Delegation with Continuation Values

Zachary Breig*

March, 2020

Latest Version: Click Here

Abstract

This paper provides sufficient conditions for a mechanism to be optimal in settings where monetary transfers are not feasible, but a principal can punish and reward the agent using other means. The setting is broad enough to encompass a variety of situations in which payments cannot be made. The conditions that are provided are compared to those which arise from standard mechanism design with transferable utility. They are then applied to dynamic delegation and delegation with money burning. The optimal dynamic delegation mechanism places a cap against the agent's bias in the second period, and varies how strict the cap is to incentivize better actions in the first period. While a more biased agent is always worse in the first period, a higher level of bias can actually improve payoffs in the second period. When money burning is added to a setting in which the agent's responsiveness to the state is different from the principal's, the principal uses this money burning as a substitute for rules.

*zbreig@ucsd.edu; Department of Economics, University of California, San Diego, 9500 Gilman Drive #0508, San Diego, CA 92093-0508. I would like to thank Joel Sobel, Simone Galperti, and James Andreoni, as well as seminar participants at the UCSD Theory, Behavioral, and Experimental Workshop for useful suggestions. All mistakes are my own.

1 Introduction

In many real-world principal-agent problems, it's impossible to promise contingent payments. As an alternative, a principal can incentivize an agent by promising more resources for future joint projects, giving the agent more flexibility in future choices, or requiring the agent to go through costly bureaucratic procedures. This paper studies how to optimally delegate choices to an agent when payments are not allowed, but the principal has some means to incentivize the agent.

The paper focuses on a setting called *delegation with continuation values*. The model builds on the original model of delegation, studied by Holmström (1977), in that an uninformed principal and an informed agent with divergent preferences must implement some action without making payments to each other. The primary addition to the model is that the principal is able to commit to actions in the future that affect both parties' payoffs. The paper gives necessary and sufficient conditions for a mechanism to be optimal. It then applies these results to two canonical settings: dynamic delegation and delegation with money burning.

The model allows the way in which the principal and agent interact in the future to be complex: it could involve the principal promising to play a particular strategy in another game, or to make more options available when the agent makes his next decision. Abstracting from this, I'll assume that there is *some* relationship between the principal's and agent's future payoffs. Thus, any commitment by the principal to provide the agent with a continuation value affects the principal's own continuation value. The principal's problem is to design an incentive scheme which improves the action that the agent takes, but isn't too costly to implement.

Optimal mechanisms in a setting with continuation values are often quite different than those found in the delegation literature. Melumad and Shibano (1991) solve for the optimal mechanism in the uniform-quadratic case, in which delegation takes the form of a "cap" that takes advantage of the agent's information when the state is low, and pools agents when the state is high. More recently, Alonso and Matouschek (2008) solve for optimal delegation with generalized quadratic loss functions and minimal conditions on the bias and state distribution. These solutions also usually include limits on the agent's actions, at which a mass of agents pool.

In contrast, the optimal mechanism with continuation values only pools agents when using incentives becomes costly enough for the principal. The key difference between classic delegation models and the one studied here is that there is *some* means to incentivize the agent, even if utility functions are not quasilinear. Thus, the core model studied here is a screening problem in which utility is not transferred linearly between the principal and the agent. In this case, the classic technique of using the envelope condition, integrating by parts, and maximizing the objective function directly cannot be used. Instead, techniques from the calculus of variations (Clarke, 1990) can solve the principal's problem.

Allowing for risk aversion on the side of the agent in settings with monetary payments introduces similar difficulties. Laffont and Rochet (1998) introduce risk aversion into a regulation model in which a regulator is screening firms with private costs. When studying a continuum of types, they restrict their analysis to a CARA utility function assume that the principal's payoff is linear in the agent's type. More recently, Garrett and Pavan (2015) introduced risk aversion into a model of dynamic managerial compensation, and use a variational approach to characterize optimal contracts.

In a technical sense, this paper is most similar to work by Koessler and Martimort (2012), which characterizes the solution to the delegation problem when there are two actions to be taken, but only one state variable. They use the agent's incentive compatibility constraints to demonstrate that the principal's problem can be reduced to solving for the optimal average action and the squared difference between the actions, which they call the spread. This spread variable enters the agent's utility function linearly and the principal's nonlinearly, so they also use optimality conditions from Clarke (1990) to define the optimal mechanism, and find that optimal delegation uses the spread of the two actions to elicit information about the state and that there's no pooling in the equilibrium. This "spread" variable acts very similar to the continuation value that appears in this paper.

The necessary and sufficient conditions in this paper focus on how the action and continuation values vary with each other and the state. The principal's primary tradeoff is between incentivizing a better action and promising a continuation contract which is more favorable for herself. These conditions are derived from the calculus of variations, which is used to characterize an indirect utility function for the agent which is optimal for the principal.

To demonstrate the usefulness of these conditions, they are applied to two canonical settings from the delegation literature: *dynamic delegation* and *delegation with money burning*. In dynamic delegation, the principal delegates two choices to the agent sequentially. In delegation with money burning, the principal can commit to making choices in the future that hurt both the principal and the agent. Both models are formulated as special cases of delegation with continuation values.

The paper characterizes the solution to a dynamic delegation problem. In the first period of optimal dynamic delegation, the mechanism implements an action which is strictly increasing and continuous in the state. The principal prevents the agent from taking actions that are too biased by changing the amount of options that the agent gets in the future.

Thus, when a principal delegates dynamically, rules that restrict which options are available should be more likely to bind at the end of a relationship, while earlier actions are more responsive to the agent's private information. The form of delegation in the first period contrasts with the results from a static delegation model. When one studies this problem from a static point of view, optimal delegation usually involves the principal placing restrictions on the choices that the agent can make, with the agent often being bound by these restrictions (for a nontrivial proportion of the realizations of the agent's private information, the agent chooses an action at the bound).

Increasing the bias of an agent in the first period of a dynamic delegation game makes the principal worse off. This is intuitive: an action which is good from the principal's point of view looks worse to an agent with a stronger bias than to an agent with a weaker bias. To incentivize the agent to take these actions, the principal would have to commit to worse allocations in the future, and would prefer not to do so.

On the other hand, increasing the bias from the second period has a dual effect. Increasing this bias makes actions taken in the second period become worse, but can also make it easier to induce large changes in the agent's continuation value at small cost to the principal. For a strong enough bias in the second period, the latter effect dominates, and increasing the bias improves the principal's payoffs.

The optimal mechanism in this dynamic delegation problem is deterministic. This is in accordance with recent work which shows that stochastic delegation is suboptimal when loss functions are quadratic (Goltsman, Hörner, Pavlov, & Squintani, 2009; Kováč & Mylovanov, 2009). However, the intuition shows that randomization is even less likely in dynamic delegation models than in static. In both the static and dynamic models, randomization is costly to both agent and the principal, and can be used to incentivize a choice which is better from the point of view of the principal. Randomization is not optimal in the static case because the changes in the choice it can induce are not worth the costs to the principal. In dynamic delegation, the costs from randomization must be low enough that the changes induced in the action must be worth these costs *and* the costs from randomization must be lower than those from giving less discretion in later periods. The principal imposes costs on the agent in the way which is least damaging to herself.

Related dynamic delegation problems are considered by Guo (2014), Guo and Hörner (2015), and Lipnowski and Ramos (2015), all of which study infinitely-repeated delegation problems. Compared to these papers, the characterization of the optimal way to promise continuation values is relatively straightforward in the setting studied here - since the second period of delegation is always a simple cap, it's easy to calculate the principals' expect utility from that delegation set. Instead, the focus of this paper is on the tradeoffs a principal faces when current payoffs are non-linear in the allocation and the state space is continuous.

Even if there is not a second period in which the principal can delegate, the principal may be able to commit to measures that are costly to both the principal and the agent. For instance, the principal could demand that the agent fill out tedious paperwork that the principal must then review, or could require that the principal and agent spend time meeting before the decision is made, wasting both parties' time. These measures are known as "money burning," because they lower both players' payoffs (Amador & Bagwell, 2013; Ambrus & Egorov, 2017; Amador & Bagwell, 2016).

An example of how money burning can be used to incentivize better actions can be found in Section 5, which studies a model of responsiveness similar to the one in Alonso and Matouschek (2008). The model includes parameters which capture both how extreme the principal is as compared to the agent, as well as how expensive money burning is from the point of view of the principal. When money burning is especially costly to the principal, the optimal mechanism is similar to Alonso and Matouschek's: an interval of intermediate points if the agent is radical, or two extreme actions if the agent is conservative. In contrast, when money burning is inexpensive for the principal, it is used as a substitute for rules: money is burnt over extreme actions if the agent is radical, and over intermediate actions if the agent is conservative.

The optimality conditions can can be applied to more general delegation problems with money burning. As one would expect, they generally imply that as the cost of money burning goes down for the principal (or the allocation becomes comparatively more important), the implemented action varies more closely with the principal's, and more money burning is used. Appendix B has a more detailed comparison to other optimal delegation mechanisms studied by Amador and Bagwell (2013) and Ambrus and Egorov (2017), both of which also allow for money burning.

The paper will proceed as follows. Section 2 presents the model and restates the problem in a way that allows for results from the calculus of variations to be used. Section 3 characterizes general results about the existence and properties of solutions to this problem. Sections 4 and 5 apply these results to the problems of dynamic delegation and delegation with money burning, respectively. Section 6 is a conclusion. Proofs of all results are in the appendices.

2 Model

A principal (she) and an agent (he) are contracting to make a decision about what action to take. While the principal has full control over what action is taken, the agent has private information about the state of the world, θ. This random variable is distributed on a compact interval Θ which without loss of generality can be set to be the unit interval. The agent wants to match the action to the state, and has utility function

$$-\frac{1}{2}(a-\theta)^2+\omega,$$

where ω is the continuation value that the principal promises to the agent. For any continuation value that the principal promises to the agent, she receives $\gamma(\omega)$, and thus has utility function

$$u^P(a,\theta)+\gamma(\omega).$$

The function γ, which is the mapping between the agent's continuation value and the principal's, is key to much of the following analysis. The slope of γ captures how costly (or beneficial) it is to either punish or reward the agent. As a baseline, it is useful to compare this to the "standard" case of mechanism design, in which there is transferable utility. In this case, the function is $\gamma(\omega)=-\omega$: one extra unit of utility for the agent comes at a cost of one unit for the principal.

For the remainder of this paper, a number of assumptions will be maintained:

Basic Assumptions *The following properties will be assumed for the entirety of the paper: (i) θ is distributed according to a distribution function $F(\theta)$ and has a density function $f(\theta)$ which is strictly positive and continuously differentiable on $(0,1)$; (ii) $u^P(\cdot,\cdot)$ is twice continuously differentiable, bounded above, and for all θ, concave in a, with first derivative $u^P_a(a,\theta)$ which converges uniformly to ∞ $(-\infty)$ as $a\to-\infty$ $(a\to\infty)$; (iii) $\gamma(\cdot)$ is concave, attains its maximum $\bar{\gamma}$, is upper semi-continuous, and if $\gamma(\cdot)>-\infty$ on some set $[a,b]$, then γ is continuously differentiable on (a,b).*

A mechanism is a message space and a pair of functions $a(\cdot)$ and $\omega(\cdot)$, which map the message space to actions and continuation values, respectively. The revelation principle implies that we can focus on direct and truthful mechanisms, which map states to actions and continuation values, subject to incentive compatibility constraints. Thus, the principal is solving the problem

$$\max_{\{a(\cdot),\omega(\cdot)\}} \int_0^1 \left[u(a(\theta),\theta)+\gamma(\omega(\theta))\right] f(\theta)d\theta \tag{P'}$$

subject to

$$\theta \in \underset{\hat{\theta}\in[0,1]}{\operatorname{argmax}} -\frac{1}{2}\left(a(\hat{\theta})-\theta\right)^2+\omega(\hat{\theta}) \tag{IC}$$

A key difference between this setting and standard mechanism design problems is that there is no participation ("individual rationality") constraint. This raises both technical and interpretative issues. On the technical side, this is what requires us to assume that $\gamma(\cdot)$ attains its maximum (ruling out, for instance, transferable utility). Otherwise, the maximization problem has no solution. Regarding interpretation, one should focus on cases in which the agent would *always* rather participate than not participate, for instance when a legislature delegates to a committee. In some cases, this model will also be applicable when there are individual rationality constraints which can be satisfied with lump sum transfers at the beginning of the relationship.[1]

This paper follows much of the literature on mechanism design in restating the problem as one of optimal control. In particular, it will define the agent's value function $W(\theta)$, and show that the principal's payoffs from an incentive compatible mechanism can be written as a function of $W(\theta)$, it's derivative, and the model's primitives. When stated in this way, the problem can be solved using results from the calculus of variations (Clarke, 1990).

For a given mechanism we can define the agent's value function $W(\theta)$ as the utility that he receives from that mechanism., i.e.

$$W(\theta)=\sup_{\hat{\theta}\in\Theta} -\frac{1}{2}\left(a\left(\hat{\theta}\right)-\theta\right)^2+\omega\left(\hat{\theta}\right),$$

which allows for a useful result:

Lemma 1 *W is absolutely continuous, and where the derivative exists,*

$$W'(\theta)=a(\theta)-\theta \tag{1}$$

[1] In particular, it's straightforward to see how the results will apply if the individual rationality constraint *always* binds. If this is true, then in searching for the optimal mechanism, one can restate the principal's preferences as

$$\hat{u}^P(a,\theta)+\hat{\gamma}(\omega)$$

where $\hat{u}^P(a,\theta)=u^P(a,\theta)-\frac{1}{2}(a-\theta)^2$, $\hat{\gamma}(\omega)=\gamma(\omega)+\omega$, and the individual rationality constraint is exactly fulfilled by the lump-sum transfer.

and

$$\omega(\theta) = W(\theta) + \frac{1}{2}W'(\theta)^2.$$

As in similar settings, incentive compatibility is equivalent to the agent's value function satisfying equation (1), the envelope condition, plus the monotonicity constraint that the action is increasing in the state. Thus, as a function of the state and the agent's value function, the principal's losses are

$$L\left(\theta, W(\theta), W'(\theta)\right) = \left[-u\left(W'(\theta)+\theta, \theta\right) - \gamma\left(W(\theta) + \frac{1}{2}W'(\theta)^2\right)\right] f(\theta).$$

and we can write the problem as

$$\min_{W(\theta)} \int_0^1 L(\theta, W(\theta), W'(\theta))d\theta \tag{P}$$

where $W(\theta)$ is absolutely continuous, and the action implied by $W(\theta)$ is monotonic. Similar to the rest of the literature, I'll solve problem (P) ignoring the monotonicity constraint, and then discuss the condition which ensures it.

3 Main Results

3.1 Existence

A key condition for the existence of solutions to problem (P) is what I will refer to as the convexity condition:

Convexity Condition: For each (θ, s), $L(\theta, s, \cdot)$ is convex.

Since this condition is critical for all of the following results, it is instructive to consider it further. In a sense, it requires that the second derivative of u^P with respect to a be "large enough" as compared to the slope of γ. This is similar to a second-order condition, and it ensures that the conditions which we later show imply optimality are valid.

The convexity condition implicitly requires that the function γ be greater than $-\infty$ for all ω less than some cutoff $\bar{\omega}$. Thus, if the principal is able to promise some continuation value, she must be able to promise any value below it. This rules out any continuation games in which payoffs are bounded below, for instance. However, two facts make this constraint less important. First, in many cases one may extend γ in a way that allows it to satisfy the convexity condition, and if an optimal mechanism is found in which ω lies completely in the region in which γ is well defined, then it is an an optimal mechanism of the original problem.[2] Second, Clarke (1977) and the references therein show the conditions under which it is sufficient to solve an adjusted problem in which one replaces $L(\theta, s, \cdot)$ with a loss function whose epigraph is a convex hull of the epigraph of $L(\theta, s, \cdot)$.

With the convexity condition and the basic assumptions, results from the calculus of variations (Clarke, 1990) imply that a solution to problem (P) exists.

Proposition 1 *Assume that the convexity condition holds. Then a solution to problem (P) exists.*

3.2 Characterization

There are many cases, including those described in Sections 4 and 5 below, in which a principal may be unable to promise a continuation value above some cutoff $\bar{\omega}$. This is accounted for in the model by setting $\gamma(\omega) > -\infty$ for $\omega \leq \bar{\omega}$ and $\gamma(\omega) = -\infty$ for $\omega > \bar{\omega}$ (in what follows, let $\bar{\omega} = \infty$ if there is no such restriction). An optimal mechanism for which $\omega(\theta) = \bar{\omega}$ for some θ is in a sense a "corner solution," and has different conditions for optimality than "interior solutions." In Propositions 2 and 3, as well as Theorem 1 below, I focus on the case in which this constraint is not binding. The more general results are presented in Propositions A.1 and A.2 in Appendix A, and an example in which this constraint is binding is considered explicitly in Section 5.

The problem (P) is somewhat nonstandard, since the loss function may not be differentiable

[2] For a more concrete example, assume that in the future, both players are playing a simple coordination game. If they cooperate, they both receive payoffs of M, but if either deviates, they both receive a payoff of 0. In the first period, the principal can commit to the probability of cooperating in the coordination game. Thus, in this case

$$\gamma(\omega) = \begin{cases} \omega \text{ if } 0 \leq \omega \leq M \\ -\infty \text{ otherwise} \end{cases},$$

where ω will be equal to the probability of cooperating multiplied by M. This is ruled out from the above analysis, because $\gamma = -\infty$ for $\omega < 0$, but $\gamma > -\infty$ for $\omega = 0$. However, if the principal solves an adjusted problem with

$$\gamma(\omega) = \begin{cases} \omega \text{ if } \omega \leq M \\ -\infty \text{ otherwise} \end{cases}$$

and none of the continuation values in the optimal mechanism are less than 0, then the mechanism also solves the original problem.

at all points. Clarke (1990) provides conditions which are necessary for optimality in such cases. If the function $\gamma(\cdot)$ is twice differentiable in the promised continuation value at some point, then at that point a solution to the problem will be a solution to a differential equation, which optimally trades off between payoffs in the first period and the continuation value.

Proposition 2 *Assume that the convexity condition holds, and that $W(\theta)$ solves problem (P). Define $a(\theta) = W'(\theta) + \theta$ and $\omega(\theta) = W(\theta) + \frac{1}{2}W'(\theta)^2$. Then for $\theta \in [0,1]$ such that $\omega(\theta) < \bar{\omega}$,*
(1) if $\theta \in \{0,1\}$,

$$W'(\theta)\gamma'\left(\omega(\theta)\right) = -u_a^P\left(a(\theta), \theta\right),$$

and

(2) if $\gamma''\left(\omega(\theta)\right)$ exists, then

$$W''(\theta) = -1 + \frac{2\gamma'\left(\omega(\theta)\right) - u_{a\theta}^P\left(a(\theta),\theta\right) - \left[u_a^P\left(a(\theta),\theta\right) + W'(\theta)\gamma'\left(\omega(\theta)\right)\right]\frac{f'(\theta)}{f(\theta)}}{u_{aa}^P\left(a(\theta),\theta\right) + W'(\theta)^2\gamma''\left(\omega(\theta)\right) + \gamma'\left(\omega(\theta)\right)}$$

almost everywhere.

Proposition 2 provides the necessary conditions for optimality, in particular for the instances when $W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega}$. The necessary conditions for optimality when $W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega}$, but $W(\theta) < \bar{\omega}$ are omitted here for clarity, but are given in Proposition A.1 in Appendix A.

Condition (1) from Proposition 2 is known as the "free endpoint condition" from the calculus of variations, and essentially states that the marginal value of changing the action taken when θ is 0 or 1 must be equal to the marginal cost of changing this action. Condition (2), known as the Euler-Lagrange equation, is also standard, and is a necessary condition for a function to minimize a functional as the solution to problem (P) must.

If $W(\theta)$ is absolutely continuous and the implied continuation values are everywhere strictly below $\bar{\omega}$, then the necessary conditions from Proposition 2 are also sufficient.

Proposition 3 *Assume that the convexity condition holds. If $W(\theta)+\frac{1}{2}W'(\theta)^2 < \bar{\omega}$, $\gamma''\left(W(\theta) + \frac{1}{2}W'(\theta)^2\right)$ exists, $W(\theta)$ is absolutely continuous, and $W(\theta)$ satisfies conditions (1) and (2) from Proposition 2 almost everywhere, then $W(\theta)$ solves problem (P).*

Again, the more general conditions for when $W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega}$ are left for the Appendix in Proposition A.2 and Corollary A.1.

Combining the conditions given in Proposition 3, along with the envelope theorem and monotonicity constraint allows us to provide sufficient conditions for a mechanism to be optimal:

Theorem 1 *Suppose that the convexity condition holds, and the following is true:*
(1) for $\theta \in \{0, 1\}$,

$$-u_a(a(\theta), \theta) = (a(\theta) - \theta)\gamma'(\omega(\theta));$$

(2) $\gamma''(\omega(\theta))$ exists almost everywhere, with

$$a'(\theta) = \frac{2\gamma'(\omega(\theta)) - u^P_{a\theta}(a(\theta), \theta) - \left[u^P_a(a(\theta), \theta) + (a(\theta) - \theta)\gamma'(\omega(\theta))\right] \frac{f'(\theta)}{f(\theta)}}{u^P_{aa}(a(\theta), \theta) + (a(\theta) - \theta)^2 \gamma''(\omega(\theta)) + \gamma'(\omega(\theta))},$$

and $\omega(\theta)$ is derived from the envelope condition;
(3) $a(\theta)$ is continuous and monotonically increasing.
Then the mechanism defined by $a(\theta)$ and $\omega(\theta)$ is optimal.

Condition (1) is an analog of the "no distortion at the top" condition from the solution to a standard mechanism design problem. If it were the case that γ were linear with a slope of -1 (i.e. transferable utility), then this condition would require that the marginal gains to the principal for raising the action at the endpoint are equal to the marginal losses for the agent. Because the transfer of utility has to first pass through the function γ, the agent's marginal utility has to be scaled by this factor. Another difference between this and the standard mechanism design setting is the fact that the condition must hold for *both* endpoints, not just the "top." This is due to the fact that there is no individual rationality constraint, so the condition can hold at both endpoints.

The equality from condition (2) can be derived from the equality

$$\frac{d}{d\theta}\left[\left(u^P_a(a(\theta), \theta) + (a(\theta) - \theta)\gamma'(\omega(\theta))\right) f(\theta)\right] = \gamma'(\omega(\theta))f(\theta). \tag{2}$$

This equation can be understood as a generalization of more standard principles of mechanism design. For instance, the first order conditions for an optimal mechanism presented by Fudenberg and Tirole (1991) are[3]

$$\frac{\partial U^P}{\partial a} + \frac{\partial U^A}{\partial a} = \frac{1 - F(\theta)}{f(\theta)} \frac{\partial^2 U^A}{\partial a \partial \theta}, \tag{3}$$

where U^P is the principal's utility function and U^A is the agent's utility function. The left hand side of equation (3) is the marginal total surplus, since it is the sum of the agent's and principal's

[3]Notation has been changed slightly for comparability.

marginal utilities. The right hand side can be interpreted as the marginal information rents that the principal must leave to higher types. Similarly, the first term in the brackets in equation (2) is the principal's marginal utility, and the second term is the agent's marginal utility, multiplied by the rate at which utility can be transferred between the principal and the agent, $\gamma'(\omega(\theta))$. Treating the right hand side as the information rents which must be left to the agent, this gives a clearer interpretation for equation (2): the principal maximizes the total surplus *which can be appropriated*, minus the cost of the information rents that must be left to the agent.

To clarify the relationship between classic optimality conditions from mechanism design and those found here, one can substitute the utility functions used in this paper in equation (3), multiply both sides by $f(\theta)$, and totally differentiate. This results in the equality

$$\frac{d}{d\theta}\left[\left(u_a^P(a(\theta),\theta) - (a(\theta)-\theta)\right) f(\theta)\right] = -f(\theta),$$

which is equivalent to substituting $\gamma'(\omega) = -1$ into equation (2). Not surprisingly, the equality from condition (2) can also be related to the conditions presented by Fudenberg and Tirole. In particular, with the assumption of quadratic losses for the agent, one can manipulate the first order conditions there (equation (7.12)) to obtain

$$\left[u_{aa}^P\left(a(\theta),\theta\right) - 1\right] a'(\theta) = \left[-u_a(a(\theta),\theta) + (a(\theta)-\theta)\right] \frac{f'(\theta)}{f(\theta)} - 2 - u_{a\theta}\left(a(\theta),\theta\right),$$

which is the same as the equation from condition (2), but with $\gamma(\omega) = -\omega$. Thus, the condition generalizes the optimality condition from standard mechanism design with transfers, but does so under a functional form assumption on the agent's utility.

Condition (3) from Theorem 1 ensures that the agent's value function is continuous at points where $\gamma''(\omega(\theta))$ does not exist, and that the action satisfies the monotonicity constraint. At points where $a(\theta)$ is continuously differentiable, the derivative is positive if and only if

$$2\gamma'(\omega(\theta)) - u_{a\theta}^P(a,\theta) - \left[u_a^P(a(\theta),\theta) + (a(\theta)-\theta)\gamma'(\omega(\theta))\right] \frac{f'(\theta)}{f(\theta)} \leq 0. \tag{4}$$

Classical results from mechanism design show that studying the "relaxed problem" is sufficient under standard assumptions along with the monotone hazard rate condition. If we were to assume $\gamma'(\omega) = -1$ and $u_{a\theta}^P(a,\theta) \geq 0$, and again substituted in equation (7.12) from Fudenberg and Tirole (1991), we would find that the monotonicity condition holds when

$$\left[\frac{1-F(\theta)}{f(\theta)}\right] \frac{f'(\theta)}{f(\theta)} \geq 0,$$

but the monotone hazard rate implies that

$$-1+\left[\frac{1-F(\theta)}{f(\theta)}\right]\frac{f'(\theta)}{f(\theta)}\geq 0.$$

Thus, Theorem 1 can be seen as an extension of the classical mechanism design results, which relaxes both the usual individual rationality constraint and the assumption of transferrable utility, while imposing additional structure on the agent's preferences.

4 Dynamic Delegation

Suppose that the principal and agent are Congress and the EPA, and that they need to make the decision about how much they should reduce SO_2 emissions in each of two years.[4] The "states" in each of the years, θ_1 and θ_2, can be thought of as factors that determine how much environmental damage SO_2 will cause. As experts in the field, EPA officials observe this, but Members of Congress do not. For the purposes of this example, let them be uniformly and independently distributed.

Assume that the EPA favors stricter environmental regulations than Members of Congress: no matter what the conditions are, EPA officials would like to reduce emissions more than Congress would (it is easy to restate the problem and get symmetric conclusions if the EPA instead wants looser environmental regulations than Members of Congress). Defining a_t as the emissions reductions in period t, the principal has per-period utility function $-\frac{1}{2}(a_t-\theta_t+b_t)^2$, and the agent has per-period utility function $-\frac{1}{2}(a_t-\theta_t)^2$, and both discount at rate δ. Assume that the principal and agent have different preferences, so $b_1\neq 0$ and $b_2\neq 0$, and without loss of generality let both be greater than 0, since the EPA wants higher emissions reductions.

This is a *dynamic delegation* problem, in which the principal sequentially delegates choices to a biased agent. Here, a direct mechanism is a pair of functions $a_1:\Theta\to\mathbb{R}$ and $a_2:\Theta\times\Theta\to\mathbb{R}$, which choose the amount of emissions reductions for any report of the state. Thus, the principal's problem is

$$\max_{a_1(\theta_1),a_2(\theta_1,\theta_2)}\mathbb{E}\left[-\frac{1}{2}(a_1(\theta_1)-\theta_1+b_1)^2-\frac{1}{2}\delta(a_2(\theta_1,\theta_2)-\theta_2+b_2)^2\right]\qquad\text{(PDD”)}$$

[4] A natural question is how the model extends to more periods. If the interaction is infinitely repeated, the model is similar to that of Guo and Hörner (2015), albeit with an allocation which affects payoffs nonlinearly. Although it's not shown here, it's likely that the T period problem could be solved similarly to the model here: first find the optimal way to promise ω utility in period T, then for $T-1$, etc.

subject to $\forall \theta_1, \theta_1' \in \Theta$,

$$\mathbb{E}\left[-\frac{1}{2}(a_1(\theta_1)-\theta_1)^2 - \frac{1}{2}\delta(a_2(\theta_1,\theta_2)-\theta_2)^2|\theta_1\right] \geq \tag{IC1DD}$$
$$\mathbb{E}\left[-\frac{1}{2}(a_1(\theta_1')-\theta_1)^2 - \frac{1}{2}\delta(a_2(\theta_1',\theta_2)-\theta_2)^2|\theta_1\right]$$

and $\forall \theta_1, \theta_2, \theta_2' \in \Theta$,

$$-\frac{1}{2}(a_2(\theta_1,\theta_2)-\theta_2)^2 \geq -\frac{1}{2}(a_2(\theta_1,\theta_2')-\theta_2)^2 \tag{IC2DD}$$

This is clearly a problem in which the principal is delegating in the first period, and there is a continuation value associated with the continuation contracts in the second period. In fact, because the two states θ_1 and θ_2 are independent, the fact that the second period must carry out the continuation value from the first period is the only connection between the periods: Congress offers choices other than optimal delegation set in the second period only if it wants to reward or punish EPA officials for their first period choice. Thus, in solving for the optimal mechanism, it will be necessary to solve for the optimal way to promise any level ω of utility.

Proposition 4 *In the uniform-quadratic setting, the principal's problem can be written as*

$$\max_{a_1(\theta_1),\omega(\theta_1)} \mathbb{E}\left[-\frac{1}{2}(a_1(\theta_1)-\theta_1+b_1)^2 + \gamma(\omega(\theta_1))\right] \tag{PDD'}$$

subject to $\forall \theta_1, \theta_1' \in \Theta$,

$$-\frac{1}{2}(a_1(\theta_1)-\theta_1)^2 + \omega(\theta_1) \geq -\frac{1}{2}(a_1(\theta_1')-\theta_1)^2 + \omega(\theta_1') \tag{IC1'}$$

where

$$\gamma(\omega) = \begin{cases} \omega - \frac{\delta}{2}b_2^2 + \frac{\delta b_2}{6}(-\frac{72}{\delta}\omega - 3)^{\frac{1}{2}} \ \textit{if}\ \omega < -\frac{\delta}{6} \\ \omega - \frac{\delta}{2}b_2^2 + \frac{\delta b_2}{2}(\frac{6}{\delta}\omega)^{\frac{2}{3}} \ \textit{if}\ -\frac{\delta}{6} \leq \omega \leq 0 \\ -\infty \ \textit{otherwise} \end{cases}.$$

Thus, this uniform-quadratic dynamic delegation setting is an example of the more general setting of delegation with continuation values considered in Section 3, where $\gamma(\cdot)$ is as given in Proposition 4. Proposition 4 greatly simplifies Congress' problem. Instead of a dynamic problem requiring it to choose an emissions reduction in the first year *and a function* for the second year when the EPA makes its first report, she can simply choose the emissions reductions and a "pseudo-transfer" for each type, where this pseudo-transfer is denoted by $\omega(\theta_1)$. In choosing to transfer some

$\omega(\theta_1)$, Congress receives $\gamma(\omega(\theta_1))$ in expected utility. Once found, this function $\gamma(\cdot)$ is what makes the results from delegation with continuation values applicable to this dynamic delegation problem.

For this specification of payoffs and distribution of states, the options that Congress gives to the EPA in the second period take a specific form. In particular, the actions that the principal makes available to the agent can be interpreted as a simple cap on the emissions reductions that the EPA can require in the second period. This is the same general form as the optimal mechanism of the static version of the game (Melumad & Shibano, 1991). The level of this cap is determined from how much expected utility the principal is promising the agent, so I'll refer to the cap conditional on a promise of ω as $y(\omega)$.

The function $\gamma(\cdot)$ in Proposition 4 is defined piecewise, with the domain split at $-\frac{\delta}{6}$. This is the expected continuation value for the agent if he is being forced to take the action $a_2 = 0$ regardless of the state in the second period. When the cap on the second period action is above 0, lowering the cap has two effects: it enlarges the set of states which are pooled at the cap, and lowers the action taken for all of those states. When the cap is below 0, lowering it further only changes the action taken (since the agent takes the maximum action available for all values of the state already). The difference between these two regions leads to a discontinuity of the second derivative of $\gamma(\cdot)$, which will be discussed below.

A version of the function γ when $b_2 = 0.5$ can be seen in Figure 1. This function provides the mapping between the EPA's continuation value and Congress' continuation value which is optimal from Congress' point of view. This function is continuously differentiable, strictly concave, and has a maximum at $\omega = -\frac{4}{3}\delta b_2^3$ for $b_2 \leq \frac{1}{2}$, and at $\omega = -\frac{1}{2}\delta b_2^2 - \frac{1}{24}\delta$ for $b_2 > \frac{1}{2}$. This maximum is the payoff that Congress would receive if it used the solution to the one period delegation problem in the second period. If the principal were only considering a single period, setting ω less than the maximizing point would be giving the agent *too little* discretion, leading to the principal not taking full advantage of the agent's information. Alternatively, setting ω greater than the maximizing point gives the agent *too much* discretion, implying that the principal doesn't properly account for the agent's bias. Thus, the principal's problem in the first period is clear: she wants to incentivize the agent to take lower actions using ω, while keeping ω near the value that maximizes $\gamma(\cdot)$.

With this simplification, the state variable in the second period, θ_2, no longer enters into the problem. Overall expect utility that arises from the mechanism in the second period is captured by $\gamma(\cdot)$. Thus, from this point on the state will be referred to as θ, which should be interpreted as

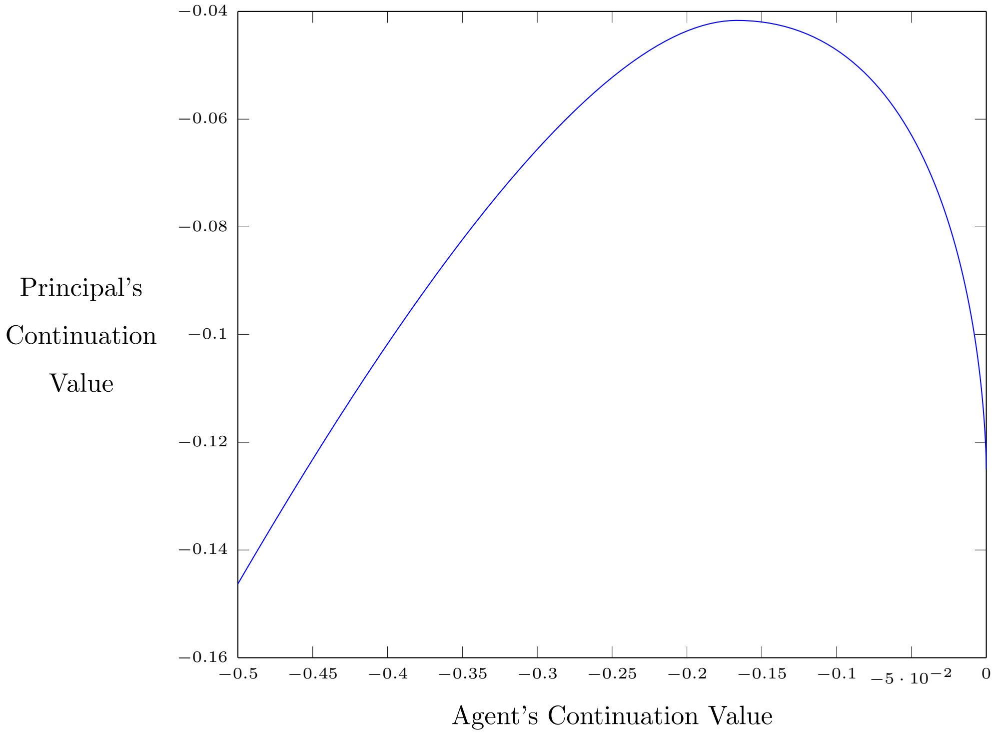


Figure 1: The function $\gamma(\cdot)$ for $\delta = 1$, $b_2 = 0.5$

the state in period 1. The loss function for any value of the state is

$$L(\theta, s, v) = \left[\frac{1}{2}(v + b_1)^2 - \gamma\left(s + \frac{1}{2}v^2\right)\right]$$

with γ defined as before, and the principal is looking for a monotonic action rule which is implied by the $W(\theta)$ that solves

$$\min_{W(\theta)} \int_0^1 \left[\frac{1}{2}(W'(\theta) + b_1)^2 - \gamma\left(W(\theta) + \frac{1}{2}W'(\theta)^2\right)\right] d\theta. \tag{PDD}$$

among absolutely continuous functions. $L(\theta, s, \cdot)$ is continuously differentiable and convex for all (θ, s) such that $s + \frac{1}{2}v^2 \leq 0$. Thus, the problem satisfies all of the requirements necessary for the use of Proposition 3, implying that if there is a solution to the differential equation and endpoints conditions found in Proposition 3, then this solution solves Problem (PDD). Proposition 5 shows that such as solution does exist, and is proven in Appendix C.

Proposition 5 *An interior solution to problem (PDD) exists, i.e. there exists a* $W(\theta)$ *which solves (PDD) such that* $\forall\theta$, $W(\theta) + \frac{1}{2}W'(\theta)^2 < \tilde{\varepsilon} < 0$.

The discontinuity of the second derivative of γ leads to a differential equation which is discontinuous, so many of the standard results regarding the existence and uniqueness of solutions to differential equations do not apply. Instead, the proof uses results from the theory of differential equations with discontinuous righthand sides (Filippov & Arscott, 1988). In particular, the proof first shows that solutions to this differential equation exist, are unique, and are continuous in initial conditions. It then continuously varies the initial conditions along the curve on which the endpoints must lie, and uses continuity and uniqueness to show that there exist values of $W(0)$ and $W'(0)$ on this curve such that $W(1)$ and $W'(1)$ will also be on the curve.

This discontinuity arises from the piecewise nature of the function $\gamma(\cdot)$ that results from delegation in the second period. Since the action in the first period can be written as a function of $W'(\theta)$, this discontinuity does not result in an action which is discontinuous in the state, but rather a jump in the rate at which the action changes with the state. This captures the fact that the rate of tradeoff between payoffs in the first and second period changes when the principal changes from setting a cap above the agent's lowest preferred action to one below it.

With the existence of a solution to the differential equation with the appropriate endpoints, it remains to show that the action implied by this function is monotonic. This result and some of the mechanism's properties are given in Theorem 2.

Theorem 2 *In the dynamic delegation game, an optimal mechanism exists and has the following properties:*

- *the action is strictly increasing and continuous in the first period's state (no pooling in the first period);*
- *the maximum action available in the second period is strictly decreasing in the first period's state;*
- *the actions taken in the first period are a strict subset of those the principal would take if she were fully informed.*

Relating this back to the example of Congress and the EPA, this result says that as the danger of environmental damage increases, when Congress is delegating optimally, they allow the EPA to impose higher emissions reductions. Usually, in the one period problem, Congress would simply set a cap, and for all values of the state above some cutoff, the EPA would set emissions reductions at that cap. In the first period of dynamic delegation, such pooling does not happen. Congress can vary the delegation set in the second period slightly, taking second order losses, but will be making first order gains from the less biased actions in the first period.

Given that the action taken in the first period is strictly increasing in the state, it is perhaps not surprising that the cap on actions in the second period is strictly decreasing. Recall that this cap is the means by which the principal incentivizes the agent to take a given action in the first period. Since the agent would in general prefer to be taking a higher action, agents who take the higher action must be given a lower continuation value, ω. This lower continuation value is in turn associated with less discretion in the second period—a lower cap.

Figure 2 shows the optimal spread of ω when $\delta = 1$ and $b_1 = b_2 = 0.5$. The principal is giving the agent the smallest continuation value when the agent reports that the state is 1, and the highest continuation value when the agent reports that the state is 0.

It may be surprising that the highest action in the first period is strictly less than the action preferred by *both* the principal and the agent (i.e. the actions taken in the first period are a strict subset of those the principal would take if she were fully informed). To understand this point, it will first be useful to consider the principal's incentives more carefully. She is using discretion in the second period, captured by $\omega(\theta)$, to incentivize the agent to choose the action closer to her own preferred action in the first period. Because there is an optimal level of discretion (the point where $\gamma(\cdot)$ is maximized) in the second period any deviation away from this comes at a cost. Thus, she'll be committing to give "too much" discretion if the agent reports a low state, and "too little" if the agent reports a high state.

If the principal were allowing the agent to take actions higher than $1 - b_1$ (the highest action the principal would ever want to take) for high values of the state, then she could lower all of these actions, and raise the continuation values promised to the agent for these states. *Both* of these changes improve the principal's payoffs, since the action taken in these states becomes closer to the principal's preferred action, and the principal is promising more discretion in the second period, which she preferred to do anyway. Similarly, the principal would never have the agent take actions below $-b_1$.

Figure 3 shows the optimal action and cap for various values of δ, b_1, and b_2, in which the characteristics described in Theorem 2 can be noted. These graphs also show the consequences of the discontinuity in the second derivative of γ (i.e. the fact that the principal shifts from delegating a set to delegating a point). At the θ where $y(\omega(\theta)) = 0$, both $a(\theta)$ and $y(\theta)$ are kinked.

The generality of these results is an interesting point to consider. They are derived using quadratic losses and a uniform distribution, both of which are restrictive assumptions. Here, the

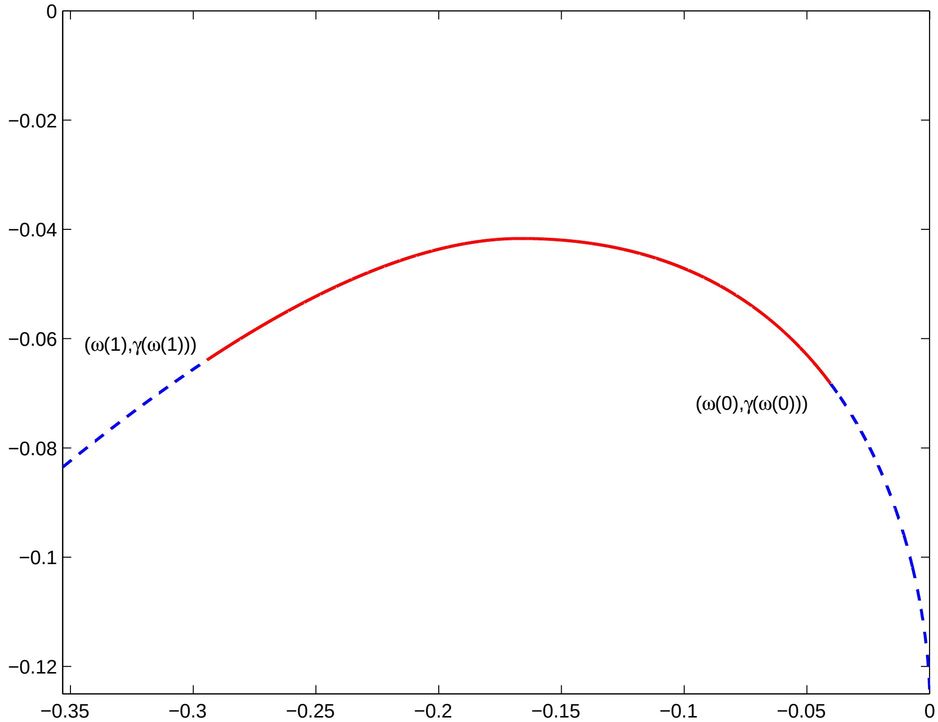


Figure 2: A plot of $\gamma(\cdot)$ with the optimal spread (in solid red) of ω for $\delta = 1$ and $b_1 = b_2 = 0.5$

principal does not need to pool different states into a single action, because she can use continuation values to vary the action smoothly. Furthermore, simply raising or lowering the cap on the second period choices is the optimal way to change the utility promise. In other settings, delegation sets at the end of the relationship are likely to be more complex, but it is to be expected that the principal will tend to make these delegation sets smaller or larger to punish or reward the agent. With the mapping from agent's continuation value to principal's that this provides, it's plausible that the principal will use these continuation values to vary the action continuously with the state as is the case in this model.

4.1 Comparative Statics and the Value of Delegating Dynamically

Intuitively, we might expect that larger differences between the principal's and agent's preferred decisions would lead to lower payoffs for the principal; it seems natural that Congress would prefer EPA officials who have preferences similar to their own. This is the case for the one period, uniform-quadradic delegation problem, until the bias becomes so high that delegation has no value at all (Melumad & Shibano, 1991).

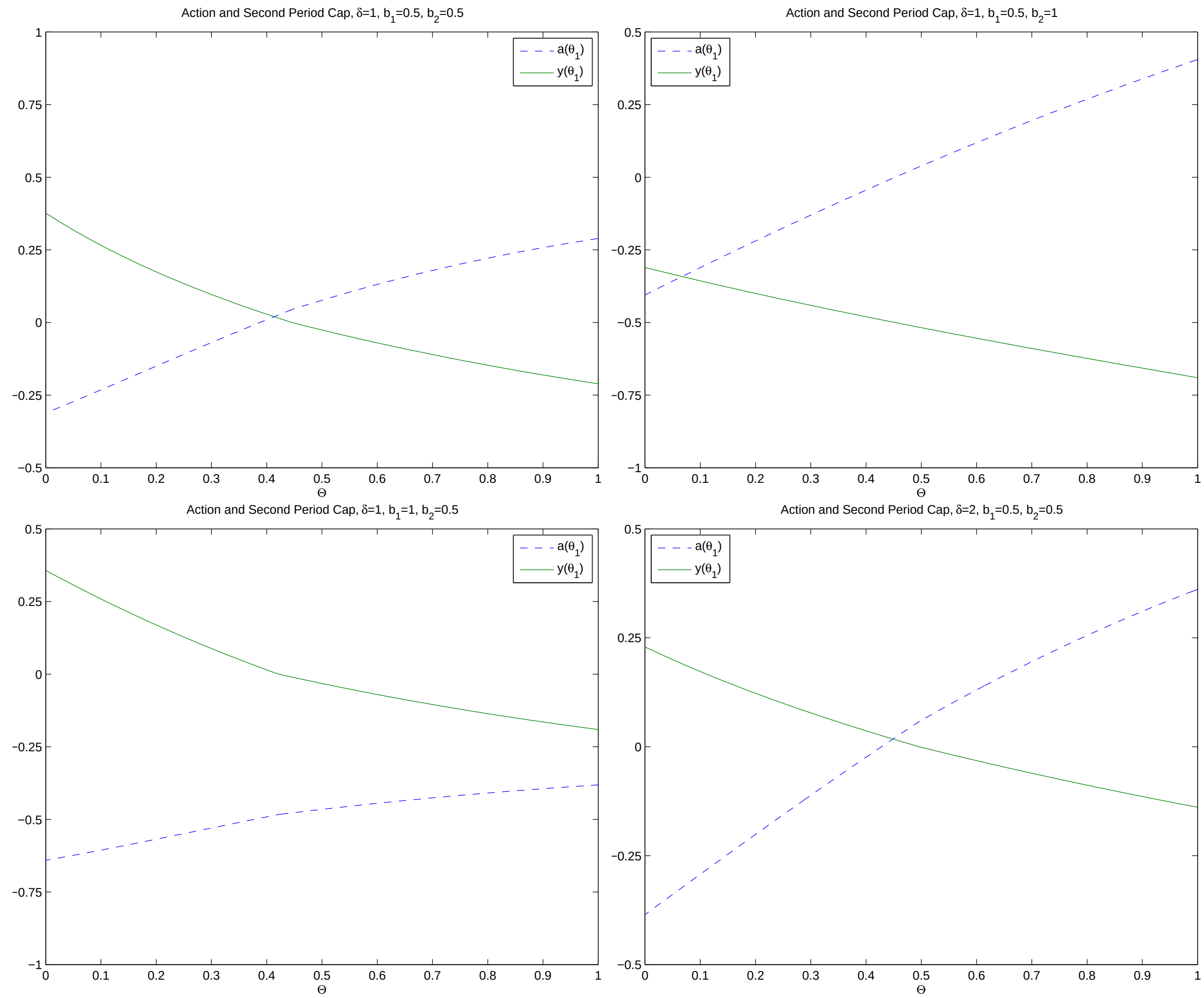


Figure 3: Optimal action and cap for various δ, b_1, and b_2

Theorem 2 demonstrates that a two period delegation problem is fundamentally different than a static one: when delegation is dynamic and biases are strictly positive, delegation is *always* valuable (at least in the first period). This is because the ability of the principal to incentivize first period actions using continuation values allows for the principal to utilize the agent's information to some extent, however small.

Even though delegation continues to be valuable in dynamic delegation, an agent who is more biased in the first period is worse for the principal, as noted in the next result.

Theorem 3 *In the dynamic delegation game, increasing the agent's bias in the first period makes the principal strictly worse off.*

Increasing the first period's bias is unambiguously bad for the principal.[5] The higher level of bias makes it necessary to increase the spread of the agent's continuation values to incentivize actions that are as good for the principal, and increasing this spread makes the principal worse off due to the concavity of $\gamma(\omega)$. Figure 4 shows how the principal's payoffs vary with b_1 for two different values of b_2, and provides the comparison to the payoffs she would receive if the optimal static mechanism for each period were used.

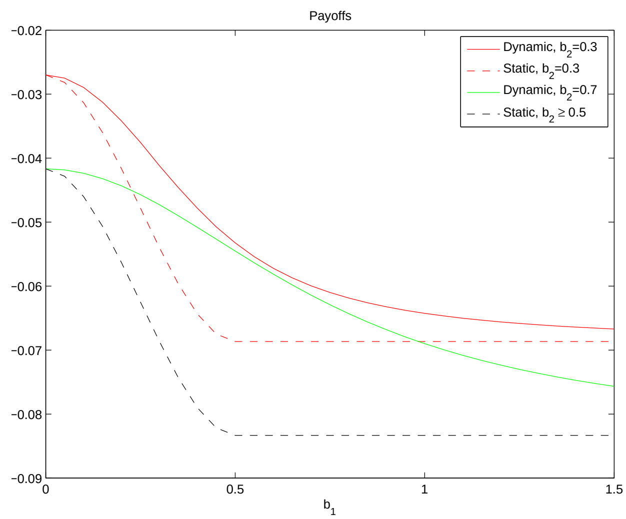


Figure 4: Payoffs for the Principal, with the optimal dynamic mechanism (solid), or repeating the optimal static mechanism (dashed)

Comparative statics for b_2 are not as obvious, because increasing b_2 has two effects. A higher b_2 means that for any delegation set in the second period, the agent will be choosing something which is weakly worse for the principal. From the point of view of period 1, though, a higher b_2 makes the agent easier to incentivize to take an action, since lowering the maximum available action in period 2 is much worse for an agent with a higher bias. In this way, increasing b_2 is similar to increasing Δ in the setting of Koessler and Martimort (2012), which also has this dual effect. Theorem 4 shows that for high enough b_2, increasing b_2 further eventually improves the principal's payoffs.

Theorem 4 *In the dynamic delegation game, for high enough second period bias, further increasing*

[5]The model above has restricted attention to cases in which both bias parameters are strictly positive. Using the result from Appendix B, it can be checked that the only case in which increasing b_1 does not make the principal strictly worse off is if $b_2 = 0$ and $b_1 \geq 0.5$, for which increasing b_1 does not change the allocation or continuation value.

this bias improves the principal's payoffs. As bias in the second period becomes arbitrarily large, the principal's payoffs approach those she would receive if she had full information in the first period, and had to make the optimal uniformed decision in the second period.

The proof of Theorem 4 shows that increasing b_2 increases the principal's payoffs if $b_2 \geq \frac{1}{2}$. For all values of b_2 greater than this, if the principal were considering the second period problem in isolation, she would prefer to not give the agent any discretion. Since the optimal static mechanism is the same in all of these cases, the maximum continuation value that the principal can receive in the first period no longer falls as the second period bias increases.

For an agent that is this biased in the second period, the principal varies the action that will be taken in the second period around her preferred uninformed action (perhaps eventually giving the agent some discretion). This change in the action taken has a higher effect on agents whose bias is higher, because the agent's loss function is convex. Thus, an equivalent change in the principal's payoff is associated with a larger change in the agent's payoff when that agent has a higher bias.

The implications of this can be seen in Figure 5. In this figure, two versions of the function γ are centered such that their maxima are at 0. The re-centered version of γ in which $b_2 = 0.7$ is everywhere above the re-centered version of γ in which $b_2 = 0.5$. Thus, as b_2 increases for values above 0.5, the principal can incentivize the exact same action profile, with the agent's continuation values simply shifted downward, and this gives a higher payoff to the principal. Figure 6 illustrates these comparative static results, and provides the comparison of what the principal would receive if she used the optimal static mechanism in each period.

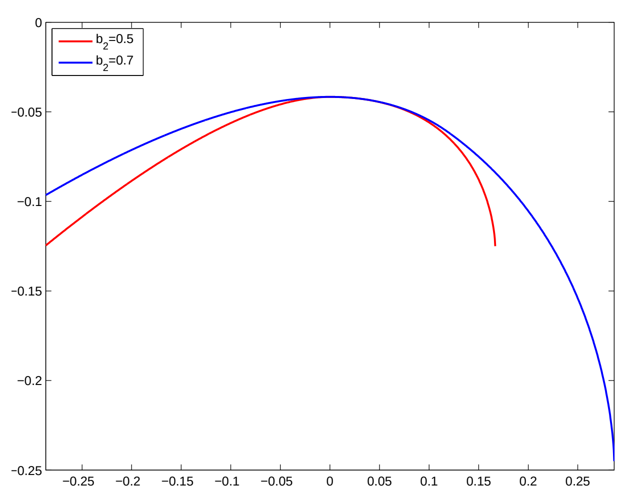


Figure 5: The function γ, centered around it's maximum, for $b_2 = 0.5$ and $b_2 = 0.7$

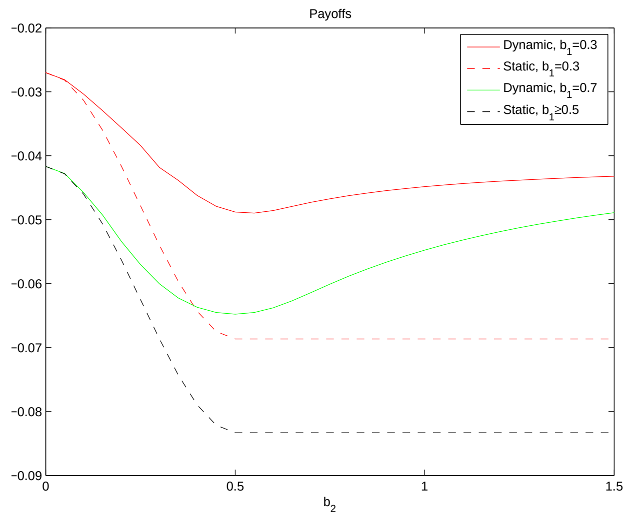


Figure 6: Payoffs for the Principal, with the optimal dynamic mechanism (solid), or repeating the optimal static mechanism (dashed)

Theorem 4 also states that as the second period bias becomes arbitrarily large, the payoffs that arise from any two fixed first period biases converge to each other. This relates to the way in which it becomes easier to incentivize the agent as b_2 increases: small changes in the promised action create large swings in the agent's continuation value, and for high enough b_2 the principal can implement her optimal action profile while essentially receiving the payoff from the optimal static mechanism in the second period.

4.2 Stochastic Mechanisms and Dynamic Delegation

The mechanism described in Theorem 2 is a deterministic one, which is to say that when the agent announces the state, a given action is executed with probability one. A natural question is whether the principal can improve her expected payoffs by instead committing to a mechanism which is stochastic.

We should first consider what it means for a mechanism to be stochastic in a delegation setting. In the original delegation problem, the setting was interpreted as the principal choosing which actions to make available to the agent. Using the idea of a delegation function through the revelation principle was simply an analytical tool to characterize the optimal mechanism. To motivate the possibility of a stochastic mechanism in their setting, Goltsman et al. (2009) consider contracting with an arbitrator, who decides which action to implement after being sent a message by the agent. Since the arbitrator does not have preferences over the implemented action, she can commit to randomizing over which action is chosen after a given report. Kováč and Mylovanov (2009), who consider a wider variety of distributions and biases, focus on stochastic mechanisms in general settings without monetary transfers rather than delegation per se.

There are three points at which a principal may incorporate randomness into the setting at hand. First, she could randomize the action taken after the report in the first period. Second, she could randomize over the delegation set that the agent will choose from in the second period. Finally, she could randomize over the action taken after the report in the second period. The next result shows that *none* of these will improve the principal's payoffs.

Theorem 5 *The optimal dynamic delegation mechanism is deterministic.*

The reason why the principal cannot improve her payoffs by randomizing over second period delegation sets is simple. To incentivize actions in the first period, the principal commits to leave some expected utility for the agent in the second period. Thus, the principal must decide whether it is more efficient to promise this utility through a single delegation set, or by randomizing over delegation sets (and thus over promises of expected utility). Because the mapping between the agent's utility and the principal's utility is concave, Jensen's Inequality implies that it is better to commit to a single delegation set.

The non-optimality of mechanisms which randomize after the agent's report arises due to the principal's aversion to variance. Quadratic losses allow payoffs to be written as a function of the average and variance of an action given some report. Thus, variance is essentially "money burning" from the point of view of both players: lowering the agent's utility by one unit by increasing the variance also lowers the principal's utility by one unit. In the first period, it's more efficient to punish the agent by reducing discretion in the second period rather than increasing variance in the first period (the slope of γ derived in Proposition 4 is always greater than 1). Similarly, the cost of incentivizing a better action in the second period by increasing variance is not worth the gains that come from that better action.

The fact that the proofs of these results rely on properties of the quadratic loss function suggests that they may not hold in more general settings. Kováč and Mylovanov (2009) demonstrate the non-optimality of deterministic mechanisms in a situation where the principal has an absolute value loss function while the agent has a quadratic loss function. In this case, the principal can use a stochastic mechanism to achieve strictly higher payoffs than any deterministic mechanism, because the principal can increase the variance, strictly lowering the agent's payoffs, at arbitrarily small cost to herself. Thus, there are likely combinations of preferences for which stochastic mechanisms dominate deterministic ones in dynamic settings.

5 Money Burning and Responsiveness

Another tool Congress might use to incentivize EPA officials is committing to some other inefficient action if the EPA chooses something that seems biased to Congress. If the cost of this action for Congress is linearly related to the cost to the EPA, then we are in the setting of *delegation with money burning* (Amador & Bagwell, 2013, 2016; Ambrus & Egorov, 2017). The action could entail Congress reducing the budget on a program that it and EPA officials agree upon, or enacting environmentally damaging policy that it otherwise wouldn't. In this case, we define the function γ to be

$$\gamma(\omega) = \begin{cases} k\omega \text{ for } \omega \leq 0 \\ -\infty \text{ otherwise} \end{cases}$$

The fact that $\gamma(\omega) = -\infty$ for $\omega > 0$ reflects the fact that there is no way for the principal to improve the agent's payoffs above 0: she can only punish the agent. Here, k is a parameter which determines how costly money burning is for the principal, relative to the costs the agent faces.[6] A high value of k indicates that punishing the agent is particularly difficult for the principal, while a low value of k implies the opposite. Amador and Bagwell (2013) interpret money burning as "wasteful administrative costs," and Ambrus and Egorov (2017) refer to it as "paperwork," but for any given setting there are a variety of other interpretations, such as the payoffs arising from a simple coordination game in the future.

Suppose that the principal and the agent prefer the same action when the state is intermediate, but in general the agent's preferred action varies with the state differently than the principal's preferred action. The principal seeks to maximize the utility function $u^P(a,\theta) = -\frac{1}{2}\left(a(\theta) - \beta\theta + \frac{1}{2}\beta - \frac{1}{2}\right)^2$. This captures the idea of designing a mechanism that will deal with either "conservatism" or "radicalism." If $\beta > 1$, the principal wants to take more extreme actions than the agent, while if $\beta < 1$, the principal wants to take less extreme actions.[7]

[6] It is equivalent to divide the principal's utility function by k, and have each unit of money burning lower the principal's payoff by one. The choice is one of interpretation: here, a change in k can be interpreted as a change in the ease of punishing the agent

[7] A similar parameterization is used by Alonso and Matouschek (2008), who study delegation *without* money burning, and use it to describe the relationship between a legislature and a committee.

We'll again assume that the state is distributed uniformly, so in this case the principal is solving

$$\max_{a(\theta),\omega(\theta)} \int_0^1 \left[-\frac{1}{2}\left(a(\theta) - \beta\theta + \frac{1}{2}\beta - \frac{1}{2} \right)^2 + k\omega(\theta) \right].$$

subject to

$$-\frac{1}{2}\left(a(\theta) - \theta\right)^2 + \omega(\theta) \geq -\frac{1}{2}\left(a(\theta') - \theta\right)^2 + \omega(\theta')$$

for all θ, θ' and

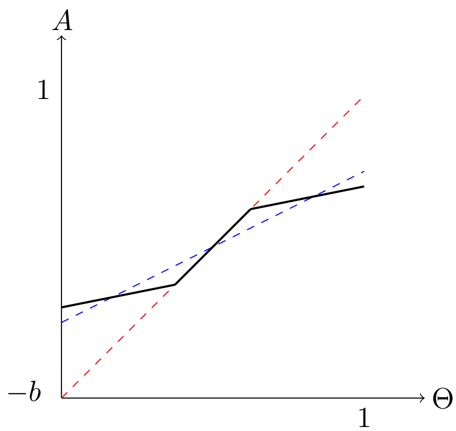
$$\omega(\theta) \leq 0.$$

As before, this problem can be transformed into a problem of delegation with continuation values. When the principal has quadratic losses in settings with money burning, the characterization provided in section 3 becomes very simple, and the optimal mechanism can often described explicitly. The fact that $u^P_{aa}(a,\theta)$, $u^P_{a\theta}(a,\theta)$, and $\gamma'(\cdot)$ (where it exists) are constant in this setting implies that when the principal is using money burning, the action varies with the state linearly, with slope $\frac{2k-\beta}{k-1}$. Then, the optimal mechanism takes one of five forms which vary with the parameters k and β.

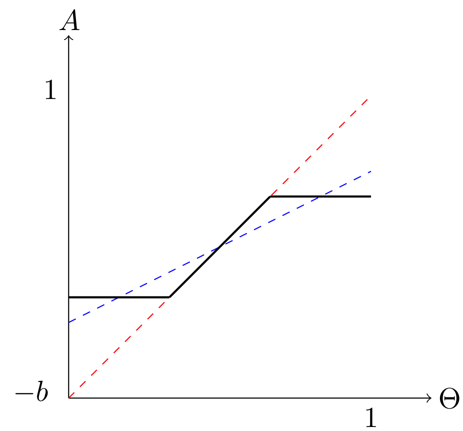
Theorem 6 *In the uniform-quadratic responsiveness setting with money burning, the optimal mechanism can take one of five forms:*

1. *Discretion with rules preventing extremes if responsiveness is high and the cost of money burning is relatively high:* $\beta < 1$ *and* $k \geq \frac{1}{2}\beta$
2. *Discretion with money burning punishing extremes if responsiveness is high and the cost of money burning is relatively low:* $\beta < 1$, $k < \frac{1}{2}\beta$
3. *Full discretion if responsiveness is intermediate and the cost of money burning is relatively high:* $1 \leq \beta \leq 2$, $k \geq \beta - 1$
4. *Money burning over intermediate actions, with rules preventing extremes if responsiveness is intermediate or low, and the cost of money burning is relatively low:* $\beta - 1 \geq k$, $k < 1$
5. *Only two points if responsiveness is low and the cost of money burning is relatively high:* $\beta > 2$, $k \geq 1$.

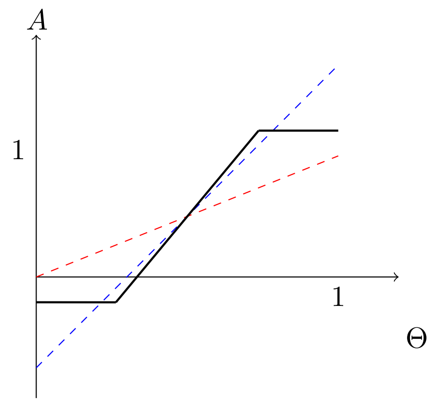
The form of optimal delegation varies in an intuitive way, and these forms can be seen in Figure 7. When β is low the principal wants to prevent the agent from taking actions which are too extreme. A high cost of money burning causes her to set a floor and a cap to prevent these extreme



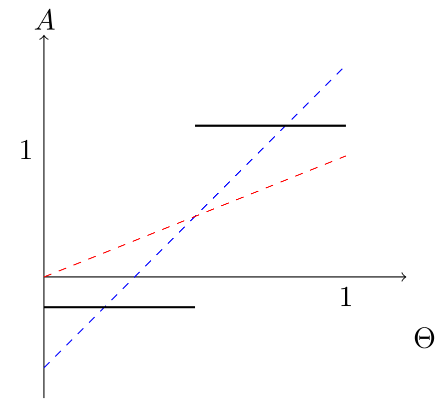
(a) Discretion with money burning at extremes: $\beta = \frac{1}{2}$, $k = \frac{1}{6}$



(b) Discretion with rules preventing extremes: $b = \frac{1}{5}$, $k = 1$



(c) Money burning over intermediate actions: $\beta = \frac{5}{2}$, $k = \frac{1}{2}$



(d) Two points: $\beta = \frac{5}{2}$, $k = 1$

Figure 7: The four possible forms of the optimal delegation with money burning mechanism with quadratic losses, a uniform distribution, cost of money burning k, and a constant bias b.

actions, but when costs are low she instead uses money burning to disincentivize them. As β increases, the agent gets more discretion, since the principal's and agent's preferences are becoming more similar, until $\beta = 1$, when the two parties' preferences are the same and the principal gives the agent full discretion. As β increases further, the principal tends to want *more* extreme actions than the principal. She does this by tying money burning to intermediate actions if costs are low enough, but otherwise simply prevents the agent from taking intermediate actions.

The way in which the form varies with parameters can be seen in Figure 8. It's obvious that the principal substitutes between using rules (which completely prevent certain actions from being taken) and money burning (which punishes certain actions for being taken) as the cost of using

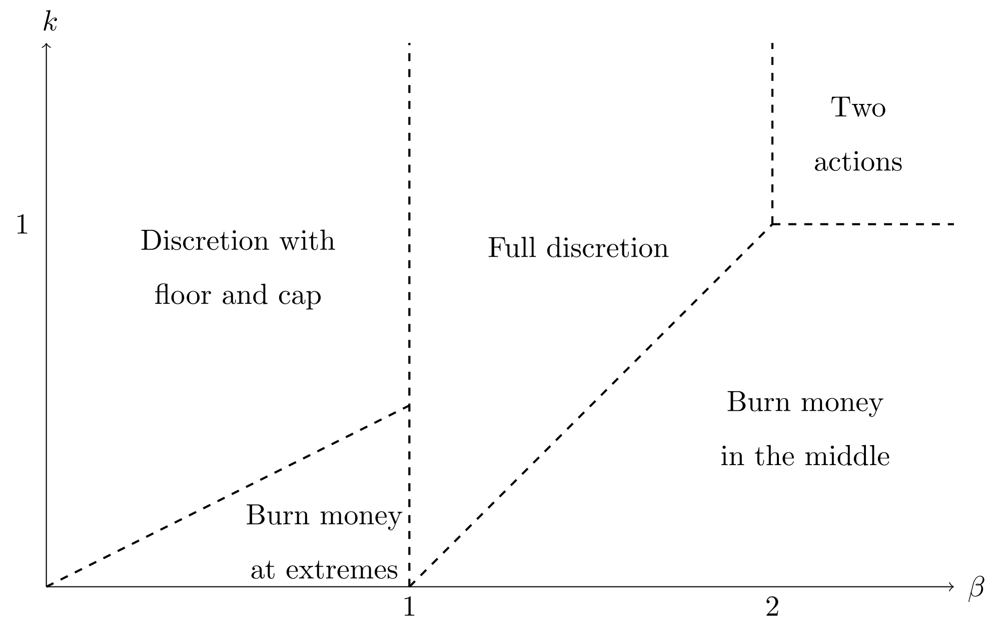


Figure 8: The form of the optimal mechanism by β and k

money burning changes.

In addition to the specific parametric model studied here, the theoretical results from Section 3 can be applied to a variety of problems in which the principal is delegating with access to money burning. In particular, Appendix B applies the results to variants of the models studied by Amador and Bagwell (2013) and Ambrus and Egorov (2017).

6 Conclusion

Dynamic contracts can overcome the barriers faced in a static setting. This paper considers the classic problem of an uninformed principal delegating an action to an informed agent when she has some means of rewarding or punishing the agent in the future, but may face a cost of doing so. In characterizing the optimal mechanism, the paper showed that the principal trades off between incentivizing a better action today and giving the agent a continuation value that is better from the point of view of the principal.

Because utility is not transferable in this setting, standard mechanism design tools cannot be used. Instead, the paper identifies techniques from the calculus of variations which can overcome these difficulties. The paper applies these results to two canonical problems from the literature, which can both be formulated as delegation problems with continuation values: dynamic delegation and delegation with money burning.

The first application is to a model in which a principal delegates two decisions dynamically. The paper first shows the optimal way to promise any continuation value, and then applies the more general results to solve for the optimal mechanism. This mechanism has no pooling in the first period, and caps the agent's action in the second period. Furthermore, one can show that the principal prefers a less biased agent in the first period, but that increasing the bias in the second period eventually improves the principal's payoffs. Finally, the optimal mechanism is deterministic.

The second application is to a delegation problem in which the principal has the option to "burn money." The results show how a principal might use money burning or rules to prevent extreme actions when the agent has more extreme preferences, or to prevent intermediate actions when the agent is more conservative.

It seems likely that these results will be useful in a variety of other problems, such as delegation when there are more than two periods, or when there is a more complicated game following the first period of delegation. A key step is to show that the principal has an optimal means of promising a continuation value, and find the explicit form of the function γ, which maps the agent's continuation value to the principal's.

References

Alonso, R., & Matouschek, N. (2008). Optimal delegation. *The Review of Economic Studies*, *75*(1), 259–293.

Amador, M., & Bagwell, K. (2013). The theory of optimal delegation with an application to tariff caps. *Econometrica*, *81*(4), 1541–1599.

Amador, M., & Bagwell, K. (2016). Money burning in the theory of delegation. *Working Paper*.

Ambrus, A., & Egorov, G. (2017). Delegation and nonmonetary incentives. *Journal of Economic Theory*(171), 101–135.

Clarke, F. H. (1977). Extremal arcs and extended hamiltonian systems. *Transactions of the American Mathematical Society*, *231*(2), 349–367.

Clarke, F. H. (1990). *Optimization and nonsmooth analysis* (Vol. 5). Siam.

Filippov, A. F., & Arscott, F. M. (1988). *Differential equations with discontinuous righthand sides: control systems* (Vol. 18). Springer Science & Business Media.

Fudenberg, D., & Tirole, J. (1991). *Game theory.* The MIT Press.

Garrett, D. F., & Pavan, A. (2015). Dynamic managerial compensation: a variational approach. *Journal of Economic Theory*, *159*, 775–818.

Gelfand, I., & Fomin, S. (1963). Calculus of variations.

Goltsman, M., Hörner, J., Pavlov, G., & Squintani, F. (2009). Mediation, arbitration and negotiation. *Journal of Economic Theory*, *144*(4), 1397–1420.

Guo, Y. (2014). *Dynamic delegation of experimentation* (Tech. Rep.). mimeo, Yale University.

Guo, Y., & Hörner, J. (2015). Dynamic mechanisms without money.

Holmström, B. R. (1977). *On incentives and control in organizations.* Stanford University.

Koessler, F., & Martimort, D. (2012). Optimal delegation with multi-dimensional decisions. *Journal of Economic Theory*, *147*(5), 1850–1881.

Kováč, E., & Mylovanov, T. (2009). Stochastic mechanisms in settings without monetary transfers: The regular case. *Journal of Economic Theory*, *144*(4), 1373–1395.

Laffont, J.-J., & Rochet, J.-C. (1998). Regulation of a risk averse firm. *Games and Economic Behavior*, *25*(2), 149–173.

Lipnowski, E., & Ramos, J. (2015). Repeated delegation. *Available at SSRN 2552926*.

Melumad, N. D., & Shibano, T. (1991). Communication in settings with no transfers. *The RAND Journal of Economics*, 173–198.

Milgrom, P., & Segal, I. (2002). Envelope theorems for arbitrary choice sets. *Econometrica*, *70*(2), 583–601.

A Proofs of Main Results

Lemma 1 *W is absolutely continuous, and where the derivative exists,*

$$W'(\theta) = a(\theta) - \theta \tag{5}$$

and

$$\omega(\theta) = W(\theta) + \frac{1}{2}W'(\theta)^2$$

Proof Standard from Milgrom and Segal (2002). □

Proposition 1 *Assume that the convexity condition holds. Then a solution to problem (P) exists.*

Proof I'll be using Theorem 4.1.3 from Clarke (1990). Note that we have sufficient assumptions such that L is measurable and $L(\theta, \cdot, \cdot)$ is lower semicontinuous. Furthermore, the convexity condition stated above immediately implies the convexity condition required for Theorem 4.1.3.

Define

$$H(\theta, s, p) = \sup\{pv - L(\theta, s, v) : v \in \mathbb{R}\}.$$

In this case, we know that

$$\begin{aligned} H(\theta, s, p) &= \sup\{pv - L(\theta, s, v) : v \in \mathbb{R}\} \\ &= \sup\left\{pv + \left[u^P(v+\theta, \theta) + \gamma\left(s + \frac{1}{2}v^2\right)\right] f(\theta) : v \in \mathbb{R}\right\} \\ &\leq \sup\left\{pv + \left[u^P(v+\theta, \theta) + \gamma\left(s + \frac{1}{2}v^2\right)\right] f(\theta) : (s, v) \in \mathbb{R}^2\right\} \\ &\leq \sup\left\{pv + u^P(v+\theta, \theta) f(\theta) + \delta\bar{\gamma} f(\theta) : v \in \mathbb{R}\right\} \end{aligned}$$

where $\bar{\gamma}$ us the upper bound on $\gamma(\cdot)$ and the last inequality is due to the fact that s enters the argument of γ linearly. Since the derivative of u^P with respect to a is unbounded and $f(\theta)$ is strictly positive, the last value is bounded by a function of p and θ, which is the first "growth condition".

The second growth condition is satisfied if it can be shown that the optimal $W(0)$ lies within some bounded set. If $\gamma(\cdot)$ reaches its maximum for all ω above or below some point, the solution to the problem is immediate (simply choose the W' which maximizes u^P for each θ, and set $W(0)$ appropriately). Thus, we only need to consider the case in which $\lim_{\omega \to -\infty} \gamma(\omega) = -\infty$, and show that fixing the endpoint $W(0) = x_0$, that the objective function does not continue to fall as $x_0 \to -\infty$ (the case for $x_0 \to \infty$ is symmetric). I'll show that this is impossible: that eventually, lowering

$W(0)$ must make the principal worse off. Define ω_1 and ω_2, respectively as the minimum and maximum ω which maximize $\gamma(\omega)$, and

$$\underline{u} = \int_0^1 [-u^P(\theta,\theta) - \bar{\gamma}] f(\theta) d\theta,$$

the losses from the principal giving the agent full discretion, and choosing the optimal continuation value. Obviously, an optimal mechanism must improve upon this. From the basic assumptions, there must exist some $\hat{v} > 0$ such that for all θ, $u_a^P(\hat{v}+\theta,\theta) < 0$, $u_a^P(-\hat{v}+\theta,\theta) > 0$, and $-u(-\hat{v}+\theta,\theta) \geq \underline{u}$. Finally, let x_0 be low enough such that $-\gamma\left(\frac{1}{2}x_0 + \frac{1}{2}\hat{v}^2\right) > \underline{u}$ and $x_0 + \frac{1}{8}x_0^2 > \omega_2$.

If the optimal mechanism with $W(0) = x_0$ improves upon giving the agent full discretion, then there must be some $\tilde{\theta} < 1$ such that $W(\tilde{\theta}) = \frac{1}{2}x_0$. Otherwise, for all θ the mechanism would either have $W(\theta) + \frac{1}{2}W'(\theta)^2 < \frac{1}{2}x_0$, or $|W'(\theta)| > \hat{v}$, giving losses that are higher than $\underline{u}$. Since $W(0) = x_0$ and $W(\tilde{\theta}) = \frac{1}{2}x_0$, there must be some mass of θ such that $W'(\theta) \geq \frac{1}{2}x_0$. Thus, we can marginally decrease any $W'(\theta) > \sqrt{2\omega_2 - W(\theta)}$ and marginally increase any $W'(\theta) < -\sqrt{2\omega_2 - W(\theta)}$, increasing the principal's payoffs with $W(0) > x_0$. Thus, for x_0 low enough, increasing $W(0)$ above x_0 increases the principal's payoffs, so the second growth condition is satisfied. $\square$

Lemma A.1 *For any $\varepsilon > 0$, fix some arc W on $[a,b]$ such that $\forall\theta \in [a,b]$, $W(\theta) \leq \bar{\omega} - \varepsilon$. Then there exists an an $\hat{\varepsilon} > 0$ and an integrable function $k(\theta)$ such that for all p, for all (θ, s_1) and (θ, s_2) in the tube $T(W;\hat{\varepsilon})$, one has*

$$|H(\theta,s_1,p) - H(\theta,s_2,p)| \leq k(\theta)(1+|p|)|s_1 - s_2|$$

Proof The basic assumptions guarantee that there must exist some v such that $pv - L(\theta,s_1,v)$ attains its supremum. Define one such v as $v_1(\theta,s_1,p)$. Next, define

$$v_2(\theta,s_1,p,\sigma) = v_1(\theta,s_1,p) - sgn\left(v_1(\theta,s_1,p)\right) \cdot \frac{2}{\sqrt{\varepsilon}}|\sigma|$$

where $\sigma = s_2 - s_1$. Defining

$$\hat{H}(\theta,s_1+\sigma,p) = pv_2(\theta,s_1,p,\sigma) + \left[u^P(v_2(\theta,s_1,p,\sigma)+\theta,\theta) + \gamma\left(s_1 + \sigma + \frac{1}{2}v_2(\theta,s_1,p,\sigma)^2\right)\right] f(\theta),$$

we can take the derivative with respect to σ at 0 from the right, giving us

$$\begin{aligned} &-p \cdot sgn\left(v_1(\theta,s_1,p)\right)\frac{2}{\sqrt{\varepsilon}} + u_a^P(v_1(\theta,s_1,p)+\theta,\theta)\left(-sgn(v_1(\theta,s_1,p))\frac{2}{\sqrt{\varepsilon}}\right) f(\theta) \\ &+ \gamma'_-\left(s_1 + \frac{1}{2}v_1(\theta,s_1,p)^2\right)\left(1 - v_1(\theta,s_1,p)\cdot sgn\left(v_1(\theta,s_1,p)\right)\frac{2}{\sqrt{\varepsilon}}\right) f(\theta) \end{aligned}$$

where γ'_- indicates the left derivative. With

$$\begin{aligned}
\underline{s} &= \min_{\theta\in[a,b]} W(\theta)\\
\bar{u}_a &= \sup\left\{u_a^P(v+\theta,\theta) | v \in \left[-\sqrt{-2\underline{s}+2\bar{\omega}}, \sqrt{-2\underline{s}+2\bar{\omega}}\right], \theta\in[a,b]\right\}\\
\bar{\gamma} &= \left|\gamma'\left(\bar{\omega}-\frac{3}{4}\varepsilon\right)\right|\\
\underline{\gamma} &= \left|\gamma'(\underline{s})\right|\\
\bar{f} &= \max_{\theta\in[0,1]} f(\theta)
\end{aligned}$$

this derivative is bounded below by

$$-\left|\frac{2}{\sqrt{\varepsilon}}|p| + \bar{u}_a\frac{2}{\sqrt{\varepsilon}}\bar{f} + \max\left(\bar{\gamma},\underline{\gamma}\right)\left(1+\sqrt{-2\underline{s}}\frac{2}{\sqrt{\varepsilon}}\right)\bar{f}\right|$$

and the derivative with respect to σ at 0 from the left can be bounded by the same value. Thus,

$$\lim_{s_2\to s_1}\frac{H(\theta,s_2,p)-H(\theta,s_1,p)}{s_2-s_1} \geq -\max\left\{\frac{2}{\sqrt{\varepsilon}}, \bar{u}_a\frac{2}{\sqrt{\varepsilon}}\bar{f} + \max\left(\bar{\gamma},\underline{\gamma}\right)\left(1+\sqrt{-2\underline{s}}\frac{2}{\sqrt{\varepsilon}}\right)\bar{f}\right\}(1+|p|)$$

Since this is true for all s_1, we get that

$$|H(\theta,s_2,p)-H(\theta,s_1,p)| \leq \max\left\{\frac{2}{\sqrt{\varepsilon}}, \bar{u}_a\frac{2}{\sqrt{\varepsilon}}\bar{f} + \max\left(\bar{\gamma},\underline{\gamma}\right)\left(1+\sqrt{-2\underline{s}}\frac{2}{\sqrt{\varepsilon}}\right)\bar{f}\right\}(1+|p|)|s_1-s_2|$$

which is the strong Lipschitz condition. □

Define

$$V_{a,b,\varepsilon}(s) = \min_{W(\theta)} \ell(W(a), W(b)+s) + \int_a^b L\left(\theta, W(\theta), W'(\theta)\right)d\theta$$

where $\varepsilon > 0$, $W(\theta)$ is absolutely continuous, L is as before, and

$$\ell(u,v) = \begin{cases} 0 \text{ if } u = v = -\varepsilon \\ -\infty \text{ otherwise}\end{cases}$$

Lemma A.2 *Assume the convexity condition holds. With the above definitions, we have that if* $V_{a,b,\varepsilon}(0) > -\infty$

$$\mathit{lim\ inf}_{s\to 0}\frac{V_{a,b,\varepsilon}(s)-V_{a,b,\varepsilon}(0)}{|s|} > -\infty$$

Proof Note that the minimization problem is the problem from before, but with endpoint conditions. Thus, a solution exists for s in the neighborhood of 0, and $V_{a,b,\varepsilon}(s)$ is well defined over this neighborhood. Define a solution to the problem implicitly defined by s as W_s, and note that $W_s(a) = W_{-s}(a)$, due to the endpoint condition. Finally, define $W_\lambda(\theta) = \lambda W_s(\theta) + (1-\lambda)W_{-s}(\theta)$. Since $L(\theta, s, v)$ is convex in (s, v),

$$\int_a^b L(\theta, W_\lambda(\theta), W_\lambda'(\theta))d\theta \leq \lambda \int_a^b L(\theta, W_s(\theta), W_s'(\theta))d\theta + (1-\lambda) \int_a^b L(\theta, W_{-s}(\theta), W_{-s}'(\theta))d\theta$$

so $V_{a,b,\varepsilon}(\cdot)$ is concave and thus locally Lipschitz. □

Lemma A.3 *Assume the convexity condition holds. Then for any function $W(\theta)$ which solves problem (P), it must be the case that $\forall\theta$ such that $W(\theta) < \bar{\omega}$, there exists an arc $p(\theta)$ such that*

$$\begin{bmatrix} -p'(\theta) \\ W'(\theta) \end{bmatrix} \in \partial H(\theta, W(\theta), p(\theta)) \ a.e.$$

Furthermore, if $W(0) < \bar{\omega}$ then $p(a) = 0$ and if $W(1) < \bar{\omega}$ then $p(b) = 0$.

Proof I will use Theorem 4.2.2 from Clarke (1990). L satisfies the basic hypotheses and the convexity condition by assumption. For any $\varepsilon > 0$, since $W(\theta)$ is continuous, the set $\{\theta \in [0,1] : \underline{s} \leq W(\theta) \leq -\varepsilon\}$ is compact. For any point a on the boundary of this set where a is not equal to 0 or 1, we have $W(a) = -\varepsilon$. Thus, fixing ε, for any $[a,b] \subset \{\theta \in [0,1] : \underline{s} \leq W(\theta) \leq -\varepsilon\}$ where a and b are in the boundary and $0 < a \leq b < 1$, W must also maximize the same loss function under the endpoint restrictions of $W(a) = W(b) = -\varepsilon$. Lemma A.2 shows that this problem with endpoint conditions is calm, and since H is the same over the interval, Lemma A.1 shows that it satisfies the strong Lipschitz condition near W. Furthermore, if a is equal to 0, then $W(\theta)$ must solve the free endpoint problem, implying that $p(a) = 0$. Similarly, $p(b) = 0$ when $b = 1$. Thus, Theorem 4.2.2 can be applied, giving the result. □

Proposition A.1 *Assume that the convexity condition holds. If $W(\theta)$ solves problem (P), then for $\theta \in [0,1]$ such that $W(\theta) < \bar{\omega}$, the following must be true almost everywhere:*

(1) if $W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega}$ and $\gamma''\left(W(\theta) + \frac{1}{2}W'(\theta)^2\right)$ exists, then

$$\begin{aligned} W''(\theta) = &-1 - \frac{\left[u_a^P\left(W'(\theta)+\theta,\theta\right) + W'(\theta)\gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right)\right]\frac{f'(\theta)}{f(\theta)}}{u_{aa}^P\left(W'(\theta)+\theta,\theta\right) + W'(\theta)^2\gamma''\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right) + \gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right)} \\ &+ \frac{2\gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right) - u_{a\theta}^P\left(W'(\theta)+\theta,\theta\right)}{u_{aa}^P\left(W'(\theta)+\theta,\theta\right) + W'(\theta)^2\gamma''\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right) + \gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right)}, \end{aligned}$$

(2) if for some interval $[a,b]$, such that $0 < a < b < 1$, $\forall\theta \in [a,b]$, $W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega}$, and

$W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega}$ in the neighborhoods of a and b, then

$$\int_a^\theta u_a^P\left(W'(a)+a,z\right)f(z)dz \geq W'(a)u_a^P\left(W'(a)+a,a\right)f(a) + W'(a)^2\gamma'_-(\bar{\omega})f(a)$$

$$- W'(\theta)u_a^P(W'(a)+a,\theta)f(\theta) - W'(\theta)^2\gamma'_-(\bar{\omega})f(\theta).$$

and this must hold with equality at $\theta = b$,

(3) if $\theta \in \{0,1\}$, if $W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega}$ then

$$W'(\theta)\gamma'\left(W(\theta) + \frac{1}{2}W'(\theta)^2\right) = -u_a^P\left(W'(\theta)+\theta,\theta\right),$$

(4) if $\forall\theta \in [0,b]$, $W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega}$, with $W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega}$ for θ in the neighborhood of b, then $\forall\theta \in [0,b]$,

$$\int_0^\theta u_a^P(W'(0),z)f(z)dz \geq \left[-W'(\theta)u_a^P(W'(\theta)+\theta,\theta) - W'(\theta)^2\gamma'_-(\bar{\omega})\right]f(\theta)$$

with equality at $\theta = b$. Similarly, if $\forall\theta \in [a,1]$, $W(\theta)+\frac{1}{2}W'(\theta)^2 = \bar{\omega}$, with $W(\theta)+\frac{1}{2}W'(\theta)^2 < \bar{\omega}$ for θ in the neighborhood of a, then $\forall\theta \in [a,1]$,

$$\int_\theta^1 u_a^P(W'(1)+1,z)f(z)dz \leq \left[W'(\theta)u_a^P(W'(\theta)+\theta,\theta) + W'(\theta)^2\gamma'_-(\bar{\omega})\right]f(\theta)$$

with equality at $\theta = a$. Finally, if $\forall\theta \in [0,1]$, $W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega}$, then $\forall\theta \in [0,1]$

$$\int_0^\theta u_a\left(W'(0),z\right)f(z)dz \geq W'(\theta)\left[-u_a\left(W'(\theta)+\theta,\theta\right) - W'(\theta)\gamma'_-(\bar{\omega})\right]g(\theta)$$

and

$$\int_0^1 u_a\left(W'(0),\theta\right)f(\theta)d\theta = 0.$$

Proof Lemma A.3 shows that under the convexity condition, $\forall\theta$ such that $W(\theta) < \bar{\omega}$, there exists an arc $p(\theta)$ such that

$$\begin{bmatrix} -p'(\theta) \\ W'(\theta) \end{bmatrix} \in \partial H(\theta, W(\theta), p(\theta)) \text{ a.e.}$$

(1) Using the definitions from Clarke (1990), when $W(\theta)+\frac{1}{2}W'(\theta)^2<\bar{\omega}$, the envelope theorem gives that

$$\frac{\partial H}{\partial s}(\theta, W(\theta), p(\theta)) = \gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right)f(\theta)$$

and that $p(\theta) = \left[-u_a(W'(\theta)+\theta,\theta) - W'(\theta)\gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right)\right]f(\theta)$. Where the second derivative of γ exists, we can write

$$\begin{aligned}
-p'(\theta) = &\left[u^P_{aa}\left(W'(\theta)+\theta,\theta\right) + W'(\theta)^2\gamma''\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right) + \gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right)\right] f(\theta)W''(\theta) \\
&+ \left[u^P_{aa}\left(W'(\theta)+\theta,\theta\right) + u^P_{a\theta}\left(W'(\theta)+\theta,\theta\right) + W'(\theta)^2\gamma''\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right)\right] f(\theta) \\
&+ \left[u^P_{a}\left(W'(\theta)+\theta,\theta\right) + W'(\theta)\gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right)\right] f'(\theta)
\end{aligned}$$

Thus, we have that

$$\begin{aligned}
&\left[u^P_{aa}\left(W'(\theta)+\theta,\theta\right) + W'(\theta)^2\gamma''\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right) + \gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right)\right] f(\theta)W''(\theta) \\
&\quad = -\left[u^P_{aa}\left(W'(\theta)+\theta,\theta\right) + u^P_{a\theta}\left(W'(\theta)+\theta,\theta\right) + W'(\theta)^2\gamma''\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right)\right] f(\theta) \\
&\quad\quad - \left[u^P_{a}\left(W'(\theta)+\theta,\theta\right) + W'(\theta)\gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right)\right] f'(\theta) \\
&\quad\quad + \gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right) f(\theta)
\end{aligned}$$

and

$$\begin{aligned}
W''(\theta) = &-\frac{\left[u^P_{aa}\left(W'(\theta)+\theta,\theta\right) + u^P_{a\theta}\left(W'(\theta)+\theta,\theta\right) + W'(\theta)^2\gamma''\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right)\right] f(\theta)}{\left[u^P_{aa}\left(W'(\theta)+\theta,\theta\right) + W'(\theta)^2\gamma''\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right) + \gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right)\right] f(\theta)} \\
&-\frac{\left[u^P_{a}\left(W'(\theta)+\theta,\theta\right) + W'(\theta)\gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right)\right] f'(\theta)}{\left[u^P_{aa}\left(W'(\theta)+\theta,\theta\right) + W'(\theta)^2\gamma''\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right) + \gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right)\right] f(\theta)} \\
&+\frac{\gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right) f(\theta)}{\left[u^P_{aa}\left(W'(\theta)+\theta,\theta\right) + W'(\theta)^2\gamma''\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right) + \gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right)\right] f(\theta)} \\
= &-1 - \frac{\left[u^P_{a}\left(W'(\theta)+\theta,\theta\right) + W'(\theta)\gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right)\right] \frac{f'(\theta)}{f(\theta)}}{u^P_{aa}\left(W'(\theta)+\theta,\theta\right) + W'(\theta)^2\gamma''\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right) + \gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right)} \\
&+\frac{2\gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right) - u^P_{a\theta}\left(W'(\theta)+\theta,\theta\right)}{u^P_{aa}\left(W'(\theta)+\theta,\theta\right) + W'(\theta)^2\gamma''\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right) + \gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right)}
\end{aligned}$$

(2) When $W(\theta)+\frac{1}{2}W'(\theta)^2=0$ but $W(\theta)<\bar{\omega}$, we have that

$$\frac{\partial H}{\partial s}(\theta, W(\theta), p(\theta)) = -\frac{1}{W'(\theta)}p(\theta) - \frac{1}{W'(\theta)}u^P_a\left(W'(\theta)+\theta,\theta\right)f(\theta).$$

If $W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega}$ for some interval $[a, b]$, then it must be the case that on that interval, $W'(\theta) = W'(a) + a - \theta$. Thus, we get the first order differential equation

$$p'(\theta) = \frac{1}{W'(a) + a - \theta} p(\theta) + \frac{1}{W'(a) + a - \theta} u_a^P \left(W'(a) + a, \theta\right) f(\theta)$$

which has the solution

$$p(\theta) = \frac{c_1}{W'(a) + a - \theta} + \frac{\int_1^\theta u_a^P \left(W'(a) + a, z\right) f(z) dz}{W'(a) + a - \theta}$$

If $W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega}$ for θ just below a, then the continuity of p implies that

$$p(a) = \left[-u_a^P \left(W'(a) + a, a\right) - W'(a)\gamma'_-(\bar{\omega})\right] f(a).$$

Plugging this in we get

$$c_1 = -W'(a)u_a^P \left(W'(a) + a, a\right) f(a) - W'(a)^2\gamma'_-(\bar{\omega}) f(a) - \int_1^a u_a^P \left(W'(a) + a, z\right) f(z) dz$$

Furthermore, with this definition of $p(\theta)$ we have that it must be the case that

$$\begin{aligned}\int_a^\theta u_a^P \left(W'(a) + a, z\right) f(z) dz \geq{}& W'(a)u_a^P \left(W'(a) + a, a\right) f(a) + W'(a)^2\gamma'_-(0) f(a) \\ & - W'(\theta)u_a^P(W'(a) + a, \theta) f(\theta) - W'(\theta)^2\gamma'_-(0) f(\theta)\end{aligned}$$

and since $W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega}$ for θ just above b,

$$p(b) = \left[-u_a^P \left(W'(b) + b, b\right) - W'(b)\gamma'_-(\bar{\omega})\right] f(b)$$

which means it must hold with equality at $\theta = b$.

(3) With the endpoint conditions given, if $W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega}$ for $\theta \in \{0, 1\}$,

$$\begin{aligned}0 &= p(\theta) \\ &= \left[-u_a \left(W'(\theta) + \theta, \theta\right) - W'(\theta)\gamma' \left(W(\theta) + \frac{1}{2}W(\theta)^2\right)\right] f(\theta)\end{aligned}$$

which simplifies to

$$W'(\theta)\gamma' \left(W(\theta) + \frac{1}{2}W'(\theta)^2\right) = -u_a^P \left(W'(\theta) + \theta, \theta\right).$$

(4) Alternatively, if $W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega}$ for $\theta \in [0, b]$, then we can use the formula for $p(\theta)$ from above to show that

$$c_1 = \int_0^1 u_a^P \left(W'(0), z\right) f(z) dz$$

giving us the inequality and that it must hold with equality at $\theta = b$. A similar exercise can be done when $W(\theta) + \frac{1}{2}W'(\theta)^2$ for $\theta \in [a, 1]$. When $W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega}$ for all θ, then $c_1 = 0$, and $p(0) = p(1) = 0$ implies the two conditions. □

Proposition 2 *Assume that the convexity condition holds, and that $W(\theta)$ solves problem (P). Define $a(\theta) = W'(\theta) + \theta$ and $\omega(\theta) = W(\theta) + \frac{1}{2}W'(\theta)^2$. Then for $\theta \in [0, 1]$ such that $\omega(\theta) < \bar{\omega}$,*

(1) if $\theta \in \{0, 1\}$,

$$W'(\theta)\gamma'(\omega(\theta)) = -u_a^P(a(\theta), \theta),$$

and

(2) if $\gamma''(\omega(\theta))$ exists, then

$$W''(\theta) = -1 + \frac{2\gamma'(\omega(\theta)) - u_{a\theta}^P(a(\theta), \theta) - \left[u_a^P(a(\theta), \theta) + W'(\theta)\gamma'(\omega(\theta))\right]\frac{f'(\theta)}{f(\theta)}}{u_{aa}^P(a(\theta), \theta) + W'(\theta)^2\gamma''(\omega(\theta)) + \gamma'(\omega(\theta))}$$

almost everywhere.

Proof This is an immediate implication of Proposition A.1. □

Proposition A.2 *Assume that the convexity condition holds. Suppose that $\forall\theta$, $W(\theta) + \frac{1}{2}W'(\theta) \leq \bar{\omega}$ and that there exists an absolutely continuous function $p : [0, 1] \to \mathbb{R}$ such that*

- $W'(\theta) \in \underset{v}{\mathit{argmax}}\ p(\theta)v - L(\theta, W(\theta), v)$
- $p(\theta) = -u_a^P(\theta, \theta)f(\theta)$ *if* $W(\theta) = \bar{\omega}$.
- $p(0) = 0$ *if* $W(0) < \bar{\omega}$, *and* $p(0) \geq 0$ *if* $W(0) = \bar{\omega}$
- $p(1) = 0$ *if* $W(1) < \bar{\omega}$, *and* $p(1) \leq 0$ *if* $W(1) = \bar{\omega}$

  and the following is true almost everywhere:

- $p'(\theta) = -\gamma'\left(W(\theta) + \frac{1}{2}W'(\theta)\right)f(\theta)$ *if* $W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega}$
- $p'(\theta) = \frac{1}{W'(\theta)}p(\theta) + \frac{1}{W'(\theta)}u_a^P(W'(\theta) + \theta, \theta)f(\theta)$ *if* $W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega}$ *and* $W(\theta) < \bar{\omega}$, *and*
- $p'(\theta) \geq -\gamma'_-(\bar{\omega})f(\theta)$ *if* $W(\theta) = \bar{\omega}$.

Then $W(\theta)$ solves problem (P).

Proof The proof will use Theorem 4.3.1 from Clarke (1990) (with $Q(\theta) = 0$ and $\varepsilon = \infty$), showing that $W(\theta)$ solves an adjusted problem, in which

$$\ell(u,v) = \begin{cases} 0 \text{ if } u \le \bar{\omega} \text{ \& } v \le \bar{\omega} \\ \infty \text{ otherwise} \end{cases} .$$

Obviously, this problem has the same solution as the problem at hand, as the loss function is infinite if $W(\theta) > \bar{\omega}$ for any θ. The first point guarantees that for all v,

$$p(\theta)W'(\theta) - L(\theta, W(\theta), W'(\theta)) \ge p(\theta)v - L(\theta, W(\theta), v).$$

As demonstrated above, if $W(\theta) < 0$,

$$\frac{\partial H}{\partial s}(\theta, W(\theta), p(\theta)) = \begin{cases} \gamma'\left(W(\theta) + \frac{1}{2}W'(\theta)\right) \text{ if } W(\theta) + \frac{1}{2}W'(\theta) < 0 \\ \frac{1}{W'(\theta)}p(\theta) + \frac{1}{W'(\theta)}u_a^P\left(W'(\theta)+\theta,\theta\right)f(\theta) \text{ if } W(\theta) + \frac{1}{2}W'(\theta)^2 = 0 \end{cases} ,$$

Furthermore, in the case when $W(\theta) = 0$ and $p(\theta) = -u_a^P(\theta,\theta)f(\theta)$,

$$\frac{\partial H}{\partial s}(\theta, W(\theta), p(\theta)) = (-\infty, \gamma'_-(0)f(\theta)].$$

$H(\theta, \cdot, W'(\theta))$ inherits concavity from $\gamma(\cdot)$, so $-p'(\theta) \in \frac{\partial H}{\partial s}(\theta, W(\theta), p(\theta))$ implies that $\forall y$,

$$H(\theta, W(\theta)+y, p(\theta)) - H(\theta, W(\theta), p(\theta)) \le -p'(\theta)y$$

For the given $W(\theta)$, $\ell(W(0), W(1)) = 0$, so if $p(0) = p(1) = 0$, it must be the case that for all u and v,

$$\ell(W(0)+u, W(1)+v) - \ell(W(0), W(1)) \ge -p(1)v + p(0)u.$$

If either $W(0) = \bar{\omega}$ and $p(0) > 0$ or $W(1) = \bar{\omega}$ and $p(1) < 0$, then the right hand side is positive only when the left hand side is infinite, so the inequality still holds. Thus, $W(\theta)$ and $p(\theta)$ fulfill all of the requirements from Theorem 4.3.1, and $W(\theta)$ solves the problem. $\square$

Corollary A.1 *Assume that the convexity condition holds. If $\forall\theta$, $W(\theta) < \bar{\omega}$, $W(\theta)$ is absolutely continuous, and $W(\theta)$ satisfies conditions (1)-(4) from Proposition A.1, then $W(\theta)$ solves problem (P).*

Proof Define a function $p : [0,1] \to \mathbb{R}$ such that $p(0) = 0$ and

$$p'(\theta) = \begin{cases} -\gamma'\left(W(\theta) + \frac{1}{2}W'(\theta)\right)f(\theta) \text{ if } W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega} \\ \frac{1}{W'(\theta)}p(\theta) + \frac{1}{W'(\theta)}u_a^P\left(W'(\theta)+\theta,\theta\right)f(\theta) \text{ otherwise} \end{cases} .$$

Then the proof of Proposition A.1 shows that conditions (1), (2), and (4) guarantee that when $W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega}$,

$$p(\theta) = \left[-u_a\left(W'(\theta) + \theta, \theta\right) - W'(\theta)\gamma'\left(W(\theta) + \frac{1}{2}W'(\theta)^2\right)\right] f(\theta),$$

and points (3) and (4) guarantee that $p(1) = 0$. Furthermore, when $W(\theta) + \frac{1}{2}W'(\theta)^2 = \bar{\omega}$,

$$p(\theta) \geq W(\theta)\left[-u_a\left(W'(\theta) + \theta, \theta\right) - W'(\theta)\gamma'\left(W(\theta) + \frac{1}{2}W'(\theta)^2\right)\right] f(\theta),$$

so $W'(\theta) \in \underset{v}{\operatorname{argmax}}\ p(\theta)v - L\left(\theta, W(\theta), v\right)$. Thus the conditions of Proposition A.2 are fulfilled. □

Proposition 3 *Assume that the convexity condition holds. If* $W(\theta)+\frac{1}{2}W'(\theta)^2 < \bar{\omega}$, $\gamma''\left(W(\theta) + \frac{1}{2}W'(\theta)^2\right)$ *exists,* $W(\theta)$ *is absolutely continuous, and* $W(\theta)$ *satisfies conditions (1) and (2) from Proposition 2 almost everywhere, then* $W(\theta)$ *solves problem (P).*

Proof All the conditions for application of Corollary A.1 apply, and $W(\theta) + \frac{1}{2}W'(\theta)^2 < \bar{\omega}$, so only conditions (1)-(2) from Proposition A.1 need to be satisfied. □

Theorem 1 *Suppose that the convexity condition holds, and the following is true:*

(1) for $\theta \in \{0, 1\}$,

$$-u_a(a(\theta), \theta) = (a(\theta) - \theta)\gamma'(\omega(\theta));$$

(2) $\gamma''(\omega(\theta))$ *exists almost everywhere, with*

$$a'(\theta) = \frac{2\gamma'(\omega(\theta)) - u^P_{a\theta}(a(\theta), \theta) - \left[u^P_a(a(\theta), \theta) + (a(\theta) - \theta)\gamma'(\omega(\theta))\right]\frac{f'(\theta)}{f(\theta)}}{u^P_{aa}\left(a(\theta), \theta\right) + \left(a(\theta) - \theta\right)^2\gamma''\left(\omega(\theta)\right) + \gamma'\left(\omega(\theta)\right)},$$

and $\omega(\theta)$ *is derived from the envelope condition;*

(3) $a(\theta)$ *is continuous and monotonically increasing.*

Then the mechanism defined by $a(\theta)$ *and* $\omega(\theta)$ *is optimal.*

Proof This simply combines the conditions of Proposition 3, with the requirement that $W(\theta)$ be absolutely continuous, the envelope condition, and the monotonicity constraint. □

To prove Proposition 4, we must solve for the optimal way to promise some amount of continuation utility, ω, in this setting. The mechanisms that the principal is solving over in problem PDD" are deterministic, in that when the agent reports the state, a particular action is taken with probability one. We could instead consider a more general setting, in which the principal commits to a *distribution* of actions after any given report. In particular, let $p(\cdot|\theta_1, \theta_2)$ be a probability density function over some set A for the action taken in the second period when the agent has reported θ_1 and θ_2. Define

$$\tilde{a}_2(\theta_1,\theta_2) = \int_A ap(a|\theta_1,\theta_2)da$$

and

$$\tilde{\sigma}^2(\theta_1,\theta_2) = \int_A (a - \tilde{a}_2(\theta_1,\theta_2))^2 p(a|\theta_1,\theta_2)da$$

as the average and variance, respectively, of the actions taken after reports θ_1 and θ_2. Furthermore, it will be useful to define $W(\theta_1,\theta_2)$ as the expected utility an agent will receive from the mechanism in the second period, after having reported θ_1 in the first period and observing θ_2. Thus, for an incentive compatible mechanism,

$$W(\theta_1,\theta_2) = \int_A -\frac{1}{2}(a-\theta_2)^2 p(a|\theta_1,\theta_2)da.$$

Lemma A.4 *A mechanism is incentive compatible in the second period if and only if*

1. $\tilde{a}_2(\theta_1,\cdot)$ *is non-decreasing and*
2. $W(\theta_1,\theta_2) = W(\theta_1,0) + \int_0^{\theta_2} [\tilde{a}(\theta_1,z) - z]\, dz$

Proof The expected payoff of the agent of type θ_2 who reports θ_1 in the first period and θ_2' in the second period is

$$\int_A -\frac{1}{2}(a-\theta_2)^2 p(a|\theta_1,\theta_2')da = -\frac{1}{2}\tilde{\sigma}^2(\theta_1,\theta_2') - \frac{1}{2}\left(\tilde{a}(\theta_1,\theta_2')-\theta_2\right)^2.$$

Thus, part 1 of the "only if" statement follows immediately from IC constraints, and part 2 is the standard envelope condition Milgrom and Segal (2002). To show the "if" statement, it suffices to show that under these conditions, for every $\theta_2,\theta_2' \in \Theta$,

$$-\frac{1}{2}\tilde{\sigma}^2(\theta_1,\theta_2) - \frac{1}{2}(\tilde{a}(\theta_1,\theta_2)-\theta_2)^2 \geq -\frac{1}{2}\tilde{\sigma}^2(\theta_1,\theta_2') - \frac{1}{2}\left(\tilde{a}(\theta_1,\theta_2')-\theta_2\right)^2$$

We can write

$$-\frac{1}{2}\tilde{\sigma}^2(\theta_1,\theta_2') - \frac{1}{2}\left(\tilde{a}(\theta_1,\theta_2')-\theta_2\right)^2 = -\frac{1}{2}\tilde{\sigma}^2(\theta_1,\theta_2') - \frac{1}{2}\left(\tilde{a}(\theta_1,\theta_2')-\theta_2'\right)^2 + \int_{\theta_2}^{\theta_2'} \left[\tilde{a}(\theta_1,\theta_2') - z\right] dz.$$

$$= W(\theta_1,\theta_2') + \int_{\theta_2}^{\theta_2'} \left[\tilde{a}(\theta_1,\theta_2') - z\right] dz$$

so

$$
\begin{aligned}
-\frac{1}{2}\tilde{\sigma}^2(\theta_1,\theta_2) - \frac{1}{2}(\tilde{a}(\theta_1,\theta_2)-\theta_2)^2 + \frac{1}{2}\tilde{\sigma}^2(\theta_1,\theta_2') + \frac{1}{2}\left(\tilde{a}(\theta_1,\theta_2')-\theta_2\right)^2& \\
&= W(\theta_1,\theta_2) - W(\theta_1,\theta_2') + \int_{\theta_2}^{\theta_2'} \left[\tilde{a}(\theta_1,\theta_2') - z\right] dz \\
&= \int_{\theta_2}^{\theta_2'} \left[\tilde{a}(\theta_1,\theta_2') - \tilde{a}(\theta_1,z)\right] dz \\
&\geq 0
\end{aligned}
$$

where the last inequality is due to the fact that $\tilde{a}_2(\theta_1,\cdot)$ is non-decreasing. Thus, the two conditions are sufficient for incentive compatibility in the second period. □

Lemma A.5 *The principal's expected payoff from the second period can be written as*

$$-b_2\left[W(\theta_1,1) - W(\theta_1,0)\right] - \frac{1}{2}b_2^2 + \int_0^1 W(\theta_1,\theta_2)d\theta_2.$$

Proof From the definitions above, we have

$$
\begin{aligned}
-\frac{1}{2}\tilde{\sigma}^2(\theta_1,\theta_2) - \frac{1}{2}(\tilde{a}(\theta_1,\theta_2)-\theta_2)^2 &= W(\theta_1,\theta_2) \\
&= W(\theta_1,0) + \int_0^{\theta_2} (\tilde{a}(\theta_1,z) - z)dz
\end{aligned}
$$

The principal's expected payoff in the second period, using the same decomposition as before, is

$$\int_0^1 \left[-\frac{1}{2}\tilde{\sigma}^2(\theta_1,\theta_2) - \frac{1}{2}\left(\tilde{a}(\theta_1,\theta_2) - \theta_2 + b_2\right)^2\right] d\theta_2.$$

Plugging in the expression for $\tilde{\sigma}^2$, this can be written as

$$\int_0^1 \left[W(\theta_1,0) + \int_0^{\theta_2} (\tilde{a}(\theta_1,z) - z)dz + \frac{1}{2}(\tilde{a}(\theta_1,\theta_2)-\theta_2)^2 - \frac{1}{2}\left(\tilde{a}(\theta_1,\theta_2) - \theta_2 + b_2\right)^2\right] d\theta_2$$

We can switch the order of integration on the second integral to get

$$
\begin{aligned}
\int_0^1 \int_0^{\theta_2} (\tilde{a}(\theta_1,z) - z)dz d\theta_2 &= \int_0^1 \int_z^1 (\tilde{a}(\theta_1,z) - z)d\theta_2 dz \\
&= \int_0^1 (1-\theta_2)(\tilde{a}(\theta_1,\theta_2) - \theta_2)d\theta_2
\end{aligned}
$$

Thus, the payoff further simplifies to

$$\int_0^1 \left[W(\theta_1,0) + (1-\theta_2)(\tilde{a}(\theta_1,\theta_2)-\theta_2) - \frac{1}{2}b_2\left(2\tilde{a}(\theta_1,\theta_2) - 2\theta_2 + b_2\right)\right] d\theta_2$$

$$= W(\theta_1,0) - \frac{1}{2}b_2^2 + \int_0^1 \tilde{a}(\theta_1,\theta_2)(1-\theta_2-b_2)d\theta_2 + \int_0^1 \left[-\theta_2 + \theta_2^2 + \theta_2 b_2\right] d\theta_2$$

$$= W(\theta_1,0) - \frac{1}{2}b_2^2 - \frac{1}{6} + \frac{1}{2}b_2 + \int_0^1 \left(\frac{\partial W(\theta_1,\theta_2)}{\partial \theta_2} + \theta_2\right)(1-\theta_2-b_2)d\theta_2$$

$$= W(\theta_1,0) - \frac{1}{2}b_2^2 + \int_0^1 \frac{\partial W(\theta_1,\theta_2)}{\partial \theta_2}(1-\theta_2-b_2)d\theta_2$$

$$= W(\theta_1,0) - \frac{1}{2}b_2^2 - b_2 W(\theta_1,1) - (1-b_2)W(\theta_1,0) + \int_0^1 W(\theta_1,\theta_2)d\theta_2$$

$$= -b_2\left[W(\theta_1,1) - W(\theta_1,0)\right] - \frac{1}{2}b_2^2 + \int_0^1 W(\theta_1,\theta_2)d\theta_2,$$

which is the expression in the lemma. □

Proposition 4 *In the uniform-quadratic setting, the principal's problem can be written as*

$$\max_{a_1(\theta_1),\omega(\theta_1)} \mathbb{E}\left[-\frac{1}{2}(a_1(\theta_1)-\theta_1+b_1)^2 + \gamma(\omega(\theta_1))\right] \tag{PDD'}$$

subject to $\forall \theta_1, \theta_1' \in \Theta$,

$$-\frac{1}{2}(a_1(\theta_1)-\theta_1)^2 + \omega(\theta_1) \geq -\frac{1}{2}(a_1(\theta_1')-\theta_1)^2 + \omega(\theta_1') \tag{IC1'}$$

where

$$\gamma(\omega) = \begin{cases} \omega - \frac{\delta}{2}b_2^2 + \frac{\delta b_2}{6}(-\frac{72}{\delta}\omega - 3)^{\frac{1}{2}} & \textit{if } \omega < -\frac{\delta}{6} \\ \omega - \frac{\delta}{2}b_2^2 + \frac{\delta b_2}{2}(\frac{6}{\delta}\omega)^{\frac{2}{3}} & \textit{if } -\frac{\delta}{6} \leq \omega \leq 0 \\ -\infty & \textit{otherwise} \end{cases}.$$

Proof I'll first show that the optimal second period utility function $W(\theta_1,\theta_2)$ conditional on the agent receiving expected utility $\frac{1}{\delta}\omega$ in the second period is

$$W(\theta_1,\theta_2) = -\sqrt{-\frac{2}{\delta}\omega - \frac{1}{12}}\theta_2 + \frac{1}{\delta}\omega + \frac{1}{2}\sqrt{-\frac{2}{\delta}\omega - \frac{1}{12}} - \frac{1}{2}\theta_2^2 + \frac{1}{2}\theta_2 - \frac{1}{12}$$

when $\omega < -\frac{\delta}{6}$ and

$$W(\theta_1,\theta_2) = \begin{cases} 0 \text{ for } \theta_2 \leq 1 + (\frac{6}{\delta}\omega)^{\frac{1}{3}} \\ -\frac{1}{2}\left(1 + (\frac{6}{\delta}\omega)^{\frac{1}{3}} - \theta_2\right)^2 \text{ otherwise} \end{cases}$$

when $-\frac{\delta}{6} \leq \omega \leq 0$.[8] Lemma A.5 shows that after accounting for IC2, the principal's payoff from the second period takes the form

$$-b_2\left[W(\theta_1,1) - W(\theta_1,0)\right] - \frac{1}{2}b_2^2 + \int_0^1 W(\theta_1,\theta_2)d\theta_2$$

Using this formula, and multiplying by the discount factor, it is easily checked that the above $W(\theta_1,\theta_2)$ generates the given function $\gamma(\cdot)$. Suppose that this second period indirect utility function were not optimal. Then there is another indirect utility function derived from an incentive compatible mechanism, $\tilde{W}(\theta_1,\theta_2)$ which gives the principal a higher payoff and the agent the same expected utility, ω. Obviously, this function must be continuous. Furthermore, the monotonicity constraint implies that it cannot be kinked downward (because this would imply a downward jump in the action) and, where twice differentiable, cannot have a second derivative lower than -2 (since this would imply a decreasing action). Since

$$\int_0^1 \tilde{W}(\theta_1,\theta_2)d\theta_2 = \omega = \int_0^1 W(\theta_1,\theta_2)d\theta_2,$$

the difference between the expected utility for the principal is

$$-b_2\left[\tilde{W}(\theta_1,1) - \tilde{W}(\theta_1,0) - W(\theta_1,1) + W(\theta_1,0)\right].$$

This quantity is positive by assumption, since $\tilde{W}$ gives the principal a higher payoff, so $\tilde{W}(\theta_1,0) - \tilde{W}(\theta_1,1) > W(\theta_1,0) - W(\theta_1,1)$. I'll show that this is impossible for any incentive compatible second period utility function by considering two cases.

[8] One can check that this is the indirect second period utility function that arises from the principal placing a cap on the second period action, and the agent choosing optimally given this cap, implying that this utility function is incentive compatible. The cap, as a function of ω, is

$$y(\omega) = \begin{cases} \frac{1}{2} - \sqrt{-\frac{2}{\delta}\omega - \frac{1}{12}} \text{ for } \omega < -\frac{1}{6} \\ 1 + \left(\frac{6}{\delta}\omega\right)^{\frac{1}{3}} \text{ otherwise} \end{cases}.$$

- Case 1 - $\tilde{W}(\theta_1,0) > W(\theta_1,0)$: Since $\int\limits_0^1 \tilde{W}(\theta_1,\theta_2)d\theta_2 = \int\limits_0^1 W(\theta_1,\theta_2)d\theta_2$, there must be a point at which $\tilde{W}(\theta_1,\theta_2)$ crosses $W(\theta_1,\theta_2)$. The restriction that $\frac{\partial^2 \tilde{W}(\theta_1,\theta_2)}{\partial \theta_2^2} \geq -2 = \frac{\partial^2 W(\theta_1,\theta_2)}{\partial \theta_2^2}$, along with the impossibility of downward kinks, implies that $\frac{\partial \tilde{W}(\theta_1,0)}{\partial \theta_2} < \frac{\partial W(\theta_1,0)}{\partial \theta_2}$, so the action associated with $\tilde{W}(\theta_1,0)$ is strictly less than that of $W(\theta_1,0)$. However, since $\tilde{W}(\theta_1,0) > W(\theta_1,0)$, this implies that the variance associated with $\tilde{W}(\theta_1,0)$ is strictly less than that which is associated with $W(\theta_1,0)$, an impossibility since the latter is 0.
- Case 2 - $\tilde{W}(\theta_1,0) \leq W(\theta_1,0)$: For $\int\limits_0^1 \tilde{W}(\theta_1,\theta_2)d\theta_2 = \int\limits_0^1 W(\theta_1,\theta_2)d\theta_2$ and $\tilde{W}(\theta_1,0) - \tilde{W}(\theta_1,1) > W(\theta_1,0) - W(\theta_1,1)$, $\tilde{W}(\theta_1,\theta_2)$ must cross $W(\theta_1,\theta_2)$ at least twice, the first time from below and the second time from above. This would require that $\frac{\partial \tilde{W}(\theta_1,\theta_2)}{\partial \theta_1} > \frac{\partial W(\theta_1,\theta_2)}{\partial \theta_2}$ at the first intersection, but the opposite at the second, and impossibility due to the restriction on kinks and the fact that $\frac{\partial^2 \tilde{W}(\theta_1,\theta_2)}{\partial \theta_2^2} \geq -2 = \frac{\partial^2 W(\theta_1,\theta_2)}{\partial \theta_2^2}$ anywhere that they could cross.

Thus, there is no second period utility function arising from an incentive compatible mechanism which gives the principal higher expected utility, and the function $\gamma(\cdot)$ provides the mapping between the agent's and principal's continuation values. □

Theorem 2 *In the dynamic delegation game, an optimal mechanism exists and has the following properties:*

- *the action is strictly increasing and continuous in the first period's state (no pooling in the first period);*
- *the maximum action available in the second period is strictly decreasing in the first period's state;*
- *the actions taken in the first period are a strict subset of those the principal would take if she were fully informed.*

Proof Proposition 5, proven in Appendix C, shows that there is an indirect utility function $W(\theta)$ which characterizes the solution to problem (PDD'), ignoring the monotonicity constraint. Thus, it will suffice to show that the utility function $a(\theta)$ implied by the solution is monotonic and then identify some of its properties. Since

$$a(\theta) = W'(\theta) + \theta,$$

and the solution to the differential equation will be continuously differentiable, $a(\theta)$ is continuous. To show monotonicity, I'll show that $\forall \theta \in [0,1]$ where it exists, $W''(\theta) > -1$.

Consider the region in which $W(\theta) + \frac{1}{2}W'(\theta)^2 < -\frac{\delta}{6}$. Here,

$$W''(\theta) = -1 + \frac{\delta^{-\frac{1}{2}}\left(-2W(\theta) - W'(\theta)^2 - \frac{1}{12}\delta\right)^{\frac{3}{2}} + 4b_2W(\theta) + 2b_2W'(\theta)^2 + \frac{1}{6}\delta b_2}{2b_2W(\theta) + \frac{1}{12}\delta b_2}$$

and Lemma C.4 showed that

$$W'(\theta)^2 = -\delta\left[\frac{2b_2W(\theta) + \frac{1}{12}\delta b_2}{W(\theta) - \lambda}\right]^2 - 2W(\theta) - \frac{1}{12}\delta$$

for some $\lambda \in \mathbb{R}$. Since in this region, $W(\theta) < -\frac{1}{6}\delta$, $W''(\theta) > -1$ if

$$\delta^{-\frac{1}{2}}\left(-2W(\theta) - W'(\theta)^2 - \frac{1}{12}\delta\right)^{\frac{3}{2}} + 4b_2W(\theta) + 2b_2W'(\theta)^2 + \frac{1}{6}\delta b_2$$

is negative. We can substitute to get

$$\delta\left[\frac{2b_2W(\theta) + \frac{1}{12}\delta b_2}{W(\theta) - \lambda}\right]^3 - 2b_2\delta\left[\frac{2b_2W(\theta) + \frac{1}{12}\delta b_2}{W(\theta) - \lambda}\right]^2,$$

and after factoring we get

$$\delta b_2\left[\frac{2b_2W(\theta) + \frac{1}{12}\delta b_2}{W(\theta) - \lambda}\right]^2\left[\frac{2W(\theta) + \frac{1}{12}\delta}{W(\theta) - \lambda} - 2\right].$$

Since $\lambda > 0$, as shown in the proof of Proposition 5, and $W(\theta) < 0$, this quantity is negative.

Suppose the solution is instead in the region in which $-\frac{\delta}{6} \leq W(\theta) + \frac{1}{2}W'(\theta)^2 \leq 0$. Then

$$W''(\theta) = -1 + \frac{\delta^{-\frac{1}{3}}\left(6W(\theta) + 3W'(\theta)^2\right)^{\frac{4}{3}} + 24b_2W(\theta) + 12b_2W'(\theta)^2}{12b_2W(\theta) + 2b_2W'(\theta)^2}.$$

Factoring, we get

$$W''(\theta) = -1 + \frac{\left(6W(\theta) + 3W'(\theta)^2\right)\left[\delta^{-\frac{1}{3}}\left(6W(\theta) + 3W'(\theta)^2\right)^{\frac{1}{3}} + 4b_2\right]}{12b_2W(\theta) + 2b_2W'(\theta)^2}$$

and

$$\left(6W(\theta) + 3W'(\theta)\right)^{\frac{1}{3}} = \frac{\delta^{\frac{1}{3}}\left[3b_2W(\theta) - \frac{1}{2}b_2W'(\theta)^2\right]}{W(\theta) - \lambda}$$

Thus, this simplifies to

$$W''(\theta) = -1 + \frac{(6W(\theta) + 3W'(\theta)^2)\left[\frac{3b_2W(\theta) - \frac{1}{2}b_2W'(\theta)^2}{W(\theta) - \lambda} + 4b_2\right]}{12b_2W(\theta) + 2b_2W'(\theta)^2}.$$

Again, since $\lambda > 0$, this $W''(\theta) > -1$ and the action in the first period is strictly increasing. From before, we know that

$$\omega(\theta) = W(\theta) + \frac{1}{2}W'(\theta)^2.$$

Thus, at points of differentiability,

$$\omega'(\theta) = W'(\theta)\left[1 + W''(\theta)\right],$$

so this is always negative. Combining this with the fact that $y(\omega)$, the cap on the second period action, is increasing in its argument gives the result that $y(\omega(\theta))$ is strictly decreasing. Finally, in the proofs from Appendix C it is shown that $W'(0) > -b_1$ and $W'(1) < -b_1$. Since

$$a(0) = W'(0)$$

and

$$a(1) = W'(1) + 1,$$

it must be the case that $a(0) > -b_1$, and $a(1) < 1 - b_1$, so the actions taken all lie strictly within the set of actions the principal would take with full information, $[-b_1, 1 - b_1]$. $\square$

Lemma A.6 *In the optimal mechanism, $\gamma(\omega(\theta))$ achieves the maximum value of $\gamma(\cdot)$ for some $\theta \in [0,1]$.*

Proof Since the action in the first period is continuous, the maximum action (and thus the continuation value $\omega(\theta)$) must also be continuous. Suppose that $\gamma(\cdot)$ didn't achieve its maximum. Since γ is single peaked and monotonic on either side of the peak, the principal could shift all continuation values either up or down, strictly increasing the principal's payoff and not affecting incentive compatibility constraints, a contradiction of optimality. $\square$

Theorem 3 *In the dynamic delegation game, increasing the agent's bias in the first period makes the principal strictly worse off.*

Proof Take some b_1 and $\hat{b}_1$, where $b_1 < \hat{b}_1$, and consider the optimal mechanism when the principal is facing an agent with types $\hat{b}_1$ and b_2, with $b_2 > 0$. This setting then defines $\hat{a}(\cdot)$ and $\hat{\omega}(\cdot)$. Furthermore, define $\bar{\theta}$ as a value of θ for which $\gamma(\hat{\omega}(\theta))$ achieves its maximum. Define $a(\theta) = \hat{a}(\theta) + \hat{b}_1 - b_1$, and let

$$\omega(\theta) = \hat{\omega}(\theta) + (\hat{b}_1 - b_1)(\hat{a}(\theta) - \hat{a}(\bar{\theta})).$$

I'll show that for b_1 near $\hat{b}_1$, the mechanism defined by a and ω is incentive compatible and strictly improves the principal's payoffs. Since the original mechanism was incentive compatible, we know that for all θ and θ',

$$-\frac{1}{2}\left(\hat{a}(\theta) - \theta\right)^2 + \hat{\omega}(\theta) \geq -\frac{1}{2}\left(\hat{a}(\theta') - \theta\right)^2 + \hat{\omega}(\theta')$$

Thus, it must be the case that for all θ and θ',

$$-\frac{1}{2}\left(a(\theta)+b_1-\hat{b}_1-\theta\right)^2+\omega(\theta)-(\hat{b}_1-b_1)(\hat{a}(\theta)-\hat{a}(\bar{\theta}))$$
$$\geq -\frac{1}{2}\left(a(\theta')+b_1-\hat{b}_1-\theta\right)^2+\omega(\theta')-(\hat{b}_1-b_1)(\hat{a}(\theta')-\hat{a}(\bar{\theta}))$$

This simplifies to

$$-\frac{1}{2}(a(\theta)-\theta)^2+\omega(\theta)\geq -\frac{1}{2}(a(\theta')-\theta)^2+\omega(\theta'),$$

so the new allocation is incentive compatible. The payoffs from the action taken are the same in both settings. Thus, any difference in payoffs arises completely from the change in continuation values. Payoffs increase weakly if for $\theta>\bar{\theta}$, $\hat{\omega}(\theta)<\omega(\theta)\leq\hat{\omega}(\bar{\theta})$, and for $\theta<\bar{\theta}$, $\hat{\omega}(\theta)>\omega(\theta)\geq\hat{\omega}(\bar{\theta})$, since this moves all continuation payoffs closer to the maximum value of γ. Since $\omega(\bar{\theta})=\hat{\omega}(\bar{\theta})$, and it's clearly the case that $\omega(\theta)>\hat{\omega}(\theta)$ if and only if $\theta\geq\bar{\theta}$, then the result holds if $\omega(\theta)$ is everywhere increasing. Notice that it's still the case that at points of differentiability,

$$\begin{aligned}\omega'(\theta)&=W'(\theta)[1+W''(\theta)]\\&=\left(\hat{W}'(\theta)+\hat{b}_1-b_1\right)\left(1+\hat{W}''(\theta)\right)\end{aligned}$$

so since $\hat{W}'(\theta)$ is bounded away from 0, ω is increasing for small b_1 near $\hat{b}_1$, showing that for any level of bias in the first period, a small decrease in bias weakly increases the principal's payoffs, so the optimal mechanism in the case of smaller bias must also improve payoffs. $\square$

Theorem 4 *In the dynamic delegation game, for high enough second period bias, further increasing this bias improves the principal's payoffs. As bias in the second period becomes arbitrarily large, the principal's payoffs approach those she would receive if she had full information in the first period, and had to make the optimal uniformed decision in the second period.*

Proof First, I'll show that for $b_2\geq 0.5$, when b_2 increases the principal can use an incentive scheme which implements the same action profile, but achieves higher continuation values for the principal. In particular, I'll show that as b_2 increases, the principal can shift the incentive scheme downward uniformly (i.e. holding $\omega(\theta)-\omega(\theta')$ constant for all θ,θ'), and that this strictly increases the principal's payoffs for all θ such that $\omega(\theta)$ isn't at the maximum of γ.

Recall that where it is defined, γ is

$$\gamma(\omega,b_2)=\begin{cases}\omega-\frac{\delta}{2}b_2^2+\frac{\delta b_2}{6}\left(-\frac{72}{\delta}\omega-3\right)^{\frac{1}{2}}\\ \omega-\frac{\delta}{2}b_2^2+\frac{\delta b_2}{2}\left(\frac{6}{\delta}\omega\right)^{\frac{2}{3}}\end{cases},$$

where we add b_2 as an argument to the function. Define $\hat{\omega} = \omega + \frac{1}{2}\delta b_2^2 + \frac{1}{24}\delta$, so that $\hat{\omega}$ is just the distance to the maximizer of $\gamma(\cdot, b_2)$ for $b_2 \geq \frac{1}{2}$. Thus, we could redefine γ as

$$\gamma(\hat{\omega}, b_2) = \begin{cases} \hat{\omega} - \delta b_2^2 - \frac{1}{24}\delta + \frac{\delta b_2}{6}\left(-\frac{72}{\delta}\hat{\omega} + 36 b_2^2\right)^{\frac{1}{2}} \\ \hat{\omega} - \delta b_2^2 - \frac{1}{24}\delta + \frac{\delta b_2}{2}\left(\frac{6}{\delta}\hat{\omega} - 3b_2^2 - \frac{1}{4}\right)^{\frac{2}{3}} \end{cases}.$$

If we then take a derivative with respect to b_2, we get

$$\gamma_{b_2}(\hat{\omega}, b_2) = \begin{cases} -2\delta b_2 + \left(-\frac{72}{\delta}\hat{\omega} + 36 b_2^2\right)^{-\frac{1}{2}}\left[12\delta b_2^2 - 12\hat{\omega}\right] \\ -2\delta b_2 + \left(-\frac{7}{2}\delta b_2^2 + 3\hat{\omega} - \frac{\delta}{8}\right)\left(\frac{6}{\delta}\hat{\omega} - 3b_2^2 - \frac{1}{4}\right)^{-\frac{1}{3}} \end{cases}$$

and we can then take a derivative with respect to $\hat{\omega}$ to get

$$\begin{aligned} \gamma_{b_2,\hat{\omega}}(\hat{\omega}, b_2) &= \begin{cases} -\frac{1}{2}\left(-\frac{72}{\delta}\hat{\omega} + 36b_2^2\right)^{-\frac{3}{2}}\left(-\frac{72}{\delta}\right)\left[12\delta b_2^2 - 12\hat{\omega}\right] - 12\left(-\frac{72}{\delta}\hat{\omega} + 36b_2^2\right)^{-\frac{1}{2}} \\ -\frac{2\delta\left(4\delta b_2^2 + \delta - 24\hat{\omega}\right)\left(-24b_2^2 + \frac{48\hat{\omega}}{\delta} - 2\right)^{\frac{2}{3}}}{\left(12\delta b_2^2 + \delta - 24\hat{\omega}\right)^2} \end{cases} \\ &= \begin{cases} \left(-\frac{72}{\delta}\hat{\omega} + 36b_2^2\right)^{-\frac{3}{2}}\left(\frac{432}{\delta}\hat{\omega}\right) \\ -\frac{2\delta\left(4\delta b_2^2 + \delta - 24\hat{\omega}\right)\left(-24b_2^2 + \frac{48\hat{\omega}}{\delta} - 2\right)^{\frac{2}{3}}}{\left(12\delta b_2^2 + \delta - 24\hat{\omega}\right)^2} \end{cases}. \end{aligned}$$

In the first region, the derivative with respect to b_2 is 0 exactly when $\hat{\omega} = 0$. Furthermore, this derivative is increasing as $\hat{\omega}$ goes away from 0. The derivative is positive whenever $\hat{\omega}$ is in the second region, i.e. $\hat{\omega} \geq \frac{1}{2}\delta b_2^2 - \frac{1}{8}$. Thus, when $b_2 \geq \frac{1}{2}$, increasing b_2 increases the principal's continuation value conditional on the agent's continuation value being the same distance from the maximum of γ. Because there is no individual rationality constraint, for a higher b_2 the principal can simply shift the promised distribution of ω downward, and this strictly increases her payoffs.

Finally, I'll show that as $b_2 \to \infty$, the principals payoffs are bounded above by $-\frac{1}{24}\delta$, and bounded below by a value that converges to $-\frac{1}{24}\delta$. Notice that for $b_2 \geq 0.5$, the function γ has a maximum value of $-\frac{1}{24}\delta$. Thus, the principal's payoffs are bounded above by what she would receive if she took exactly her preferred action in the first period (receiving 0) and a continuation value of $-\frac{1}{24}\delta$. Next, consider the action profile $a(\theta) = \theta - b_1$, so principal's preferred action profile is chosen in the first period. This action profile can be supported by $\omega(\theta)$ such that

$$\omega(\theta) = \omega(0) - b_1\theta$$

In particular, set $\omega(\theta) = -\frac{1}{2}\delta b_2^2 - \frac{1}{24} - b_1\theta + \frac{1}{2}b_1$. Then for any θ, when b_2 is large, the principal has payoffs of 0 in the first period, and continuation value of

$$\gamma(\omega(\theta)) = -\delta b_2^2 - \frac{1}{24}\delta - b_1\theta + \frac{1}{2}b_1 + \frac{\delta b_2}{6}\left(36b_2^2 + \frac{72}{\delta}b_1\theta - \frac{36}{\delta}b_1\right)^{\frac{1}{2}}.$$

As $b_2 \to \infty$, this approaches

$$-\frac{1}{24}\delta - b_1\theta + \frac{1}{2}b_1$$

which in expectation is $-\frac{1}{24}\delta$. Since this mechanism is incentive compatible, and approaches the upper bound on the principal's payoffs, the optimal mechanism's payoffs must also approach these payoffs, and they are the same as those that the principal would receive if she had full information in the first period, and were completely uninformed with no agent to work with in the second period. $\square$

Theorem 5 *The optimal dynamic delegation mechanism is deterministic.*

Proof The non-optimality of randomizing in the second period was already shown in the proof of Proposition 4, and the non-optimality of randomizing over delgation sets offered in the second period follows immediately from the concavity of the function $\gamma(\cdot)$. Thus, it remains to show that it is not optimal to randomize the action after the report in the first period.

Suppose that the principal is randomizing the action taken after the agent's report in the first period. Then the principal's and agent's payoffs can again be decomposed into a function of the average and variance of the action taken in each state. Define these as $\bar{a}_1(\theta)$ and $\sigma^2(\theta)$, respectively and let $\omega(\theta)$ be the continuation value promised to the agent. Suppose that $\sigma^2(\theta) > 0$ for some θ, and define

$$\begin{aligned}
\hat{\bar{a}}(\theta) &= \bar{a}(\theta) \\
\hat{\sigma}^2(\theta) &= 0 \\
\hat{\omega}(\theta) &= \omega(\theta) - \frac{1}{2}\sigma^2(\theta)
\end{aligned}$$

Since the original allocation was incentive compatible, we have that

$$-\frac{1}{2}(\bar{a}(\theta)-\theta)^2 - \frac{1}{2}\sigma^2(\theta) + \omega(\theta) \geq -\frac{1}{2}(\bar{a}(\theta')-\theta)^2 - \frac{1}{2}\sigma^2(\theta') + \omega(\theta')$$
$$-\frac{1}{2}(\hat{\bar{a}}(\theta)-\theta)^2 + \hat{\omega}(\theta) \geq -\frac{1}{2}(\hat{\bar{a}}(\theta')-\theta)^2 + \hat{\omega}(\theta')$$

so the new allocation is incentive compatible. Furthermore, since $\gamma'(\omega) < -1$, , $-\sigma^2(\theta) + \gamma(\omega(\theta)) < \gamma\left(\omega(\theta) - \sigma^2(\theta)\right)$, and

$$-(\bar{a}(\theta)-\theta)^2 - \sigma^2(\theta) + \gamma(\omega(\theta)) < -(\hat{\bar{a}}(\theta)-\theta)^2 - \hat{\sigma}^2(\theta) + \gamma(\omega(\theta)).$$

This new incentive compatible mechanism improves the principal's payoffs, so randomizing in the first period can't be optimal. $\square$

Theorem 6 *In the uniform-quadratic responsiveness setting with money burning, the optimal mechanism can take one of five forms:*

1. *Discretion with rules preventing extremes if responsiveness is high and the cost of money burning is relatively high:* $\beta < 1$ *and* $k \geq \frac{1}{2}\beta$
2. *Discretion with money burning punishing extremes if responsiveness is high and the cost of money burning is relatively low:* $\beta < 1$, $k < \frac{1}{2}\beta$
3. *Full discretion if responsiveness is intermediate and the cost of money burning is relatively high:* $1 \leq \beta \leq 2$, $k \geq \beta - 1$
4. *Money burning over intermediate actions, with rules preventing extremes if responsiveness is intermediate or low, and the cost of money burning is relatively low:* $\beta - 1 \geq k$, $k < 1$
5. *Only two points if responsiveness is low and the cost of money burning is relatively high:* $\beta > 2$, $k \geq 1$.

Proof First, notice that if for any combination of b and k, if the optimal mechanism doesn't include any money burning, then in all settings with the same level of responsiveness and higher k, the optimal mechanism is the same. This is because any alternative mechanism must improve payoffs either with money burning or with some other incentive compatible allocation. This implies that there is a contradiction, since the same mechanism must have been incentive compatible for lower k, and would have given the principal weakly higher payoffs.

Consider the case with $\beta < 1$ and $k \leq \frac{1}{2}\beta$. The proposed action profile is

$$a(\theta) = \begin{cases} \frac{2k-\beta}{k-1}\theta + \frac{1-\beta}{2(1-k)} \text{ if } \theta \leq \frac{1-\beta}{2(k-\beta+1)} \\ \theta \text{ if } \frac{1-\beta}{2(k-\beta+1)} < \theta \leq \frac{2k-\beta+1}{2k-2\beta+2} \\ \frac{2k-\beta}{k-1}\theta + \frac{\beta-2k-1}{2(k-1)} \text{ otherwise} \end{cases}.$$

The envelope condition implies that when $\theta \leq \frac{1-\beta}{2(k-\beta+1)}$

$$\begin{aligned} \omega'(\theta) &= (a(\theta) - \theta)a'(\theta) \\ &= \left(\frac{k-\beta+1}{k-1}\theta + \frac{1-\beta}{2(1-k)}\right)\left(\frac{2k-\beta}{k-1}\right), \end{aligned}$$

and with the condition that $\omega\left(\frac{1-\beta}{2(k-\beta+1)}\right) = 0$, we get

$$\omega(\theta) = \frac{(\beta-2k)(2(k+1-\beta)\theta + \beta - 1)^2}{8(k-1)^2(\beta-k-1)}.$$

Making the symmetric computations for $\theta \geq \frac{2k-\beta+1}{2k-2\beta+2}$, we get that the agent's indirect utility function is

$$W(\theta)=\begin{cases}\frac{k-\beta+1}{2(k-1)}\theta^2+\frac{1-\beta}{2(1-k)}\theta+\frac{(1-\beta)^2}{8(k-1)(k-\beta+1)} \text{ if } \theta\le\frac{1-\beta}{2(k-\beta+1)}\\ 0 \text{ if } \frac{1-\beta}{2(k-\beta+1)}<\theta\le\frac{2k-\beta+1}{2k-2\beta+2}\\ \frac{k-\beta+1}{2(k-1)}\theta^2+\frac{\beta-2k-1}{2(k-1)}\theta+\frac{(2k-\beta+1)^2}{8(k-1)(k-\beta+1)} \text{ otherwise}\end{cases}$$

Next, define

$$p(\theta)=\begin{cases}-k\theta \text{ if } \theta\le\frac{1-\beta}{2(k-\beta+1)}\\ (1-\beta)\theta+\frac{1}{2}\beta-\frac{1}{2} \text{ if } \frac{1-\beta}{2(k-\beta+1)}<\theta\le\frac{2k-\beta+1}{2k-2\beta+2}\\ -k(\theta-1) \text{ otherwise}\end{cases},$$

which satisfies all of the conditions required in Proposition A.2. Finally, we must check that

$$W'(\theta)\in\operatorname*{argmax}_v\ p(\theta)v-L(\theta,W(\theta),v),$$

which in the first region has first order conditions

$$-k\theta-(v+(1-\beta)\theta+\frac{1}{2}\beta-\frac{1}{2})+kv=0$$

and in third region has first order conditions

$$-k(\theta-1)-\left(v+(1-\beta)\theta+\frac{1}{2}\beta-\frac{1}{2}\right)+kv=0.$$

Thus, $W'(\theta)$ solves the problem, and the proposed mechanism is optimal.

Next consider the case of $\beta>1$, $k\le\beta-1$ and $k<1$. The proposed action profile has

$$a(\theta)=\begin{cases}\frac{k^2+1-\beta+\sqrt{(\beta-2k)(k)(2\beta-2-\beta k)}}{2(k-1)^2} \text{ if } \theta\le\frac{k(\beta-2k)-\sqrt{(\beta-2k)(k)(2\beta-2-\beta k)}}{2(\beta-2k)(k-1)}\\ \frac{2k-\beta}{k-1}\theta+\frac{\beta-1-k}{2(k-1)} \text{ if } \frac{k(\beta-2k)-\sqrt{(\beta-2k)(k)(2\beta-2-\beta k)}}{2(\beta-2k)(k-1)}<\theta\le 1-\frac{k(\beta-2k)-\sqrt{(\beta-2k)(k)(2\beta-2-\beta k)}}{2(\beta-2k)(k-1)}\\ 1+\frac{\beta-k^2-1-\sqrt{(\beta-2k)(k)(2\beta-2-\beta k)}}{2(k-1)^2} \text{ otherwise}\end{cases}$$

We'll define

$$p(\theta)=\begin{cases}\frac{-\theta[\beta(\theta-1)(k-1)^2+\beta-k^2-1-\sqrt{(\beta-2k)(k)(2\beta-2-\beta k)}+(k-1)^2]}{\beta-k^2-1-\sqrt{(\beta-2k)(k)(2\beta-2-\beta k)}+2\theta(k-1)^2} \text{ if } \theta\le\frac{k(\beta-2k)-\sqrt{(\beta-2k)(k)(2\beta-2-\beta k)}}{2(\beta-2k)(k-1)}\\ \frac{k}{2}-k\theta \text{ if } \frac{k(\beta-2k)-\sqrt{(\beta-2k)(k)(2\beta-2-\beta k)}}{2(\beta-2k)(k-1)}<\theta\le 1-\frac{k(\beta-2k)-\sqrt{(\beta-2k)(k)(2\beta-2-\beta k)}}{2(\beta-2k)(k-1)}\\ \frac{(\theta-1)[(\beta\theta-1)(k-1)^2-\beta+k^2+1+\sqrt{(\beta-2k)(k)(2\beta-2-\beta k)}}{\beta-k^2-1-\sqrt{(\beta-2k)(k)(2\beta-2-\beta k)}+2(1-\theta)(k-1)^2} \text{ otherwise}\end{cases}$$

With this action profile,

$$W'(\theta) = \begin{cases} \frac{k^2+1-\beta+\sqrt{(\beta-2k)(k)(2\beta-2-\beta k)}}{2(k-1)^2} - \theta \text{ if } \theta \leq \frac{k(\beta-2k)-\sqrt{(\beta-2k)(k)(2\beta-2-\beta k)}}{2(\beta-2k)(k-1)} \\ \frac{k-\beta+1}{k-1}\theta + \frac{\beta-1-k}{2(k-1)} \text{ if } \frac{k(\beta-2k)-\sqrt{(\beta-2k)(k)(2\beta-2-\beta k)}}{2(\beta-2k)(k-1)} < \theta \leq 1 - \frac{k(\beta-2k)-\sqrt{(\beta-2k)(k)(2\beta-2-\beta k)}}{2(\beta-2k)(k-1)} \\ 1 + \frac{\beta-k^2-1-\sqrt{(\beta-2k)(k)(2\beta-2-\beta k)}}{2(k-1)^2} - \theta \text{ otherwise} \end{cases}$$

Next, we must check that

$$W'(\theta) \in \underset{v}{\operatorname{argmax}}\ p(\theta)v - L(\theta, W(\theta), v),$$

In the middle region, this has first order conditions

$$\frac{k}{2} - k\theta - \left(v + (1-\beta)\theta + \frac{1}{2}\beta - \frac{1}{2}\right) + kv = 0,$$

which simplifies to

$$v = \frac{\beta - 1 - k}{1-k}\theta - \frac{k-\beta+1}{2(k-1)}$$

In the first region, if we define $c = -\frac{k^2+1-\beta+\sqrt{(\beta-2k)(k)(2\beta-2-\beta k)}}{2(k-1)^2}$ (the negative of the floor on the agent's action), take the derivative with respect to v and plug in that $v = -c - \theta$, we get

$$\frac{-\theta(\beta(\theta-1)+2c+1)}{2(c+\theta)} + c - kc + \beta\theta - k\theta - \frac{1}{2}\beta + \frac{1}{2}.$$

This expression is negative for $\theta \leq \frac{k(\beta-2k)-\sqrt{(\beta-2k)(k)(2\beta-2-\beta k)}}{2(\beta-2k)(k-1)}$, which shows that the v implied by that action profile in this region is the maximizer. Furthermore, the analysis in the third region is exactly symmetric, so all the conditions for optimization are fulfilled. $\square$

Finally, we can check that a delegation set of two points is optimal when $\beta > 2$ and $k = 1$. The proposed action profile is

$$a(\theta) = \begin{cases} -\frac{1}{4}\beta + \frac{1}{2} \text{ if } \theta \leq \frac{1}{2} \\ \frac{1}{4}\beta + \frac{1}{2} \text{ otherwise} \end{cases}.$$

Here, we'll define

$$p(\theta) = \begin{cases} \frac{-\theta\left(\beta\theta - \frac{1}{2}\beta\right)}{\frac{1}{2}\beta - 1 + 2\theta} \text{ if } \theta \leq \frac{1}{2} \\ \frac{(\theta-1)\left(\beta\theta-\frac{1}{2}\beta\right)}{2\left(\frac{1}{4}\beta - \theta + \frac{1}{2}\right)} \text{ otherwise} \end{cases}$$

It's easy to check that $p(\theta)$ satisfies the derivative conditions given in Proposition A.2, and we must only check that the given action profile solves the maximization problem. If we take the derivative of the objective function and plug in the action profile when $\theta \leq \frac{1}{2}$, we get

$$\frac{-\theta\beta\left(\theta - \frac{1}{2}\right)}{\frac{1}{2}\beta - 1 + 2\theta} + (\beta - 1)\left(\theta - \frac{1}{2}\right)$$

This simplifies to

$$\left[\frac{\frac{1}{2}\beta^2 - \frac{3}{2}\beta + \theta\beta + 1 - 2\theta}{\frac{1}{2}\beta - 1 + 2\theta}\right]\left(\theta - \frac{1}{2}\right),$$

which is always negative for $\theta \in \left[0, \frac{1}{2}\right]$ when $\beta > 2$. For $\theta \in \left(\frac{1}{2}, 1\right]$, the symmetric expression simplifies to

$$\left[\frac{\frac{1}{2}\beta^2 - \frac{1}{2}\beta - \beta\theta + 2\theta - 1}{\frac{1}{2}\beta - 2\theta + 1}\right]\left(\theta - \frac{1}{2}\right)$$

which is always positive. Thus, this action profile satisfies all of the conditions of Proposition A.2, and it solves the principal's problem. $\square$

B Delegation with Money Burning Comparisons

B.1 Comparison to Amador and Bagwell (2013)

Amador and Bagwell (2013) follows the previous literature on delegation in focusing on *interval delegation*, in which the principal allows the agent to make choices from a single interval (Holmström, 1977; Alonso & Matouschek, 2008). In particular, they provide conditions both in delegation settings with and without money burning for when interval delegation is optimal. For the setting with money burning, interval delegation is defined as allowing the agent to choose from an interval *and* there being no money burning. In a sense, this is a "corner solution," in which the principal is not utilizing one of the instruments that is available to her: money burning is at the upper bound of 0 for all values of the state.

When looking for conditions that ensure such interval delegation is optimal when money burning is a possibility, it is obvious that the main results from Section 3 won't apply, since those refer to solutions such that the continuation value is never at its upper bound. The continuation values in interval delegation are *always* at their upper bound: by definition, interval delegation includes no money burning. Instead, I'll use the more general results found in Proposition A.2 in Appendix A, which account for these corner solutions.

To use the results developed in this paper, one has to make stronger assumptions than those that are made by Amador and Bagwell (2013). There, they allow the agent's payoffs to vary flexibly with the allocation variable, as long as payoffs are still affine in the state variable. Here, I put more structure on preferences in assuming that the relationship between payoffs and the allocation variable is quadratic. In their setting, this is equivalent to setting $b(\pi) = -\frac{1}{2}\pi^2$. With this assumption, I can state the result, which can be compared to their Proposition 1 (b).

Theorem B.1 *Assume that the convexity condition holds. When money burning is feasible and equally costly to the principal and the agent, the following conditions imply that the optimal mechanism is one in which money burning is everywhere* 0*, and the actions taken are on an interval* $[\theta_L, \theta_H]$.

(1) $\forall \theta \in [0, \theta_L]$, $\int_0^\theta u_a^P(\theta_L, z) f(z) dz \geq \left[u_a^P(\theta_L, \theta) + (\theta_L - \theta)\right](\theta - \theta_L) f(\theta)$, *with equality at* θ_L,

(2) if $\theta_L = 0$, $u_a^P(0,0) \leq 0$,

(3) $F(\theta) - u_a^P(\theta, \theta) f(\theta)$ *is nondecreasing for* $\theta \in [\theta_L, \theta_H]$,

(4) if $\theta_H = 1$, $u_a^P(1,1) \geq 0$, *and*

(5) $\forall\theta \in [\theta_H, 1]$, $\int\limits_{\theta}^{1} u_a^P(\theta, z)f(z) \leq \left[-u_a^P(\theta_H, \theta) - (\theta_H - \theta)\right](\theta - \theta_H)f(\theta)$, *with equality at* θ_H.

Proof The described mechanism has

$$W(\theta) = \begin{cases} \theta_L\theta - \frac{1}{2}\theta^2 - \frac{1}{2}\theta_L^2 \text{ if } \theta < \theta_L \\ 0 \text{ if } \theta_L \leq \theta \leq \theta_H \\ \theta_H\theta - \frac{1}{2}\theta^2 - \frac{1}{2}\theta_H^2 \text{ if } \theta_H < \theta \end{cases}$$

I'll define a function $p(\theta)$ and show that if $W(\theta)$ has the assumed properties, then $p(\theta)$ and $W(\theta)$ satisfy the requirements of Proposition A.2. In particular,

$$p(\theta) = \begin{cases} \frac{\int\limits_0^{\theta} u_a^P(\theta_L, z)f(z)dz}{\theta_L - \theta} \text{ if } \theta < \theta_L \\ -u_a^P(\theta, \theta)f(\theta) \text{ if } \theta_L \leq \theta \leq \theta_H \\ \frac{\int\limits_{\theta}^{1} u_a^P(\theta_H, z)f(z)dz}{\theta - \theta_H} \text{ if } \theta_H < \theta \end{cases}$$

Since $\int\limits_0^{\theta_L} u_a^P(\theta_L, z)f(z)dz = 0$,

$$\begin{aligned} \lim_{\theta \uparrow \theta_L} p(\theta) &= \lim_{\theta \uparrow \theta_L} -\frac{\int\limits_0^{\theta} u_a^P(\theta_L, z)f(z)dz - \int\limits_0^{\theta_L} u_a^P(\theta_L, z)f(z)dz}{\theta - \theta_L} \\ &= -u_a^P(\theta_L, \theta_L)f(\theta_L). \end{aligned}$$

Similarly, $\lim_{\theta \downarrow \theta_H} p(\theta) = -u_a^P(\theta_H, \theta_H)f(\theta_H)$, so $p(\theta)$ is continuous, and fulfills all of the requirements from Proposition A.2 for the level of $p(\theta)$. $p(\theta)$ was defined in such a way that

$$p'(\theta) = \frac{1}{W'(\theta)}p(\theta) + \frac{1}{W'(\theta)}u_a^P(W'(\theta) + \theta, \theta)f(\theta),$$

for $\theta \notin [\theta_L, \theta_H]$, and for $\theta \in [\theta_L, \theta_H]$,

$$p'(\theta) \geq -\gamma'_-(0)f(\theta)$$

is implied by the derivative of

$$F(\theta) - u_a^P(\theta, \theta)f(\theta)$$

being positive, which is the same as $F(\theta) - u_a^P(\theta, \theta)f(\theta)$ being nondecreasing. $\square$

The conditions of Theorem B.1 ensure that the principal cannot improve on the interval she is delegating. Conditions (2) and (4) are the easiest to interpret. They state that if the principal is allowing the agent to have flexibility at the "extremes," then the principal must want even more extreme actions when the state is at its highest or lowest. For instance, if the principal wanted an action lower than one when the state is equal to one, then she could simply place a cap on the available actions which was slightly less than one, improving her payoffs without affecting incentive compatibility.

Condition (3) establishes that there is no profitable way to change incentives for the agent on the interval where the principal is giving the agent full discretion. The condition is equivalent to the quantity

$$\frac{d}{d\theta}\left[u_a^P(\theta,\theta)f(\theta)\right] - f(\theta)$$

being weakly negative. Intuitively, the principal is considering starting to use money burning to incentivize the agent to take actions lower than the agent's preferred action. Thus, at a given value of the state, the gains from using money burning are related to how costly the agent taking his preferred action is over the states immediately above the current one: if costs are sharply increasing, money burning is valuable. Money burning at a given state comes at a cost of $f(\theta)$ to the principal, so the principal doesn't have an incentive to deviate from giving the agent discretion over the interval as long as the benefits are lower than the cost.

Conditions (1) and (5) ensure optimality on the intervals in which the principal is giving the agent no discretion (the floor and the cap, respectively). To interpret this, consider condition (5). Where $\theta \neq \theta_H$, the condition can be restated as

$$\left[u_a^P(\theta_H,\theta) + (\theta_H - \theta)\right] f(\theta) \leq -\frac{\int_\theta^1 u_a^P(\theta,z)f(z)}{\theta - \theta_H}.$$

The principal could deviate from the proposed cap in an incentive compatible way by allowing the agent to take a slightly higher action, but require money burning for the agent to do so. $u_a^P(\theta_H,\theta)$ is the benefit of raising the action one unit, and $\theta - \theta_H$ is the money burning required to do so. For this change to be incentive compatible without using money burning at any other values of the state, the actions taken at all states above θ must increase by an amount proportional to the inverse of $(\theta - \theta_H)$. Thus, condition (5) states that the benefit of increasing the action taken has to be weighed against the costs of burning money and the changes in payoffs that arise from the incentive compatibility constraint.

These results are closely related to the sufficiency conditions found by Amador and Bagwell. When the settings are made comparable, conditions (2)-(4) here are the same as conditions (c1), (c2'), and (c3') there. In fact, the conditions in Theorem B.1 imply the conditions found in Amador and Bagwell (2013).

Remark 1 *The conditions from Theorem B.1 imply those in used in Proposition 1(b) by Amador and Bagwell (2013) when applied to this setting.*

Proof After translating the setting of this paper into that of Amador and Bagwell, we find that the convexity condition implies that $\kappa = 1$. It's easy then to see that conditions (c1), (c2'), and (c3') are the same as conditions (2)-(4) above. Suppose that condition (c3) were violated at the point θ_2. Since both sides of the inequality are continuous, then $\exists \hat{\theta}_1$, with $\hat{\theta}_1 < \hat{\theta}_2 \leq \theta_L$ such that

$$\int_0^{\theta} u_a^P(\theta_L, z) f(z) dz < (\theta - \theta_L) F(\theta)$$

for $\theta \in (\hat{\theta}_1, \hat{\theta}_2]$ and

$$\int_0^{\hat{\theta}_1} u_a^P(\theta_L, z) f(z) dz = (\hat{\theta}_1 - \theta_L) F(\hat{\theta}_1),$$

implying that

$$\frac{\int_0^{\hat{\theta}_2} u_a^P(\theta_L, z) f(z) dz - (\hat{\theta}_2 - \theta_L) F(\hat{\theta}_2) - \int_0^{\hat{\theta}_1} u_a^P(\theta_L, z) dz + (\hat{\theta}_1 - \theta_L) F(\hat{\theta}_2)}{\hat{\theta}_2 - \hat{\theta}_1} < 0.$$

The mean value theorem then implies that there exists a $\hat{\theta}_3 \in (\hat{\theta}_1, \hat{\theta}_2)$ such that

$$u_a^P(\theta_L, \hat{\theta}_3) f(\hat{\theta}_3) - (\hat{\theta}_3 - \theta_L) f(\hat{\theta}_3) - F(\hat{\theta}_3) < 0$$

Thus,

$$\begin{aligned} \int_0^{\hat{\theta}_3} u_a^P(\theta_L, z) f(z) dz &< (\hat{\theta}_3 - \theta_L) F(\theta) \\ &< \left[u_a^P(\theta_L, \hat{\theta}_3) + (\theta_L - \hat{\theta}_3) \right] f(\hat{\theta}_3)(\hat{\theta}_3 - \theta_L), \end{aligned}$$

which violates the condition (1) from above. Similarly, suppose that condition (c2) is violated at the point $\hat{\theta}_1$. Again, there must exist $\hat{\theta}_2$, with $\theta_H < \hat{\theta}_1 < \hat{\theta}_2$ such that

$$\int_{\theta}^{1} u_a^P(\theta_H, z) f(z) dz > (1 - F(\theta))(\theta - \theta_H)$$

for $\theta \in [\hat{\theta}_1, \hat{\theta}_2)$ with

$$\int_{\hat{\theta}_2}^{1} u_a^P(\theta_H, z) f(z) dz = (1 - F(\hat{\theta}_2))(\hat{\theta}_2 - \theta_h).$$

This gives us that

$$\frac{\int_{\hat{\theta}_2}^{1} u_a^P(\theta_H, z) f(z) dz - (1 - F(\hat{\theta}_2))(\hat{\theta}_2 - \theta_H) - \int_{\hat{\theta}_1}^{1} u_a^P(\theta_H, z) f(z) dz + (1 - F(\hat{\theta}_1))(\hat{\theta}_1 - \theta_H)}{\hat{\theta}_2 - \hat{\theta}_1} < 0$$

Then another use of the mean value theorem shows that $\exists \hat{\theta}_3 \in (\hat{\theta}_1, \hat{\theta}_2)$ such that

$$-u_a^P(\theta_H, \hat{\theta}_3) f(\hat{\theta}_3) - (1 - F(\hat{\theta}_3)) - (\hat{\theta}_3 - \theta_H)(-f(\hat{\theta}_3)) < 0,$$

so $(1 - F(\hat{\theta}_3)) > [-u_a^P(\theta_H, \hat{\theta}_3) - (\theta_H - \hat{\theta}_3)] f(\hat{\theta}_3)$, and

$$\begin{aligned} \int_{\hat{\theta}_3}^{1} u_a^P(\theta_H, z) f(z) dz &> (1 - F(\hat{\theta}_3))(\hat{\theta}_3 - \theta_H) \\ &> [-u_a^P(\theta_H, \hat{\theta}_3 - (\theta_H - \hat{\theta}_3)](\hat{\theta}_3 - \theta_H) f(\hat{\theta}_3), \end{aligned}$$

showing that this must violate condition (5) from above. □

It's unclear how much more general Amador and Bagwell's result is. The results are derived using similar tools (the costate equation used in Proposition A.2 can be interpreted as Lagrange multipliers). One possible difference is that the conditions in Theorem B.1 guarantee that the interval described is optimal *without* imposing monotonicity. In particular, conditions (1) and (5) ensured that even variations that lead to a non-monotonic action profile aren't beneficial to the principal, and these are precisely the conditions for which Theorem B.1 and Amador and Bagwell's Proposition 1 (b) differ.

B.2 Comparison to Ambrus and Egorov (2017)

It is instructive to compare the results on money burning and responsiveness from Section 5 to what optimal delegation looks like when there is a constant bias parameter, similar to Section 4. Drawing on the intuition from the problem of money burning and responsiveness, we might expect that the principal will either use money burning or a cap to prevent the most biased actions. Again, the clearest is one in which the principal faces quadratic losses, and has the utility function

$u^P(a,\theta) = -\frac{1}{2}\left(a(\theta) - \theta + b\right)^2$. With a uniform distribution, the principal is solving

$$\max_{a(\theta),\omega(\theta)} \int_0^1 \left[-\frac{1}{2}\left(a(\theta) - \theta + b\right)^2 + k\omega(\theta) \right] d\theta$$

subject to

$$-\frac{1}{2}\left(a(\theta) - \theta\right)^2 + \omega(\theta) \geq -\frac{1}{2}\left(a(\theta') - \theta\right)^2 + \omega(\theta')$$

for all θ, θ' and

$$\omega(\theta) \leq 0.$$

This is similar to the parametric uniform-quadratic model studied by Ambrus and Egorov (2017), and the differences will be made clearer below.

Once again, $u^P_{aa}(a,\theta)$, $u^P_{a\theta}(a,\theta)$, and $\gamma'(\cdot)$ are constant so when the principal uses money burning, the action varies with the state linearly, now with slope $\frac{2k-1}{k-1}$. Applying the sufficient conditions, one finds that the optimal mechanism takes one of four forms, as given in Theorem B.2.

Theorem B.2 *In the uniform-quadratic delegation setting with money burning, the optimal mechanism can take one of four forms:*

1. *Discretion for low actions, with a cap preventing high actions when* $b \leq \frac{1}{2}$ *and* $k \geq \frac{1}{2}$
2. *Discretion for low actions, and money burning for high actions when* $b \leq k < \frac{1}{2}$
3. *Constant action for low states and money burning for high states when* $b \geq k$ *and* $k \leq \frac{1}{2b+1}$
4. *The optimal uninformed action otherwise*

Proof Similar to the proof of Theorem 6, this result will rely on the fact that if the optimal mechanism doesn't include any money burning, then in all settings with the same bias and higher k, the optimal mechanisms is the same.

Consider the case with $b \leq k \leq \frac{1}{2}$. The proposed action profile is

$$a(\theta) = \begin{cases} \theta \text{ if } \theta \leq \frac{k-b}{k} \\ \frac{2k-1}{k-1}\theta + \frac{b-k}{k-1} \text{ otherwise} \end{cases}.$$

For $\theta > \frac{k-b}{k}$, the envelope condition gives that

$$\begin{aligned} \omega'(\theta) &= (a(\theta) - \theta)a'(\theta) \\ &= \frac{2k^2 - k}{(k-1)^2}\theta + \frac{(b-k)(2k-1)}{(k-1)^2} \end{aligned}$$

Since $\omega\left(\frac{k-b}{k}\right)=0$, for $\theta \geq \frac{k-b}{k}$,

$$\omega(\theta)=\frac{2k^2-k}{2(k-1)^2}\theta^2+\frac{(b-k)(2k-1)}{(k-1)^2}\theta+\frac{2(k-b)(2k-1)-(2k-1)(k-b)^2}{2k(k-1)}.$$

This gives indirect utility function

$$W(\theta)=\begin{cases}0 \text{ if } \theta \leq \frac{k-b}{k} \\ \frac{k}{2(k-1)}\theta^2+\frac{b-k}{k-1}\theta-\frac{(b-k)^2}{2(k-1)^2}+\frac{2(k-b)(2k-1)-(2k-1)(k-b)^2}{2k(k-1)} \text{ otherwise}\end{cases}$$

Finally define

$$p(\theta)=\begin{cases}b \text{ for } \theta \leq \frac{k-b}{k} \\ k-k\theta \text{ otherwise}\end{cases}$$

Notice that $p(\theta)$ satisfies all of the conditions required in Proposition A.2. The final condition to check is that

$$W'(\theta)\in \operatorname*{argmax}_v \; p(\theta)v-\frac{1}{2}(v+b)^2+\frac{1}{2}kv^2$$

which implies that $W'(\theta)=0$ when $\theta<\frac{k-b}{k}$ and $W'(\theta)=\frac{k}{k-1}\theta+\frac{b-k}{k-1}$ for $\theta>\frac{k-b}{k}$, which is the case for the indirect utility function given above. When $k=\frac{1}{2}$ in this solution, the mechanism involves no money burning, and has simple cap on the action at $1-2b$. Thus, this mechanism is also optimal for any $k \geq \frac{1}{2}$.

Next, consider the case with $b>k$ and $k \leq \frac{1}{2b+1}$. Here the proposed action has

$$a(\theta)=\begin{cases}c \text{ if } \theta \leq d \\ \frac{2k-1}{k-1}\theta+\frac{b-k}{k-1} \text{ otherwise}\end{cases},$$

with c and d determined below, $a(\theta)$ continuous, and money burning for $\theta>d$. To show that such a mechanism is optimal, I will find a $p(\theta)$ which satisfies all of the conditions from Proposition A.2, including $p(0)=p(1)=0$. For the strictly increasing portion of the action profile, $p'(\theta)=-k$, so $p(\theta)=(1-\theta)k$ on $(d,1]$. On the flat portion of the profile, it must be the case that

$$p'(\theta)=\frac{1}{c-\theta}p(\theta)-\frac{c-\theta+b}{c-\theta}$$

which with the initial condition $p(0)=0$ has solution $p(\theta)=\frac{\theta(\theta-2b-2c)}{2(c-\theta)}$. Thus, we are solving a system of nonlinear equations:

$$c=\frac{2k-1}{k-1}d+\frac{b-k}{k-1}$$
$$(1-d)k=\frac{d(d-2b-2c)}{2(c-d)}.$$

The equations are simply the requirements that both the action profile and the function $p(\theta)$ be continuous at the point $\theta = d$. Solving this system, we get

$$(2k^2 - 3k + 1)d^2 + (2k - 4k^2)d - 2bk + 2k^2 = 0.$$

For $b > k$, $0 < k < \frac{1}{1+2b}$, this is negative when $d = 1$ and positive when $d = 0$. Thus, there is a unique value for d in $[0, 1]$. Thus, with this c and d defined, we have that

$$p(\theta) = \begin{cases} \frac{\theta(\theta-2b-2c)}{2(c-\theta)} \text{ if } \theta \le d \\ (1-\theta)k \text{ otherwise} \end{cases}$$

Here, $p(0) = p(1) = 0$ and by construction $p'(\theta)$ satisfies the necessary constraints. The last condition to check is that $W'(\theta)$ is the solution to the maximization condition in Proposition A.2. For $\theta \in [0, d]$, naive first order conditions (ignoring the boundary) give that

$$v^* = \frac{\theta(\theta - 2b - 2c)}{2(c-\theta)(1-k)} - \frac{b}{1-k}$$

Since $W'(\theta) = c - \theta$ is on the boundary and the problem is convex, we need to check that

$$\frac{\theta(\theta - 2b - 2c)}{2(c-\theta)(1-k)} - \frac{b}{1-k} \le c - \theta$$

which simplifies to

$$(1 + 2k)\theta^2 - (2c + 4ck)\theta + 2c^2 - 2c^2k + 2bc \le 0.$$

The values c and d are defined such that this holds with equality at $\theta = d$ and is not binding at $\theta = 0$. Since the left hand side of this function is quadratic with a positive coefficient on the quadratic term, the inequality holds everywhere. We also must check that $W'(\theta)$ is the maximizer of

$$(1-\theta)kv - \frac{1}{2}(v+b)^2 + \frac{1}{2}kv^2,$$

for $\theta \ge d$, which it is. Thus, the proposed mechanism is optimal. Furthermore, for $b \ge \frac{1}{2}$, along the boundary $k = \frac{1}{2b+1}$, the optimal c and d are $\frac{1}{2} - b$ and 1, respectively, which implies that the action is flat at $\frac{1}{2} - b$. Since this doesn't use any money burning, for any $k > \frac{1}{2b+1}$ the same mechanism is optimal. $\square$

As b and k vary, the form of the optimal mechanism changes qualitatively. All four forms can be seen in Figure 9. When k is high enough, the principal doesn't use any money burning, because the cost of incentivizing the agent are not worth the benefits of a lower action. In this case, the principal sets an upper bound on the action which can be taken. For high enough bias, this bound is always binding and is set at $\frac{1}{2} - b$, the principal's optimal uninformed action. When the agent is less biased, the principal takes advantage of the agent's superior information by giving him discretion to choose among low actions, and prevents him from taking actions above $1 - 2b$.

When k, the price of burning money, is low enough, the principal *does* use money burning. She uses money burning at higher values of the state to incentivize better actions, and doesn't use any money burning at lower values of the state. Money burning isn't used at lower states because using money burning to lower actions at these values of θ forces the principal to burn more money at higher θ. Thus, at low values of the state, the principal gives the agent discretion when $b \leq k$, and holds the agent at a single action otherwise.

The way in which the optimal mechanism varies with the parameters can be seen in Figure 10. The comparative statics that are implicit within the figure are intuitive. In all cases, when you hold b constant and lower k, the cost of money burning, the principal uses money burning more. Holding k constant and lowering the agent's bias leads to the principal giving the agent more discretion, and using less money burning.

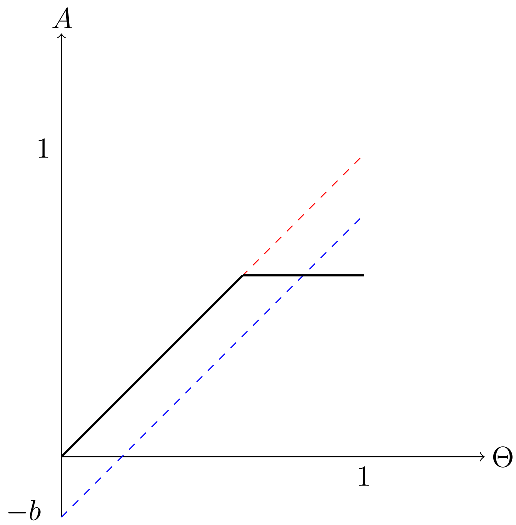
Inspection of the optimal mechanism suggests that there is a relationship between the optimal mechanism found here and that found in the uniform-quadratic section of Ambrus and Egorov (2017). The two mechanisms share the same forms for various combinations of parameters. The clearest comparison occurs when their minimum wage is not binding, but the ex-ante individual rationality constraint is. In their case the principal's payoffs from the allocation are $-A(a-\theta)^2$ and the agents are $-(a-\theta-b)^2$, where A is interpreted as the weight the principal puts on the allocation as compared to money burning. This model can be restated in such a way that it satisfies the assumptions here by instead assuming that the agent is maximizing utility

$$-\frac{1}{2}(a(\theta)-\theta)^2 + \omega(\theta)$$

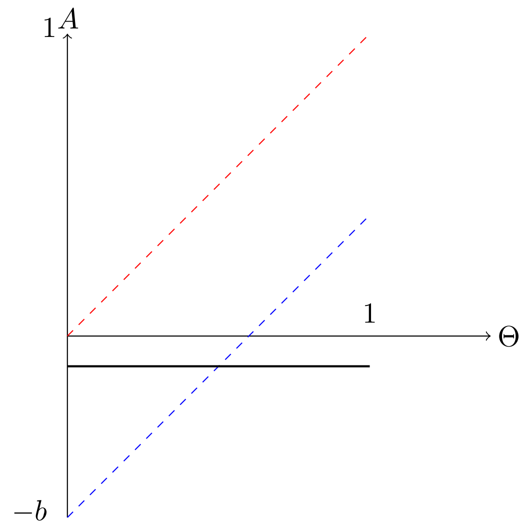
and the principal is maximizing

$$-A(a-\theta+b)^2 - (a-\theta)^2 + 2\omega(\theta),$$

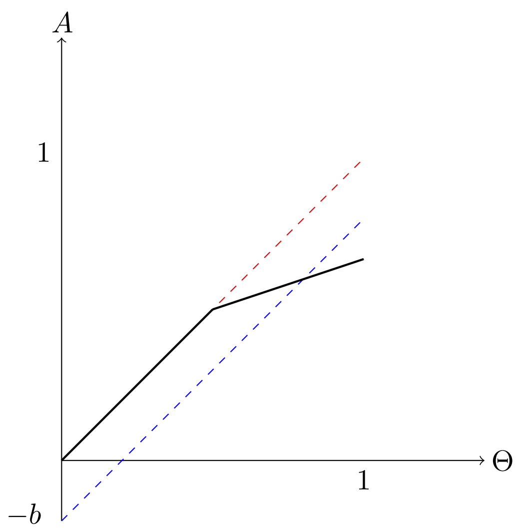
where $\omega(\theta)$ is required to be weakly negative. The principal's preferences here take into account that when the individual rationality constraint in the model studied by Ambrus and Egorov is



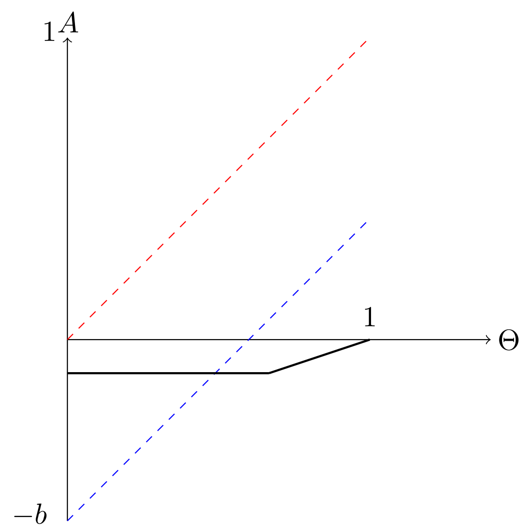
(a) Discretion with a cap: $b = \frac{1}{5}$, $k = 1$



(b) Optimal uninformed action: $b = \frac{3}{5}$, $k = 1$



(c) Discretion with money burning: $b = \frac{1}{5}$, $k = \frac{2}{5}$



(d) Floor with money burning: $b = \frac{3}{5}$, $k = \frac{2}{5}$

Figure 9: The four possible forms of the optimal delegation with money burning mechanism with quadratic losses, a uniform distribution, cost of money burning k, and a constant bias b.

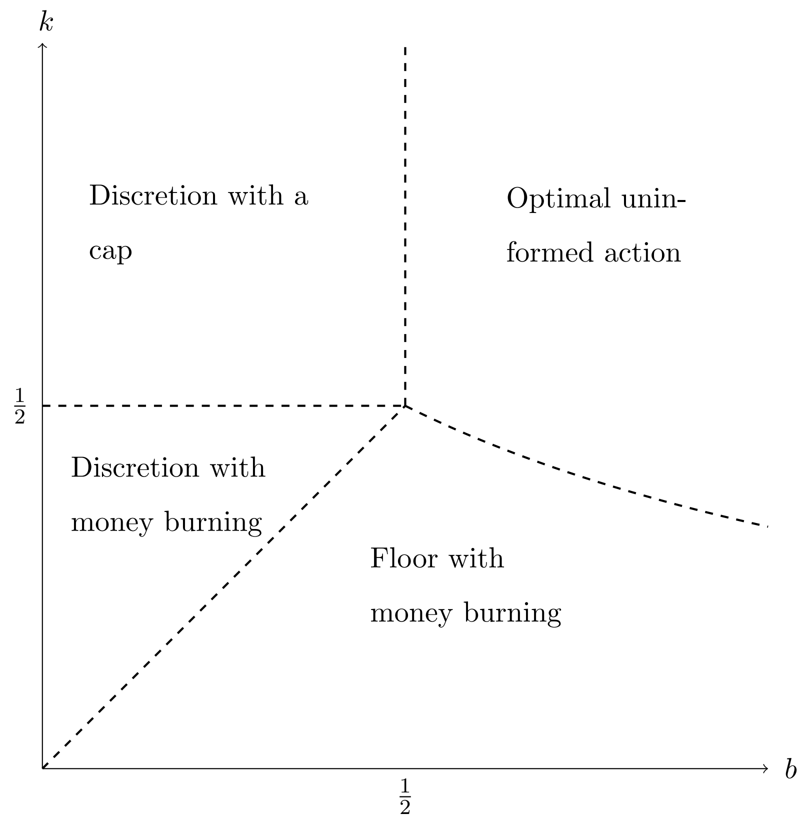


Figure 10: The form of the optimal mechanism by b and k

binding, the principal must make up for any losses that the agent will face from the allocation through the ex-ante transfer. Thus, when solving for the optimal mechanism, the principal will take into account the agent's preferred action when designing the mechanism. Again, because u^P_{aa}, $u^P_{a\theta}$, and γ' are constant, the slope of the action taken when money burning is being used is

$$\begin{aligned} a'(\theta) &= \frac{2\gamma' - u_{a\theta}}{u_{aa} + \gamma'} \\ &= 1 - \frac{1}{A}, \end{aligned}$$

which matches the slope found by Ambrus and Egorov.

C Differential Equation Proofs

This appendix is contains the proofs of various lemmas that are used to prove Proposition 5. From Proposition 2 and the calculations in the main text, we get that the solution to the dynamic delegation problem is characterized by the differential equation

$$W''(\theta) = \begin{cases} -1 + \frac{\delta^{-\frac{1}{2}}\left(-2W(\theta)-W'(\theta)^2-\frac{1}{12}\delta\right)^{\frac{3}{2}}+4b_2W(\theta)+2b_2W'(\theta)^2+\frac{1}{6}\delta b_2}{2b_2W(\theta)+\frac{1}{12}\delta b_2} & \text{if } W(\theta)+\frac{1}{2}W'(\theta)^2 < -\frac{\delta}{6} \\ -1 + \frac{\delta^{-\frac{1}{3}}\left(6W(\theta)+3W'(\theta)^2\right)^{\frac{4}{3}}+24b_2W(\theta)+12b_2W'(\theta)^2}{12b_2W(\theta)+2b_2W'(\theta)^2} & \text{if } -\frac{\delta}{6} \le W(\theta)+\frac{1}{2}W'(\theta)^2 \le 0 \end{cases} \tag{3}$$

with endpoints $(W(0), W'(0))$ and $(W(1), W'(1))$ given by

$$W'(\theta)\gamma'\left(W(\theta)+\frac{1}{2}W'(\theta)^2\right) = -u_a^P\left(W'(\theta)+\theta,\theta\right) \tag{4}$$

for $\theta \in \{0,1\}$. Since equation (3) is defined piecewise, it will be useful in the future to refer to the region $W(\theta)+\frac{1}{2}W'(\theta)^2 < -\frac{\delta}{6}$ as the first region, and $-\frac{\delta}{6} \le W(\theta)+\frac{1}{2}W'(\theta)^2 \le 0$ as the second region.

Lemma C.1 *If the graph of a solution to equation (3) lies in the domain*

$$G \equiv \left\{(W(\theta), W'(\theta) : W(\theta)+\frac{1}{2}W'(\theta)^2 \le -\varepsilon, W(\theta) \ge c\right\}$$

for some $c < 0$ and $\varepsilon > 0$, then the solution is unique.

Proof As in §10 of Filippov and Arscott (1988), define further

$$G^+ \equiv \left\{(W(\theta), W'(\theta) : W(\theta)+\frac{1}{2}W'(\theta)^2 < -\frac{\delta}{6}, W(\theta) \ge c\right\}$$

and

$$G^- \equiv \left\{(W(\theta), W'(\theta)) : -\varepsilon \ge W(\theta)+\frac{1}{2}W'(\theta)^2 \ge -\frac{\delta}{6}, W(\theta) \ge c\right\}$$

so that G^- and G^+ separate G into two domains by the smooth surface $S_1 = \{(W(\theta), W'(\theta)) : W(\theta)+\frac{1}{2}W'(\theta)^2 = -\frac{\delta}{6}\}$. Note that the partial derivatives of $W''(W(\theta), W'(\theta))$ are continuous up to the boundaries of G^- and G^+. Define $W''^-(W(\theta), W'(\theta))$ and $W''^+(W(\theta), W'(\theta))$ as the limiting values of the function $W''(W(\theta), W'(\theta))$ at the point $W(W(\theta), W'(\theta)) \in S_1$ from the regions G^- and G^+ respectively. Define $W_N^-(W(\theta), W'(\theta))$ and $W_N^+(W(\theta), W'(\theta))$ to be the projections of the vectors $[W'(\theta), W''^-(W(\theta), W'(\theta))]$ and $[W'(\theta), W''^+(W(\theta), W'(\theta))]$ onto the normal to S_1 directed from G^- to G^+ at the point $(W(\theta), W'(\theta))$. That is,

$$W_N^-\left(W(\theta), W'(\theta)\right) = \frac{[-1, -W'(\theta)] \cdot [W'(\theta), W''^-(W(\theta), W'(\theta))]}{|[-1, -W'(\theta)]|}$$

We can first find that

$$
\begin{aligned}
&[-1, -W'(\theta)] \cdot [W'(\theta), W''^{-}(W(\theta), W'(\theta))] \\
&\qquad = -W'(\theta) + W'(\theta) + \frac{-W'(\theta)\left[\delta^{-\frac{1}{3}}\left(6W(\theta) + 3W'(\theta)^2\right)^{\frac{4}{3}} + 24b_2W(\theta) - 12b_2W'(\theta)^2\right]}{12B_2W(\theta) + 2b_2W'(\theta)^2} \\
&\qquad = \frac{-W'(\theta)\left[\delta^{-\frac{1}{3}}(-\delta)^{\frac{4}{3}} - 4\delta b_2\right]}{-6b_2W'(\theta)^2 - 2b_2\delta + 2b_2W'(\theta)^2} \\
&\qquad = \frac{W'(\theta)(1-4b_2)}{b_2(4W'(\theta)^2 + 2\delta)}
\end{aligned}
$$

so

$$
W_N^-\left(W(\theta), W'(\theta)\right) = \frac{W'(\theta)(1-4b_2)}{b_2(4W'(\theta)^2 + 2\delta)\sqrt{1 + W'(\theta)^2}}
$$

Similarly,

$$
\begin{aligned}
&[-1, -W'(\theta)] \cdot [W'(\theta), W''^{+}(W(\theta), W'(\theta))] \\
&\qquad = \frac{-W'(\theta)\left[\delta^{-\frac{1}{2}\left(\frac{1}{4}\delta\right)^{\frac{3}{2}} - \frac{1}{2}b_2\delta}\right]}{-b_2\left[W'(\theta)^2 + \frac{1}{4}\delta\right]} \\
&\qquad = \frac{\delta W'(\theta)\left[\frac{1}{8} - \frac{1}{2}b_2\right]}{b_2\left[W'(\theta)^2 + \frac{1}{4}\delta\right]} \\
&\qquad = \frac{\delta W'(\theta)[1-4b_2]}{b_2[4W'(\theta)^2 + \delta]}
\end{aligned}
$$

These have the same sign at any point, thus from Theorem 2 of §10 of Filippov and Arscott (1988), uniqueness occurs at all points in the domain G such that $W'(\theta) \neq 0$.[9,10] □

Lemma C.2 *If the graph of a solution to equation (3) exists in G for $\theta \in [0, 1]$, then the solution is continuous in initial conditions for $\theta \in [0, 1]$.*

Proof This follows immediately from uniqueness and Theorem 2 of §8 in Filippov and Arscott (1988). □

It will be useful to have the point at which both $W''(\theta) = 0$ and $W'(\theta) = 0$. Plugging in the formula for $W''(\theta)$, we get $W(\theta) = -\frac{4}{3}\delta b_2^3$ or $W(\theta) = -\frac{1}{2}b_2^2\delta - \frac{1}{24}\delta$, depending on which of the equation this point is located in.

[9]The theorem requires that the functions W_N^- and W_N^+ be strictly positive or negative. In this case, they can both be 0 for $b_2 = \frac{1}{4}$, bu this value of b_2 actually makes the second order differential equation continuous, implying uniqueness through standard methods.

[10]In fact, the theorem gives conditions for *right uniqueness*, but since this differential equation is autonomous, one can reverse the process (i.e. "run time backwards") to show left uniqueness.

Define $S_2 \equiv \{(s,v) : s = q_1(v)\}$ where

$$q_1(W'(\theta)) = \begin{cases} -\frac{\delta}{2}\left(\frac{b_2}{b_1}\right)^2 W'(\theta)^2 - \frac{1}{2}W'(\theta)^2 - \frac{1}{24}\delta \text{ if} \\ \frac{4}{3}\delta\left(\frac{b_2}{b_1}\right)^3 W'(\theta)^3 - \frac{1}{2}W'(\theta)^2 \text{ otherwise} \end{cases},$$

and

$$q_2(b_2) = \begin{cases} -\frac{4}{3}\delta b_2^3 \text{ for } b_2 \leq \frac{1}{2} \\ -\frac{1}{2}\delta b_2^2 - \frac{1}{24}\delta \text{ otherwise} \end{cases}$$

Lemma C.3 *There exists a solution to equation (3) with either (a)* $(W(1), W'(1)) \in S_2$ *and* $W(\theta_1) \leq q_1(W'(\theta_1))$ *for all* $\theta \in [0,1]$*, or (b)* $(W(0), W'(0)) \in S_1$ *and* $W(\theta) \leq q_1(W'(\theta))$ *for all* $\theta \in [0,1]$*.*

Proof We found above that when $W(\theta) = q_2(b_2)$ and $W'(\theta) = 0$, equation (3) is equal to 0. Thus, $W(\theta) = q_2(b_2)$ is a solution to equation (3) for all θ. For all $W(\theta) \in G$ such that $W(\theta) < q_2(b_2)$, W'' is negative, so if $W(0) = q_2(b_2)$ and $W'(0) < 0$, then $W(\theta) < q_2(b_2)$ and $W'(\theta) < 0$ for all $\theta > 0$ for which a solution exists. Start at $(W(0), W'(0)) = (q_2(b_2), 0)$. By decreasing $W'(0)$ (and holding $W(0)$ constant), it's clear from continuity that one of three things must be true: (a) or (b) above, or both $W(0) < q_1(W'(0))$, $W(1) < q_1(W'(1))$, and $W(\theta) \geq q_1(W'(\theta))$ for some set of θ. Suppose that the last is the case. Then it must be the case that for some θ_a, $W(\theta_1) = q_1(W'(\theta_a))$ and

$$W''(\theta_a) < \begin{cases} \frac{1}{4\delta\left(\frac{b_2}{b_1}\right)^3 W'(\theta)-1} \text{ if } W'(\theta_1) \geq -\frac{b_1}{2b_2} \\ -\frac{1}{\delta\left(\frac{b_2}{b_1}\right)^2+1} \text{ otherwise} \end{cases}$$

The right hand side of the above equation is the second order differential equation for a function whose graph stays along the curve q_1, so to pass through the curve at θ_1 the second derivative of W must be less than it. Similarly, there must be some $\theta_b > \theta_a$ such that $W(\theta_b) = q_1(W'(\theta_b))$ and

$$W''(\theta_b) > \begin{cases} \frac{1}{4\delta\left(\frac{b_2}{b_1}\right)^3 W'(\theta)-1} \text{ if } W'(\theta_1) \geq -\frac{b_1}{2b_2} \\ -\frac{1}{\delta\left(\frac{b_2}{b_1}\right)^2+1} \text{ otherwise} \end{cases}$$

We can plug in $q_1(W'(\theta))$ to equation (3) to determine under what conditions these inequalities hold:

$$W''(\theta) = -1 + \frac{\delta^{-\frac{1}{3}}\left[8\delta\left(\frac{b_2}{b_1}\right)^3 - 3W'(\theta)^2 + 3W'(\theta)^2\right]^{\frac{4}{3}} + 24b_2\left(\frac{4}{3}\delta\left(\frac{b_2}{b_1}\right)^3 W'(\theta)^3\right)}{12b_2\left(\frac{4}{3}\delta\left(\frac{b_2}{b_1}\right)^3 W'(\theta)^3\right) - 6b_2W'(\theta)^2 + 2b_2W'(\theta)^2}$$

$$= \frac{4\delta\left(\frac{b_2}{b_1}\right)^4 W'(\theta)^2 + 4\delta b_2\left(\frac{b_2}{b_1}\right)^3 W'(\theta) + b_2}{4\delta b_2\left(\frac{b_2}{b_1}\right)^3 W'(\theta) - b_2}.$$

Thus, the inequality is fulfilled when

$$\frac{4\delta\left(\frac{b_2}{b_1}\right)^4 W'(\theta)^2 + 4\delta b_2\left(\frac{b_2}{b_1}\right)^3 W'(\theta) + b_2}{4\delta b_2\left(\frac{b_2}{b_1}\right)^3 W'(\theta) - b_2} < \frac{1}{4\delta\left(\frac{b_2}{b_1}\right)^3 W(\theta) - 1}$$

$$4\delta\left(\frac{b_2}{b_1}\right)^4 W'(\theta)^2 + 4\delta b_2\left(\frac{b_2}{b_1}\right)^3 W'(\theta) > 0$$

$$\frac{1}{b_1}W'(\theta) < -1$$

$$W'(\theta) < -b_1$$

In the other region, we can also solve for $W''(\theta)$:

$$W''(\theta) = -1 + \frac{\delta^{-\frac{1}{2}}\left(\delta\left(\frac{b_2}{b_1}\right)^2 W'(\theta)^2\right)^{\frac{3}{2}} - 2\delta b_2\left(\frac{b_2}{b_1}\right)^2 W'(\theta)^2 - 2b_2W'(\theta)^2 - \frac{1}{6}\delta b_2 + 2b_2W'(\theta)^2 + \frac{1}{6}\delta b_2}{-\delta b_2\left(\frac{b_2}{b_1}\right)^2 W'(\theta)^2 - b_2W'(\theta)^2}$$

$$= -1 + \frac{\delta^{-\frac{1}{2}}\left(\delta\left(\frac{b_2}{b_1}\right)^2 W'(\theta)^2\right)^{\frac{3}{2}} - 2\delta b_2\left(\frac{b_2}{b_1}\right)^2 W'(\theta)^2}{-\delta b_2\left(\frac{b_2}{b_1}\right)^2 W'(\theta)^2 - b_2W'(\theta)^2}$$

$\left(W'(\theta)^2\right)^{\frac{3}{2}}$ is positive, so

$$W''(\theta) = \frac{\delta b_2\left(\frac{b_2}{b_1}\right)^2 W'(\theta)^2 + b_2W'(\theta)^2 - \delta\left(\frac{b_2}{b_1}\right)^3 W'(\theta)^3 - 2\delta b_2\left(\frac{b_2}{b_1}\right)^2 W'(\theta)^2}{-\delta b_2\left(\frac{b_2}{b_1}\right)^2 W'(\theta)^2 - b_2W'(\theta)^2}$$

$$= \frac{-\delta\left(\frac{b_2}{b_1}\right)^3 W'(\theta)^3 - \delta b_2\left(\frac{b_2}{b_1}\right)^2 W'(\theta)^2 + b_2W'(\theta)^2}{-\delta b_2\left(\frac{b_2}{b_1}\right)^2 W'(\theta)^2 - b_2W'(\theta)^2}$$

$$= \frac{-\delta\left(\frac{b_2}{b_1}\right)^3 W'(\theta) - \delta b_2\left(\frac{b_2}{b_1}\right)^2 + b_2}{-\delta b_2\left(\frac{b_2}{b_1}\right)^2 - b_2}$$

Thus, we can again check the inequality,

$$\frac{-\delta\left(\frac{b_2}{b_1}\right)^3 W'(\theta) - \delta b_2\left(\frac{b_2}{b_1}\right)^2 + b_2}{-\delta b_2\left(\frac{b_2}{b_1}\right)^2 - b_2} < \frac{1}{-\delta\left(\frac{b_2}{b_1}\right)^2 - 1}$$

$$-\delta\left(\frac{b_2}{b_1}\right)^3 W'(\theta) > \delta b_2\left(\frac{b_2}{b_1}\right)^2$$

$$W'(\theta) < -b_1$$

so along the surface, the first inequality holds if and only if $W'(\theta) < -b_1$. Since $\theta_b > \theta_a$ implies that $W'(\theta_b) < W'(\theta_a)$, the above inequalities cannot both hold. □

Lemma C.4 *In the first region, solutions to equation (3) must satisfy*

$$W'(\theta)^2 = -\delta\left[\frac{2b_2W(\theta) + \frac{\delta b_2}{12}}{W(\theta) - \lambda}\right]^2 - 2W(\theta) - \frac{\delta}{12} \tag{5}$$

while in the second region, they must satisfy

$$\frac{\delta^{\frac{1}{3}}\left[3b_2W(\theta) - \frac{1}{2}b_2W'(\theta)^2\right]}{(6W(\theta) + 3W'(\theta)^2)^{\frac{1}{3}}} = W(\theta) - \lambda \tag{6}$$

for some $\lambda \in \mathbb{R}$.

Proof From Gelfand and Fomin (1963, p.19), we see that a solution must satisfy

$$L(\theta, W(\theta), W'(\theta)) - W'(\theta)L_{W'(\theta)}(\theta, W(\theta), W'(\theta)) = C.$$

Letting $C = -\lambda + \frac{1}{2}b_1^2 + \frac{\delta}{2}b_2^2$, in the first region this gives

$$b_1W'(\theta) + \frac{1}{2}b_1^2 - W(\theta) + \frac{\delta}{2}b_2^2 - \frac{\delta b_2}{6}\left(-\frac{72}{\delta}W(\theta) - \frac{36}{\delta}W'(\theta)^2 - 3\right)^{\frac{1}{2}}$$
$$- b_1W'(\theta) - 6b_2W'(\theta)^2\left(-\frac{72}{\delta}W(\theta) - \frac{36}{\delta}W'(\theta)^2 - 3\right)^{-\frac{1}{2}} = -\lambda + \frac{1}{2}b_1^2 + \frac{\delta}{2}b_2^2$$

and this simplifies, to

$$W'(\theta)^2 = -\delta\left[\frac{2b_2W(\theta) + \frac{\delta b_2}{12}}{W(\theta) - \lambda}\right]^2 - 2W(\theta) - \frac{\delta}{12}$$

In the other region, we have

$$b_1W'(\theta) - W(\theta) + \frac{1}{2}b_1^2 + \frac{\delta}{2}b_2^2 - \frac{\delta b_2}{2}\left(\frac{6}{\delta}W(\theta) + \frac{3}{\delta}W'(\theta)^2\right)^{\frac{2}{3}}$$
$$- b_1W'(\theta) + 2b_2W'(\theta)^2\left(\frac{6}{\delta}W(\theta) + \frac{3}{\delta}W'(\theta)^2\right)^{-\frac{1}{3}} = C,$$

which simplifies to

$$\frac{\delta^{\frac{1}{3}}\left[\frac{1}{2}b_2W'(\theta)^2 - 3b_2W(\theta)\right]}{(6W(\theta)+3W'(\theta)^2)^{\frac{1}{3}}} = W(\theta) - \lambda,$$

and these are the equations above. □

Define v_a and v_b as the first and second roots of $4\delta\left(\frac{b_2}{b_1}\right)^3 v^3 - 3\left(\frac{1}{2} - 2\delta b_1\left(\frac{b_2}{b_1}\right)^3\right)v^2 - 3b_1v - 2\delta b_2^3$, respectively. Also, define v_c as the second root of $4\delta\left(\frac{b_2}{b_1}\right)^3 v^3 - 3\left(\frac{1}{2} - 2\delta b_1\left(\frac{b_2}{b_1}\right)^3\right)v^2 - 3b_1v + \frac{3}{2}b_2\delta + \frac{1}{2}\delta$ and

$$v_d = \begin{cases} v_a \text{ if } v_a \geq -\frac{b_1}{2b_2} \\ \frac{-b_2\delta\left(\frac{b_2}{b_1}\right)^2 - b_2 - \sqrt{\delta^2 b_2^2\left(\frac{b_2}{b_1}\right)^2 + 2b_2^2\delta\left(\frac{b_2}{b_1}\right)^2 + b_2^2 - \left(2\delta\left(\frac{b_2}{b_1}\right)^3 + 2\frac{b_2}{b_1}\right)\left(\frac{1}{24}\delta\frac{b_2}{b_1} + \frac{b_2}{b_1}\left(\frac{2}{3}\delta b_2^3\right)\right)}}{\delta\left(\frac{b_2}{b_1}\right)^3 + \frac{b_2}{b_1}} & \text{otherwise} \end{cases}$$

Lemma C.5 *For $b_2 \leq \frac{1}{2}$, the graph of the solution to equation (3) passing through $(q_1(v_d), v_d)$ approaches $(q_2(b_2), 0)$.*

Proof Suppose that a solution to equation 1 has initial conditions $(W(0), W'(0)) = (q_1(v_d), v_d)$. If this is in the second region, then this implies that

$$\frac{\delta^{\frac{1}{3}}\left[\frac{1}{2}b_2W'(\theta)^2 - 3b_2W(\theta)\right]}{(6W(\theta)+3W'(\theta)^2)^{\frac{1}{3}}} = W(\theta) - \lambda$$

Using the fact that $W(0) = q_1(W'(0)) = \frac{4}{3}\delta\left(\frac{b_2}{b_1}\right)^3 W'(0)^3 - \frac{1}{2}W'(0)^2$, we get

$$\frac{\delta^{\frac{1}{3}}\left[\frac{1}{2}b_2W'(0)^2 - 4\delta b_2\left(\frac{b_2}{b_1}\right)^3 W'(0)^3 + \frac{3}{2}b_2W'(0)^2\right]}{\left(8\delta\left(\frac{b_2}{b_1}\right)^3 W'(0)^3 - 3W'(0)^2 + 3W'(0)^2\right)^{\frac{1}{3}}} = \frac{4}{3}\delta\left(\frac{b_2}{b_1}\right)^3 W'(0)^3 - \frac{1}{2}W'(0)^2 - \lambda$$

We can be simplified to get

$$4\delta\left(\frac{b_2}{b_1}\right)^3 W'(0)^3 - 3\left(\frac{1}{2} - 2\delta b_1\left(\frac{b_2}{b_1}\right)^3\right)W'(0)^2 - 3b_1W'(0) - 3\lambda = 0$$

The only λ for which v_d satisfies this equation is $\lambda = \frac{2}{3}\delta b_2^3$. Alternatively, if $(W(0), W'(0)) = (q_1(v_d), v_d)$ is in the first region, it must be the case that

$$W'(\theta)^2 = -\delta\left[\frac{2b_2W(\theta) + \frac{\delta b_2}{12}}{W(\theta) - \lambda}\right]^2 - 2W(\theta) - \frac{\delta}{12}$$

We can use the fact that here, $W(0) = q_1(W'(0)) = -\frac{\delta}{2}\left(\frac{b_2}{b_1}\right)^2 W'(0)^2 - \frac{1}{2}W'(0)^2 - \frac{1}{24}\delta$, so

$$W'(0)^2 = -\delta \left[\frac{-b_2\delta\left(\frac{b_2}{b_1}\right)^2 W'(0)^2 - b_2 W'(0)^2 - \frac{\delta b_2}{12} + \frac{\delta b_2}{12}}{-\frac{\delta}{2}\left(\frac{b_2}{b_1}\right)^2 W'(0)^2 - \frac{1}{2}W'(0)^2 - \frac{1}{24}\delta - \lambda}\right]^2 + \delta\left(\frac{b_2}{b_1}\right)^2 W'(0)^2 + W'(0)^2 + \frac{1}{12}\delta - \frac{1}{12}\delta,$$

and

$$\left[\frac{\delta}{2}\left(\frac{b_2}{b_1}\right)^3 + \frac{1}{2}\frac{b_2}{b_1}\right] W'(0)^2 + \left[\delta b_2\left(\frac{b_2}{b_1}\right)^2 + b_2\right] W'(0) + \frac{1}{24}\delta\frac{b_2}{b_1} + \frac{b_2}{b_1}\lambda = 0.$$

Again, the only λ for which v_d satisfies this equation is $\lambda = \frac{2}{3}\delta b_2^3$. Since $\lambda = \frac{2}{3}\delta b_2^3$, we can show that there is not point at which $W'(\theta) = 0$ in the first region.

$$-\delta\left[\frac{2b_2 W(\theta) + \frac{\delta b_2}{12}}{W(\theta) - \lambda}\right]^2 - 2W(\theta) - \frac{\delta}{12} = 0$$

$$\frac{\delta b_2^2\left(2W(\theta) + \frac{\delta}{12}\right)^2}{\left(W(\theta) - \frac{2}{3}\delta b_2^3\right)^2} + 2W(\theta) + \frac{\delta}{12} = 0$$

$$\left(W(\theta) - \frac{2}{3}\delta b_2^3\right)^2 + \delta b_2^2\left(2W(\theta) + \frac{\delta}{12}\right) = 0$$

We can simplify this to get

$$W(\theta)^2 + \left(2\delta b_2^2 - \frac{4}{3}\delta b_2^3\right) W(\theta) + \frac{4}{9}\delta^2 b_2^6 + \frac{\delta^2}{12} b_2^2 = 0$$

Note that by the quadratic formula, this doesn't have any solutions if $4b_2^2 - \frac{16}{3}b_2^3 - \frac{1}{3}$ is less than zero, which is exactly when $b_2 \leq \frac{1}{2}$. On the other region, $W'(\theta) = 0$ when $W(\theta) = -\frac{4}{3}\delta b_2^3$, since the solution must satisfy

$$-\frac{\delta^{\frac{1}{3}}\left(3b_2 W(\theta)\right)}{\left(6W(\theta)\right)^{\frac{1}{3}}} = W(\theta) - \frac{2}{3}\delta b_2^3$$

$$-\delta^{\frac{1}{3}} 6^{-\frac{1}{3}} 3b_2 W(\theta)^{\frac{2}{3}} = W(\theta) - \frac{2}{3}\delta b_2^3$$

$$W(\theta) + \delta^{\frac{1}{3}} 6^{-\frac{1}{3}} 3b_2 W(\theta)^{\frac{2}{3}} - \frac{2}{3}\delta b_2^3 = 0$$

One can take the derivative of this to see that it reaches its local maximum at $W(\theta) = -\frac{4}{3}\delta b_2^3$, which is $q_2(b_2)$. Thus, there is only one point less than $q_1(v_d)$ at which $W'(\theta) = 0$ when $\lambda = \frac{2}{3}\delta b_2^3$. Furthermore, there are no values of $W(\theta)$ between $q_1(v_d)$ and $q_2(b_2)$ such that the graph of the solution intersects S_2. Thus, the graph of the solution must approach $(q_2(b_2), 0)$. $\square$

Lemma C.6 *For $b_2 \leq \frac{1}{2}$, the graph of the solution to equation (3) that approaches $(q_2(b_2), 0)$ passes through $(q_1(v_b), v_b)$.*

Proof Since $b_2 \leq \frac{1}{2}$, $q_2(b_2)$ is in the second region. Thus, we know that it satisfies the equation

$$\frac{\delta^{\frac{1}{3}}\left[\frac{1}{2}b_2 W'(\theta)^2 - 3b_2 W(\theta)\right]}{(6W(\theta) + 3W'(\theta)^2)^{\frac{1}{3}}} = W(\theta) - \lambda$$

Since we are considering the graph of the solution that approaches $(q_2(b_2), 0)$, it must be the case that

$$\begin{aligned}
\frac{3b_2\delta^{\frac{1}{3}}q_2(b_2)}{(6q_2(b_2))^{\frac{1}{3}}} &= q_2(b_2) - \lambda \\
\frac{3b_2\delta^{\frac{1}{3}}\left(-\frac{4}{3}\delta b_2^3\right)}{\left(8\delta b_2^3\right)^{\frac{1}{3}}} &= -\frac{4}{3}\delta b_2^3 - \lambda \\
\frac{-4\delta^{\frac{4}{3}}b_2^4}{2\delta^{\frac{1}{3}}b_2} &= -\frac{4}{3}\delta b_2^3 - \lambda \\
-2\delta b_2^3 &= -\frac{4}{3}\delta b_2^3 - \lambda \\
\lambda &= \frac{2}{3}\delta b_2^3
\end{aligned}$$

Since $\lambda = \frac{2}{3}\delta b_2^3$, then the second region, the solution must satisfy

$$\frac{\delta^{\frac{1}{3}}\left[\frac{1}{2}b_2 W'(\theta)^2 - 3b_2 W(\theta)\right]}{(6W(\theta) + 3W'(\theta)^2)^{\frac{1}{3}}} = W(\theta) - \frac{2}{3}\delta b_2^3$$

This intersects the curve $q_1(W'(\theta)) = \frac{4}{3}\delta\left(\frac{b_2}{b_1}\right)^3 W'(\theta)^3 - \frac{1}{2}W'(\theta)^2$ when

$$4\delta\left(\frac{b_2}{b_1}\right)^3 W'(\theta)^2 - 3\left(\frac{1}{2} - 2\delta b_2\left(\frac{b_2}{b_1}\right)^3\right)W'(\theta)^2 - 3b_1 W'(\theta) - 2\delta b_2^3 = 0$$

The second root of this equation must be in the second region, so the graph must approach $(q_1(v_b), v_b)$. $\square$

Lemma C.7 *For $b_2 > \frac{1}{2}$, the graph of the solution to equation (3) passing through*

$$\left(-\frac{1}{2}b_1^2\left(\left(\delta\left(\frac{b_2}{b_1}\right)^2 + 1\right)^{\frac{1}{2}} + 1\right)^2 - \frac{1}{24}\delta, -b_1\left(1 + \left(\delta\left(\frac{b_2}{b_1}\right)^2 + 1\right)^{-\frac{1}{2}}\right)\right)$$

approaches $\left(-\frac{1}{2}\delta b_2^2 - \frac{1}{24}\delta, 0\right)$.

Proof Suppose that the graph of the solution passes through

$$\left(-\frac{1}{2}b_1^2\left(\left(\delta\left(\frac{b_2}{b_1}\right)^2 + 1\right)^{\frac{1}{2}} + 1\right)^2 - \frac{1}{24}\delta, -b_1\left(1 + \left(\delta\left(\frac{b_2}{b_1}\right)^2 + 1\right)^{-\frac{1}{2}}\right)\right).$$

For $b_2 > \frac{1}{2}$, this is in the first region, so

$$b_1^2\left[1+\left(\delta\left(\frac{b_2}{b_1}\right)^2+1\right)^{-\frac{1}{2}}\right]^2 = -\delta\left[\frac{-b_1^2 b_2\left(\left(\delta\left(\frac{b_2}{b_1}\right)^2+1\right)^{\frac{1}{2}}+1\right)^2}{-\frac{1}{2}b_1^2\left(\left(\delta\left(\frac{b_2}{b_1}\right)^2+1\right)^{\frac{1}{2}}+1\right)^2-\frac{1}{24}\delta-\lambda}\right]^2$$
$$+b_1^2\left(\left(\delta\left(\frac{b_2}{b_1}\right)^2+1\right)^{\frac{1}{2}}+1\right)^2,$$

and

$$\pm\frac{1}{1+\delta\left(\frac{b_2}{b_1}\right)^2} = \frac{b_1^2\left(\left(\delta\left(\frac{b_2}{b_1}\right)^2+1\right)^{\frac{1}{2}}+1\right)}{\frac{1}{2}b_1^2\left(\left(\delta\left(\frac{b_2}{b_1}\right)^2+1\right)^{\frac{1}{2}}+1\right)^2+\frac{1}{24}\delta+\lambda},$$

which results in $\lambda = \frac{1}{2}\delta b_2^2 - \frac{1}{24}$ or $\lambda = -\frac{3}{2}\delta b_2^2 - 2b_1^2 - 2b_1^2\left(\delta\left(\frac{b_2}{b_1}\right)^2+1\right)^{\frac{1}{2}} - \frac{1}{24}\delta$. The latter of these is associated with positive $W'(\theta)$, so it must be the case that $\lambda = \frac{1}{2}\delta b_2^2 - \frac{1}{24}\delta$. Next, note that $W(\theta) = -\frac{1}{2}\delta b_2^2 - \frac{1}{24}\delta$ is the only solution to

$$0 = -\delta\left[\frac{2b_2 W(\theta)+\frac{1}{12}\delta b_2}{W(\theta)-\lambda}\right]^2 - 2W(\theta) - \frac{1}{12}\delta$$

in this region. The solution also only intersects the curve $q_1(W'(\theta))$ at one other point,

$$-\frac{1}{2}b_1^2\left[\left(1+\delta\left(\frac{b_2}{b_1}\right)^2\right)^{\frac{1}{2}}-1\right]^2 - \frac{1}{24}\delta > -\frac{1}{2}\delta b_2^2 - \frac{1}{24}\delta.$$

Finally, I'll note that the only point at which the graph of the solution could intersect the curve $W(\theta)+\frac{1}{2}W'(\theta)^2 = -\frac{1}{6}\delta$ is when

$$W(\theta) = \frac{\frac{1}{2}\delta b_2^2 + \frac{1}{6}\delta b_2 - \frac{1}{24}\delta}{1-4b_2}$$

which is greater than $-\frac{1}{2}\delta b_2^2 - \frac{1}{24}\delta$ for all $b_2 > \frac{1}{2}$. Thus, for the graph of a solution to equation (3) to pass through those initial conditions, it must then approach $\left(-\frac{1}{2}\delta b_2^2 - \frac{1}{24}\delta, 0\right)$. □

Lemma C.8 *For $b_2 > \frac{1}{2}$, the graph of the solution to equation (3) passing through $(q_1(v_c), v_c)$ passes through $\left(-\frac{1}{6}\delta, 0\right)$.*

Proof Similar to the proof above, the graph of the solution with these initial conditions is in the second region, so it will be the case that

$$4\delta\left(\frac{b_2}{b_1}\right)^3 W'(0)^3 - 3\left(\frac{1}{2} - 2\delta b_1\left(\frac{b_2}{b_1}\right)^3\right) W'(0)^2 - 3b_1 W'(0) - 3\lambda.$$

The only λ for which v_c satisfies this equation is $\lambda = \frac{1}{2}b_2\delta - \frac{1}{6}\delta$. Suppose that $W(\theta) = -\frac{1}{3}\delta$. This implies that

$$\frac{\delta^{\frac{1}{3}}\left(\frac{1}{2}b_2 W'(\theta)^2 + \frac{1}{2}\delta b_2\right)}{(3W'(\theta)^2 - \delta)^{\frac{1}{3}}} = -\frac{1}{2}\delta b_2,$$

and the only $W'(\theta)$ for which this is true is $W'(\theta) = 0$. With $\lambda = \frac{1}{2}\delta b_2 - \frac{1}{6}\delta$ and $W'(\theta) = 0$, there is no other negative solution for $W(\theta)$. Finally, if the solution passes through the curve $(q_1(W'(\theta)), W'(\theta))$, then it must be the case that

$$4\delta\left(\frac{b_2}{b_1}\right)^3 W'(0)^3 - 3\left(\frac{1}{2} - 2\delta b_1\left(\frac{b_2}{b_1}\right)^3\right) W'(0)^2 - 3b_1 W'(0) - \frac{3}{2}b_2\delta + \frac{1}{6}\delta = 0,$$

but this has two negative roots, only one of which is greater than $-2b_1$, so the solution can't pass through the curve again before passing through $\left(-\frac{1}{6}\delta, 0\right)$. Thus, the graph of the solution must pass through $\left(-\frac{1}{6}\delta, 0\right)$. $\square$

Lemma C.9 *If a solution to equation (3) has* $\left(W(\hat{\theta}), W'(\hat{\theta})\right) = (q_1(-b_1), -b_1)$ *for some* $\hat{\theta}$, *then* $W(\theta) > q_1(W'(\theta))$ *for all* $\theta < \hat{\theta}$ *for which the solution exists.*

Proof For $b_2 \leq \frac{1}{2}$, $\left(W(\hat{\theta}, W'(\hat{\theta}))\right) = (q_1(-b_1), -b_1)$ implies that $\lambda = \frac{2}{3}\delta b_2^3 + \frac{1}{2}b_1^2$. For $W'(\theta) > -b_1$, $(q_1(W'(\theta), W'(\theta)))$ lies completely in the first region, and there is no other point at which the graph of the solution to equation (3) intersects S_1. Consider when $b_2 > \frac{1}{2}$. This implies that $\lambda = \frac{1}{2}\delta b_2^2 + \frac{1}{2}b_1^2 - \frac{1}{24}\delta$, so there is no other point at which the graph of the solution to equation (5) intersects S_2. Suppose that the solution to equation (3) passed into the second region. The normal to S_1 in this region is the vector $\left[-1, 4\delta\left(\frac{b_2}{b_1}\right)^3 W'^2 - W'\right]$. Since along the curve

$$\begin{aligned}
&\left[-1, 4\delta\left(\frac{b_2}{b_1}\right)^3 W'^2 - W'\right] \cdot \left[W'. -1 + \frac{\delta^{-\frac{1}{3}}\left[6W + 3W'^2\right]^{\frac{4}{3}} + 24b_2 W + 12b_2 W'^2}{12b_2 W + 2b_2 W'^2}\right] \\
&\qquad = \left[-1, 4\delta\left(\frac{b_2}{b_1}\right)^3 W'^2 - W'\right] \cdot \left[W', -1 + \frac{16\delta\left(\frac{b_2}{b_1}\right)^4 W'^4 + 32\delta b_2\left(\frac{b_2}{b_1}\right)^3 W'^3}{16\delta b_2\left(\frac{b_2}{b_1}\right)^3 W'^3 - 4b_2 W'^2}\right] \\
&\qquad = 4\delta\frac{1}{b_1}\left(\frac{b_2}{b_1}\right)^3 W'^3 + 4\delta\left(\frac{b_2}{b_1}\right)^3 W'^2 \\
&\qquad = 4\delta\frac{1}{b_1}\left(\frac{b_2}{b_1}\right)^3 W'^2\left[W' + b_1\right]
\end{aligned}$$

and this is always positive for $W' > -b_1$, the graphs of the solution can only pass through S_1 from below to above. □

Proposition 5 *An interior solution to problem (PDD) exists, i.e. there exists a* $W(\theta)$ *which solves (PDD) such that* $\forall\theta$, $W(\theta) + \frac{1}{2}W'(\theta)^2 < \tilde{\varepsilon} < 0$.

Proof Suppose that case (a) from Lemma C.3 were true. The proof actually shows that in this case, there exists a solution to equation (3) whose graph has an endpoint $(W(1), W'(1)) \in S_2$, $W(0) = q_2(b_2)$, and $W'(0) < 0$. Consider modifying the initial conditions, increasing $W(1)$ but keeping $(W(1), W'(1)) \in S_2$. Since the solution with $(W(1), W'(1)) = (q_1(-b_1), -b_1)$ has $W(\theta) > q_1(W'(\theta))$ for all $\theta < 1$, by continuity in initial conditions (proven in Lemma C.2) there must be a point at which $W(0) = q_1(W'(0))$ or a point at which $(W(\theta), W'(\theta))$ is on the boundary of G for $0 < \theta < 1$. Because $W''(W(\theta), 0) > 0$ for all $W(\theta) > q_2(b_2)$, the only way that $(W(\theta), W'(\theta))$ could be on the boundary of G is for $W(1) = -\tilde{\varepsilon} - \frac{1}{2}W'(1)^2$. For $b_2 \leq \frac{1}{2}$, the graph of such a solution would intersect the graph of the solution found in Lemma C.6 and for $b_2 > \frac{1}{2}$, the graph of such a solution would intersect the graph of the solution in Lemma C.8, violating uniqueness. In this case the solution exists with $(W(\theta), W'(\theta)) \in G$ for all θ, and with the given endpoints.

Suppose instead that case (b) from Lemma C.3 is true. Again, shifting the initial conditions along S_2 (decreasing $W(0)$), $W(1)$ must intersect either the graph of the solution identified in Lemma C.5 (for $b_2 \leq \frac{1}{2}$), the graph of the solution identified in Lemma C.7 (for $b_2 > \frac{1}{2}$), or S_2. Uniqueness guarantees that the former two are impossible, implying that there exists initial conditions $(W(0), W'(0)) \in S_2$ such that $(W(1), W'(1))$ lies in S_2. Thus, the solution exists. This also implies that when $b_2 \leq \frac{1}{2}$, the λ associated with the solution is strictly greater than $\frac{2}{3}\delta b_2^3$, and when $b_2 > \frac{1}{2}$, the λ associated with the solution is strictly greater than $\frac{1}{2}b_2\delta - \frac{1}{6}\delta$, i.e. λ must be strictly positive. □