31210 Acoustic signal processing
Exercises to lecture on orthogonality, *Oct 11, 2019*

Exercise 4 (Optional): Transforming signals into random, orthonormal spaces

The DFT is one of infinitely many possible basis transformations between the space of complex signals. It's singular power is that in Fourier space convolutions turn into multiplications. In this exercise, we create a random unitary transformation that can be used similarly to the DFT, although its interpretation is not as simple.

1. Create a random, complex matrix $\mathbf{A}$. Check if it is unitary, i.e. it's column vectors are orthonormal (for a unitary matrix $\mathbf{U}^H\mathbf{U} = \mathbf{I}$).

   If it is not unitary, you can use the QR-factorization (`qr` in MATLAB) or `orth` (takes much longer) to compute a unitary matrix from $\mathbf{A}$.
2. Repeat the analysis of the pure tone and/or the bass signal, but this time use our new transformation matrix. Transform and reconstruct the audio signals. Plot and listen to them.
3. Try to compress (set to zero) the coefficients in the new space, which are smaller than 20% of the signals average. How does the compressed, reconstructed signal compare to the compressed, reconstructed signal of the DFT?

Exercise 5 (for the curious): analytic signals, signal envelopes, and the Hilbert-Transform

The Fourier Transform of real signals $s(t)$ have Hermitian symmetry around $f = 0$, i.e. $S(-f) = S(f)^*$. Thus, the information in the negative frequencies are in principle superfluous for the characterization of any real signal. Disregarding this redundant information by setting the negative frequency spectrum to zero, one arrives at the spectrum of an equivalent signal representation

$$S_a(f) = 2\Theta(f)S(f) = (1 + \mathrm{sgn}(f))S(f),$$

where Θ and $\mathrm{sgn}(f)$ are the Heaviside step function and sign function, respectively:

$$\Theta(f) = \begin{cases} 0, & f < 0 \\ \frac{1}{2}, & f = 0 \\ 1, & f > 0 \end{cases}, \qquad \mathrm{sgn}(f) = \begin{cases} -1, & f < 0 \\ 0, & f = 0 \\ 1, & f > 0 \end{cases}.$$

The corresponding time signal $s_a(t) = \mathcal{F}^{-1}\{S_a(f)\}$ is called the analytic representation of $s(t)$. It belongs to the class of *analytic signals*, which are complex valued function with some interesting properties.

1. Compute the analytic signal of the provided audio file and compare it to the original real signal. Look at both the Cartesian and polar components of the complex signal. Which relations can you make out from visual inspection?

Let's try to understand what is happening.

2. Starting from the definition of $s_a(t)$ show that the real part of the analytic signal is the original real signal and the imaginary part is its Hilbert transform defined as $\mathcal{H}[s(t)] = \frac{1}{\pi t} * s(t)$. It might be practical to know that $\mathcal{F}^{-1}\{\mathrm{sgn}(f)\} = j\frac{1}{\pi t}$.

In frequency domain, the Hilbert transform acts as a phase shift of the negative frequency components by $+90°$ while positive frequencies are shifted by $-90°$, i.e. $\mathcal{F}\{\mathcal{H}[s(t)]\} = -j\mathrm{sgn}(f)S(f)$.

3. Given this knowledge, try to interpret what the magnitude and phase of the analytic signal represent.

As the real valued signal and it's analytic representation are equivalent representations of the same information one might think that this transformation implies another basis, just like the Fourier Transform is defined using the basis of harmonic functions.

4. Analogous to the DFT matrix, build the matrix of the analytic signal transformation. You may want to use that the discrete infinite impulse response of the Hilbert transform $\mathcal{H}(s[n]) = h[n] * s[n]$ is

   $$h[n] = \begin{cases} 0, \text{n even} \\ \frac{2}{\pi n}, \text{n odd} \end{cases}.$$

   You may use an FIR approximation of the IIR filter by truncating it.

5. Calculate the signal envelope using this matrix and plot it together with the real signal. What happens with the envelope close to the ends of the signal?

Even though the transformation to analytic signals is quite interesting, the resulting signals do not span the whole space of complex signals and thus the transformation does not give us a new basis.

6. Show numerically that the columns or rows of this matrix *do not* define an orthogonal basis of the complex discrete signal space.
7. Find two examples of complex signals that have the same analytic signal.