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ME570 HW2
Professor Tron
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Q1.1:

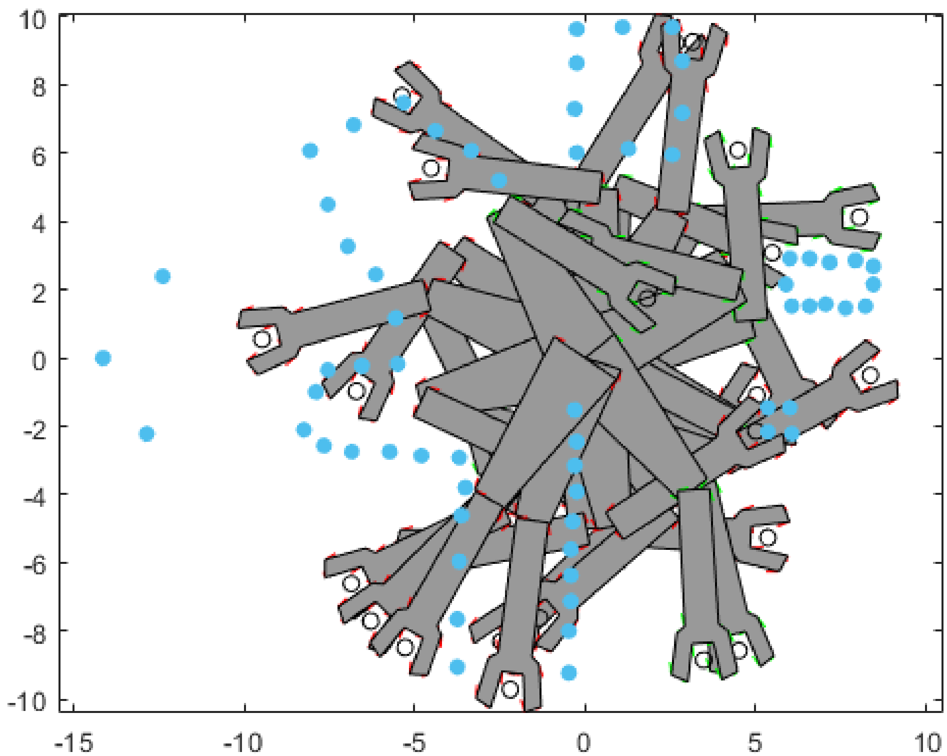
1. $R_1(\theta)$ represents a 2-D CCW rotation on the x_2 x_3 plane.
2. $R_2(\theta)$ represents a 2-D CCW rotation on the x_1 x_3 plane.
3. $R_3(\theta)$ represents a 2-D CCW rotation on the x_1 x_2 plane.
4. $R_4(\theta)$ represents a 2-D CCW rotation on the x_1 x_2 plane, which is then negated (in other words, an additional π rad CCW atop the theta input).

Q2.1:

1. ${}^{W}p = {}^{W}R_{B_1}\,{}^{B_1}p + {}^{W}T_{B_1} = {}^{W}R_{B_1}\,{}^{B_1}p$
2. ${}^{W}p = {}^{W}R_{B_1}\,{}^{B_1}p$ and ${}^{B_1}p = {}^{B_1}R_{B_2}\,{}^{B_2}p + {}^{B_1}T_{B_2} \rightarrow$

$${}^{W}p = {}^{W}R_{B_1}\left({}^{B_1}R_{B_2}\,{}^{B_2}p + {}^{B_1}T_{B_2}\right) = {}^{W}R_{B_1}\,{}^{B_1}R_{B_2}\,{}^{B_2}p + {}^{W}R_{B_1}\,{}^{B_1}T_{B_2}$$

Q2.2:



Q4.1:

$SO(d) \equiv \{R \in \mathbb{R}^{d\times d}: R^T R = I, \det(R) = 1\}$

$R_{2D}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}$

$R_{2D}^T(\theta) R_{2D}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$

$\det\left(R_{2D}(\theta)\right) = cos^2(\theta) + sin^2(\theta) = 1$

$\therefore R_{2D}(\theta) \in SO(2) \rightarrow R_{2D}(\theta)$ is a rotation

(Also demonstrable by comparing to mapping between complex numbers and angle, I believe).

$\varphi_{circle}(\theta) = R_{2D}(\theta) \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

$\mathbb{S}^n \equiv \{x \in \mathbb{R}^{n+1}: \|x\| = 1\}$

Show: $\varphi_{circle}(\theta) \in \mathbb{S}^1 \quad \forall\, \theta \in \mathbb{R}$

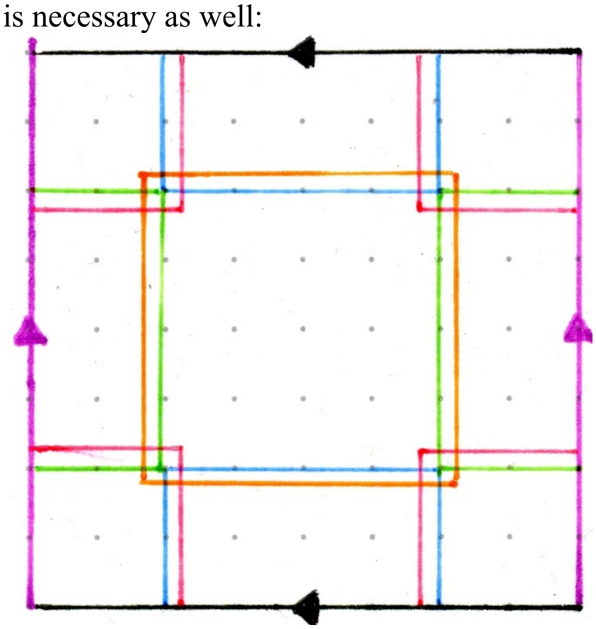
$\varphi_{circle}(\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} \cos(\theta) \\ \sin(\theta) \end{bmatrix}$

$\|\varphi_{circle}(\theta)\| = \sqrt{\cos^2(\theta) + \sin^2(\theta)} = \sqrt{1} = 1$

$\therefore \varphi_{circle}(\theta) \in \mathbb{S}^1 \quad \forall\, \theta \in \mathbb{R}$

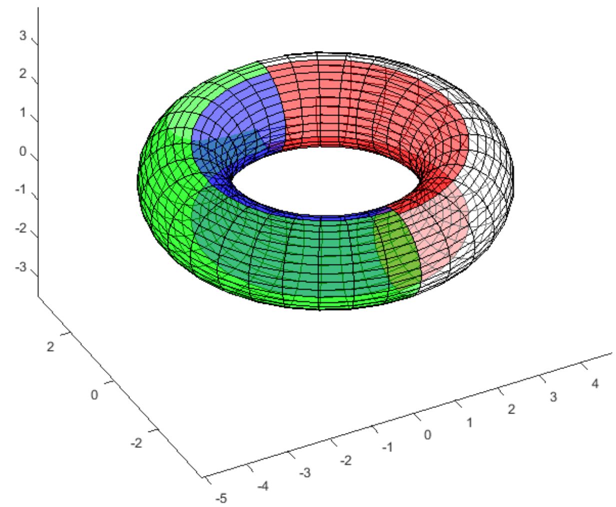
Q5.1:

As $\mathbb{T}^2 = \mathbb{S}^1 \times \mathbb{S}^1$, and $\mathbb{S}^1$ requires 2 charts, it follows easily that for $\mathbb{T}^2$, 4 is sufficient. Pictured on a flat torus and with the restriction of only using square regions within $\mathbb{R}^2$, it becomes evident that 4 is necessary as well:



Q5.2:

The same charts, as applied to the surface of the torus in 3D, with the colors green, blue, red, and white, respectively.



Q5.3:

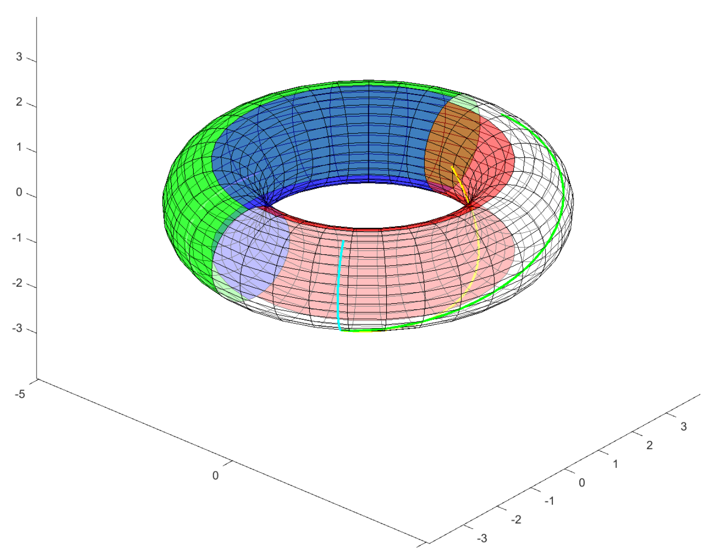
If charts self-overlapped, the mapping would no longer be diffeomorphic; the same point on the surface could be represented multiple ways on a single chart, defeating the chart's purpose. If sections of the torus were uncovered, one would not have a full atlas of the topology; there would exist points in the space that one could not map to, again defeating the purpose.

Q5.4:

$\vec{\theta}(t) = [a(1) * t + b(1) \quad a(2) * t + b(2)] \in \mathbb{R}^2 \quad \forall t \in [tMin, tMax]$

$\dot{\theta}(t) = [a(1) \quad a(2)]$

Q5.5: Torus with curves.



Q6.1:

From Q2.1, ${}^{W}p = {}^{W}R_{B_1}\,{}^{B_1}R_{B_2}\,{}^{B_2}p + {}^{W}R_{B_1}\,{}^{B_1}T_{B_2}$

Thereby, ${}^{W}p_{eff} = {}^{W}R_{B_1}\,{}^{B_1}R_{B_2}\begin{bmatrix}5\\0\end{bmatrix} + {}^{W}R_{B_1}\begin{bmatrix}5\\0\end{bmatrix}$. Both ${}^{W}R_{B_1}$ and ${}^{B_1}R_{B_2}$ are functions of $\theta(t)$.

To find $\frac{d}{dt}({}^{W}p_{eff})$, I utilized the MATLAB symbolic math toolbox, which yielded

$$\frac{d}{dt}({}^{W}p_{eff}) =$$

$$\begin{bmatrix} -5\sin(\theta_1)\,\dot{\theta}_1 - 5\cos(\theta_1)\sin(\theta_2)\,\dot{\theta}_1 - 5\cos(\theta_2)\sin(\theta_1)\,\dot{\theta}_1 - 5\cos(\theta_1)\sin(\theta_2)\,\dot{\theta}_2 - 5\cos(\theta_2)\sin(\theta_1)\dot{\theta}_2 \\ 5\cos(\theta_1)\,\dot{\theta}_1 + 5\cos(\theta_1)\cos(\theta_2)\,\dot{\theta}_1 + 5\cos(\theta_1)\cos(\theta_2)\,\dot{\theta}_2 - 5\sin(\theta_1)\sin(\theta_2)\,\dot{\theta}_1 - 5\sin(\theta_1)\sin(\theta_2)\dot{\theta}_2 \end{bmatrix}$$

Q6.2:

Utilizing the program written for *code Q6.1*, with the equation detailed in *report 6.1*, with the inputs:

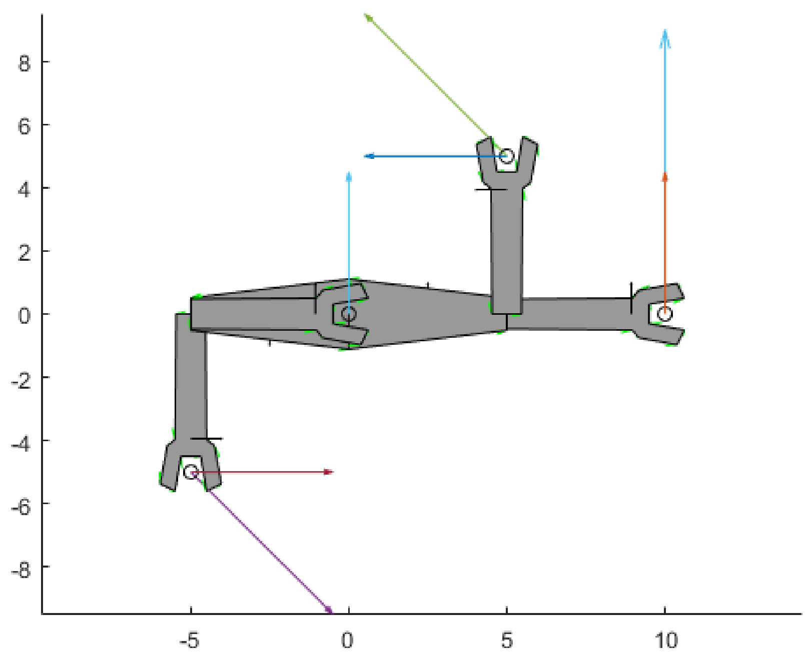
$$\theta = \begin{bmatrix} 0 & 0 & \pi & \pi & 0 & 0 & \pi & \pi \\ 0 & \pi/2 & \pi/2 & \pi & 0 & \pi/2 & \pi/2 & \pi \end{bmatrix}$$

$$\dot{\theta} = a = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}$$

We get the result:

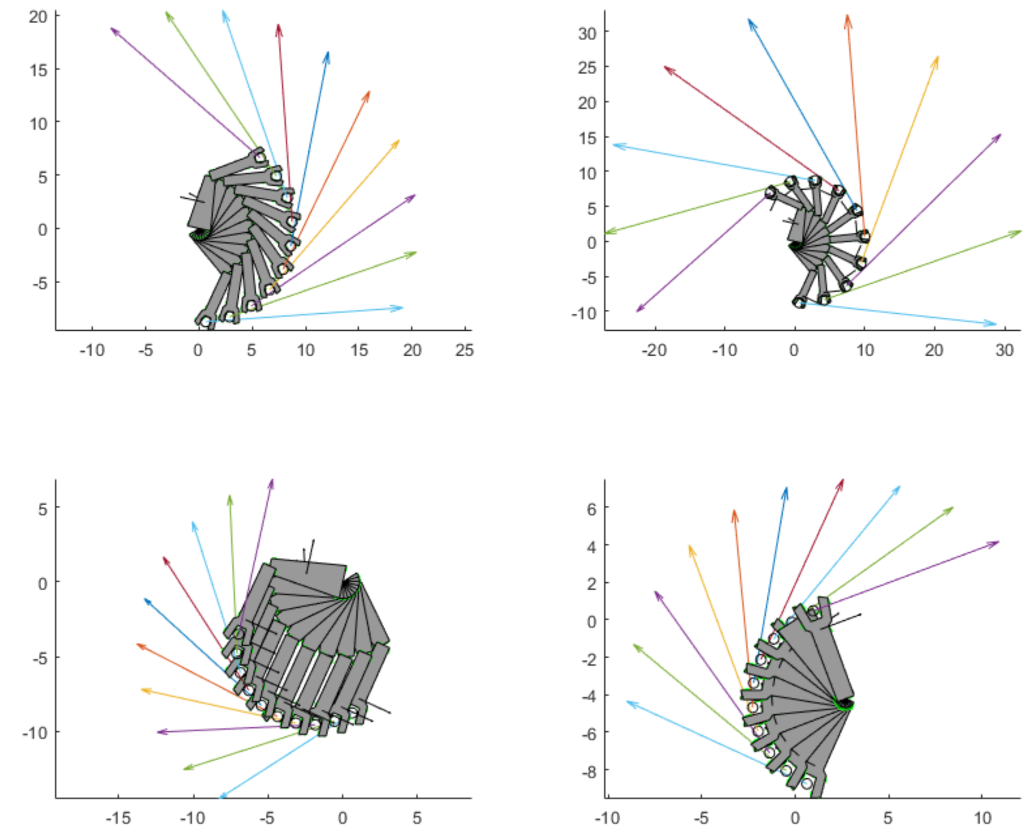
$$\frac{d}{dt}({}^{W}p_{eff}) = \begin{bmatrix} 0 & -5 & 5 & 0 & 0 & -5 & 5 & 0 \\ 10 & 5 & -5 & 0 & 5 & 0 & 0 & 5 \end{bmatrix}$$

Using twolink_plot.m, with the states superimposed, we get:



Likely the oddest feature of the tangents is how they remain the same for some, but not all configurations with different $\dot{\theta}$s.

Q6.3:



Q6.4:

The arms traversing the XY plane corresponds to lines travelling along the surface of the torus in *provided 5.2*, mapped via the two arm angles. As such, the tangents are interrelated between the two spaces.

Q7.1:

This homework took approximately 12 hours for me to complete, the majority of that attributable to this being my first homework for this class (late join), MATLAB brush-up work required (far greater familiarity with other languages such as Python–the last time I used MATLAB intensively was 3ish years ago), and a steep personal learning curve for some of the linear algebra, due to a less intensive background in it. Altogether, once done, the concepts and work make sense and appear fairly trivial, syntax was truly the sticking point.