Homework 8: Estimation Review

Zach Neveu

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1 Least Squares

$$\hat{\theta}_{LS} = \frac{1}{\boldsymbol{g'g}}\boldsymbol{g'y}$$

2 MVUB

$$\hat{\theta}_{MVUB} = \frac{1}{\boldsymbol{g'C_z^{-1}g}}\boldsymbol{g'C_z^{-1}y} = \frac{1}{g'C_z^{-1}g}g'C_z^{-1}[g\theta + z] = \theta + \frac{1}{g'C_z^{-1}g}g'C_z^{-1}z \tag{1}$$

$$Var(\theta) = E\{|\theta - \hat{\theta}|^2\} = E\{|c'z|^2\} = E\{(c'z)(c'z)^*\} = \boldsymbol{c'C_zc} = \frac{1}{\boldsymbol{g'C_z^{-1}g}} \tag{2}$$

3 Linear MMSE

$$\hat{\theta}_{lin.MMSE} = \frac{R_\theta}{1 + R_\theta \boldsymbol{g'C_z^{-1}g}}\boldsymbol{g'C_z^{-1}y}$$

$$MMSE = \boldsymbol{R_\theta} - \boldsymbol{r}'\boldsymbol{R}^{-1}\boldsymbol{r} = \frac{R_\theta}{1 + R_\theta g'C_z^{-1}g}$$

$$r = E\{\boldsymbol{y}\theta\} = E\{[\boldsymbol{g}\theta + \boldsymbol{z}]\theta^*\} = \boldsymbol{g}E\{\theta^2\} + 0$$

4 ML Estimate

If $\boldsymbol{Z} \sim \mathcal{N}(0, \boldsymbol{C_z})$, then $\hat{\theta}_{ML} = \hat{\theta}_{MVUB}$

$$\hat{\theta}_{ML} = \hat{\theta}_{MVUB} = \frac{1}{\boldsymbol{g'C_z^{-1}g}}\boldsymbol{g'C_z^{-1}y}$$

5 ML Detection of θ

This problem can be thought of in two steps. The first step is to determine the ML estimate of theta as in section **??**. The second step is to classify our estimate of θ as either θ_0 or θ_1. From section **??**, we know that our estimator for θ is unbiased. This reduces the detection problem to simply setting a threshold at $\gamma = \frac{\theta_0+\theta_1}{2}$. Assuming $\theta_0 > \theta_1$, the detection problem can be stated as

$$\hat{\theta}_{ML} = \hat{\theta}_{MVUB} = \frac{1}{\boldsymbol{g'C_z^{-1}g}}\boldsymbol{g'C_z^{-1}y} \underset{\theta_1}{\overset{\theta_0}{\gtrless}} \frac{\theta_1+\theta_0}{2}$$

6 Gaussian Noise and Complex Gaussian θ

The mean of $\theta|\boldsymbol{y}$ here is simply the linear MMSE estimate of θ. This makes sense. With both $\boldsymbol{Z}$ and θ having gaussian distributions, $\theta|\boldsymbol{y}$ will also be gaussian, and the mean of $\theta|\boldsymbol{y}$ will be our best prediction of θ. It also makes sense that $\theta|\boldsymbol{y}$ has a complex normal distribution, as θ is complex valued. Intuitively, it also makes sense that the covariance of $\theta|\boldsymbol{y}$ is proportional to $\boldsymbol{C_z}$, and is equal to σ_θ^2 in the noiseless case.

6.1 MMSE

$$\hat{\theta}_{MMSE} = E\{\theta|\boldsymbol{y}\}$$

Since $\theta|\boldsymbol{y}$ is gaussian, it's expected value is just the mean, which is

$$\hat{\theta}_{MMSE} = \frac{\sigma_\theta^2}{1+\sigma_\theta^2\boldsymbol{g'C_z^{-1}g}}\boldsymbol{g'C_z^{-1}y}$$

6.2 ABS

$$\hat{\theta}_{ABS} = median(\theta|\boldsymbol{y})$$

Since $\theta|\boldsymbol{y}$ is gaussian, the median is the same as the mean, so once again

$$\hat{\theta}_{ABS} = \frac{\sigma_\theta^2}{1+\sigma_\theta^2\boldsymbol{g'C_z^{-1}g}}\boldsymbol{g'C_z^{-1}y}$$

6.3 MAP

$$\hat{\theta}_{MAP} = argmax_\theta f(\theta|\boldsymbol{y})$$

Yet again, the gaussian shape of $\theta|\boldsymbol{y}$ makes our job easier. The peak of the distribution is also the mean, so

$$\hat{\theta}_{MAP} = \frac{\sigma_\theta^2}{1+\sigma_\theta^2\boldsymbol{g'C_z^{-1}g}}\boldsymbol{g'C_z^{-1}y}$$