When Factorization Stalls at dim = 8: What Constraints to Try Next

Context: 6-point gravity scattering-form search on the De Concini–Procesi wonderful compactification (OS/DCP) of the 6-pt channel arrangement {s_S=0, 2<=|S|<=3}. Current runs report repeated **status=HIT** with **null_dim=8** after intersecting factorization boundaries, i.e., factorization does not reduce the candidate space.

1) What the logs mean

Each INTERSECT step imposes one boundary constraint (a factorization residue condition on a divisor s_S=0). Repeated lines like **Result: status=HIT null_dim=8 bad=0** and **new_dim=8** indicate that the current candidate subspace already satisfies every tested single-boundary factorization condition, so no new linear constraints are being added.

2) Core issue

Tree-level factorization on physical poles is necessary but can be insufficient to uniquely fix the 6-pt gravity form. In practice, factorization can carve out a nontrivial linear subspace (here: dimension 8). To isolate the physical solution (ideally dim 1, up to scale), add constraints that are not equivalent to single-pole factorization.

3) Priority constraints to try (recommended order)

3.1 Enforce full S6 invariance (hard enforcement)

Why: gravity is fully permutation invariant. If the code only *reports* the S6-invariant dimension but does not project/intersect with the invariant subspace, you can remain in a larger space.

- **Linear constraint:** for each generator σ of S6, impose (M_σ – I) c = 0 on coefficient vector c in your chosen basis.
- Intersect these invariance equations with the current candidate space after boundary intersections.
- **Expected impact:** often reduces 8 → smaller quickly; sometimes all the way to 1.

3.2 Use all physical channels (include |S|=2 and |S|=3)

Why: if you are only checking the 10 three-particle channels, you are missing two-particle poles and/or additional boundary strata. For n=6 there are 15 two-particle channels and 10 three-particle channels (mod complements), i.e., 25 physical divisors total.

- Loop over all S with |S|=2 or 3 (identifying S with its complement where appropriate).
- Impose the same residue/factorization structure constraints already implemented.
- **Expected impact:** can cut the space further than 3-particle-only.

3.3 Add iterated residue consistency (boundary-of-boundary constraints)

Why: single-boundary factorization can pass while double residues fail. Physics requires compatibility of sequential factorizations: different orders of taking residues on intersecting divisors must agree (up to sign) and match products of lower-point forms.

- Choose many compatible channel pairs (A,B) whose divisors intersect in the DCP boundary stratification (nested/disjoint as appropriate).
- Impose Res_{s_A=0} Res_{s_B=0} Ω = ± Res_{s_B=0} Res_{s_A=0} Ω and that the result matches the expected wedge product.
- Implement by computing iterated residues in charts and translating into linear constraints on coefficients.
- **Expected impact:** strong new constraints beyond single-pole factorization.

4) Physics-selective constraints (beyond linear factorization)

4.1 Soft limits (gravitational soft theorem)

Why: soft behavior is not equivalent to ordinary factorization and is highly selective for gravity. Take one leg soft (e.g., p6 → ε q) and require the candidate scales with the universal gravity soft factor times the 5-pt object.

- Sample kinematics with p6 = ε q and take ε → 0; compare leading term to S^grav · Ω5 (up to scale).
- Alternative: encode as scaling constraints near a chart degeneration that corresponds to softness.
- **Expected impact:** often collapses remaining ambiguity (8 → 1).

4.2 4D Gram determinant / 4D kinematics restriction

Why: OS/DCP constructions in abstract kinematic space can admit solutions not realizable by 4D massless momenta. Restricting to the 4D Gram locus can eliminate spurious solutions.

- Evaluate candidates on random 4D kinematic points (spinor-helicity or explicit 4-vectors).
- Impose agreement with known physical behavior on the 4D locus, or eliminate solutions that disagree/vanish incorrectly.
- **Expected impact:** further dimension reduction when factorization alone is insufficient.

4.3 BCFW large-z scaling

Why: gravity typically has stronger large-z falloff than Yang–Mills. This UV-style constraint is not encoded by locality + factorization alone.

- Apply a BCFW shift (i,j), sample large z, and require scaling consistent with gravity (commonly ~1/z^2 for good shifts).
- Use as a filter to remove factorizing-but-wrong UV behavior solutions.

5) Ground-truth projection constraints (fastest practical isolator)

If you can compute a reference value for the 6-pt MHV gravity amplitude (e.g., Hodges det'Φ in 4D, or CHY/KLT evaluation), you can directly project the 8D space onto the 1D physical line.

- Pick several random rational 4D kinematic points.
- Compute reference amplitude A_ref using Hodges or KLT (up to an overall scale).
- Evaluate each basis candidate f_i on the same points; solve linear system for coefficients c_i matching A_ref.
- Verify the recovered combination matches across additional random points and charts.

6) Diagnostic reasons dim might not drop (even if constraints are 'right')

If every boundary reports HIT without shrinking dimension, the test itself may be underdetermined or accidentally degenerate on the sampled charts. Before concluding physics non-uniqueness, strengthen the testing protocol.

- Require matching of the **full residue form**, not just a subset of components or a boolean 'structure ok' check.
- Use multiple independent charts per boundary (not only the single 'best chart') and randomize seeds.
- Increase independent sample points per boundary to avoid landing in loci where differences vanish.
- Log which linear constraints are being added (rank increments) to ensure intersections are actually applied.

7) Recommended action plan (fastest path to dim = 1)

Phase	Add constraints	Expected effect
A (linear) easiest	1) Enforce full S6 invariance 2) Include all |S|=2,3 channels 3) Add iterated residue consistency	Major reduction; often enough to reach dim=1.
B (physics-selective)	4) Soft theorem 5) 4D Gram-locus restriction	Selects physical line when factorization allows spurious solutions.
C (projection) verify	6) Match Hodges/CHY/KLT at random points. Cross-validate across more points/charts.	Pins the unique physical combination and confirms robustness.

Appendix: Minimal invariance equation (pasteable)

If the current candidate space is represented by a basis matrix B (columns are basis vectors in a global coefficient basis), then enforcing invariance under σ can be implemented by solving for coefficients x such that:

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(M_σ · B − B) x = 0
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for all generators σ, then replace the candidate basis by B · Nullspace(stacked_constraints).