Repeated Contracting without Commitment

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Abstract

I study a dynamic model of monopoly sales in which a monopolist without commitment power interacts with a consumer whose valuation is private. I characterize equilibria of this game and show how the seller's strategy varies with initial beliefs. I find that the seller's payoffs under spot contracting can be higher than under commitment with renegotiation and that random delivery contracts can improve payoffs beyond posted prices.

Keywords: Commitment; Mechanism design; Posted prices; Renegotiation; Spot contracting

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1 Introduction

This paper provides a solution to a classic problem in economics: how a monopolist sells a perishable good to a buyer with private valuation when the monopolist cannot commit to future prices. The interaction involves a finite time horizon, and the consumer's valuation takes one of two possible values. I fully characterize equilibria that are best from the point of view of the seller and show that several results from the literature are incorrect.

In the model I study, a monopolist seller (she) of a perishable good interacts with a buyer (he) whose private valuation can be either high or low. This interaction is finitely repeated. The seller offers a price in each period and the buyer chooses whether or not to purchase, leading the seller to update her beliefs. I contrast this spot contracting setting, in which the seller cannot commit to price offers in the future, to the full commitment and commitment with renegotiation settings whose solutions are well known. The spot contracting model is equivalent to the "rental model under non-commitment" that was presented in Hart and Tirole (1988). Limiting commitment has the effect of delaying the seller's learning. Identifying a low valuation buyer leads the seller to charge him a low price in all periods, reducing the surplus which can be extracted from high valuation buyers. Spot contracting introduces an additional reverse incentive compatibility constraint in which low types have the incentive to purchase at a low price and then never interact with the seller again. Optimal equilibria for the seller take both of these effects into account to determine the optimal sequence of prices to offer.

In equilibria that are optimal from the point of view of the seller, there are three outcomes that can occur in a given period. When the buyer is unlikely to have high valuation, the seller pools both types of buyers and charges a low price. When the buyer is very likely to have a high valuation, the seller charges a high price at which a high valuation buyer either purchases or randomizes between purchasing or not. Finally, for intermediate beliefs, the seller sometimes sets a low price at which a low valuation buyer randomizes the decision to purchase and at which a high valuation buyer either purchases or randomizes.

The fact that inducing low valuation buyers to mix can be optimal has been well-documented by the more recent literature on mechanism design with imperfect commitment (see, for instance, Bester and Strausz (2001) and Bester and Strausz (2007)), but it contradicts results from Hart and

Tirole (1988), which claimed that low valuation buyers never mix. In the spot contracting setting, inducing low valuation buyers to mix can increase profits for the seller because it can lead to higher posteriors (which are associated with higher continuation profits) at a relatively small cost in the current period. However, low valuation buyers have no strict incentive to mix at the rate that is optimal for the seller, and my results do not rule out the possibility that equilibria without low types mixing exist even when an equilibrium with low types mixing is optimal for the seller.

There are some cases in which a spot contracting monopolist makes strictly higher profits than a monopolist who commits with renegotiation. This contradicts Proposition 6 of Hart and Tirole (1988), which claims that in this setting spot contracting is always worse. Payoffs in both settings are lower than under full commitment due to the seller's opportunistic behavior in later periods. In the cases in which spot contracting gives higher payoffs than commitment with renegotiation, the extra constraints actually restrict this opportunistic behavior in future periods, thus allowing the monopolist to make higher payoffs from the point of view of the first period.

I also show that restricting a seller to posted prices can lower the seller's profits under spot contracting. Sometimes, when a low valuation buyer randomizes, a high valuation buyer strictly prefers to purchase. When this is the case, the seller can improve payoffs by allowing low valuation buyers to randomize between a guaranteed delivery contract and a random delivery contract. This increases the seller's profits without violating the high valuation buyer's incentive compatibility constraints.

This paper's results show that giving a seller the ability to commit subject to renegotiation can lower the seller's payoffs. This relates to previous work which shows how improving contracting within or across periods can lead to worse outcomes (Baker, Gibbons, & Murphy, 1994; Schmidt & Schnitzer, 1995; Kovrijnykh, 2013; Breig, 2019). However, the mechanisms in this previous work revolve around how improving commitment increases the payoffs of the punishment equilibrium in an infinitely repeated game, making deviations from an implicit contract more tempting. Implicit contracts play no role in this paper because the interaction is finitely repeated.

The specific spot contracting model which is studied in this paper was first solved by Schmidt (1993) in an equivalent setting for the case in which both the buyer's and seller's discount rates are equal to one. This assumption on the discount rates implies that the reverse incentive compatibility constraint is *always* binding. This simplifies the problem and leads to equilibria in which low types never mix and spot contracting payoffs are equal to commitment with renegotiation payoffs. Neither of these results hold when allowing for discount rates strictly less than one. Devanur, Peres, and Sivan (2019) also study the finitely repeated version of this game with no discounting, but focus on continuous distributions of types.

A number of other papers study closely related models. Beccuti and Möller (2018) studies a case in which the seller is more patient than the buyer and is one of the few other papers that finds random delivery to be optimal. Gerardi and Maestri (2020) studies a related model of employment contracting with limited commitment but assumes that if the employee rejects contract offers in a particular period, they can never again interact with the employer. Beccuti (2020) uses a mechanism design approach in the same economic setting as this paper and shows that when discount factors are sufficiently high, the amount of learning that is possible in a given period (the spread of posteriors) is limited. Because of these limits to the principal's learning, random delivery dominates screening with posted prices only when full pooling dominates random delivery. These results complement the example I give of random delivery improving payoffs because the example relies on intermediate discount factors.

The remainder of the paper proceeds as follows: Section 2 of this paper presents the underlying economic framework. Section 3 characterizes optimal equilibria of the spot contracting setting for the seller and shows that the seller can sometimes improve profits by using a random delivery contract. Section 4 shows that the seller can earn higher profits under spot contracting than commitment with renegotiation. Section 5 is a conclusion. Proofs of the main results are in Appendix A while proofs of additional lemmas and corollaries can be found in Online Appendix A.

2 Model

A seller (she) and buyer (he) interact for $T < \infty$ periods. In each period, the seller can produce a perishable consumption good at a normalized cost of 0.[1] The buyer has unit demand for the good in each period. The buyer's value of consumption is $b \in \{\underline{b}, \bar{b}\}$, where $0 < \underline{b} < \bar{b}$. This valuation is constant, known to the buyer, and unknown to the seller. The probability that a buyer is of the

[1]The fact that the good is perishable makes the problem different from the literature on bargaining over durable good pricing (Skreta, 2006, 2015; Doval & Skreta, 2020). In particular, the reverse incentive compatibility constraint discussed below can only become binding if the buyer and seller continue interacting after the buyer's type is revealed.

high type is $P(b = \bar{b}) = \mu$. Both the buyer and the seller have discount factor δ which is strictly between $\frac{1}{2}$ and 1.[2]

The seller's strategy space in any given period depends on the commitment structure she faces. This paper will compare equilibria of the *spot contracting* game to equilibria of the *full commitment* and *commitment with renegotiation* games.

In each period t of the spot contracting game, the seller can post a price p_t.[3] The buyer can then choose whether or not to purchase in that period at that price. The seller cannot make any commitments about future periods.

In the full commitment game, the seller makes a single offer at the beginning of the first period. The offer specifies consumption levels $q_\tau \in \{0, 1\}$ and prices p_τ for all periods $\tau \in \{1, \ldots, T\}$. The buyer can choose to accept or reject this offer. If he accepts, then the offer's consumption and prices are implemented. If the buyer rejects the offer, he consumes nothing and pays nothing in all periods.

In the commitment with renegotiation game, the seller makes an offer in each period t. The offer specifies consumption levels $q_{t,\tau} \in \{0, 1\}$ and prices $p_{t,\tau}$ for the current and all future periods, $\tau \in \{t, \ldots, T\}$. The buyer can choose to accept or reject this offer. If he accepts, then the offer's consumption and price in period t are implemented. If he rejects the current offer, then the consumption level and price for period t from the most recently accepted offer are implemented.[4]

The seller's payoffs are the expected discounted sum of payments which are implemented in equilibrium. The buyer's payoffs are the expected discounted sum of consumption utility minus payments.

I will study the Perfect Bayesian Equilibria of these games and focus on the equilibria which are optimal from the point of view of the seller. The equilibrium concept requires that given beliefs about the other player's strategy and the buyer's type, both the buyer and the seller are maximizing their expected payoffs. The seller's beliefs about the buyer's type is required to satisfy Bayes' rule

[2]For discount rates that are less than or equal to $\frac{1}{2}$, the reverse incentive compatibility constraint never binds, so the spot contracting equilibrium is the same as the commitment with renegotiation equilibrium.

[3]As will be discussed in Section 3.1, restricting the seller to posted prices is not without loss of generality.

[4]If no offer has yet been accepted (which is always the case in the first period) and the offer is rejected, then the implemented consumption and price in period t are both zero.

whenever possible and only update when the buyer takes an action.[5]

In Sections 3 and 4, I document properties of equilibria of the spot contracting game for particular parameterizations. Example 1 assumes that $T = 4$, $\delta = 0.6$, $\underline{b} = 1$, and $\bar{b} = 3$. Example 2 assumes that $T = 4$, $\delta = 0.7$, $\underline{b} = 1$, and $\bar{b} = 2$. I specify the strategies which are part of the seller-optimal equilibria for these examples in Online Appendices B.1 and B.2, respectively.

3 Results

The consequences of allowing for renegotiation in screening problems are well known. Hart and Tirole (1988) describes the equilibrium of the commitment with renegotiation game in their Proposition 2 and Theorem 1. These results are restated for comparison in Section 4 of this paper.

One of the key characteristics of the equilibrium of the commitment with renegotiation game is that as compared to a game with full commitment, allowing for renegotiation slows down the seller's learning process. The seller cannot extract full information about the buyer's type in the first period because she would then have an incentive to offer the product to low valuation buyers at a low price. This in turn prevents the seller from charging the high valuation buyer a high price in the first period, because he knows that he could wait and receive the good in all future periods at a low price. As such, the seller engages in what is known as "Coasian Bargaining," which involves charging a series of high prices that high valuation buyers accept with some probability in each period. For T large enough, all high valuation buyers purchase the good by a period that is independent of T, and the low valuation buyers purchase in subsequent periods as well.

The equilibrium of the commitment with renegotiation game cannot be implemented under spot contracting when T and μ are high enough. In the spot contracting game, when the seller is certain that the buyer is a high type, she charges the high type's valuation in every period. Thus, in any period that the buyer reveals his type, he needs to be provided with his full information rents. In some cases, this is not implementable because the price that the seller would need to charge to provide the high type buyer with sufficient information rents is lower than the low type's valuation. If the seller offered such a price, the low type buyer has a strict incentive to purchase, making it

[5]If the seller ever observes an event which should happen with probability zero in equilibrium, I assume that she updates her beliefs to one. This leads to continuation payoffs of zero for both types of buyer.

impossible to separate high and low valuation buyers. Thus, the seller faces a constraint that the price she charges cannot be strictly less than $\underline{b}$. This is sometimes known as the reverse incentive compatibility constraint or the "take-the-money-and-run" constraint (Laffont & Tirole, 1987).

The first step towards characterizing equilibria of the spot contracting game involves ruling out some of the potential outcomes in a given period by showing that they are either infeasible or suboptimal. Lemma 1 guarantees that regardless of the continuation equilibrium, the seller need only compare the optimal payoffs from three types of outcomes.

Lemma 1 *In any period of any equilibrium of the spot contracting game, the seller posts a price such that either*

1. *both types purchase with probability one,*
2. *low types purchase with probability strictly between zero and one, and high types purchase with probability strictly greater than zero, or*
3. *low types do not purchase, and high types purchase with probability strictly greater than zero.*

This lemma uses the fact that I restrict the seller to offering posted prices. If the seller had a broader set of options available to her (for instance, offering contracts with random delivery), the optimal set of outcomes in a period could be quite different. In fact, random delivery can relax incentive constraints or strictly improve payoffs as will be discussed in Section 3.1.

An equilibrium of the spot contracting game that is optimal for the seller will involve the seller using the option from Lemma 1 that gives her the highest payoffs. I will show cases in which each of the options are optimal. In finding optimal equilibria, I construct the payoffs that the seller receives if she optimally implements each of these options.[6] With the simplification given by Lemma 1, I can characterize equilibria of the spot contracting game. This characterization holds for all histories,

[6]To be more precise, the payoffs for low types mixing that I describe in Lemma A.2 and show in Figures 1 and 2 are those that arise from low types purchasing with a probability that is not "too close" to one. I formalize this in the proof of Lemma A.2 and show that this approach does not rule out any equilibria that would be optimal for the seller, given that I am also considering full pooling equilibria. This restriction is for expositional purposes, in order to not conflate low types mixing and full pooling, because payoffs from the former approach payoffs from the latter as the probabilities that each type purchases approach one.

not just those on the equilibrium path.

Proposition 1 *There exist numbers $\underline{\mu}_t$ and $\hat{\mu}_t$, $0 < \underline{\mu}_t \leq \hat{\mu}_t < 1$ such that in period t of an optimal equilibrium of the spot contracting game,*

- *both types purchase with probability one for $\mu_t < \underline{\mu}_t$,*
- *low types purchase with probability strictly greater than zero, and high types purchase with higher probability than low types for $\mu_t \in (\underline{\mu}_t, \hat{\mu}_t)$, and*
- *the low types do not purchase and the high types purchase with probability strictly greater than zero for $\mu_t > \hat{\mu}_t$.*

The proof uses backwards induction starting from period $T-1$. It shows that if continuation values starting from period $t+1$ (as a function of beliefs in period $t+1$) take a particular form, then an optimal outcome in that period takes the form described in the proposition and continuation values starting from period t take the same form as those starting from period $t+1$. While the outcome in which both types purchase is straightforward, the other two potential outcomes merit a discussion.[7][8]

For high enough beliefs in any period t of the spot contracting game, only high types purchase with positive probability. This can involve either high types mixing or high types purchasing with probability one, but in the remainder of this paper it will often be referred to as "high types mixing" in order to be more concise. If the seller observes the buyer purchase the item, she updates her beliefs to 1, while she updates her beliefs downward otherwise. Given that upon purchasing the seller knows the type of the high valuation buyer, she charges $\bar{b}$ in all future periods. Thus, the buyer must receive surplus in period t exactly equal to the continuation value he would receive from choosing not to purchase, which pins down the price. The tradeoffs that the seller faces between higher prices and higher continuation values are very similar to those found in the commitment with renegotiation case discussed by Hart and Tirole (1988).

[7] Despite important differences between the models, the *form* of the equilibria here are substantively similar to those that were found in Laffont and Tirole (1987).

[8] Solving for both the "high types mixing" and "low types mixing" equilibria involve finding rates of mixing which maximize expected payoffs for the seller given equilibrium constraints. This could alternatively be framed as optimally choosing *posterior beliefs* as in Doval and Skreta (2021).

When current beliefs are low it may be impossible to have high types mix and low types not purchase. This is because upon observing the buyer not purchase, beliefs must fall even further, which leads to higher continuation values for the high valuation buyers. If these continuation values are high enough, then the price the seller would have to charge in the current period to incentivize high types to purchase would need to be lower than $\underline{b}$, violating the reverse incentive compatibility constraint.

For some intermediate levels of beliefs, it can be optimal for the seller to have low valuation buyers mix between purchasing and not purchasing while high valuation buyers either purchase the item with probability one or mix.[9] The tradeoffs that the seller faces relative to full pooling with this type of allocation are straightforward. The seller loses profits in the current period because not all buyers are purchasing. The benefits to the seller come from mixing over continuation payoffs, which are increasing and convex in beliefs.[10]

The optimal payoffs in each period can be calculated by taking the upper envelope of payoffs from the three possible outcomes. In the proof of Proposition 1, I show that when beliefs are low enough, pooling dominates, while when beliefs are high enough, having high valuation buyers mix or purchase with probability one dominates. It *can* be the case that for intermediate beliefs, having low types mix maximizes payoffs.

Figure 1 shows the payoffs from each possible outcome when $T = 3$, $\delta = 0.6$, $\underline{b} = 1$, and $\bar{b} = 3$. This corresponds to the subgame which begins in the *second* period of the spot contracting game in Example 1.[11] In this case, having low types mix is never beneficial to the seller. For beliefs that are low enough (less than $\frac{1}{3}$) the seller pools all buyers at price $\underline{b}$ in the first period. When beliefs are higher than $\frac{1}{3}$, she either partially or fully separates buyers in the first period.

[9] For the low valuation buyer to be willing to mix, the price that is charged must be $\underline{b}$. This implies that the high type's incentive compatibility constraints must be satisfied using continuation values. This leads to restrictions on the probabilities of each type purchasing, because continuation values for the high valuation buyer depend on the seller's beliefs in the next period. This is discussed further in the proof of Lemma A.2 in Online Appendix A.

[10] A natural question is why it is never beneficial to implement this type of mixing under commitment with renegotiation. The reason is that for any "low mixing" outcome that is feasible, there is a "high mixing" outcome in which the posteriors upon observing the buyer not purchase are the same as the "low mixing" outcome, but the posteriors upon observing the buyer purchase are 1. This increases efficiency while keeping payoffs for the buyer the same, leading to higher profits for the seller. This is proven in Proposition 5 of Hart and Tirole (1988).

[11] The equilibrium associated with these parameter values is fully described in Online Appendix B.1.

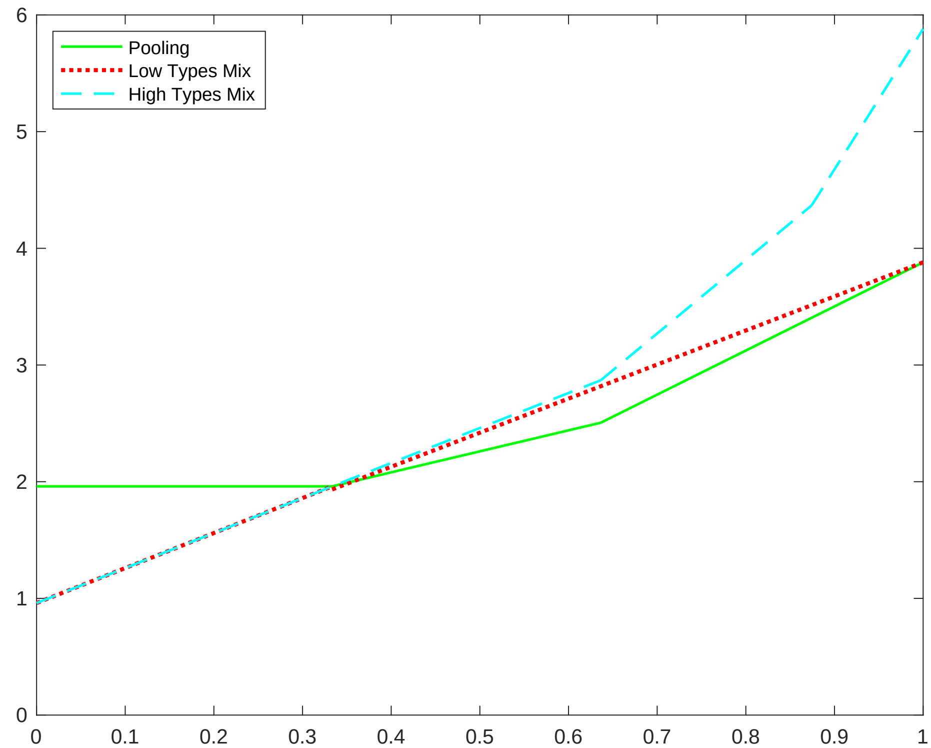

Figure 1: Payoffs for the seller in the first period as a function of μ when $T = 3$, $\delta = 0.6$, $\underline{b} = 1$, and $\bar{b} = 3$ for various potential outcomes. Pooling agents is optimal for low beliefs, while having high types mix is optimal for high beliefs.

Figure 2 shows the payoffs from each possible outcome for Example 1, in which $T = 4$, $\delta = 0.6$, $\underline{b} = 1$, and $\bar{b} = 3$. It should be noted that the upper envelope of payoffs from Figure 1 form the continuation payoffs to the seller in the first period of a four period game (for instance, the "Pooling" payoffs in Figure 2 are the upper envelope from Figure 1 multiplied by $\delta = 0.6$ and added to $\underline{b} = 1$). In this case, having low types mix dominates the other options for intermediate levels of beliefs.

The equilibrium paths described by Proposition 1 can involve randomization on the part of the seller and both types of buyer. A buyer that randomizes in any period is made indifferent by the price which is being charged and the continuation equilibrium that he expects. When the seller is indifferent between two continuation equilibria, she can randomize between prices that are

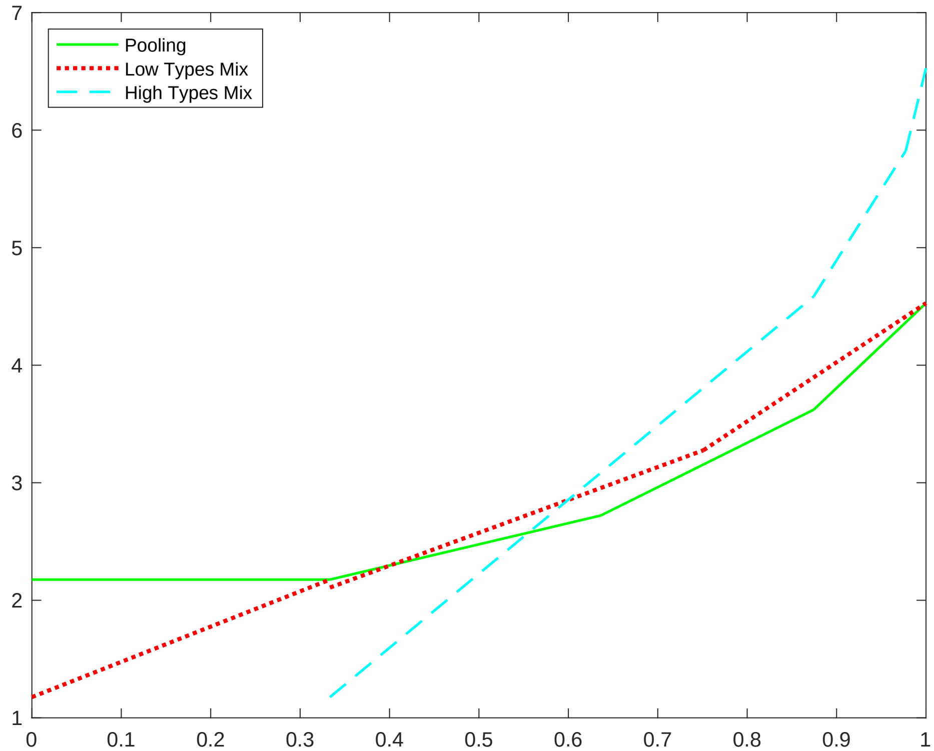

Figure 2: Payoffs for the seller in the first period as a function of μ when $T = 4$, $\delta = 0.6$, $\underline{b} = 1$, and $\bar{b} = 3$ for various potential outcomes. Pooling agents is optimal for low beliefs, having low types mix is optimal for intermediate beliefs, and having high types mix is optimal for high beliefs.

consistent with these equilibria. Randomizing in this way allows her to give high valuation buyers any convex combination of continuation values that are consistent with these equilibria. In this way, she can credibly promise in earlier periods that the high valuation buyer will receive that particular convex combination of continuation values. This is particularly relevant in equilibria in which both low types and high types mix. To induce low types to mix, the price being charged must be $\underline{b}$. Thus, a high valuation buyer's incentive compatibility constraints must be satisfied by promising particular continuation values that can only be provided through randomizing between continuation equilibria. Gul, Sonnenschein, and Wilson (1986) shows that this type of randomization is also used by durable good monopolists without commitment, and Hart and Tirole (1988) demonstrates that the outcome of the commitment with renegotiation game described here coincides with that of the durable good model without commitment.

The existence of an equilibrium in which it is optimal for the seller to have low types mix while high types purchase with probability one leads to the first contradiction between the results here and those found in Hart and Tirole (1988).

Observation 1 *Suppose that s satisfies $\delta + \cdots + \delta^{T-s} > 1$. Contrary to Lemma 1 of Hart and Tirole (1988), there can exist a $t \leq s$ such that $\mu_{t+1} < \frac{\underline{b}}{\bar{b}}$ with positive probability.*

Lemma 1 of Hart and Tirole (1988) states that in the spot contracting setting, beliefs stay above the cutoff $\frac{\underline{b}}{\bar{b}}$ with probability one until a period which is a fixed distance from the end of the game. With the parameters in Example 1, this means that beliefs remain above $\frac{1}{3}$ until period three. This is why low types mixing while high types are purchasing contradicts the lemma; when the seller observes the buyer not purchase the good in period one, her beliefs fall to zero in period two.

The problem with the proof of Lemma 1 of Hart and Tirole (1988) lies with the claim that low valuation buyers must purchase with probability one in period t when $p_t \leq \underline{b}$ ($r_t \leq \underline{b}$ in their notation). This is true for $p_t < \underline{b}$, but low valuation buyers will be indifferent when $p_t = \underline{b}$. The example shows that it can be valuable to the seller for low valuation buyers to randomize in this case.[12] While the possibility of low types mixing being optimal was known at the time (see, for instance, Laffont and Tirole (1987)), more recent work in the field of mechanism design without commitment has investigated the conditions under which both upwards and downwards incentive compatibility constraints bind (Bester & Strausz, 2001, 2007; Kumar & Langberg, 2009; Goltsman, 2011).

When comparing the payoffs from inducing low types to mix to those from full pooling or inducing (only) high types to mix, one can show that the seller only induces low types to mix when the end of the interaction is at least three periods away.

Corollary 1 *Low types mix in period t only if $T - t \geq 3$.*

When the seller is inducing low types to mix, the outcome can be thought of as a randomization between two options: receiving nothing in the current period and the continuation payoffs associated

[12] While this shows that the characterization of the spot contracting equilibrium found in Hart and Tirole (1988) is not correct, it can still be shown that the equilibrium dynamics of the spot contracting game are not Coasian, which is the result found Hart and Tirole's Proposition 4.

with lower beliefs going forward or receiving $\underline{b}$ and the continuation payoffs of higher beliefs going forward. This second option gives exactly the payoff that she receives from pooling both types at those higher beliefs. The proof of Corollary 1 constructs the continuation payoffs starting from periods T and $T-1$ to show that for these continuation payoffs, one can always improve on an allocation in which low types are mixing by either fully pooling buyers or inducing only high types to mix or purchase. These improvements rely on the specific demand and belief structures that are studied in this paper, and there is no reason to believe the result will hold in more general economic settings. For instance, in a closely related model with no commitment and convex costs, Laffont and Tirole (1987) finds that both types mix in a two period model.

I can now comment on the uniqueness of equilibrium paths in the spot contracting setting. It has been noted before that in the commitment with renegotiation setting, there are multiple equilibrium paths for a set of prior beliefs with measure zero (Hart & Tirole, 1988). Because the equilibrium of the spot contracting game coincides with that of the commitment with renegotiation game for some parameter values, there are also multiple equilibrium paths for a measure zero set of prior beliefs for those parameter values. However, the equilibrium of Example 1 gives a stronger result: for some values of δ, $\bar{b}$, and $\underline{b}$, there is a set of prior beliefs with strictly positive measure for which there are multiple strategies which can be implemented on the equilibrium path. When initial beliefs are on the range in which low types randomize, there is a *range* of continuation values that are consistent with the posterior going into period 2 and which satisfy the high valuation buyer's incentive compatibility constraint. These continuation values correspond to the seller placing different probabilities on the prices she offers in period 2.

3.1 Posted Prices

This paper follows much of the previous literature in assuming that the seller is restricted to posting prices. The important restriction is that in each period a buyer either receives the item or does not; the seller does not have the ability to randomize whether the item will be delivered. In the spot contracting setting described in this paper, I can show that the seller may strictly improve her payoffs at some points by using random delivery of the good.

Proposition 2 *Assuming that the seller posts prices is with loss of generality.*

Proposition 2 is proven by using the equilibrium of Example 1. The key feature is that when beliefs are between $\frac{167}{416}$ and $\frac{3507}{5851}$, low valuation buyers randomize and high valuation buyers *strictly* prefer to purchase.[13] Thus, rather than posting a single low price, the seller can offer a menu with two options: a buyer can receive the item with probability one at a price of $\underline{b}$ or receive the item with probability $1-\delta-\delta^2$ at a price of $(1-\delta-\delta^2)\underline{b}$. The low valuation buyer remains indifferent between his options and mixes at the same rate (although instead of sometimes purchasing nothing, he now sometimes purchases the random delivery contract). The high valuation buyer is made indifferent, and the seller's payoffs strictly increase.

4 Comparison to Commitment with Renegotiation

To facilitate comparison between spot contracting and commitment with renegotiation, here I restate the characterization of the equilibrium of the commitment with renegotiation game given by Hart and Tirole (1988).

Proposition 3 *The equilibrium path of the commitment and renegotiation game is generically unique and takes the following form: there exists a sequence of numbers* $0 = \bar{\mu}_0 < \bar{\mu}_1 < \cdots < \bar{\mu}_T < 1$ *such that*

(i) If current posterior beliefs μ_t *at date* t *belong to the interval* $[\bar{\mu}_i, \bar{\mu}_{i+1})$ *for* $i \leq T-t+1$*, the seller will sell only to high types for* i *more periods including the current one. Posterior beliefs are* $\bar{\mu}_{i-1}$ *at* $t+1$*,* $\bar{\mu}_{i-2}$ *at* $t+2$ *and so on. The discounted sum of prices charged in one of these periods is such that the high type buyer is indifferent between purchasing and waiting for the low type's contract.*

(ii) If current beliefs are such that $\mu_t \geq \bar{\mu}_{T-t+1}$*, only high types purchase in every period, and the discounted sum of prices charged is such that the high type is indifferent between his allocation and not purchasing.*

Proof This follows directly from Proposition 2 and Theorem 1 in Hart and Tirole (1988). $\square$

[13] The fact that the example relies on a strict preference to purchase on the part of the high valuation buyer shows why this type of random delivery contract does not improve payoffs in the commitment with renegotiation setting of Hart and Tirole (1988). There, a high valuation buyer is exactly indifferent when he is not fully pooling with low valuation buyers.

Hart and Tirole (1988) also compares the profits that the seller earns under commitment with renegotiation to those she earns from spot contracting. In Proposition 6, it claims that the seller's expected payoffs under spot contracting are weakly higher than the expected payoffs under commitment with renegotiation.

The payoff structure as a function of beliefs that arises from commitment with renegotiation takes the same form as the payoff structures from spot contracting. The payoff function is weakly increasing, piecewise linear, and convex. Furthermore, the payoffs from all three commitment settings are equal to each other when beliefs are equal to one (the seller is sure that all buyers are high types) and when beliefs are below $\frac{\underline{b}}{\overline{b}}$ (when the commitment outcome is to pool all buyers in all periods). Payoffs for the three types of games can be found in Figure 3 for the parameterization given in Example 2.[14]

As is to be expected, the payoffs from full commitment are always weakly higher than the other two settings: with full commitment, the seller could always commit to carry out exactly the same sequence of prices that she would carry out with imperfect commitment, ensuring that she receives the same payoffs. However, the payoffs of spot contracting and commitment with renegotiation cannot consistently be ranked: the payoffs from Example 2 show that for different ranges of prior beliefs, either commitment structure can dominate the other. This fact contradicts Proposition 6 from Hart and Tirole (1988).

Observation 2 *Contrary to Proposition 6 of Hart and Tirole (1988), payoffs from the commitment with renegotiation setting can be strictly below those from the spot contracting setting.*

Observation 2 may be surprising. One may think that commitment with renegotiation offers "more" commitment than spot contracting and should lead to higher payoffs, but this is incorrect. What is true is that for a fixed set of potential continuation equilibria (conditional on beliefs) in period $t+1$ that are consistent with incentive compatibility, the seller will earn weakly higher profits in period t in the commitment with renegotiation setting than in the spot contracting setting. That is because out of this set of potential continuation equilibria, the seller can promise to implement any that are Pareto undominated in the commitment with renegotiation setting, but in the spot contracting setting the seller can only promise continuation equilibria which maximize her own

[14] The equilibrium associated with these parameter values is fully described in Online Appendix B.2.

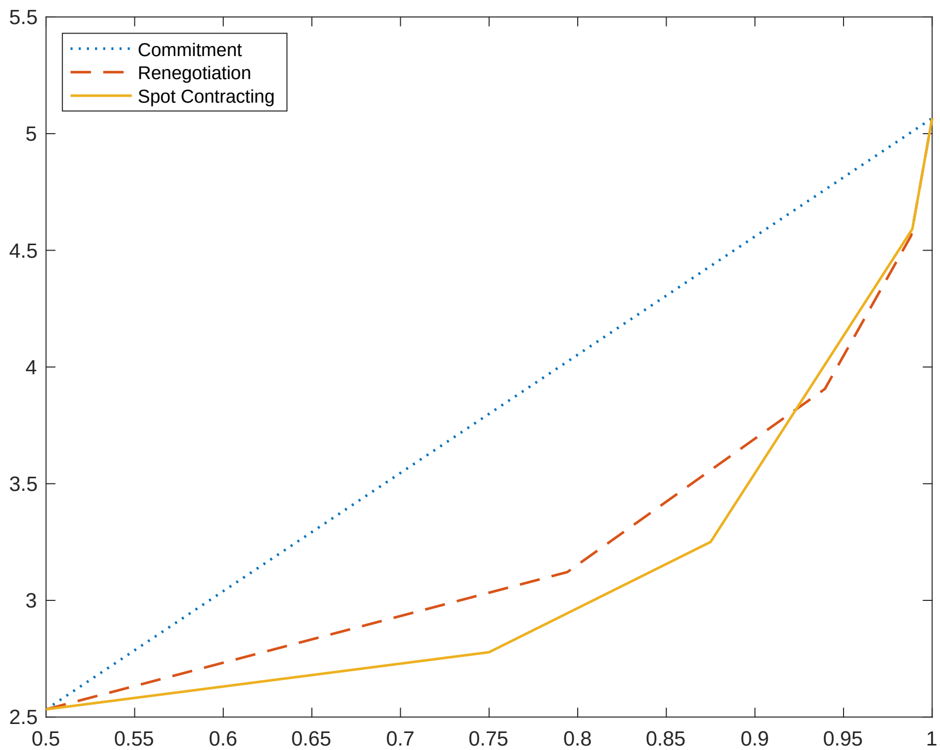

Figure 3: Payoffs for the sellers of various commitment types as a function of μ when $T = 4$, $\delta = 0.7$, $\underline{b} = 1$, and $\bar{b} = 2$. Full commitment must always give higher payoffs than other commitment types, but payoff rankings otherwise depend on the specific parameters used.

payoffs. In the more general game, this argument breaks down because it is not the case that the set of potential continuation equilibria are the same in the two settings.

It is instructive to work through how this logic applies to Example 2. Continuation equilibria for the two models are the same starting from period $t = 3$, so I will begin with a focus on period 2.[15] The optimal contract under commitment with renegotiation prescribes that for $\mu_2 \in \left[\frac{1}{2}, \frac{27}{39}\right]$, all high types should receive the good in the current and all future periods. This can be implemented

[15]This logic is actually more general. For games of any length, the reverse incentive compatibility constraint never binds in periods T or $T-1$, implying that the equilibria of the spot contracting and commitment with renegotiation games are the same in those periods. This means that period $T-2$ is the first period in which equilibria can be different, so period $T-3$ is the first period in which spot contracting can have higher payoffs than commitment with renegotiation.

in the commitment with renegotiation setting because the seller is able to promise to charge prices strictly less than $\bar{b}$ in future periods. It is *not* implementable in the spot contracting setting because the seller cannot commit to charging prices below $\bar{b}$ in future periods. Because of this, the seller would have to charge a price strictly below $\underline{b}$ in the current period to induce high types to purchase, violating the reverse incentive compatibility constraint. Instead, in period 2 of the spot contracting setting, it is optimal for the seller to pool all buyers for $\mu_2 \in \left[\frac{1}{2}, \frac{3}{4}\right]$ and to induce high types to mix such that $\mu_3 = \frac{1}{2}$ if $\mu_2 \in \left[\frac{3}{4}, \frac{27}{39}\right]$.[16]

Given the equilibria starting from period $t = 2$ in this example, we can see that the continuation payoffs that are available to the seller in period 1 are not the same in the two settings. The continuation equilibria that are particularly relevant are those that arise when $\mu_2 = \frac{3}{4}$. When this is the case in the renegotiation setting, starting from period 2 the seller receives expected payoffs of $\frac{269}{100}$ and a high valuation buyer receives expected payoffs of $\frac{119}{100}$. In the spot contracting setting, these beliefs correspond to the seller receiving payoffs of $\frac{127}{50}$ and the high valuation buyer receiving payoffs anywhere between $\frac{49}{100}$ and $\frac{149}{100}$ (corresponding to the seller inducing high types to mix and the seller fully pooling, respectively).[17] This lower potential continuation value for high valuation buyers when $\mu_2 = \frac{3}{4}$ in the spot contracting setting allows the seller to charge higher prices and earn higher profits for some μ in the first period.

It is important to note that the contradiction of Proposition 6 of Hart and Tirole (1988) does *not* rely on the earlier contradiction pointed out in Observation 1. In fact, for the parameters used in Example 2, inducing low valuation buyers to mix is never beneficial and the possibility of them mixing does not affect payoffs.

The proof of Hart and Tirole's Proposition 6 is completed by backwards induction, with an inductive hypothesis that (in the terminology used here) for any period t, beliefs at that date, and continuation

[16]Payoffs of pooling and inducing high types to mix are the same when $\mu_2 = \frac{3}{4}$.

[17]One may have the intuition that the seller should be able to implement the spot contracting allocation in the commitment with renegotiation setting by, for instance, committing to extreme consumption levels or transfers and then renegotiating to the spot contracting allocation (similar arguments have previously been made in the reverse, in which two-period contracts can replicate longer term contracts with renegotiation, as in Rey and Salanie (1996)). This example shows why this cannot be the case. Regardless of what has been previously agreed to, the seller would never renegotiate to the spot contracting outcome in period 2 because the commitment with renegotiation outcome gives strictly higher payoffs to *both* the seller and the high valuation buyer.

equilibrium in the spot contracting setting, there exists a renegotiation proof outcome which gives the same utilities to both types of buyer and weakly higher payoffs to the seller. They then claim that given that the inductive hypothesis holds for period $t + 1$, one can construct a renegotiation proof outcome which dominates any spot contracting outcome. This construction uses the spot outcome from the current period and the renegotiation outcome starting from period $t + 1$. While the proof claims that this construction must also be renegotiation proof, in actuality it need not be. In Example 2, when beliefs are between $\frac{3}{4}$ and $\frac{27}{34}$ in the second period, it is optimal in the spot contracting game for high valuation buyers to mix such that the posterior is either $\frac{1}{2}$ or 1 in the third period. However, in the commitment with renegotiation game, this is not renegotiation proof because both the seller and the buyer receive higher payoffs from high valuation buyers purchasing the item with probability one.

5 Conclusion

In this paper I characterize seller-optimal equilibria of a spot contracting game between a monopolist and a consumer with private information. Some of the results contradict previous claims which have been made about the same model. In every period, one of three outcomes occurs: both types are purchasing, low types are randomizing and high types are purchasing or randomizing, or only high types are purchasing or randomizing. Payoffs in the spot contracting game can be higher than a game with commitment and renegotiation, and the seller can improve her payoffs by not restricting herself to posted prices.

Further work should study under what conditions limited commitment leads to posted prices not being optimal and what form these more general contracts take. This paper shows that simple random delivery contracts can improve profits, but a full mechanism design approach may lead to contracting dynamics which are not seen here.

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A Proofs of Main Results

A.1 Proof of Lemma 1

First, I note that in any period the seller will charge a price no lower than $\underline{b}$. I show this using backwards induction. It is well known that in the last period of the game, the seller will charge either $\underline{b}$ or $\bar{b}$. Suppose now that in period t the seller were charging a price that was strictly lower than $\underline{b}$ and that she will never charge a price strictly lower than $\underline{b}$ in all future periods. Because purchasing in the current period gives strictly positive surplus to the low type, all low types will strictly prefer to purchase in period t. This implies that the high type purchases as well, because not purchasing would lead to the seller updating her posterior beliefs to one (which implies payoffs of zero for the high valuation buyer). Given that both types purchase in period t with probability one, the seller could increase the price to $\underline{b}$. This strictly increases profits without violating either type's incentive compatibility or individual rationality constraints.

Thus, in a given period, there will be at most two options for the buyer to choose from: purchasing at a price weakly greater than $\underline{b}$ or not purchasing. For a given price offer, each type can either not purchase (N), mix between purchasing and not purchasing (M), or purchase with probability 1 (B). Thus, all of the possibilities in a given period can be described by $\{\underline{N}, \underline{M}, \underline{B}\} \times \{\bar{N}, \bar{M}, \bar{B}\}$, where the underline describes the behavior of low types and the bar describes behavior of high types. The lemma then claims that we can focus on $(\underline{B}, \bar{B})$, $(\underline{M}, \bar{B})$, $(\underline{M}, \bar{M})$, $(\underline{N}, \bar{M})$, and $(\underline{N}, \bar{B})$.

The seller will not fully pool both types not purchasing because she can strictly increase profits by selling in the current period to both types at price $\underline{b}$ and using the same continuation equilibrium. Neither type's incentive constraints are violated. This rules out $(\underline{N}, \bar{N})$.

There cannot be an equilibrium in which the low type purchases with strictly positive probability and the high type does not purchase. Suppose that there were. In this case the price being charged must be no higher than $\underline{b}$ if the low type is purchasing. The posterior after observing the buyer purchase in that period would be 0, leading to a price of $\underline{b}$ in all future periods. Thus, the high type could receive $\bar{b} - \underline{b}$ in all periods, which is his maximal payoff, and he would want to purchase. This rules out $(\underline{B}, \bar{N})$ and $(\underline{M}, \bar{N})$.

Suppose that the high type was randomizing and the low type was purchasing with probability one.

Then when the seller observes a buyer not purchase, she knows he is a high type and will charge price $\bar{b}$ in all future periods. This would leave the high valuation buyer who did not purchase with 0 surplus. Since the low valuation buyer is purchasing, the price can be no higher than $\underline{b}$ so a high type would receive strictly positive surplus from purchasing in the current period. Thus, the high type would purchase with probability one in the current period. This rules out $(\underline{B}, \bar{M})$.

Thus, we have ruled out $(\underline{N}, \bar{N})$, $(\underline{B}, \bar{N})$, $(\underline{M}, \bar{N})$, and $(\underline{B}, \bar{M})$ and in any period of any equilibrium one of the remaining options must occur. Item 1 of the lemma (which I refer to as both types purchasing) corresponds to the outcome $(\underline{B}, \bar{B})$. Item 2 of the lemma (which I refer to as low types mixing) corresponds to the outcomes $(\underline{M}, \bar{B})$ and $(\underline{M}, \bar{M})$. Finally, item 3 of the lemma (which I refer to as high types mixing) corresponds to the outcomes $(\underline{N}, \bar{M})$, and $(\underline{N}, \bar{B})$. $\square$

A.2 Proof of Proposition 1

To prove Proposition 1 I will use a series of lemmas before showing the main result. Define $V_{t+1}(\mu_{t+1})$ as the seller's optimal payoffs starting in period $t+1$ as a function of beliefs in period $t+1$ and $W_{t+1}(\mu_{t+1})$ as the correspondence between beliefs in period $t+1$ and all possible expected payoffs for high valuation buyers starting in period $t+1$, given that the seller is optimizing her payoffs.

It will be useful to define (V, W) as a Spot Payoff Pair with cutoffs $\{\mu_{t,1}, \ldots, \mu_{t,N_t}\}$ if

- $\mu_{t,1} = 0$, $\mu_{t,N_t} = 1$, and $\mu_{t,i} < \mu_{t,j}$ if $i < j$,
- V is an increasing, convex, and piecewise linear function mapping the unit interval to $\mathbb{R}$ with changes in slope at $\mu_{t,2}, \ldots, \mu_{t,N_t-1}$, and
- W is a correspondence mapping the unit interval to closed and convex subsets of $\mathbb{R}$ such that for $\bar{W}(\mu) = \max W(\mu)$ and $\underline{W}(\mu) = \min W(\mu)$, both $\bar{W}(\mu)$ and $\underline{W}(\mu)$ are decreasing step functions with discontinuities at $\mu_{t,2}, \ldots, \mu_{t,N_t-1}$. Furthermore, $\underline{W}(\mu_{t,i-2}) \geq \bar{W}(\mu_{t,i}) \geq \underline{W}(\mu_{t,i-1})$.

Lemma 1 shows that there are three possible outcomes in any period. I will show the properties of the seller's and high-type buyer's payoffs conditional on carrying out each of these outcomes. Thus, given continuation values V_{t+1} and W_{t+1}, define $V_t^{\mathrm{FP}}(\mu_t)$, $V_t^{\mathrm{LM}}(\mu_t)$, and $V_t^{\mathrm{HM}}(\mu_t)$ as the payoffs to the seller when the seller is optimally fully pooling buyers, inducing low types to mix, and inducing only high types to mix respectively.[18] I then define $W_t^{\mathrm{FP}}(\mu_t)$, $W_t^{\mathrm{LM}}(\mu_t)$, and $W_t^{\mathrm{HM}}(\mu_t)$ as the sets of payoffs for high valuation buyers which are consistent with the seller optimally fully pooling buyers, inducing low types to mix, and inducing only high types to mix respectively. It will sometimes be the case that in period t for a given μ_t, it is impossible to induce high types to randomize when low types are purchasing with probability zero. In these cases, I define $V_t^{HM}(\mu_t) = -\infty$ and leave $W_t^{HM}(\mu_t)$ undefined.

With these definitions and the results from Lemma 1, we find that the seller's payoffs in period t are given by

$$V_t(\mu_t) = \max\{V_t^{FP}(\mu_t), V_t^{LM}(\mu_t), V_t^{HM}(\mu_t)\},$$

and $W_t(\mu_t)$ is the set of equilibrium payoffs for high valuation buyers which are consistent with the seller receiving $V_t(\mu_t)$.

I now state several lemmas which will be useful to prove Proposition 1. The proofs of these lemmas can be found in Online Appendix A.

For continuation values that form a Spot Payoff Pair, Lemma A.1 gives the payoffs to the seller and high valuation buyer when the seller is optimally pooling both types of buyer.

Lemma A.1 *Suppose that (V_{t+1}, W_{t+1}) is a Spot Payoff Pair with cutoffs $\{\mu_{t+1,1}, \ldots, \mu_{t+1,N_{t+1}}\}$. Then*

$$V_t^{FP}(\mu_t) = \underline{b} + \delta V_{t+1}(\mu_t)$$

and W_t^{FP} is a closed- and convex-valued correspondence such that for $\underline{W}_t^{FP}(\mu_t) = \min W_t^{FP}(\mu_t)$ and $\bar{W}_t^{FP}(\mu_t) = \max W_t^{FP}(\mu_t)$,

$$\underline{W}_t^{FP}(\mu_t) = \bar{b} - \underline{b} + \delta \underline{W}_{t+1}(\mu_t)$$
$$\bar{W}_t^{FP}(\mu_t) = \bar{b} - \underline{b} + \delta \bar{W}_{t+1}(\mu_t)$$

and both $\underline{W}_t^{FP}(\mu_t)$ and $\bar{W}_t^{FP}(\mu_t)$ are decreasing step functions.

[18] To be more precise, the payoffs I define as $V_t^{\mathrm{LM}}(\mu_t)$ are those that arise from low types mixing such that the higher posterior is weakly higher than $\mu_{t+1,j}$, where $\mu_{t+1,j}$ is the lowest cutoff which is strictly higher than μ_t. I show in the proof of Lemma A.2 that this restriction does not lower the seller's payoffs.

For continuation values that form a Spot Payoff Pair, Lemma A.2 and Corollary A.1 give the payoffs to the seller and high valuation buyer, respectively when the seller is optimally inducing low valuation buyers to mix.

Lemma A.2 *Suppose that (V_{t+1}, W_{t+1}) is a Spot Payoff Pair with cutoffs $\{\mu_{t+1,1}, \ldots, \mu_{t+1,N_{t+1}}\}$. Then V_t^{LM} is piecewise linear, and*

$$\begin{aligned} V_t^{LM}(\mu_t) = \max_{i,j} & \left(\frac{\mu_{t+1,j} - \mu_t}{\mu_{t+1,j} - \mu_{t+1,i}}\right) [\delta V_{t+1}(\mu_{t+1,i})] + \left(\frac{\mu_t - \mu_{t+1,i}}{\mu_{t+1,j} - \mu_{t+1,i}}\right) [\underline{b} + \delta V_{t+1}(\mu_{t+1,j})] \\ & \text{subject to } \bar{b} - \underline{b} + \delta \bar{W}_{t+1}(\mu_{t+1,j}) \geq \delta \underline{W}_{t+1}(\mu_{t+1,i}), \\ & \bar{b} - \underline{b} + \delta \underline{W}_{t+1}(\mu_{t+1,j}) \leq \delta \bar{W}_{t+1}(\mu_{t+1,i}) \text{ if } \mu_{t+1,i} > 0, \\ & \text{and } \mu_{t+1,i} \leq \mu_t < \mu_{t+1,j}. \end{aligned} \tag{1}$$

Low types mixing optimally for beliefs μ_t only involves low (and potentially high types) mixing such that posteriors are $\mu_{t+1,i}$ and $\mu_{t+1,j}$ for i and j that solve problem (1). Furthermore, the only discontinuities in $V_t^{LM}(\mu_t)$ are downward jump discontinuities which occur for some $\mu_t \in \{\mu_{t+1,2}, \ldots, \mu_{t+1,N_{t+1}-1}\}$.

Corollary A.1 *Suppose that (V_{t+1}, W_{t+1}) is a Spot Payoff Pair with cutoffs $\{\mu_{t+1,1}, \ldots, \mu_{t+1,N_{t+1}}\}$ and let $I_t^{LM}(\mu_t)$ be the set of (i,j) which solve problem (1). Then W_t^{LM} is a correspondence mapping the unit interval to closed and convex subsets of $\mathbb{R}$. Furthermore, for $\underline{W}_t^{LM}(\mu_t) = \min W_t^{LM}(\mu_t)$ and $\bar{W}_t^{LM}(\mu_t) = \max W_t^{LM}(\mu_t)$,*

$$\begin{aligned} \underline{W}_t^{LM}(\mu_t) &= \min_{(i,j) \in I_t^{LM}(\mu_t)} \left[\max\{\bar{b} - \underline{b} + \delta \underline{W}_{t+1}(\mu_{t+1,j}), \delta \underline{W}_{t+1}(\mu_{t+1,i})\}\right] \\ \bar{W}_t^{LM}(\mu_t) &= \max_{(i,j) \in I_t^{LM}(\mu_t)} \left[\bar{b} - \underline{b} + \delta \bar{W}_{t+1}(\mu_{t+1,j})\right] \end{aligned}$$

and both $\underline{W}_t^{LM}(\mu_t)$ and $\bar{W}_t^{LM}(\mu_t)$ are decreasing step functions.

For continuation values that form a Spot Payoff Pair, Lemma A.3 and Corollary A.2 give the payoffs to the seller and high valuation buyer, respectively when the seller is optimally inducing high valuation buyers to mix.

Lemma A.3 *Suppose that (V_{t+1}, W_{t+1}) is a Spot Payoff Pair with cutoffs $\{\mu_{t+1,1}, \ldots, \mu_{t+1,N_{t+1}}\}$ and define $\underline{i}$ as the minimum value of i such that $\delta \underline{W}_{t+1}(\mu_{t+1,i}) \leq \bar{b} - \underline{b}$. Then for $\mu_t < \mu_{t+1,\underline{i}}$,*

$V_t^{HM}(\mu_t) = -\infty$ and for $\mu_t \in [\mu_{t+1,\underline{i}}, 1]$, $V_t^{HM}(\mu_t)$ is piecewise linear, convex, and

$$V_t^{HM}(\mu_t) = \max_{i \geq \underline{i}} \left(\frac{1-\mu_t}{1-\mu_{t+1,i}}\right)[\delta V_{t+1}(\mu_{t+1,i})] + \left(\frac{\mu_t - \mu_{t+1,i}}{1-\mu_{t+1,i}}\right)\left[\bar{b} - \delta \underline{W}_{t+1}(\mu_{t+1,i}) + \delta V_{t+1}(1)\right] \quad \text{subject to } \mu_{t+1,i} \leq \mu_t. \tag{2}$$

High types mixing optimally for beliefs μ_t only involves high types mixing such that posteriors are $\mu_{t+1,i}$ and 1 for i that solve problem (2).

Corollary A.2 *Suppose that (V_{t+1}, W_{t+1}) is a Spot Payoff Pair with cutoffs $\{\mu_{t+1,1}, \ldots, \mu_{t+1,N_{t+1}}\}$ and let $I_t^{HM}(\mu_t)$ be the set of i which solve problem (2). Then W_t^{HM} is a correspondence mapping the set of μ for which V_t^{HM} is defined to closed and convex subsets of $\mathbb{R}$. Furthermore, for $\underline{W}_t^{HM}(\mu_t) = \min W_t^{HM}(\mu_t)$ and $\bar{W}_t^{HM}(\mu_t) = \max W_t^{HM}(\mu_t)$,*

$$\underline{W}_t^{HM}(\mu_t) = \min_{i \in I_t^{HM}(\mu_t)} \delta \underline{W}_{t+1}(\mu_{t+1,i}),$$

$$\bar{W}_t^{HM}(\mu_t) = \max_{i \in I_t^{HM}(\mu_t)} \delta \underline{W}_{t+1}(\mu_{t+1,i})$$

and both $\underline{W}_t^{HM}(\mu_t)$ and $\bar{W}_t^{HM}(\mu_t)$ are decreasing step functions.

For continuation values that form a Spot Payoff Pair, Lemma A.4 shows that for μ_t high enough, optimally inducing high types to mix gives the seller higher payoffs than pooling or inducing low types to mix.

Lemma A.4 *Suppose that (V_{t+1}, W_{t+1}) is a Spot Payoff Pair with cutoffs $\{\mu_{t+1,1}, \ldots, \mu_{t+1,N_{t+1}}\}$. Then there exists a $\hat{\mu}_t \in (0,1)$ such that $V_t^{HM}(\mu_t) > \max\{V_t^{FP}(\mu_t), V_t^{LM}(\mu_t)\}$ if $\mu_t > \hat{\mu}_t$ and $V_t^{HM}(\mu_t) < \max\{V_t^{FP}(\mu_t), V_t^{LM}(\mu_t)\}$ if $\mu_t < \hat{\mu}_t$.*

For continuation values that form a Spot Payoff Pair, Lemma A.5 shows that for μ_t low enough, full pooling gives the seller higher payoffs than either low or high types mixing.

Lemma A.5 *Suppose that (V_{t+1}, W_{t+1}) is a Spot Payoff Pair with cutoffs $\{\mu_{t+1,1}, \ldots, \mu_{t+1,N_{t+1}}\}$. Then there exists a $\underline{\mu}_t$ such that $V_t^{FP}(\mu_t) > \max\{V_t^{LM}(\mu_t), V_t^{HM}(\mu_t)\}$ if $\mu_t < \underline{\mu}_t$.*

Lemma A.6 finally shows that if the continuation values starting from period $t+1$ form a Spot Payoff Pair, then the value functions starting from period t form a Spot Payoff Pair.

Lemma A.6 *Suppose that (V_{t+1}, W_{t+1}) is a Spot Payoff Pair with cutoffs $\{\mu_{t+1,1}, \ldots, \mu_{t+1,N_{t+1}}\}$.*

Then (V_t, W_t) is a Spot Payoff Pair.

We can now prove Proposition 1.

Proof of Proposition 1 The proof will use backwards induction starting from period T.

Basis Step: Notice first that in period T, the monopolist faces a single period screening problem and thus chooses prices to maximize profits in that period given μ_T. To do so, she sets a price of $\underline{b}$ and sells to both the high types and the low types if $\mu_T < \frac{\underline{b}}{\bar{b}}$ or sets a price of $\bar{b}$ and sells only to the remaining unidentified high types if $\mu_T > \frac{\underline{b}}{\bar{b}}$. When $\mu_T = \frac{\underline{b}}{\bar{b}}$ the seller is indifferent between these two options.

Consider the payoffs to each type of player as a function of μ_T. Buyers with valuation $\underline{b}$ always receive payoffs equal to 0. For $\mu_T \in [0, \frac{\underline{b}}{\bar{b}})$, the seller receives payoffs equal to $\underline{b}$ while the high valuation buyer receives payoffs equal to $\bar{b} - \underline{b}$. For $\mu_T \in (\frac{\underline{b}}{\bar{b}}, 1]$, the seller receives payoffs equal to $\mu_T \bar{b}$ while high valuation buyers receive payoffs of 0. When $\mu_T = \frac{\underline{b}}{\bar{b}}$, the seller receives $\underline{b}$ and the buyer can receive any payoff between 0 and $\bar{b} - \underline{b}$. Thus,

$$V_T(\mu_T) = \begin{cases} \underline{b} \text{ if } \mu_T \leq \frac{\underline{b}}{\bar{b}} \\ \mu_T \bar{b} \text{ otherwise} \end{cases}$$

$$W_T(\mu_T) = \begin{cases} \bar{b} - \underline{b} \text{ if } \mu_T < \frac{\underline{b}}{\bar{b}} \\ [0, \bar{b} - \underline{b}] \text{ if } \mu_T = \frac{\underline{b}}{\bar{b}} \\ 0 \text{ otherwise} \end{cases},$$

and (V_T, W_T) is a Spot Payoff Pair with cutoffs $\left\{0, \frac{\underline{b}}{\bar{b}}, 1\right\}$.

Inductive Step: Now suppose that (V_{t+1}, W_{t+1}) is a Spot Payoff Pair with cutoffs $\{\mu_{t+1,1}, \ldots, \mu_{t+1,N_{t+1}}\}$.

Lemma A.4 shows that for $\mu_t > \hat{\mu}_t$, the seller receives her highest payoffs when low types do not purchase and high types purchase or mix between purchasing and not purchasing. Thus, on the range $(\hat{\mu}_t, 1]$, $V_t(\mu_t) = V_t^{HM}(\mu_t)$ and $W_t(\mu_t) = W_t^{HM}(\mu_t)$. Similarly, Lemma A.5 shows that for $\mu_t < \underline{\mu}_t$, the seller receives her highest payoffs when both types purchase. Thus, on the range $[0, \underline{\mu}_t)$, $V_t(\mu_t) = V_t^{FP}(\mu_t)$ and $W_t(\mu_t) = W_t^{FP}(\mu_t)$. On the range $(\underline{\mu}_t, \hat{\mu}_t)$, Lemma A.4 shows that

$V_t(\mu_t) > V_t^{HM}(\mu_t)$, so the seller receives her highest payoffs when either both types purchase or when low types do not purchase and high types purchase or mix between purchasing and not purchasing. Furthermore, in Lemma A.2, the fact that $\mu_{t+1,j} > \mu_{t+1,i}$ implies that high types purchase with higher probability than low types on the range $(\underline{\mu}_t, \hat{\mu}_t)$. Lemma A.6 shows that (V_t, W_t) is a spot payoff pair, completing the inductive step. □

A.3 Proof of Observation 1

Any time the high types purchase with probability one and low types mix between purchasing and not purchasing, beliefs fall to 0 when the seller observes the buyer not purchasing. In the equilibrium of Example 1, this outcome occurs in the first period for beliefs μ such that $\frac{167}{416} \leq \mu < \frac{3507}{5851}$. Since in this case $\delta + \delta^2 + \delta^3 > 1$, Lemma 1 of Hart and Tirole (1988) claims that beliefs cannot fall below $\frac{\underline{b}}{\bar{b}}$ before period 3. Thus, the equilibrium of Example 1 provides a counterexample. □

A.4 Proof of Corollary 1

The proof will show that when $t = T$, $t = T - 1$, or $t = T - 2$, the seller receives higher profits from either high types mixing or from full pooling.

First, notice that in period T, there are no continuation payoffs. Thus, charging a price of $\underline{b}$ and selling only to a proportion of buyers must give strictly lower payoffs than fully pooling at a price of $\underline{b}$ and it is never valuable to have low types mix when $t = T$.

Next, I will show that for all t, $V_t^{HM}(\mu_t) > V_t^{LM}(\mu_t)$ for $\mu_t \in [\mu_{t+1,N_{t+1}-1}, 1]$. On this range, Lemma A.2 shows that

$$V_t^{LM}(\mu_t) = \left(\frac{1-\mu_t}{1-\mu_{t+1,i}}\right)[\delta V_{t+1}(\mu_{t+1,i})] + \left(\frac{\mu_t - \mu_{t+1,i}}{1-\mu_{t+1,i}}\right)[\underline{b} + \delta V_{t+1}(1)]$$

for some i such that $\delta \underline{W}_{t+1}(\mu_{t+1,i}) \leq \bar{b} - \underline{b}$. But if this is the case, then Lemma A.3 shows that

$$\begin{aligned} V_t^{HM}(\mu_t) &= \max_{i \geq \underline{i}} \left(\frac{1-\mu_t}{1-\mu_{t+1,i}}\right)[\delta V_{t+1}(\mu_{t+1,i})] + \left(\frac{\mu_t - \mu_{t+1,i}}{1-\mu_{t+1,i}}\right)[\bar{b} - \delta \underline{W}_{t+1}(\mu_{t+1,i}) + \delta V_{t+1}(1)] \\ &> \left(\frac{1-\mu_t}{1-\mu_{t+1,i}}\right)[\delta V_{t+1}(\mu_{t+1,i})] + \left(\frac{\mu_t - \mu_{t+1,i}}{1-\mu_{t+1,i}}\right)[\underline{b} + \delta V_{t+1}(1)] \\ &= V_t^{\text{LM}}(\mu_t) \end{aligned}$$

So high mixing payoffs are higher than low mixing payoffs.[19]

[19]The strict inequality actually requires that $\delta \underline{W}_{t+1}(\mu_{t+1,i}) < \bar{b} - \underline{b}$. However, if $\delta \underline{W}_{t+1}(\mu_{t+1,i}) = \bar{b} - \underline{b}$, then the

Suppose now that for some μ_t,

$$V_t^{LM}(\mu_t) = \left(\frac{\mu_{t+1,j} - \mu_t}{\mu_{t+1,j} - \mu_{t+1,i}}\right)[\delta V_{t+1}(\mu_{t+1,i})] + \left(\frac{\mu_t - \mu_{t+1,i}}{\mu_{t+1,j} - \mu_{t+1,i}}\right)[\underline{b} + \delta V_{t+1}(\mu_{t+1,j})]$$

for $\mu_{t+1,j} < 1$. Because V_t^{FP} is convex and piecewise linear with $V_t^{FP}(\mu_{t+1,j}) = \underline{b} + \delta V_{t+1}(\mu_{t+1,j})$, $V_t^{LM}(\mu_t)$ is only higher than $V_t^{FP}(\mu_t)$ if $V_t^{LM}(\mu_{t+1,j-1}) > V_t^{FP}(\mu_{t+1,j-1})$.[20] I will show that this is impossible when $t = T-1$ or $t = T-2$.

Consider the case when $t = T-1$. The solution to the single period screening problem is well known, with

$$V_T(\mu_T) = \begin{cases} \underline{b} \text{ if } \mu_T \leq \frac{\underline{b}}{\bar{b}} \\ \mu_T \bar{b} \text{ otherwise} \end{cases} .$$

Thus, the cutoffs that are that are used in Lemma A.2 to compute V_{T-1}^{LM} are $\mu_{T,1} = 0$, $\mu_{T,2} = \frac{\underline{b}}{\bar{b}}$, and $\mu_{T,3} = 1$. It obviously cannot be the case then when $\mu_{T,j} < 1$ that $V_{T-1}^{LM}(\mu_{T,j-1}) > V_{T-1}^{FP}(\mu_{T,j-1})$, because $\mu_{T,j} < 1$ implies that $\mu_{T,j-1} = 0$, and $V_{T-1}^{LM}(0) = \delta V_T(0) < \underline{b} + \delta V_T(0) = V_{T-1}^{FP}(0)$.

Next, consider the case when $t = T-2$. I just showed that low types never mix in period $T-1$, so the seller's value function in period $T-1$ is the upper envelope of the profits she receives from fully pooling and the profits she receives from high types mixing. When a high type mixes, he does so such that posteriors in period T are either $\mu_{T,1} = 0$ or $\mu_{T,2} = \frac{\underline{b}}{\bar{b}}$. Using the same computations that are found in Hart and Tirole (1988), we find that

$$V_{T-1}(\mu_{T-1}) = \begin{cases} (1+\delta)\underline{b} \text{ if } \mu_{T-1} < \frac{\underline{b}}{\bar{b}} \\ \mu_{T-1}\bar{b} + \delta \underline{b} \text{ if } \frac{\underline{b}}{\bar{b}} \leq \mu_{T-1} < \frac{(1+\delta)\bar{b}\underline{b} - \delta \underline{b}^2}{\delta \bar{b}^2 + (1-\delta)\bar{b}\underline{b}} \\ \frac{(1+\delta)\mu_{T-1}\bar{b}^2 - \delta \mu_{T-1}\bar{b}\underline{b} - \bar{b}\underline{b}}{\bar{b} - \underline{b}} \text{ if } \frac{(1+\delta)\bar{b}\underline{b} - \delta \underline{b}^2}{\delta \bar{b}^2 + (1-\delta)\bar{b}\underline{b}} \leq \mu_{T-1} \end{cases} ,$$

so the cutoffs that are that are used in Lemma A.2 to compute V_{T-2}^{LM} are $\mu_{T-1,1} = 0$, $\mu_{T-1,2} = \frac{\underline{b}}{\bar{b}}$, $\mu_{T-1,3} = \frac{(1+\delta)\bar{b}\underline{b} - \delta \underline{b}^2}{\delta \bar{b}^2 + (1-\delta)\bar{b}\underline{b}}$, and $\mu_{T-1,4} = 1$. It cannot then be the case that $V_{T-2}^{LM}(\mu_{T-1,2}) > V_{T-2}^{FP}(\mu_{T-1,2})$, because $V_{T-2}^{FP}(\mu_{T-1,2}) = (1+\delta+\delta^2)\underline{b}$, which is equal to the *full commitment* payoffs at those beliefs. Thus, low types mixing can never be optimal for the seller when $t = T-2$. $\square$

high mixing outcome in which the posterior after no purchase is $\mu_{t+1,i}$ is equivalent to the above low mixing outcome, because the price charged is $\underline{b}$.

[20]To see this, note that the function defined by the RHS of the above equation for V_t^{LM} is linear, and can only be strictly greater than V_t^{LM} at some point if it passes through V_t^{LM} exactly twice. Because it passes through V_t^{LM} at $\mu_{t+1,j}$ and V_t^{LM} is linear on the interval from $\mu_{t+1,j-1}$ to $\mu_{t+1,j}$, the RHS is strictly higher than V_t^{LM} only if it passes through V_t^{LM} at some point below $\mu_{t+1,j-1}$, which would make it strictly higher than V_t^{LM} *at* $\mu_{t+1,j-1}$.

A.5 Proof of Proposition 2

Take any equilibrium in which the high type purchases with probability one and the low type mixes. The strategies of such an equilibrium are given in Online Appendix B.1. Generically in these cases a low valuation buyer receives 0 regardless of his choice and a high valuation buyer strictly prefers to purchase. In this case, the seller could improve profits by offering a menu of contracts. One contract charges price $\underline{b}$ to receive the good with certainty, while the other contract charges price $\varepsilon\underline{b}$ for likelihood ε of receiving the good. For ε small enough, the low valuation buyer can mix at the same rate, no incentive constraints are violated, and payoffs for the seller strictly increase. $\square$

Zachary Breig

Repeated Contracting without Commitment

Online Appendix

A Additional Proofs

A.1 Proof of Lemma A.1

Since both types of buyer receive the same allocation, they receive the same price which is optimally set at $\underline{b}$ to satisfy the low valuation buyer's individual rationality constraint. Beliefs do not change going into the next period. The seller receives her discounted continuation value plus the price $\underline{b}$ and a high valuation buyer receives his discounted continuation value plus the difference between his consumption value and the price. $\underline{W}_t^{FP}(\mu_t)$ and $\bar{W}_t^{FP}(\mu_t)$ are decreasing step functions because, by assumption, $\underline{W}_{t+1}(\mu_t)$ and $\bar{W}_{t+1}(\mu_t)$ are decreasing step functions. $\square$

A.2 Proof of Lemma A.2

If low types are mixing between purchasing and not purchasing, then low types' incentive compatibility constraints imply that the price the seller charges must be $\underline{b}$. Thus, the seller's problem amounts to determining the rate at which each type mixes into purchasing. Defining $\underline{x}_t$ and $\bar{x}_t$ as the probabilities with which low types and high types purchase, this problem can be written as

$$\begin{aligned}\max_{\underline{x}_t,\bar{x}_t}\ & [(1-\mu_t)(1-\underline{x}_t)+\mu_t(1-\bar{x}_t)]\left[\delta V_{t+1}\left(\frac{\mu_t(1-\bar{x}_t)}{(1-\mu_t)(1-\underline{x}_t)+\mu_t(1-\bar{x}_t)}\right)\right] \\ & +[(1-\mu_t)\underline{x}_t+\mu_t\bar{x}_t]\left[\underline{b}+\delta V_{t+1}\left(\frac{\mu_t\bar{x}_t}{(1-\mu_t)\underline{x}_t+\mu_t\bar{x}_t}\right)\right]\end{aligned}$$

subject to

$$\underline{x}_t,\bar{x}_t\in[0,1]$$

$$\bar{b}-\underline{b}+\delta\bar{W}_{t+1}\left(\frac{\mu_t\bar{x}_t}{(1-\mu_t)\underline{x}_t+\mu_t\bar{x}_t}\right)\geq\delta\underline{W}_{t+1}\left(\frac{\mu_t(1-\bar{x}_t)}{(1-\mu_t)(1-\underline{x}_t)+\mu_t(1-\bar{x}_t)}\right)$$

and, if $\bar{x}_t<1$,

$$\bar{b}-\underline{b}+\delta\underline{W}_{t+1}\left(\frac{\mu_t\bar{x}_t}{(1-\mu_t)\underline{x}_t+\mu_t\bar{x}_t}\right)\leq\delta\bar{W}_{t+1}\left(\frac{\mu_t(1-\bar{x}_t)}{(1-\mu_t)(1-\underline{x}_t)+\mu_t(1-\bar{x}_t)}\right).$$

This second constraint is the high type buyer's incentive compatibility constraint, which is that purchasing in the given period must be giving high type buyers weakly higher payoffs than not purchasing.[21] However, the payoffs from purchasing and not purchasing must be equal if the high valuation buyer is randomizing between the two options. The second and third constraints, in combination, guarantee that a continuation value exists that makes the payoffs equal.

Rather than solving for the rates at which each type randomizes into purchasing, it will be useful to instead solve directly for the posteriors that result from this randomization. Using this logic, we can define $\bar{\mu}$ as the posterior beliefs the seller has after observing a purchase and $\underline{\mu}$ as the beliefs after observing the buyer not purchase. With these definitions, the problem can be restated as[22]

$$\max_{\underline{\mu},\bar{\mu}} \left(\frac{\bar{\mu}-\mu_t}{\bar{\mu}-\underline{\mu}}\right)\left[\delta V_{t+1}(\underline{\mu})\right] + \left(\frac{\mu_t-\underline{\mu}}{\bar{\mu}-\underline{\mu}}\right)\left[\underline{b}+\delta V_{t+1}(\bar{\mu})\right]$$

subject to

$$0 \leq \underline{\mu} \leq \mu_t \leq \bar{\mu} \leq 1$$
$$\bar{b}-\underline{b}+\delta\bar{W}_{t+1}\left(\bar{\mu}\right) \geq \delta\underline{W}_{t+1}\left(\underline{\mu}\right)$$

and, if $\underline{\mu} > 0$,

$$\bar{b}-\underline{b}+\delta\underline{W}_{t+1}\left(\bar{\mu}\right) \leq \delta\bar{W}_{t+1}\left(\underline{\mu}\right).$$

Where it is differentiable with respect to $\underline{\mu}$, the derivative of the objective function is

$$\left(\frac{\bar{\mu}-\mu_t}{(\bar{\mu}-\underline{\mu})^2}\right)\delta V_{t+1}(\underline{\mu}) + \left(\frac{\bar{\mu}-\mu_t}{\bar{\mu}-\underline{\mu}}\right)\delta V'_{t+1}(\underline{\mu}) - \left(\frac{\bar{\mu}-\mu_t}{(\bar{\mu}-\underline{\mu})^2}\right)\left[\underline{b}+\delta V_{t+1}(\bar{\mu})\right]$$

Because $V_{t+1}(\underline{\mu})$ is convex and increasing, $V'_{t+1}(\underline{\mu}) \leq \frac{V_{t+1}(\bar{\mu})-V_{t+1}(\underline{\mu})}{\bar{\mu}-\underline{\mu}}$, so the derivative is less than

$$-\left(\frac{\bar{\mu}-\mu_t}{(\bar{\mu}-\underline{\mu})^2}\right)\underline{b}$$

[21] The particular use of $\underline{W}$ and $\bar{W}$ ensures that *there exist* continuation values that are consistent with incentive compatibility and the chosen allocation. If the inequalities do not hold, then there are no continuation values consistent with equilibrium that the seller could promise high valuation buyers in order to induce them to purchase when low valuation buyers are purchasing at rate $\underline{x}_t$. This same reasoning applies to the formulations of the seller's problem in the proofs that follow.

[22] The constraint that $\underline{\mu} \leq \mu_t \leq \bar{\mu}$ implies that high valuation buyers purchase with a higher likelihood than low valuation buyers. While the assumptions on $\underline{W}_{t+1}$ have not ruled out the possibility that low valuation buyers purchase with a higher likelihood than high valuation buyers, the incentive compatibility constraints imply that both posteriors must be on an interval $[\mu_{t+1,i-1}, \mu_{t+1,i}]$ for some i. One can show that such mixing can never be optimal.

and the objective function is decreasing in $\underline{\mu}$. Because W_{t+1} is part of a Spot Payoff Pair, both $\underline{W}_{t+1}(\mu_{t+1})$ and $\bar{W}_{t+1}(\mu_{t+1})$ are decreasing step functions. For a fixed $\bar{\mu}$, decreasing $\underline{\mu}$ does not affect the constraints unless $\underline{\mu}$ is equal to some $\mu_{t+1,i}$. Thus, it is always optimal for $\underline{\mu}$ to be equal to the lowest $\mu_{t+1,i}$ for which the constraints are not violated.

For $\bar{\mu}$ and $\mu_{t+1,j}$ such that $\mu_{t+1,j} < \bar{\mu} < \mu_{t+1,j+1}$, the objective function is twice continuously differentiable in $\bar{\mu}$ with a first derivative of

$$\begin{aligned} &\frac{\mu_t - \underline{\mu}}{(\bar{\mu} - \underline{\mu})^2}\delta V_{t+1}(\underline{\mu}) + \frac{\underline{\mu} - \mu_t}{(\bar{\mu} - \underline{\mu})^2}[\underline{b} + \delta V_{t+1}(\bar{\mu})] + \frac{\mu_t - \underline{\mu}}{\bar{\mu} - \underline{\mu}}\delta V'_{t+1}(\bar{\mu}) \\ &= \frac{\underline{\mu} - \mu_t}{\bar{\mu} - \underline{\mu}}\left[\frac{\underline{b} + \delta V_{t+1}(\bar{\mu}) - \delta V_{t+1}(\underline{\mu})}{\bar{\mu} - \underline{\mu}} - \delta V'_{t+1}(\bar{\mu})\right] \end{aligned}$$

Because $V_{t+1}(\mu_{t+1})$ is linear on the range $(\mu_{t+1,j}, \mu_{t+1,j+1})$, for any $\bar{\mu}'$ and $\bar{\mu}''$ on that range,

$$V_{t+1}(\bar{\mu}'') = V_{t+1}(\bar{\mu}') + (\bar{\mu}'' - \bar{\mu}')V'_{t+1}(\bar{\mu}')$$

so the derivative of the objective function with respect to $\bar{\mu}$ at $\bar{\mu}''$ is

$$\begin{aligned} &\frac{\underline{\mu} - \mu_t}{\bar{\mu}'' - \underline{\mu}}\left[\frac{\underline{b} + \delta V_{t+1}(\bar{\mu}'') - \delta V_{t+1}(\underline{\mu})}{\bar{\mu}'' - \underline{\mu}} - \delta V'_{t+1}(\bar{\mu}'')\right] \\ &= \frac{\underline{\mu} - \mu_t}{\bar{\mu}'' - \underline{\mu}}\left[\frac{\underline{b} + \delta V_{t+1}(\bar{\mu}') + \delta(\bar{\mu}'' - \bar{\mu}')V'_{t+1}(\bar{\mu}') - \delta V_{t+1}(\underline{\mu})}{\bar{\mu}'' - \underline{\mu}} - \delta V'_{t+1}(\bar{\mu}'')\right] \\ &= \frac{\underline{\mu} - \mu_t}{\bar{\mu}'' - \underline{\mu}}\left[\left(\frac{\underline{b} + \delta V_{t+1}(\bar{\mu}') - \delta V_{t+1}(\underline{\mu})}{\bar{\mu}' - \underline{\mu}}\right)\left(\frac{\bar{\mu}' - \underline{\mu}}{\bar{\mu}'' - \underline{\mu}}\right) + \frac{\bar{\mu}'' - \bar{\mu}'}{\bar{\mu}'' - \underline{\mu}}\delta V'_{t+1}(\bar{\mu}') - \delta V'_{t+1}(\bar{\mu}')\right] \\ &= \left(\frac{\bar{\mu}' - \underline{\mu}}{\bar{\mu}'' - \underline{\mu}}\right)^2\left(\frac{\underline{\mu} - \mu_t}{\bar{\mu}' - \underline{\mu}}\right)\left[\frac{\underline{b} + \delta V_{t+1}(\bar{\mu}') - \delta V_{t+1}(\underline{\mu})}{\bar{\mu}' - \underline{\mu}} - \delta V'_{t+1}(\bar{\mu}')\right]. \end{aligned}$$

This is the derivative of the objective function with respect to $\bar{\mu}$ at $\bar{\mu}'$ multiplied by $\left(\frac{\bar{\mu}'-\underline{\mu}}{\bar{\mu}''-\underline{\mu}}\right)^2$. Thus, the derivative of the objective function with respect to $\bar{\mu}$ at $\bar{\mu}'$ is strictly positive (negative) if and only if the derivative at $\bar{\mu}''$ is strictly positive (negative). Phrased differently, this means that the *sign* of the derivative is constant on the interval $(\mu_{t+1,j}, \mu_{t+1,j+1})$. For a fixed $\underline{\mu}$, increasing $\bar{\mu}$ does not affect the constraints unless $\bar{\mu}$ is equal to some $\mu_{t+1,j}$. Thus, it will either be strictly optimal to set $\bar{\mu}$ equal to some $\mu_{t+1,j}$ or the objective function is constant in $\bar{\mu}$ on the interval $[\mu_{t+1,j}, \mu_{t+1,j+1}]$.

Finally, note that there will exist a $\underline{\mu}$ such that setting $\bar{\mu} = \mu_t$ satisfies all of the problem's constraints. This would imply that all buyers purchase the good with probability one, which is equivalent to the full pooling setting. While this may be the optimal "low mixing" outcome, for the purpose of exposition I rule this out, as the optimal outcome for the current period will already take into account the possibility of full pooling. I also rule out setting $\bar{\mu}$ equal to *any* value on the interval $[\mu_t, \mu_{t+1,j})$ where $\mu_{t+1,j}$ is the lowest cutoff which is strictly higher than μ. Due to the above proven fact that the sign of the derivative is constant between any two cutoffs, setting $\bar{\mu}$ equal to μ_t or $\mu_{t+1,j}$ will always dominate setting $\bar{\mu} \in (\mu_t, \mu_{t+1,j})$.

Given the fact that for any μ_t, $\underline{\mu}$ and $\bar{\mu}$ will be set equal to the cutoffs in the next period, V_t^{LM} is the upper envelope of a finite set of line segments connecting $\delta V_{t+1}(\mu_{t+1,i})$ and $\underline{b} + \delta V_{t+1}(\mu_{t+1,j})$ for some i and j. This envelope must itself be piecewise linear.

I now show that the only discontinuities in $V_t^{\text{LM}}(\mu_t)$ are downward jump discontinuities which occur for some $\mu_t \in \{\mu_{t+1,2}, \ldots, \mu_{t+1,N_{t+1}-1}\}$. Because $V_t^{\text{LM}}(\mu_t)$ is made up of the upper envelope of a set of line segments, it can only have a discontinuity at one of the endpoints of those line segments. Furthermore, each segment extends from $\delta V_{t+1}(\mu_{t+1,i})$ to $\underline{b} + \delta V_{t+1}(\mu_{t+1,j})$ for $\mu_{t+1,i} < \mu_{t+1,j}$, so because $\underline{b} > 0$ the discontinuities can only be downward jumps. $\square$

A.3 Proof of Corollary A.1

First, I will show that there exists a "low mixing" outcome in period t in which the seller receives $V_t^{LM}(\mu_t)$ and a high type buyer receives

$$\min_{(i,j)\in I_t^{LM}(\mu_t)} \left[\max\{\bar{b} - \underline{b} + \delta \underline{W}_{t+1}(\mu_{t+1,j}), \delta \underline{W}_{t+1}(\mu_{t+1,i})\}\right].$$

For any (i^*, j^*) that is an element of the arg min of the above problem, define

$$\omega = \begin{cases} \delta \underline{W}_{t+1}(\mu_{t+1,i^*}) \text{ if } i^* = 1 \\ \max\{\bar{b} - \underline{b} + \delta \underline{W}_{t+1}(\mu_{t+1,j^*}), \delta \underline{W}_{t+1}(\mu_{t+1,i^*})\} \text{ otherwise} \end{cases}$$

and

$$\omega' = \max\{\delta \underline{W}_{t+1}(\mu_{t+1,j^*}), \delta \underline{W}_{t+1}(\mu_{t+1,i^*}) - \bar{b} + \underline{b}\}.$$

The seller receives payoffs of $V_t^{LM}(\mu_t)$ by charging a price of $\underline{b}$ and promising high-type continuation values of ω if the buyer does not purchase the good in period t and ω' if the buyer does purchase in period t. In this case, the buyers are mixing such that the seller's posteriors are μ_{t+1,i^*} when observing the buyer not purchase and μ_{t+1,j^*} when observing the buyer purchase. The seller receives $V_t^{LM}(\mu_t)$ because $(i^*, j^*) \in I_t^{LM}(\mu_t)$. The promised continuation values are incentive compatible for high types because $\omega \leq \omega' + \bar{b} - \underline{b}$ when $i^* = 1$ and $\omega = \omega' + \bar{b} - \underline{b}$ when $i^* > 1$. We know that $\omega \geq \delta \underline{W}_{t+1}(\mu_{t+1,i^*})$ by definition, and the fact that $(i^*, j^*) \in I_t^{LM}(\mu_t)$ guarantees that $\omega \leq \delta \bar{W}_{t+1}(\mu_{t+1,i^*})$.[23] Thus, $\frac{1}{\delta}\omega \in W_{t+1}(\mu_{t+1,i^*})$. Similarly, the definition of ω' and the fact that $(i^*, j^*) \in I_t^{LM}(\mu_t)$ guarantees that $\frac{1}{\delta}\omega' \in W_{t+1}(\mu_{t+1,j^*})$. Thus, for this (i^*, j^*), there exists an incentive compatible allocation with associated continuation values with which the high type buyer receives

$$\max\{\bar{b} - \underline{b} + \delta \underline{W}_{t+1}(\mu_{t+1,j^*}), \delta \underline{W}_{t+1}(\mu_{t+1,i^*})\}$$

while the seller solves her problem.

Next, I will show that there exists a "low mixing" outcome in period t in which the seller receives $V_t^{LM}(\mu_t)$ and a high type buyer receives

$$\max_{(i,j) \in I_t^{LM}(\mu_t)} \left[\bar{b} - \underline{b} + \delta \bar{W}_{t+1}(\mu_{t+1,j})\right]$$

For any (i^*, j^*) that is an element of the arg max of the above problem, define

$$\omega = \begin{cases} \delta \underline{W}_{t+1,}(\mu_{t+1,i^*}) \text{ if } i^* = 1 \\ \delta \bar{W}_{t+1}(\mu_{t+1,j^*}) + \bar{b} - \underline{b} \text{ otherwise} \end{cases}$$

and

$$\omega' = \delta \bar{W}_{t+1}(\mu_{t+1,j^*}).$$

The seller receives payoffs of $V_t^{LM}(\mu_t)$ by charging a price of $\underline{b}$ and promising high-type continuation values of ω if the buyer does not purchase the good in period t and ω' if the buyer does purchase in period t. In this case, the buyers are mixing such that the seller's posteriors are μ_{t+1,i^*} when observing the buyer not purchase and μ_{t+1,j^*} when observing the buyer purchase. The seller receives $V_t^{LM}(\mu_t)$ because $(i^*, j^*) \in I_t^{LM}(\mu_t)$. The promised continuation values are incentive compatible for high types because $\omega \leq \omega' + \bar{b} - \underline{b}$ when $i^* = 1$ and $\omega = \omega' + \bar{b} - \underline{b}$ when $i^* > 1$. The definition of ω combined with the fact that $(i^*, j^*) \in I_t^{LM}(\mu_t)$ implies that $\frac{1}{\delta}\omega \in W(\mu_{t+1,i^*})$ and the definition of ω' implies that $\frac{1}{\delta}\omega' \in W_{t+1}(\mu_{t+1,j^*})$. Thus, for this (i^*, j^*), there exists an incentive compatible allocation with associated continuation values with which the high type buyer receives

$$\bar{b} - \underline{b} + \delta \bar{W}_{t+1}(\mu_{t+1,j^*})$$

[23] This is immediate if $\omega = \delta \underline{W}_{t+1}(\mu_{t+1,i^*})$. If instead $\omega = \bar{b} - \underline{b} + \delta \underline{W}_{t+1}(\mu_{t+1,j^*})$, then $i^* > 1$ and the constraints in the problem stated in (1) guarantee that if $i^* > 1$, $\bar{b} - \underline{b} + \delta \underline{W}_{t+1}(\mu_{t+1,j^*}) \leq \delta \bar{W}_{t+1}(\mu_{t+1,i^*})$.

while the seller solves her problem.

Lemma A.2 shows that the seller only maximizes her payoffs conditional on low types mixing if buyers mix such that posteriors are μ_{t+1,i^*} and μ_{t+1,j^*} for $(i^*,j^*) \in I_t^{LM}(\mu_t)$. Because a high type receives payoffs of $\bar{b}-\underline{b}$ from purchasing in period t, the only way he can receive payoffs higher than $\max_{(i,j)\in I_t^{LM}(\mu_t)} \left[\bar{b}-\underline{b}+\delta\bar{W}_{t+1}(\mu_{t+1,j})\right]$ is if the seller promises him continuation payoffs higher than $\max_{(i,j)\in I_t^{LM}(\mu_t)} \delta\bar{W}_{t+1}(\mu_{t+1,j})$. However, the posteriors that would make such a promise sequentially rational would lead to lower payoffs in the current period. Similarly, high type payoffs which are lower than

$$\min_{(i,j)\in I_t^{LM}(\mu_t)} \left[\max\{\bar{b}-\underline{b}+\delta\underline{W}_{t+1}(\mu_{t+1,j}), \delta\underline{W}_{t+1}(\mu_{t+1,i})\}\right]$$

are inconsistent with posterior beliefs that correspond to maximal payoffs for the seller. Thus, there are no high type payoffs outside of the given range which are consistent with the seller maximizing her payoffs.

Given that high type payoffs of

$$\min_{(i,j)\in I_t^{LM}(\mu_t)} \left[\max\{\bar{b}-\underline{b}+\delta\underline{W}_{t+1}(\mu_{t+1,j}), \delta\underline{W}_{t+1}(\mu_{t+1,i})\}\right]$$

and

$$\max_{(i,j)\in I_t^{LM}(\mu_t)} \left[\bar{b}-\underline{b}+\delta\bar{W}_{t+1}(\mu_{t+1,j})\right]$$

can be promised when the seller is solving problem (1), the seller can promise any payoffs between these values by randomizing between the two equilibria.[24] Thus, $W_t^{LM}(\mu_t)$ is closed and convex.

The fact that $V_t^{LM}(\mu_t)$ is piecewise linear with the set of (i,j) which solve Problem (1) remaining constant on the interior of each segment implies that both $\underline{W}_t^{LM}(\mu_t)$ and $\bar{W}_t^{LM}(\mu_t)$ are step functions.

Finally, I show that $\underline{W}_t^{LM}(\mu_t)$ and $\bar{W}_t^{LM}(\mu_t)$ are decreasing. For μ'' and μ' such that $\mu' > \mu''$, take any $(i,j) \in I_t^{LM}(\mu'')$ and $(i',j') \in I_t^{LM}(\mu')$, and suppose that $j' < j$. Incentive compatibility when

[24]Here, because the seller is charging a price of $\underline{b}$ in two different continuation equilibria that both have a chance to be played, I assume that either the seller can directly communicate which equilibria she expects to be played (and thus how her beliefs will evolve conditional on the buyer's action) or that she can use a public randomization device to select the continuation equilibrium.

beliefs are μ'' implies that $\bar{b} - \underline{b} + \delta\bar{W}_{t+1}(\mu_{t+1,j}) \geq \delta\underline{W}_{t+1}(\mu_{t+1,i})$, so because $\bar{W}_{t+1}$ is decreasing, $\bar{b}-\underline{b}+\delta\bar{W}_{t+1}(\mu_{t+1,j'}) \geq \delta\underline{W}_{t+1}(\mu_{t+1,i})$. Thus, it is incentive compatible for the buyers to randomize such that the posteriors after buying or not buying are $\mu_{t+1,j'}$ or $\mu_{t+1,i}$, respectively. Since a lower $\mu_{t+1,i'}$ improves the seller's payoffs, low types can only be mixing optimally when $\mu_t = \mu'$ if $\mu_{t+1,i'}$ is no higher than $\mu_{t+1,i}$, or in other words, $i' \leq i$. But if $(i', j') \in I_t^{LM}(\mu')$, then

$$\begin{aligned}
&\left(\frac{\mu_{t+1,j'} - \mu'}{\mu_{t+1,j'} - \mu_{t+1,i'}}\right)\left[\delta V_{t+1}(\mu_{t+1,i'})\right] + \left(\frac{\mu' - \mu_{t+1,i'}}{\mu_{t+1,j'} - \mu_{t+1,i'}}\right)\left[\underline{b} + \delta V_{t+1}(\mu_{t+1,j'})\right] \\
&\geq \left(\frac{\mu_{t+1,j} - \mu'}{\mu_{t+1,j} - \mu_{t+1,i}}\right)\left[\delta V_{t+1}(\mu_{t+1,i})\right] + \left(\frac{\mu' - \mu_{t+1,i}}{\mu_{t+1,j} - \mu_{t+1,i}}\right)\left[\underline{b} + \delta V_{t+1}(\mu_{t+1,j})\right].
\end{aligned}$$

Algebraic manipulations of this inequality give

$$\begin{aligned}
&\left(\frac{\mu_{t+1,j'} - \mu''}{\mu_{t+1,j'} - \mu_{t+1,i'}}\right)\left[\delta V_{t+1}(\mu_{t+1,i'})\right] + \left(\frac{\mu'' - \mu_{t+1,i'}}{\mu_{t+1,j'} - \mu_{t+1,i'}}\right)\left[\underline{b} + \delta V_{t+1}(\mu_{t+1,j'})\right] \\
&+ (\mu' - \mu'')\delta\left[\frac{V_{t+1}(\mu_{t+1,j'}) - V_{t+1}(\mu_{t+1,i'})}{\mu_{t+1,j'} - \mu_{t+1,i'}} - \frac{V_{t+1}(\mu_{t+1,j}) - V_{t+1}(\mu_{t+1,i})}{\mu_{t+1,j} - \mu_{t+1,i}}\right] \\
&\geq \left(\frac{\mu_{t+1,j} - \mu''}{\mu_{t+1,j} - \mu_{t+1,i}}\right)\left[\delta V_{t+1}(\mu_{t+1,i})\right] + \left(\frac{\mu'' - \mu_{t+1,i}}{\mu_{t+1,j} - \mu_{t+1,i}}\right)\left[\underline{b} + \delta V_{t+1}(\mu_{t+1,j})\right].
\end{aligned}$$

Because V_{t+1} is convex, the term

$$\left[\frac{V_{t+1}(\mu_{t+1,j'}) - V_{t+1}(\mu_{t+1,i'})}{\mu_{t+1,j'} - \mu_{t+1,i'}} - \frac{V_{t+1}(\mu_{t+1,j}) - V_{t+1}(\mu_{t+1,i})}{\mu_{t+1,j} - \mu_{t+1,i}}\right] < 0,$$

so

$$\begin{aligned}
&\left(\frac{\mu_{t+1,j'} - \mu''}{\mu_{t+1,j'} - \mu_{t+1,i'}}\right)\left[\delta V_{t+1}(\mu_{t+1,i'})\right] + \left(\frac{\mu'' - \mu_{t+1,i'}}{\mu_{t+1,j'} - \mu_{t+1,i'}}\right)\left[\underline{b} + \delta V_{t+1}(\mu_{t+1,j'})\right] \\
&> \left(\frac{\mu_{t+1,j} - \mu''}{\mu_{t+1,j} - \mu_{t+1,i}}\right)\left[\delta V_{t+1}(\mu_{t+1,i})\right] + \left(\frac{\mu'' - \mu_{t+1,i}}{\mu_{t+1,j} - \mu_{t+1,i}}\right)\left[\underline{b} + \delta V_{t+1}(\mu_{t+1,j})\right].
\end{aligned}$$

This contradicts the assumption that $(i, j) \in I_t^{LM}(\mu'')$. Thus, it must be the case that $i' \geq i$ and $j' \geq j$. This in combination with the facts that $\underline{W}_{t+1}$ and $\bar{W}_{t+1}$ are decreasing and the given form of $\underline{W}_t^{LM}$ and $\bar{W}_t^{LM}$ implies that $\underline{W}_t^{LM}(\mu'') \geq \underline{W}_t^{LM}(\mu')$ and $\bar{W}_t^{LM}(\mu'') \geq \bar{W}_t^{LM}(\mu')$, so both functions are decreasing. □

A.4 Proof of Lemma A.3

When optimizing conditional on low types not purchasing and high types purchasing with probability greater than 0, the seller's problem amounts to choosing a price which will determine the proportion of high types that purchase in period t. This problem can be written as

$$\max_{p_t,\bar{x}_t} \; (1-\bar{x}_t\mu_t)\,\delta V_{t+1}\left(\frac{(1-\bar{x}_t)\mu_t}{1-\bar{x}_t\mu_t}\right)+(\bar{x}_t\mu_t)\left[p_t+\delta V_{t+1}(1)\right]$$

subject to

$$\begin{aligned}
&p_t \le \bar{b}-\delta\underline{W}_{t+1}\left(\frac{(1-\bar{x}_t)\mu_t}{1-\bar{x}_t\mu_t}\right)\\
&p_t \ge \bar{b}-\delta\bar{W}_{t+1}\left(\frac{(1-\bar{x}_t)\mu_t}{1-\bar{x}_t\mu_t}\right)\ \text{ if } \bar{x}_t<1\\
&p_t \ge \underline{b}\\
&\bar{x}_t \in [0,1]
\end{aligned}$$

where $\bar{x}_t$ is the probability that the high valuation types purchase. The first constraint is due to the fact that in order to be willing to purchase, a high value buyer must receive a payoff that is weakly higher than what he would receive if he did not purchase in the current period and instead imitated a low valuation buyer. The second constraint requires that when the high valuation buyer is supposed to randomize, the seller must be able to promise him a continuation value that makes him indifferent between purchasing and not purchasing. The third constraint is the "reverse incentive compatibility constraint." Since the seller's objective function is increasing in p_t, this problem can be rewritten as

$$\max_{\bar{x}_t} \; (\bar{x}_t\mu_t)\left[\bar{b}-\delta\underline{W}_{t+1}\left(\frac{(1-\bar{x}_t)\mu_t}{1-\bar{x}_t\mu_t}\right)+\delta V_{t+1}(1)\right]+(1-\bar{x}_t\mu_t)\,\delta V_{t+1}\left(\frac{(1-\bar{x}_t)\mu_t}{1-\bar{x}_t\mu_t}\right) \tag{3}$$

subject to

$$\begin{aligned}
&\bar{b}-\delta\underline{W}_{t+1}\left(\frac{(1-\bar{x}_t)\mu_t}{1-\bar{x}_t\mu_t}\right)\ge \underline{b}\\
&\bar{x}_t \in [0,1].
\end{aligned}$$

Where this problem has a solution, define the seller's profits as $V_t^{\mathrm{HM}}(\mu_t)$ (otherwise, let $V_t^{\mathrm{HM}}(\mu_t) = -\infty$).

Problem (3) does not have a solution when $\delta\underline{W}_{t+1}(\mu_{t+1}) > \bar{b}-\underline{b}$. Because $\underline{W}$ is decreasing, this implies that for all $\mu_t < \mu_{t+1,\underline{i}}$, $V_t^{HM}(\mu_t) = -\infty$.

Similar to the proof of A.2, instead of solving for the rate at which the high type is randomizing into purchasing, we can instead focus on the posteriors $\underline{\mu}$ the seller has after observing the buyer not purchase. Thus, the problem can be written as

$$\max_{\underline{\mu}} \left(\frac{1-\mu_t}{1-\underline{\mu}}\right)\left[\delta V_{t+1}(\underline{\mu})\right] + \left(\frac{\mu_t-\underline{\mu}}{1-\underline{\mu}}\right)\left[\bar{b} - \delta \underline{W}_{t+1}(\underline{\mu}) + \delta V_{t+1}(1)\right]$$

$$\text{subject to } \mu_{t+1,\underline{i}} \leq \underline{\mu} \leq \mu_t.$$

Where it is differentiable with respect to $\underline{\mu}$, the derivative of the objective function is

$$\left(\frac{1-\mu_t}{(1-\underline{\mu})^2}\right)\delta V_{t+1}(\underline{\mu}) + \left(\frac{1-\mu_t}{1-\underline{\mu}}\right)\delta V'_{t+1}(\underline{\mu}) - \left(\frac{1-\mu_t}{(1-\underline{\mu})^2}\right)\left[\bar{b} - \delta \underline{W}_{t+1}(\underline{\mu}) + \delta V_{t+1}(1)\right].$$

Because $V_{t+1}(\underline{\mu})$ is convex and increasing, $V'_{t+1}(\underline{\mu}) \leq \frac{V_{t+1}(1)-V_{t+1}(\underline{\mu})}{1-\underline{\mu}}$, so the derivative of the objective function is less than

$$-\left(\frac{\bar{\mu}-\mu_t}{(\bar{\mu}-\underline{\mu})^2}\right)\left[\bar{b} - \delta \underline{W}_{t+1}(\underline{\mu})\right]$$

and the objective function is decreasing in $\underline{\mu}$. However, the objective function has discrete jumps upward when $\underline{\mu}$ is equal to the points at which $\underline{W}_{t+1}(\underline{\mu})$ has discrete jumps downward, which are exactly the cutoffs of the Spot Payoff Pair. Thus, the seller's problem is solved when the high type buyer randomizes such that the seller's posteriors are equal to the payoff maximizing cutoff from the Spot Payoff Pair which is less than current beliefs μ and greater than the minimal cutoff $\mu_{t+1,\underline{i}}$.

Given the fact that for any μ_t, $\underline{\mu}$ will be set equal to the cutoffs in the next period, V_t^{HM} is the upper envelope of a finite set of line segments connecting $\delta V_{t+1}(\mu_{t+1,i})$ and $\bar{b} - \delta \underline{W}_{t+1}(\mu_{t+1,i}) + \delta V_{t+1}(1)$ for some i. This envelope must itself be piecewise linear. Furthermore, because all of these line segments are increasing, the envelope must be convex. $\square$

A.5 Proof of Corollary A.2

Lemma A.3 shows that conditional on the "high mixing" outcome being the one implemented, the seller has posteriors of 1 conditional on observing the buyer purchase and some $\mu_{t+1,i}$ on observing the buyer not purchase. In this case, the seller is charging the buyer the highest price such that the buyer is indifferent between purchasing and not purchasing in period t, given these posteriors and their associated continuation values in period $t+1$. This highest price corresponds to promising the lowest possible continuation value conditional on not purchasing, which is exactly $\delta \underline{W}_{t+1}(\mu_{t+1,i})$. When there are multiple cutoffs in the next period which solve the seller's problem, the seller can promise the highest or lowest continuation values which are consistent with these cutoffs. Furthermore, because the seller can choose to randomize among continuation equilibria, she is able to offer any continuation in the convex hull of these continuation values.

There are no attainable payoffs for the high type buyer which are outside of the range

$$\left[\min_{i\in I_t^{HM}(\mu_t)} \delta \underline{W}_{t+1}(\mu_{t+1,i}), \max_{i\in I_t^{HM}(\mu_t)} \delta \underline{W}_{t+1}(\mu_{t+1,i})\right].$$

If the high type buyer were to receive payoffs outside this range, Lemma A.3 implies that would imply that the seller is not receiving her maximal equilibrium payoffs in period t.

Given that the payoffs of $\min_{i\in I_t^{HM}(\mu_t)} \delta \underline{W}_{t+1}(\mu_{t+1,i})$ and $\max_{i\in I_t^{HM}(\mu_t)} \delta \underline{W}_{t+1}(\mu_{t+1,i})$ can be promised to the high type when the seller is solving problem (2), the seller can promise any payoffs between these values by randomizing between the two equilibria. Thus, $W_t^{HM}(\mu_t)$ is closed and convex.

The fact that $V_t^{HM}(\mu_t)$ is piecewise linear with the set of i that solve Problem (2) remaining constant on the interior of each segment implies that both $\underline{W}_t^{HM}(\mu_t)$ and $\underline{W}_t^{HM}(\mu_t)$ are step functions.

Finally, I show that $\underline{W}_t^{HM}(\mu_t)$ and $\bar{W}_t^{HM}(\mu_t)$ are decreasing. For μ'' and μ' such that $\mu' > \mu''$, take any $i \in I_t^{HM}(\mu'')$ and $i' \in I_t^{HM}(\mu')$, and suppose that $i' < i$. Then

$$\begin{aligned}
&\left(\frac{1-\mu'}{1-\mu_{t+1,i'}}\right)\left[\delta V_{t+1}(\mu_{t+1,i'})\right] + \left(\frac{\mu'-\mu_{t+1,i'}}{1-\mu_{t+1,i'}}\right)\left[\bar{b} - \delta \underline{W}_{t+1}(\mu_{t+1,i'}) + \delta V_{t+1}(1)\right] \\
&\geq \left(\frac{1-\mu'}{1-\mu_{t+1,i}}\right)\left[\delta V_{t+1}(\mu_{t+1,i})\right] + \left(\frac{\mu'-\mu_{t+1,i}}{1-\mu_{t+1,i}}\right)\left[\bar{b} - \delta \underline{W}_{t+1}(\mu_{t+1,i}) + \delta V_{t+1}(1)\right]
\end{aligned}$$

Algebraic manipulations of this term give that

$$\begin{aligned}
&\left(\frac{1-\mu''}{1-\mu_{t+1,i'}}\right)\left[\delta V_{t+1}(\mu_{t+1,i'})\right] + \left(\frac{\mu''-\mu_{t+1,i'}}{1-\mu_{t+1,i'}}\right)\left[\bar{b} - \delta \underline{W}_{t+1}(\mu_{t+1,i'}) + \delta V_{t+1}(1)\right] \\
&+ (\mu''-\mu')\left[\frac{\delta(V_{t+1}(1)-V_{t+1}(\mu_{t+1,i}))}{1-\mu_{t+1,i}} - \frac{\delta(V_{t+1}(1)-V_{t+1}(\mu_{t+1,i'}))}{1-\mu_{t+1,i'}}\right] \\
&+ (\mu''-\mu')\left[\frac{\bar{b}-\delta \underline{W}_{t+1}(\mu_{t+1,i})}{1-\mu_{t+1,i}} - \frac{\bar{b}-\delta \underline{W}_{t+1}(\mu_{t+1,i'})}{1-\mu_{t+1,i'}}\right] \\
&\geq \left(\frac{1-\mu''}{1-\mu_{t+1,i}}\right)\left[\delta V_{t+1}(\mu_{t+1,i})\right] + \left(\frac{\mu''-\mu_{t+1,i}}{1-\mu_{t+1,i}}\right)\left[\bar{b} - \delta \underline{W}_{t+1}(\mu_{t+1,i}) + \delta V_{t+1}(1)\right].
\end{aligned}$$

Combining the maintained assumptions that $\mu' > \mu''$ and $\mu_{t+1,i'} < \mu_{t+1,i}$ with the fact that V_{t+1} is convex, we find that

$$(\mu''-\mu')\left[\frac{\delta(V_{t+1}(1)-V_{t+1}(\mu_{t+1,i}))}{1-\mu_{t+1,i}} - \frac{\delta(V_{t+1}(1)-V_{t+1}(\mu_{t+1,i'}))}{1-\mu_{t+1,i'}}\right] < 0.$$

Similarly when we combine the maintained assumptions that $\mu' > \mu''$ and $\mu_{t+1,i'} < \mu_{t+1,i}$ with the fact that $\underline{W}_{t+1}$ is decreasing, we find that

$$(\mu'' - \mu')\left[\frac{\bar{b} - \delta \underline{W}_{t+1}(\mu_{t+1,i})}{1 - \mu_{t+1,i}} - \frac{\bar{b} - \delta \underline{W}_{t+1}(\mu_{t+1,i'})}{1 - \mu_{t+1,i'}}\right] < 0.$$

Thus,

$$\begin{aligned}
&\left(\frac{1-\mu''}{1-\mu_{t+1,i'}}\right)\left[\delta V_{t+1}(\mu_{t+1,i'})\right] + \left(\frac{\mu'' - \mu_{t+1,i'}}{1-\mu_{t+1,i'}}\right)\left[\bar{b} - \delta \underline{W}_{t+1}(\mu_{t+1,i'}) + \delta V_{t+1}(1)\right] \\
&> \left(\frac{1-\mu''}{1-\mu_{t+1,i}}\right)\left[\delta V_{t+1}(\mu_{t+1,i})\right] + \left(\frac{\mu'' - \mu_{t+1,i}}{1-\mu_{t+1,i}}\right)\left[\bar{b} - \delta \underline{W}_{t+1}(\mu_{t+1,i}) + \delta V_{t+1}(1)\right],
\end{aligned}$$

but this contradicts the fact that $i \in I_t^{HM}(\mu'')$. Thus, for μ'' and μ' such that $\mu' > \mu''$ and any $i \in I_t^{HM}(\mu'')$ and $i' \in I_t^{HM}(\mu')$, it must be the case that $i' \geq i$. This shows that

$$\min_{i \in I_t^{HM}(\mu'')} \delta \underline{W}_{t+1}(\mu_{t+1,i}) \geq \min_{i' \in I_t^{HM}(\mu')} \delta \underline{W}_{t+1}(\mu_{t+1,i'})$$

and

$$\max_{i \in I_t^{HM}(\mu'')} \delta \underline{W}_{t+1}(\mu_{t+1,i}) \geq \max_{i' \in I_t^{HM}(\mu')} \delta \underline{W}_{t+1}(\mu_{t+1,i'}),$$

so both $\underline{W}_t^{HM}(\mu_t)$ and $\bar{W}_t^{HM}(\mu_t)$ are decreasing. $\square$

A.6 Proof of Lemma A.4

Define $\hat{\mu}_t$ as the infimum of the set $\{\mu_t : V_t^{HM}(\mu_t) > \max\{V_t^{FP}(\mu_t), V_t^{LM}(\mu_t)\}\}$. The lemma states that $\hat{\mu}_t \in (0,1)$ and that for all $\mu_t > \hat{\mu}_t$, $V_t^{HM}(\mu_t) > \max\{V_t^{FP}(\mu_t), V_t^{LM}(\mu_t)\}$.

First I will show that $\hat{\mu}_t \in (0,1)$. Notice that

$$\begin{aligned}
V_t^{HM}(\mu_{t+1,\underline{i}}) &= \delta V_{t+1}(\mu_{t+1,\underline{i}}) \\
&< \underline{b} + \delta V_{t+1}(\mu_{t+1,\underline{i}}) \\
&= V_t^{FP}(\mu_{t+1,\underline{i}}) \\
&\leq \max\{V_t^{FP}(\mu_{t+1,\underline{i}}), V_t^{LM}(\mu_{t+1,\underline{i}})\}.
\end{aligned}$$

The first line follows from Lemma A.3, the second from the fact that $\underline{b} > 0$, and the third line from Lemma A.1. It is also the case that

$$\begin{aligned}
V_t^{HM}(1) &= \bar{b} + \delta V_{t+1}(1) \\
&> \underline{b} + \delta V_{t+1}(1) \\
&= \max\{V_t^{FP}(\mu_{t+1,\underline{i}}), V_t^{LM}(\mu_{t+1,\underline{i}})\}.
\end{aligned}$$

The first line follows from the fact that when the buyer is a high type with probability one, the seller can charge price of $\bar{b}$ in every period and the buyer will purchase. The third line follows from the definitions of V_t^{FP} and V_t^{LM} given in Lemmas A.1 and A.2. Thus, $\mu_{t+1,\underline{i}} < \hat{\mu}_t < 1$.

Next, I will show that for any μ'' such that

$$V_t^{HM}(\mu'') > \max\{V_t^{FP}(\mu''), V_t^{LM}(\mu'')\},$$

and $\mu' > \mu''$,

$$V_t^{HM}(\mu') > \max\{V_t^{FP}(\mu'), V_t^{LM}(\mu')\}.$$

Suppose that $i \in I_t^{HM}(\mu'')$. Then by Lemma A.3, because $\mu' > \mu''$,

$$\begin{aligned}
V_t^{HM}(\mu') &\geq \left(\frac{1-\mu'}{1-\mu_{t+1,i}}\right)[\delta V_{t+1}(\mu_{t+1,i})] + \left(\frac{\mu'-\mu_{t+1,i}}{1-\mu_{t+1,i}}\right)\left[\bar{b} - \delta \underline{W}_{t+1}(\mu_{t+1,i}) + \delta V_{t+1}(1)\right] \\
&= \left(\frac{1-\mu'}{1-\mu''}\right)\left[V_t^{HM}(\mu'')\right] + \left(\frac{\mu'-\mu''}{1-\mu''}\right)\left[\bar{b} - \delta \underline{W}_{t+1}(\mu_{t+1,i}) + \delta V_{t+1}(1)\right] \\
&\geq \left(\frac{1-\mu'}{1-\mu''}\right)\left[V_t^{HM}(\mu'')\right] + \left(\frac{\mu'-\mu''}{1-\mu''}\right)\left[\underline{b} + \delta V_{t+1}(1)\right] \\
&= \left(\frac{1-\mu'}{1-\mu''}\right)\left[V_t^{HM}(\mu'')\right] + \left(\frac{\mu'-\mu''}{1-\mu''}\right)\left[V_t^{FP}(1)\right].
\end{aligned}$$

Because V_t^{FP} is convex,

$$V_t^{FP}(\mu') \leq \left(\frac{1-\mu'}{1-\mu''}\right)\left[V_t^{FP}(\mu'')\right] + \left(\frac{\mu'-\mu''}{1-\mu''}\right)\left[V_t^{FP}(1)\right]$$

and since by assumption $V_t^{FP}(\mu'') < V_t^{HM}(\mu'')$, $V_t^{HM}(\mu') > V_t^{FP}(\mu')$.

Lemma A.2 states that there exist $(\hat{\imath}, \hat{\jmath}) \in I_t^{LM}(\mu')$ such that

$$V_t^{LM}(\mu') = \left(\frac{\mu_{t+1,\hat{\jmath}} - \mu'}{\mu_{t+1,\hat{\jmath}} - \mu_{t+1,\hat{\imath}}}\right)\left[\delta V_{t+1}(\mu_{t+1,\hat{\imath}})\right] + \left(\frac{\mu' - \mu_{t+1,\hat{\imath}}}{\mu_{t+1,\hat{\jmath}} - \mu_{t+1,\hat{\imath}}}\right)\left[\underline{b} + \delta V_{t+1}(\mu_{t+1,\hat{\jmath}})\right].$$

If $\mu_{t+1,\hat{\imath}} \geq \mu''$,

$$\begin{aligned}
V_t^{LM}(\mu') &= \left(\frac{\mu_{t+1,\hat{\jmath}} - \mu'}{\mu_{t+1,\hat{\jmath}} - \mu_{t+1,\hat{\imath}}}\right)\left[\delta V_{t+1}(\mu_{t+1,\hat{\imath}})\right] + \left(\frac{\mu' - \mu_{t+1,\hat{\imath}}}{\mu_{t+1,\hat{\jmath}} - \mu_{t+1,\hat{\imath}}}\right)\left[\underline{b} + \delta V_{t+1}(\mu_{t+1,\hat{\jmath}})\right] \\
&< \left(\frac{\mu_{t+1,\hat{\jmath}} - \mu'}{\mu_{t+1,\hat{\jmath}} - \mu_{t+1,\hat{\imath}}}\right)\left[\underline{b} + \delta V_{t+1}(\mu_{t+1,\hat{\imath}})\right] + \left(\frac{\mu' - \mu_{t+1,\hat{\imath}}}{\mu_{t+1,\hat{\jmath}} - \mu_{t+1,\hat{\imath}}}\right)\left[\underline{b} + \delta V_{t+1}(\mu_{t+1,\hat{\jmath}})\right] \\
&\leq \left(\frac{1-\mu'}{1-\mu''}\right)\left[\underline{b} + \delta V_{t+1}(\mu'')\right] + \left(\frac{\mu'-\mu''}{1-\mu''}\right)\left[\underline{b} + \delta V_{t+1}(1)\right] \\
&= \left(\frac{1-\mu'}{1-\mu''}\right)\left[V_t^{FP}(\mu'')\right] + \left(\frac{\mu'-\mu''}{1-\mu''}\right)\left[V_t^{FP}(1)\right].
\end{aligned}$$

The second line follows because we have added a positive term, the third line follows because $\mu'' \leq \mu_{t+1,\hat{i}} < \mu_{t+1,\hat{j}} \leq 1$ and V_{t+1} is convex. Above, I showed that

$$V_t^{HM}(\mu') \geq \left(\frac{1-\mu'}{1-\mu''}\right)\left[V_t^{HM}(\mu'')\right] + \left(\frac{\mu'-\mu''}{1-\mu''}\right)\left[V_t^{FP}(1)\right],$$

so this implies that $V_t^{HM}(\mu') > V_t^{LM}(\mu')$ because by assumption, $V_t^{FP}(\mu'') < V_t^{HM}(\mu'')$.

Finally, we consider the case in which $\mu_{t+1,\hat{i}} < \mu''$. Here,

$$\begin{aligned}
V_t^{LM}(\mu') &= \left(\frac{\mu_{t+1,\hat{j}}-\mu'}{\mu_{t+1,\hat{j}}-\mu_{t+1,\hat{i}}}\right)\left[\delta V_{t+1}(\mu_{t+1,\hat{i}})\right] + \left(\frac{\mu'-\mu_{t+1,\hat{i}}}{\mu_{t+1,\hat{j}}-\mu_{t+1,\hat{i}}}\right)\left[\underline{b}+\delta V_{t+1}(\mu_{t+1,\hat{j}})\right] \\
&= \left(\frac{\mu_{t+1,\hat{j}}-\mu'}{\mu_{t+1,\hat{j}}-\mu''}\right)\left[\left(\frac{\mu_{t+1,\hat{j}}-\mu''}{\mu_{t+1,\hat{j}}-\mu_{t+1,\hat{i}}}\right)[\delta V_{t+1}(\mu_{t+1,\hat{i}})] + \left(\frac{\mu''-\mu_{t+1,\hat{i}}}{\mu_{t+1,\hat{j}}-\mu_{t+1,\hat{i}}}\right)[\underline{b}+\delta V_{t+1}(\mu_{t+1,\hat{j}})]\right] \\
&\quad + \left(\frac{\mu'-\mu''}{\mu_{t+1,\hat{j}}-\mu''}\right)\left[\underline{b}+\delta V_{t+1}(\mu_{t+1,\hat{j}})\right] \\
&\leq \left(\frac{\mu_{t+1,\hat{j}}-\mu'}{\mu_{t+1,\hat{j}}-\mu''}\right) V_t^{LM}(\mu'') + \left(\frac{\mu'-\mu''}{\mu_{t+1,\hat{j}}-\mu''}\right) V_t^{FP}(\mu_{t+1,\hat{j}}) \\
&\leq \left(\frac{\mu_{t+1,\hat{j}}-\mu'}{\mu_{t+1,\hat{j}}-\mu''}\right)\left[\max\{V_t^{FP}(\mu''), V_t^{LM}(\mu'')\}\right] + \left(\frac{\mu'-\mu''}{\mu_{t+1,\hat{j}}-\mu''}\right) V_t^{FP}(\mu_{t+1,\hat{j}}) \\
&\leq \left(\frac{1-\mu'}{1-\mu''}\right)\left[\max\{V_t^{FP}(\mu''), V_t^{LM}(\mu'')\}\right] + \left(\frac{\mu'-\mu''}{1-\mu''}\right) V_t^{FP}(1) \\
&< V_t^{HM}(\mu').
\end{aligned}$$

The second line is algebraic manipulation of the first. The third line uses the fact that $V_t^{FP}(\mu_{t+1,\hat{j}}) = \underline{b} + \delta V_{t+1}(\mu_{t+1,\hat{j}})$ and Lemma A.2 in that

$$\begin{aligned}
V_t^{LM}(\mu'') = &\max_{i,j} \left(\frac{\mu_{t+1,j}-\mu''}{\mu_{t+1,j}-\mu_{t+1,i}}\right)[\delta V_{t+1}(\mu_{t+1,i})] + \left(\frac{\mu''-\mu_{t+1,i}}{\mu_{t+1,j}-\mu_{t+1,i}}\right)[\underline{b}+\delta V_{t+1}(\mu_{t+1,j})] \\
&\quad \text{subject to } \bar{b}-\underline{b}+\delta \bar{W}_{t+1}(\mu_{t+1,j}) \geq \delta \underline{W}_{t+1}(\mu_{t+1,i}), \\
&\quad \bar{b}-\underline{b}+\delta \underline{W}_{t+1}(\mu_{t+1,j}) \leq \delta \bar{W}_{t+1}(\mu_{t+1,i}) \text{ if } \mu_{t+1,i} > 0, \\
&\quad \text{and } \mu_{t+1,i} \leq \mu'' < \mu_{t+1,j} \\
\geq &\left(\frac{\mu_{t+1,\hat{j}}-\mu''}{\mu_{t+1,\hat{j}}-\mu_{t+1,\hat{i}}}\right)[\delta V_{t+1}(\mu_{t+1,\hat{i}})] + \left(\frac{\mu''-\mu_{t+1,\hat{i}}}{\mu_{t+1,\hat{j}}-\mu_{t+1,\hat{i}}}\right)[\underline{b}+\delta V_{t+1}(\mu_{t+1,\hat{j}})].
\end{aligned}$$

The fourth line is immediate from the max operator, and the fifth line follows from the fact that V_t^{FP} is convex. The final line follows from the previously shown fact that

$$V_t^{HM}(\mu') \geq \left(\frac{1-\mu'}{1-\mu''}\right)\left[V_t^{HM}(\mu'')\right] + \left(\frac{\mu'-\mu''}{1-\mu''}\right)\left[V_t^{FP}(1)\right].$$

Thus, if for some μ'' and $\mu' > \mu''$ we have $V_t^{HM}(\mu'') > \max\{V_t^{FP}(\mu''), V_t^{LM}(\mu'')\}$, then we have $V_t^{HM}(\mu') > V_t^{FP}(\mu')$ and $V_t^{HM}(\mu') > V_t^{LM}(\mu')$, so

$$V_t^{HM}(\mu') > \max\{V_t^{FP}(\mu'), V_t^{LM}(\mu')\}.$$

Because $\hat{\mu}_t$ is the infimum of the set $\{\mu_t : V_t^{HM}(\mu_t) > \max\{V_t^{FP}(\mu_t), V_t^{LM}(\mu_t)\}\}$, this implies that for all $\mu_t > \hat{\mu}_t$, $V_t^{HM}(\mu_t) > \max\{V_t^{FP}(\mu_t), V_t^{LM}(\mu_t)\}$. $\square$

A.7 Proof of Lemma A.5

Lemmas A.1, A.2, and A.3 show that $V_t^{FP}(\mu_t)$, $V_t^{LM}(\mu_t)$, and $V_t^{HM}(\mu_t)$ are piecewise linear functions. Lemma A.1 shows that $V_t^{FP}(0) = \underline{b} + \delta V_{t+1}(0)$. Because $\mu_{t+1,1} = 0$ is the only $\mu_{t+1,i}$ that is weakly lower than 0, Lemma A.2 shows that $V_t^{LM}(0) = \delta V_{t+1}(0)$ and Lemma A.3 shows that either $V_t^{HM} = \delta V_{t+1}(0)$ or $V_t^{HM} = -\infty$. Thus, $V_t^{FP}(0) > \max\{V_t^{LM}(0), V_t^{HM}(0)\}$, so there is some $\underline{\mu}_t$ such that $V_t^{FP}(\mu_t) > \max\{V_t^{LM}(\mu_t), V_t^{HM}(\mu_t)\}$ if $\mu_t < \underline{\mu}_t$. $\square$

A.8 Proof of Lemma A.6

First, because $V_t(\mu_t)$ is the maximum of a finite set of piecewise linear functions, $V_t(\mu_t)$ itself is piecewise linear.

Next, I will show that $V_t(\mu_t)$ is continuous and increasing. Because

$$V_t(\mu_t) = \max\{V_t^{FP}(\mu_t), V_t^{LM}(\mu_t), V_t^{HM}(\mu_t)\},$$

$V_t(\mu_t)$ can only be discontinuous or decreasing at the points at which $V_t^{FP}(\mu_t)$, $V_t^{LM}(\mu_t)$,or $V_t^{HM}(\mu_t)$ are discontinuous or decreasing. Because V_{t+1} is continuous and increasing, Lemma A.1 shows that $V_t^{FP}(\mu_t)$ is continuous and increasing on $[0, 1]$. Lemma A.2 shows that $V_t^{LM}(\mu_t)$ has discontinuous jumps downward only at points where $V_t^{LM}(\mu_t) = V_t^{FP}(\mu_t)$. Thus, because $V_t^{FP}(\mu_t)$ is continuous and increasing at those points, $\max\{V_t^{FP}(\mu_t), V_t^{LM}(\mu_t)\}$ is continuous and increasing at these points. Finally, Lemma A.3 shows that $V_t^{HM}(\mu_t)$ discontinuous at most at the single point $\mu_{t+1,\underline{i}}$. However, at this point $V_t^{HM}(\mu_t) = \delta V_{t+1}(\mu_t) < \underline{b} + \delta V_{t+1}(\mu_t) = V_t^{FP}(\mu_t)$, so $\max\{V_t^{FP}(\mu_t), V_t^{LM}(\mu_t), V_t^{HM}(\mu_t)\}$ must be continuous and increasing there. Thus, $V_t(\mu_t)$ is continuous and increasing.

Next, I will show that $V_t(\mu_t)$ is convex. Because $V_t(\mu_t)$ is piecewise linear and continuous, it is concave only if at some point $\hat{\mu}$ where the derivative of $V_t(\mu_t)$ is discontinuous,

$$\lim_{\mu_t \to \hat{\mu}^-} V_t'(\mu_t) > \lim_{\mu_t \to \hat{\mu}^+} V_t'(\mu_t).$$

I will show that this is impossible through three cases.

1. Suppose that $\lim_{\mu_t \to \hat{\mu}^-} V_t'(\mu_t) > \lim_{\mu_t \to \hat{\mu}^+} V_t'(\mu_t)$ and $V_t(\mu_t) = V_t^{FP}(\mu_t)$ on the range $[\hat{\mu}, \hat{\mu} + \varepsilon)$ for ε small enough. Lemma A.1 shows that $V_t^{FP}(\mu_t)$ is increasing and convex. This implies that $\lim_{\mu_t \to \hat{\mu}^-} V_t^{FP\prime}(\mu_t) \leq \lim_{\mu_t \to \hat{\mu}^+} V_t'(\mu_t)$, which in turn implies that $V_t^{FP}(\mu_t) > V_t(\mu_t)$ on $(\hat{\mu} - \varepsilon, \hat{\mu})$. This is a contradiction because $V_t(\mu_t) = \max\{V_t^{FP}(\mu_t), V_t^{LM}(\mu_t), V_t^{HM}(\mu_t)\}$.

2. Suppose that $\lim_{\mu_t \to \hat{\mu}^-} V_t'(\mu_t) > \lim_{\mu_t \to \hat{\mu}^+} V_t'(\mu_t)$ and $V_t(\mu_t) = V_t^{LM}(\mu_t)$ on the range $[\hat{\mu}, \hat{\mu} + \varepsilon)$ for ε small enough. Lemma A.2 shows that at any point $\hat{\mu}$, $V_t^{LM}(\mu_t)$ is either linear on the range $(\hat{\mu} - \varepsilon, \hat{\mu} + \varepsilon)$, discontinuous at $\hat{\mu}$ with $\lim_{\mu_t \to \hat{\mu}^+} V_t^{LM}(\mu_t) < \underline{b} + \delta V_{t+1}(\hat{\mu})$, or piecewise linear with $\lim_{\mu_t \to \hat{\mu}^-} V_t^{LM\prime}(\mu_t) \leq \lim_{\mu_t \to \hat{\mu}^+} V_t^{LM\prime}(\mu_t)$.[25] The first and third cases imply that $V_t^{LM}(\mu_t) > V_t(\mu_t)$ on $(\hat{\mu} - \varepsilon, \hat{\mu})$, which is a contradiction because $V_t(\mu_t) = \max\{V_t^{FP}(\mu_t), V_t^{LM}(\mu_t), V_t^{HM}(\mu_t)\}$. The second case contradicts the assumption that $V_t(\mu_t) = V_t^{LM}(\mu_t)$ because on $[\hat{\mu}, \hat{\mu} + \varepsilon)$, $V_t^{LM}(\mu_t) < \underline{b} + \delta V_{t+1}(\hat{\mu}) = V_t^{FP}(\mu_t) \leq \max\{V_t^{FP}(\mu_t), V_t^{LM}(\mu_t), V_t^{HM}(\mu_t)\} = V_t(\mu_t)$.

3. Suppose that $\lim_{\mu_t \to \hat{\mu}^-} V_t'(\mu_t) > \lim_{\mu_t \to \hat{\mu}^+} V_t'(\mu_t)$ and $V_t(\mu_t) = V_t^{HM}(\mu_t)$ on the range $[\hat{\mu}, \hat{\mu} + \varepsilon)$ for ε small enough. Lemma A.3 shows that above some cutoff, $V_t^{HM}(\mu_t)$ is increasing and convex while below the cutoff, it is equal to negative infinity. At the cutoff itself, $V_t^{HM}(\mu_t) = \delta V_{t+1}(\mu_t)$. This implies that $\hat{\mu}$ is strictly higher than the cutoff, and that $\lim_{\mu_t \to \hat{\mu}^-} V_t^{HM\prime}(\mu_t) \leq \lim_{\mu_t \to \hat{\mu}^+} V_t'(\mu_t)$, which in turn implies that $V_t^{HM}(\mu_t) > V_t(\mu_t)$ on $(\hat{\mu} - \varepsilon, \hat{\mu})$. This is a contradiction because $V_t(\mu_t) = \max\{V_t^{FP}(\mu_t), V_t^{LM}(\mu_t), V_t^{HM}(\mu_t)\}$.

Since $V_t(\mu_t)$ must be equal to either $V_t^{FP}(\mu_t)$, $V_t^{LM}(\mu_t)$, or $V_t^{HM}(\mu_t)$, this implies that at any point $\hat{\mu}$ where the derivative of $V_t(\mu_t)$ is discontinuous,

$$\lim_{\mu_t \to \hat{\mu}^-} V_t'(\mu_t) < \lim_{\mu_t \to \hat{\mu}^+} V_t'(\mu_t).$$

and that $V_t(\mu_t)$ is convex.

Next, I must show that W_t takes the required form for (V_t, W_t) to be a Spot Payoff Pair. $W_t(\mu_t)$ is the set of payoffs the high type can receive given that the seller is optimizing her payoffs. Thus,

[25] This last case follows from the fact that V_t^{LM} is the upper envelope of a set of increasing line segments, so the slope must increase at any kink.

where $V_t^{HM}(\mu_t) > -\infty$,

$$\begin{aligned}\bar{W}_t(\mu_t) &= \max W_t(\mu_t) \\ &= \max_{i\in\{FP,LM,HM\}} \bar{W}_t^i(\mu_t) \\ &\quad \text{subject to } V_t^i(\mu_t) = V_t(\mu_t) \\ \underline{W}_t(\mu_t) &= \min W_t(\mu_t) \\ &= \min_{i\in\{FP,LM,HM\}} \underline{W}_t^i(\mu_t) \\ &\quad \text{subject to } V_t^i(\mu_t) = V_t(\mu_t)\end{aligned}$$

When $V_t^{HM}(\mu_t) = -\infty$, the maxima and minima are instead only taken over the payoffs from full pooling and low types mixing. These values exist because Lemma A.1, Corollary A.1, and Corollary A.2 show that $W_t^{FP}(\mu_t)$, $W_t^{LM}(\mu_t)$, and $W_t^{HM}(\mu_t)$ are closed and bounded. Finally, we know that $W_t(\mu_t) = [\underline{W}_t(\mu_t), \bar{W}_t(\mu_t)]$ because at the beginning of the period the seller can randomize which continuation equilibrium to play, so $W_t(\mu_t)$ is closed and convex.

Lemma A.1, Corollary A.1, and Corollary A.2 show that $\underline{W}_t^i(\mu_t)$ and $\bar{W}_t^i(\mu_t)$ are decreasing step functions for $i \in \{FP, LM, HM\}$. This implies that on any interval where the set of outcomes that gives the seller maximal payoffs is constant, $\underline{W}_t(\mu_t)$ and $\bar{W}_t(\mu_t)$ are decreasing step functions. Thus, to show that $\underline{W}_t(\mu_t)$ and $\bar{W}_t(\mu_t)$ are decreasing step functions, I need only show that $\underline{W}_t(\mu_t)$ and $\bar{W}_t(\mu_t)$ are decreasing at the belief cutoffs where the optimal outcome for the seller changes. Lemmas A.4 and A.5 imply that there are four possible types of such cutoffs: 1) LM payoffs overtaking FP payoffs, 2) FP payoffs overtaking LM payoffs, 3) HM payoffs overtaking FP payoffs, or 4) HM payoffs overtaking LM payoffs.

First, consider the case in which LM payoffs overtake FP payoffs at some point μ'. Then there exists an $\varepsilon > 0$ such that for $\mu_t \in (\mu' - \varepsilon, \mu' + \varepsilon)$, Corollary A.1 implies that

$$\bar{W}_t^{LM}(\mu_t) = \bar{b} - \underline{b} + \delta\bar{W}_{t+1}(\mu_{t+1,j})$$

for some $\mu_{t+1,j} > \mu'$. Because $\bar{W}_t^{FP}(\mu') = \bar{b} - \underline{b} + \delta\bar{W}_{t+1}(\mu')$ and $\bar{W}_{t+1}$ is decreasing, $\bar{W}_t^{LM}(\mu') \leq \bar{W}_t^{FP}(\mu')$. Corollary A.1 also shows that for $\mu_t \in (\mu' - \varepsilon, \mu' + \varepsilon)$,

$$\underline{W}_t^{LM}(\mu_t) = \max\{\bar{b} - \underline{b} + \delta\underline{W}_{t+1}(\mu_{t+1,j}), \delta\underline{W}_{t+1}(\mu_{t+1,i})\}.$$

for some $\mu_{t+1,i} < \mu'$ and $\mu_{t+1,j} > \mu'$. Again, because $\underline{W}_t^{FP}(\mu') = \bar{b} - \underline{b} + \delta \underline{W}_{t+1}(\mu')$ and $\underline{W}_{t+1}$ is decreasing, $\delta \underline{W}_{t+1}(\mu_{t+1,i}) \leq \bar{W}_t^{FP}(\mu')$. Furthermore,

$$\begin{aligned}
\delta \underline{W}_{t+1}(\mu_{t+1,i}) &\leq \bar{b} - \underline{b} + \delta \bar{W}_{t+1}(\mu_{t+1,j}) \\
&\leq \bar{b} - \underline{b} + \delta \underline{W}_{t+1}(\mu_{t+1,j-2}) \\
&\leq \delta W_{t+1}(\mu') + \bar{b} - \underline{b}
\end{aligned}$$

where the first inequality is implied by i and j satisfying the constraints in Problem 1, the second inequality following from the maintained assumptions about W_{t+1} being part of a Spot Payoff Pair, and the third inequality following from the facts that $W_{t+1}(\mu_{t+1})$ is a decreasing step function and that if V_t^{LM} overtakes V_t^{FP} at μ', μ' has to be strictly less than $\mu_{t+1,j-1}$.

Next, consider the case in which FP payoffs overtake LM payoffs at some point μ'. Lemma A.1 shows that FP payoffs are convex in μ_t and equal to $\underline{b} + \delta V_{t+1}(\mu_t)$, while Lemma A.2 shows that LM payoffs are a convex combination of points $\delta V_{t+1}(\mu_{t+1,i})$ and $\underline{b} + \delta V_{t+1}(\mu_{t+1,j})$. This implies that FP payoffs can only overtake LM payoffs where μ' is equal to the higher of the cutoffs that are being mixed in the LM payoffs, $\mu_{t+1,j}$. At that point, Lemma A.1 and Corollary A.1 show that

$$\begin{aligned}
\underline{W}_t^{FP}(\mu_{t+1,j}) &= \bar{b} - \underline{b} + \delta \underline{W}_{t+1}(\mu_{t+1,j}) \\
&\leq \max\{\bar{b} - \underline{b} + \delta \underline{W}_{t+1}(\mu_{t+1,j}), \delta \underline{W}_{t+1}(\mu_{t+1,i})\} \\
&= \underline{W}_t^{LM}(\mu_{t+1,j}).
\end{aligned}$$

Thus, there exists an $\varepsilon > 0$ such that $\underline{W}_t(\mu') \leq \underline{W}_t(\mu_t)$ for $\mu_t \in (\mu' - \varepsilon, \mu')$ and $\underline{W}_t(\mu') \geq \underline{W}_t(\mu_t)$ for $\mu_t \in (\mu', \mu' + \varepsilon)$. Lemma A.1 and Corollary A.1 also show that

$$\begin{aligned}
\bar{W}_t^{FP}(\mu_{t+1,j}) &= \bar{b} - \underline{b} + \delta \bar{W}_{t+1}(\mu_{t+1,j}) \\
&= \bar{W}_t^{LM}(\mu_{t+1,j})
\end{aligned}$$

so there exists an $\varepsilon > 0$ such that $\underline{W}_t(\mu') \leq \underline{W}_t(\mu_t)$ for $\mu_t \in (\mu' - \varepsilon, \mu')$ and $\underline{W}_t(\mu') \geq \underline{W}_t(\mu_t)$ for $\mu_t \in (\mu', \mu' + \varepsilon)$. Thus, $\underline{W}_t(\mu_t)$ and $\bar{W}_t(\mu_t)$ are decreasing around any point where FP payoffs overtake LM payoffs.

Finally, consider the point $\hat{\mu}_t$ where HM payoffs overtake either FP or LM payoffs. Lemma A.1, and Corollary A.1 show that for all μ_t, $\bar{W}_t^{FP}$, $\underline{W}_t^{FP}$, $\bar{W}_t^{LM}$, and $\underline{W}_t^{LM}$ are all weakly greater than $\bar{b} - \underline{b}$. When high types are mixing, some high types are purchasing at a price weakly higher than $\underline{b}$

and receive no continuation payoffs, so $\bar{W}_t^{HM}$ and $\underline{W}_t^{HM}$ are weakly less than $\bar{b} - \underline{b}$. Thus, $\underline{W}_t(\mu_t)$ and $\bar{W}_t(\mu_t)$ are decreasing around any point where HM payoffs overtake either FP or LM payoffs.

Next, I show that $\underline{W}_t(\mu_{t,i-2}) \geq \bar{W}_t(\mu_{t,i})$. Suppose that $\bar{W}_t(\mu_{t,i}) > \underline{W}_t(\mu_{t,i-2})$. This means that in period t, there are seller-optimal allocations which give high valuation buyers the same payoffs when $\mu_t = \mu_{t,i-2}$ and when $\mu_t = \mu_{t,i}$. In other words, the high valuation buyer is indifferent between these allocations. If that is the case, then it also must be incentive compatible for high valuation buyers to randomize *between* these allocations (as always, low valuation buyers are also indifferent). Given that all buyers are indifferent and would be willing to randomize, then for all $\mu_t \in [\mu_{t,i-2}, \mu_{t,i}]$, payoffs of

$$\frac{\bar{\mu} - \mu_t}{\bar{\mu} - \underline{\mu}} V_t(\underline{\mu}) + \frac{\mu_t - \underline{\mu}}{\bar{\mu} - \underline{\mu}} V_t(\bar{\mu})$$

must be feasible. This is because a buyer could use a (potentially) two-step randomization, where in the first step he chooses whether he wants the allocation consistent with either $\mu_{t,i-2}$ or $\mu_{t,i}$ and in the second step he makes whatever choice he would have made if beliefs were originally at those cutoffs. This is a contradiction because if V_t is convex and piecewise linear with kinks at the given cutoffs, the above payoffs would be higher than the optimal payoffs of $V_t(\mu_{t,i-1})$.

Finally, I show that $\bar{W}_t(\mu_{t,i}) \geq \underline{W}_t(\mu_{t,i-1})$. Because $V_t(\mu_t)$ is the continuous upper envelope of piecewise linear functions, $\mu_{t,i-1}$ is exactly the level of beliefs where the seller is indifferent between implementing the allocation that is used when $\mu' \in (\mu_{t,i-2}, \mu_{t,i-1})$ and the allocation that is used when $\mu'' \in (\mu_{t,i-1}, \mu_{t,i})$. Thus, for any such beliefs, $\underline{W}_t(\mu_{t,i}) = \min\{\underline{W}_t(\mu'), \underline{W}_t(\mu'')\}$. Similarly $\mu_{t,i}$ is exactly the level of beliefs where the seller is indifferent between implementing the allocation that is used when $\mu'' \in (\mu_{t,i-1}, \mu_{t,i})$ and the allocation that is used when $\mu''' \in (\mu_{t,i}, \mu_{t,i+1})$. Thus, $\bar{W}_t(\mu_{t,i}) = \max\{\underline{W}_t(\mu''), \underline{W}_t(\mu''')\}$. Thus, for these beliefs, we have

$$\begin{aligned}
\underline{W}_t(\mu_{t,i-1}) &= \min\{\underline{W}_t(\mu'), \underline{W}_t(\mu'')\} \\
&\leq \underline{W}_t(\mu'') \\
&\leq \bar{W}_t(\mu'') \\
&\leq \max\{\underline{W}_t(\mu''), \underline{W}_t(\mu''')\} \\
&= \bar{W}_t(\mu_{t,i}),
\end{aligned}$$

so $\bar{W}_t(\mu_{t,i}) \geq \underline{W}_t(\mu_{t,i-1})$. $\square$

B Equilibria of Examples

B.1 Example 1

In example 1 we have $\underline{b} = 1$, $\bar{b} = 3$, $\delta = 0.6$, and $T = 4$.

B.1.1 Seller's Strategy

The seller's strategy in each period is to offer a (potentially random) price in each period as a function of the history. The price offered in period t will be defined as p_t, the history in period t as h^t, and the beliefs as μ_t. The randomizations that are being used at certain beliefs are designed to make the buyer strictly prefer randomizing with the appropriate probability in earlier periods. I write the strategies in terms of primitive parameters ($\bar{b}$, $\underline{b}$, and δ) in order to make the reason for the strategies clearer, but the form of the strategies is not invariant to the specific values of the parameters that I have assumed.

$$p_4(h^4) = \begin{cases} \bar{b} \text{ if } \mu_4 > \frac{1}{3} \\ (\bar{b}, \underline{b}) \text{ with prob. } (\pi_{4,1}, 1-\pi_{4,1}) \text{ if } \mu_4 = \frac{1}{3} \text{ and } \mu \neq \frac{1}{3} \\ \underline{b} \text{ otherwise} \end{cases}$$

$$p_3(h^3) = \begin{cases} \bar{b} \text{ if } \mu_3 > \frac{7}{11} \\ (\bar{b}, \bar{b} - (\bar{b} - \underline{b})\delta) \text{ with prob. } (\pi_{3,2}, 1-\pi_{3,2}) \text{ if } \mu_3 = \frac{7}{11} \\ \bar{b} - (\bar{b} - \underline{b})\delta \text{ if } \frac{7}{11} > \mu_3 > \frac{1}{3} \\ (\bar{b} - (\bar{b} - \underline{b})\delta, \underline{b}) \text{ with prob. } (\pi_{3,1}, 1-\pi_{3,1}) \text{ if } \mu_3 = \frac{1}{3} \text{ and } \mu \neq \frac{1}{3} \\ \underline{b} \text{ otherwise} \end{cases}$$

$$p_2(h^2) = \begin{cases} \bar{b} \text{ if } \mu_2 > \frac{167}{191} \\ (\bar{b}, \bar{b} - \delta^2(\bar{b} - \underline{b})) \text{ with prob. } (\pi_{2,3}, 1-\pi_{2,3}) \text{ if } \mu_2 = \frac{167}{191} \\ \bar{b} - \delta^2(\bar{b} - \underline{b}) \text{ if } \frac{167}{191} > \mu_2 > \frac{7}{11} \\ (\bar{b} - \delta^2(\bar{b} - \underline{b}), \bar{b} - (\delta + \delta^2)(\bar{b} - \underline{b})) \text{ with prob. } (\pi_{2,2}, 1-\pi_{2,2}) \text{ if } \mu_2 = \frac{7}{11} \\ \bar{b} - (\delta + \delta^2)(\bar{b} - \underline{b}) \text{ if } \frac{7}{11} > \mu_2 > \frac{1}{3} \\ (\bar{b} - (\delta + \delta^2)(\bar{b} - \underline{b}), \underline{b}) \text{ with prob. } (\pi_{2,1}, 1-\pi_{2,1}) \text{ if } \mu_2 = \frac{1}{3} \text{ and } \mu \neq \frac{1}{3} \\ \underline{b} \text{ otherwise} \end{cases}$$

$$p_1 = \begin{cases} \bar{b} \text{ if } \mu > \frac{18631}{19063} \\ \bar{b} - \delta^3(\bar{b} - \underline{b}) \text{ if } \frac{18631}{19063} \geq \mu > \frac{167}{191} \\ \bar{b} - (\delta^2 + \delta^3)(\bar{b} - \underline{b}) \text{ if } \frac{167}{191} \geq \mu > \frac{3507}{5851} \\ \underline{b} \text{ if } \frac{3507}{5851} \geq \mu \end{cases}$$

where $\pi_{2,1} = \max\{\min\{1 + \delta + \delta^2 - \frac{\bar{b}-p_1}{\delta(\bar{b}-\underline{b})}, 1\}, 0\}$, $\pi_{2,2} = \max\{\min\{1 + \delta - \frac{\bar{b}-p_1}{\delta(\bar{b}-\underline{b})}, 1\}, 0\}$, $\pi_{2,3} = \max\{\min\{1 - \frac{\bar{b}-p_1}{\delta^3(\bar{b}-\underline{b})}, 1\}, 0\}$, $\pi_{3,1} = \max\{\min\{1 + \delta - \frac{\bar{b}-p_2}{\delta(\bar{b}-\underline{b})}, 1\}, 0\}$, $\pi_{3,2} = \max\{\min\{1 - \frac{\bar{b}-p_2}{\delta^2(\bar{b}-\underline{b})}, 1\}, 0\}$, and $\pi_{4,1} = \max\{\min\{1 - \frac{\bar{b}-p_3}{\delta(\bar{b}-\underline{b})}, 1\}, 0\}$.

B.1.2 Low Valuation Buyer's Strategy

$$\underline{x}_4(h^4) = \begin{cases} 0 \text{ if } \underline{b} < p_4 \\ 1 \text{ otherwise} \end{cases}$$

$$\underline{x}_3(h^3) = \begin{cases} 0 \text{ if } \underline{b} < p_3 \\ 1 \text{ otherwise} \end{cases}$$

$$\underline{x}_2(h^2) = \begin{cases} 0 \text{ if } \underline{b} < p_2 \\ 1 \text{ otherwise} \end{cases}$$

$$\underline{x}_1(h^1) = \begin{cases} 0 \text{ if } p_1 > \underline{b} \\ \frac{24\mu}{167-167\mu} \text{ if } p_1 = \underline{b} \text{ and } \mu_1 > \frac{167}{416} \\ 1 \text{ otherwise} \end{cases}$$

B.1.3 High Valuation Buyer's Strategy

$$\bar{x}_4(h^4) = \begin{cases} 0 \text{ if } \bar{b} < p_4 \\ 1 \text{ otherwise} \end{cases}$$

If $\mu_3 < \frac{1}{3}$,

$$\bar{x}_3(h^3) = \begin{cases} 0 \text{ if } \bar{b} - \delta(\bar{b} - \underline{b}) < p_3 \\ 1 \text{ otherwise} \end{cases}$$

otherwise if $\mu_3 \geq \frac{1}{3}$

$$\bar{x}_3(h^3) = \begin{cases} 0 \text{ if } \bar{b} < p_3 \\ \frac{3\mu_3-1}{2\mu_3} \text{ if } \bar{b} - \delta(\bar{b} - \underline{b}) < p_3 \leq \bar{b} \\ 1 \text{ otherwise} \end{cases} .$$

If $\mu_2 < \frac{1}{3}$,

$$\bar{x}_2(h^2) = \begin{cases} 0 \text{ if } \bar{b} - (\delta + \delta^2)(\bar{b} - \underline{b}) < p_4 \\ 1 \text{ otherwise} \end{cases} ,$$

while if $\frac{1}{3} \leq \mu_2 < \frac{7}{11}$,

$$\bar{x}_2(h^2) = \begin{cases} 0 \text{ if } \bar{b} - \delta^2(\bar{b} - \underline{b}) < p_2 \\ \frac{3\mu_2 - 1}{2\mu_2} \text{ if } \bar{b} - (\delta + \delta^2)(\bar{b} - \underline{b}) < p_2 \leq \bar{b} - \delta^2(\bar{b} - \underline{b}) \\ 1 \text{ otherwise} \end{cases}$$

and if $\frac{7}{11} \leq \mu_2$,

$$\bar{x}_2(h^2) = \begin{cases} 0 \text{ if } \bar{b} < p_2 \\ \frac{11\mu_2 - 7}{4\mu_2} \text{ if } \bar{b} - \delta^2(\bar{b} - \underline{b}) < p_2 \leq \bar{b} \\ \frac{3\mu_2 - 1}{2\mu_2} \text{ if } \bar{b} - (\delta + \delta^2)(\bar{b} - \underline{b}) < p_2 \leq \bar{b} - \delta^2(\bar{b} - \underline{b}) \\ 1 \text{ otherwise} \end{cases}$$

If $\mu < \frac{1}{3}$,

$$\bar{x}_1(h^1) = \begin{cases} 0 \text{ if } \underline{b} < p_1 \\ 1 \text{ otherwise} \end{cases},$$

if $\frac{1}{3} \leq \mu < \frac{7}{11}$,

$$\bar{x}_1(h^1) = \begin{cases} 0 \text{ if } \bar{b} - (\delta^2 + \delta^3)(\bar{b} - \underline{b}) < p_1 \\ \frac{3\mu - 1}{2\mu} \text{ if } \underline{b} < p_1 \leq \bar{b} - (\delta^2 + \delta^3)(\bar{b} - \underline{b}) \\ 1 \text{ otherwise} \end{cases}$$

if $\frac{7}{11} \leq \mu < \frac{167}{191}$,

$$\bar{x}_1(h^1) = \begin{cases} 0 \text{ if } \bar{b} - \delta^3(\bar{b} - \underline{b}) < p_1 \\ \frac{11\mu - 7}{4\mu} \text{ if } \bar{b} - (\delta^2 + \delta^3)(\bar{b} - \underline{b}) < p_1 \leq \bar{b} - \delta^3(\bar{b} - \underline{b}) \\ \frac{3\mu - 1}{2\mu} \text{ if } \underline{b} < p_1 \leq \bar{b} - (\delta^2 + \delta^3)(\bar{b} - \underline{b}) \\ 1 \text{ otherwise} \end{cases}$$

and if $\frac{167}{191} \leq \mu$,

$$\bar{x}_1(h^1) = \begin{cases} 0 \text{ if } \bar{b} < p_1 \\ \frac{191\mu - 167}{24\mu} \text{ if } \bar{b} - \delta^3(\bar{b} - \underline{b}) < p_1 \leq \bar{b} \\ \frac{11\mu - 7}{4\mu} \text{ if } \bar{b} - (\delta^2 + \delta^3)(\bar{b} - \underline{b}) < p_1 \leq \bar{b} - \delta^3(\bar{b} - \underline{b}) \\ \frac{3\mu - 1}{2\mu} \text{ if } \underline{b} < p_1 \leq \bar{b} - (\delta^2 + \delta^3)(\bar{b} - \underline{b}) \\ 1 \text{ otherwise} \end{cases}$$

B.1.4 Value Functions

Here, I adopt the notation used in Appendix A. Specifically, $\underline{W}_t(\mu_t) = \min\{W_t(\mu_t)\}$ and $\bar{W}_t(\mu_t) = \max\{W_t(\mu_t)\}$. Furthermore, for any set A and constant x, I define $A + x$ as the set containing the sum of x and each element of A (so $A + x = \{a + x : a \in A\}$).

Buyer Value Functions:

$$W_4(\mu_4) = \begin{cases} 0 \text{ if } \mu_4 > \frac{1}{3} \\ [0, \bar{b} - \underline{b}] \text{ if } \mu_4 = \frac{1}{3} \\ \bar{b} - \underline{b} \text{ otherwise} \end{cases}$$

$$= \begin{cases} 0 \text{ if } \mu_4 > \frac{1}{3} \\ [0, 2] \text{ if } \mu_4 = \frac{1}{3} \\ 2 \text{ otherwise} \end{cases}$$

$$W_3(\mu_3) = \begin{cases} \delta \underline{W}_4\left(\frac{1}{2}\right) \text{ if } \mu_3 > \frac{7}{11} \\ [\delta \underline{W}_4\left(\frac{1}{2}\right), \delta \underline{W}_4(0)] \text{ if } \mu_3 = \frac{7}{11} \\ \delta \underline{W}_4(0) \text{ if } \frac{7}{11} > \mu_3 > \frac{1}{3} \\ [\delta \underline{W}_4(0), \bar{b} - \underline{b} + \delta \bar{W}_4(\mu_3)] \text{ if } \mu_3 = \frac{1}{3} \\ \bar{b} - \underline{b} + \delta W_4(\mu_3) \text{ otherwise} \end{cases}$$

$$= \begin{cases} 0 \text{ if } \mu_3 > \frac{7}{11} \\ \left[0, \frac{6}{5}\right] \text{ if } \mu_3 = \frac{7}{11} \\ \frac{6}{5} \text{ if } \frac{7}{11} > \mu_3 > \frac{1}{3} \\ \left[\frac{6}{5}, \frac{16}{5}\right] \text{ if } \mu_3 = \frac{1}{3} \\ \frac{16}{5} \text{ otherwise} \end{cases}$$

$$W_2(\mu_2) = \begin{cases} \delta \underline{W}_3\left(\frac{7}{11}\right) \text{ if } \mu_2 > \frac{167}{191} \\ \left[\delta \underline{W}_3\left(\frac{7}{11}\right), \delta \underline{W}_3\left(\frac{1}{3}\right)\right] \text{ if } \mu_2 = \frac{167}{191} \\ \delta \underline{W}_3\left(\frac{1}{3}\right) \text{ if } \frac{167}{191} > \mu_2 > \frac{7}{11} \\ \left[\delta \underline{W}_3\left(\frac{1}{3}\right), \delta \underline{W}_3(0)\right] \text{ if } \mu_2 = \frac{7}{11} \\ \delta \underline{W}_3(0) \text{ if } \frac{7}{11} > \mu_2 > \frac{1}{3} \\ \left[\delta \underline{W}_3(0), \bar{b} - \underline{b} + \delta \bar{W}_3(\mu_2)\right] \text{ if } \mu_2 = \frac{1}{3} \\ \bar{b} - \underline{b} + \delta W_3(\mu_2) \text{ otherwise} \end{cases}$$

$$= \begin{cases} 0 \text{ if } \mu_2 > \frac{167}{191} \\ \left[0, \frac{18}{25}\right] \text{ if } \mu_2 = \frac{167}{191} \\ \frac{18}{25} \text{ if } \frac{167}{191} > \mu_2 > \frac{7}{11} \\ \left[\frac{18}{25}, \frac{48}{25}\right] \text{ if } \mu_2 = \frac{7}{11} \\ \frac{48}{25} \text{ if } \frac{7}{11} > \mu_2 > \frac{1}{3} \\ \left[\frac{48}{25}, \frac{98}{25}\right] \text{ if } \mu_2 = \frac{1}{3} \\ \frac{98}{25} \text{ otherwise} \end{cases}$$

$$W_1(\mu) = \begin{cases} \delta\underline{W}_2\left(\frac{167}{191}\right) \text{ if } \mu > \frac{18631}{19063} \\ \left[\delta\underline{W}_2\left(\frac{167}{191}\right), \delta\underline{W}_2\left(\frac{7}{11}\right)\right] \text{ if } \mu = \frac{18631}{19063} \\ \delta\underline{W}_2\left(\frac{7}{11}\right) \text{ if } \frac{18631}{19063} > \mu > \frac{167}{191} \\ \left[\delta\underline{W}_2\left(\frac{7}{11}\right), \delta\underline{W}_2\left(\frac{1}{3}\right)\right] \text{ if } \mu = \frac{167}{191} \\ \delta\underline{W}_2\left(\frac{1}{3}\right) \text{ if } \frac{167}{191} > \mu > \frac{3507}{5851} \\ \left[\delta\underline{W}_2\left(\frac{1}{3}\right), \bar{b} - \underline{b} + \delta\bar{W}_2\left(\frac{167}{191}\right)\right] \text{ if } \mu = \frac{3507}{5851} \\ \left[\max\left\{\bar{b} - \underline{b} + \delta\underline{W}_2\left(\frac{167}{191}\right), \delta\underline{W}_2(0)\right\}, \bar{b} - \underline{b} + \delta\bar{W}_2\left(\frac{167}{191}\right)\right] \text{ if } \frac{3507}{5851} > \mu > \frac{167}{416} \\ \left[\max\left\{\bar{b} - \underline{b} + \delta\underline{W}_2\left(\frac{167}{191}\right), \delta\underline{W}_2(0)\right\}, \bar{b} - \underline{b} + \delta\bar{W}_2(\mu)\right] \text{ if } \mu = \frac{167}{416} \\ \bar{b} - \underline{b} + \delta W_2(\mu) \text{ otherwise} \end{cases}$$

$$= \begin{cases} 0 \text{ if } \mu > \frac{18631}{19063} \\ \left[0, \frac{54}{125}\right] \text{ if } \mu = \frac{18631}{19063} \\ \frac{54}{125} \text{ if } \frac{18631}{19063} > \mu > \frac{167}{191} \\ \left[\frac{54}{125}, \frac{144}{125}\right] \text{ if } \mu = \frac{167}{191} \\ \frac{144}{125} \text{ if } \frac{167}{191} > \mu > \frac{3507}{5851} \\ \left[\frac{144}{125}, \frac{304}{125}\right] \text{ if } \mu = \frac{3507}{5851} \\ \left[\frac{294}{125}, \frac{304}{125}\right] \text{ if } \frac{3507}{5851} > \mu > \frac{167}{416} \\ \left[\frac{294}{125}, \frac{394}{125}\right] \text{ if } \mu = \frac{167}{416} \\ \frac{394}{125} \text{ if } \frac{167}{416} > \mu > \frac{1}{3} \\ \left[\frac{394}{125}, \frac{544}{125}\right] \text{ if } \mu = \frac{1}{3} \\ \frac{544}{125} \text{ otherwise} \end{cases}$$

Seller Value Functions:

$$V_4(\mu_4) = \begin{cases} \bar{b}\mu_4 \text{ if } \mu_4 > \frac{1}{3} \\ \underline{b} \text{ otherwise} \end{cases}$$

$$= \begin{cases} 3\mu_4 \text{ if } \mu_4 > \frac{1}{3} \\ 1 \text{ otherwise} \end{cases}$$

$$V_3(\mu_3) = \begin{cases} \left(\frac{1-\mu_3}{1-\frac{1}{3}}\right)\left[\delta V_4\left(\frac{1}{3}\right)\right] + \left(\frac{\mu_3-\frac{1}{3}}{1-\frac{1}{3}}\right)\left[\bar{b} - \delta \underline{W}_4\left(\frac{1}{3}\right) + \delta V_4(1)\right] \text{ if } \mu_3 > \frac{7}{11} \\ \left(\frac{1-\mu_3}{1-0}\right)\left[\delta V_4(0)\right] + \left(\frac{\mu_3-0}{1-0}\right)\left[\bar{b} - \delta \underline{W}_4(0) + \delta V_4(1)\right] \text{ if } \frac{7}{11} \geq \mu_3 > \frac{1}{3} \\ \underline{b} + \delta V_4(\mu_3) \text{ otherwise} \end{cases}$$

$$= \begin{cases} 6.3\mu_3 - 1.5 \text{ if } \mu_3 > \frac{7}{11} \\ 3\mu_3 + 0.6 \text{ if } \frac{7}{11} \geq \mu_3 > \frac{1}{3} \\ 1.6 \text{ otherwise} \end{cases}$$

$$V_2(\mu_2) = \begin{cases} \left(\frac{1-\mu_2}{1-\frac{7}{11}}\right)\left[\delta V_3\left(\frac{7}{11}\right)\right] + \left(\frac{\mu_2-\frac{7}{11}}{1-\frac{7}{11}}\right)\left[\bar{b} - \delta \underline{W}_3\left(\frac{7}{11}\right) + \delta V_3(1)\right] \mu_2 > \frac{167}{191} \\ \left(\frac{1-\mu_2}{1-\frac{1}{3}}\right)\left[\delta V_3\left(\frac{1}{3}\right)\right] + \left(\frac{\mu_2-\frac{1}{3}}{1-\frac{1}{3}}\right)\left[\bar{b} - \delta \underline{W}_3\left(\frac{1}{3}\right) + \delta V_3(1)\right] \text{ if } \frac{167}{191} \geq \mu_2 > \frac{7}{11} \\ \left(\frac{1-\mu_2}{1-0}\right)\left[\delta V_3(0)\right] + \left(\frac{\mu_2-0}{1-0}\right)\left[\bar{b} - \delta \underline{W}_3(0) + \delta V_3(1)\right] \text{ if } \frac{7}{11} \geq \mu_2 > \frac{1}{3} \\ \underline{b} + \delta V_3(\mu_2) \text{ otherwise} \end{cases}$$

$$= \begin{cases} 12.03\mu_2 - 6.15 \text{ if } \mu_2 > \frac{167}{191} \\ 6.3\mu_2 - 1.14 \text{ if } \frac{167}{191} \geq \mu_2 > \frac{7}{11} \\ 3\mu_2 + 0.96 \text{ if } \frac{7}{11} \geq \mu_2 > \frac{1}{3} \\ 1.96 \text{ otherwise} \end{cases}$$

$$V_1(\mu) = \begin{cases} \left(\frac{1-\mu}{1-\frac{167}{191}}\right)\left[\delta V_2\left(\frac{167}{191}\right)\right] + \left(\frac{\mu-\frac{167}{191}}{1-\frac{167}{191}}\right)\left[\bar{b} - \delta \underline{W}_2\left(\frac{167}{191}\right) + \delta V_2(1)\right] \text{ if } \mu > \frac{18631}{19063} \\ \left(\frac{1-\mu}{1-\frac{7}{11}}\right)\left[\delta V_2\left(\frac{7}{11}\right)\right] + \left(\frac{\mu-\frac{7}{11}}{1-\frac{7}{11}}\right)\left[\bar{b} - \delta \underline{W}_2\left(\frac{7}{11}\right) + \delta V_2(1)\right] \text{ if } \frac{18631}{19063} \geq \mu > \frac{167}{191} \\ \left(\frac{1-\mu}{1-\frac{1}{3}}\right)\left[\delta V_2\left(\frac{1}{3}\right)\right] + \left(\frac{\mu-\frac{1}{3}}{1-\frac{1}{3}}\right)\left[\bar{b} - \delta \underline{W}_2\left(\frac{1}{3}\right) + \delta V_2(1)\right] \text{ if } \frac{167}{191} \geq \mu > \frac{3507}{5851} \\ \left(\frac{\frac{167}{191}-\mu}{\frac{167}{191}-0}\right)\left[\delta V_2(0)\right] + \left(\frac{\mu-0}{\frac{167}{191}-0}\right)\left[\underline{b} + \delta V_2\left(\frac{167}{191}\right)\right] \text{ if } \frac{3507}{5851} \geq \mu > \frac{167}{416} \\ \underline{b} + \delta V_2(\mu) \text{ otherwise} \end{cases}$$

$$= \begin{cases} \frac{31093}{1000}\mu - \frac{24565}{1000} \text{ if } \mu > \frac{18631}{19063} \\ 12.03\mu - \frac{2967}{500} \text{ if } \frac{18631}{19063} \geq \mu > \frac{167}{191} \\ 6.3\mu - 0.924 \text{ if } \frac{167}{191} \geq \mu > \frac{3507}{5851} \\ \frac{467}{167}\mu + 1.176 \text{ if } \frac{3507}{5851} \geq \mu > \frac{167}{416} \\ 1.8\mu + 1.576 \text{ if } \frac{167}{416} \geq \mu > \frac{1}{3} \\ 2.176 \text{ otherwise} \end{cases}$$

B.1.5 Evolution of Beliefs

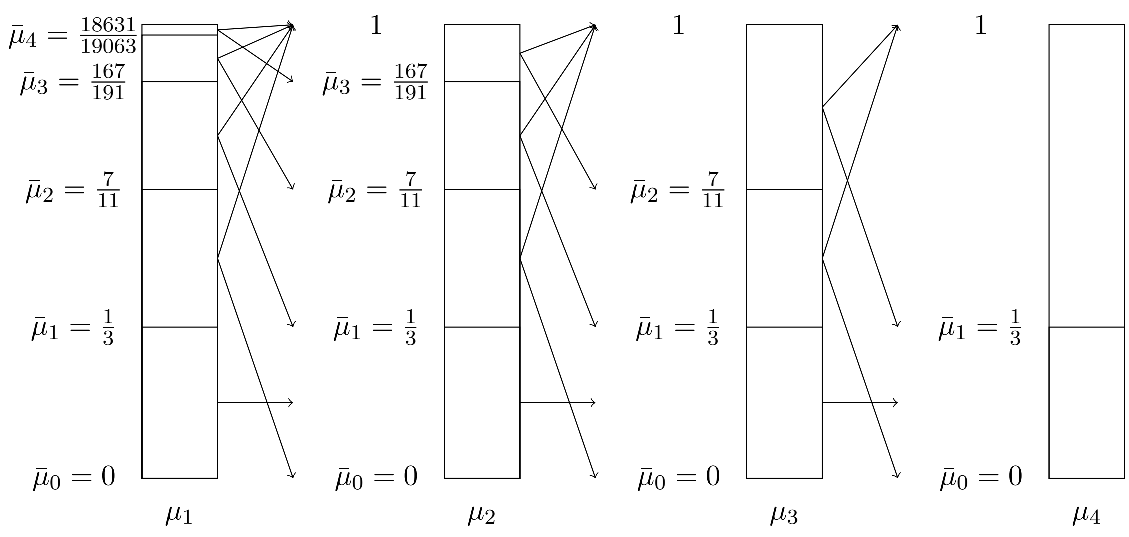

Figure 4: This figure describes the path of beliefs in the renegotiation setting when $T = 4$, $\delta = 0.6$, $\underline{b} = 1$, and $\bar{b} = 3$. When beliefs are above $\bar{\mu}_1$, the seller offers prices such that high types mix. When the seller observes a purchase, beliefs update to one and otherwise fall to a lower value in the next period. If beliefs are below $\bar{\mu}_1$, the seller always charges $\underline{b}$, both types of buyers purchase with probability one, and the seller does not update.

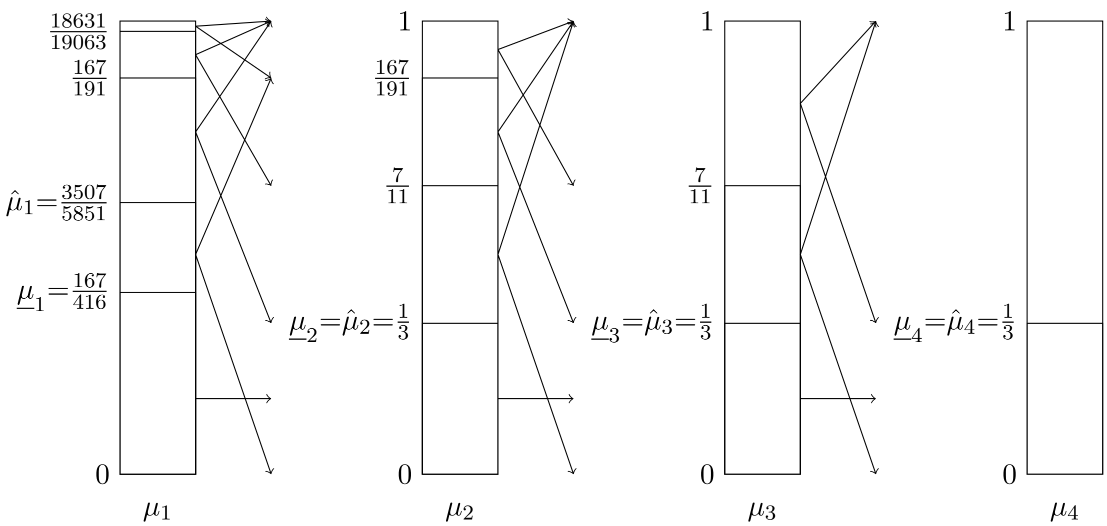

Figure 5: This figure describes the path of beliefs in the spot contracting setting when $T = 4$, $\delta = 0.6$, $\underline{b} = 1$, and $\bar{b} = 3$. When beliefs are below $\underline{\mu}_t$ the seller pools all buyers and beliefs do not update. When beliefs are above $\hat{\mu}_t$, the low valuation buyer does not purchase and the high valuation buyer purchases with positive probability, implying that beliefs either increase to one or decrease. In the intermediate case in period one, low types are mixing and high types are purchasing with probability one, so beliefs either increase or fall to zero.

B.1.6 Equilibrium Verification for $\mu \in \left(\frac{167}{416}, \frac{3507}{5851}\right)$

In these games, it can be onerous to verify that a proposed strategy profile is an equilibrium. This section shows that no player has a profitable deviation along the equilibrium path described by the above strategies when initial beliefs are on the range where inducing low types to mix is optimal.

The next few paragraphs describe the equilibrium path.

In the first period, the seller offers a price of $\underline{b}$. Low types mix at a rate of $\frac{24\mu}{167-167\mu}$ and high types purchase. This implies that beliefs in the second period are 0 (when the buyer does not purchase) or $\frac{167}{191}$ (when the buyer does purchase).

In the second period, the seller offers a price of $\underline{b}$ when beliefs are 0 and $\bar{b} - \delta^2(\bar{b} - \underline{b})$ when beliefs are $\frac{167}{191}$. At the lower price, both types purchase so beliefs in the third period are zero. At the high price, low types do not purchase and high types randomize at rate $\frac{11\mu_2-7}{4\mu_2}$. Thus, beliefs in the third period are 0, $\frac{7}{11}$ or 1.

In the third period, the seller offers a price of $\underline{b}$ when beliefs are 0, $\bar{b} - \delta(\bar{b} - \underline{b})$ when beliefs are $\frac{7}{11}$, and $\bar{b}$ when beliefs are 1. At the price of $\underline{b}$, both types purchase and beliefs remain at 0. At the price of $\bar{b} - \delta(\bar{b} - \underline{b})$, low types do not purchase and high types randomize at rate $\frac{3\mu_3-1}{2\mu_3}$, leading to beliefs of either $\frac{1}{3}$ or 1. At the price of $\bar{b}$, all high types purchase and beliefs stay at 1. Thus, beliefs in the fourth period are 0, $\frac{1}{3}$, or 1.

In the fourth period, the seller offers a price of $\underline{b}$ when beliefs are 0 or $\frac{1}{3}$ and $\bar{b}$ when beliefs are 1. Both types purchase when the price is $\underline{b}$ and only high types purchase when the price is $\bar{b}$.

I now verify that players are maximizing their payoffs given equilibrium strategies.

Low Valuation Buyer:

In all periods and for all histories, the low valuation buyer is never offered a price that is strictly below his value. Thus, he has no option which gives him strictly positive surplus. In the proposed equilibrium, he only chooses options that give him zero surplus. Thus, the low valuation buyer is optimizing given all other players' strategies.

High Valuation Buyer:

On the equilibrium path in the final period, the high valuation buyer purchases for every price offered to him. This gives him surplus of 0 when the price is $\bar{b}$ and $\bar{b} - \underline{b}$ when the price is $\underline{b}$. Because not purchasing gives the high valuation buyer a surplus of 0, it always maximizes payoffs to purchase.

Now consider the equilibrium path in the third period:

- When the price offered is $\underline{b}$, all buyers are expected to purchase. If the high valuation buyer deviated and did not purchase, he would receive nothing in the current period, and beliefs would update to one in the next period. If he does purchase, he receives payoffs of $\bar{b} - \underline{b}$ in the current period and the continuation payoffs associated with beliefs of 0 in period 4. Thus, purchasing gives total payoffs of $(1 + \delta)(\bar{b} - \underline{b})$ and not purchasing gives total payoffs of 0, so the high valuation buyer is maximizing by purchasing.

- When the price offered is $\bar{b} - \delta(\bar{b} - \underline{b})$ in the third period, the high valuation buyer is expected to mix while the low valuation buyer does not purchase. If the high valuation buyer purchases, he receives payoffs of $\delta(\bar{b} - \underline{b})$ in the current period and continuation payoffs associated with beliefs of 1 (which are 0) in the next period. If the high valuation buyer does not purchase, he receives payoffs of 0 in the current period and continuation payoffs associated with beliefs of $\frac{1}{3}$ (which are $\delta(\bar{b} - \underline{b})$) in the next period. Since the payoffs of both purchasing and not purchasing are the same, the high valuation buyer is indifferent between his options.

- When the price offered is $\bar{b}$, only the high valuation buyer is expected to purchase. If the high valuation buyer did not purchase, he would receive nothing in the current period and beliefs would remain at one in the next period (implying continuation values of 0). If the high valuation buyer purchases, he receives no surplus in the current period and beliefs would remain at one in the next period (again implying continuation values of 0). Thus, the high valuation buyer is indifferent and is maximizing his payoffs when he purchases.

Now consider the equilibrium path in the second period:

- When the price offered is $\underline{b}$, all buyers are expected to purchase. If the high valuation buyer deviated and did not purchase, he would receive nothing in the current period, and beliefs would update to one in the next period. If he does purchase, he receives payoffs of $\bar{b} - \underline{b}$ in the current period and the continuation payoffs associated with beliefs of 0 (which are $(\delta + \delta^2)(\bar{b} - \underline{b})$) in period 3. Thus, purchasing gives total payoffs of $(1 + \delta + \delta^2)(\bar{b} - \underline{b})$ and not purchasing gives total payoffs of 0, so the high valuation buyer is maximizing by purchasing.

- When the price offered is $\bar{b} - \delta^2(\bar{b} - \underline{b})$ in the third period, the high valuation buyer is expected to mix while the low valuation buyer does not purchase. If the high valuation buyer purchases, he receives payoffs of $\delta^2(\bar{b} - \underline{b})$ in the current period and continuation payoffs associated with beliefs of 1 (which are 0) in the next period. If the high valuation buyer does not purchase, he receives payoffs of 0 in the current period and continuation payoffs associated with beliefs of $\frac{7}{11}$ (which are $\delta^2(\bar{b} - \underline{b})$) in the next period. Since the payoffs of both purchasing and not purchasing are the same, a high valuation buyer is indifferent between his options.

- When the price offered is $\bar{b}$, only the high valuation buyer is expected to purchase. If the high valuation buyer did not purchase, he would receive nothing in the current period and beliefs would remain at one in the next period (implying continuation values of 0). If the high valuation buyer purchases, he receives no surplus in the current period and beliefs would remain at one in the next period (again implying continuation values of 0). Thus, the high valuation buyer is indifferent and is maximizing his payoffs when he purchases.

Finally, we can consider the first period. The seller offers a price of $\underline{b}$. If the high valuation buyer purchases, he receives $\bar{b} - \underline{b}$ in the current period and continuation values consistent with beliefs of $\frac{167}{191}$ (which are $\delta^3(\bar{b} - \underline{b})$) in the next period. If the high valuation buyer deviated, he would receive 0 in the current period and continuation values consistent with beliefs of 0 (which are $(\delta + \delta^2 + \delta^3)(\bar{b} - \underline{b})$) in the next period. Because $\bar{b} - \underline{b} + \delta^3(\bar{b} - \underline{b})$ is equal to $\frac{304}{125}$ and $(\delta + \delta^2 + \delta^3)(\bar{b} - \underline{b})$ is equal to $\frac{294}{125}$, the high valuation buyer strictly prefers to follow his equilibrium strategy and purchase.

Seller:[26]

Notice first that when beliefs are equal to 0 or 1 in any period, it is clearly optimal to charge a price of $\underline{b}$ or $\bar{b}$ (respectively) in all future periods. Thus, we will only consider intermediate beliefs.

[26]For this section, I rely partially on the seller's continuation value functions computed in Section B.1.4 above.

In the fourth period, the only intermediate beliefs on the equilibrium path are $\mu_4 = \frac{1}{3}$. When this is the case, the buyers' purchasing choices only change when prices increase above $\underline{b} = 1$ or $\bar{b} = 3$, so it is optimal to set the price to one of those levels. When the price is 1 all buyers purchase, giving payoffs of 1. When the price is 3, all high types purchase, giving payoffs of $\frac{1}{3}(3) = 1$. Thus, the seller is indifferent between these two prices and is optimizing with any mixture between them.

In the third period, the only intermediate beliefs on the equilibrium path are $\mu_3 = \frac{7}{11}$. When this is the case, the buyer's purchasing choices only change when the price increases above $\underline{b} = 1$, $\bar{b} - \delta(\bar{b} - \underline{b}) = 1.8$, and $\bar{b} = 3$. Thus, the optimal price will be one of those levels.

- When the price is 1, all buyers purchase, giving current period payoffs of 1 and continuation payoffs of $\frac{63}{55}$, for a total of $\frac{118}{55}$.
- When the price is 1.8, high types purchase with probability one, giving current period payoffs of $\frac{7}{11}(1.8) = \frac{63}{55}$ and continuation payoffs of $\frac{7}{11}(1.8) + \frac{4}{11}(0.6) = \frac{15}{11}$ for a total of $\frac{138}{55}$.
- When the price is 3, high types purchase with probability $\frac{5}{7}$, giving current period payoffs of $3\left(\frac{7}{11}\right)\left(\frac{5}{7}\right) = \frac{15}{11}$ and continuation payoffs of $1.8\left(\frac{7}{11}\right)\left(\frac{5}{7}\right) + 0.6\left(1 - \left(\frac{7}{11}\right)\left(\frac{5}{7}\right)\right) = \frac{63}{55}$ for total payoffs of $\frac{138}{55}$.

Thus, the seller is indifferent between the prices of 1.8 and 3 and is optimizing with any mixture between them.

In the second period, the only intermediate beliefs on the equilibrium path are $\frac{167}{191}$. When this is the case, the buyers' purchasing choices only change when the price increases above $\underline{b} = 1$, $\bar{b} - (\delta + \delta^2)(\bar{b} - \underline{b}) = 1.08$, $\bar{b} - \delta^2(\bar{b} - \underline{b}) = 2.28$, and $\bar{b} = 3$. Thus, the optimal price will be one of those levels.

- When the price is 1, all buyers purchase, giving current period payoffs of 1 and continuation payoffs of $0.6\left(\frac{20859}{4775}\right) \approx 2.6210$ for a total of ≈ 3.6210.
- When the price is 1.08, high types purchase with probability one, giving current period payoffs of $1.08\left(\frac{167}{191}\right) = \frac{4509}{4775}$ and continuation payoffs of $\frac{167}{191}\left(\frac{72}{25}\right) + \frac{24}{191}\left(\frac{24}{25}\right) = \frac{504}{191}$ for a total of $\frac{17109}{4775}$.
- When the price is 2.28, high types purchase with probability $\frac{155}{167}$, giving current period payoffs

of 2.28 $\left(\frac{155}{167}\right)\left(\frac{167}{191}\right) = \frac{1767}{955}$ and continuation payoffs of $\left(\frac{155}{167}\right)\left(\frac{167}{191}\right)\left(\frac{72}{25}\right)+\left(1-\left(\frac{155}{167}\right)\left(\frac{167}{191}\right)\right)\left(\frac{24}{25}\right) = \frac{12024}{4775}$ for a total of $\frac{20859}{4775}$.

- When the price is 3, high types purchase with probability $\frac{125}{167}$, giving current period payoffs of $3\left(\frac{125}{167}\right)\left(\frac{167}{191}\right) = \frac{375}{191}$ and continuation payoffs of $\left(\frac{125}{167}\right)\left(\frac{167}{191}\right)\left(\frac{72}{25}\right)+\left(1-\left(\frac{125}{167}\right)\left(\frac{167}{191}\right)\right)\left(\frac{414}{275}\right) = \frac{11484}{4775}$ for a total of $\frac{20859}{4775}$.

Thus, the seller is indifferent between the prices of 2.28 and 3 and is optimizing with any mixture between them.

In the first period, I have assumed that beliefs take some value between $\frac{167}{416}$ and $\frac{3507}{5851}$. When this is the case, the buyers' purchasing choices only change when the price increases above $\underline{b} = 1$ and $\bar{b} - (\delta^2 + \delta^3)(\bar{b} - \underline{b}) = 1.848$. Thus, the optimal price will be one of those levels.

- When the price is 1, the high valuation buyer purchases and the low valuation buyer purchases with probability $\frac{24\mu}{167-167\mu}$, giving current period payoffs of $\left(\mu + (1-\mu)\left(\frac{24\mu}{167-167\mu}\right)\right) = \frac{191\mu}{167}$ and continuation payoffs of $\left(\frac{191\mu}{167}\right)\left(\frac{62577}{23875}\right) + \left(\frac{167-191\mu}{167}\right)\left(\frac{147}{125}\right) = \frac{276\mu}{167} + \frac{147}{125}$ for a total of $\frac{467}{167}\mu + \frac{147}{125}$.

- When the price is 1.848, the high valuation buyer purchase with probability $\frac{3\mu-1}{2\mu}$, giving current period payoffs of $1.848\left(\frac{3\mu-1}{2}\right)$ and continuation payoffs of $\left(\frac{3\mu-1}{2}\right)\left(\frac{441}{125}\right)+\left(\frac{3-3\mu}{2}\right)\left(\frac{147}{125}\right) = \frac{441}{125}\mu$ for a total of $6.3\mu - 0.924$.

Because $\frac{467}{167}\mu + \frac{147}{125} > 6.3\mu - 0.924$ for $\mu \in \left(\frac{167}{416}, \frac{3507}{5851}\right)$, the seller strictly prefers to charge a price of $\underline{b}$ in this region.

B.2 Example 2

In Example 2 we have $\underline{b} = 1$, $\bar{b} = 2$, $\delta = 0.7$, and $T = 4$.

B.2.1 Equilibrium with Renegotiation

Hart & Tirole's Theorem 1 shows that the outcome of the rental model with commitment and renegotiation coincides with that of the durable good model without commitment. This equilibrium takes the following form: there are a series of cutoffs, $\bar{\mu}_i$ for $i = 0, \ldots, \infty$, such that if $\mu \in [\bar{\mu}_i, \bar{\mu}_{i+1})$, then the seller sells to only high types for the first i periods, including period 1, then all buyers thereafter. Once the buyer has bought once, he buys in all future periods. The total amount charged to a low type is his full valuation for the periods in which he consumes, and the total amount charged to the high type is just low enough to make him indifferent between his allocation and mimicking the low type.

These cutoffs $\bar{\mu}_i$ can be found by calculating indifference points between separating buyers over $i-1$ periods and i periods. Regardless of δ, $\bar{\mu}_0 = 0$ and $\bar{\mu}_1 = \frac{\underline{b}}{\bar{b}} = \frac{1}{2}$. It will be useful to calculate $\bar{\mu}_2$ through $\bar{\mu}_4$ for the purposes of this example. For instance, the first equality is

$$
\begin{aligned}
[\bar{\mu}_2 - (1-\bar{\mu}_2)] \left[\bar{b} + \delta\bar{b} + (\delta^2+\delta^3)\underline{b}\right] + (1-\bar{\mu}_2)\left[\delta\bar{b} + (\delta^2+\delta^3)\underline{b}\right] + (1-\bar{\mu}_2)(\delta^2+\delta^3)\underline{b} \\
= \bar{\mu}_2[\bar{b} + (\delta+\delta^2+\delta^3)\underline{b}] + (1-\bar{\mu}_2)(\delta+\delta^2+\delta^3)\underline{b} \\
(2\bar{\mu}_2 - 1)\left(\frac{4233}{1000}\right) + (1-\bar{\mu}_2)\left(\frac{2233}{1000}\right) + (1-\bar{\mu}_2)\left(\frac{833}{1000}\right) = \bar{\mu}_2\left(\frac{3533}{1000}\right) + (1-\bar{\mu}_2)\left(\frac{1533}{1000}\right) \\
\frac{27}{5}\bar{\mu}_2 - \frac{1167}{1000} = 2\bar{\mu}_2 + \frac{1533}{1000} \\
\bar{\mu}_2 = \frac{27}{34}
\end{aligned}
$$

Plugging in $\bar{\mu}_2$, we can get $\bar{\mu}_3 = \frac{5323}{5666}$ and $\bar{\mu}_4 = \frac{9825789}{9943438}$ similarly. Thus, for these parameter values, the seller's profit as a function of beliefs is

$$
\pi_{RP}(\mu) = \begin{cases}
\frac{7285989}{171500}\mu - \frac{6417170}{171500} \text{ if } \mu > \frac{9825789}{9943438} \\
\frac{4723}{350}\mu - \frac{61399}{7000} \text{ if } \frac{9825789}{9943438} \geq \mu > \frac{5323}{5666} \\
\frac{27}{5}\mu - \frac{1167}{1000} \text{ if } \frac{5323}{5666} \geq \mu > \frac{27}{34} \\
2\mu + \frac{1533}{1000} \text{ if } \frac{27}{34} \geq \mu > \frac{1}{2} \\
\frac{2553}{1000} \text{ otherwise}
\end{cases}
$$

B.2.2 Seller's Strategy with Spot Contracting

The seller's strategy in each period is to offer a (potentially random) price in each period as a function of the history. The price offered in period t will be defined as p_t, the history in period t as h^t, and the beliefs as μ_t. The randomizations that are being used at certain beliefs are designed to make the buyer strictly prefer randomizing with the appropriate probability in earlier periods. I write the strategies in terms of primitive parameters ($\bar{b}$, $\underline{b}$, and δ) in order to make the reason for the strategies clearer, but the form of the strategies is not invariant to the specific values of the parameters that I have assumed.

$$p_4(h^4) = \begin{cases} \bar{b} \text{ if } \mu_4 > 0.5 \\ (\bar{b}, \underline{b}) \text{ with prob. } (\pi_{4,1}, 1-\pi_{4,1}) \text{ if } \mu_4 = \frac{1}{2} \\ \underline{b} \text{ otherwise} \end{cases}$$

$$p_3(h^3) = \begin{cases} \bar{b} \text{ if } \frac{27}{34} < \mu_3 \\ \left(\bar{b}, \bar{b} - (\bar{b} - \underline{b})\delta\right) \text{ with prob. } (\pi_{3,2}, 1-\pi_{3,2}) \text{ if } \mu_3 = \frac{27}{34} \\ \bar{b} - (\bar{b} - \underline{b})\delta \text{ if } \frac{1}{2} < \mu_3 < \frac{27}{34} \\ (\bar{b} - (\bar{b} - \underline{b})\delta, \underline{b}) \text{ with prob. } (\pi_{3,1}, 1-\pi_{3,1}) \text{ if } \mu_3 = \frac{1}{2} \\ \underline{b} \text{ if } \mu_3 < \frac{1}{2} \end{cases}$$

$$p_2(h^2) = \begin{cases} \bar{b} \text{ if } \frac{5323}{5666} < \mu_2 \\ \left(\bar{b}, \bar{b} - (\bar{b} - \underline{b})\delta^2\right) \text{ with prob. } (\pi_{2,2}, 1-\pi_{2,2}) \text{ if } \mu_2 = \frac{5323}{5666} \\ \bar{b} - (\bar{b} - \underline{b})\delta^2 \text{ if } \frac{3}{4} < \mu_2 < \frac{5323}{5666} \\ \left(\bar{b} - (\bar{b} - \underline{b})\delta^2, \underline{b}\right) \text{ with prob. } (\pi_{2,1}, 1-\pi_{2,1}) \text{ if } \mu_2 = \frac{3}{4} \\ \underline{b} \text{ if } \mu_2 < \frac{3}{4} \end{cases}$$

$$p_1(h^1) = \begin{cases} \bar{b} \text{ if } \frac{10413789}{10531438} \leq \mu \\ \bar{b} - (\bar{b} - \underline{b})\delta^3 \text{ if } \frac{7}{8} \leq \mu < \frac{10413789}{10531438} \\ \underline{b} \text{ if } \mu < \frac{7}{8} \end{cases}$$

where $\pi_{2,1} = \max\left\{\min\left\{\frac{1+\delta+\delta^2}{1+\delta} - \frac{\bar{b}-p_1}{\delta(\bar{b}-\underline{b})}, 1\right\}, 0\right\}$, $\pi_{2,2} = \max\left\{\min\left\{1 - \frac{\bar{b}-p_1}{\delta^3(\bar{b}-\underline{b})}, 1\right\}, 0\right\}$, $\pi_{3,1} = \max\left\{\min\left\{1 + \delta - \frac{\bar{b}-p_2}{\delta(\bar{b}-\underline{b})}, 1\right\}, 0\right\}$, $\pi_{3,2} = \max\left\{\min\left\{1 - \frac{\bar{b}-p_2}{\delta^2(\bar{b}-\underline{b})}, 1\right\}, 0\right\}$, and $\pi_{4,1} = \max\left\{\min\left\{1 - \frac{\bar{b}-p_3}{\delta(\bar{b}-\underline{b})}, 1\right\}, 0\right\}$.

B.2.3 Low Valuation Buyer's Strategy with Spot Contracting

The low valuation buyer chooses to purchase the item in period t if and only if $p_t \leq \underline{b}$.

B.2.4 High Valuation Buyer's Strategy with Spot Contracting

$$\bar{x}_4(h^4) = \begin{cases} 0 \text{ if } \bar{b} < p_4 \\ 1 \text{ otherwise} \end{cases}$$

If $\mu_3 < \frac{1}{2}$

$$\bar{x}_3(h^3) = \begin{cases} 0 \text{ if } \bar{b} - \delta(\bar{b} - \underline{b}) < p_3 \\ 1 \text{ otherwise} \end{cases}$$

while if $\frac{1}{2} \leq \mu_3$,

$$\bar{x}_3(h^3) = \begin{cases} 0 \text{ if } \bar{b} < p_3 \\ \frac{2\mu_3 - 1}{\mu_3} \text{ if } \bar{b} - \delta(\bar{b} - \underline{b}) < p_3 \leq \bar{b} \\ 1 \text{ otherwise} \end{cases}$$

If $\mu_2 < \frac{1}{2}$,

$$\bar{x}_2(h^2) = \begin{cases} 0 \text{ if } \underline{b} < p_2 \\ 1 \text{ otherwise} \end{cases},$$

while for $\frac{1}{2} \leq \mu_2 < \frac{27}{34}$, the buyer uses

$$\bar{x}_2(h^2) = \begin{cases} 0 \text{ if } \bar{b} - \delta^2(\bar{b} - \underline{b}) < p_2 \\ \frac{2\mu_2 - 1}{\mu_2} \text{ if } \underline{b} < p_2 \leq \bar{b} - \delta^2(\bar{b} - \underline{b}) \\ 1 \text{ otherwise} \end{cases}$$

and if $\frac{27}{34} \leq \mu_2$,

$$\bar{x}_2(h^2) = \begin{cases} 0 \text{ if } \bar{b} < p_2 \\ \frac{34\mu_2 - 27}{7\mu_2} \text{ if } \bar{b} - \delta^2(\bar{b} - \underline{b}) < p_2 \leq \bar{b} \\ \frac{2\mu_2 - 1}{\mu_2} \text{ if } \underline{b} < p_2 \leq \bar{b} - \delta^2(\bar{b} - \underline{b}) \\ 1 \text{ otherwise} \end{cases}.$$

Finally, in the first period, when $\mu_1 < \frac{1}{2}$, the high type's strategy is

$$\bar{x}_1(h^1) = \begin{cases} 0 \text{ if } \underline{b} < p_1 \\ 1 \text{ otherwise} \end{cases}$$

For $\frac{3}{4} \leq \mu_1 \leq \frac{5323}{5666}$, the strategy is

$$\bar{x}_1(h^1) = \begin{cases} 0 \text{ if } \bar{b} - \delta^3(\bar{b} - \underline{b}) < p_1 \\ \frac{4\mu - 3}{\mu} \text{ if } \underline{b} < p_1 \leq \bar{b} - \delta^3(\bar{b} - \underline{b}) \\ 1 \text{ otherwise} \end{cases},$$

and if $\mu_1 > \frac{5323}{5666}$,

$$x_1(h^1) = \begin{cases} 0 \text{ if } \bar{b} < p_1 \\ \frac{5666\mu - 5323}{343\mu} \text{ if } \bar{b} - \delta^3(\bar{b} - \underline{b}) < p_1 \leq \bar{b} \\ \frac{4\mu - 3}{\mu} \text{ if } \underline{b} < p_1 \leq \bar{b} - \delta^3(\bar{b} - \underline{b}) \\ 1 \text{ otherwise} \end{cases}.$$

B.2.5 Value Functions with Spot Contracting

Here, I adopt the notation used in Appendix A. Specifically, $\underline{W}_t(\mu_t) = \min\{W_t(\mu_t)\}$ and $\bar{W}_t(\mu_t) = \max\{W_t(\mu_t)\}$. Furthermore, for any set A and constant x, I define $A + x$ as the set containing the sum of x and each element of A (so $A + x = \{a + x : a \in A\}$).

Buyer Value Functions:

$$W_4(\mu_4) = \begin{cases} 0 \text{ if } \mu_4 > \frac{1}{2} \\ [0, \bar{b} - \underline{b}] \text{ if } \mu_4 = \frac{1}{2} \\ \bar{b} - \underline{b} \text{ otherwise} \end{cases}$$

$$= \begin{cases} 0 \text{ if } \mu_4 > \frac{1}{2} \\ [0, 1] \text{ if } \mu_4 = \frac{1}{2} \\ 1 \text{ otherwise} \end{cases}$$

$$W_3(\mu_3) = \begin{cases} \delta\underline{W}_4\left(\frac{1}{2}\right) \text{ if } \mu_3 > \frac{27}{34} \\ [\delta\underline{W}_4\left(\frac{1}{2}\right), \delta\underline{W}_4\left(0\right)] \text{ if } \mu_3 = \frac{27}{34} \\ \delta\underline{W}_4\left(0\right) \text{ if } \frac{27}{34} > \mu_3 > \frac{1}{2} \\ [\delta\underline{W}_4\left(0\right), \bar{b} - \underline{b} + \delta\bar{W}_4(\mu_3)] \text{ if } \mu_3 = \frac{1}{2} \\ \bar{b} - \underline{b} + \delta W_4(\mu_3) \text{ otherwise} \end{cases}$$

$$= \begin{cases} 0 \text{ if } \mu_3 > \frac{27}{34} \\ [0, \frac{7}{10}] \text{ if } \mu_3 = \frac{27}{34} \\ \frac{7}{10} \text{ if } \frac{27}{34} > \mu_3 > \frac{1}{2} \\ [\frac{7}{10}, \frac{17}{10}] \text{ if } \mu_3 = \frac{1}{2} \\ \frac{17}{10} \text{ otherwise} \end{cases}$$

$$W_2(\mu_2) = \begin{cases} \delta\underline{W}_3\left(\frac{27}{34}\right) \text{ if } \mu_2 > \frac{5323}{5666} \\ [\delta\underline{W}_3\left(\frac{27}{34}\right), \delta\underline{W}_3\left(\frac{1}{2}\right)] \text{ if } \mu_2 = \frac{5323}{5666} \\ \delta\underline{W}_3\left(\frac{1}{2}\right) \text{ if } \frac{5323}{5666} > \mu_2 > \frac{3}{4} \\ [\delta\underline{W}_3\left(\frac{1}{2}\right), \bar{b} - \underline{b} + \delta\bar{W}_3\left(\mu_3\right)] \text{ if } \mu_2 = \frac{3}{4} \\ \bar{b} - \underline{b} + \delta W_3(\mu_2) \text{ otherwise} \end{cases}$$

$$= \begin{cases} 0 \text{ if } \mu_2 > \frac{5323}{5666} \\ [0, \frac{49}{100}] \text{ if } \mu_2 = \frac{5323}{5666} \\ \frac{49}{100} \text{ if } \frac{5323}{5666} > \mu_2 > \frac{3}{4} \\ \left[\frac{49}{100}, \frac{149}{100}\right] \text{ if } \mu_2 = \frac{3}{4} \\ \frac{149}{100} \text{ if } \frac{3}{4} > \mu_2 > \frac{1}{2} \\ \left[\frac{149}{100}, \frac{219}{100}\right] \text{ if } \mu_2 = \frac{1}{2} \\ \frac{219}{100} \text{ otherwise} \end{cases}$$

$$W_1(\mu) = \begin{cases} \delta \underline{W}_2\left(\frac{5323}{5666}\right) \text{ if } \mu > \frac{10413789}{10531438} \\ \left[\delta \underline{W}_2\left(\frac{5323}{5666}\right), \delta \underline{W}_2\left(\frac{3}{4}\right)\right] \text{ if } \mu = \frac{10413789}{10531438} \\ \delta \underline{W}_2\left(\frac{3}{4}\right) \text{ if } \frac{10413789}{10531438} > \mu > \frac{7}{8} \\ \left[\delta \underline{W}_2\left(\frac{3}{4}\right), \bar{b} - \underline{b} + \delta \bar{W}_2(\mu)\right] \text{ if } \mu = \frac{7}{8} \\ \bar{b} - \underline{b} + \delta W_2(\mu) \text{ otherwise} \end{cases}$$

$$= \begin{cases} 0 \text{ if } \mu > \frac{10413789}{10531438} \\ \left[0, \frac{343}{1000}\right] \text{ if } \mu = \frac{10413789}{10531438} \\ \frac{343}{1000} \text{ if } \frac{10413789}{10531438} > \mu > \frac{7}{8} \\ \left[\frac{343}{1000}, \frac{1343}{1000}\right] \text{ if } \mu = \frac{7}{8} \\ \frac{1343}{1000} \text{ if } \frac{7}{8} > \mu > \frac{3}{4} \\ \left[\frac{1343}{1000}, \frac{2043}{1000}\right] \text{ if } \mu = \frac{3}{4} \\ \frac{2043}{1000} \text{ if } \frac{3}{4} > \mu > \frac{1}{2} \\ \left[\frac{2043}{1000}, \frac{2533}{1000}\right] \text{ if } \mu = \frac{1}{2} \\ \frac{2533}{1000} \text{ otherwise} \end{cases}$$

Seller value functions:

$$V_4(\mu_4) = \begin{cases} \bar{b}\mu_4 \text{ if } \mu_4 > \frac{1}{2} \\ \underline{b} \text{ otherwise} \end{cases}$$

$$= \begin{cases} 2\mu_4 \text{ if } \mu_4 > \frac{1}{2} \\ 1 \text{ otherwise} \end{cases}$$

$$V_3(\mu_3) = \begin{cases} \left(\frac{1-\mu_3}{1-\frac{1}{2}}\right)\left[\delta V_4\left(\frac{1}{2}\right)\right] + \left(\frac{\mu_3-\frac{1}{2}}{1-\frac{1}{2}}\right)\left[\bar{b} - \delta \underline{W}_4\left(\frac{1}{2}\right) + \delta V_4(1)\right] \text{ if } \mu_3 > \frac{27}{34} \\ \left(\frac{1-\mu_3}{1-0}\right)\left[\delta V_4\left(0\right)\right] + \left(\frac{\mu_3-0}{1-0}\right)\left[\bar{b} - \delta \underline{W}_4\left(0\right) + \delta V_4(1)\right] \text{ if } \frac{27}{34} \geq \mu_3 > \frac{1}{2} \\ \underline{b} + \delta V_4(\mu_3) \text{ otherwise} \end{cases}$$

$$= \begin{cases} \frac{27}{5}\mu_3 - 2 \text{ if } \mu_3 > \frac{27}{34} \\ 2\mu_3 + 0.7 \text{ if } \frac{27}{34} \geq \mu_3 > \frac{1}{2} \\ 1.7 \text{ otherwise} \end{cases}$$

$$V_2(\mu_2) = \begin{cases} \left(\frac{1-\mu_2}{1-\frac{27}{34}}\right)\left[\delta V_3\left(\frac{27}{34}\right)\right] + \left(\frac{\mu_2-\frac{27}{34}}{1-\frac{27}{34}}\right)\left[\bar{b} - \delta \underline{W}_3\left(\frac{27}{34}\right) + \delta V_3(1)\right] \text{ if } \mu_2 > \frac{5323}{5666} \\ \left(\frac{1-\mu_2}{1-\frac{1}{2}}\right)\left[\delta V_3\left(\frac{1}{2}\right)\right] + \left(\frac{\mu_2-\frac{1}{2}}{1-\frac{1}{2}}\right)\left[\bar{b} - \delta \underline{W}_3\left(\frac{1}{2}\right) + \delta V_3(1)\right] \text{ if } \frac{5323}{5666} \geq \mu_2 > \frac{3}{4} \\ \underline{b} + \delta V_3(\mu_2) \text{ otherwise} \end{cases}$$

$$= \begin{cases} \frac{4723}{350}\mu_2 - \frac{319}{35} \text{ if } \mu_2 > \frac{5323}{5666} \\ 5.4\mu_2 - 1.51 \text{ if } \frac{5323}{5666} \geq \mu_2 > \frac{3}{4} \\ 1.4\mu_2 + 1.49 \text{ if } \frac{3}{4} \geq \mu_2 > \frac{1}{2} \\ \frac{219}{100} \text{ otherwise} \end{cases}$$

$$V_1(\mu) = \begin{cases} \left(\frac{1-\mu}{1-\frac{5323}{5666}}\right)\left[\delta V_2\left(\frac{5323}{5666}\right)\right] + \left(\frac{\mu-\frac{5323}{5666}}{1-\frac{5323}{5666}}\right)\left[\bar{b} - \delta \underline{W}_2\left(\frac{5323}{5666}\right) + \delta V_2(1)\right] \text{ if } \mu > \frac{10413789}{10531438} \\ \left(\frac{1-\mu}{1-\frac{3}{4}}\right)\left[\delta V_2\left(\frac{3}{4}\right)\right] + \left(\frac{\mu-\frac{3}{4}}{1-\frac{3}{4}}\right)\left[\bar{b} - \delta \underline{W}_2\left(\frac{3}{4}\right) + \delta V_2(1)\right] \text{ if } \frac{10413789}{10531438} \geq \mu > \frac{7}{8} \\ \underline{b} + \delta V_2(\mu) \text{ otherwise} \end{cases}$$

$$= \begin{cases} \frac{7285989}{171500}\mu_1 - \frac{641717}{17150} \text{ if } \mu > \frac{10413789}{10531438} \\ \frac{589}{50}\mu - \frac{7057}{1000} \text{ if } \frac{10413789}{10531438} \geq \mu > \frac{7}{8} \\ \frac{189}{50}\mu - \frac{57}{1000} \text{ if } \frac{7}{8} \geq \mu > \frac{3}{4} \\ \frac{49}{50}\mu + \frac{2043}{1000} \text{ if } \frac{3}{4} \geq \mu > \frac{1}{2} \\ \frac{2533}{1000} \text{ otherwise} \end{cases}$$

B.2.6 Evolution of Beliefs

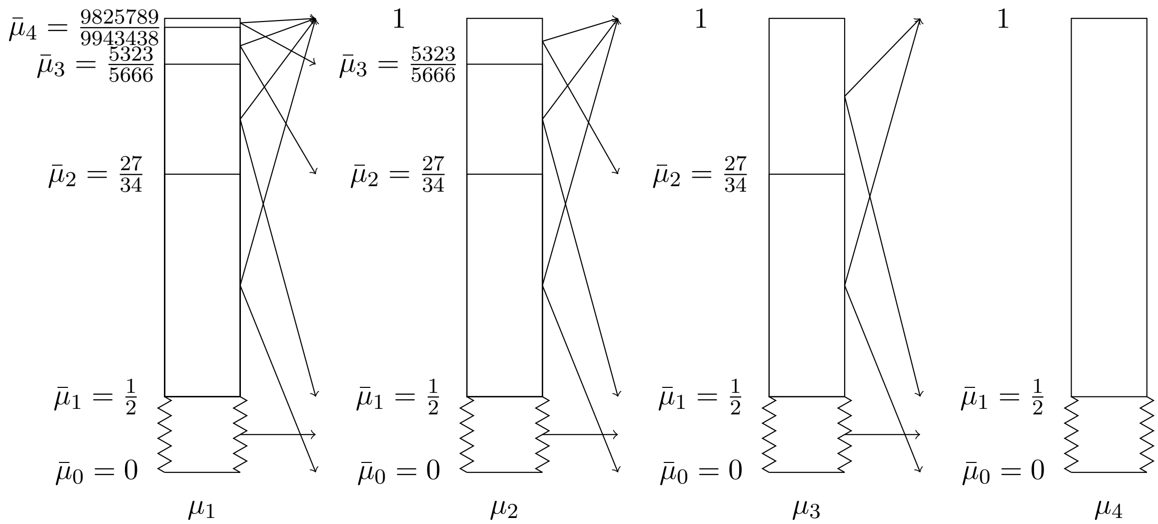

Figure 6: This figure describes the path of beliefs in the renegotiation setting when $T = 4$, $\delta = 0.7$, $\underline{b} = 1$, and $\bar{b} = 2$. When beliefs are above $\bar{\mu}_1$, the seller offers prices such that high types mix. When the seller observes a purchase, beliefs update to one and otherwise fall to a lower value in the next period. If beliefs are below $\bar{\mu}_1$, the seller always charges $\underline{b}$, both types of buyers purchase with probability one, and the seller does not update.

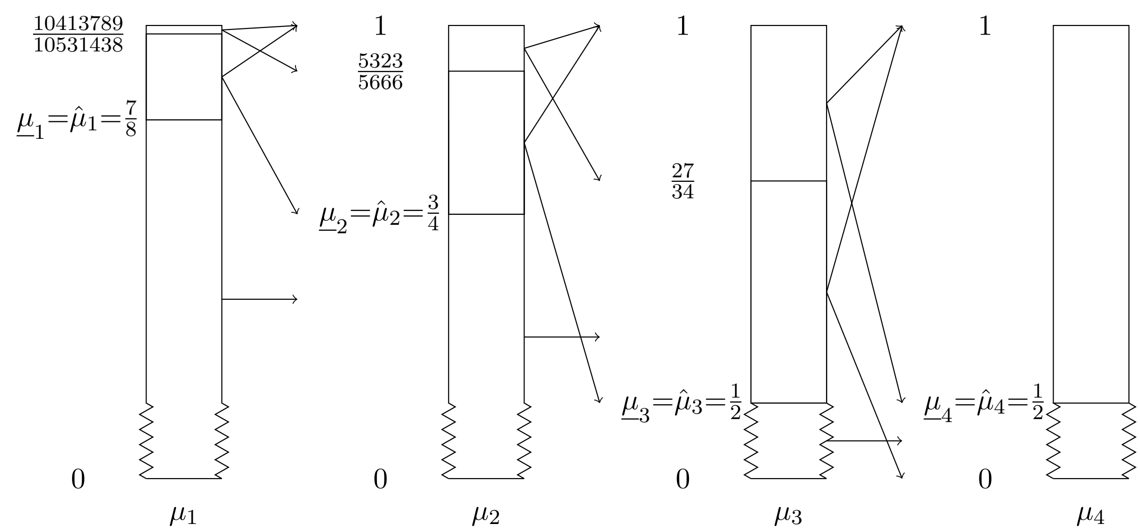

Figure 7: This figure describes the path of beliefs in the spot contracting setting when $T = 4$, $\delta = 0.7$, $\underline{b} = 1$, and $\bar{b} = 2$. When beliefs are below $\underline{\mu}_t$ the seller pools all buyers and beliefs do not update. When beliefs are above $\hat{\mu}_t$, the low valuation buyer does not purchase and the high valuation buyer purchases with positive probability, implying that beliefs either increase to one or decrease.

B.2.7 Equilibrium Verification for $\mu = 0.95$

In these games, it can be onerous to verify that a proposed strategy profile is an equilibrium. This section shows that no player has a profitable deviation along the equilibrium path described by the above strategies when initial beliefs are 0.95, which is on the range for which spot contracting payoffs are higher than commitment with renegotiation payoffs.

The next few paragraphs describe the equilibrium path.

In the first period, the seller offers a price of $\bar{b} - \delta^3(\bar{b} - \underline{b})$. Low types do not purchase and high types mix at a rate of $\frac{4\mu-3}{\mu}$. This implies that beliefs in the second period are $\frac{3}{4}$ (when the buyer does not purchase) or 1 (when the buyer does purchase).

In the second period, the seller offers a a price of $\bar{b} - \delta^2(\bar{b} - \underline{b})$ when beliefs are $\frac{3}{4}$ and $\bar{b}$ when beliefs are 1. At the low price, low types do not purchase and high types mix at a rate of $\frac{2}{3}$. At the high price, all high types purchase and beliefs stay at 1. Thus, beliefs in the third period are $\frac{1}{2}$ or 1.

In the third period, the seller offers a price of $\bar{b} - \delta(\bar{b} - \underline{b})$ when beliefs are $\frac{1}{2}$ and $\bar{b}$ when beliefs are 1. At the low price, low types do not purchase and high types purchase. At the high price, all high types purchase and beliefs stay at 1. Thus, beliefs in the fourth period are 0 or 1.

In the fourth period, the seller offers a price of $\underline{b}$ when beliefs are 0 and $\bar{b}$ when beliefs are 1. At the low price, both low and high types purchase. At the high price, only high types purchase.

I now verify that players are maximizing their payoffs given equilibrium strategies.

Low Valuation Buyer:

In all periods and for all histories, the low valuation buyer is never offered a price that is strictly below his value. Thus, he has no option which gives him strictly positive surplus. In the proposed equilibrium, he only chooses options that give him zero surplus. Thus, the low valuation buyer is optimizing given all other players' strategies.

High Valuation Buyer:

On the equilibrium path in the fourth period, the high valuation buyer purchases for every price offered to him. This gives him surplus of 0 when the price is $\bar{b}$ and $\bar{b} - \underline{b}$ when the price is $\underline{b}$. Because not purchasing gives the high valuation buyer a surplus of 0, it always maximizes payoffs to purchase.

Now, consider the equilibrium path in the third period:

- When the price offered is $\bar{b} - \delta(\bar{b} - \underline{b})$ in the third period, the high valuation buyer is expect to mix while the low valuation buyer does not purchase. If the high valuation buyer purchases, he receives payoffs of $\delta(\bar{b} - \underline{b})$ in the current period and continuation payoffs associated with beliefs of 1 (which are 0) in the next period. If the high valuation buyer does not purchase, he receives payoffs of 0 in the current period and continuation payoffs associated with beliefs of 0 (which are $\delta(\bar{b} - \underline{b})$) in the next period. Since the payoffs of both purchasing and not purchasing are the same, the high valuation buyer is indifferent between his options.

- When the price offered is $\bar{b}$, only the high valuation buyer is expected to purchase. If the high valuation buyer did not purchase, he would receive nothing in the current period and beliefs would remain at one in the next period (implying continuation values of 0). If the high valuation buyer purchases, he receives no surplus in the current period and beliefs would remain at one in the next period (again implying continuation values of 0). Thus, the high valuation buyer is indifferent and is maximizing his payoffs when he purchases.

Now consider the equilibrium path in the second period:

- When the price offered is $\bar{b} - \delta^2(\bar{b} - \underline{b})$ in the third period, the high valuation buyer is expect to mix while the low valuation buyer does not purchase. If the high valuation buyer purchases, he receives payoffs of $\delta^2(\bar{b} - \underline{b})$ in the current period and continuation payoffs associated with beliefs of 1 (which are 0) in the next period. If the high valuation buyer does not purchase, he receives payoffs of 0 in the current period and continuation payoffs associated with beliefs of $\frac{1}{2}$ (which are $\delta^2(\bar{b} - \underline{b})$) in the next period. Since the payoffs of both purchasing and not purchasing are the same, the high valuation buyer is indifferent between his options.

- When the price offered is $\bar{b}$, only the high valuation buyer is expected to purchase. If the high valuation buyer did not purchase, he would receive nothing in the current period and beliefs would remain at one in the next period (implying continuation values of 0). If the high valuation buyer purchases, he receives no surplus in the current period and beliefs would remain at one in the next period (again implying continuation values of 0). Thus, the high valuation buyer is indifferent and is maximizing his payoffs when he purchases.

Finally, we can consider the first period. The seller offers a price of $\bar{b} - \delta^3(\bar{b} - \underline{b})$ and the high valuation buyer is expect to mix while the low valuation buyer does not purchase. If the high valuation buyer purchases, he receives payoffs of $\delta^3(\bar{b} - \underline{b})$ in the current period and continuation payoffs associated with beliefs of 1 (which are 0) in the next period. If the high valuation buyer does not purchase, he receives payoffs of 0 in the current period and continuation payoffs associated with beliefs of $\frac{3}{4}$ (which are $\delta^3(\bar{b} - \underline{b})$) in the next period. Since the payoffs of both purchasing and not purchasing are the same, the high valuation buyer is indifferent between his options.

Seller:[27]

Notice first that when beliefs are equal to 0 or 1 in any period, it is clearly optimal to charge a price of $\underline{b}$ or $\bar{b}$ (respectively) in all future periods. Thus, we will only consider intermediate beliefs.

In the fourth period there are no intermediate beliefs, so the fact that the seller is optimizing is immediate.

In the third period, the only intermediate beliefs on the equilibrium path are $\frac{1}{2}$. When this is the case, the buyer's purchasing choices only change when the price increases above $\underline{b} = 1$, $\bar{b} - \delta(\bar{b} - \underline{b}) = 1.3$, and $\bar{b} = 2$. Thus, the optimal price will be one of those levels.

- When the price is 1, all buyers purchase, giving the seller current period payoffs of 1 and continuation payoffs of 0.7 for a total of 1.7.

- When the price is 1.3, the high valuation buyer purchases with probability one, giving the seller current period payoffs of $\frac{1}{2}(1.3)$ and continuation payoffs of $\frac{1}{2}(1.4) + \frac{1}{2}(.7) = 1.05$ for a total of 1.7.

- When the price is greater than 1.3, neither type purchases, giving the seller current period payoffs of 0 and continuation payoffs of 0.7 for a total of 0.7.

[27] For this section, I rely partially on the seller's continuation value functions computed in Section B.2.5 above.

Thus, the seller is indifferent between the prices of 1 and 1.3 and is optimizing with any mixture between them.

In the second period, the only intermediate beliefs on the equilibrium path are $\frac{3}{4}$. When this is the case, the buyers' purchasing choices only change when the price increases above $\underline{b} = 1$ and $\bar{b} - \delta^2(\bar{b} - \underline{b}) = 1.51$.

- When the price is 1, all buyers purchase, giving the seller current period payoffs of 1 and continuation payoffs of 1.54 for a total of 2.54.

- When the price is 1.51, the high valuation buyer purchases with probability $\frac{2}{3}$, giving the seller current period payoffs of $1.51\left(\frac{2}{3}\right)\left(\frac{3}{4}\right) = \frac{151}{200}$ and continuation payoffs of $\frac{1}{2}(2.38) + \frac{1}{2}(1.19) = \frac{357}{200}$ for a total of 2.54.

Thus the seller is indifferent between the prices of 1 and 1.51 and is optimizing with any mixture between them.

In the first period, I have assume that beliefs are 0.95. When this is the case, the buyers' purchasing choices only change when the price increases above $\underline{b} = 1$, $\bar{b} - \delta^3(\bar{b} - \underline{b}) = 1.657$, and $\bar{b} = 2$.

- When the price is 1, all buyers purchase, giving the seller current period payoffs of 1 and continuation payoffs of 2.5937 for a total of 3.5937.

- When the price is 1.657, the high valuation buyer purchases with probability $\frac{16}{19}$, giving the seller current period payoffs of $1.657\left(\frac{19}{20}\right)\left(\frac{16}{19}\right) = \frac{1657}{1250}$ and continuation payoffs of $(3.066)\left(\frac{19}{20}\right)\left(\frac{16}{19}\right) + (1.778)\left(1 - \left(\frac{19}{20}\right)\left(\frac{16}{19}\right)\right) = \frac{7021}{2500}$ for a total of $\frac{2067}{500}$.

- When the price is 2, the high valuation buyer purchases with probability $\frac{1194}{6517}$, giving the seller current period payoffs of $2\left(\frac{19}{20}\right)\left(\frac{1194}{6517}\right) = \frac{597}{1715}$ and continuation payoffs of approximately $(3.066)\left(\frac{19}{20}\right)\left(\frac{1194}{6517}\right) + (2.4942)\left(1 - \left(\frac{19}{20}\right)\left(\frac{1194}{6517}\right)\right) \approx 2.5937$ for a total of approximately 2.9418.

Thus, for initial beliefs of 0.95, the seller strictly prefers to charge the price of 1.657, that is prescribed by the equilibrium.