31210 Acoustic signal processing
Exercises to lecture on orthogonality, *Oct 11, 2019*

Exercise 1: Linear Basis

Consider the basis in 3D space

$$\mathbf{d}_1 = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \; \mathbf{d}_2 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}, \; \mathbf{d}_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{1}$$

1. Is this basis orthogonal?
2. Is it orthonormal?
3. Estimate the expansion coefficients a_1, a_2, a_3 of the vector $\mathbf{x} = \begin{bmatrix} 1 & 3 & 0 \end{bmatrix}^T$ in that basis.

Exercise 2: orthogonality of $\cos$ and $\exp$

In this exercise we want to show that the set of harmonic functions is orthogonal.

1. Check that the inner product of two cosines of the form $\cos(nx), \cos(mx)$ with $n, m \in \mathbb{Z}$ is zero in the domain $x \in X = [-\pi, \pi)$, unless $n = m$. You can do this analytically or by numerical integration (check MATLAB's documentation on numerical integration). It may be useful to remember that $2\cos(\phi)\cos(\psi) = \cos(\psi - \phi) + \cos(\psi + \phi)$ and $\cos^2(x) = \frac{1}{2}\cos(2x) + \frac{1}{2}$.
   To visualize you can plot the individual functions and their products for $n, m = (1, 2), (0, 2), (1, 1), (2, 2)$.
2. Repeat this for the *inner* product of the basis of complex exponentials e^{jnx}, $n \in \mathbb{Z}$.

Exercise 3: analysis and reconstruction as matrix multiplications

As you have seen, any linear transformation between two finite-dimensional vector spaces can be represented by a matrix. Thinking of the DFT as a matrix multiplication that implements a change of basis can be useful as most basis transformations are implemented in this way, as e.g. transformation into spherical harmonics or plane wave bases.

1. *Constructing the DFT matrix*
   - Implement a matrix $\mathbf{F}$ of size N^2 with which you can calculate the DFT by a simple matrix multiplication, i.e. given a signal $\mathbf{x} \in \mathbb{R}^N$ you could compute the DFT coefficients $\mathbf{a} = \mathbf{F}^H\mathbf{x}$.
   - Prove numerically that the columns of $\mathbf{F}$ are orthogonal. You can do this by computing the inner product of the columns, $< \mathbf{f}_n, \mathbf{f}_m >$, or by computing the Gram matrix $\mathbf{F}^H\mathbf{F}$. Try to avoid for-loops by exploiting the way MATLAB computes products of arrays.
   - How do you have to normalize $\mathbf{F}$ such that this transformation is unitary?
2. *Analyze and reconstruct a pure tone signal using your DFT matrix*
   - Generate 10 periods of a C_2 pure tone (65 Hz) and transform it using $\mathbf{F}^H$. Inspect the resulting spectrum.
   - Reconstruct the tone using $\mathbf{F}$ and compare the original and reconstructed signals. Are the two signals exactly equal? If not, why?
3. *Analyze, reconstruct and compress an audio file using your DFT matrix*
   - Load to the provided audio file using `audioread`. You can use `sound(x,fs); pause(length(x)/fs);` to listen to the song and halt execution of following code at the same time.
   - Estimate (via matrix multiplication) the signal's spectrum, and plot it.
   - Reconstruct and reproduce the signal (using your DFT matrix).
   - Compress the file by discarding the coefficients with amplitude less than 20% of the mean of the signal. Plot against the original time signal and listen to it. Try to explain the differences that you hear, e.g. what happens at the end of the reconstructed signal?