Formal Models: Section 1 Exercises*

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Exercise 1.a

As a group, write down and define the following sets:

- N, which is the set composed of your group members
- A, which is the set composed of your group members' ages, in years
- M, which is the set composed of just yourself
- U, which is the set composed of everyone you collaborated on this question with
- T, which is the set defined as myself and Sean
- G, which is the set composed of everyone you've heard lecture for this class so far.

Imagine I was working on this problem set with a group of three people, Andrew, Tom and myself. Then:

$$N = \{\text{"Andrew"}, \text{"Tom"}, \text{"Zach"}\}$$
$$A = \{27, 26, 28\}$$
$$M = \{\text{"Zach"}\}$$
$$U = \{\text{"Andrew"}, \text{"Tom"}\}$$
$$T = \{\text{"Sean"}, \text{"Zach"}\}$$
$$G = \{\text{"Zach"}, \text{"Sean"}\}$$

Note one small trick here: assuming you collaborated with everyone in your group, you might think that we should say $U = \{\text{"Andrew"}, \text{"Tom"}, \text{"Zach"}\}$, but it's not correct to say that individuals collaborated with themselves so we omit yourself from the set.

*While many exercises are original work, some draw on materials from Tak-Huen Chau. Any errant mistakes are mine alone.

Exercise 1.b

- Does $N = M$? Does $N = U$? Does $T = G$?
- Is U a subset of N?
- Is M a subset of N?
- Is T a subset of G?
- Is A a subset of $\mathbb{R}$?

- From above, and recalling the definition of **set equality** (review the section handout if needed), we can see that there are elements of N that are not included in either M or U. So, $N \neq M$ and $N \neq U$.
- Again, review the definition of **subset** if necessary. Checking the sets defined above (essentially, assuming you collaborated with your group members), we see that every element of U is included in N, so $U \subset N$.
- Similarly, you are one of your group members. So, every element of M is included in N and $M \subset N$.
- This is a small trick: note that T and G are equal, so it remains true that every element of T is included in G and thus $T \subset G$. Subsets can be smaller **or equal** to the original set.
- Every element in A should be an integer, and thus A is a subset of $\mathbb{R}$.

Exercise 2.a

Translate the following statements into plain English:

- 2.1.1: $\forall x \in \mathbb{R}, \exists y \in \mathbb{R}$ such that $y > x$
- 2.1.2: $A \subseteq B \land B \subseteq C \implies A \subseteq C$
- 2.1.3: $\forall x \in [0,1], x^2 \in [0,1]$

- 2.1.1: “For all x in the set of real numbers, there exists some y in the set of real numbers such that y is greater than x.”
- 2.1.2: “The set A being a subset of B and the set B being a subset of C implies that A is a subset of C.”
- 2.1.3: “For all x in the closed interval from zero to one, x^2 is within the closed interval from zero to one.”

Exercise 2.b

Translate the following statements into formal math:

- 2.2.1: “There exists a real number that is greater than 5 and less than 6.”
- 2.2.2: “If x is an element of set A, and A is a subset of B, then x is an element of set B.”
- 2.2.3: “For all elements in the open interval from negative 1 to 1, the absolute value of that element is less than 1.”

- 2.2.1: $\exists x \in \mathbb{R}$ such that $5 < x < 6$
- 2.2.2: $x \in A \land A \subseteq B \implies x \in B$
- 2.2.3: $\forall x \in (-1, 1), |x| < 1$

Exercise 3

Define a binary relation L over a set of all humans, where aLb means that a "loves" b. Is L a preference relation?

No. L is not a preference relation.

Strategy: To show that L is not a preference relation, we need to show that at least one of the two required properties (completeness or transitivity) fails to hold. To do this, we use proof by counterexample. This is a strategy used in proofs to show some statement is not always true. Here, the statement would be "L is a preference relation." To disprove it, we first remember the assumptions necessary for L to be a preference relation.

Recall: From lecture/our handout, a preference relation must satisfy both:

- **Completeness:** For any two elements a, b in the set, either aLb or bLa (or both).
- **Transitivity:** If aLb and bLc, then aLc.

Proof: So, to disprove this using proof by counterexample, we need to find at least one example for which L is not complete (ie. find two elements for which it is not true that aLb or bLa) or for which L is not transitive (ie. find a set of three elements for which it is not true that if aLb and bLc, then aLc). Remember also that the elements here are people. Arbitrarily, I will show that L violates completeness.

Consider two humans: myself and Cameron Winter. Both of us are elements of the set of all humans. However:

- I do not love Cameron Winter, so me $\not L$ Cameron Winter
- Cameron Winter does not love me, so Cameron Winter $\not L$ me

This violates completeness. For L to be complete, it is required that for *any* pair of humans, we must be able to say that either aLb or bLa.

Since we found a counterexample where neither relation holds, L is not complete. Since L fails to satisfy completeness (one of the two required properties), L is not a preference relation.

Note: We could also show that L violates transitivity. For example, if Taylor Swift loves Travis Kelce, and Travis Kelce loves Patrick Mahomes, it doesn't necessarily follow that Taylor Swift loves Patrick Mahomes. Either violation is sufficient to prove L is not a preference relation.