B481 / Fall 2022 – Homework 04

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A.

3D Geometry

1. Assuming that the R_x rotation denotes a rotation about the x-axis, R_y about the y-axis, and so on (that information is not provided in the question, but I think that's what it means), we have:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \cos\Theta_x & -\sin\Theta_x & 0 \\ 0 & \sin\Theta_x & \cos\Theta_x & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\Theta_y & 0 & \sin\Theta_y & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\Theta_y & 0 & \cos\Theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\Theta_z & -\sin\Theta_z & 0 & 0 \\ \sin\Theta_z & \cos\Theta_z & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} \cos\Theta_y \cos\Theta_z & -\cos\Theta_y \sin\Theta_z & \sin\Theta_y & 0 \\ \sin\Theta_x \sin\Theta_y \cos\Theta_z + \cos\Theta_x \sin\Theta_z & \cos\Theta_x \cos\Theta_z - \sin\Theta_x \sin\Theta_y \sin\Theta_z & -\sin\Theta_x \cos\Theta_y & 0 \\ \sin\Theta_x \sin\Theta_z - \cos\Theta_x \cos\Theta_z \sin\Theta_y & \cos\Theta_x \sin\Theta_y \sin\Theta_z + \sin\Theta_x \cos\Theta_z & \cos\Theta_x \cos\Theta_y & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

2. Trivially, yes, but in a general sense no. In other words, the claim that you can obtain the same result by changing the order of transformations is neither universally true, or universally untrue. For example, when translating by $(0, 0, 0)$, mathematically, the translation matrix becomes $T = I_4$ (the identity matrix), which commutes: $T \cdot R \cdot S = I_4 \cdot R \cdot S = R \cdot I_4 \cdot S = R \cdot T \cdot S$. The same applies when scaling by $(1, 1, 1)$ or rotating by 0.

   This is, of course, not always true; to disprove that this always holds, we only need existential proof (i.e. a counterexample). Imagine we translate by $(1, 0, 0)$, rotate by 0 (so $R = I_4$ regardless of the axis of rotation), and scale by $(2, 0, 0)$:

$$T \cdot R \cdot S = T \cdot I_4 \cdot S = T \cdot S = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$$

$$S \cdot R \cdot T = S \cdot I_4 \cdot T = S \cdot T = \begin{bmatrix} 2 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 0 & 0 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \neq T \cdot R \cdot S$$

B.

3D Transformations

1. $\begin{bmatrix} \cos\Theta & 0 & \sin\Theta & 0 \\ 0 & 1 & 0 & 0 \\ -\sin\Theta & 0 & \cos\Theta & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

2. $\begin{bmatrix} \cos\Theta & -\sin\Theta & 0 & 0 \\ \sin\Theta & \cos\Theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$

3. First, find the unit vector representing the direction of the line: $\langle 0, 0, 1\rangle$. To rotate a given point about the line segment, perform the following operations: translate by $(-1, -1, 0)$, rotate about $\langle 0, 0, 1\rangle$ (which happens to be the z-axis, making this problem significantly easier), then translate back by $(1, 1, 0)$.

$$M = \begin{bmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 & 0 \\ \sin\Theta & \cos\Theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{1}$$

$$= \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 & 1 \\ \sin\Theta & \cos\Theta & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & -1 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{2}$$

$$= \begin{bmatrix} \cos\Theta & -\sin\Theta & 0 & \sin\Theta - \cos\Theta + 1 \\ \sin\Theta & \cos\Theta & 0 & 1 - \sin\Theta - \cos\Theta \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \tag{3}$$