Transformations Between the Vector Bases of Spherical Coordinate Systems

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April 2018

Let $\vec{\mathbf{v}}(\hat{\mathbf{s}})$ be a tangent vector field on the unit sphere $\mathbb{S}^2 = \{\hat{\mathbf{s}} \ : \ \hat{\mathbf{s}} \in \mathbb{R}^3, |\hat{\mathbf{s}}| = 1\}$. If we choose a set of Cartesian coordinates (q_1, q_2, q_3) for $\mathbb{R}^3$ then we can define a coordinate chart for the embedded sphere by the spherical coordinates with $\theta \in (0, \pi)$, $\phi \in (0, 2\pi)$

$$\hat{\mathbf{q}} \longleftrightarrow (\theta, \phi) = \left(\cos^{-1}\left(\frac{q_3}{|\vec{\mathbf{q}}|}\right), \text{atan2}(q_2, q_1)\right) \tag{1}$$

where

$$\begin{aligned} \hat{\mathbf{q}} &= \begin{pmatrix} \cos(\phi)\sin(\theta) \\ \sin(\phi)\sin(\theta) \\ \cos(\theta) \end{pmatrix} &(2)\\ &= \cos(\phi)\sin(\theta)\hat{\mathbf{q}}_1 + \sin(\phi)\sin(\theta)\hat{\mathbf{q}}_2 + \cos(\theta)\hat{\mathbf{q}}_3, &(3) \end{aligned}$$

$$\hat{\mathbf{q}}_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{q}}_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad \hat{\mathbf{q}}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{4}$$

From these spherical coordinates we obtain an orthonormal basis for $\vec{\mathbf{v}}$:

$$\begin{aligned} \hat{\mathbf{e}}_\theta &= \frac{\partial \hat{\mathbf{q}}}{\partial \theta} &(5)\\ &= \cos(\phi)\cos(\theta)\hat{\mathbf{q}}_1 + \sin(\phi)\cos(\theta)\hat{\mathbf{q}}_2 - \sin(\theta)\hat{\mathbf{q}}_3 &(6)\\ \hat{\mathbf{e}}_\phi &= \frac{\frac{\partial \hat{\mathbf{q}}}{\partial \phi}}{\left|\frac{\partial \hat{\mathbf{q}}}{\partial \phi}\right|} &(7)\\ &= -\sin(\phi)\hat{\mathbf{q}}_1 + \cos(\phi)\hat{\mathbf{q}}_2 &(8) \end{aligned}$$

It is straightforward to see explicitly that

$$\hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{e}}_\theta = 1, \qquad \hat{\mathbf{e}}_\phi \cdot \hat{\mathbf{e}}_\phi = 1, \qquad \hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{e}}_\phi = 0. \tag{9}$$

and that these vector fields are in the tangent space of the sphere at every point:

$$\hat{\mathbf{q}} \cdot \hat{\mathbf{e}}_\theta = 0, \qquad \hat{\mathbf{q}} \cdot \hat{\mathbf{e}}_\phi = 0. \tag{10}$$

Then the vector field $\hat{\mathbf{v}}$ may be represented as

$$\hat{\mathbf{v}} = v_\theta \hat{\mathbf{e}}_\theta + v_\phi \hat{\mathbf{e}}_\phi \tag{11}$$

where the component functions v_θ and v_ϕ are by definition

$$v_\theta \equiv \hat{\mathbf{e}}_\theta \cdot \hat{\mathbf{v}}, \tag{12a}$$
$$v_\phi \equiv \hat{\mathbf{e}}_\phi \cdot \hat{\mathbf{v}}. \tag{12b}$$

Now we choose a different set of Cartesian coordinates $(p_1, p_2, p_3) = \vec{\mathbf{p}}$ such that they are related to the first set of coordinates $(q_1, q_2, q_3) = \vec{\mathbf{q}}$ by a rotation matrix $\boldsymbol{R}$:

$$\vec{\mathbf{p}} = \boldsymbol{R}\vec{\mathbf{q}} \tag{13}$$

We can then define a different set of spherical coordinates (β, α) with the same domain $\beta \in (0, \pi)$ and $\alpha \in (0, 2\pi)$ by

$$\hat{\mathbf{p}} \longleftrightarrow (\beta, \alpha) = \left(\cos^{-1}\left(\frac{p_3}{|\vec{\mathbf{p}}|}\right), \operatorname{atan2}(p_2, p_1)\right), \tag{14}$$

$$\hat{\mathbf{p}} = \begin{pmatrix} \cos(\alpha)\sin(\beta) \\ \sin(\alpha)\sin(\beta) \\ \cos(\beta) \end{pmatrix} \tag{15}$$
$$= \cos(\alpha)\sin(\beta)\hat{\mathbf{p}}_1 + \sin(\alpha)\sin(\beta)\hat{\mathbf{p}}_2 + \cos(\beta)\hat{\mathbf{p}}_3. \tag{16}$$

Again, we obtain a second basis for the tangent space of the sphere by

$$\hat{\mathbf{e}}_\beta = \frac{\partial \hat{\mathbf{p}}}{\partial \beta} \tag{17}$$
$$= \cos(\alpha)\cos(\beta)\hat{\mathbf{p}}_1 + \sin(\alpha)\cos(\beta)\hat{\mathbf{p}}_2 - \sin(\beta)\hat{\mathbf{p}}_3 \tag{18}$$
$$\hat{\mathbf{e}}_\alpha = \frac{\frac{\partial \hat{\mathbf{p}}}{\partial \alpha}}{\left|\frac{\partial \hat{\mathbf{p}}}{\partial \alpha}\right|} \tag{19}$$
$$= -\sin(\alpha)\hat{\mathbf{p}}_1 + \cos(\alpha)\hat{\mathbf{p}}_2 \tag{20}$$

and the vector field $\vec{\mathbf{v}}$ can be decomposed in this basis by

$$\hat{\mathbf{v}} = (\hat{\mathbf{e}}_\beta \cdot \vec{\mathbf{v}})\hat{\mathbf{e}}_\beta + (\hat{\mathbf{e}}_\alpha \cdot \vec{\mathbf{v}})\hat{\mathbf{e}}_\alpha \tag{21}$$
$$= v_\beta \hat{\mathbf{e}}_\beta + v_\alpha \hat{\mathbf{e}}_\alpha \tag{22}$$

We now have two different representations of the vector field $\vec{\mathbf{v}}$ by the components (v_θ, v_ϕ) and the components (v_β, v_α). These two representations are related to each other by a linear transformation. From the definition of the component functions (Equations 12, 21) we compute

$$\begin{aligned} v_\beta &= \hat{\mathbf{e}}_\beta \cdot \vec{\mathbf{v}} & v_\alpha &= \hat{\mathbf{e}}_\alpha \cdot \vec{\mathbf{v}} \\ &= \hat{\mathbf{e}}_\beta \cdot (v_\theta \hat{\mathbf{e}}_\theta + v_\phi \hat{\mathbf{e}}_\phi) & &= \hat{\mathbf{e}}_\alpha \cdot (v_\theta \hat{\mathbf{e}}_\theta + v_\phi \hat{\mathbf{e}}_\phi) \\ &= (\hat{\mathbf{e}}_\beta \cdot \hat{\mathbf{e}}_\theta) v_\theta + (\hat{\mathbf{e}}_\beta \cdot \hat{\mathbf{e}}_\phi) v_\phi & &= (\hat{\mathbf{e}}_\alpha \cdot \hat{\mathbf{e}}_\theta) v_\theta + (\hat{\mathbf{e}}_\alpha \cdot \hat{\mathbf{e}}_\phi) v_\phi \end{aligned} \tag{23}$$

These equations can be combined in the matrix equation

$$\begin{bmatrix} v_\beta \\ v_\alpha \end{bmatrix} = \begin{bmatrix} \hat{\mathbf{e}}_\beta \cdot \hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_\beta \cdot \hat{\mathbf{e}}_\phi \\ \hat{\mathbf{e}}_\alpha \cdot \hat{\mathbf{e}}_\theta & \hat{\mathbf{e}}_\alpha \cdot \hat{\mathbf{e}}_\phi \end{bmatrix} \begin{bmatrix} v_\theta \\ v_\phi \end{bmatrix} \tag{24}$$

Since this represents a mapping between orthonormal bases we already know it is a rotation matrix[1]. For each point $\hat{\mathbf{s}}$ on the sphere - except for the 4 points corresponding to the poles of each coordinate system - this is a rotation in the tangent plane of the sphere at that point:

$$\boldsymbol{\mathcal{U}}(\hat{\mathbf{s}}) = \begin{bmatrix} \hat{\mathbf{e}}_\beta(\hat{\mathbf{s}}) \cdot \hat{\mathbf{e}}_\theta(\hat{\mathbf{s}}) & \hat{\mathbf{e}}_\beta(\hat{\mathbf{s}}) \cdot \hat{\mathbf{e}}_\phi(\hat{\mathbf{s}}) \\ \hat{\mathbf{e}}_\alpha(\hat{\mathbf{s}}) \cdot \hat{\mathbf{e}}_\theta(\hat{\mathbf{s}}) & \hat{\mathbf{e}}_\alpha(\hat{\mathbf{s}}) \cdot \hat{\mathbf{e}}_\phi(\hat{\mathbf{s}}) \end{bmatrix} \tag{25}$$

$$= \begin{bmatrix} \cos(\chi(\hat{\mathbf{s}})) & \sin(\chi(\hat{\mathbf{s}})) \\ -\sin(\chi(\hat{\mathbf{s}})) & \cos(\chi(\hat{\mathbf{s}})) \end{bmatrix} \tag{26}$$

Computing the Transformation

Assume we are given some data specifying the components $v_\theta(\hat{\mathbf{q}}), v_\phi(\hat{\mathbf{q}})$ of the vector field $\vec{\mathbf{v}}$ in the $\hat{\mathbf{e}}_\theta, \hat{\mathbf{e}}_\phi$ basis at some points $\hat{\mathbf{q}}$ and we would like to obtain the components $v_\beta(\hat{\mathbf{q}}), v_\alpha(\hat{\mathbf{q}})$. To do this we must know the rotation matrix $\boldsymbol{R}$ that relates the two coordinate systems. The unit vectors of two spherical coordinate systems can be related by the rotation matrix that connects the coordinates:

$$\hat{\mathbf{e}}_\beta(\hat{\mathbf{q}}, \boldsymbol{R}) = \cos(\alpha(\hat{\mathbf{q}}))\cos(\beta(\hat{\mathbf{q}}))\boldsymbol{R}^T\hat{\mathbf{q}}_1 + \sin(\alpha(\hat{\mathbf{q}}))\cos(\beta(\hat{\mathbf{q}}))\boldsymbol{R}^T\hat{\mathbf{q}}_2 - \sin(\beta(\hat{\mathbf{q}}))\boldsymbol{R}^T\hat{\mathbf{q}}_3 \tag{27}$$

$$\hat{\mathbf{e}}_\alpha(\hat{\mathbf{q}}, \boldsymbol{R}) = -\sin(\alpha(\hat{\mathbf{q}}))\boldsymbol{R}^T\hat{\mathbf{q}}_1 + \cos(\alpha(\hat{\mathbf{q}}))\boldsymbol{R}^T\hat{\mathbf{q}}_2 \tag{28}$$

Here $\alpha(\hat{\mathbf{q}}) = \alpha(\theta, \phi)$ and $\beta(\hat{\mathbf{q}}) = \beta(\theta, \phi)$ implicitly denote the transition maps between the spherical coordinates. We may now view the transformation matrix as a function of the direction $\hat{\mathbf{q}}$ and the relative rotation matrix $\boldsymbol{R}$,

$$\boldsymbol{\mathcal{U}}(\hat{\mathbf{q}}, \boldsymbol{R}) = \begin{bmatrix} \mathcal{U}_{\beta\theta}(\hat{\mathbf{q}}, \boldsymbol{R}) & \mathcal{U}_{\beta\phi}(\hat{\mathbf{q}}, \boldsymbol{R}) \\ \mathcal{U}_{\alpha\theta}(\hat{\mathbf{q}}, \boldsymbol{R}) & \mathcal{U}_{\alpha\phi}(\hat{\mathbf{q}}, \boldsymbol{R}) \end{bmatrix} \tag{29}$$

$$= \begin{bmatrix} \hat{\mathbf{e}}_\beta(\hat{\mathbf{q}}, \boldsymbol{R}) \cdot \hat{\mathbf{e}}_\theta(\hat{\mathbf{q}}) & \hat{\mathbf{e}}_\beta(\hat{\mathbf{q}}, \boldsymbol{R}) \cdot \hat{\mathbf{e}}_\phi(\hat{\mathbf{q}}) \\ \hat{\mathbf{e}}_\alpha(\hat{\mathbf{q}}, \boldsymbol{R}) \cdot \hat{\mathbf{e}}_\theta(\hat{\mathbf{q}}) & \hat{\mathbf{e}}_\alpha(\hat{\mathbf{q}}, \boldsymbol{R}) \cdot \hat{\mathbf{e}}_\phi(\hat{\mathbf{q}}) \end{bmatrix} \tag{30}$$

$$= \begin{bmatrix} \cos(\chi(\hat{\mathbf{q}}, \boldsymbol{R})) & \sin(\chi(\hat{\mathbf{q}}, \boldsymbol{R})) \\ -\sin(\chi(\hat{\mathbf{q}}, \boldsymbol{R})) & \cos(\chi(\hat{\mathbf{q}}, \boldsymbol{R})) \end{bmatrix}. \tag{31}$$

[1]and for the moment, both bases have the same handedness so the rotation is proper

The components of $\vec{\mathbf{v}}(\hat{\mathbf{q}})$ in the $\hat{\mathbf{e}}_\beta, \hat{\mathbf{e}}_\alpha$ basis are then

$$v_\beta(\hat{\mathbf{q}}) = \mathcal{U}_{\beta\theta}(\hat{\mathbf{q}}, \boldsymbol{R}) v_\theta(\hat{\mathbf{q}}) + \mathcal{U}_{\beta\phi}(\hat{\mathbf{q}}, \boldsymbol{R}) v_\phi(\hat{\mathbf{q}}) \tag{32}$$

$$v_\alpha(\hat{\mathbf{q}}) = \mathcal{U}_{\alpha\theta}(\hat{\mathbf{q}}, \boldsymbol{R}) v_\theta(\hat{\mathbf{q}}) + \mathcal{U}_{\alpha\phi}(\hat{\mathbf{q}}, \boldsymbol{R}) v_\phi(\hat{\mathbf{q}}). \tag{33}$$

Additional Notes

- Because the atan2 function is not uniquely defined it is important make sure that the implementation matches the definition of coordinate maps.
- The matrix field $\boldsymbol{\mathcal{U}}(\hat{\mathbf{s}}, \boldsymbol{R})$ has several properties that are useful to note as consistency checks:
  1. The matrix is a rotation, so it should have determinant 1 everywhere it is defined. Additionally, because the matrix is fully specified by $\cos(\chi)$ and $\sin(\chi)$, two of the dot-products between unit vectors should be redundant (up to a sign).
  2. The components $\cos(\chi)$ and $\sin(\chi)$ have four points at which they are not continuous corresponding to the four poles of the two coordinate systems.
  3. Locally near a pole of either coordinate system, one of the coordinate systems looks like a set of plane-cartesian coordinate curves while the other looks like a set of plane-polar curves. Thus the angle χ is approximately the azimuthal angle about that pole i.e. near $\hat{\mathbf{p}}_3 = (0, 0, \pm 1)$ we have $\chi \sim \alpha$ while near $\hat{\mathbf{q}}_3 = (0, 0, \pm 1)$ we have $\chi \sim \phi$.

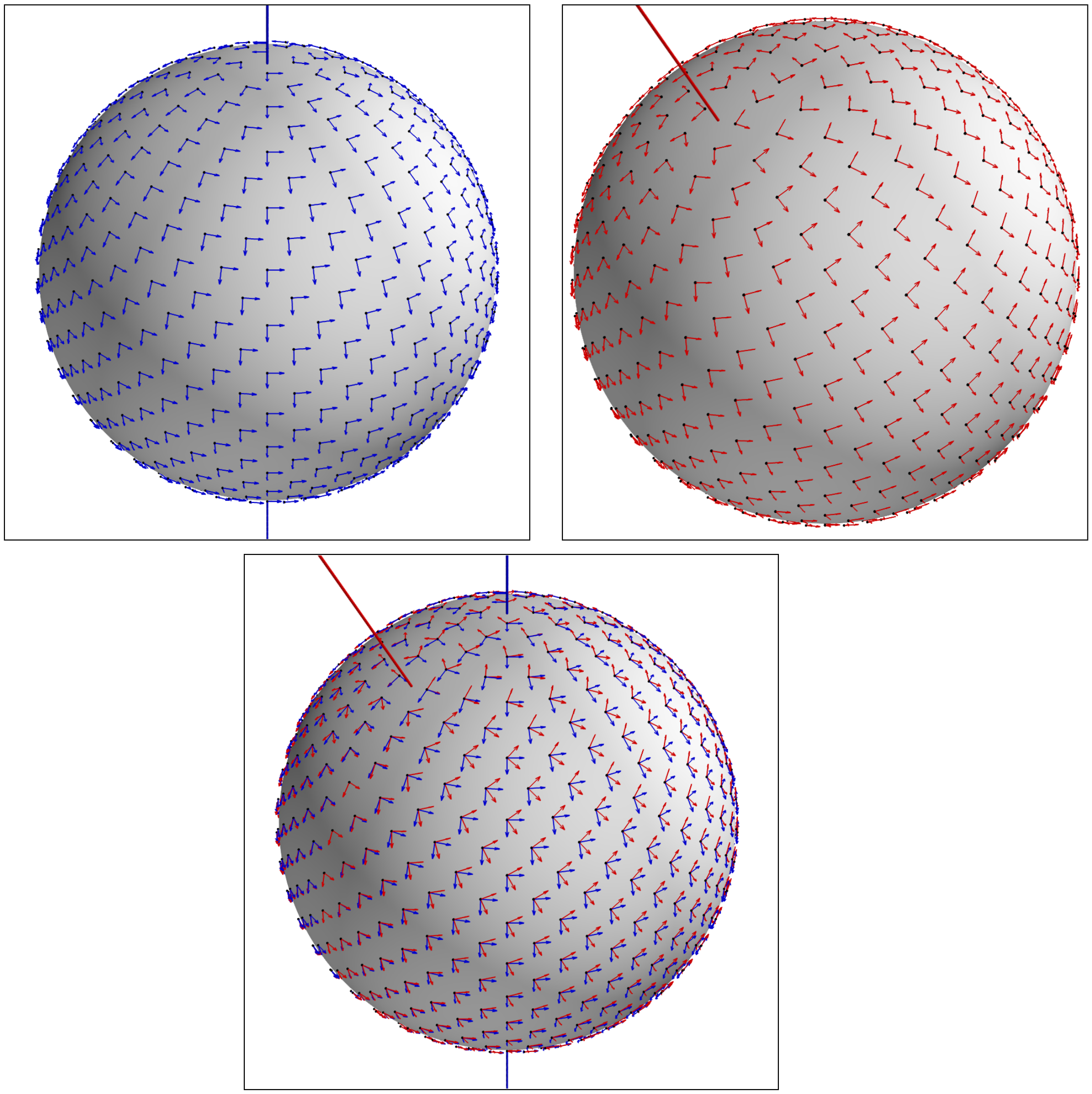

Figure 1: Visualization of the relation between the bases of two different spherical coordinate systems on the sphere. The discrete point-set is a HEALPix map.

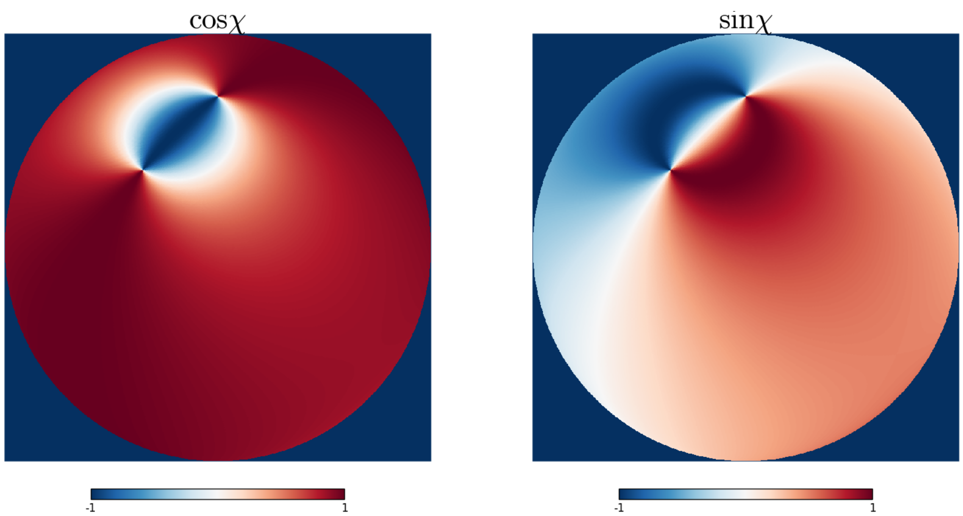

Figure 2: The elements of the matrix field $\mathcal{U}(\hat{\mathbf{q}})$ that transforms from the blue basis to the red basis in Figure 1.