Formal Models: Section 2 Exercises*

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Exercise 1

A lottery pays \$25 with probability 0.6 and \$0 with probability 0.4.

For each decision-maker with the following utility functions:

- (a) Calculate the expected monetary value
- (b) Calculate the expected utility
- (c) Find the certainty equivalent
- (d) Classify their risk attitude

Part i: $u(x) = 3x + 5$

1.i.a: Calculate the expected monetary value.

Recall that the expected monetary value (EMV) of a lottery is $\sum_{i=1}^{n} p_i \cdot x_i$, where p_i is the probability of outcome x_i.

We have two outcomes: the DM wins \$25 with probability 0.6, or wins \$0 with probability 0.4. So:

$$\text{EMV} = 0.6(25) + 0.4(0) = 15 + 0 = 15$$

The lottery has an expected monetary value of \$15.

1.i.b: Calculate the expected utility.

Recall also that the expected utility is $\mathbb{E}[u] = \sum_{i=1}^{n} p_i \cdot u(x_i)$. We need to first calculate how much utility this particular decision-maker gets from each outcome using their utility function, then take the probability-weighted average of those utilities.

First, find the utility of each outcome.

- If they win \$25, $x = 25$ so: $u(25) = 3(25) + 5 = 75 + 5 = 80$ jollies
- If they win \$0, $x = 0$ and: $u(0) = 3(0) + 5 = 0 + 5 = 5$ jollies

*While many exercises are original work, some draw on materials from Tak-Huen Chau. Any errant mistakes are mine alone.

Then, we use these utilities in combination with the probability each outcome happens to calculate the expected utility:

$$\mathbb{E}[u] = 0.6 \cdot u(25) + 0.4 \cdot u(0) = 0.6(80) + 0.4(5) = 48 + 2 = 50$$

This person expects to receive 50 jollies from entering the lottery.

1.i.c: Find the certainty equivalent.

Remember from the handout that the *certainty equivalent* is the certain amount of money that gives the decision-maker exactly the same utility as the lottery. In other words, we're looking for the dollar amount c where the person is indifferent between receiving c with certainty and entering the lottery.

We need to find c such that $u(c) = \mathbb{E}[u]$. We just calculated that $\mathbb{E}[u] = 50$, so:

$$\begin{aligned} u(c) &= 50 \\ 3c + 5 &= 50 \\ 3c &= 45 \\ c &= 15 \end{aligned}$$

The certainty equivalent is \$15. This person is indifferent between receiving \$15 with certainty and entering the lottery.

1.i.d: Classify their risk attitude.

From the above, we can conclude that this person is **risk-neutral**.

How do we know? There are a few different ways to see this:

1. **From the shape of the utility function:** The utility function $u(x) = 3x + 5$ is linear (it has the form $u(x) = a + bx$). From the handout, we know that linear utility functions always represent risk-neutral preferences.

2. **By comparing EMV and certainty equivalent:** The expected monetary value is \$15, and the certainty equivalent is also \$15. We defined risk neutrality as this exact situation: a risk-neutral decision-maker values the risky lottery exactly the same as receiving its expected value with certainty. They neither need extra payoff to take on risk (risk-averse) nor are they willing to pay to take on risk (risk-loving).

3. For linear utility functions, the second derivative $u''(x) = 0$, which is another way to identify risk neutrality.

Part ii: $u(x) = x^{\frac{1}{2}} + 2$

1.ii.a: Calculate the expected monetary value.

The lottery itself hasn't changed, so:

$$\text{EMV} = 0.6(25) + 0.4(0) = 15$$

The expected monetary value is still \$15.

1.ii.b: Calculate the expected utility.

Again, use this person's utility function to calculate their jollies for each outcome:

- If they win \$25: $u(25) = 25^{\frac{1}{2}} + 2 = 5 + 2 = 7$ jollies
- If they win \$0: $u(0) = 0^{\frac{1}{2}} + 2 = 0 + 2 = 2$ jollies

Remember that $x^{\frac{1}{2}} = \sqrt{x}$, so $25^{\frac{1}{2}} = \sqrt{25} = 5$.

Now calculate the expected utility:

$$\mathbb{E}[u] = 0.6 \cdot u(25) + 0.4 \cdot u(0) = 0.6(7) + 0.4(2) = 4.2 + 0.8 = 5$$

This person expects to receive 5 jollies from the lottery.

1.ii.c: Find the certainty equivalent.

Recall that to solve this, we need to find the certain amount c such that $u(c) = \mathbb{E}[u]$. We just calculated that $\mathbb{E}[u] = 5$, so:

$$\begin{aligned} u(c) &= 5 \\ c^{\frac{1}{2}} + 2 &= 5 \\ \sqrt{c} + 2 &= 5 \\ \sqrt{c} &= 3 \\ c &= 3^2 = 9 \end{aligned}$$

The certainty equivalent is \$9. This person is indifferent between receiving \$9 with certainty and entering the lottery.

1.ii.d: Classify their risk attitude.

This person is **risk-averse**. As before, there are a few different ways to get here:

1. **From the shape of the utility function:** The utility function $u(x) = x^{\frac{1}{2}} + 2 = \sqrt{x} + 2$ is concave (curves downward). We can verify this by taking derivatives with respect to x:
   - First derivative: $u'(x) = \frac{1}{2}x^{-\frac{1}{2}} = \frac{1}{2\sqrt{x}} > 0$ for $x > 0$
   - Second derivative: $u''(x) = -\frac{1}{4}x^{-\frac{3}{2}} = -\frac{1}{4x^{\frac{3}{2}}} < 0$ for $x > 0$

Since $u''(x) < 0$, the function is concave. This means the person has diminishing marginal utility — each additional dollar is worth less to them than the previous dollar. Note that the constant (the $+2$) in the utility function doesn't affect attitudes towards risk, it just shifts all utilities up by 2 jollies. What matters for risk attitudes is the curvature of the function, which comes from the $\sqrt{x}$ part.

2. **By comparing EMV and the certainty equivalent:** The expected monetary value is \$15, but the certainty equivalent is only \$9. The certainty equivalent is *less than* the EMV. This means the person would accept considerably less money with certainty (giving up \$6 in expected value!) rather than face the risk of the lottery. This is the defining characteristic of risk aversion.

Part iii: $u(x) = x^2$

1.iii.a: Calculate the expected monetary value.

This is still the same lottery:

$$\text{EMV} = 0.6(25) + 0.4(0) = 15$$

1.iii.b: Calculate the expected utility.

Using this person's utility function:

- If they win \$25: $u(25) = 25^2 = 625$ jollies
- If they win \$0: $u(0) = 0^2 = 0$ jollies

Now calculate the expected utility:

$$\mathbb{E}[u] = 0.6(625) + 0.4(0) = 375 + 0 = 375$$

This person expects to receive 375 jollies from the lottery.

1.iii.c: Find the certainty equivalent.

We need to find c such that $u(c) = 375$:

$$\begin{aligned} c^2 &= 375 \\ c &= \sqrt{375} = \sqrt{25 \cdot 15} = 5\sqrt{15} \approx 19.36 \end{aligned}$$

The certainty equivalent is approximately \$19.36. This person would need to be offered about \$19.36 with certainty before they'd be willing to give up the lottery.

1.iii.d: Classify their risk attitude.

This person is **risk-loving** (or risk-tolerant/risk-acceptant). Again, we prove this in a few ways:

1. **From the shape of the utility function:** The utility function $u(x) = x^2$ is convex (curves upward). We can verify this:

   - First derivative: $u'(x) = 2x$
   - Second derivative: $u''(x) = 2 > 0$

   Since $u''(x) > 0$, the function is convex, which means the person has increasing marginal utility. Each additional dollar is worth *more* to them than the previous dollar, which creates risk-loving behavior.

2. **By comparing EMV and certainty equivalent:** The expected monetary value is \$15, but the certainty equivalent is about \$19.36. The certainty equivalent *exceeds* the EMV! This means you'd have to pay this person more than the lottery's expected value to get them to give up the gamble. They actually enjoy the risk — they're willing to give up certain money to take on uncertainty. This is the defining characteristic of risk-loving behavior.

Exercise 2

A voter supports a policy reform if their personal benefit B_i exceeds the cost C. The benefit has an idiosyncratic component: $B_i = \beta + \epsilon_i$, where β is a base benefit and ϵ_i follows a distribution with CDF G and density g (symmetric around 0, single-peaked).

The probability voter i supports reform is: $P_i = 1 - G(C - \beta)$

Let's take a second to restate what we know here. Voter i supports the reform if their benefit $B_i = \beta + \epsilon_i$ exceeds the cost C. That is, they support it if:

$$\begin{aligned} \beta + \epsilon_i &> C \\ \epsilon_i &> C - \beta \end{aligned}$$

Since ϵ_i is random (following distribution G), the probability of support is:

$$\begin{aligned} P_i &= P(\epsilon_i > C - \beta) \\ &= 1 - P(\epsilon_i \leq C - \beta) \\ &= 1 - G(C - \beta) \end{aligned}$$

This is the formula we're given. Now we want to understand how this probability changes when we vary β or C.

(a) What is $\frac{\partial P_i}{\partial \beta}$? Calculate and interpret it.

$\frac{\partial P_i}{\partial \beta}$ tells us how the probability of supporting reform changes as β, or benefits to the voter, increases. We need to take the derivative of $P_i = 1 - G(C - \beta)$ with respect to β. Since G is a function of $(C - \beta)$, and $(C - \beta)$ is itself a function of β, we'll need to use the chain rule.

The chain rule states that if we have a composite function $f(g(x))$, then:

$$\frac{df}{dx} = \frac{df}{dg} \cdot \frac{dg}{dx}$$

We also will rely on a useful property flagged in the handout: for a CDF G, its derivative is the corresponding PDF: $\frac{d}{dx}G(x) = g(x)$.

Now let's apply this:

$$\begin{aligned} \frac{\partial P_i}{\partial \beta} &= \frac{\partial}{\partial \beta}[1 - G(C - \beta)] \\ &= -\frac{\partial}{\partial \beta}[G(C - \beta)] \end{aligned}$$

Now use the chain rule. Let $z = C - \beta$, so we have $G(z)$ where z depends on β:

$$\begin{aligned} &= -g(C - \beta) \cdot \frac{\partial (C - \beta)}{\partial \beta} \\ &= -g(C - \beta) \cdot (-1) \\ &= g(C - \beta) \end{aligned}$$

This looks like a final answer. But hwat do we make of it? Since g is a probability density function, we know that $g(x) > 0$ for all x (densities are always strictly positive). Therefore:

$$\frac{\partial P_i}{\partial \beta} = g(C - \beta) > 0$$

This derivative is **strictly positive**. As the base benefit β increases, the probability of supporting reform increases. This should track with your intuition; if the baseline benefit of the reform goes up (while cost stays fixed), voters become more likely to support it. The magnitude of this effect is given by the density $g(C - \beta)$.

(b) What is $\frac{\partial P_i}{\partial C}$? Calculate and interpret it.

$\frac{\partial P_i}{\partial C}$ tells us how the probability of supporting reform changes as C, or costs to the voter, increases. This is very similar to part (a), but now we're differentiating with respect to C instead of β.

$$\begin{aligned}\frac{\partial P_i}{\partial C} &= \frac{\partial}{\partial C}[1 - G(C - \beta)] \\ &= -\frac{\partial}{\partial C}[G(C - \beta)]\end{aligned}$$

Using the chain rule again, where we now consider how $(C - \beta)$ changes with C:

$$\begin{aligned}&= -g(C - \beta) \cdot \frac{\partial (C - \beta)}{\partial C} \\ &= -g(C - \beta) \cdot (1) \\ &= -g(C - \beta)\end{aligned}$$

This derivative is **strictly negative** (since $g(C - \beta) > 0$):

$$\frac{\partial P_i}{\partial C} = -g(C - \beta) < 0$$

So, as the cost C increases, the probability of supporting reform decreases. This might fit your intuition: if the cost of reform goes up (while benefits stay fixed), voters become less likely to support the reform.

Exercise 3

A voter decides whether to vote or abstain. Voting costs $c > 0$ jollies. If the voter votes, they are pivotal with probability $p \in (0, 1)$ and can secure their preferred outcome worth v jollies. If they vote but are not pivotal, the outcome is decided by a coin flip: their preferred outcome occurs with probability 0.5, giving them v jollies, and their non-preferred outcome occurs with probability 0.5, giving them 0 jollies. If they abstain, the outcome is also decided by a coin flip with the same probabilities: their preferred outcome occurs with probability 0.5, giving them v jollies, and their non-preferred outcome occurs with probability 0.5, giving them 0 jollies.

(a) Write the expected utility of voting and the expected utility of abstaining.

For each choice, we need to calculate what the voter expects to get. Recall that expected utility is the probability-weighted average of utilities across all possible outcomes.

Expected utility of voting:

The voter faces two scenarios: being pivotal (probability p) and not being pivotal (probability $1 - p$).

- If pivotal: They secure their preferred outcome, getting v jollies
- If not pivotal: The outcome is random. They get v with probability 0.5 and 0 with probability 0.5, so expected utility is $0.5v + 0.5(0) = 0.5v$

Putting this together:

$$\begin{aligned} EU(\text{vote}) &= p \cdot v + (1 - p) \cdot (0.5v + 0.5 \cdot 0) - c \\ &= pv + (1 - p)(0.5v) - c \\ &= pv + 0.5v - 0.5pv - c \\ &= 0.5pv + 0.5v - c \end{aligned}$$

Expected utility of abstaining:

If they abstain, the outcome is decided by coin flip:

$$\begin{aligned} EU(\text{abstain}) &= 0.5v + 0.5 \cdot 0 \\ &= 0.5v \end{aligned}$$

(b) What condition determines whether the voter turns out?

The voter votes if the expected utility of voting is at least as high as the expected utility of abstaining. So we need to find when $EU(\text{vote}) \geq EU(\text{abstain})$.

$$\begin{aligned} 0.5pv + 0.5v - c &\geq 0.5v \\ 0.5pv - c &\geq 0 \\ 0.5pv &\geq c \\ pv &\geq 2c \end{aligned}$$

The turnout condition is $pv \geq 2c$. We can also write this as $p \geq \frac{2c}{v}$ or $v \geq \frac{2c}{p}$.

Why does the benefit need to cover *twice* the cost? If the voter abstains, they still get $0.5v$ from the coin flip. By voting, they improve their chances from 0.5 to $0.5 + 0.5p$, which gives expected benefit of $0.5pv$. This needs to exceed cost c, giving us $0.5pv \geq c$, or equivalently $pv \geq 2c$.

(c) How does an increase in p affect the turnout decision? Interpret.

Look at the turnout condition $pv \geq 2c$. The left side is pv, which is increasing in p (since $v > 0$). As p increases, the left side gets larger while the right side ($2c$) stays constant. This makes the inequality easier to satisfy.

As the probability of being pivotal increases, voters are more likely to turn out. If your vote is more likely to determine the outcome, you're more willing to pay the cost to cast it.

For a concrete example, suppose $v = 10$ and $c = 1$. Then:

- If $p = 0.1$: $pv = 1 < 2$, so the voter abstains
- If $p = 0.3$: $pv = 3 > 2$, so the voter turns out

When the probability of being pivotal increases from 0.1 to 0.3, the expected benefit rises enough to justify the cost. This helps explain why turnout is often higher in close elections (where p is perceived to be higher) than in landslides (where p is very small).