Topics in Theoretical Computer Science: Randomized Algorithms

EPFL, Fall semester, 2025

Homework 2 Solutions

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1 Problem 1 (Only Connect!)

Given an undirected (unweighted) graph $G = (V, E)$, let $G(p)$ be the random graph where we retain each edge of G independently with probability p. In lecture #4, we saw that setting $p \geq c\frac{\log n}{\lambda}$, where λ is the min-cut value in G, the graph $G(p)$ is a cut-approximator for G with probability $1 - o(1)$. In particular, we get the simpler fact: if G is connected, then G_p is also connected whp. Let's prove this simpler fact in a different way that does not use the cut-counting lemma. Consider the following process:

> Initialize $G_0 = G$, and define $L = 100 \log n$. For each $i = 1, 2, \ldots, L$, let S_i be a set where we pick each edge in G_{i-1} independently with probability $1/\lambda$. Contract all the edges from S_i in the graph G_{i-1} (and remove self-loops) to get G_i.

Analyze it as follows:

(a) For any vertex v in G_{i-1}, let $\mathcal{G}_{v,i}$ be the event that the set S_i contains at least one edge incident to v. Show that $\Pr[\mathcal{G}_{v,i}] \geq 1 - 1/e$. Btw, are $G_{v,i}$ and $G_{u,i}$ independent?

(b) Let N_i be the number of vertices in G_i, so that $N_0 = n$. Define the event $\mathcal{E}_i$ if $N_i \leq N_{i-1} \cdot 3/4$. Show that $\Pr[\mathcal{E}_i] \geq c$, for some absolute constant $c > 0$.

(c) Use a Chernoff bound to show that $|N_L| = 1$ with probability at least $1 - 1/\text{poly}(n)$. Please clearly state what random variables are you summing over, and why they are independent and bounded.

(d) Finally, define $S = \cup_{i=1}^{L} S_i$, and note that each edge in G belongs to S with probability at most L/λ. Infer that sampling each edge of G with probability $p := L/\lambda$ gives us a connected graph with high probability.

1.1 Solution

1.1.1 Part A

Let $\mathcal{G}_{v,i}$ denote the event that a vertex $v \in G_{i-1}$ is relaxed at the step i. That is, at least one of its incident edges is selected as S_i.

$$\Pr[\mathcal{G}_{v,i}] = \Pr[\text{at least one } e \in \delta(v) \text{ is selected in } S_i]$$

Where $\delta(v)$ denotes the set of edges incident to v.

We aim to show that:

$$\Pr[\mathcal{G}_{v,i}] \geq 1 - \frac{1}{e}$$

Proof. The event $\mathcal{G}_{v,i}$ fails to occur only if none of the edges incident to v is selected. Since each edge is included in S_i independently with probability $p = \frac{1}{\lambda}$,

$$\Pr[\mathcal{G}_{v,i}] = 1 - \Pr[\text{no edge in } \delta(v) \text{ is selected}] = 1 - (1-p)^{|\delta(v)|} = 1 - \left(1 - \frac{1}{\lambda}\right)^{|\delta(v)|}$$

Because the minimum degree in the graph is at least the min-cut value, we have $|\delta(v)| \geq \lambda$. Hence

$$\Pr[\mathcal{G}_{v,i}] \geq 1 - \left(1 - \frac{1}{\lambda}\right)^{\lambda}$$

Using the standard inequality $(1-p)^x \leq e^{-px}$ with $p = 1/\lambda$ and $x = \lambda$,

$$\Pr[\mathcal{G}_{v,i}] \geq 1 - e^{-\frac{1}{\lambda}\lambda} = 1 - e^{-1}$$

By the bound $(1-p)^x \leq e^{-px}$ □

Two events $\mathcal{G}_{v,i}$ and $\mathcal{G}_{u,i}$ are independent if and only if the sets of edges on which they depend are disjoint:

$$\mathcal{G}_{v,i} \perp \mathcal{G}_{u,i} \iff \delta(v) \cap \delta(u) = \emptyset$$

That is:

$$\delta(v) \cap \delta(u) = \begin{cases} \emptyset & \text{independent,} \\ \{(u,v)\} & \text{dependent} \end{cases}$$

If v and u are adjacent, the common edge (u,v) affects both events, creating a positive correlation. Knowing that u is relaxed increases the probability that v is also relaxed, since the shared edge may have been selected.

1.1.2 Part B

Let ε_i be the event when the number of vertices N_i in G_i is smaller than or equal to 3/4 of the previous number of vertices N_{i-1}.

$$\Pr[\varepsilon_i] = \Pr\left[N_i \leq N_{i-1}\frac{3}{4}\right]$$

We show that $\Pr[\varepsilon_i] \geq c$ for an absolute constant $c > 0$.

Proof. By complement and Markov's inequality,

$$\Pr[\varepsilon_i] = 1 - \Pr\left[N_i > N_{i-1}\frac{3}{4}\right] \geq 1 - \Pr\left[N_i \geq N_{i-1}\frac{3}{4}\right] \geq 1 - \frac{\mathbb{E}[N_i]}{\frac{3}{4}N_{i-1}}$$

Thus it suffices to upper bound $\mathbb{E}[N_i]$. Let A_i be the number of activated vertices (those incident to at least one sample edge in S_i). From part (a),

$$\mathbb{E}[A_i] = \sum_{v \in G_{i-1}} \Pr[\mathcal{G}_{v,i}] \geq (1 - \frac{1}{e})N_{i-1}$$

The number of vertices removed at each step is the number of components in the graph $H_i = (V_i, S_i)$. This number is upper-bounded by $A_i/2$ which occur when all selected vertices have degree 1. Hence

$$N_i \le N_{i-1} - \frac{A_i}{2}$$

With expectation

$$\mathbb{E}[N_i] \le N_{i-1} - \frac{\mathbb{E}[A_i]}{2} \le \left(1 - \frac{1}{2}(1 - \frac{1}{e})\right) N_{i-1} = \frac{1+\frac{1}{e}}{2} N_{i-1}$$

Plugging into Markov,

$$\Pr\left[N_i \ge N_{i-1}\frac{3}{4}\right] \le \frac{\frac{1+\frac{1}{e}}{2}N_{i-1}}{\frac{3}{4}N_{i-1}} = \frac{2}{3}\left(1+\frac{1}{e}\right)$$

so

$$\Pr[\varepsilon_i] \ge 1 - \frac{2}{3}\left(1+\frac{1}{e}\right) = \frac{1}{3} - \frac{2}{3e} = c > 0$$

□

1.1.3 Part C

We show that the number of vertices remaining after L steps is 1 with high probability:

$$\Pr[N_L = 1] \ge 1 - \frac{1}{\text{poly}(n)}$$

Proof. Let X denote the total number of *good shrink* steps, that is the number of indices $i \in L$ for which the vertex set shrinks by at leas a factor $\frac{3}{4}$:

$$X_i = \begin{cases} 1 & \text{if } N_i \le N_{i-1}\frac{3}{4} \\ 0 & \text{otherwise} \end{cases} \qquad \text{and} \qquad X = \sum_{i=1}^{L} X_i$$

Knowing the number of good shrink helps to upper bound the number of remaining vertices

$$N_i \le n\left(\frac{3}{4}\right)^t$$

To ensures that $N_L \le 1$ (and therefor exactly 1 since a graph has a positive number of vertices), it requires

$$n\left(\frac{3}{4}\right)^t \le 1 \Rightarrow t \ge \frac{\log(n)}{\log(4/3)}$$

Define the threshold

$$t_\star := \left\lceil \frac{\log(n)}{\log(4/3)} \right\rceil$$

Thus, having at least $t_\star$ good shrink events guarantees $N_L = 1$:

$$X_L \ge t_\star \Rightarrow N_L = 1$$

Applying the Multiplicative Chernoff's bound. Let $\mu = \mathbb{E}[X_L]$. From part (b), the events $\{\varepsilon_i\}$ are independent, and each occurs with constant probability c. hence

$$\mu = \sum_{i=1}^{L} \Pr[\varepsilon_i] = cL \quad \text{with} \quad c = \frac{1}{3} - \frac{2}{3e} \quad \text{and} \quad L = 100 \log n$$

For any $\delta > 0$,

$$\Pr[X_L \leq (1-\delta)\mu] \leq \exp\left(-\frac{\delta^2 \mu}{2}\right)$$

Set $(1-\delta)\mu = t_\star$ so that the left-hand event corresponds to "fewer than $t_\star$ good shrinks". This gives

$$\delta = 1 - \frac{t_\star}{cL} \approx 1 - \frac{1}{100c \cdot \log(\frac{4}{3})} \approx \frac{cL-1}{cL}$$

Plugging this into the Chernoff bound:

$$\Pr[X_L \leq (1-\delta)\mu] \leq \exp\left(-\frac{\delta^2 \mu}{2}\right) = \exp\left(-\frac{\left(\frac{cL-1}{cL}\right)^2 cL}{2}\right) = \exp\left(-\Theta(\log n)\right) = n^{-\Theta(1)}$$

Finally, with the probability at least $1 - n^{-\Omega(1)}$, we have $X_L \geq t_\star$, and therefore $N_L = 1$. Hence:

$$\Pr[N_L = 1] = \Pr[N_L \geq 1] - \Pr[N_L \geq 2] = 1 - \Pr[X_L < t_\star] \geq 1 - n^{-\Omega(1)}$$

□

1.1.4 Part D

Given

$$S = \bigcup_{i=1}^{L} S_i \quad \text{and} \quad \Pr[e \in S] = 1 - \left(1 - \tfrac{1}{\lambda}\right)^L \leq \frac{L}{\lambda} =: p$$

We claim that sampling edge of G independently with probability $p := \frac{L}{\lambda}$ yields a connected graph with high probability.

Proof. From part (c), we have shown that with high probability, after L rounds, the remaining graph has a single contracted vertex:

$$N_L = 1 \Rightarrow G \text{ is connected under edges set } S$$

Hence the, the random subgraph $G' = (V, S \subseteq E)$ is connected with probability $1 - n^{-\Omega(1)}$.

Now, consider the Erdős-Rényi random graph $G(p)$, where each edge of G is included independently with probability p. Since $\Pr[e \in S] \leq p$, the distribution of G' is stochastically dominated by $G(p)$. Because de graph connectivity is monotone property, adding edges cannot destroy connectivity:

$$\Pr[G(p) \text{ is connected}] \geq \Pr[G' \text{ is connected}] = 1 - n^{-\Omega(1)}$$

Therefore, sampling edges of G independently with probability $p = \frac{L}{\lambda}$ results in a connected graph with high probability. □

2 Problem 2 (Nearly Orthonormal Vectors)

Call a set of unit vectors "near-orthonormal" if the inner product of any two of them is close to zero. In this problem we will show that while there are at most d orthonormal vectors in $\mathbb{R}^d$, there can be exponentially many near-orthonormal vectors! For vectors $x, y \in \mathbb{R}^d$, we use $\langle x, y \rangle = \sum_{i=1}^{d} x_i y_i$ to denote the inner product.

(a) Let $x = (x_1, x_2, \ldots, x_d)$ and $y = (y_1, y_2, \ldots, y_d)$ be two independently and uniformly chosen vectors in $\{-1, 1\}^d$. (I.e., each bit x_i and y_i in each vector is independently and uniformly chosen from $\{-1, 1\}$.) Show that

$$\Pr[|\langle x, y \rangle| \geq \varepsilon d] \leq 2 \exp\left(-\varepsilon^2 d/6\right)$$

(b) Given parameter $\varepsilon > 0$, a set S of unit vectors is called *ε-orthonormal* if for all $\vec{x}, \vec{y} \in S$,

$$|\langle \vec{x}, \vec{y} \rangle| \leq \varepsilon.$$

Show that there exists a constant $c > 0$ and constant d_0, such that for any $\varepsilon \leq 1/2$ (say) and any $d \geq d_0$, if you sample $N := \exp(c\varepsilon^2 d)$ random vectors independently and uniformly from the set $\{-\frac{1}{\sqrt{d}}, +\frac{1}{\sqrt{d}}\}^d$, this sampled set is ε-orthonormal with probability at least $1/2$.

2.1 Solution

2.2 Part A

Proof. Let $X_i = x_i y_i$. Because each bit x_i and y_i are chosen uniformly and independently, it follows that $X_1, ..., X_d$ are iid Rademacher random variables. Define

$$X = \sum_{i=1}^{d} X_i = \langle \vec{x}, \vec{y} \rangle$$

By direct computation:

$$M_{X_i}(\lambda) = \mathbb{E}\left[e^{\lambda X_i}\right] = \mathbb{P}[X_i = 1]e^{\lambda} + \mathbb{P}[X_i = -1]e^{-\lambda} = \frac{1}{2}(e^{\lambda} + e^{-\lambda}) = \cosh \lambda$$

Because the X_i are iid variables, we have

$$M_X(\lambda) = \prod_{i=1}^{d} M_{X_i}(\lambda) = \left(M_{X_i}(\lambda)\right)^d = (\cosh \lambda)^d$$

Furthermore, the Taylor expansion of $\cosh \lambda$ yields the following inequality:

$$\cosh \lambda = 1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \ ... \ \leq \sum_{k \geq 0} \frac{(\lambda^2/2)^k}{k!} = e^{\lambda^2/2}$$

Thus:

$$M_X(\lambda) = (\cosh \lambda)^d \leq e^{\lambda^2 d/2}$$

By Markov, we get that

$$\mathbb{P}[X \geq a] \leq \frac{\mathbb{E}\left[e^{\lambda X}\right]}{e^{\lambda a}} = M_X(\lambda) e^{-\lambda a},\ \forall \lambda > 0$$

Using our bound for the MGF yields

$$\mathbb{P}[X \geq a] \leq \exp\left(-\lambda a + \frac{\lambda^2 d}{2}\right)$$

We can optimize this expression over λ:

$$\frac{d}{d\lambda}\left(-\lambda a + \frac{\lambda^2 d}{2}\right) = 0 \Longrightarrow \tilde{\lambda} = \frac{a}{d}$$

It follows that

$$\mathbb{P}[X \geq a] \leq \exp\left(-\tilde{\lambda} a + \frac{\tilde{\lambda}^2 d}{2}\right) = \exp\left(-\frac{a^2}{2d}\right)$$

By using the same bound for $-X$ and taking the union bound,

$$\mathbb{P}[|X| \geq a] \leq 2\exp\left(-\frac{a^2}{2d}\right)$$

Finally, setting $a = \varepsilon d$ gives

$$\mathbb{P}\left[|\langle x, y\rangle| \geq \varepsilon d\right] \leq 2\exp\left(\frac{-\varepsilon^2 d}{2}\right)$$

Which relaxes to

$$\mathbb{P}\left[|\langle x, y\rangle| \geq \varepsilon d\right] \leq 2\exp\left(\frac{-\varepsilon^2 d}{6}\right)$$

□

2.3 Part B

Proof. Let $\varepsilon \in]0, \frac{1}{2}[$, and sample $N = \exp(c\varepsilon^2 d)$ vectors $u^{(1)}, ..., u^{(N)}$ iid and uniformly from the discrete set $\{\pm 1/\sqrt{d}\}^d$. Consider a fixed pair of these sampled vectors $\left(u^{(i)}, u^{(j)}\right)$ with $i \neq j$. We have:

$$\langle u^{(i)}, u^{(j)}\rangle = \sum_{k=1}^{d} u_k^{(i)} u_k^{(j)} = \frac{1}{d}\sum_k X_k$$

Where X_k are iid Rademacher variables. Consider vectors x and y sampled uniformly from $\{-1,1\}$, as in part a). From our results in part a), we get that

$$\langle u^{(i)}, u^{(j)} \rangle = \frac{1}{d}\langle x, y \rangle \implies \mathbb{P}\left[|\langle u^{(i)}, u^{(j)} \rangle| \geq \varepsilon\right] = \mathbb{P}\left[|\langle x, y \rangle| \geq \varepsilon d\right] \leq 2\exp\left(-\frac{\varepsilon^2 d}{6}\right) \tag{1}$$

Let G_{ij} be the event that the desired bound holds for two fixed pair of sampled vectors, and G the even that the desired bound holds for all pairs:

$$G_{ij} = \{|\langle u^{(i)}, u^{(j)} \rangle| \leq \varepsilon\}, \quad G = \bigcap_{i \neq j} G_{ij}.$$

Union bounding over all $\binom{N}{2}$ pairs of vectors in 1 yields:

$$\begin{aligned}
\mathbb{P}(G^C) &\leq \mathbb{P}\left[\bigcup_{i \neq j} G_{ij}^C\right] \\
&= \mathbb{P}\left[\exists i \neq j : |\langle u^{(i)}, u^{(j)} \rangle| \geq \varepsilon\right] \\
&\leq \binom{N}{2} \cdot 2\exp\left(-\frac{\varepsilon^2 d}{6}\right) \\
&\leq \frac{N^2}{2} \cdot 2\exp\left(-\frac{\varepsilon^2 d}{6}\right) \\
&= \exp\left[\left(2c - \frac{1}{6}\right)\varepsilon^2 d\right]
\end{aligned}$$

We notice that it suffices to pick $c < 1/12$ for the exponent to become negative. In what follows, we shall consider $c = 1/24 < 1/12$ for simplicity. Furthermore, the case $N < 2$ trivially yields an ε-orthonormal set, as in that case, we do not have any two pairs of sampled vectors. Thus, we shall consider the case:

$$N \geq 2 \implies \exp(c\varepsilon^2 d) \geq 2 \iff \varepsilon^2 d \geq \frac{\log 2}{c}$$

In which case our above bound becomes, with $c = 1/24$:

$$\mathbb{P}(G^C) \leq \exp\left[\left(2c - \frac{1}{6}\right)\varepsilon^2 d\right] \leq \exp\left[\left(2c - \frac{1}{6}\right)\frac{\log 2}{c}\right] = 2^{2-\frac{1}{6c}} = 2^{2-4} = \frac{1}{4}$$

Thus, taking the complement, we get that the probability that the bound holds for all pairs $\left(u^{(i)}, u^{(j)}\right)$ is:

$$\mathbb{P}(G) = 1 - \mathbb{P}(G^C) \geq 1 - \frac{1}{4} = \frac{3}{4} \geq \frac{1}{2}$$

$\square$

3 Problem 3 (Packet Scheduling with Randomized Delays)

We consider the problem of routing k packets in a network, where packet i goes from s_i to t_i, along a fixed s_i-t_i path P_i. (These paths may be chosen, e.g., using randomized rounding as seen in class, or some other approach—but this does not matter for us.) We assume a synchronous model where time proceeds in discrete steps ($t = 0, 1, 2, \ldots$). For feasibility, each edge should be used by at most one packet per timestep. Define two key parameters based on the given paths:

- **Congestion (C)**: The maximum number of paths P_i that use any single edge.
- **Dilation (D)**: The maximum length (number of edges) of any path P_i.

In the *naïve schedule*, packet i tries to cross the j^{th} edge of P_i at time j. This takes time at most D to complete, but could be infeasible: as many as C packets may try to use an edge at the same time. Let us see how to get schedules which take slightly longer to complete, but have less congestion.

Consider the following randomized approach for defining a *nominal schedule*.

> **Random Delay:** Each packet i independently chooses an initial integer delay Δ_i uniformly at random from the set $\{0, 1, 2, \ldots, C\}$. (Note: There are $C + 1$ choices.)
>
> **Nominal Schedule:** Suppose path $P_i = (e_{i,1}, e_{i,2}, \ldots)$. In the nominal schedule, packet i crosses edge $e_{i,j}$ at time $\Delta_i + j$.

I.e., each packet waits for a random amount Δ_i, and then moves one edge per timestep after that. Of course, this may still cause packets to use an edge at the same time. Define the *nominal congestion* $X_{e,t}$ as the number of packets whose nominal schedule requires them to cross edge e at time t. We want to show that the maximum nominal congestion is small:

(a) Argue that $X_{e,t}$ is a sum of independent random variables, and use a Chernoff bound plus union bound to show that with high probability, the maximum nominal congestion $R_{\max} = \max_{e,t} X_{e,t}$ is $O(\log(k \cdot D \cdot C))$.

Remark: Starting with the nominal schedule, greedily scheduling the packets (where packets wait if an edge is busy) can be shown to result in a schedule length bounded using this result for $R_{\max}$, specifically $O(C + D \cdot R_{\max})$. This result is relatively close to the lower bound $\max(C, D)$. A landmark result by Leighton, Maggs, and Rao (1988) used a much more powerful probabilistic tool, the Lovász Local Lemma (which we will cover in lecture #5), to prove that there exists a choice of delays such that the schedule length is $O(C + D)$. This is a constant factor approximation of the optimal schedule.

3.1 Solution

3.2 Part A

Proof. Consider a fixed edge e and fixed time t. Let

$$I_i^{(e,t)} \stackrel{\text{def}}{=} \mathbb{I}\{\Delta_i = t - j\}$$

Be the indicator variable denoting that e is the the j-th position of packet i at time t. Thus:

$$X_{e,t} = \sum_i I_i^{(e,t)}$$

We note that, because Δ_i are iid, the indicator variables $\{I_i^{(e,t)}\}_i$ are also independent. And since $\Delta_i \sim U\{0, ..., C\}$, we have:

$$\mathbb{P}\left[I_i^{(e,t)} = 1\right] = \mathbb{P}[\Delta_i = t - j] = \frac{1}{C+1}$$

Let n_e be the number of paths that use edge e. By definition, of congestion, we have that $n_e \leq C$. We therefore have:

$$\mu = \mathbb{E}\left[X_{e,t}\right] = \sum_{i=1}^{n_e} \mathbb{E}\left[I_i^{(e,t)}\right] = n_e \cdot \frac{1}{C+1} \leq \frac{C}{C+1} < 1$$

For any $\lambda > 0$, Markov's inequality yields:

$$\mathbb{P}\left[X_{e,t} \geq \alpha\right] = \mathbb{P}\left[e^{\lambda X_{e,t}} \geq e^{\lambda\alpha}\right] \leq \mathbb{E}\left[e^{\lambda X_{e,t}}\right] e^{-\lambda\alpha} \tag{2}$$

By independence, bounding the MGF gives

$$\begin{aligned}
\mathbb{E}\left[e^{\lambda X_{e,t}}\right] &= \prod_{i=1}^{n_e} \mathbb{E}\left[e^{\lambda I_i^{(e,t)}}\right] \\
&= \prod_i \left[(1 - p_i) + p_i e^{\lambda}\right] \\
&= \prod_i \left[1 + p_i(e^{\lambda} - 1)\right] \\
&\leq \prod_i \exp\left[p_i(e^{\lambda} - 1)\right] \\
&= \exp\left[(e^{\lambda} - 1)\sum_i p_i\right] \\
&= \exp\left(\mu(e^{\lambda} - 1)\right)
\end{aligned}$$

Where the inequality comes from the property $(1 + x) \leq e^x$, $\forall x \in \mathbb{R}$, which is a direct result of the Taylor expansion for e^x. Thus, (2) becomes:

$$\mathbb{P}\left[X_{e,t} \geq \alpha\right] \leq \exp\left(\mu(e^{\lambda} - 1) - \lambda\alpha\right) \tag{3}$$

Minimizing the exponent w.r.t $\lambda > 0$ yields $\tilde{\lambda} = \log\frac{\alpha}{\mu}$. And plugging back into (3), we have

$$\mathbb{P}\left[X_{e,t} \geq \alpha\right] \leq \exp\left[\alpha - \mu - \alpha \log \frac{\alpha}{\mu}\right] \leq \exp\left(\alpha - \alpha \log \frac{\alpha}{\mu}\right) = \left(\frac{e\mu}{\alpha}\right)^{\alpha} \leq \left(\frac{e}{\alpha}\right)^{\alpha} \tag{4}$$

Where we use the fact that $\mu < 1$ in the last inequality.

Define N as the number of relevant pairs (e, t). Each of the k packets crosses at most D edges at $(C+1)$ possible time steps. Thus

$$N \leq kD(C+1)$$

By taking the union bound, equation (4) becomes:

$$\mathbb{P}\left[\max_{e,t} X_{e,t} \geq \alpha\right] \leq N \left(\frac{e}{\alpha}\right)^{\alpha}$$

Picking $\alpha = c \log N$ for a suitable constant c, we finally have:

$$M \left(\frac{e}{\alpha}\right)^{\alpha} = \exp\left[\log N + \alpha(1 - \log \alpha)\right] = \exp\left(\log N \left[1 + c - c \log(c \log N)\right]\right) = o(1)$$

Thus

$$\mathbb{P}[R_{\max} \geq c \log N] \leq o(1) \implies \mathbb{P}[R_{\max} \leq c \log N] = 1 - o(1)$$

Which implies

$$R_{\max} = \max_{e,t} X_{e,t} = O(\log N) = O(\log(kDC)) \quad \text{w.h.p}$$

$\square$