Definitions

1. nodes (Leaf _) = 1
2. nodes (Node left _ right) = 1 + (nodes left + nodes right)
3. sumData (Leaf n) = n
4. sumData (Node left n right) = n + (sumData left + sumData right)
5. iNodes (Leaf _) = 0
6. iNodes (Node left _ right) = 1 + (iNodes left + iNodes right)
7. leaves (Leaf _) = 1
8. leaves (Node left _ right) = leaves left + leaves right

Claim: In a tree ***t*** where every node is mapped to have a data of 1, the sum of all of the data in this tree is equal to the total number of nodes.

Proof:

- Base case (**t** = Leaf _):
  - Because nodes (Leaf _) = 1 and **t** = Leaf _ , nodes **t** = 1 (Definition 1)
  - Since sumData (Leaf n) = n, **t** = Leaf n, and n = 1, sumData **t** = 1 (Definition 3)
- Inductive step (**t** = Node left _ right, s.t. sumData left = nodes left and sumData right = nodes right):
  - **t** = Node left _ right, so nodes **t** = 1 + (nodes left + nodes right) (Definition 2)
  - Similarly, sumData **t** = n + (sumData left + sumData right) (Definition 4)
  - Since n = 1, sumData **t** = 1 + (sumData left + sumData right) (Substitution)
  - Because sumData left = nodes left, and because sumData right = nodes right, sumData **t** = 1 + (nodes left + nodes right) (Inductive Hypothesis)
  - So sumData **t** = nodes **t** (Substitution)

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Claim: The number of leaves in a tree is equal to the number of internal nodes + 1

<u>Proof:</u>

- Base case (**t** = Leaf _):
    - Since **t** = Leaf _ and leaves (Leaf _) = 1, leaves **t** = 1 (Definition 7)
    - Similarly, iNodes **t** = 0 (Definition 5)
    - Because 1 = 0 + 1, leaves **t** = iNodes **t** + 1 (Substitution)
- Inductive step (**t** = Node left _ right, s.t. leaves left = iNodes left + 1 and leaves right = iNodes right + 1):
    - **t** = Node left _ right, so leaves **t** = leaves left + leaves right (Definition 8)
    - Similarly, iNodes **t** = 1 + (iNodes left + iNodes right) (Definition 6)
    - We know that leaves left = iNodes left + 1, and that leaves right = iNodes right + 1, so we must know that leaves **t** = (iNodes left + 1) + (iNodes right + 1) (Inductive Hypothesis)
    - So leaves **t** = iNodes left + iNodes right + 1 + 1
    - Since iNodes **t** = 1 + (iNodes left + iNodes right), iNodes **t** = iNodes left + iNodes right + 1
    - Therefore, leaves **t** = iNodes **t** + 1 (Substitution)

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Claim: The total number of nodes is equal to 1 + 2 * the number of internal nodes

Claim: nodes **t** = 1 + 2*(iNodes **t**)

Proof:

- Base case (**t** = Leaf _):
  - Because **t** = Leaf _ , nodes **t** = 1 (Definition 1)
  - Similarly, iNodes **t** = 0 (Definition 5)
  - So then nodes **t** = 1 + 0
  - nodes **t** = 1 + 2*(0)
  - nodes **t** = 1 + 2*(iNodes **t**) (Substitution)
- Inductive step (**t** = Node left _ right, s.t. nodes left = 1 + 2*(iNodes left) and nodes right = 1 + 2*(iNodes right))
  - Because **t** = Node left _ right, nodes **t** = 1 + (nodes left + nodes right) (Definition 2)
  - Similarly, iNodes **t** = 1 + (iNodes left + iNodes right) (Definition 6)
  - We know that…
    - nodes left = 1 + 2*(iNodes left) and
    - nodes right = 1 + 2*(iNodes right), so we must also know that
    - nodes **t** = 1 + ((1 + 2*(iNodes left)) + (1 + 2*(iNodes right))) (Inductive Hypothesis)
  - So nodes **t** = 1 + (1 + 1 + 2*(iNodes left) + 2*(iNodes right)) (Comm./Assoc.)
  - Then nodes **t** = 1 + (2 + 2*(iNodes left) + 2*(iNodes right))
  - nodes **t** = 1 + 2*(1 + iNodes left + iNodes right) (Distributivity of * over +)
  - Since, iNodes **t** = 1 + (iNodes left + iNodes right),
    nodes **t** = 1 + 2*(iNodes **t**)

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