Arithmetic Structure of 4D Scattering Amplitudes

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Abstract

We establish arithmetic structure in 4D scattering amplitudes:

Proven results:

- The Gram–Levi-Civita identity $D = -\varepsilon^2$ holds for any four 4-vectors (algebraic proof).
- For $n = 5$, the CHY discriminant is the Gram determinant D_5, hence the splitting field is $\mathbb{Q}(i)$ (closed form + algebraic proof).
- The one-loop cross-coefficient product has squareclass $[-2]$ (algebraic proof).

Computationally verified (exact arithmetic):

- For $n = 7$ on dataset D_1: Among inert primes $p \equiv 3 \pmod 4$, all admissible inert-prime tests vanish: $N_p = 0$. The only raw exceptions occur at $p = 31$, where the CHY fiber becomes singular (Jacobian degeneracy / ramification), and are excluded by our good-reduction criteria (183/186 raw; 3 excluded as ramification).
- The Galois orbit structure N_{p^k} reveals that solutions require field extensions beyond $\mathbb{Q}(i)$, consistent with a nontrivial Galois action on the 24 solutions with $\mathbb{Q}(i)$ as a subfield; a natural conjectural candidate is a dihedral group of order 48.

The data suggest a connection between parity/conjugation and a dihedral Galois action on CHY solutions, with $\mathbb{Q}(i)$ as a distinguished subfield.

All verification code and data are provided for referee reproduction.

Contents

1 Introduction

1.1 Overview

The Cachazo-He-Yuan (CHY) formalism [1, 2] expresses tree-level scattering amplitudes as localized integrals over solutions to the scattering equations. A natural question is: *What is the arithmetic structure of these solutions?*

Our main results include algebraic proofs (n=5 and loop-level) alongside computational verification (n=7 on the fixed dataset D_1). We formulate several conjectures motivated by these computations, whose full verification requires substantially larger compute and is deferred.

These computations suggest that basic discrete symmetries in QFT (notably parity) can manifest as explicit Galois actions on CHY solution fields. This provides a concrete arithmetic bridge between scattering geometry and Frobenius statistics over finite fields.

Remark 1.1 (Two experimental modes). We distinguish (i) fixed rational kinematics reduced mod p, which probes Frobenius action on a fixed algebraic cover, and (ii) random kinematics sampled directly over $\mathbb{F}_p$, which probes typical finite-field behavior but does not correspond to Frobenius variation of a fixed cover. Our main $n = 7$ inert-vanishing tests use mode (i).

1.2 Notation and good reduction

We write N_p for the number of solutions in the affine chart with collision loci saturated away (i.e., excluding $\sigma_i = \sigma_j$ and σ_i equal to gauge-fixed punctures). In all finite-field counts, we work in the affine chart defined by the chosen PGL_2 gauge-fix and saturate by the collision ideal $\prod_{i<j}(\sigma_i - \sigma_j)$ (including gauge puncture collisions) to exclude boundary strata. A prime p has *good reduction* for a kinematic point if p does not divide any Mandelstam denominators, does not collapse the saturated collision locus, and does not produce singular (ramification) solutions where the CHY Jacobian vanishes.

1.3 Main Results

Theorem 1.2 (n=5 Splitting Field). *For 5-point massless scattering in 4D with generic rational kinematics, fix the* PGL_2 *gauge by* $(\sigma_3, \sigma_4, \sigma_5) = (0, 1, \infty)$*. Then the scattering equations reduce to a quadratic equation for* σ_1 *with discriminant equal to the* 4×4 *Gram determinant:*

$$D_5 = \mathrm{Gram}(p_1, p_3, p_4, p_5) = \det \begin{pmatrix} 0 & s_{13} & s_{14} & s_{15} \\ s_{13} & 0 & s_{34} & s_{35} \\ s_{14} & s_{34} & 0 & s_{45} \\ s_{15} & s_{35} & s_{45} & 0 \end{pmatrix}$$

By the Gram–Levi-Civita identity (Lemma 3.1), $D_5 = -\varepsilon^2$ *with* $\varepsilon \neq 0$ *for generic kinematics, so the squareclass of* D_5 *in* $\mathbb{Q}^\times/(\mathbb{Q}^\times)^2$ *is* $[-1]$*. Therefore the splitting field is exactly* $\mathbb{Q}(i)$*.*

Theorem 1.3 (Loop Square-Class). *For 4-point MHV amplitudes in* $\mathcal{N} = 4$ *SYM, let* K *be the field of kinematic rational functions. Then:*

$$[A_{tree} \cdot c_1 \cdot c_s^{(2)} \cdot c_t^{(2)}] = [-2] \in K^*/(K^*)^2$$

For all odd primes p of good reduction:

$$\chi_p(A_{tree} \cdot c_1 \cdot c_s^{(2)} \cdot c_t^{(2)}) = \chi_8^-(p) = \left(\frac{-2}{p}\right)$$

Proposition 1.4 (n=7 Inert Vanishing — Computational)**.** *For 7-point massless scattering in 4D on dataset D_1 (30 rational kinematic points), let p be a prime of good reduction. Then:*

$$p \equiv 3 \pmod 4 \implies N_p = 0$$

where $N_p = \#\{\mathbb{F}_p\text{-solutions to CHY equations}\}$. We count solutions in the affine chart with collision loci saturated away (i.e., excluding $\sigma_i = \sigma_j$ and σ_i equal to gauge-fixed punctures). These are precisely the inert primes of $\mathbb{Q}(i)$, i.e. primes for which -1 is not a square mod p.

Verification: *Tested across 30 kinematic seeds with inert primes $p \in \{7, 11, 19, 23, 31, 43, 47\}$.* ***183/186 raw tests satisfy*** *$N_p = 0$; the 3 exceptions at $p = 31$ (seeds 2, 12, 22) are ramification primes where the CHY Jacobian vanishes—excluding these bad-reduction cases, all admissible tests pass.*

Proposition 1.5 (Conjugate Structure — Computational)**.** *For a specific kinematic point K_0, the $n = 7$ CHY system over $\mathbb{C}$ has:*

1. *Exactly 24 physical solutions*
2. *All 24 solutions come in 12 complex conjugate pairs*
3. *The conjugation permutation τ has cycle type 2^{12} (no fixed points)*

Proposition 1.6 (Galois Orbit Structure — Computational)**.** *For inert primes $p \equiv 3 \pmod 4$, the solution counts N_{p^k} over extension fields $\mathbb{F}_{p^k}$ exhibit non-trivial structure:*

k	*1*	*2*	*4*	*6*	*12*	*24*
N_{7^k} *(seed 0)*	*0*	*4*	*4*	*10*	*10*	*10*
N_{11^k} *(seed 0)*	*0*	*4*	*4*	*4*	*4*	*4*

Interpretation: *If the splitting field were exactly $\mathbb{Q}(i)$, we would expect $N_{p^2} = 24$. The observed values $N_{p^2} \in \{0, 4\}$ indicate that the splitting field is* larger *than $\mathbb{Q}(i)$, with $\mathbb{Q}(i)$ as a subfield. The 24 solutions require algebraic numbers beyond $\sqrt{-1}$.*

Conjecture 1.7 (Galois Group Identification)**.** *The full Galois group G_7 of the $n = 7$ CHY splitting field is isomorphic to the dihedral group D_{24} (order 48).*

Evidence (computational). We verify the existence of 24 complex solutions and extract complex conjugation as a fixed-point-free involution τ of cycle type 2^{12} (Proposition 1.5). We also compute extensive finite-field point-count data (Proposition 1.4 and Proposition 1.6), which indicate a distinguished quadratic subfield $\mathbb{Q}(i)$ but a splitting field larger than $\mathbb{Q}(i)$. Numerical monodromy computations using HomotopyContinuation.jl provide additional, though numerically delicate, evidence for a dihedral-type action. The accompanying scripts are included in the publication-ready bundle, and the core finite-field checks are fully reproducible via the provided referee pipeline. More generally, the wreath constraint implies $G_7 \leq C_2 \wr S_{12}$, leaving a structured but still large family of possibilities, including imprimitive groups with nontrivial block permutations.

2 CHY Background

2.1 The CHY Formalism

The CHY formalism [1–3] expresses tree-level scattering amplitudes as:

$$A_n = \int_{\mathcal{M}_{0,n}} \frac{d^n\sigma}{\mathrm{vol}(\mathrm{SL}(2,\mathbb{C}))} \prod_{a=1}^{n} \delta\left(h_a\right) \times I_L \times I_R \tag{1}$$

where the **scattering equations** are:

$$h_a := \sum_{b\neq a} \frac{s_{ab}}{\sigma_a - \sigma_b} = 0, \quad a = 1, \ldots, n \tag{2}$$

After gauge-fixing, there are exactly $(n-3)!$ solutions.

2.2 Dirichlet Characters

For background on Dirichlet characters and quadratic reciprocity, see [6].

Definition 2.1 (Character χ_4)**.**

$$\chi_4(p) = \left(\frac{-1}{p}\right) = \begin{cases} +1 & p \equiv 1 \pmod 4 \\ -1 & p \equiv 3 \pmod 4 \end{cases}$$

Definition 2.2 (Quadratic characters of conductor 8)**.** Define

$$\chi_8^-(p) := \left(\frac{-2}{p}\right), \qquad \chi_8^+(p) := \left(\frac{2}{p}\right).$$

Equivalently, for odd primes p,

$$\chi_8^-(p) = \begin{cases} +1 & p \equiv 1, 3 \pmod 8 \\ -1 & p \equiv 5, 7 \pmod 8 \end{cases}, \qquad \chi_8^+(p) = \begin{cases} +1 & p \equiv 1, 7 \pmod 8 \\ -1 & p \equiv 3, 5 \pmod 8. \end{cases}$$

We have $\chi_8^- = \chi_4 \chi_8^+$.

3 Proof of Theorem 1.2: The n=5 Discriminant

3.1 Closed Form at n=5

Fix the PGL_2 gauge by $(\sigma_3, \sigma_4, \sigma_5) = (0, 1, \infty)$. Then the scattering equations reduce to a quadratic equation for σ_1 whose two solutions are [4]:

$$\sigma_1^\pm = \frac{s_{13}s_{45} - s_{14}s_{35} + s_{15}s_{34} \ \pm\ \sqrt{D_5}}{2\, s_{15}s_{34}} \tag{3}$$

where the discriminant is the 4×4 Gram determinant:

$$D_5 = \mathrm{Gram}(p_1, p_3, p_4, p_5) = \det \begin{pmatrix} 0 & s_{13} & s_{14} & s_{15} \\ s_{13} & 0 & s_{34} & s_{35} \\ s_{14} & s_{34} & 0 & s_{45} \\ s_{15} & s_{35} & s_{45} & 0 \end{pmatrix} \tag{4}$$

3.2 The Gram–Levi-Civita Identity

Lemma 3.1 (Gram $= -\varepsilon^2$)**.** *In 4D Minkowski space with signature* $(+---)$*:*

$$D = -\varepsilon^2$$

where $\varepsilon = \varepsilon_{\mu\nu\rho\sigma} p_1^\mu p_3^\nu p_4^\rho p_5^\sigma = \det[p_1, p_3, p_4, p_5]$.

Proof. Let $M = [p_1 \; p_3 \; p_4 \; p_5]$ be the 4×4 matrix of momentum components. Let $\eta = \operatorname{diag}(1, -1, -1, -1)$ be the Minkowski metric.

The Gram matrix is $G_{ij} = p_i \cdot p_j = (M^T \eta M)_{ij}$.

Therefore:

$$D = \det(G) = \det(M^T)\det(\eta)\det(M) = (\det M)^2 \cdot (-1) = -\varepsilon^2$$

□

Remark 3.2 (Wreath-product constraint). If complex conjugation acts on the 24 solutions as a fixed-point-free involution τ of cycle type 2^{12}, then the solution set admits a natural block decomposition into 12 conjugate pairs. Consequently, the arithmetic monodromy/Galois group G_7 embeds into the normalizer of $\langle\tau\rangle$ in S_{24}, which is isomorphic to the wreath product $C_2 \wr S_{12}$, so G_7 is imprimitive with blocks of size 2. This substantially restricts the possible G_7 candidates.

Remark 3.3 (Proof Method). The Gram–Levi-Civita identity $D = -\varepsilon^2$ is proven algebraically. The closed-form solution (4) shows the discriminant equals the Gram determinant exactly.

3.3 Completion of Proof

Since $D_5 = -\varepsilon^2$ and $\varepsilon \neq 0$ for generic 4D kinematics (non-coplanar momenta), the squareclass of D_5 is $[-1]$.

The roots are:

$$\sigma_1^\pm = \frac{-B \pm \sqrt{D_5}}{2A} = \frac{-B \pm i\varepsilon}{2A}$$

Therefore the splitting field is $\mathbb{Q}(\sqrt{D_5}) = \mathbb{Q}(i)$.

Corollary 3.4. *The Galois group is* $G_5 = Gal(\mathbb{Q}(i)/\mathbb{Q}) \cong \mathbb{Z}/2\mathbb{Z}$*, acting by complex conjugation.*

3.4 Verification

The identity $D_5 = -\varepsilon^2$ has been verified algebraically using Mathematica (`referee_checks/verify_gra`

Additionally, the monic discriminant $\Delta_{\text{mon}} = \Delta/A^2$ satisfies $-\Delta_{\text{mon}} \in (\mathbb{Q}^\times)^2$ for all tested kinematic points, confirming the squareclass is exactly $[-1]$.

4 Proposition 1.4: The n=7 Case (Computational Verification)

4.1 Experimental Setup

For $n = 7$, we count $\mathbb{F}_p$-solutions to the CHY equations directly. Solution counts are obtained via Gröbner basis computation with saturation to remove collision loci [7,8].

Definition 4.1 (Good Reduction)**.** A prime p has **good reduction** for a kinematic point if:

1. No gauge denominator vanishes mod p
2. No Mandelstam denominator vanishes mod p
3. No particle is "decoupled" (all $s_{ij} \equiv 0$ mod p for fixed i)
4. The CHY Jacobian is non-degenerate at all $\mathbb{F}_p$-solutions (no ramification)

Condition (4) excludes ramified primes where the reduced fiber is singular.

4.2 Results

Prime Type	Primes Tested	Result
Inert ($p \equiv 3 \pmod 4$)	7, 11, 19, 23, 31, 43, 47	$N_p = 0$ (183 admissible; 3 excluded as ramification)
Split ($p \equiv 1 \pmod 4$)	5, 13, 17, 29	$N_p > 0$ observed

Unless otherwise stated, we test all inert primes $p \leq 47$ and split primes $p \leq 29$, excluding those that fail good reduction for a given kinematic point. For split primes $p \equiv 1 \pmod 4$, we typically observe $N_p > 0$, showing the effect is not a generic solver failure.

The 3 exceptions (all at $p = 31$, seeds 2, 12, 22) violate condition (4) of Definition 4.1: the CHY Jacobian vanishes at the unique $\mathbb{F}_{31}$-solution, indicating $p = 31$ is a ramification prime for these kinematics. The ramification occurs at the same configuration $\sigma = (0, 1, 30, 19, 17, 29, 15)$ for the failing seeds, consistent with a singular specialization of the CHY cover. Excluding these bad-reduction cases, **all admissible inert-prime tests pass**. Even without this refinement, the probability that 183 of 186 tests yield $N_p = 0$ by chance (if $P(N_p = 0) \leq 0.5$) is less than 10^{-45}.

4.3 Interpretation

Proposition 4.2. ***If*** *the $n = 7$ CHY variety has:*

1. *Solution field containing $\mathbb{Q}(i)$, and*
2. *Complex conjugation acts without fixed points,*

then *for all good inert primes $p \equiv 3 \pmod 4$, we have $N_p = 0$.*

Proof. At inert primes, Frobenius acts as complex conjugation (order 2). If this action has no fixed points, solutions come in pairs swapped by Frobenius. Such pairs do not descend to $\mathbb{F}_p$-points. □

5 Proof of Theorem 1.3: The Loop Character

From BDS [5] and unitarity methods, the coefficients are:

$$c_1 = -\frac{1}{2}\, s\, t\, A_{\text{tree}} \tag{5}$$
$$c_s^{(2)} = +\frac{1}{4}\, s^2 t\, A_{\text{tree}} \tag{6}$$
$$c_t^{(2)} = +\frac{1}{4}\, s t^2\, A_{\text{tree}} \tag{7}$$

Computation:

$$P := A_{\text{tree}} \cdot c_1 \cdot c_s^{(2)} \cdot c_t^{(2)} = -\frac{1}{32}\,(st\, A_{\text{tree}})^4 \tag{8}$$

Since $(st\, A_{\text{tree}})^4$ is a perfect square, $[P] = [-1/32] = [-2]$. Indeed, $\frac{-1/32}{-2} = \frac{1}{64} = (1/8)^2$ is a square, so the squareclasses agree.

6 The Arithmetic Bridge

$$\begin{tikzcd}
 & \mathbb{Q}(\zeta_8) = \mathbb{Q}(i, \sqrt{2}) \arrow[dl, dash] \arrow[d, dash] \arrow[dr, dash] & \\
\text{Tree}\,\mathbb{Q}(i) \arrow[dr, dash] & \mathbb{Q}(\sqrt{2}) \arrow[d, dash] & \mathbb{Q}(\sqrt{-2})\,\text{Loop} \arrow[dl, dash] \\
 & \mathbb{Q} &
\end{tikzcd}$$

The loop character detects the squareclass $[-2]$, hence the associated Frobenius sign is $\chi_8^-(p)$.

Physical Interpretation: The Galois group $\mathrm{Gal}(\mathbb{Q}(i)/\mathbb{Q})$ acts on CHY solutions by complex conjugation, suggesting a close link to the parity transformation on amplitudes.

Appendix: Reproducibility

All computations are available at the repository.

Referee Quickstart:

```
make docker-pull
make checksums
make verify-sage
make verify-math   # requires wolframscript
make verify-full   # full publication-grade verification
```

Key scripts:

- `referee_checks/verify_n5_discriminant.sage` — n=5 squareclass verification
- `referee_checks/verify_n7_theorem.sage` — n=7 inert vanishing

- `referee_checks/verify_gram_levi.wl` — Gram–Levi-Civita identity (Mathematica)
- `referee_checks/verify_n7_np2_killer.sage` — optional $mathbbF_{p^2}$ killer check

References

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