# B481 / Fall 2022 – Homework 03

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## B.

### Splines

1. Defining a single-segment 3D cubic spline takes several independent constant parameters. A cubic spline requires 4 control points; in 3D space, each of those points requires 3 independent constant parameters (i.e. an $x$, $y$, and $z$ coordinate). As such, you would need 12 independent **constant** parameters.

   *Note:* you also need an independent parameter (usually $t$), but in this situation $t$ is not constant, so I am not including it in the count.

2. 

   (a) 2nd derivative

   (b) Since it is guaranteed that a Bezier curve will always pass through the first and last control point, the two segments would have to share one vertex. Vertices would have to be numbered, for example, 0,1,2,3 for the first spline, then 3,4,5,6 for the second. No derivative is matched automatically between continuous segments.

   (c) With Catmull-Rom cubic splines, the curve would pass through the middle two points, meaning that in order to be continuous, the two segments would share three vertices. For example, 0,1,2,3 for the first spline and 1,2,3,4 for the second. The first derivative is matched automatically.

3. Using the De Casteljau method (AKA a Bezier curve), we would first perform a linear interpolation between points $\mathbf{P}_0$ and $\mathbf{P}_1$, given by $\mathbf{P}_A(t) = t\mathbf{P}_1 + (1-t)\mathbf{P}_0$. Then, we describe a linear interpolation between $\mathbf{P}_1$ and $\mathbf{P}_2$ in terms of the same parameter: $\mathbf{P}_B(t) = t\mathbf{P}_2 + (1-t)\mathbf{P}_1$. Lastly, to describe the resulting curve, we interpolate between the points given by the other linear interpolations:

$$
\begin{aligned}
B(t) &= t\mathbf{P}_B(t) + (1-t)\mathbf{P}_A(t) && (1)\\
&= t(t\mathbf{P}_2 + (1-t)\mathbf{P}_1) + (1-t)(t\mathbf{P}_1 + (1-t)\mathbf{P}_0) && (2)\\
&= t^2\mathbf{P}_2 + t(1-t)\mathbf{P}_1 + (1-t)(t\mathbf{P}_1 + \mathbf{P}_0 - t\mathbf{P}_0) && (3)\\
&= t^2\mathbf{P}_2 + t\mathbf{P}_1 - t^2\mathbf{P}_1 + t\mathbf{P}_1 + \mathbf{P}_0 - t\mathbf{P}_0 - t^2\mathbf{P}_1 - t\mathbf{P}_0 + t^2\mathbf{P}_0 && (4)\\
&= t^2\mathbf{P}_2 - 2t^2\mathbf{P}_1 + 2t\mathbf{P}_1 + t^2\mathbf{P}_0 - 2t\mathbf{P}_0 + \mathbf{P}_0 && (5)\\
&= t^2\mathbf{P}_2 + (2t - 2t^2)\mathbf{P}_1 + (t^2 - 2t + 1)\mathbf{P}_0 && (6)\\
B(t) &= t^2\mathbf{P}_2 + 2(t - t^2)\mathbf{P}_1 + (t-1)^2\mathbf{P}_0 && (7)
\end{aligned}
$$

**C.**

## Clipping

1. Let the vertices of the rectangle be defined by $\mathbf{P}_0 = (50, 10)$, $\mathbf{P}_1 = (200, 10)$, $\mathbf{P}_2 = (200, 130)$, and $\mathbf{P}_3 = (50, 130)$. In other words, the rectangle's vertices are ordered counter-clockwise, and $\mathbf{P}_0$ is located in the lower-left corner. Then the following lines of the form $\vec{n} \cdot (\mathbf{P} - \mathbf{P}_n) = 0$ (where $\mathbf{P}$ is an arbitrary point in $\mathbb{R}^2$) describe the sides of the rectangle:

   $l_0 :\ \langle 0, 1\rangle \cdot (\mathbf{P} - \langle 50, 10\rangle) = 0$
   $l_1 :\ \langle -1, 0\rangle \cdot (\mathbf{P} - \langle 200, 10\rangle) = 0$
   $l_2 :\ \langle 0, -1\rangle \cdot (\mathbf{P} - \langle 200, 130\rangle) = 0$
   $l_3 :\ \langle 1, 0\rangle \cdot (\mathbf{P} - \langle 50, 130\rangle) = 0$

   The subject line through $(0, 0)$ and $(250, 150)$ is described by $\mathbf{P} = \mathbf{P}_A + t\vec{v}$, where $\mathbf{P}_A = \langle 0, 0\rangle$ and $\vec{v} = \langle 250, 150\rangle$. So $l_s(t) = t\langle 250, 150\rangle$.

   First we classify for each line whether our subject line ($l_s$) falls in its positive or negative half-space. This can be accomplished by testing the sign of the inner product between the subject line's direction vector and each line's normal vector. We know that for any two vectors $\vec{v}$ and $\vec{u}$ [and $\Theta$, the angle between them], if $\Theta < 90^\circ$, then $\vec{u} \cdot \vec{v} > 0$. If $\Theta > 90$, then $\vec{u} \cdot \vec{v} < 0$. In other words, if the dot product between a line's normal and the subject direction is positive, then they form an acute angle and thus point into the same half-space.

   $l_0 :\ \langle 0, 1\rangle \cdot \langle 250, 150\rangle = (0 \cdot 250) + (1 \cdot 150) = 150 > 0$
   $l_1 :\ \langle -1, 0\rangle \cdot \langle 250, 150\rangle = (-1 \cdot 250) + (0 \cdot 150) = -250 < 0$
   $l_2 :\ \langle 0, -1\rangle \cdot \langle 250, 150\rangle = (0 \cdot 250) + (-1 \cdot 150) = -150 < 0$
   $l_3 :\ \langle 1, 0\rangle \cdot \langle 250, 150\rangle = (1 \cdot 250) + (0 \cdot 150) = 250 > 0$

   Next, we need to find all intersections of the rectangle's side lines with $l_s$ (recall $\mathbf{P}_A = \langle 0, 0\rangle$, so this term is discarded):

   $$l_0 :\ t = \frac{\vec{n} \cdot (\mathbf{P}_1 - \mathbf{P}_A)}{\vec{n} \cdot \vec{v}} = \frac{\langle 0, 1\rangle \cdot \langle 50, 10\rangle}{150} = \frac{1}{15}$$
   $$l_1 :\ t = \frac{\vec{n} \cdot (\mathbf{P}_1 - \mathbf{P}_A)}{\vec{n} \cdot \vec{v}} = \frac{\langle -1, 0\rangle \cdot \langle 200, 10\rangle}{-250} = \frac{20}{25} = \frac{4}{5}$$
   $$l_2 :\ t = \frac{\vec{n} \cdot (\mathbf{P}_1 - \mathbf{P}_A)}{\vec{n} \cdot \vec{v}} = \frac{\langle 0, -1\rangle \cdot \langle 200, 130\rangle}{-150} = \frac{13}{15}$$
   $$l_3 :\ t = \frac{\vec{n} \cdot (\mathbf{P}_1 - \mathbf{P}_A)}{\vec{n} \cdot \vec{v}} = \frac{\langle 1, 0\rangle \cdot \langle 50, 130\rangle}{250} = \frac{5}{25} = \frac{1}{5}$$

   We lastly pick our $t$ values. For our clipped start point we have the positive-pointing $l_0$ and $l_3$, as well as of course the original (naturally, $t = 0$ at the subject segment's original start point). As such, we pick the maximum of 0, $\frac{1}{15}$, and $\frac{1}{5}$, arriving at $\frac{1}{5}$. So we know that $l_3$ clips $l_s$ at $t = \frac{1}{5}$. Similarly, for our clipped end point we pick the minimum $t$ from $l_1$, $l_2$, and the original end point ($t = 1$). So our line segment is clipped on the end by $l_1$ at $t = \frac{4}{5}$. Finally, with our knowledge that $\frac{1}{5} \le t \le \frac{4}{5}$ in hand, we can calculate the start and end point for the clipped segment:

   $\mathbf{P}_s = \frac{1}{5}(250, 150) = (50, 30)$
   $\mathbf{P}_e = \frac{4}{5}(250, 150) = (200, 120)$

2.

(a) For the following vertices and the clipping rectangle given, these are their codes:

$\mathbf{A} = (79, 0) \to \begin{array}{|c|c|c|c|}\hline 0 & 0 & 0 & 1 \\ \hline\end{array}$

$\mathbf{B} = (240, 200) \to \begin{array}{|c|c|c|c|}\hline 0 & 1 & 1 & 0 \\ \hline\end{array}$

$\mathbf{C} = (40, 40) \to \begin{array}{|c|c|c|c|}\hline 1 & 0 & 0 & 0 \\ \hline\end{array}$

(b) For the following pairs...

**Pair A:** Impossible. The code $\begin{array}{|c|c|c|c|}\hline 0 & 1 & 0 & 1 \\ \hline\end{array}$ would specify that the point is simultaneously above and below, which is impossible.

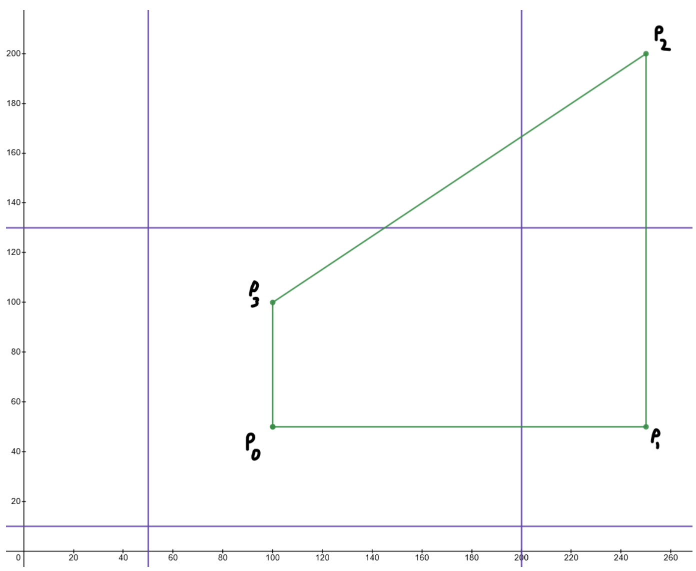
**Pair B:** Could be either partially or completely invisible; more processing is needed.

**Pair C:** Completely visible.

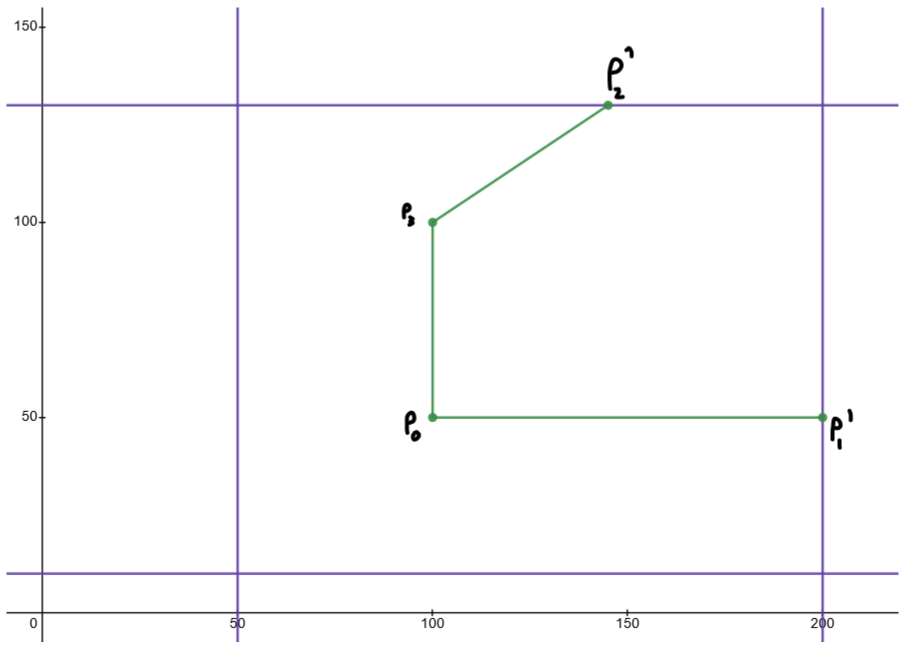
**Pair D:** Impossible. The code $\begin{array}{|c|c|c|c|}\hline 1 & 0 & 1 & 0 \\ \hline\end{array}$ means the point is both to the left and right of the clipping rectangle. That is impossible.

**Pair E:** Partially visible.

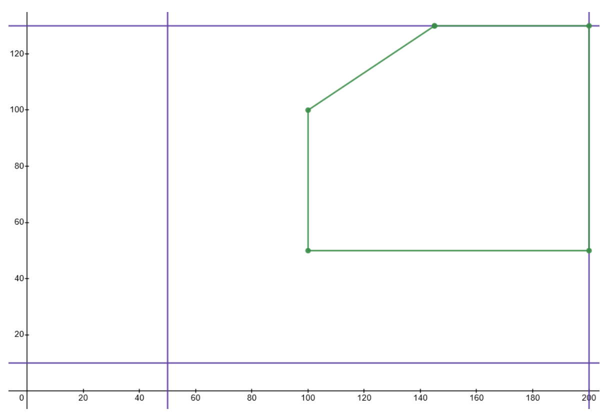
3. Consider the following picture of the polygon and the clipping rectangle:




Clearly the segment $\overline{\mathbf{P}_0\mathbf{P}_1}$ has its end clipped at $(200, 50)$. We also find that the segment $\overline{\mathbf{P}_2\mathbf{P}_3}$ has its start clipped to $(145, 130)$. $\overline{\mathbf{P}_1\mathbf{P}_2}$ is discarded entirely since both of its vertices are to the right of the clip rectangle (i.e. it never crosses the "viewport"). We now have something like the following:




Then we close things up by using the upper-right corner of the clipping rectangle as a new vertex:



We then have the new vertices of our polygon as (in counter-clockwise order),

(100, 50),
(200, 50),
(200, 130),
(145, 130),
(100, 100)