Homework 2

May 19, 2019

1. Definitions

   (a) The Complexity Class NPC: The set of all problems in NP for which every problem in NP is reducible to each of these problems.

   (b) Reduction: A problem, α is reducible to another problem β if an instance of α can be converted to an instance of β in polynomial-time such that the instances of α and β will have identical decisions.

2. NP Membership

   (a) TSP decision problem: in NP because given a candidate cycle and cost, it can be verified in polynomial-time.

   (b) # of tours with cost $\leq k = 1000000$: Not in NP. Say the certificate was 1 million tours, it would be possible to verify that each tour is valid, however verifying that more tours do not exist is not possible using known polynomial-time algorithms.

   (c) tours with cost $\leq k \geq 1000000$: In NP. A **yes** instance is verifiable in polynomial-time, by simply verifying that each of the 1000000+ tours found has cost $\leq k$.

   (d) N^{th}-shortest tour $\leq k$: In NP. Proving that the N^{th} tour has cost $\leq k$ does not necessarily involve finding all N cheapest tours. For a **yes** instance, a certificate consisting of N tours, the greatest of which has cost $\leq k$ would suffice.

   (e) N^{th}-shortest tour $\geq k$: Not in NP. Verifying a **yes** instance of this problem would require proving that more short tours do not exist, a task which has no known solution in polynomial-time.

3. $HC \leq TSP$

   (a) For TSP, set $k = \infty$. A graph with a Hamiltonian Cycle (a **yes** instance of HC) then also has a shortest Hamiltonian Cycle with cost $\leq \infty$ (a **yes** instance of TSP).

   (b) A graph with no Hamiltonian Cycles (a **no** instance of HC) also has no shortest Hamiltonian Cycle with cost $\leq \infty$ making it a **no** instance of TSP.

4. $SP \leq SS$

   (a) For Subset Sum, set the target size to $\frac{1}{2}$ the total size.

   (b) If a set can divided into two partitions with equal value, then the value of each must be $\frac{1}{2}$ the total value. This means that at least one subset exists with value equal to $\frac{1}{2}$ the total, meaning that a **yes** instance of SP is also a **yes** instance of SS

   (c) If a set has a subset with value $\frac{1}{2}$ of the total, then the remaining items also have the value $\frac{1}{2}$ the total value. This means that a **yes** instance of SS is also a **yes** instance of SP, completing the proof that $SP \leq SS$.

5. Informally, $A_1 \leq A_2$ means that A_2 is harder than or equal to A_1. More formally, A_1 can be solved by performing a polynomial amount of work to transform it into an instance of A_2. If A_1 cannot be solved in polynomial-time, then transforming it then solving it will also be impossible in polynomial-time.

6. If $A_1 \in NPC$, this means that all problems in NP reduce to A_1. Reduction is transitive, so if $A_1 \leq A_2$, this means that $NP \leq A_2$. By definition, this makes A_2 part of NPC.

7. If $A_1 \in NPC$ then $NP \leq A_1$. Likewise if $A_2 \in NPC$, then $NP \leq A_2$. Since $A_1, A_2 \in NP$, $A_1 \leq A_2$ and $A_2 \leq A_1$

8. If $A_1 \in NPC$, then $NP \leq A_1$. If also $A_1 \in P$, then $NP \in P$. Because $A_2 \in NP$, $A_2 \in P$.

9. By the definition of NPC, all NPC problems are reducible to each other. This means that if a single NPC problem were solved in polynomial-time, all NPC problems would be solvable in polynomial-time. Thousands of NPC problems have been found, and not a single one has been solved in polynomial-time. This seems to indicate that $NPC \notin P$.

10. The proof of this problem lies in showing that $\Pi' \leq \Pi$. As Π' is a sub-problem of Π, this is trivial. The transformation from an instance of Π' to an instance of Π is not needed, as instances of Π' are already instances of Π. By the definition of a sub-problem, a **yes** instance of Π' will be a yes instance of Π, and a **no** instance of Π' will be a **no** instance of Π. This proves that $\Pi' \leq \Pi$. If $\Pi' \in NPC$, then $NP \leq \Pi' \leq \Pi$. This means that $NP \leq \Pi$, which by the definition of NPC proves that $\Pi \in NPC$.