{Hom}_{Space}(\mathrm{Spec}\, B/\mathfrak{m}, \mathrm{Spec}\, B \times_X \mathcal{U}) \simeq \{ \varphi \in \mathrm{Hom}_{CRing}(B, B/\mathfrak{m}) :$$

$$\varphi(I_{\mathcal{U}} B)\, B/\mathfrak{m} = B/\mathfrak{m} \}$$

$$\| $$

$$(\mathrm{Spec}\, B \times_X \mathcal{U})(B/\mathfrak{m})$$

$$B \twoheadrightarrow B/\mathfrak{m} \rightsquigarrow \mathrm{Spec}\, B/\mathfrak{m} \to \mathrm{Spec}\, B \xrightarrow{f} \mathrm{Spec}\, A \quad \sim \varphi : A \to B$$

$$\sim \psi : A \to B/\mathfrak{m}$$

ψ factors though φ. Hence, $\psi(I_{\mathcal{U}}) B/\mathfrak{m} = B/\mathfrak{m}$.

$\Rightarrow \varphi(I_{\mathcal{U}} B) B/\mathfrak{m} = B/\mathfrak{m}$.

$$I_{\mathcal{U}} \trianglelefteq A \qquad \psi(I_{\mathcal{U}}) = \varphi(I_{\mathcal{U}} B) \text{ in } B/\mathfrak{m}$$

(iv) $\Rightarrow$ (ii): $\sum_{\mathcal{U} \in \mathfrak{U}} I_{\mathcal{U}} = A \Rightarrow 1_A = f_1 + \cdots + f_n$ for $f_i \in I_{\mathcal{U}_i}$ w/ $\mathcal{U}_i \in \mathfrak{U}$. Hence, $\{ \mathcal{U}_1, \ldots, \mathcal{U}_n \}$ is an open cov. of X because $\sum_{1 \le i \le n} I_{\mathcal{U}_i} = A$. $\square$

One implication is that every cover $\mathfrak{U}$ has a subcovering

$\{D(I_1), \ldots, D(I_n)\}$. $I_1, \ldots, I_n$ may not be fin. gen.,

but we can still take a collective set of generators $\{f_t\}$.

Hence, we get covering $\underbrace{\{D(f_1), \ldots, D(f_c)\}}_{\text{principal open covering}}$.