ADIC SPACES, RIGID ANALYTIC GEOMETRY, AND PERFECTOID SHEAFINESS

ZACHARY BERENS

Abstract. In this note, I'll go through through the main part of the proof of perfectoid sheafiness. I'll also touch on the reinterpretaion of affinoid spaces and Berkovich spaces in the language of adic spaces.

1. Adic Spaces, Affinoid Spaces, and Berkovich Spaces

Affinoid spaces and Berkovich spaces can be reinterpreted in the language of adic spaces. In the case of affinoid spaces, the actual topology of the adic spectrum corresponds to the G-topology on affinoids. And Berkovich spaces can be understood as maximal Hausdorff quotients of (Tate/analytic) adic spaces.

1.1. **Affinoid Spaces.** Let A be an affinoid algebra. Taking the spectrality of adic spaces for granted, it is relatively straightforward to prove that the category of sheaves of sets on $\mathrm{Spa}(A, A^\circ)$ coincides with the category of sheaves of sets for the G-topology on $\mathrm{Sp}(A)$. We have a fully faithful functor $(-)^{\mathrm{ad}}$ from the category of rigid spaces to the category of adic spaces given by sending $\mathrm{Sp}(A)$ to $\mathrm{Spa}(A, A^\circ)$. Open covers by rational domains on the adic side correspond to admissible covers on the rigid analytic side:

Theorem 1.1.1. *For $X = \mathrm{Sp}(A)$ and $X^{\mathrm{ad}} = \mathrm{Spa}(A, A^\circ)$, a collection $\{U_i\}$ of admissible opens of X is an admissible cover if and only if $\{U_i^{\mathrm{ad}}\}$ is a cover of X^{ad}.*

But there are more opens on the adic side:

Proposition 1.1.1. *$V \subset X^{\mathrm{ad}}$ is not necessarily of the form U^{ad} for $U \subset X$ an admissible open.*

Proof. Let $X = \mathrm{Sp}(k\langle T\rangle)$ with $k = \overline{k}$. We have $X^{\mathrm{ad}} = \mathrm{Spa}(k\langle T\rangle, k^\circ\langle T\rangle)$. Remove the closure of the Gaußpoint $\eta \in X^{\mathrm{ad}}$. We get an open $V = X^{\mathrm{ad}} \setminus \overline{\eta} \subset X^{\mathrm{ad}}$ which is *not* quasicompact (you need an infinite union of disks of different radii). V is readily seen to be covered by the pairwise disjoint opens

$$D_{\overline{a}} = \{|T - a| < 1\}^{\mathrm{ad}}$$

for $a \in k^\circ$ a representative of $\overline{a} \in \widehat{k} = k^\circ/k^{\circ\circ}$. Hence $V \simeq Y^{\mathrm{ad}}$ for

$$Y := \coprod_{\overline{a}} \{|T - a| < 1\}$$

Assume $V \simeq U^{\mathrm{ad}}$ with $U \subset X$ admissible. Then the isomorphism $Y^{\mathrm{ad}} \simeq U^{\mathrm{ad}}$ respecting inclusions into X^{ad} would have to arise from an isomorphism of rigid spaces $Y \simeq U$ respecting inclusions into X. But V clearly contains all Type-I points. So $U = X$ i.e., Y would express X as a highly disconnected space which contradicts the connectedness of rigid spaces. □

We note that V is in the essential image of $(-)^{\mathrm{ad}}$; it just doesn't come from an admissible open of X. In particular if f^{ad} is an open immersion, f need not be an open immersion. The converse, however, is true i.e., the adification functor sends open immersions to open immersions. For more, see [ConradL16].

1.2. **Berkovich Spaces.** The following is also from Conrad's notes.

Theorem 1.2.1. *Let A be a k-affinoid algebra. We have a continuous quotient map $q : \mathrm{Spa}(A, A^\circ) \longrightarrow \mathcal{M}(A)$ given by sending x to its unique rank-1 generization η_x. In particular, $\mathcal{M}(A)$ is the maximal Hausdorff quotient of $\mathrm{Spa}(A, A^\circ)$.*

Proof. x and η_x induce the same topology on their common residue field $\kappa(x)$ (they have the same residue field because η_x is a vertical generization of x). A subbase for the topology on $\mathcal{M}(A)$ is given by open sets of the form $U_{f,g} = \{z \in \mathcal{M}(A) : |f|_z < |g|_z\}$. We see that

$$q^{-1}(U_{f,g}) = \{x \in X : |f|_{\eta_x} < |g|_{\eta_x}\}$$

The inequality $|f|_{\eta_x} < |g|_{\eta_x}$ implies that $|f|_{\eta_x}/|g|_{\eta_x} < 1$. And $|f|_{\eta_x}/|g|_{\eta_x} < 1$ implies that $|f|_{\eta_x}/|g|_{\eta_x}$ is topologically nilpotent for the η_x-topology on $\kappa(x)$ because η_x is rank-1. So $|f|_{\eta_x}/|g|_{\eta_x}$ is topologically nilpotent for the x-topology on $\kappa(x)$. Therefore, $|f|_x/|g|_x < 1$. Note that these fractions make sense because $|g|_{\eta_x} \neq 0$. Loci of the form $\{x \in X : |f|_x < |g|_x\}$ are generally not open in the adic world because of the presence of type-V points.[1]. We'll get around this by using k. To show that $q^{-1}(U_{f,g})$ is open, it suffices to find an open neighborhood V of each point in $q^{-1}(U_{f,g})$ such that $V \subset q^{-1}(U_{f,g})$. Let $x_0 \in q^{-1}(U_{f,g})$ be arbitrary. Take $c \in k$ such that $0 < |c| < 1$. Since $|f|_{x_0}/|f|_{x_0}$ is topologically nilpotent in $\kappa(x_0)$, there exists an n such that $c^{-1}(|f|_{x_0}/|f|_{x_0})^n \in R_{x_0}$ where R_{x_0} is the valuation subring of $\kappa(x_0)$ for x_0. Indeed, cR_{x_0} is open in $\kappa(x_0)$. Define

$$\Omega = \{x \in X : |f^n|_x \leq |cg^n|_x\}$$

which is open in X. We see that $x_0 \in \Omega$ by construction. We'd like to show that $\Omega \subset q^{-1}(U_{f,g})$. Let $x \in \Omega$. The image $|g|_x$ of g in $\kappa(x)$ is nonzero. Thus $|f|_x^n/|c|_x|g|_x^n$ makes sense and lies in the valuation ring R_x of x so it is power-bounded for the x-topology. It is then also power-bounded for the η_x-topology which is rank-1. Thus, the power-bounded elements are precisely those elements in the valuation ring i.e., $R^\circ_{\eta_x} = \{f \in \kappa(x) : |f|_x \leq 1\}$. Thus, $|f^n|_{\eta_x} \leq |cg^n|_{\eta_x} \neq 0$. But $|c|_{\eta_x} < 1$ because c is topologically nilpotent, so $|cg^n|_{\eta_x} < |g^n|_{\eta_x}$. Therefore, $|f|_{\eta_x} < |g|_{\eta_x}$.

Since q is continuous, it sends closed subsets of X to compact subsets of $\mathcal{M}(A)$ which are closed in $\mathcal{M}(A)$ because $\mathcal{M}(A)$ is Hausdorff. So q is a topological quotient map. □

2. Perfectoid Sheafiness

Perfectoid sheafiness is rather non-trivial. This is especially true for the original proof given by Scholze. Buzzard-Verberkmoes found a simplification using the stably uniform criterion for sheafiness. This is the criterion we'll be working with. Perfectoid spaces are uniform by definition. To show that they're stably uniform, we need to show that they are uniform on rational opens. To do this, one proves the following stronger fact: the algebra of functions on a rational domain is perfectoid.

Heuristically, functions on the integral level are almost given by the obvious algebra which is made by adjoining all p^{th} roots of the fractions defining the rational domain to the ring of integral elements and then completing. The category of perfectoid algebras over A is equivalent to the category of "almost perfectoid" algebras over A^+. Scholze uses this equivalence to prove the aforementioned result. Morel, on the other hand, does not.

Fix a prime p for eternity. Let's recall a number of facts about perfectoid rings. A *perfectoid ring* is a complete, uniform, Tate, Huber ring A with pseudouniformizer ϖ such that $\varpi^p|p$ and Frobenius $A^\circ/\varpi A^\circ \longrightarrow A^\circ/\varpi^p A^\circ$ is an isomorphism (in particular, a surjection). Recall that uniform means A° is bounded. We can form the tilt $A^\flat$ of a perfectoid ring by taking an inverse limit over Frobenius. All pseudouniformizers ϖ are first-component projections of a pseudouniformizer $\varpi^\flat \in A^\flat$ up to multiplication by a unit. In particular, we may take ϖ so that it has a system of p^{th} roots in A.

An adic space is a sheafy pre-adic space i.e., a pre-adic space for which the structure presheaf $\mathcal{O}_X$ is a sheaf.

Theorem 2.0.1. *$X = \mathrm{Spa}(A, A^+)$ is an adic space if any of the following hold:*

(1) $\widehat{A}$ is discrete (the schemes case)

(2) A is strongly noetherian i.e., $\widehat{A}\langle X_1, \dots, X_n\rangle$ is noetherian for all n (the rigid analytic case)

[1]Here is an example: take $X = \mathrm{Spa}(k[T], k^\circ[T])$ with k algebraically closed, take $c \in k$ with $0 < |c| < 1$, and consider the set $\{x \in X : |c|_x < |T|_x\}$. This set is not open because its complement $\{x \in X : |T|_x \leq |c|_x\}$ is not closed. Indeed, the type-II point $\nu_{c,|c|}$ has closure containing the type-V point $\nu_{c,|c|^+}$ but $\nu_{c,|c|^+}$ cannot be contained in our set because $\nu_{c,|c|^+}(c) < \nu_{c,|c|^+}(T)$.

(3) $\widehat{A}$ has a noetherian ring of definition (the noetherian formal scheme case)
(4) A is stably uniform i.e., $\mathcal{O}_X(U)$ is uniform for all rational $U \subseteq X$

We're concerned with case 4.

An affinoid perfectoid space is an affinoid adic space $\mathrm{Spa}(A, A^+)$ where (A, A^+) is a perfectoid Huber pair (A is perfectoid and A^+ an open and integrally closed subring of A°).

2.1. **Almost Mathematics, Almost Demystified.** Let A be a perfectoid ring. An A^+-module is called *almost zero* if it's annihilated by $A^{\circ\circ}$. More generally, an *almost setup* is a ring R with ideal I such that $I^2 = I$. In the perfectoid world, we take $R = A^+$ and $I = A^{\circ\circ}$. It's equivalent to take $R = A^\circ$ because, as we'll elaborate upon in the next section, $A^+ \overset{a}{=} A^\circ$. Let $\mathcal{A}$ denote the full subcategory of Mod_R spanned by modules M such that $I \otimes_R M \cong M$. $\mathcal{A}$ is an abelian category. We have the following diagram of categories

$$\begin{tikzcd} \mathrm{Mod}_{R/I} \arrow[r, "i_*"] & \mathrm{Mod}_R \arrow[l, bend right, "i^*"'] \arrow[l, bend left, "i^!"] \arrow[r, "j^*"] & \mathcal{A} \arrow[l, bend right, "j_!"'] \arrow[l, bend left, "j_*"] \end{tikzcd}$$

The functors are given by

$$i^*M = R/I \otimes_R M \qquad j_!N = N$$
$$i^*M = M \qquad j^*M = I \otimes_R M$$
$$i^!M = \mathrm{Hom}_R(R/I, M) \qquad j_*N = \mathrm{Hom}_R(I, N)$$

The composition of any row is zero, any functor is right adjoint to the one above it, $j_!$ is exact, and i_*, $j_!$, and j_* are fully faithful. We call j_* the "almostification" functor and will from now on denote it by $(-)^a$. We call j_*N "the module of almost elements of N" and denote it by $(-)_*$. For torsionfree M, we have $M_* = \{m \in M \otimes_{A^+} A \,|\, \varepsilon M \in M, \, \forall \varepsilon \in A^{\circ\circ}\}$. In $\mathcal{A}$, being zero is equivalent to being almost zero. We determine other properties similarly: a morphism is *almost injective* if its kernel is annihilated by $A^{\circ\circ}$. In particular, a morphism is an *almost isomorphism* if it is both almost injective and almost surjective.

2.2. **Characterizing the Adic Presheaf on Perfectoid Spaces.** Fix $U = U\left(\frac{f_1, \ldots, f_{n-1}, \varpi^N}{g}\right)$ a rational domain of $X = \mathrm{Spa}(A, A^+)$ where (A, A^+) is a perfectoid Huber pair and $\varpi \in A^+$ is a pseudouniformizer of A with $\varpi = (\varpi^\flat)^\sharp$ for $\varpi^\flat$ a pseudouniformizer of $A^\flat$. One can prove that

$$\mathcal{O}_X(U)^{\circ a} \simeq A^+\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \ldots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle^a$$

in two ways. One way makes use of the equivalence of categories $\mathrm{Perf}_A \simeq \mathrm{Perf}_{A^{+a}}$. The former category comprises perfectoid $A-$algebras and the latter category comprises almost perfectoid A^+-algebras with A^+-algebra morphisms up to almost equality. An almost perfectoid A^+-algebra B is a ϖ-adically complete and separated, ϖ-torsionfree,[2] A^+-algebra on which Frobenius induces an almost isomorphism

$$\Phi : B/\varpi^{1/p} \overset{a}{\simeq} B/\varpi$$

It is equivalent to take perfectoid $A^{+a}-$algebras with $A^{+a} = A^+ \otimes_{A^+} A^{\circ\circ}-$algebra morphisms up to equality. These two ways of thinking about almost algebras are the same because everything almost equal becomes equal after tensoring with $A^{\circ\circ}$. It comes down to whether or not you prefer asking that your morphisms are taken up to almost equality or if you prefer tensoring your objects with $A^{\circ\circ}$: almost morphisms or almost objects. Note that $A^{\circ\circ}$ is flat over A^+ so the almostification functor $(-)^a$, given by tensoring with $A^{\circ\circ}$, is exact. Also note that A°/A^+ is almost zero because $A^{\circ\circ} \subset A^+$, thus $A^{\circ a} = A^{+a}$ i.e.,

[2]Scholze works with $K^{\circ a}-$algebras with K a perfectoid field. The definition given here is equivalent to his definition when you restrict to $A^+ = K^\circ$: he requires flatness which is equivalent to $\varpi-$torsionfree-ness. It is the ϖ-torsionfree-ness that makes the equivalence of categories work because we rely on the explicit description of the module of almost elements of a ϖ-torsionfree $K^\circ-$module.

$A^\circ \overset{a}{=} A^+$. So, the ring of integral elements is almost the ring of power bounded elements. The equivalence of categories, taking the "almost morphism" perspective, is given by the two functors

$$\mathrm{Perf}_A \longrightarrow \mathrm{Perf}_{A^{+a}}$$

$$B \longmapsto B^\circ$$

and

$$\mathrm{Perf}_{A^{+a}} \longrightarrow \mathrm{Perf}_A$$

$$B_0 \longmapsto B_0\left[\frac{1}{\varpi}\right]$$

Note that the functors are different if one considers almost objects instead of almost morphisms.[3] By this equivalence of categories, if two objects in Perf_A are isomorphic, they are isomorphic on the almost integral level (i.e., almost isomorphic on the integral level). So, the strategy is to show that $\mathcal{O}_X(U)$ is isomorphic to the perfectoid $A-$algebra

$$A^+\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle\left[\frac{1}{\varpi}\right]$$

It would then follow that the almost integral extensions[4] of $\mathcal{O}_X(U)$ and

$$A^+\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle\left[\frac{1}{\varpi}\right]$$

are isomorphic. One can then conclude that

$$\mathcal{O}_X(U) \overset{a}{\simeq} A^+\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle$$

by showing that the image of $A^+\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle\left[\frac{1}{\varpi}\right]$ along the functor $\mathrm{Perf}_A \longrightarrow \mathrm{Perf}_{A^{+a}}$ is

$$A^+\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle$$

The other way to prove this is to make use of [Morel, Proposition V.1.5.10] which allows you to conclude the same thing (in fact you get almost equality) without making use of the full technology of the equivalence

[3]We are taking $\mathrm{Perf}_{A^{+a}}$ to have almost perfectoid A^+-algebras as objects with morphisms up to almost equality. If we were instead taking perfectoid $A^{+a}-$algebras with morphisms up to equality, the functors would be

$$B \longmapsto B^{\circ a}$$

and

$$B_0 \longmapsto (B_0)_*\left[\frac{1}{\varpi}\right]$$

where $(-)_*$ the functor taking an $A^{+a}-$algebra to its module of almost elements $\mathrm{Hom}_{A^+}(A^{\circ\circ}, B_0)$. This functor is right adjoint to the almostification functor $(-)^a$.

[4]We note here that the almost integral level of a perfectoid $A-$algebra B (as an object in the category $\mathrm{Perf}_{A^{+a}}$) can be thought of in four different ways:

(1) It is the almostification of the integral closure of A^+ in B with morphisms up to equality.
(2) It is the integral closure of A^+ in B with morphisms up to almost equality.
(3) It is $B^{\circ a}$ with morphisms up to equality.
(4) It is B° with morphisms up to almost equality.

In particular, there are four different, yet equivalent functors one can take from Perf_A to $\mathrm{Perf}_{A^{+a}}$. This demonstrates the two perspectives one can take on $\mathrm{Perf}_{A^{+a}}$ as well as the fact that taking integral closures is almost the same as taking power-bounded elements.

of categories $\mathrm{Perf}_A \simeq \mathrm{Perf}_{A^{+a}}$ (but still using most of the equivalence).[5] Instead, it suffices to show that

$$A^+\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle$$

is an almost perfectoid A^+-subalgebra of $\mathcal{O}_X(U)$. Both proofs require the same amount of effort, but the prerequisites are easier for the latter. There is a mistake in Morel's proof of the following theorems: she mistakenly assumes that $A^+\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle$ is integral perfectoid. One can use almost mathematics to get around the problem. In particular, we can take completions as long as we're working with indeterminates. Let's begin.

Theorem 2.2.1. *Let (A, A^+) be a perfectoid Huber pair of characteristic p, let $X = \mathrm{Spa}(A, A^+)$ and let $U = U(\frac{f_1,\dots,f_n}{g})$ be a rational domain of X with $f_n = \varpi$ and $f_i, g \in A^+$ for all i. Then $\mathcal{O}_X(U) = A\left\langle \frac{f_1}{g}, \dots, \frac{f_{n-1}}{g}, \frac{\varpi^N}{g} \right\rangle$ is perfectoid and*

$$\mathcal{O}_X(U)^\circ \overset{a}{=} A^+\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle$$

Proof. We would like to use [Morel, Proposition V.1.5.10] with $B = \mathcal{O}_X(U)$ and

$$B_0 = A^+\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle$$

So, we need to show that $A^+\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle$ is an almost perfectoid A^+-subalgebra of $\mathcal{O}_X(U)$. It is clear by definition that it is an A^+-algebra. Recall that perfectoid of characteristic p implies perfect (in particular, a Tate and Huber ring of characteristic p is perfectoid if and only if it is complete and perfect). First we show that $A^+\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle$ is a subring of $\mathcal{O}_X(U)$. We have the inclusion

$$A^+\left[\frac{f_1}{g}, \dots, \frac{f_n}{g}\right] \subset A^+\left[\left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty}\right]$$

We claim that

$$\varpi^{nN} A^+\left[\left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty}\right] \subset A^+\left[\frac{f_1}{g}, \dots, \frac{f_n}{g}\right]$$

Indeed: any element of $A^+\left[\left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty}\right]$ is a sum of products of the form $\prod_{i=1}^n \left(\frac{f_i}{g}\right)^{1/p^{m_i}}$ with A^+-coefficients. Hence, it suffices to show that $\varpi^{nN} \prod_{i=1}^n \left(\frac{f_i}{g}\right)^{1/p^{m_i}}$ is an element of $A^+\left[\frac{f_1}{g}, \dots, \frac{f_n}{g}\right]$. Noting that $f_n = \varpi^N$, we see that

$$\begin{aligned}
\varpi^{nN} \prod_{i=1}^n \left(\frac{f_i}{g}\right)^{1/p^{m_i}} &= \prod_{i=1}^n \varpi^N \left(\frac{f_i}{g}\right)^{1/p^{m_i}} \\
&= \prod_{i=1}^n (f_i)^{1/p^{m_i}} (g)^{-1/p^{m_i}} f_n \frac{g}{g} \\
&= \prod_{i=1}^n ((f_i)^{1/p^{m_i}} (g)^{1-1/p^{m_i}}) \frac{f_n}{g}
\end{aligned}$$

[5] We note that this proposition is missing the ϖ-torsionfree condition. It needs this condition to hold true and it is implicitly assumed in the proof that this condition holds: the explicit description of $(B_0)_*$ would not work otherwise.

which is an element of $A^+\left[\frac{f_1}{g},\dots,\frac{f_n}{g}\right]$. So, passing to ϖ-adic completions we have

$$\begin{aligned}\varpi^{nN}A^+\left\langle\left(\frac{f_1}{g}\right)^{1/p^\infty},\dots,\left(\frac{f_n}{g}\right)^{1/p^\infty}\right\rangle &\subset A^+\left\langle\frac{f_1}{g},\dots,\frac{f_n}{g}\right\rangle \\ &\subset A^+\left\langle\left(\frac{f_1}{g}\right)^{1/p^\infty},\dots,\left(\frac{f_n}{g}\right)^{1/p^\infty}\right\rangle\end{aligned}$$

Thus, inverting ϖ we get

$$\begin{aligned}A^+\left\langle\left(\frac{f_1}{g}\right)^{1/p^\infty},\dots,\left(\frac{f_n}{g}\right)^{1/p^\infty}\right\rangle\left[\frac{1}{\varpi}\right] &= A^+\left\langle\frac{f_1}{g},\dots,\frac{f_n}{g}\right\rangle\left[\frac{1}{\varpi}\right] \\ &= A\left\langle\frac{f_1}{g},\dots,\frac{f_n}{g}\right\rangle \\ &= \mathcal{O}_X(U)\end{aligned}$$

So,

$$A^+\left\langle\left(\frac{f_1}{g}\right)^{1/p^\infty},\dots,\left(\frac{f_n}{g}\right)^{1/p^\infty}\right\rangle \subset A^+\left\langle\left(\frac{f_1}{g}\right)^{1/p^\infty},\dots,\left(\frac{f_n}{g}\right)^{1/p^\infty}\right\rangle\left[\frac{1}{\varpi}\right] = \mathcal{O}_X(U)$$

Therefore, if we can show that $A^+\left\langle\left(\frac{f_1}{g}\right)^{1/p^\infty},\dots,\left(\frac{f_n}{g}\right)^{1/p^\infty}\right\rangle$ is almost perfectoid, we'll immediately get that $\mathcal{O}_X(U)$ is perfectoid.[6] We need to show that the Frobenius

$$\Phi : A^+\left\langle\left(\frac{f_1}{g}\right)^{1/p^\infty},\dots,\left(\frac{f_n}{g}\right)^{1/p^\infty}\right\rangle/\varpi^{1/p} \longrightarrow A^+\left\langle\left(\frac{f_1}{g}\right)^{1/p^\infty},\dots,\left(\frac{f_n}{g}\right)^{1/p^\infty}\right\rangle/\varpi$$

is an almost isomorphism. We'd like to simply say that

$$A^+\left\langle\left(\frac{f_1}{g}\right)^{1/p^\infty},\dots,\left(\frac{f_n}{g}\right)^{1/p^\infty}\right\rangle$$

is perfect, so Frobenius is an isomorphism. But this doesn't work. The problem is that arbitrary completions of perfectoid algebras like $A[g_1^{1/p^\infty},\dots,g_n^{1/p^\infty}]$ are unpredictable. In particular, completing can introduce almost ambiguity and we'll end up with something not perfectoid. So we must instead consider

$$A^+\big[X_1^{1/p^\infty},\dots,X_n^{1/p^\infty}\big]/I$$

where $I = \big((g)^{1/p^m}X_i^{1/p^m} - (f_i)^{1/p^m} \mid i \in \{1,\dots,n\}, m \in \mathbb{N}\big)$. There are no issues when we have variables instead of elements. That is to say $A\langle X_1^{1/p^\infty},\dots,X_n^{1/p^\infty}\rangle$ *is* perfectoid. Take the map

$$\begin{aligned}\psi : A^+\big[X_1^{1/p^\infty},\dots,X_n^{1/p^\infty}\big] &\longrightarrow A^+\left[\left(\frac{f_1}{g}\right)^{1/p^\infty},\dots,\left(\frac{f_n}{g}\right)^{1/p^\infty}\right] \\ X_i^{1/p^m} &\longmapsto \left(\frac{f_i}{g}\right)^{1/p^m}\end{aligned}$$

[6]We see that

$$A^+\left\langle\left(\frac{f_1}{g}\right)^{1/p^\infty},\dots,\left(\frac{f_n}{g}\right)^{1/p^\infty}\right\rangle$$

is open in $\mathcal{O}_X(U)$ because it contains $A^+\left\langle\frac{f_1}{g},\dots,\frac{f_n}{g}\right\rangle$, an open subring of $\mathcal{O}_X(U)$. So, since it is also ϖ-adically separated, $A^+\left\langle\left(\frac{f_1}{g}\right)^{1/p^\infty},\dots,\left(\frac{f_n}{g}\right)^{1/p^\infty}\right\rangle$ is an open and bounded subring of $\mathcal{O}_X(U)$, and is hence a ring of definition. This follows from an easy application of the open mapping mapping theorem see [Morel, Lemma V.1.5.11].

Clearly we have $I \subset \ker(\psi)$. Let $\overline{\psi}$ denote the surjection

$$\overline{\psi} : A^+[X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}]/I \longrightarrow A^+\left[\left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty}\right]$$

Inverting ϖ gives us an isomorphism $\overline{\psi}[\frac{1}{\varpi}]$ because $f_n = \varpi$, so inverting ϖ inverts g. Thus, $\ker(\overline{\psi})$ is $(\varpi)^\infty$-torsion. Consider an element $f \in \ker(\overline{\psi}) \subset A^+[X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}]/I$. There exists an M such that $(\varpi)^M f = 0$. Then for all $r \in \mathbb{N}$

$$\begin{aligned}(\varpi)^M f \cdot f^{p^r-1} = (\varpi)^M f^{p^r} = 0 &\Rightarrow ((\varpi)^{M/p^r} f)^{p^r} = 0\\ &\Rightarrow (\varpi)^{M/p^r} f = 0\end{aligned}$$

The last implication comes from the fact that $A^+[X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}]/I$ is perfect because

$$A^+[X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}]$$

and I are perfect. Thus, $\ker(\overline{\psi})$ is almost zero, so $\overline{\psi}$ is an almost isomorphism. This is retained after completing, so

$$A^+\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle$$

is almost perfect, hence Frobenius is an almost isomorphism on it. Thus, $A^+\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle$ is almost perfectoid so [Morel, Proposition V.1.5.10] tells us that

$$\mathcal{O}_X(U)^\circ \stackrel{a}{=} A^+\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle$$

In particular,

$$\mathcal{O}_X(U)^\circ = A^+\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle_*$$

where

$$A^+\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle_* = \mathrm{Hom}_{A^+}\left(A^{\circ\circ}, A^+\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle\right)$$

is the A^+-algebra of almost elements of

$$A^+\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle$$

□

Theorem 2.2.2. *Let (A, A^+) be a perfectoid Huber pair of characteristic 0, let $X = \mathrm{Spa}(A, A^+)$, and let $U = R\left(\frac{f_1, \dots, f_n}{g}\right) \subset X^\flat = \mathrm{Spa}(A^\flat, A^{\flat +})$ be a rational domain of $X^\flat$ with $f_n = (\varpi^\flat)^N$ for some N and $f_i^\sharp, g^\sharp \in A^+$ for all i.*[7] *Then $\mathcal{O}_X(U^\sharp) = A\left\langle \frac{f_1^\sharp}{g^\sharp}, \dots, \frac{f_{n-1}^\sharp}{g^\sharp}, \frac{\varpi^N}{g^\sharp} \right\rangle$ is perfectoid and*

$$\mathcal{O}_X(U^\sharp)^\circ \stackrel{a}{=} A^+\left\langle \left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \dots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty} \right\rangle$$

[7] It is possible to ensure that the sharps are in A^+ without changing the rational domain because we can just multiply everything by a high enough power of a pseudouniformizer.

Proof. We would like to show that

$$A^+\left\langle \left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \ldots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty} \right\rangle$$

is an almost perfectoid A^+−subalgebra of $\mathcal{O}_X(U^\sharp)$. The trouble is showing that Frobenius is an almost isomorphism. We need to make use of the results in characteristic p by tilting. We'd like to simply say that

$$A^+\left\langle \left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \ldots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty} \right\rangle/\varpi \cong A^{\flat+}\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \ldots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle/\varpi^\flat$$

to get that Frobenius is an almost isomorphism on

$$A^+\left\langle \left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \ldots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty} \right\rangle/\varpi$$

because it is an almost isomorphism in characteristic p as shown in Theorem 2.2.1. However, this does not work! Indeed, we have almost ambiguity: we can't use tilting that easily. We must instead consider

$$A^+\big[X_1^{1/p^\infty}, \ldots, X_n^{1/p^\infty}\big]/I$$

where $I = \big((g^\sharp)^{1/p^m} X_i^{1/p^m} - (f_i^\sharp)^{1/p^m} \mid i \in \{1, \ldots, n\}, m \in \mathbb{N}\big)$ which does give us an isomorphism on completions

$$A^+\langle X_1^{1/p^\infty}, \ldots, X_n^{1/p^\infty}\rangle/(I, \varpi) \cong A^{\flat+}\langle X_1^{1/p^\infty}, \ldots, X_n^{1/p^\infty}\rangle/(I^\flat, \varpi^\flat)$$

We then know, from Theorem 2.2.1, that we have an almost isomorphism

$$A^{\flat+}\langle X_1^{1/p^\infty}, \ldots, X_n^{1/p^\infty}\rangle/I^\flat \overset{a}{\simeq} A^{\flat+}\left\langle \left(\frac{f_1}{g}\right)^{1/p^\infty}, \ldots, \left(\frac{f_n}{g}\right)^{1/p^\infty} \right\rangle$$

Hence, frobenius is an almost isomorphism on

$$A^+\left\langle \left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \ldots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty} \right\rangle/\varpi$$

We begin: it is clear by definition that

$$A^+\left\langle \left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \ldots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty} \right\rangle$$

is an A^+−algebra. We also see that it is ϖ-adically complete and separated and ϖ−torsionfree by construction. It is a subring of $\mathcal{O}_X(U^\sharp)$ by the exact same argument given in Theorem 2.2.1.: ϖ^{nN} annihilates the cokernel of the inclusion

$$A^+\left[\frac{f_1^\sharp}{g^\sharp}, \ldots, \frac{f_n^\sharp}{g^\sharp}\right] \longrightarrow A^+\left[\left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \ldots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty}\right]$$

Note that $U^\sharp$ is the inverse image of U along the map $\flat : X \longrightarrow X^\flat$ given by $|\cdot|_x \longmapsto |\cdot|_{x^\flat} = |(-)^\sharp|_x$ so it is a rational domain because $f_n^\sharp = \varpi^N$ is a unit (the $f_i^\sharp$'s will still generate A as an ideal). Let

$$b_0 : A^+\left[\left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \ldots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty}\right] \longrightarrow \mathcal{O}_X^+(U^\sharp)$$

be the canonical map induced by the universal property of $\mathcal{O}_X^+(U^\sharp)$. Consider

$$\theta : W(\mathcal{O}_{X^\flat}^+(U)) \longrightarrow \mathcal{O}_X^+(U^\sharp)$$

$$[a] \longmapsto a^\sharp$$

where $a \in \mathcal{O}_{X^\flat}^+(U)$. Let $J = \ker(\theta)$, and let

$$(A, A^+) \longrightarrow (S, S^+) = \big((W(\mathcal{O}_{X^\flat}^+(U))/J)[\frac{1}{[\varpi^\flat]}], W(\mathcal{O}_{X^\flat}^+(U)/J)\big)$$

be the untilt of $(A^\flat, A^{\flat+}) \longrightarrow (\mathcal{O}_{X^\flat}(U), \mathcal{O}^+_{X^\flat}(U))$. That is to say, let (S, S^+) be the untilt *over* (A, A^+) of $(\mathcal{O}_{X^\flat}(U), \mathcal{O}^+_{X^\flat}(U))$ over $(A^\flat, A^{\flat+})$. Different rings may give the same Witt vectors, but the recovery of those rings from the Witt vectors via the untilting procedure is parameterized by certain principal ideals. We see that g divides f_i in $\mathcal{O}_{X^\flat}(U)$ so $g^\sharp$ divides $f_i^\sharp$ in S^+ (using the multiplicativity of the $\sharp$ map). Thus, the map $\mathrm{Spa}(S, S^+) \longrightarrow \mathrm{Spa}(A, A^+)$ factors over $U^\sharp$ and the universal property of $U^\sharp$ implies that $(A, A^+) \longrightarrow (S, S^+)$ extends to a map $(\mathcal{O}_X(U^\sharp), \mathcal{O}^+_X(U^\sharp)) \longrightarrow (S, S^+)$. Consider the map $c : \mathcal{O}^+_X(U^\sharp) \longrightarrow S^+$. Let

$$d_0 = c \circ b_0 : A^+\left[\left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \dots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty}\right] \longrightarrow S^+$$

Taking ϖ-adic completions, we get the following diagram (a_0 and d_0 extend to maps a and d because they're continuous):

$$\begin{tikzcd}
A^+[X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}]/I \arrow[r, "a_0"] \arrow[d] & A^+\left[\left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \dots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty}\right] \arrow[r, "d_0"] \arrow[d] & S^+ \arrow[d, equal] \\
A^+\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\rangle/I \arrow[r, "a"] & A^+\left\langle\left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \dots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty}\right\rangle \arrow[r, "d"] & S+
\end{tikzcd}$$

Consider the maps in the bottom row modulo ϖ. We have

$$A^+\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\rangle/(I, \varpi) \cong A^{\flat+}\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\rangle/(I^\flat, \varpi^\flat)$$

and

$$S^+/\varpi \cong S^{\flat^+}/\varpi^\flat = \mathcal{O}^+_{X^\flat}(U)/\varpi^\flat$$

From Theorem 2.2.1., we have almost isomorphisms

$$A^{\flat+}\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\rangle/I^\flat \overset{a}{\simeq} A^{\flat+}\left\langle\left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty}\right\rangle \overset{a}{\simeq} \mathcal{O}^+_{X^\flat}(U) = S^{\flat^+}$$

Thus we have

$$\begin{tikzcd}
A^{\flat+}\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\rangle/I^\flat \arrow[r, "\overset{a}{\sim}"] \arrow[d] & A^{\flat+}\left\langle\left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty}\right\rangle \arrow[r, "\overset{a}{\sim}"] \arrow[d] & S^{\flat+} \arrow[d] \\
A^{\flat+}\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\rangle/(I^\flat, \varpi^\flat) \arrow[r, "\overset{a}{\sim}"] \arrow[d, "\wr"] & A^{\flat+}\left\langle\left(\frac{f_1}{g}\right)^{1/p^\infty}, \dots, \left(\frac{f_n}{g}\right)^{1/p^\infty}\right\rangle/\varpi^\flat \arrow[r, "\overset{a}{\sim}"] \arrow[d, "\wr a"] & S^{\flat+}/\varpi^\flat \arrow[d, "\wr"] \\
A^+\langle X_1^{1/p^\infty}, \dots, X_n^{1/p^\infty}\rangle/(I, \varpi) \arrow[r, "\overset{a}{\sim}"] & A^+\left\langle\left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \dots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty}\right\rangle/\varpi \arrow[r, "\overset{a}{\sim}"] & S^+/\varpi
\end{tikzcd}$$

where $\overset{a}{\sim}$ and $\wr a$ denote almost isomorphisms. Frobenius is an isomorphism on S^+/ϖ which is almost isomorphic to

$$A^+\left\langle\left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \dots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty}\right\rangle/\varpi$$

by the above diagram. Thus, Frobenius is an almost isomorphism on

$$A^+\left\langle \left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \dots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty} \right\rangle/\varpi$$

so $A^+\left\langle \left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \dots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty} \right\rangle$ is an almost perfectoid A^+-algebra. Hence, we are done: [Morel, Proposition V.1.5.10] tells us that

$$\mathcal{O}_X(U^\sharp)^\circ \stackrel{a}{=} A^+\left\langle \left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \dots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty} \right\rangle$$

In particular,

$$\mathcal{O}_X(U^\sharp)^\circ = A^+\left\langle \left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \dots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty} \right\rangle_*$$

where

$$A^+\left\langle \left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \dots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty} \right\rangle_* = \mathrm{Hom}_{A^+}\left(A^{\circ\circ}, A^+\left\langle \left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \dots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty} \right\rangle\right)$$

is the A^+-algebra of almost elements of

$$A^+\left\langle \left(\frac{f_1^\sharp}{g^\sharp}\right)^{1/p^\infty}, \dots, \left(\frac{f_n^\sharp}{g^\sharp}\right)^{1/p^\infty} \right\rangle$$

$\square$

We need to show that there's a homeomorphism $X \longrightarrow X^\flat$ that sends rational domains to rational domains. This requires a rather torturous yet self-contained approximation lemma. Taking that for granted, we conclude that every rational domain of $\mathrm{Spa}(A, A^+)$ with A in characteristic 0 is of the form $U^\sharp$ for some $U \subset X^\flat$. Stably uniform Huber pairs are sheafy. Thus, perfectoids are sheafy because they are uniform by definition (and the algebra of functions on rational domains is also perfectoid, hence uniform).