Formal Models: Section 4*

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Review: from last week

We started last week by reviewing some additional **CDF properties** that will be useful as we calculate probabilities using distributions. First, we can find $\mathbb{P}(X \geq x)$ for some CDF $F(x)$ using $1 - F(x)$;[1] we can calculate the probability $\mathbb{P}(b < x \leq a)$ as $F(b) - F(a)$; and for symmetric distributions with mean μ, we have the useful property that $F(\mu - x) = 1 - F(\mu + x)$.

Perhaps more importantly, last week was a large step forward in how we approach modeling decisions. While previously we modeled decision-makers as individuals whose outcomes depended only on their own actions (and some randomness), last week we expanded our framework to consider **outcomes that depend on the choices of multiple decision-makers**, in what we call a *strategic environment*.

To do so, we introduced the concept of **strategic-form games**, which are characterized by three components: a finite set of *players* N, a collection of *strategy sets* S_i for each player i, and *utility functions* v_i that assign payoffs to every possible strategy profile. For now, we focus on **pure strategies** (deterministic plans like "always play Rock") rather than mixed strategies (randomized plans).

To describe these strategies, we use **strategy profiles** that describe a particular combination of each player's choices. In a two player game of "rock paper scissors" this might be $(\text{Rock}, \text{Paper})$, or $(\text{Rock}, \text{Rock})$. Our games will specify how every strategy profile corresponds to outcomes, and payoffs, for each player, and we begin by drawing the game as a matrix whose cells display a tuple of these payoffs (p_1, p_2) for each player.

Finally, we note that for analytical purposes we assume that these games are **static** (ie. players choose their actions 'simultaneously', in the sense that they choose actions without knowledge of which action the other player(s) have chosen), and that they are games of **complete information**, in which all players have *common knowledge* of the game's structure. While we note that this assumption is somewhat unrealistic, it provides the bulk of our analytical power when we start to construct simple models of strategic interactions, and we will relax it as we progress further in the class.

*I drew on materials from Tak-Huen Chau. All writing (including its poor sense of humor and any errant mistakes) are mine.

[1] In section, someone raised the point that the complement of $\mathbb{P}(X \leq x)$ is $\mathbb{P}(X > x)$, not $\mathbb{P}(X \geq x)$, so how can we say this is equivalent? The simplest reasoning takes advantage of the fact that for any continuous random variable X, $\mathbb{P}(X = x) = 0$. As a result, $\mathbb{P}(X > x)$ and $\mathbb{P}(X \geq x)$ are equivalent.

A brief example: Actions map to outcomes, preferences map to utilities

Just to refresh your brain juices,[2] I want to explicitly connect games to our original framework. Recall from our Section 2 handout:

> …**in each problem we are faced with, there should be a mathematical solution through which we can determine the action through which actors maximize their utility.** Actors should be aware of their sets of possible actions and outcomes, how their actions relate to outcomes, and what their preferences are over outcomes; and then choose the action that will realize their most-preferred outcome.

This is still the case! Now, the actions available to an individual i are defined with the strategy set S_i.[3] The outcomes are said to be a set X composed of individual outcomes $x \in X$. These actions correspond to outcomes via some mapping $\mu : S \to X$. Similarly, the utility function is how we map the outcomes x onto preferences, $v : x \to \mathbb{R}$. Hopefully, it's clear how this is an extension of our previous framework to the case where interactions are *strategic*, or those situations in which outcomes depend on the actions of multiple actors.

An example, with apologies to Rousseau

To illustrate this, I want to talk through an example game that is a classic example ported to a more modern application. In this game, there are two people enmeshed in what our generation[4] might call a ~situationship~. Each person has two choices:

1. Confess *feelings*.
2. Play it cool.

If both players manage to be vulnerable and confess their feelings, they have successfully coordinated and can reap a valuable reward: a mutually-fulfilling relationship. This is risky, of course! If one player confesses their feelings, but the other plays it cool, the player who confesses goes home dejected while the one who plays it cool manages to scrape by. If both players play it cool, though, no one has to deal with the brutal pains of rejection and both can manage to reap the modest reward of keeping the status quo going. We'll say that in this game, each player i receives the same number of jollies from a given outcome x.

So, having gotten the structure of the game out of the way, we say that the strategy profile for player 1 is $S_1 = \{C, P\}$ - just their actions written in order, though you'll note we often abbreviate to single letters. Similarly, player 2 has the strategy profile $S_2 = \{C, P\}$. Furthermore, there are three possible outcomes for this game: "get a partner," "get snubbed," and "get by with the status quo." Remember that our outcomes need to map to the actions that players take! So, we can *formalize* this with the following mapping:

[2] Sorry.

[3] Generally, people will use "action" and "strategy" interchangeably when talking about static strategic-form games. This was covered last week, but remember that this is because under pure strategies, every chosen strategy will map onto an action with probability 1.

[4] Or my generation, anyways, acknowledging against my will that most of you were born in the 21st century and I was born late in the 20th thus technically making us from different generations, and making me specifically "chopped" and "unc."

$$\text{outcomes: } \begin{cases} C, C & \rightarrow \text{ Both get a partner! :)} \\ C, P & \rightarrow \text{ 1 gets snubbed :(, 2 gets by with the status quo :/} \\ P, C & \rightarrow \text{ 1 gets by with the status quo :/, 2 gets snubbed :(} \\ P, P & \rightarrow \text{ Both get by with the status quo :/} \end{cases}$$

We also need to write the utilities that each player receives, based on each outcome. Since, by design of the game, we say that each player receives the same number of jollies as the other player depending on which outcome they are experiencing, we can write both players' utilities generally as:

$$v_i(x) = \begin{cases} \text{Get a partner} & \rightarrow 3 \text{ jollies} \\ \text{Get by with the status quo} & \rightarrow 1 \text{ jollies} \\ \text{Get snubbed} & \rightarrow 0 \text{ jollies} \end{cases}$$

From this, we can construct the game as so:

$$\text{outcomes: } \begin{cases} C, C & \rightarrow \text{ Both get a partner! :)} \rightarrow (3,3) \\ C, P & \rightarrow \text{ 1 gets snubbed :(, 2 gets by with the status quo :/} \rightarrow (0,1) \\ P, C & \rightarrow \text{ 1 gets by with the status quo :/, 2 gets snubbed :(} \rightarrow (1,0) \\ P, P & \rightarrow \text{ Both get by with the status quo :/} \rightarrow (1,1) \end{cases}$$

And, using all of this, we can finally construct the game in a 2x2 matrix:

	C_2	P_2
C_1	(3,3)	(0,1)
P_1	(1,0)	(1,1)

This game, which is a modified version of the stag hunt described by Jean-Jacques Rousseau in his *Discourse on Inequality*, I refer to as **"Love Hunt."**

Pause for understanding

Complete Worksheet Exercise 1.

Analyzing strategic-form games

Remember that in this class, we deal in rational actors who are single-minded utility maximizers. This means that we can predict and prescribe actions to the decision-makers, if we are able to find a **solution concept.** For a solution concept, we attempt to restrict the set of *all possible outcomes* into a subset of actions that decision-makers view as more reasonable than the other available actions.

We call any individual strategy profile that emerges as one of the strategy profiles predicted by a *solution concept* an "**equilibrium**".

Definition (Equilibrium): An *equilibrium* is an strategy profile (s_1^*, s_2^*) such that for all players i and all strategies s_i: $u_i(s_i^*, s_j^*) \geq u_i(s_i, s_j^*)$ What, in plain terms, does this mean? We are saying that, conditional on player j sticking to the action s_j^*, the utility for Player i for strategy s_i^* is better or at least as good as any other action s_i.

You can think of an equilibrium as the likely prediction of our theory: what actors ought to do, given the constraints of the game. So, to analyze games, we need to be able to understand how actors compare and select strategies. We start with the concept of **strict dominance**.

Definition (Strict dominance): Recall that a decision-maker i has a set of possible strategies S_i. Given two strategies, $s_i, s_i' \in S_i$, we say that s_i' is *strictly dominated* by s_i if, for every possible combination of the other players' strategies $s_{-i} \in S_{-i}$, the utility that i receives from strategy s_i', $v_i(s_i', s_{-i})$ is strictly less than the utility i receives from s_i, $v_i(s_i, s_{-i})$. In other words, s_i' is *strictly dominated* by s_i if $v_i(s_i, s_{-i}) > v_i(s_i', s_{-i}) \forall s_{-i} \in S_{-i}$.

We can see that strict dominance can be used to relate strategies to each other; furthermore, we can state that a strategy itself is strictly dominant by extending the definition:

Definition (Strictly dominant strategy): A strategy s_i is a *strictly dominant strategy* if and only if s_i strictly dominates all $s_i' \in S_i$ such that $s_i' \neq s_i$.

In other words, the strategy s_i is itself a strictly dominant strategy if i receives strictly greater utility from s_i than all other possible strategies, **regardless of the other players' actions.** In some games, strictly dominant strategies exist and provide nice clean solutions. Recall the Prisoner's Dilemma:

	Cooperate$_2$	Defect$_2$
Cooperate$_1$	(-2,-2)	(-5,-1)
Defect$_1$	(-1,-5)	(-4,-4)

Here, for both players, Defect$_i$ is a strictly dominant strategy. Regardless of whether $-i$ plays Cooperate$_{-i}$ or Defect$_{-i}$, i receives more jollies from Defect$_i$. Strict dominance requires that i receives more utility from the strategy *regardless of the other players' actions.* But, in **"Love Hunt,"** it's not so simple. Player i should only confess if $-i$ is confessing; otherwise, they ought to play it cool. This gets at what makes games different from simple decision problems: sometimes, other player's actions *must* be considered.

In short, we have an important fact that will guide our analysis throughout the class: for decision-makers in a game to undergo rational utility optimizing, they must choose a strategy that is a **best response** to the strategies chosen by the other decision-makers. This leads us to the definition of best response:

Definition (Best response): A strategy $s_i \in S_i$ is player i's *best response* to the strategies of all not-i players, $s_{-i} \in S_{-i}$, if $v_i(s_i, s_{-i}) \geq v_i(s_i', s_{-i}) \forall s_i' \in S_i$. In words, the strategy s_i is the best response if the utility i receives from that strategy is greater than or equal to the utility they would receive from all other strategies, holding the strategies of all other players constant.

This concept is **critical** to our work. After all, our definition of rationality implied that decision-makers will choose their best responses. More specifically, this also implies that players will choose their best action, *given their beliefs about the other players' behavior.* This helps us delineate the concepts of *best response* and *strict dominance.* After all, if a strategy is strictly dominant, it is that player's best response **no matter what the opponent plays**. Strictly dominant strategies are always a best response![5]

Best respond like no one is watching

But when no strategy is strictly dominant, rational players must still find a best response. Furthermore, if a player is in a two-player game where their *opponent* has no strictly dominant strategy, it may not be clear what their best response is! To prove this to yourself, consider yourself as Player 1 in *Love Hunt*. Does it make more sense to confess or to play it cool, based purely on the structure of the game? Well, your payoffs from your actions depend entirely on whether you think Player 2 confesses or not. In short, this requires players to use their *beliefs*:

Definition (Belief): A *belief* of player i is some possible strategy profile of their opponents' strategies, $s_{-i} \in S_{-i}$.

Less formally, a *belief* is a guess (or expectation) that a player has about what strategies all other players choose. You might naturally ask yourself, on what basis are these beliefs formed? Consider *Love Hunt*. Perhaps you have learned in past conversations that the other player has an avoidant attachment style[6], perhaps the other player is known to have a Ted Mosby-like predilection to confessions of love, perhaps you have experienced unfortunate outcomes when making your own confessions in the past. In short, an assumption that we will make moving forward in this class is that *players form their beliefs from past experiences playing the game, which lead players to be sure of their opponents' actions.*[7]

Note that this assumption is compatible with our view of these games as one-shot games, that is, the assumption that beliefs are formed from experience has no bearing on how the player approaches the static strategic-form game in question. Players view each play of the game as its own isolated iteration, they do not condition actions on the *specific* identity of the player they face, nor (for now!) does the player expect their current action to affect the future actions of their opponents.

This can be a bit of a confusing or grating assumption at first. It might help to analogize these games to the set up of an online game platform, ie. chess.com. A game of chess has two players, $White$ and *Black*. For each "player" on chess.com there is a wider population of millions of decision-makers - the individual users who, at any point, may assume the role of each "player" in a game of chess. Thus, for any given game of chess, players have been selected somewhat-randomly,[8] and an individual person i plays chess repeatedly against a large multitude of different opponents. This experience leads people to build their beliefs about the action of an abstract, "typical" opponent that they hold when facing a new specific opponent, but these beliefs are of course not about the actual person that they face.

Given this, we can see that rationality (as we have thus far defined it in this class) demands that in our games, we expect that each player must play a *best response* based on their *beliefs*. In our pursuit of analytical solutions, we add one final ingredient. We assume not only that players hold

[5] Sorry if this point is feeling a bit belabored, but it is an important one.

[6] In other words, an aversion to emotional intimacy. See this primer, if you're unfamiliar (and hopefully you are; if you're already familiar I hope for your sake it was indirect experience).

[7] We will return later to the question of *how* these experiences lead players to *correct* beliefs about their opponents.

[8] Of course, I believe chess.com uses ELO to match players. But I beg you to pardon the abstraction for sake of analogy.

beliefs about their opponents, but also *that these beliefs are correct.* Immediately, this second requirement may stand out as a tremendous leap that flies in the face of external validity.

Why do we impose such a demanding requirement? Without correct beliefs, we would have a situation where players are each doing their best *given what they think* is happening, but their plans don't fit together. In that world, Player A optimizes against what they believe Player B will do, but Player B is actually doing something else entirely. If that is the case, then at least one player will discover they're not playing optimally and have an incentive to change their strategy. As a result, we would not have a solution.

Complete information and correct beliefs work together to give us stable solutions. To see why *complete information* matters, consider the battle of wits scene from *The Princess Bride*. For those who haven't seen it yet, the conniving Vizzini is holding the Princess Buttercup hostage. Vizzini is challenged by a heroic man in black to a battle of wits: the man in black has poisoned one of two goblets, and Vizzini, who did not observe which goblet was poisoned, must pick the goblet free of poison.

"But it's so simple," Vizzini reasons elaborately: "All I have to do is divine from what I know of you…" and yet he has committed a classic blunder: his beliefs about his opponent are not correct, nor does he have complete information about the structure of the game. Because (spoiler alert?) both goblets are poisoned, his carefully-reasoned choice isn't actually optimal given reality, and there's no stable outcome to his reasoning process. The correct beliefs requirement ensures we identify situations that are genuinely stable — where if players expect a particular outcome, it will occur, and no player will discover they made a mistake. The chess.com analogy similarly helps here: through repeated experience against the population, beliefs about what a "typical" opponent does converge to what typical opponents actually do.

A beautiful [k]ind (of solution)

With both requirements in hand — players choose best responses given their beliefs, and these beliefs are correct — we can now define our central solution concept, a special kind of equilibrium that we call a **Nash equilibrium.**

Definition (Nash equilibrium): A strategy profile $s^* = (s_1^*, s_2^*, \dots, s_n^*)$ is a *Nash equilibrium* if no player can unilaterally improve their utility by deviating from their strategy, given the strategies of the other players. Formally, for each player i, $u_i(s_i^*, s_{-i}^*) \geq u_i(s_i, s_{-i}^*)$ for all $s_i \in S_i$, where s_{-i}^* represents the strategy profiles of all players except player i.

We can think of Nash[9] equilibria in a few ways. First, a Nash equilibrium is a sort of stable "social norm": if every player maintains the Nash equilibrium, no individual player should wish to deviate from it. This implies a second way of thinking of Nash equilibria: as a strategy with mutual best responses. I note also that, as stated a few times, we are dealing only with pure strategies for now. So, we impose a final constraint to define the concept of a **pure strategy Nash equilibrium**:

Definition (Pure strategy Nash equilibrium): A strategy profile $s^* = (s_1^*, s_2^*, \dots, s_n^*)$ is a *pure strategy Nash equilibrium* if all elements in s^* are pure strategies.

[9] Capitalized as it is named after mathematician John F. Nash Jr., a prolific mathematician whose struggles with schizophrenia inspired the unauthorized 1998 biography *A Beautiful Mind* and its 2001 film adaptation by Ron Howard starring Russell Crowe; the title of this section is a forced reference to those works.

Applying and finding Nash equilibria

Now that we have defined Nash equilibrium, let's apply it to **Love Hunt** to see how we can identify these special strategy profiles. Recall the game:

	C_2	P_2
C_1	(3,3)	(0,1)
P_1	(1,0)	(1,1)

To find Nash equilibria, we need to check each possible strategy profile and ask: could either player improve their payoff by unilaterally changing their strategy? If the answer is no, we have a Nash equilibrium.

Is (C, C) a Nash equilibrium?

Suppose both players are confessing. Could Player 1 do better by deviating to P while Player 2 sticks with C? If Player 1 switches to P, their payoff drops from 3 jollies to 1 jolly. So Player 1 has no incentive to deviate. Similarly, if Player 2 switches to P while Player 1 sticks with C, Player 2's payoff drops from 3 jollies to 1 jolly. Neither player wants to deviate. (C, C) is a Nash equilibrium.

Is (C, P) a Nash equilibrium?

Suppose Player 1 confesses and Player 2 plays it cool. Player 1 gets 0 jollies while Player 2 gets 1 jolly. Could Player 1 do better? If Player 1 switches to P, their payoff increases from 0 to 1 jolly. Player 1 has an incentive to deviate! (C, P) is *not* a Nash equilibrium.

Is (P, C) a Nash equilibrium?

By symmetry, this is just the mirror image of (C, P). Player 2 gets 0 jollies and would prefer to switch to P to get 1 jolly instead. (P, C) is *not* a Nash equilibrium.

Is (P, P) a Nash equilibrium?

Suppose both players play it cool. Each gets 1 jolly. Could Player 1 do better by switching to C? If they do, they get 0 jollies instead of 1. Could Player 2 do better by switching to C? Same story—they'd get 0 jollies instead of 1. Neither player wants to deviate. (P, P) is a Nash equilibrium.

So **Love Hunt** has *two* pure strategy Nash equilibria: (C, C) and (P, P). This multiplicity of equilibria is actually quite common and raises an interesting question: which equilibrium should we expect to observe? Both are stable—if players coordinate on either one, no one wants to deviate—but they lead to very different outcomes. The mutual confession equilibrium yields 3 jollies for each player, while the mutual cool-playing equilibrium yields only 1 jolly each. We'll return to questions of equilibrium selection later in the course.

A shortcut: circling best responses

Checking every strategy profile can be tedious, especially as games get larger. A useful shortcut is to identify best responses directly in the matrix by underlining payoffs. For each player, we look at each possible action by their opponent(s) and circle the payoff from their best response.

For Player 1 (who chooses rows):

- If Player 2 plays C_2: Player 1's best response is C_1 $(3 > 1)$, so circle the 3
- If Player 2 plays P_2: Player 1's best response is P_1 $(1 > 0)$, so circle the 1

For Player 2 (who chooses columns):

- If Player 1 plays C_1: Player 2's best response is C_2 $(3 > 1)$, so circle the 3
- If Player 1 plays P_1: Player 2's best response is P_2 $(1 > 0)$, so circle the 1

	C_2	P_2
C_1	(3,3)	(0,1)
P_1	(1,0)	(1,1)

A Nash equilibrium occurs at any cell where *both* payoffs are circled—these are the cells where both players are playing best responses to each other. Sure enough, we find both (C, C) and (P, P) have both payoffs circled, confirming our earlier analysis.

Pause for understanding

Complete Worksheet Exercises 2 and 3.

A word of caution

As a final note, avoid a commonly-made error and do not refer to the payoffs as the solution. You should always describes solutions in terms of the strategies that the players will choose. Recall our base language: strategies are a set of actions by the players, and payoffs are a result of the outcome. When we talk about predictions, or equilibria, we will always refer to what players do as the equilibrium, *not* their payoffs.

Summary

Following the precedent request for an overly-short summary I'd say that **Nash equilibrium represents strategy profiles where each player's choice is a best response to others' choices, requiring not just that players optimize given their beliefs, but that these beliefs correctly reflect what others actually do.**

Less tersely, this week we developed our central solution concept for analyzing strategic-form games. We began by recognizing that not all games have strictly dominant strategies—in games like Love Hunt, the best action for player i depends on what player $-i$ actually does. This led us to the concept of best responses: a strategy s_i is a best response to s_{-i} if it maximizes player i's utility given the other players' strategies.

But best responses alone aren't enough to generate predictions. Players must form beliefs about what others will do, typically based on past experience playing similar games. Here we imposed a critical assumption: beliefs must be correct. Without this requirement, we'd have situations where players optimize given what they think is happening, but their plans don't fit together—at least one player would discover they're not playing optimally and want to revise. The two requirements—best responses plus correct beliefs—define a Nash equilibrium: a strategy profile where no player can improve their payoff by unilaterally deviating. We explored how to find Nash equilibria systematically by checking each strategy profile or by identifying mutual best responses (cells where both payoffs are circled). Love Hunt revealed that games can have multiple Nash equilibria, raising questions about equilibrium selection that we'll return to later.