ÉTALE RECONSTRUCTION OF $\mathbb{F}_p(t)$-SCHEMES

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ABSTRACT. Let k be a field that is finitely generated over $\mathbb{Q}$. Voevodsky proved that morphisms out of a normal k-scheme of finite type can be reconstructed from admissible morphisms out of its étale site. In particular a normal k-scheme of finite type can be reconstructed from its étale site. Let K be a field that is finitely generated over $\mathbb{F}_p(t)$. Grothendieck conjectured that one should be able to reconstruct perfections of finite type K-schemes. In this paper, I adapt Voevodsky's methods to prove this.

CONTENTS

1. Introduction

Let k be an absolutely finitely generated field of characteristic 0 i.e., a field that is finitely generated over $\mathbb{Q}$. Let X and Y be finite type k-schemes. Let $\mathrm{Shv}(X_{\text{ét}})$ and $\mathrm{Shv}(Y_{\text{ét}})$ be the étale topoi of X and Y respectively. In his letter to Faltings [LTF], Grothendieck conjectured that, if X is normal, there is a bijection

$$\mathrm{Mor}_k(X,Y) \longrightarrow \mathrm{Mor}_{\mathrm{Shv}(k_{\text{ét}})}(\mathrm{Shv}(X_{\text{ét}}), \mathrm{Shv}(Y_{\text{ét}}))$$

where the right-hand side denotes the set of $\mathrm{Shv}(k_{\text{ét}})$-morphisms of étale topoi up to (2-)isomorphism, and the map sends a morphism of schemes to the induced morphism of topoi given by pullback.[1]

For a scheme X, let $X_{\text{ét}}$ be the quasicompact and separated étale site of X whose opens are étale, quasicompact, and separated morphisms $U \longrightarrow X$. Voevodsky proved the following:[2]

Theorem 1.1 (Voe, Theorem 3.1)**.** *If X and Y are k-schemes of finite type and X is normal, there is a bijection*

$$\mathrm{Mor}_k(X,Y) \longrightarrow \mathrm{Mor}^{\bullet}_{k_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}})$$

where the right-hand side is the set of (2-)*isomorphism classes of "admissible"* $\mathrm{Spec}(k)_{\text{ét}}$*-morphisms from $X_{\text{ét}}$ to $Y_{\text{ét}}$.*

As before, the map is given by pullback. An admissible morphism is a morphisms of sites whose induced map on topological spaces sends closed points to closed points (Definition 2.5). An immediate corollary of Voevodsky's result is that two normal schemes X and Y of finite type over k are isomorphic over k if and only if there is a $\mathrm{Spec}(k_{\text{ét}})$-equivalence of sites $X_{\text{ét}} \simeq Y_{\text{ét}}$ [Voe, Corollary 3.1]. In particular, a normal scheme can be reconstructed from its quasicompact and separated étale site. It turns out that morphisms of quasicompact and separated sites are the same as morphisms of topoi in this setting [CHW, A.13 Corollary]. And the topos for $X_{\text{ét}}$ is equivalent to the topos for the full étale site if X is qs [CHW, A.4 Lemma]. So Voevodsky showed that Grothendieck's conjecture is true after restricting to admissible morphisms of sites.

Let S be an absolutely finitely generated (AFG) field of any characteristic i.e., a field that is finitely generated over its prime field. In *Esquisse d'un Programme* [EDP], Grothendieck gave a more general conjecture (that is also intimated in his Letter to Faltings). Following Voevodsky, we have to restrict to admissible morphisms of sites/topoi. Then Grothendieck's conjecture becomes

Conjecture 1.2 (The Toposic Hom Conjecture)**.** *Let $\mathscr{C} = \mathbf{Sch}_K^{\mathrm{ft}}[\mathrm{UH}^{-1}]$ be the localization of the category of finite type K-schemes at the universal homeomorphisms. Let $X, Y \in \mathrm{Ob}(\mathscr{C})$. Then the natural map*

$$\mathrm{Mor}_{\mathscr{C}}(X,Y) \longrightarrow \mathrm{Mor}^{\bullet}_{\mathrm{Shv}(K_{\text{ét}})}(\mathrm{Shv}(X_{\text{ét}}), \mathrm{Shv}(Y_{\text{ét}}))$$

is a bijection.

This is basically the hom form of the anabelian main conjecture with étale fundamental groups replaced with étale topoi. For this reason, it makes sense to call it the toposic hom conjecture. The étale fundamental group factors through the étale topos so that we have functors

$$\mathbf{Sch}_K^{\mathrm{ft}} \longrightarrow \mathbf{Topoi} \longrightarrow \mathbf{Groups}$$

The hom conjecture cares about the situations in which the composite is fully faithful. We are interested in the situations in which the first functor is fully faithful. The idea is that, in nice cases, universal homeomorphisms should be the only morphisms that the étale topos can't see. A universal homeomorphism $X \longrightarrow Y$ is a map for which $|X \times_Y Z| \cong |Z|$ for all $Z \longrightarrow Y$. Equivalently, a universal homeomorphism is integral, surjective, and universally injective [Stacks, Tag 04DF]. Inverting universal homeomorphisms is necessary because the étale topology cannot distinguish between schemes that are universally homeomorphic—this is

[1] Grothendieck mentions that there can be at most one isomorphism between any two $\mathrm{Shv}(k_{\text{ét}})$-morphisms of sites. This rigidity property implies that $\mathrm{Mor}_{\mathrm{Shv}(k_{\text{ét}})}(\mathrm{Shv}(X_{\text{ét}}), \mathrm{Shv}(Y_{\text{ét}}))$ is a set so that we can say "bijection." I do not know of a proof of this, and I do not know if it is true. But [CHW, 2.22 Proposition] shows it for admissible morphisms (morphisms of topoi for which the induced morphism on underlying topological spaces sends closed points to closed points) which is all we need.

[2] Voevodsky mistakenly assumes that X need only be nonsingular in codimension 1. See Remark 1.3.

the "topological invariance of the étale site" [Stacks, Tag 04DY]. In particular, if $f : X \longrightarrow Y$ is a universal homeomorphism, the induced morphism between étale sites given by pullback is an equivalence.

A scheme is absolutely weakly normal if any universal homeomorphism into it is an isomorphism. Any scheme X admits a canonical absolute weak normalization $X^{\mathrm{awn}} \longrightarrow X$ which is the initial object in the category of universal homeomorphisms $Y \longrightarrow X$ [Stacks, Tag 0EUK]. Over $\mathbb{Q}$, the absolute weak normalization is the seminormalization [Rydh, Remark B.8] (see Definition 1.4; Intuition 1.5). Over $\mathbb{F}_p$, it is the perfection (the limit over the absolute Frobenius) [BS, Lemma 3.8/Remark 3.9]. Via the functor that sends a scheme to its absolute weak normalization, the localization of the category of finite type K-schemes at the universal homeomorphisms is equivalent to the full subcategory of K^{awn}-schemes that are absolute weak normalizations of a finite type K-scheme [BGH, 14.3.3 Corollary; CHW, 1.32 Lemma]. Since seminormalization preserves being of finite type, and since $\mathrm{Spec}(K)$ is seminormal, this localization is equivalent to the category of seminormal K-schemes when $\mathrm{char}(K) = 0$. The perfection of a scheme of finite type, on the other hand, is almost never of finite type. So when $\mathrm{char}(K) > 0$, we work with the category of perfect $K^{\mathrm{awn}} = K^{p^{-\infty}}$−schemes that are universally homeomorphic to a finite type K-scheme.

At the beginning of [Voe], Voevodsky suggests that, "with some modifications," his proof "seems to apply to schemes over finitely generated fields of characteristic $p > 0$ [that] have transcendence degree ≥ 1." He was right when it comes to perfections of finite type schemes. Let K be an infinite AFG field.

Theorem 3.10. *Let $X = X_0^{\mathrm{perf}}$ be the perfection of a finite type K-scheme and let Y be any finite type $K^{p^{-\infty}}$−scheme. Then the natural map*

$$\mathrm{Mor}_{K^{p^{-\infty}}}(X, Y) \longrightarrow \mathrm{Mor}^{\bullet}_{K_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}})$$

is a bijection.

Corollary 3.11. *Let X and Y be perfections of K-schemes of finite type. Then the natural map*

$$\mathrm{Mor}_{K^{p^{-\infty}}}(X, Y) \longrightarrow \mathrm{Mor}^{\bullet}_{K_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}})$$

is bijective.

Corollary 3.12. *Let X and Y be perfections of schemes of finite type over K. Then $X \cong Y$ if and only if $X_{\text{ét}} \simeq Y_{\text{ét}}$ (over $K^{p^{-\infty}}$ and $\mathrm{Spec}(K)_{\text{ét}}$ respectively).*

This settles the toposic hom conjecture in positive characteristic for all but the case of finite fields. During the course of this project, I found out that Carlson, Haine, and Wolf independently show this in a similar way [CHW, 6.25 Corollary]. They also show that seminormal schemes of finite type over an AFG field of characteristic 0 can be reconstructed from their étale sites, so the toposic hom conjecture holds over AFG fields of characteristic 0 [CHW, 0.3 Theorem] (see Remark 1.11).

Remark 1.3. Let k be an AFG field of characteristic 0. For [Voe, Theorem, 3.1], Voevodsky actually only assumes that X is nonsingular in codimension 1 (R_1). This is not quite sufficient. In his proof, he uses a lemma that says that Picard groups of R_1 schemes of finite type over k are finitely generated. But the proof of this lemma works only if X is S_2 in addition to being R_1. There are R_1 schemes of finite type over k that fail to be étale reconstructible. For example, let X be the scheme obtained from $\mathbb{A}^2_k$ by collapsing two linearly independent tangent vectors, creating a cusp. Because of the cusp, X is not seminormal. So it is not normal. But it is R_1, so it is not S_2. Its seminormalization $\mathbb{A}^2_k \longrightarrow X$ induces an equivalence on étale sites $\mathbb{A}^2_{k,\text{ét}} \xrightarrow{\sim} X_{\text{ét}}$ whose inverse does not come from a morphism of schemes. So there exists an étale open U of X such that $\mathrm{Pic}(U)$ has infinitely divisible torsion elements. We can show that the Picard group of $X = \mathrm{Spec}(A)$ is not finitely generated. Consider the Milnor pullback square [Wei, Example 2.6]

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "\iota"'] & k[x,y] \arrow[d, "\tilde{\iota}"] \\ k \arrow[r, "\overline{f}"] & k[x,y]/(x,y)^2 \end{tikzcd}$$

One computes that $A = k[x^2, xy, y^2, x^3, y^3, xy^3, yx^3]$. By [Wei, Theorem 3.10], we get a Mayer-Vietoris sequence

$$1 \longrightarrow A^\times \longrightarrow k^\times \times k[x,y]^\times \longrightarrow (k[x,y]/(x,y)^2)^\times \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\mathbb{A}^2_k) \times \mathrm{Pic}(k) \longrightarrow \dots$$

which yields

$$1 \longrightarrow k^\times \longrightarrow k^\times \times k^\times \longrightarrow k^\times \times k^+ \times k^+ \longrightarrow \mathrm{Pic}(X) \longrightarrow 1$$

The map $k^\times \longrightarrow k^\times \times k^\times$ is the diagonal. And the map $k^\times \times k^\times \longrightarrow k^\times \times k^+ \times k^+$ is given by

$$(a,b) \mapsto (\tilde{\iota}(b)\overline{f}(a)^{-1}, 0, 0)$$

We conclude that $\mathrm{Pic}(X) \cong k^+ \times k^+$ (c.f. Sanity Check 1.6). The two k^+'s correspond to the two tangent vectors we collapsed to create X. Intuitively, a line bundle on X is given by a line bundle on $\mathbb{A}^2_k$ with gluing data on the two tangent vectors.

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I. **Characteristic** 0**.** Let k be an AFG field of characteristic 0. As mentioned before, absolute weak normality is the same as seminormality when we are working over k. There's a more concrete and geometric criterion for a scheme to be seminormal. The idea is that we want to avoid having cusp-like singularities (e.g., A_n singularities are forbidden).

Definition 1.4 (Stacks, Tag 0EUK)**.** An affine k-scheme $X = \mathrm{Spec}(A)$ is *seminormal* if for all $x, y \in A$ such that $x^2 = y^3$, there exists an $a \in A$ such that $a^2 = y$ and $a^3 = x$.

One immediately sees from this definition that the cuspidal cubic $\mathrm{Spec}\, k[x,y]/(y^2 - x^3)$ is not seminormal.

Intuition 1.5 (SS). Seminormal schemes are schemes for which all non-normality comes from maximally transverse gluing. Any non-normal affine scheme X of finite type over k can be constructed as the pushout of the diagram

$$\begin{tikzcd} Y \arrow[r, two heads, "\text{finite}"] \arrow[d, hook, "\text{closed}"'] & Z \\ \widetilde{X} & \end{tikzcd}$$

where $\widetilde{X}$ is the normalization of X, Z is the non-normal locus of X, and Y is the inverse image of the non-normal locus in $\widetilde{X}$. If X is seminormal, Y is reduced. So there is no "undue gluing of tangent spaces" in the non-normal locus of a seminormal scheme. The cuspidal cubic is not seminormal because it is given by gluing $\widetilde{X} = \mathbb{A}^1$ to $Z = \mathrm{Spec}(k)$ along $Y = \mathrm{Spec}(k[\varepsilon]/\varepsilon^2)$. The nodal cubic, on the other hand, *is* seminormal because it is given by gluing $\widetilde{X} = \mathbb{A}^1$ to $Z = \mathrm{Spec}(k)$ along $Y = \mathrm{Spec}(k) \coprod \mathrm{Spec}(k)$.

Let X and Y be finite type k-schemes with X normal. Consider morphisms $\mathrm{Mor}_k(X,Y)$. Voevodsky's proof of Theorem 1.1 proceeds in three steps:

(1) Show that it suffices to (a) replace Y with $\mathbb{G}_m$ and (b) show that, for every affine, geometrically connected scheme U that is étale over X and every $\varphi \in \mathrm{Mor}^\bullet_{k_{\text{ét}}}(U_{\text{ét}}, \mathbb{G}_{m,\text{ét}})$, there exists a map of k-schemes $U \longrightarrow \mathbb{G}_m$ that coincides with φ on geometric points (φ is admissible so it gives a map on geometric points: see Proposition 2.10).

(2) We may replace U with X because schemes étale over a normal scheme are also normal. We construct maps

$$\begin{tikzcd} \widehat{\mathscr{O}}^*(\overline{X}) \arrow[dr, "\chi"] & \\ & H^1_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}}(1)) \\ \mathscr{O}^*_{\text{top}}(X) \arrow[ur, "\chi_{\text{top}}"'] & \end{tikzcd}$$

where

$$\widehat{\mathscr{O}}^*(\overline{X}) = \varprojlim_n \mathscr{O}^*(\overline{X})/\mathscr{O}^*(\overline{X})^n$$
$$\mathscr{O}^*_{\text{top}}(X) = \mathrm{Mor}^{\bullet}_{k_{\text{ét}}}(X_{\text{ét}}, \mathbb{G}_{m,\text{ét}})$$

and show that it suffices for the image of χ_{top} to lie in the image of χ.

(3) Show that the image of χ_{top} lies in the image of χ when X is normal by showing that the Picard group of a normal k-scheme has bounded torsion i.e., that the Tate module $\widehat{T}(\mathrm{Pic}(X))$ vanishes.[3] This follows from Mordell-Weil-Néron-Lang which implies that $\mathrm{Pic}(X)$ is finitely generated. This is the only point at which normality is used.

Sanity Check 1.6. Regarding (3), I'll note that there are non-seminormal schemes with Picard groups that have bounded torsion. For example, the Picard group of the cuspidal cubic over k is k^+. For reconstruction, one would need to show that the Picard groups of all affine, geometrically connected schemes that are étale over the cuspidal cubic have bounded torsion. But there are étale opens of the cuspidal cubic with Picard groups of unbounded torsion. An example is the once-punctured cuspidal cupic X (with a puncture at $(1,1)$, away from the cusp). Some computation shows that $X = \mathrm{Spec}(k[x, y, \frac{y+1}{x-1}]/(y^2 - x^3))$ (note this this is a non-distinguished affine open of the cuspidal cubic). The seminormalization of X is $\mathbb{G}_m$. So the natural map

$$\mathrm{Mor}_k(X, \mathbb{G}_m) \longrightarrow \mathrm{Mor}^{\bullet}_{k_{\text{ét}}}(X_{\text{ét}}, \mathbb{G}_{m,\text{ét}})$$

can't be surjective because the equivalence $X_{\text{ét}} \xrightarrow{\sim} \mathbb{G}_{m,\text{ét}}$, given by the inverse of the equivalence induced by the seminormalization, is not realized by an isomorphism of schemes. One can compute the Picard group of X as follows: Consider the Milnor square [Wei, Example 2.6]

$$\begin{tikzcd} k\left[t^2, t^3, \frac{t^3+1}{t^2-1}\right] \arrow[r] \arrow[d] & k[(t-1)^{\pm 1}] \arrow[d] \\ k \arrow[r] & k[\varepsilon]/\varepsilon^2 \end{tikzcd}$$

The top map records the seminormalization of X and the map on the right is given by $(t-1)^n \mapsto 1 + n\varepsilon$. We get a Mayer-Vietoris sequence

$$1 \longrightarrow k\left[t^2, t^3, \frac{t^3+1}{t^2-1}\right]^\times \longrightarrow k[(t-1)^{\pm 1}]^\times \times k^\times \longrightarrow (k[\varepsilon]/\varepsilon^2)^\times \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\mathbb{G}_m) \times \mathrm{Pic}(k) \longrightarrow \dots$$

The last term is trivial. We find that

$$k\left[t^2, t^3, \frac{t^3+1}{t^2-1}\right]^\times = k^\times$$

and

$$(k[\varepsilon]/\varepsilon^2)^\times \cong \{a + b\varepsilon : a \in k^\times, b \in k\} \cong k^\times \times k^+$$

So we have

$$1 \longrightarrow k^\times \longrightarrow k[(t-1)^{\pm 1}]^\times \times k^\times \longrightarrow k^\times \times k^+ \longrightarrow \mathrm{Pic}(X) \longrightarrow 1$$

[3] I will note that Voevodsky makes a typo when he says that $\widehat{T}(\mathrm{Pic}(X)) = 0$ is equivalent to saying that $\mathrm{Pic}(X)$ has no infinitely divisible elements. He means no infinitely divisible *torsion* elements.

and the image of $k[(t-1)^{\pm 1}]^\times$ in k^+ is $\mathbb{Z}$. We conclude that $\mathrm{Pic}(X) \cong k^+/\mathbb{Z}$.

Normal schemes are seminormal, but there are non-normal seminormal schemes e.g., the nodal cubic. It turns out that normality is indeed a suboptimal assumption for reconstructing schemes from their étale sites. Voevodsky's proof gives us a method for proving generalizations in characteristic 0. We start with a scheme X satisfying some property P. Then we show that, if $U \longrightarrow X$ is étale, U satisfies P. Then we show that the Picard groups of P schemes have bounded torsion. Using this strategy and results of [GJRW], we will conclude that seminormal, S_2 schemes of finite type over k are étale reconstructible.

Lemma 1.7 (Rush, Lemma 1.10)**.** *If $U \longrightarrow X$ is étale and X is a seminormal k-scheme of finite type, then U is also seminormal.*

The following two lemmas are well known. We give proofs for completeness.

Lemma 1.8. *Being S_2 is an étale-local property.*

Proof. Let $X = \mathrm{Spec}(A)$. Let $U = \mathrm{Spec}(B) \longrightarrow X$ be étale. Take $\mathfrak{q} \in \mathrm{Spec}(B)$ lying above $\mathfrak{p} \in \mathrm{Spec}(A)$. Then $A_\mathfrak{p} \longrightarrow B_\mathfrak{q}$ is an étale map of local rings, so it is faithfully flat. Hence all primes in B of height ≥ 2 lie above the primes in A of height ≥ 2. Since $A_\mathfrak{p} \longrightarrow B_\mathfrak{q}$ is étale, $\mathrm{depth}(A_\mathfrak{p}) = \mathrm{depth}(B_\mathfrak{q})$. Conversely, if $U \longrightarrow X$ is étale and surjective and U is S_2 then X is S_2 by the same argument. □

Lemma 1.9. *Let X be a Noetherian S_2 scheme. Let $Z \subset X$ be a closed subscheme of codimension ≥ 2. Then we get an injection $\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}(X \smallsetminus Z)$.*

Proof. By the cohomological characterization of S_2, we have $H^0_Z(X, \mathscr{F}) = H^1_Z(X, \mathscr{F}) = 0$ for any coherent sheaf $\mathscr{F}$ with support on X (in general, a scheme is S_n if and only if $H^i_Z(X, \mathscr{F}) = 0$ for all $i < n$ for all closed $Z \subset X$ of codimension $\geq n$). So the long exact sequence in cohomology with closed support Z

$$H^0_Z(X, \mathscr{F}) \longrightarrow H^0(X, \mathscr{F}) \longrightarrow H^0(X \smallsetminus Z, \mathscr{F}) \longrightarrow H^1_Z(X, \mathscr{F}) \longrightarrow \dots$$

yields an isomorphism

$$\mathrm{res}_{X \smallsetminus Z}(\mathscr{F}) : H^0(X, \mathscr{F}) \xrightarrow{\sim} H^0(X \smallsetminus Z, \mathscr{F})$$

Let $\mathscr{L} \in \mathrm{Pic}(X)$ be a line bundle. Let $\mathscr{H} = \mathscr{H}\!om(\mathscr{L}, \mathscr{O}_X)$ and $\mathscr{H}' = \mathscr{H}\!om(\mathscr{O}_X, \mathscr{L})$. Assume that $\mathscr{L}|_{X \smallsetminus Z} \cong \mathscr{O}_{X \smallsetminus Z}$ and choose an isomorphism $s \in H^0(X \smallsetminus Z, \mathscr{H})$. Its inverse s^{-1} is in $H^0(X \smallsetminus Z, \mathscr{H}')$. So $s \circ s^{-1} = \mathrm{id}_{\mathscr{L}|_{X \smallsetminus Z}}$ and $s^{-1} \circ s = \mathrm{id}_{\mathscr{O}_{X \smallsetminus Z}}$. Since $\mathrm{res}_{X \smallsetminus Z}(\mathscr{H})$ and $\mathrm{res}_{X \smallsetminus Z}(\mathscr{H}')$ are isomorphisms, we get an isomorphism $\mathscr{L} \cong \mathscr{O}_X$ from the inverse images of s and s^{-1} in $H^0(X, \mathscr{H})$ and $H^0(X, \mathscr{H}')$. □

Theorem 1.10. *Let X be a seminormal, S_2 scheme of finite type over k. Let Y be any k-scheme of finite type. The map*

$$\mathrm{Mor}_k(X, Y) \longrightarrow \mathrm{Mor}^\bullet_{k_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}})$$

is a bijection.

Proof. It suffices to show that $\mathrm{Pic}(X)$ has bounded torsion. As mentioned above in (1), we may reduce to the case in which X is affine. We reproduce the argument given in [GJRW, Theorem 6.5] which shows that

$$\mathrm{Pic}(X) \cong \text{finite group} \oplus \text{countably generated free abelian group}$$

Let $\mathcal{C}$ be the class of groups that can be written as the direct sum of a finite group and a countably generated free abelian group. Let $X = \mathrm{Spec}(A)$ and let $\widetilde{X} = \mathrm{Spec}(\widetilde{A})$ be the normalization of X. Let $I = \{x \in A : x\widetilde{A} \subset A\}$ be the conductor of A in $\widetilde{A}$ (which is an ideal of both A and $\widetilde{A}$); the non-normal locus of X is $Z = \mathrm{Spec}(A/I)$. The non-normal locus Q of Z is of codimension ≥ 2 in X (because Q is of codimension ≥ 1 in Z, and Z is of codimension ≥ 1 in X). It is also closed in X. So by Lemma 1.9,

$\mathrm{Pic}(X) \hookrightarrow \mathrm{Pic}(X \smallsetminus Q)$. Hence, we may replace X with $X \smallsetminus Q$ and proceed under the assumption that Z is normal. In particular, $\mathrm{Pic}(Z)$ is finitely generated. We have a pushout square by [Wei, Example 2.6]

$$\begin{tikzcd} Y \arrow[r, two heads, "\text{finite}"] \arrow[d, hook, "\text{closed}"'] & Z \arrow[d] \\ \widetilde{X} \arrow[r] & X \end{tikzcd}$$

where Y is the inverse image of Z in $\widetilde{X}$. Since X is seminormal, Y is reduced. The pushout is a Milnor square, so we get a Mayer–Vietoris sequence

$$\ldots \longrightarrow \widetilde{A}^{\times} \oplus (A/I)^{\times} \longrightarrow (\widetilde{A}/I)^{\times} \longrightarrow \mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\widetilde{X}) \oplus \mathrm{Pic}(Z) \longrightarrow \mathrm{Pic}(Y) \longrightarrow \ldots$$

Let Λ denote the integral closure of k in $\widetilde{A}/I$. Let E_1 denote the integral closure of k in A/I. And let E_2 denote the image in Λ of the integral closure of k in A. Let D denote the kernel of the map $\mathrm{Pic}(X) \longrightarrow \mathrm{Pic}(\widetilde{X}) \oplus \mathrm{Pic}(Z)$. Since Y is reduced, [GJRW, Proposition 1.6] gives us the finitely generated term in the exact sequence

$$\text{finitely generated} \longrightarrow \Lambda^*/E_1^*E_2^* \longrightarrow D \longrightarrow 0.$$

By [GJRW, Theorem 6.4(2)], $\Lambda^*/E_1^*E_2^* \in \mathcal{C}$ because Y is reduced. Hence, $D \in \mathcal{C}$. The Picard groups of $\widetilde{X}$ and Z are finitely generated because $\widetilde{X}$ and Z are normal. One sees that $\mathcal{C}$ is equivalent to the class of groups given by countably many generators and finitely many relations. Then it is not hard to show that $\mathcal{C}$ is closed under extensions. So $\mathrm{Pic}(X) \in \mathcal{C}$. We conclude that $\widehat{T}(\mathrm{Pic}(X)) = 0$. □

Remark 1.11. I do not know if the Picard groups of seminormal k-schemes have bounded torsion. In [CHW, Section 4], it is shown that one can prove reconstruction for seminormal schemes by using h-descent and de Jong's alterations to reduce to the case of regular schemes which are étale reconstructible by Voevodsky's theorem. In particular, the toposic hom conjecture is true over AFG fields of characteristic 0.

II. **Positive Characteristic.** Fix a prime p for the rest of this paper. Recall that an $\mathbb{F}_p$-scheme is perfect if the absolute Frobenius endomorphism Φ—given by $f \mapsto f^p$ on functions and the identity on the underlying topological space—is an isomorphism. Any $\mathbb{F}_p$-scheme X admits a perfection $X^{\mathrm{perf}} := \lim_{\Phi} X$ (we write lim for the projective limit and colim for the inductive limit).[4] For an affine scheme $\mathrm{Spec}(A)$, we have $\mathrm{Spec}(A)^{\mathrm{perf}} = \mathrm{Spec}(A^{p^{-\infty}})$ where $A^{p^{-\infty}} = \mathrm{colim}_{\Phi} A$ is the perfect closure of A.[5] As mentioned before, the perfection of an $\mathbb{F}_p$-scheme is its absolute weak normalization. Therefore, by the topological invariance of the étale site, we have $X_{\text{ét}} \simeq X^{\mathrm{perf}}_{\text{ét}}$.[6] The upshot is that reconstructing perfect schemes is the best hope in positive characteristic.

The positive characteristic case differs from the characteristic 0 case in a number of ways:

- We work with perfections of schemes of finite type over an infinite, AFG field of positive characteristic. These schemes are almost never of finite type. We need the base field to be infinite for Proposition 3.8.
- We have to work with prime-to-p parts of every projective limit. We take $\chi = \varprojlim_{p \nmid n} \chi_n$ where $\chi_n : \mathscr{O}^*(\overline{X})/\mathscr{O}^*(\overline{X})^n \longrightarrow H^1_{\text{ét}}(\overline{X}, \mu_n)$ is the coboundary map.
- For a base field K, $\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)$ is isomorphic to $\widehat{\mathbb{Z}}$ if $\mathrm{char}(K) = 0$, but this is not the case if $\mathrm{char}(K) > 0$. So we need a new χ_{top} which we will construct by passing to the tame fundamental group of $\mathbb{G}_m$.

[4] Note that $|X^{\mathrm{perf}}| \cong |X|$.

[5] This can get a little confusing because there is also a notion of perfection for a ring which is given by the *limit* over Frobenius.

[6] For a point, topological invariance in characteristic p is equivalent to the fact that $\mathrm{Aut}(E^{\mathrm{sep}}/E) \cong \mathrm{Aut}(E^{\mathrm{sep}}/E^{\mathrm{perf}})$. In particular, universal injectivity requires residue field extensions to be purely inseparable [Stacks, Tag 0BR5]. In characteristic 0, the fact that all fields are perfect is equivalent to the fact that a point is seminormal.

- Decompletion (Proposition 3.8) doesn't necessarily go through if X is imperfect (whereas in characteristic 0 it works for any finite type scheme over an AFG field of characteristic 0).[7]

The overall strategy of the proof is to show that $\operatorname{Im}\chi_{\text{top}} \subset \operatorname{Im}\chi$ suffices if X is the perfection of scheme of finite type over an infinite, AFG field of positive characteristic.

2. Preparations

Fix a perfect field K of any characteristic. We let $\Gamma_E := \mathrm{Gal}(E^{\text{sep}}/E)$ for any field E. We will work with sites as Voevodsky did.[8]

Remark 2.1. We work over a perfect field until the very end where we have to work with a scheme that starts life over an infinite, AFG field of positive characteristic. The absolute finite generation allows us to conclude that the Picard group has bounded torsion. Similarly, Voevodsky's use of an AFG field containing $\mathbb{Q}$ is basically only used to apply Mordell-Weil-Néron-Lang. For this reason, reconstruction in characteristic 0 is not limited to AFG fields. For example, one can take $k = k_0(x_1, x_2, \ldots)$ with $k_0 \supset \mathbb{Q}$ AFG and Mordell-Weil-Néron-Lang still holds. One also needs the fact that the natural map

$$\overline{k}^{\times} \longrightarrow \varinjlim_{E \supset k} \varprojlim_{n} E^{\times}/E^{\times n}$$

is injective (that is to say that $k^{\times}$ has no infinitely divisible elements; see the proof of Theorem 3.10). The divisible elements of k are the same as the divisible elements of k_0 because $k^{\times} \cong k_0[x_1, x_2, \ldots]^{\times} \oplus \bigoplus_I \mathbb{Z}$. Since k_0 is AFG, it has no divisible elements. Taken with Remark 1.11, this means the toposic hom conjecture generalizes to k.

I. **Generalities on Étale Sites.** A morphism of sites $f : X \longrightarrow Y$ is a functor $f^{-1} : Y \longrightarrow X$ that sends covers to covers and preserves fiber products. The 2-morphisms are natural transformations $f^{-1} \Rightarrow g^{-1}$. Let **Site** denote the 2-category of sites. We would like to work with schemes over a base. So we have to work with sites over a base. Let $\mathbf{Site}_{/S}$ denote the 2-category of sites over a base S. The objects are S-sites. A morphism $f : X \longrightarrow Y$ is a pair (f^{-1}, α) where $f^{-1} : Y \longrightarrow X$ is a functor as before and $\alpha : p_X^{-1} \Rightarrow f^{-1} \circ p_Y^{-1}$ is a natural isomorphism where $p_{(-)}$ denotes the structure map from $(-)$ to S. And a 2-morphism $(f^{-1}, \alpha_1) \Rightarrow (g^{-1}, \alpha_2)$ is a natural transformation $f^{-1} \Rightarrow g^{-1}$ such that

$$\begin{tikzcd}
 & f^{-1} \circ p_Y^{-1} \arrow[dd, Rightarrow] \\
p_X^{-1} \arrow[ur, Rightarrow, "\alpha_1"] \arrow[dr, Rightarrow, "\alpha_2"'] & \\
 & g^{-1} \circ p_Y^{-1}
\end{tikzcd}$$

commutes.

Let X be a scheme. Its quasicompact and separated étale site $X_{\text{ét}}$ has as open sets étale $U \longrightarrow X$ that are quasicompact and separated. The coverings are the jointly surjective families. A morphism of schemes $\varphi : X \longrightarrow Y$ induces a morphism of sites $\varphi_{\text{ét}} : X_{\text{ét}} \longrightarrow Y_{\text{ét}}$ given by pullback along φ. So we get a functor $(-)_{\text{ét}} : \mathbf{Sch} \longrightarrow \mathbf{Site}$. In fact, we get a functor $(-)_{\text{ét}} : \mathbf{Sch}_S \longrightarrow \mathbf{Site}_{/S_{\text{ét}}}$. Indeed, take $f \in \mathrm{Mor}_S(X, Y)$ and consider its étalification $f_{\text{ét}}$. To construct α_f to get a morphism of $S_{\text{ét}}$-sites, take an étale open $U \in S_{\text{ét}}$. We see that $p_X^{-1}(U)$ is the étale open $U \times_S X \in X_{\text{ét}}$. And $(f_{\text{ét}}^{-1} \circ p_Y^{-1})(U)$ is the étale open $(U \times_S Y) \times_f X \in X_{\text{ét}}$. Then $\alpha_f(U)$ is the isomorphism $U \times_S X \longrightarrow (U \times_S Y) \times_f X$ given by $(\mathrm{pr}_U \times (f \circ \mathrm{pr}_X)) \times \mathrm{pr}_X$ where $\mathrm{pr}_{(-)}$ denotes the projection to $(-)$.

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[7]Being prime to the characteristic in characteristic 0 is the same as being prime to 1 with 1 being the exponential characteristic. In particular, we invert 1 when we complete the group of units in Proposition 3.8. This realization allows [CHW] to redo Voevodsky's proof in a way that is independent of the characteristic of the base field.

[8][CHW] works with topoi.

Let X and Y be arbitrary K-schemes. We let $\mathrm{Mor}_{K_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}})$ denote the set of $\mathrm{Spec}(K)_{\text{ét}}$-morphisms of étale sites up to (2-)isomorphism.

Proposition 2.2 (Voe, Proposition 1.2)**.** *Let X and Y be K-schemes with Y of finite type. Then for any separable extension $E \supset K$, we get a functor*

$$(-)_E : \mathrm{Mor}_{K_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}}) \longrightarrow \mathrm{Mor}_{E_{\text{ét}}}(X_{E,\text{ét}}, Y_{E,\text{ét}})$$

that is natural relative to morphisms of $X_{\text{ét}}$ and $Y_{\text{ét}}$. And the diagram

$$\begin{tikzcd} X_{E,\text{ét}} \arrow[r, "\varphi_E"] \arrow[d, "(\mathrm{pr}_X)_{\text{ét}}"'] & Y_{E,\text{ét}} \arrow[d, "(\mathrm{pr}_Y)_{\text{ét}}"] \\ X_{\text{ét}} \arrow[r, "\varphi"] & Y_{\text{ét}} \end{tikzcd}$$

commutes.

Proof. Given $\varphi = (\varphi^{-1}, \alpha)$ we define $\varphi_E = (\varphi_E^{-1}, \alpha_E)$ as follows: Let $E \in K_{\text{ét}}$ be the étale open coming from $\mathrm{Spec}\, E \longrightarrow \mathrm{Spec}\, K$. Then $X_{E,\text{ét}} = p_X^{-1}(E)_{\text{ét}}$ and $Y_{E,\text{ét}} = p_Y^{-1}(E)_{\text{ét}}$. We take $\varphi_E^{-1} = \alpha(E)_{\text{ét}}^{-1} \circ \varphi^{-1}|_{Y_E}$ where $\alpha(E)_{\text{ét}}^{-1}$ is the functor from $\varphi^{-1}(Y_E)_{\text{ét}} \longrightarrow X_{E,\text{ét}}$ given by the étalification of $\alpha(E) : X_E \xrightarrow{\sim} \varphi^{-1}(Y_E)$. For every étale open $V \in E_{\text{ét}}$, we need to give an isomorphism $\alpha_E(V) : p_{X_E}^{-1}(V) \to (\varphi_E^{-1} \circ p_{Y_E}^{-1})(V)$. Assume that $V = \mathrm{Spec}(L)$. Then we have

$$\begin{aligned} (\varphi_E^{-1} \circ p_{Y_E}^{-1})(V) &= (\alpha(E)_{\text{ét}}^{-1} \circ \varphi^{-1}|_{Y_E} \circ p_{Y_E}^{-1})(\mathrm{Spec}(L)) \\ &= \alpha(E)_{\text{ét}}^{-1}(\varphi^{-1}|_{Y_E}(Y_L)) \\ &= X_E \times_{\varphi^{-1}|_{Y_E}(Y_E)} \varphi^{-1}|_{Y_E}(Y_L) \end{aligned}$$

Since we have isomorphisms $\alpha(L) : X_L \xrightarrow{\sim} \varphi^{-1}|_{Y_E}(Y_L)$ and $\alpha(E) : X_E \xrightarrow{\sim} \varphi^{-1}|_{Y_E}(Y_E)$, we get

$$\begin{tikzcd} p_{X_E}^{-1}(V) = X_L \arrow[drr, bend left, "\sim"', "\alpha(L)"] \arrow[dr, dotted, "\sim", "\alpha_E(V)"'] \arrow[ddr, bend right] & & \\ & X_E \times_{\varphi^{-1}|_{Y_E}(Y_E)} \varphi^{-1}|_{Y_E}(Y_L) \arrow[r, "\sim"] \arrow[d] & \varphi^{-1}|_{Y_E}(Y_L) \arrow[d] \\ & X_E \arrow[r, "\sim", "\alpha(E)"'] & \varphi^{-1}|_{Y_E}(Y_E) \end{tikzcd}$$

In particular, we have found our isomorphism $\alpha_E(V)$. It is straightforward to extend to the case where $\pi_0(V) > 1$. □

Morphisms of étale sites give rise to morphisms on underlying topological spaces. This leads us to the notion of admissibility. The motivation behind the following is as follows: if $\varphi : X \longrightarrow Y$ is a morphism of topological spaces with Y sober, it is uniquely determined by what it does on preimages of opens of Y by $\varphi(x) = \text{Generic point of } (Y \setminus \bigcup_{x \notin \varphi^{-1}(U)} U)$.

Lemma 2.3 (Voe, Proposition 1.1)**.** *Let $\varphi : X_{\text{ét}} \longrightarrow Y_{\text{ét}}$ be a morphism of étale sites. There exists a unique morphism $|\varphi| : |X| \longrightarrow |Y|$ such that $|\varphi|^{-1}(U) = |\operatorname{Im} \varphi^{-1}(U)|$ for any open set $U \subset |Y|$.*

Proof. Take $x \in |X|$. There is a maximal open subset $V_x \subset |Y|$ such that $|\operatorname{Im} \varphi^{-1}(V_x)| \subset |X| \setminus \overline{\{x\}}$. Since $|\operatorname{Im} \varphi^{-1}(V)|$ is open, this is equivalent to saying that V_x is the maximal open such that $x \notin |\operatorname{Im} \varphi^{-1}(V_x)|$. It is given by

$$V_x = \bigcup_{x \notin |\operatorname{Im} \varphi^{-1}(U)|} U$$

It turns out that $|Y| \smallsetminus V_x$ is irreducible. To see this, let $V_x = U_1 \cap U_2$ and assume that neither U_1 nor U_2 is a subset of V_x. Then we have $x \in |\operatorname{Im} \varphi^{-1}(U_i)|$ for $i = 1, 2$. But $U_1 \cap U_2 = V_x$, so, since φ^{-1} preserves fiber products and covers, we have

$$x \in |\operatorname{Im} \varphi^{-1}(U_1)| \cap |\operatorname{Im} \varphi^{-1}(U_2)| = |\operatorname{Im} \varphi^{-1}(V_x)| \subset X \smallsetminus \overline{\{x\}}.$$

This is a contradiction.

With this we define

$$|\varphi|(x) = \text{Generic point of } (|Y| \smallsetminus V_x)$$

which we may do because $|Y|$ is sober and $|Y| \smallsetminus V_x$ is irreducible.

Let $V \subset |Y|$. We would like to show that $|\varphi|^{-1}(V) = |\operatorname{Im} \varphi^{-1}(V)|$. We have

$$\begin{aligned} |\varphi|^{-1}(V) &= \{x \in |X| : |\varphi|(x) \in V\} \\ &= \left\{ x \in |X| : \text{Generic point of} \left(|Y| \smallsetminus \bigcup_{x \notin |\operatorname{Im} \varphi^{-1}(U)|} U \right) \in V \right\} \\ &= \left\{ x \in |X| : x \in |\operatorname{Im} \varphi^{-1}(V)| \right\} \\ &= |\operatorname{Im} \varphi^{-1}(V)| \end{aligned}$$

□

Remark 2.4. If $\varphi : X \longrightarrow Y$ is a morphism of schemes and $|\varphi| : |X| \longrightarrow |Y|$ is the map it gives on topological spaces, then $|\varphi_{\text{ét}}| = |\varphi|$.

Definition 2.5. A morphism $\varphi \in \operatorname{Mor}_{K_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}})$ is *admissible* if $|\varphi|$ sends closed points to closed points.[9]

Remark 2.6. Admissibility is a reasonable condition because any morphism $X \longrightarrow Y$ of K-schemes with X of finite type sends closed points to closed points.

Given this, we will let $\operatorname{Mor}^{\bullet}_{K_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}})$ denote the subset of $\operatorname{Mor}_{K_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}})$ comprising admissible $\operatorname{Spec}(K)_{\text{ét}}$-morphisms of étale sites up to (2-)isomorphism.

Lemma 2.7 (Voe, Proposition 1.3)**.** *Let X and Y be $\overline{K}$-schemes with Y of finite type and pick some $\varphi = (\varphi^{-1}, \alpha) \in \operatorname{Mor}^{\bullet}_{\overline{K}_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}})$. Then, for any étale open $U \longrightarrow Y$ and any closed point $x \in X$, the map $|\varphi_U|$ is a bijection from $|\varphi^{-1}(U)_x|$ to $|U_{|\varphi|(x)}|$ where $\varphi^{-1}(U)_x$ is the fiber of $\varphi^{-1}(U)$ over x, and $U_{|\varphi|(x)}$ is the fiber of U over $|\varphi|(x)$.*

Proof. To show that $|\varphi_U|$ sends $|\varphi^{-1}(U)_x|$ to $|U_{|\varphi|(x)}|$, it suffices to show that the following diagram commutes

$$\begin{tikzcd} {|\varphi^{-1}(U)|} \arrow[r, "|\varphi_U|"] \arrow[d] & {|U|} \arrow[d] \\ {|X|} \arrow[r, "|\varphi|"] & {|Y|} \end{tikzcd}$$

This just follows from the functoriality of the association $X_{\text{ét}} \rightsquigarrow |X|, \varphi \rightsquigarrow |\varphi|$ applied to the commutative diagram

$$\begin{tikzcd} \varphi^{-1}(U)_{\text{ét}} \arrow[r, "\varphi_U"] \arrow[d] & U_{\text{ét}} \arrow[d] \\ X_{\text{ét}} \arrow[r, "\varphi"] & Y_{\text{ét}} \end{tikzcd}$$

Surjectivity: Let $y = |\varphi|(x)$. Pick a point $z \in U_y$ and assume that it doesn't lie in the image of $|\varphi_U|$. Let $U^z = U \smallsetminus (U_y \smallsetminus z)$. Then $|\varphi^{-1}|(U^z)_x = \varnothing$. But U^z is a covering in some Zariski neighborhood V of y. So

[9] In [CHW] admissible morphisms are called *pinned* morphisms.

$\varphi^{-1}(U^z)$ must be a covering in some neighborhood $\varphi^{-1}(V)$ of $|\varphi|^{-1}(y) = x$ because φ^{-1} sends coverings to coverings. This is expressed in the following diagram:

$$\begin{tikzcd}
 & {|\varphi^{-1}(U^z)|} \arrow[dl, two heads] \arrow[d] & {|U^z|} \arrow[d] \arrow[dr, two heads] & \\
{|\varphi^{-1}(V)|} \arrow[dr] & {|\varphi^{-1}(U)|} \arrow[r, "|\varphi_U|"] \arrow[d] & {|U|} \arrow[d] & {|V|} \arrow[dl] \\
 & {|X|} \arrow[r, "|\varphi|"] & {|Y|} &
\end{tikzcd}$$

So the fiber $|\varphi^{-1}|(U^z)_x$ is non-empty.

Injectivity: Consider $\mathrm{pr}_1 : U^z \times_Y U^z \longrightarrow U^z$. Since $\overline{K}$ is algebraically closed, Y is of finite type, and z is closed, $\mathrm{pr}_1^{-1}(z) = (z, z)$. So the diagonal is a covering in some neighborhood of $\mathrm{pr}_1^{-1}(z)$. Hence, the diagonal $\varphi^{-1}(U^z) \longrightarrow \varphi^{-1}(U^z) \times_X \varphi^{-1}(U^z)$ is a covering in some neighborhood of $\mathrm{pr}_1^{-1}(|\varphi|^{-1}(z))$ where $\mathrm{pr}_1 : \varphi^{-1}(U^z) \times_X \varphi^{-1}(U^z) \longrightarrow \varphi^{-1}(U^z)$ is the projection onto the first component. So there is a unique point in the fiber $\varphi^{-1}(U^z)_x$. □

II. **Étalified Rational Points.** Fix a perfect field of positive characteristic. Minor modifications need to be made to the statements in [Voe, §2]. We will reproduce/produce proofs here for convenience.

Lemma 2.8 (Rigidity of Geometric Points of Étale Sites, Voe, Proposition 2.1)**.** *Let X be a $\overline{K}$-scheme of finite type and pick two morphisms $(\varphi_1^{-1}, \alpha_1), (\varphi_2^{-1}, \alpha_2) \in \mathrm{Mor}_{\overline{K}_{\text{ét}}}(\mathrm{Spec}(\overline{K})_{\text{ét}}, X_{\text{ét}})$. If $\eta : \varphi_1^{-1} \overset{\sim}{\Rightarrow} \varphi_2^{-1}$ is a natural isomorphism, we get a 2-isomorphism $(\varphi_1^{-1}, \alpha_1) \overset{\sim}{\Rightarrow} (\varphi_2^{-1}, \alpha_2)$.*

Proof. Take $U \in \overline{K}_{\text{ét}}$. It is a disjoint union of $\mathrm{Spec}(\overline{K})$'s. We would like to show that

$$\begin{tikzcd}
 & (\varphi_1^{-1} \circ p_X^{-1})(U) \arrow[dd, "\eta(p_X^{-1}(U))"] \\
U \arrow[ur, "\alpha_1(U)"] \arrow[dr, "\alpha_2(U)"'] & \\
 & (\varphi_2^{-1} \circ p_X^{-1})(U)
\end{tikzcd}$$

commutes. We get an automorphism

$$\alpha_2(U)^{-1} \circ \eta(U) \circ \alpha_1(U) : U \longrightarrow U$$

Since, α_1, α_2, η are natural isomorphisms, they commute with the action of $\mathrm{Aut}(U/\operatorname{Spec}\overline{K}) = S_{\pi_0(U)}$ (the symmetric group on $\pi_0(U)$ elements). So $\alpha_2(U)^{-1} \circ \eta(U) \circ \alpha_1(U) = \mathrm{id}_U$ if $\pi_0(U) \geq 3$ because $S_{\pi_0(U)}$ has trivial center when $\pi_0(U) \geq 3$. If $\pi_0(U) < 3$, we may embed U into an étale open V with $\pi_0(V) \geq 3$ and deduce the same result by functoriality. Hence, $\alpha_2(U) = \eta(U) \circ \alpha_1(U)$. □

Lemma 2.9 (Voe, Proposition 2.1)**.** *Let $\overline{X}$ be a finite type $\overline{K}$-scheme. The natural map*

$$\overline{X}(\overline{K}) \longrightarrow \mathrm{Mor}^{\bullet}_{\overline{K}_{\text{ét}}}(\mathrm{Spec}(\overline{K})_{\text{ét}}, \overline{X}_{\text{ét}})$$

is bijective.

Proof. Injectivity: Over $\overline{K}$, a $\overline{K}$-point is determined by its image.

Surjectivity: Let $\varphi \in \mathrm{Mor}^{\bullet}_{\overline{K}_{\text{ét}}}(\mathrm{Spec}(\overline{K})_{\text{ét}}, \overline{X}_{\text{ét}})$. Consider the $\overline{K}$-point $\overline{x}$ corresponding to $\mathrm{Im}\,|\varphi|$. Lemma 2.7 implies that $\overline{x}_{\text{ét}} = \varphi$ in $\mathrm{Mor}^{\bullet}(\mathrm{Spec}(\overline{K})_{\text{ét}}, X_{\text{ét}})$. And Lemma 2.8 implies that $\overline{x}_{\text{ét}} = \varphi$ in $\mathrm{Mor}^{\bullet}_{\overline{K}_{\text{ét}}}(\mathrm{Spec}(K)_{\text{ét}}, X_{\text{ét}})$. □

Proposition 2.10 (Voe, Proposition 2.2)**.** *Let X be a finite type K-scheme. The map*
$$\begin{aligned}\beta : \overline{X}(\overline{K}) &\longrightarrow \mathrm{Mor}^{\bullet}_{K_{\text{ét}}}(\mathrm{Spec}(\overline{K})_{\text{ét}}, X_{\text{ét}})\\ \overline{x} &\mapsto (\mathrm{pr}_X)_{\text{ét}} \circ \overline{x}_{\text{ét}}\end{aligned}$$
is bijective ($\mathrm{pr}_X : \overline{X} \longrightarrow X$ *is the projection*).

Proof. Consider $\varphi \in \mathrm{Mor}^{\bullet}_{K_{\text{ét}}}(\mathrm{Spec}(\overline{K})_{\text{ét}}, X_{\text{ét}})$. By Proposition 2.2, we get a morphism $\varphi_{\overline{K}}$ fitting in a diagram
$$\begin{tikzcd}(\mathrm{Spec}(\overline{K}) \times_K \mathrm{Spec}(\overline{K}))_{\text{ét}} \arrow[r, "\varphi_{\overline{K}}"] \arrow[d] & \overline{X}_{\text{ét}} \arrow[d] \\ \mathrm{Spec}(\overline{K})_{\text{ét}} \arrow[r, "\varphi"] & X_{\text{ét}}\end{tikzcd}$$
Let $\triangle : \mathrm{Spec}(\overline{K}) \longrightarrow \mathrm{Spec}(\overline{K}) \times_K \mathrm{Spec}(\overline{K})$ be the diagonal and set $\beta^{-1}(\varphi) = \mathrm{Im}\,|\varphi_{\overline{K}} \circ \triangle_{\text{ét}}|$ (Lemma 2.9). For surjectivity, we have
$$\begin{aligned}(\beta \circ \beta^{-1})(\varphi) &= \beta(\mathrm{Im}\,|\varphi_{\overline{K}} \circ \triangle_{\text{ét}}|)\\ &= (\mathrm{pr}_X)_{\text{ét}} \circ \varphi_{\overline{K}} \circ \triangle_{\text{ét}}\\ &= \varphi\end{aligned}$$
by the commutativity of the diagram (note that we are conflating $\mathrm{Im}\,|\varphi_{\overline{K}} \circ \triangle_{\text{ét}}|$ with the geometric point corresponding to it). For injectivity, we have
$$\begin{aligned}(\beta^{-1} \circ \beta)(\overline{x}) &= \beta^{-1}((\mathrm{pr}_X)_{\text{ét}} \circ \overline{x}_{\text{ét}})\\ &= \mathrm{Im}\,|((\mathrm{pr}_X)_{\text{ét}} \circ \overline{x}_{\text{ét}})_{\overline{K}} \circ \triangle_{\text{ét}}|\\ &= \mathrm{Im}\,|\overline{x}_{\text{ét}}|\\ &= \overline{x}\end{aligned}$$
□

Proposition 2.11. *Let $X = X_0^{\text{perf}}$ be the perfection of a K-scheme of finite type. Let Y be a finite type K-scheme. the natural map*
$$\mathrm{Mor}_K(X, Y) \longrightarrow \mathrm{Mor}^{\bullet}_{K_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}})$$
is injective.

Proof. First, notice that $X \cong X_{0,\text{red}}^{\text{perf}}$. For $X_{0,\text{red}}$ and Y of finite type over K, a K-morphism is determined by what it does on geometric points. This still holds if we replace $X_{0,\text{red}}$ with its perfection because $X_{0,\text{red}}^{\text{perf}}$ is reduced and $|X_{0,\text{red}}| \cong |X_{0,\text{red}}^{\text{perf}}|$ (in particular, being of finite type is relevant only for the density of closed points which still holds for the perfection). Consider the composite
$$\mathrm{Mor}_K(X, Y) \longrightarrow \mathrm{Mor}(X(\overline{K}), Y(\overline{K})) \longrightarrow \mathrm{Mor}(\mathrm{Mor}^{\bullet}_{K_{\text{ét}}}(\mathrm{Spec}\,\overline{K}_{\text{ét}}, X_{\text{ét}}), \mathrm{Mor}^{\bullet}_{K_{\text{ét}}}(\mathrm{Spec}\,\overline{K}_{\text{ét}}, Y_{\text{ét}}))$$
Proposition 2.10 gives us
$$\mathrm{Mor}(\mathrm{Mor}^{\bullet}_{K_{\text{ét}}}(\mathrm{Spec}\,\overline{K}_{\text{ét}}, X_{\text{ét}}), \mathrm{Mor}^{\bullet}_{K_{\text{ét}}}(\mathrm{Spec}\,\overline{K}_{\text{ét}}, Y_{\text{ét}})) \cong \mathrm{Mor}(\overline{X}(\overline{K}), \overline{Y}(\overline{K}))$$
So we would like to show that the composite
$$\mathrm{Mor}_K(X, Y) \longrightarrow \mathrm{Mor}(X(\overline{K}), Y(\overline{K})) \longrightarrow \mathrm{Mor}(\overline{X}(\overline{K}), \overline{Y}(\overline{K}))$$
is injective. Take $f \in \mathrm{Mor}_K(X, Y)$. By the universal properties of $\overline{X}$ and $\overline{Y}$, every $f \in \mathrm{Mor}_K(X, Y)$ extends to a unique $\overline{f} \in \mathrm{Mor}_{\overline{K}}(\overline{X}, \overline{Y})$ such that for every $\overline{x} \in X(\overline{K})$
$$\begin{tikzcd} & \overline{X} \arrow[r, "\overline{f}"] \arrow[d] & \overline{Y} \arrow[d] \\ \mathrm{Spec}(\overline{K}) \arrow[ur, "\overline{\overline{x}}"] \arrow[r, "\overline{x}"] & X \arrow[r, "f"] & Y\end{tikzcd}$$

commutes. The map that sends a geometric point $\overline{x} \in \overline{X}(\overline{K})$ to $\overline{f} \circ \overline{x}$ is the image of f in $\mathrm{Mor}(\overline{X}(\overline{K}), \overline{Y}(\overline{K}))$. Take $f, g \in \mathrm{Mor}_K(X, Y)$ such that $f \circ \overline{x} \neq g \circ \overline{x}$ for some $\overline{x} \in X(\overline{K})$. We want to show that $\overline{f} \circ \overline{\overline{x}} \neq \overline{g} \circ \overline{\overline{x}}$. Assume otherwise. Then, by the commutativity of the above diagram, $f \circ \overline{x} = g \circ \overline{x}$, a contradiction. □

Remark 2.12. Observe that injectivity also holds if X is a reduced K-scheme of finite type and/or Y is the perfection of a finite type K-scheme.

Remark 2.13. It is shown in [CHW, Section 3] that injectivity holds if the base is any Jacobson scheme.

Notation 2.14. Let X be the perfection of a finite type K-scheme. Given any $\varphi \in \mathrm{Mor}^{\bullet}_{K_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}})$, we let $\overline{\varphi} \in \mathrm{Mor}(X(\overline{K}), Y(\overline{K}))$ denote the corresponding map on geometric points coming from Proposition 2.10. Note that if φ is a morphism of schemes and φ^* is the map it induces on geometric points, we have $\overline{\varphi_{\text{ét}}} = \varphi^*$.

Proposition 2.15 (Voe, Proposition 2.3)**.** *Let X be a reduced, finite type K-scheme or the perfection of a finite type K-scheme. Let Y be a finite type K-scheme. Let $\varphi \in \mathrm{Mor}^{\bullet}_{K_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}})$. Assume that for all étale $U \longrightarrow Y$, there exists some $\widetilde{\varphi}_U \in \mathrm{Mor}_K(\varphi^{-1}(U), U)$ such that $\widetilde{\varphi}_U(\overline{x}) = \overline{\varphi_U}(\overline{x})$ for all $\overline{x} \in \varphi^{-1}(U)(\overline{K})$. Then φ comes from a morphism of schemes.*

Proof. We will prove the theorem for a reduced, finite type K-scheme X. As in Proposition 2.11, we may replace X with its perfection and the proof goes through just the same. We would like to show that the étalification of $\widetilde{\varphi}_Y = \widetilde{\varphi}$ is $\varphi = (\varphi^{-1}, \alpha)$, that is, we want to show that there is a 2-isomorphism $\widetilde{\varphi}_{\text{ét}} = (\widetilde{\varphi}_{\text{ét}}^{-1}, \widetilde{\alpha}) \cong_2 (\varphi^{-1}, \alpha)$. Consider the diagram

$$\begin{tikzcd}
\varphi^{-1}(U) \arrow[drr, "\widetilde{\varphi}_U"] \arrow[ddr] & & \\
& \widetilde{\varphi}_{\text{ét}}^{-1}(U) \arrow[r] \arrow[d] & U \arrow[d] \\
& X \arrow[r, "\widetilde{\varphi}"] & Y
\end{tikzcd}$$

The square commutes by construction because $\widetilde{\varphi}_{\text{ét}}^{-1}(U) = X \times_{\widetilde{\varphi}} U$. By assumption, the whole diagram commutes on geometric points. Since X is reduced, $\varphi^{-1}(U)$ is reduced because it is étale over X. Everything in sight is of finite type (étale morphisms are locally of finite presentation). Hence, the diagram commutes. So we get a diagram

$$\begin{tikzcd}
\varphi^{-1}(U) \arrow[drr, "\widetilde{\varphi}_U"] \arrow[ddr] \arrow[dr, dotted, "\exists!"] & & \\
& \widetilde{\varphi}_{\text{ét}}^{-1}(U) \arrow[r] \arrow[d] & U \arrow[d] \\
& X \arrow[r, "\widetilde{\varphi}"] & Y
\end{tikzcd}$$

We may (compatibly) do this for all U, so we get a natural transformation $\eta : \varphi^{-1} \Rightarrow \widetilde{\varphi}_{\text{ét}}^{-1}$. We would like to show that η is a natural isomorphism. Lift the diagram to $\overline{K}$. To show that $\eta_{\overline{K}}(U)$ is an isomorphism, it suffices to show that there is a bijection $|(\varphi^{-1}(U)_{\overline{K}})_x| \longrightarrow |(\widetilde{\varphi}_{\text{ét}}^{-1}(U)_{\overline{K}})_x|$ between the fibers over the closed points x of $\overline{X}$. Indeed, $\varphi^{-1}(U)_{\overline{K}}$ and $\widetilde{\varphi}_{\text{ét}}^{-1}(U)_{\overline{K}}$ are flat and quasicompact over $\overline{X}$, so if $\eta_{\overline{K}}(U)$ induces a bijection on fibers over closed points of $\overline{X}$, $\eta_{\overline{K}}(U)$ is an isomorphism by [Stacks, Tag 06NC]. Proposition 2.7 tells us that there is in fact a bijection on the fibers.

It remains to show that η is a 2-isomorphism over $\mathrm{Spec}(K)_{\text{ét}}$. It suffices to show that, for every finite extension $E \supset K$, the following diagram commutes:

$$\begin{tikzcd} & \varphi^{-1}(Y_E) \arrow[dr, "\eta(Y_E)"] & \\ X_E \arrow[ur, "\alpha(E)"] \arrow[rr, "\widetilde{\alpha}(E)"] & & \widetilde{\varphi}_{\text{ét}}^{-1}(Y_E) \end{tikzcd}$$

Indeed, we just need to check commutativity with every étale open of the base (recall that a 2-morphism involves composing with p_X^{-1} and p_Y^{-1}). We see that X_E is reduced because X is reduced over a perfect field. And everything is of finite type over K. So the diagram commutes if and only if it commutes on geometric points. Details to come. $\square$

Proposition 2.16. *Let X and Y be K-schemes of finite type with X reduced. Let $\varphi \in \mathrm{Mor}_K^\bullet(X_{\text{ét}}, Y_{\text{ét}})$ and let E be an algebraic extension of K. If there exists some $\widetilde{\varphi}_E \in \mathrm{Mor}_E(X_E, Y_E)$ such that $\widetilde{\varphi}_E(\overline{x}) = \overline{\varphi_E}(\overline{x})$ for all $\overline{x} \in X_E(\overline{K})$, then there exists some $\widetilde{\varphi} \in \mathrm{Mor}_K(X, Y)$ such that $\widetilde{\varphi}(\overline{x}) = \overline{\varphi}(\overline{x})$ for all $\overline{x} \in X(\overline{K})$.*

Proof. By fpqc descent we have an equalizer diagram

$$\mathrm{Mor}_K(X, Y) \longrightarrow \mathrm{Mor}_E(X_E, Y_E) \rightrightarrows \mathrm{Mor}_K(X_E \times_X X_E, Y)$$

The equalizer condition is determined by what happens on geometric points because X and Y are of finite type and X is reduced. So the equalizer condition for $\widetilde{\varphi}_E \in \mathrm{Mor}_E(X_E, Y_E)$ is says that $\widetilde{\varphi}_E(\overline{x}) = \overline{\varphi_E}(\overline{x})$ for all $x \in X_E(\overline{K})$. The equalizer is $\mathrm{Mor}_K(X, Y)$ so there exists a $\widetilde{\varphi} \in \mathrm{Mor}_K(X, Y)$ with $\widetilde{\varphi}(\overline{x}) = \overline{\varphi}(\overline{x})$ for all $\overline{x} \in X(\overline{K})$. $\square$

Remark 2.17. As per usual, Proposition 2.16 holds if X is the perfection of a finite type K-scheme.

Lemma 2.18 (Voe, Proposition 2.5)**.** *Let X be a reduced, finite type K-scheme. Let E be an algebraic extension of K. The natural map*

$$\mathrm{Mor}_K(\mathrm{Spec}(E), X) \longrightarrow \mathrm{Mor}_{K_{\text{ét}}}^\bullet(\mathrm{Spec}(E)_{\text{ét}}, X_{\text{ét}})$$

is a bijection.

Proof. Proposition 2.11 gives us injectivity. Take $\varphi \in \mathrm{Mor}_{K_{\text{ét}}}(\mathrm{Spec}(E)_{\text{ét}}, X_{\text{ét}})$. By Proposition 2.2, we get $\varphi_{\overline{K}}$ and a commutative diagram

$$\begin{tikzcd} (\mathrm{Spec}(E) \times_K \mathrm{Spec}(\overline{K}))_{\text{ét}} \arrow[r, "\varphi_{\overline{K}}"] \arrow[d] & \overline{X}_{\text{ét}} \arrow[d] \\ \mathrm{Spec}(E)_{\text{ét}} \arrow[r, "\varphi"] & X_{\text{ét}} \end{tikzcd}$$

Now $\mathrm{Spec}(E) \times_K \mathrm{Spec}(\overline{K})$ is just a disjoint union of $\mathrm{Spec}(\overline{K})$'s. So, by Lemma 2.9, $\varphi_{\overline{K}}$ comes from a morphism on geometric points. Thus, by Proposition 2.16, φ coincides with some morphism of schemes on geometric points. We may conclude the same for φ_U for all $U \in X_{\text{ét}}$. So we conclude that φ comes from a morphism of schemes by Proposition 2.15. $\square$

This gives us an analogue of [Voe, Proposition 2.8]:

Proposition 2.19. *Let $X = X_0^{\text{perf}}$ be the perfection of a geometrically connected K-scheme of finite type. Assume that $X(K) \neq \varnothing$. Let Y be a finite type K-scheme. Then for any $\varphi \in \mathrm{Mor}_{K_{\text{ét}}}^\bullet(X_{\text{ét}}, Y_{\text{ét}})$, we have a* *commutative diagram*

$$\begin{tikzcd} X(\overline{K}) \arrow[r, "\overline{\varphi}"] \arrow[d] & \overline{K}^{\times} \arrow[d] \\ \varinjlim_{E\supset K} H^1\big(\Gamma_E, \pi_1^{\text{ét}}(\overline{X},\overline{x})\big) \arrow[r] & \varinjlim_{E\supset K} H^1\big(\Gamma_E, \pi_1^{\text{ét}}(\overline{Y},\overline{\varphi}(\overline{x}))\big) \end{tikzcd}$$

where the colimit is indexed over finite extensions of K.

Proof. Observe that $X(E) \cong X_{0,\mathrm{red}}(E)$ for all $E \supset K$ because E is perfect and $X \cong X_{0,\mathrm{red}}^{\mathrm{perf}}$. The maps to H^1 come from sections of the homotopy exact sequence (see e.g., [Stix, Proposition 8/Definition 20]). Let $\varphi = (\varphi^{-1}, \alpha) : X_{\text{ét}} \longrightarrow Y_{\text{ét}}$ be an admissible morphism. We know that $\overline{\varphi}$ sends geometric points to geometric points, so we get a map $\pi(\varphi) : \pi_1^{\text{ét}}(X, \overline{x}) \longrightarrow \pi_1^{\text{ét}}(Y, \overline{\varphi}(\overline{x}))$. Since φ^{-1} sends coverings to coverings, we get a morphism of homotopy exact sequences

$$\begin{tikzcd} & \pi_1^{\text{ét}}(\overline{X},\overline{x}) \arrow[r] \arrow[d, "\pi(\varphi_{\overline{K}})"] & \pi_1^{\text{ét}}(X,\overline{x}) \arrow[d, "\pi(\varphi)"] \arrow[dr] & & \\ 1 \arrow[ur] \arrow[dr] & & & \Gamma_K \arrow[r] & 1 \\ & \pi_1^{\text{ét}}(\overline{Y},\overline{\varphi}(\overline{x})) \arrow[r] & \pi_1^{\text{ét}}(Y,\overline{\varphi}(\overline{x})) \arrow[ur] & & \end{tikzcd}$$

So $\pi(\varphi_{\overline{K}})$ is Γ_K-equivariant (the exact sequence splits because $X(K) \neq \varnothing$). In particular, we get a map $H^1\big(\Gamma_E, \pi_1^{\text{ét}}(\overline{X}, \overline{x})\big) \longrightarrow H^1\big(\Gamma_E, \pi_1^{\text{ét}}(\overline{Y}, \overline{\varphi}(\overline{x}))\big)$ for every finite extension $E \supset K$. Then Lemma 2.18 guarantees that nothing goes wrong when we take a colimit over extensions of K, so we get a map

$$\varinjlim_{E\supset K} H^1\big(\Gamma_E, \pi_1^{\text{ét}}(\overline{X},\overline{x})\big) \longrightarrow \varinjlim_{E\supset K} H^1\big(\Gamma_E, \pi_1^{\text{ét}}(\overline{Y},\overline{\varphi}(\overline{x}))\big)$$

The diagram commutes by construction. ☐

3. Now We Feast

I. **χ and χ_{top}.** Fix a perfect field K of positive characteristic. We get an analogue of [Voe, Proposition 3.1]:

Proposition 3.1. *Let X_0 be a scheme of finite type over K. Let $X = X_0^{\mathrm{perf}}$. Let Y be a scheme of finite type over K. Let E be a finite extension of K. Assume that, for all geometrically connected and affine U that are étale over X with $U(E) \neq \varnothing$, and all $\varphi \in \mathrm{Mor}^{\bullet}_{E_{\text{ét}}}(U_{E,\text{ét}}, (\mathbb{G}_{m,E})_{\text{ét}})$, there exists a morphism $\widetilde{\varphi} \in \mathrm{Mor}_E(U_E, \mathbb{G}_{m,E})$ such that $\widetilde{\varphi}(\overline{x}) = \overline{\varphi}(\overline{x})$ for all $\overline{x} \in U_E(\overline{K})$. Then*

$$\mathrm{Mor}_K(X, Y) \longrightarrow \mathrm{Mor}^{\bullet}_{K_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}})$$

is surjective (and hence bijective).

Proof. By Proposition 2.15, it suffices to show that, for any admissible $\varphi : X_{\text{ét}} \longrightarrow Y_{\text{ét}}$ and any $U \in Y_{\text{ét}}$, the map $\varphi_U : \varphi^{-1}(U) \longrightarrow U$ coincides with some morphism of schemes on geometric points of $\varphi^{-1}(U)$. Furthermore, Proposition 2.16 tells us that coincidence over E is enough, and we may pass to finite $E \supset K$ such that $X(E) \neq \varnothing$. So without loss of generality, we will take $E = K$. We may reduce to the case where $X = \mathrm{Spec}(A)$ and $Y = \mathrm{Spec}(B)$ by gluing. And we may further assume that X is geometrically connected by working on connected components. The following argument works if we replace X and Y with $\varphi^{-1}(U)$ and U respectively, so we may do everything on X and Y.

Consider $\varphi \in \mathrm{Mor}^{\bullet}_{K_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}})$. We will take $b \in B = \mathrm{Mor}_K(Y, \mathbb{A}^1_K)$ and conflate it with the morphism it induces on geometric points. We get a morphism $\overline{b_{\text{ét}} \circ \varphi} = b \circ \overline{\varphi} : X(\overline{K}) \longrightarrow \mathbb{A}^1_K(\overline{K})$. Consider the two sets $X_0(\overline{K}) = X(\overline{K}) \setminus (b \circ \overline{\varphi})(0)^{-1}$ and $X_1(\overline{K}) = X(\overline{K}) \setminus (b \circ \overline{\varphi})(1)^{-1}$. We have $X(\overline{K}) = X_0(\overline{K}) \cup X_1(\overline{K})$, and, by assumption, $(b \circ \overline{\varphi})|_{X_0(\overline{K})} : X_0(\overline{K}) \longrightarrow \mathbb{A}^1_K(\overline{K}) \setminus \{0\} = \mathbb{G}_m(\overline{K})$ agrees with some morphism of schemes on $X_0(\overline{K})$, and $(b\circ\overline{\varphi})|_{X_1(\overline{K})} : X_1(\overline{K}) \longrightarrow \mathbb{A}^1_K(\overline{K}) \smallsetminus \{1\} = \mathbb{G}_m(\overline{K})$ agrees with some morphism of schemes on $X_1(\overline{K})$. We see that $(b\circ\overline{\varphi})|_{X_0(\overline{K})}$ and $(b\circ\overline{\varphi})|_{X_1(\overline{K})}$ coincide on $X_0(\overline{K})\cap X_1(\overline{K})$. Therefore, there exists a $\psi_b : X \longrightarrow \mathbb{A}^1_K$ that agrees with $b_{\text{ét}}\circ\varphi$ on $X(\overline{K})$. So we get a map

$$\begin{aligned}\varphi^* : B = \mathrm{Mor}_K(Y,\mathbb{A}^1_K) &\longrightarrow \mathrm{Mor}_K(X,\mathbb{A}^1_K) = A\\ b &\mapsto \psi_b\end{aligned}$$

of sets. One checks that φ^* is a ring homomorphism. And $\mathrm{Spec}(\varphi^*)$ is a map of schemes that coincides with φ on $X(\overline{K})$. □

From now on, we let X_0 be a geometrically connected, affine scheme of finite type over K with $X_0(\overline{K}) \neq \varnothing$ and let $X = X_0^{\text{perf}}$. The following lemma is well-known:

Lemma 3.2. *If $f : U \longrightarrow X$ is étale and X is perfect, U is perfect.*

Proof. Let $\Phi_{(-)}$ denote the absolute Frobenius on $(-)$. Let $U^{(p/X)} = U \times_{\Phi_X} X$ and let $\Phi_{U/X} : U \longrightarrow U^{(p/X)}$ denote the relative Frobenius of U over X [Stacks, Tag 0CC9]. We see that $\mathrm{pr}_U : U^{(p/X)} \longrightarrow U$ is an isomorphism because Φ_X is. The composition $U \xrightarrow{\Phi_{U/X}} U^{(p/X)} \xrightarrow{\mathrm{pr}_X} X$ is étale because it is equal to f. And pr_X is étale. So $\Phi_{U/X}$ is étale by cancellation. By [Stacks, Tag 0CCB], $\Phi_{U/X}$ is a universal homeomorphism. It is also étale, so it is an isomorphism by [Stacks, Tag 025G]. We conclude that $\Phi_U = \mathrm{pr}_U\circ\Phi_{U/X}$ is an isomorphism. □

We have the Kummer short exact sequence on the étale site of $\overline{X}$

$$1 \longrightarrow \mu_{n,\overline{X}} \longrightarrow \mathbb{G}_{m,\overline{X}} \xrightarrow{t\mapsto t^n} \mathbb{G}_{m,\overline{X}} \longrightarrow 1$$

for all n. In general, we only get it for n prime-to$-p$, but Frobenius is an isomorphism, so, by Lemma 3.2, every section is étale locally a p^m-th root for all m. The long exact sequence in étale cohomology gives us

$$1 \longrightarrow \mathscr{O}^*(\overline{X})/\mathscr{O}^*(\overline{X})^n \xrightarrow{\chi_n} H^1_{\text{ét}}(\overline{X},\mu_n) \longrightarrow \mathrm{Pic}(\overline{X})[n] \longrightarrow 1$$

where $\mathrm{Pic}(\overline{X})[n]$ denotes n-torsion in the Picard group. Taking the limit over n, we have

$$1 \longrightarrow \widehat{\mathscr{O}}^*(\overline{X}) \xrightarrow{\chi} H^1_{\text{ét}}(\overline{X},\widehat{\mathbb{Z}}(1)) \longrightarrow \widehat{T}(\mathrm{Pic}(\overline{X})) \longrightarrow 1$$

Note that $\mathscr{O}^*(\overline{X})/\mathscr{O}^*(\overline{X})^{p^m} = 0$ for all m. So $\widehat{\mathscr{O}}^*(\overline{X}) = \widehat{\mathscr{O}}^*(\overline{X})_{\neq p}$ (where $\neq p$ denotes taking the prime-to$-p$ limit) and $H^1_{\text{ét}}(\overline{X},\mu_{p^m}) \cong \mathrm{Pic}(\overline{X})[p^m]$ for all m. The following lemma is well-known:

Lemma 3.3 (BS, Lemma 3.5)**.** *Let X be a qcqs $\mathbb{F}_p$-scheme. Pullback along $X^{\text{perf}} \to X$ induces an isomorphism* $\mathrm{Pic}(X^{\text{perf}}) \cong \mathrm{Pic}(X)[1/p]$.

Proof. $X^{\text{perf}} = \lim_\Phi X$ so $\mathrm{Pic}(X^{\text{perf}}) \cong \operatorname{colim}\mathrm{Pic}(X)$ where the colimit is taken over pullbacks of Frobenius which raise each line bundle to its p-th power. □

So $\mathrm{Pic}(\overline{X})[p^m] = H^1_{\text{ét}}(\overline{X},\mu_{p^m}) = 0$ for all m because p is invertible in $\mathrm{Pic}(\overline{X}) = \mathrm{Pic}(\overline{X}^{\text{perf}})$. Therefore $H^1_{\text{ét}}(\overline{X},\widehat{\mathbb{Z}}(1))$ is the same as its prime-to$-p$ part.

Let $\mathscr{O}^*_{\text{top}}(\overline{X}) = \mathrm{Mor}_{\overline{K}_{\text{ét}}}(\overline{X}_{\text{ét}},(\mathbb{G}_{m,\overline{K}})_{\text{ét}})$. Fix a geometric point $\overline{x}$ of X. By the functoriality of $\pi_1^{\text{ét}}$, we get a map

$$\mathscr{O}^*_{\text{top}}(\overline{X}) \longrightarrow \mathrm{Hom}_{\text{cts}}(\pi_1^{\text{ét}}(\overline{X},\overline{x}),\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}},1))$$

If K is an AFG field of characteristic 0 and X is a finite-type K-scheme, we have $\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}},1) \cong \widehat{\mathbb{Z}}$, so we get an isomorphism

$$\mathrm{Hom}_{\text{cts}}(\pi_1^{\text{ét}}(\overline{X},\overline{x}),\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}},1)) \xrightarrow{\sim} H^1_{\text{ét}}(\overline{X},\widehat{\mathbb{Z}}(1))$$

and the composite

$$\mathscr{O}^*_{\text{top}}(X) \longrightarrow \mathscr{O}^*_{\text{top}}(\overline{X}) \longrightarrow \mathrm{Hom}_{\text{cts}}(\pi_1^{\text{ét}}(\overline{X},\overline{x}),\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}},1)) \xrightarrow{\sim} H^1_{\text{ét}}(\overline{X},\widehat{\mathbb{Z}}(1))$$

is our χ_{top} where the map $\mathscr{O}^*_{\mathrm{top}}(X) \longrightarrow \mathscr{O}^*_{\mathrm{top}}(\overline{X})$ comes from Proposition 2.2. The image of χ_{top} lies in $H^1_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}}(1))^{\Gamma_K}$ (where the action of Γ_K comes from the splitting of the homotopy exact sequence).

In characteristic p, $\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)$ is a bit unwieldy (every p-group is a quotient of its p-part). So if we want an analogue of χ_{top}, we should pass to the tame fundamental group $\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)^t \cong \widehat{\mathbb{Z}}_{\neq p} \cong \prod_{\ell \neq p} \mathbb{Z}_\ell$. The tame fundamental group is a quotient of $\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)$ which sits in an exact sequence

$$1 \longrightarrow \pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)^w \longrightarrow \pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1) \longrightarrow \pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)^t \longrightarrow 1$$

where $\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)^w$—the wild fundamental group—is pro$-p$. In this case, the tame fundamental group is the same as the prime-to$-p$ fundamental group. The exact sequence splits by a profinite version of the Schur-Zassenhaus Theorem [RZ, Theorem 2.3.15]. The tame fundamental group is the "Kummer part" of $\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)$ ($t \mapsto t^n$, $p \nmid n$) while $\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)^w$ keeps track of Artin-Schreier covers ($t \mapsto t^n - t - f(t)$, $p \nmid \deg(f)$). We want to isolate the Kummer part of $\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)$ because the image of the boundary map $\chi : \widehat{\mathscr{O}}^*(\overline{X}) \longrightarrow H^1_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}}(1))$ comprises prime-to$-p$ projective limits of Kummer classes. Recall that the Kummer classes in $H^1_{\text{ét}}(\overline{X}, \mu_n)$ are the classes associated to the étale μ_n-torsors that are given by extracting an n-th root of a unit; so if $\overline{X} = \mathrm{Spec}(A)$, such torsors are given by $\mathrm{Spec}(A[T]/T^n - u)$, $u \in \mathscr{O}^*(\overline{X})$.

Given this, we will define χ_{top} as the composite

$$\mathscr{O}^*_{\mathrm{top}}(X) \longrightarrow \mathscr{O}^*_{\mathrm{top}}(\overline{X}) \longrightarrow \mathrm{Hom}_{\mathrm{cts}}(\pi_1^{\text{ét}}(\overline{X}, \overline{x}), \pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)^t) \xrightarrow{\sim} H^1_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}}(1))$$

where the first map comes from Proposition 2.2, and the second map comes from the functoriality of $\pi_1^{\text{ét}}$ and the composition of maps $\pi_1^{\text{ét}}(\overline{X}, \overline{x}) \longrightarrow \pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)$ with the quotient $\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1) \longrightarrow \pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)^t$. With this, we get an analogue of [Voe, Proposition 3.2]:

Lemma 3.4. *We have a commutative diagram*

$$\begin{tikzcd}
 & \widehat{\mathscr{O}}^*(\overline{X}) \arrow[dr, "\chi"] & \\
\mathscr{O}^*(X) \arrow[ur] \arrow[dr] & & H^1_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}}(1)) \\
 & \mathscr{O}^*_{\mathrm{top}}(X) \arrow[ur, "\chi_{\mathrm{top}}"'] &
\end{tikzcd}$$

Proof. Let $X = \mathrm{Spec}(A)$. We would like to show that at finite level n prime-to$-p$, the bottom composition sends a unit u to the class corresponding to the étale μ_n-torsor $(A \otimes_K \overline{K})[T]/(T^n - u)$ (which is invariant under the Galois action because $u \in \mathscr{O}^*(X)$). After tensoring up to $\overline{K}$, finite level n sees only the degree$-n$ Kummer cover $\mathbb{G}_m \xrightarrow{x \mapsto x^n} \mathbb{G}_m$. The degree$-n$ Kummer cover gets sent to $A[T]/(T^n - u)$ along $u_{\text{ét}}$. And the induced map on étale fundamental groups gives the desired torsor. $\square$

X_0 is of finite type over K and has a K-rational point. So we have a split exact sequence

$$1 \longrightarrow \pi_1^{\text{ét}}(\overline{X_0}, \overline{x}) \longrightarrow \pi_1^{\text{ét}}(X_0, \overline{x}) \longrightarrow \Gamma_K \longrightarrow 1$$

By topological invariance, we get a split exact sequence

$$1 \longrightarrow \pi_1^{\text{ét}}(\overline{X}, \overline{x}) \longrightarrow \pi_1^{\text{ét}}(X, \overline{x}) \longrightarrow \Gamma_K \longrightarrow 1$$

which gives an action of Γ_K on $\pi_1^{\text{ét}}(\overline{X}, \overline{x})$. There is also an action of Γ_K on $\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)^t$ inherited from the action of Γ_K on $\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)$ because $\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)^t$ is a subgroup of $\pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)$. As in characteristic 0, we have $\mathrm{Im}\,\chi_{\mathrm{top}} \subset H^1_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}}(1))^{\Gamma_K}$ (as mentioned in the proof of Lemma 3.4).

Remark 3.5. Everything we have done in this subsection works if X is a geometrically connected scheme of finite type over K as long as we take prime-to$-p$ limits.

II. **The Main Theorem.** We can now obtain characteristic p analogues of Voevodsky's key propositions to prove the main theorem.

Proposition 3.6. *Any class $\xi \in H^1_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}}(1))^{\Gamma_K}$ gives rise to a map*

$$f_\xi : X(\overline{K}) \longrightarrow \varinjlim_{E \supset K} \varprojlim_{p \nmid n} E^\times / E^{\times n}$$

where the colimit is taken over finite extensions E of K. Let

$$i : \overline{K}^\times \longrightarrow \varinjlim_{E \supset K} \varprojlim_{p \nmid n} E^\times / E^{\times n}$$

be the natural map. Then for any $\overline{y} \in X(\overline{K})$ and $\varphi \in \mathscr{O}^*_{\text{top}}(X)$ such that $\overline{\varphi}(\overline{x}) = 1$ we have

$$i(\overline{\varphi}(\overline{y})) = f_{\chi_{\text{top}}(\varphi)}(\overline{y})$$

where $\overline{\varphi} : X(\overline{K}) \longrightarrow \mathbb{G}_m(\overline{K}) = \overline{K}^\times$ is the map on geometric points corresponding to φ.

Proof. Consider the Kummer sequence for $\overline{K}^\times$

$$1 \longrightarrow \mu_n \longrightarrow \overline{K}^\times \longrightarrow \overline{K}^\times \longrightarrow 1$$

which exists for all n. Taking the long exact sequence in Galois cohomology, we get

$$1 \longrightarrow (\overline{K}^\times)^{\Gamma_K} / (\overline{K}^{\times n})^{\Gamma_K} \longrightarrow H^1(\Gamma_K, \mu_n) \longrightarrow H^1(\Gamma_K, \overline{K}^\times) \longrightarrow 1$$

We have $H^1(\Gamma_K, \overline{K}^\times) = 0$ by Hilbert 90, and $(\overline{K}^\times)^{\Gamma_K} = K^\times$. Hence, $K^\times / K^{\times n} \cong H^1(\Gamma_K, \mu_n)$ and we conclude that

$$H^1(\Gamma_K, \pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)^t) \cong H^1(\Gamma_K, \widehat{\mathbb{Z}}) \cong \varprojlim_{p \nmid n} K^\times / K^{\times n}$$

noting that $H^1(\Gamma_K, \mu_{p^n}) = 0$ for all n. A class $\xi \in H^1_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}}(1))^{\Gamma_K}$ corresponds to a (continuous) Γ_K-equivariant map $\pi_1^{\text{ét}}(\overline{X}, \overline{x}) \longrightarrow \pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)^t$. So we get a map

$$H^1(\Gamma_K, \pi_1^{\text{ét}}(\overline{X}, \overline{x})) \longrightarrow H^1(\Gamma_K, \pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)^t)$$

Composing with the map

$$X(K) \longrightarrow H^1(\Gamma_K, \pi_1^{\text{ét}}(\overline{X}, \overline{x}))$$

coming from sections of the homotopy exact sequence induced by rational points and passing to the colimit over finite extensions $E \supset K$ gives us f_ξ. By Proposition 2.19, we have a commutative diagram

$$\begin{tikzcd}
X(\overline{K}) \arrow[r, "\overline{\varphi}"] \arrow[d] & \overline{K}^\times \arrow[d] \\
\varinjlim_{E \supset K} H^1(\Gamma_E, \pi_1^{\text{ét}}(\overline{X}, \overline{x})) \arrow[r] & \varinjlim_{E \supset K} H^1(\Gamma_E, \pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1))
\end{tikzcd}$$

By composing with

$$\varinjlim_{E \supset K} H^1(\Gamma_E, \pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)) \longrightarrow \varinjlim_{E \supset K} H^1\big(\Gamma_E, \pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}}, 1)^t\big) \cong \varinjlim_{E \supset K} \varprojlim_{p \nmid n} E^\times / E^{\times n}$$

we get a commutative diagram

$$\begin{tikzcd}
X(\overline{K}) \arrow[r, "\overline{\varphi}"] \arrow[d] & \overline{K}^\times \arrow[d] \\
\varinjlim_{E \supset K} H^1\big(\Gamma_E, \pi_1^{\text{ét}}(\overline{X}, \overline{x})\big) \arrow[r] & \varinjlim_{E \supset K} \varprojlim_{p \nmid n} E^\times / E^{\times n}
\end{tikzcd}$$

The fact that the composite

$$\overline{K}^\times \longrightarrow \varinjlim_{E\supset K} H^1(\Gamma_E, \pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}},1)) \longrightarrow \varinjlim_{E\supset K} H^1\big(\Gamma_E, \pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}},1)^t\big) \cong \varinjlim_{E\supset K} \varprojlim_{p\nmid n} E^\times / E^{\times n}$$

is equal to i follows from [Stix, Corollary 71] after restricting to sections of the tame fundamental group and taking the prime-to$-p$ limit. In particular, the map $E^\times \longrightarrow H^1(\Gamma_E, \pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}},1)^t)$ sends u to the prime-to$-p$ projective limit of the mod$-n$ Kummer torsors given by extracting an n-th root of u. Then

$$H^1(\Gamma_E, \pi_1^{\text{ét}}(\mathbb{G}_{m,\overline{K}},1)^t) \xrightarrow{\sim} \varprojlim_{p\nmid n} E^\times / E^{\times n}$$

sends this profinite Kummer torsor to the prime-to$-p$ projective limit of $u \bmod E^{\times n}$. We conclude that $i(\overline{\varphi}(\overline{x})) = f_{\chi_{\text{top}}(\varphi)}(\overline{x})$ by the second commutative diagram. □

The following lemma guarantees that our main argument works étale-locally on X.

Lemma 3.7. *Let $X = X_0^{\text{perf}}$ with X_0 of finite type over K and let $U \longrightarrow X$ be étale. Then $U \cong U_0^{\text{perf}}$ for some U_0 étale over X_0.*

Proof. Topological invariance tells us that pullback along $X \longrightarrow X_0$ induces an equivalence $X_{0,\text{ét}} \simeq X_{\text{ét}}$. Let $U \longrightarrow X$ be étale. By essential surjectivity, there exists an étale $U_0 \longrightarrow X_0$ such that

$$\begin{aligned} U &\cong U_0 \times_{X_0} X \\ &\cong (U_0 \times_{X_0} X)^{\text{perf}} \\ &\cong U_0^{\text{perf}} \times_X X \\ &\cong U_0^{\text{perf}} \end{aligned}$$

where the second isomorphism comes from Lemma 3.2. □

Now we assume that $\operatorname{trdeg}(K) \geq 1$. The following argument is the only thing that breaks when working over a finite field. We need K to admit a nontrivial valuation.

Proposition 3.8 (Decompletion)**.** *If $\xi \in H^1_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}}(1))^{\Gamma_K} \cap \operatorname{Im}\chi$ and $\operatorname{Im} f_\xi \subset \operatorname{Im} i$, then $\xi \in \operatorname{Im} \mathscr{O}^*(X)$.*

Proof. We would like to arrange for $x \mapsto 1$ so that we can use the construction of f_ξ from Proposition 3.6 (doing so will also get rid of the constants in the group of units). To this end, we let $\mathscr{O}_1^*(\overline{X})$ denote the subgroup of maps in $\mathscr{O}^*(\overline{X})$ that take the value 1 at x and note that the images in $H^1_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}}(1))$ of both groups coincide.

$\mathscr{O}_1^*(\overline{X_0})$ is a finitely generated free abelian group (because X_0 is of finite type). Its perfect closure is the finitely generated free $\mathbb{Z}[1/p]$-module $\mathscr{O}_1^*(\overline{X})$. Let $g_1, \ldots, g_n$ be its generators. Then $\widehat{\mathscr{O}}_1^*(\overline{X}) \cong \widehat{\mathscr{O}}_1^*(\overline{X})_{\neq p}$ is isomorphic to $\widehat{\mathbb{Z}}^n_{\neq p}$ and has $\widehat{\mathbb{Z}}_{\neq p}$-module generators $g_1, \ldots, g_n$. A unit gives a map on geometric points

$$X(\overline{K}) \longrightarrow \overline{K}^\times$$

So every $\hat{u} \in \widehat{\mathscr{O}}_1^*(\overline{X})$ gives a map

$$X(\overline{K}) \longrightarrow \varinjlim_{E\supset K} \varprojlim_{p\nmid n} E^\times / E^{\times n}$$

We have $\hat{u} = g_1^{\varepsilon_1} \ldots g_n^{\varepsilon_n}$ where $\varepsilon_i \in \widehat{\mathbb{Z}}_{\neq p}$. And $\hat{u} = f_{\chi(\hat{u})}$. Therefore, $\operatorname{Im} f_{\chi(\hat{u})} \subset \operatorname{Im} i$ is equivalent to $\hat{u}(X(\overline{K})) \subset i(\overline{K}^\times)$. Assuming that $\hat{u}(X(\overline{K})) \subset i(\overline{K}^\times)$, we would like to show that $\hat{u}$ lies in the image of $\mathscr{O}^*(X)$ i.e., that $\varepsilon_i \in \mathbb{Z}[1/p]$ for all i. We are guaranteed that it would lie in the image of $\mathscr{O}^*(X)$ and not just $\mathscr{O}^*(\overline{X})$ because $\xi \in H^1_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}}(1))^{\Gamma_K}$ and not just $H^1_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}}(1))$.

Let E be a finite extension of K. Since $\operatorname{trdeg}(E) \geq 1$, there exists a non-trivial valuation $\nu : E^\times \longrightarrow \mathbb{Z}$. It extends to a map $\hat{\nu} : \varprojlim_{p \nmid n} E^\times/E^{\times n} \longrightarrow \widehat{\mathbb{Z}}_{\neq p}$. Consider E-rational points $x_1, \ldots, x_n$. For every x_j, we have

$$\sum_i \varepsilon_i \hat{\nu}(g_i(x_j)) = \hat{\nu}(\hat{u}(x_j))$$

Let

$$N = \begin{pmatrix} \hat{\nu}(g_1(x_1)) & \ldots & \hat{\nu}(g_n(x_1)) \\ \vdots & \ddots & \vdots \\ \hat{\nu}(g_1(x_n)) & \ldots & \hat{\nu}(g_n(x_n)) \end{pmatrix}$$

By Cramer's Rule we have

$$\det(N)\varepsilon_i = P_i(\hat{\nu}(g_i(x_1)), \ldots, \hat{\nu}(g_i(x_n)), \hat{\nu}(\hat{u}(x_1)), \ldots, \hat{\nu}(\hat{u}(x_n))) =: P_i$$

where the right-hand side is some polynomial in $\hat{\nu}(g_i(x_1)), \ldots, \hat{\nu}(g_i(x_n)), \hat{\nu}(\hat{u}(x_1)), \ldots, \hat{\nu}(\hat{u}(x_n))$. We assume that $\hat{u}(X(\overline{K})) \subset i(\overline{K}^\times)$. In particular we have $\det(N)$, $P_i \in \mathbb{Z}[1/p] \subset \widehat{\mathbb{Z}}_{\neq p}$ because $\nu(i(E^\times)) \subset \mathbb{Z}[1/p]$. If $\varepsilon \in \widehat{\mathbb{Z}}_{\neq p}$ and $n, n\varepsilon \in \mathbb{Z}[1/p]$, then $\varepsilon \in \mathbb{Z}[1/p]$. So we may conclude that $\varepsilon_i \in \mathbb{Z}[1/p]$ if we can show that $\det(N) \neq 0$. To show this, we use induction on n.

For $n = 1$, we have $\det(N) = \hat{\nu}(g_1(x_1))$. Since g_1 is non-constant, it is open. In particular, we may choose x_1 so that $\hat{\nu}(g_1(x_1)) \neq 0$. Assume we may choose $n-1$ x_j's so that the determinant is nonzero. We have

$$\det(N) = \sum (-1)^{i+1} \hat{\nu}(g_i(x_1)) D_i$$

where D_i is the determinant of the matrix obtained by deleting the top row and i-th column of N. By assumption, we may find $x_2, \ldots, x_n$ such that $D_i \neq 0$ for all i. The inverse image of $\det(N)$ along $\hat{\nu}$ is

$$g' = g_1^{D_1} g_2^{-D_2} \ldots g_n^{(-1)^{n+1} D_n}$$

Since the g_i are generators of $\mathscr{O}_1^*(\overline{X})$ and the D_i are nonzero, g' is non-constant. Hence, there exists some x_1 such that $\hat{\nu}(g'(x_1)) \neq 0$. We conclude that $\det(N) \neq 0$. □

From now on we let K be an infinite, AFG field of positive characteristic.

Theorem 3.9 (GJRW, Theorem 6.6)**.** *Let X_0 be a finite type K-scheme. Then* $\operatorname{Pic}(X_0)$ *is of the form*

(Countably Generated Free Abelian Group) $\oplus$ (Infinite Direct Sum of $\mathbb{Z}/p^n\mathbb{Z}$ for some n) $\oplus$ (Finite Group)

Theorem 3.10. *Let $X = X_0^{\mathrm{perf}}$ be the perfection of a finite type K-scheme and let Y be any finite type $K^{p^{-\infty}}$−scheme. Then the natural map*

$$\operatorname{Mor}_{K^{p^{-\infty}}}(X, Y) \longrightarrow \operatorname{Mor}^\bullet_{K_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}})$$

is a bijection.

Proof. We already know that the map is injective by Proposition 2.11 (noting that $X_0^{\mathrm{perf}} \cong X_{0,K^{p^{-\infty}}}^{\mathrm{perf}}$). By Proposition 3.1, it suffices to show that for all $U \in X_{\text{ét}}$ and all $\varphi \in \mathscr{O}^*_{\mathrm{top}}(U)$, there exists a $\widetilde{\varphi} : U \longrightarrow \mathbb{G}_m$ coinciding with φ on geometric points. By Lemma 3.7, the following argument applies to all étale opens over X. In particular, it suffices to just consider X. Pick some $\varphi \in \mathscr{O}^*_{\mathrm{top}}(X)$. We may assume without loss of generality that $\overline{\varphi}(\overline{x}) = 1$. Proposition 3.8 tells us that $\chi_{\mathrm{top}}(\varphi)$ comes from a unit if we can show that $\chi_{\mathrm{top}}(\varphi) \in H^1_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}}(1))^{\Gamma_K} \cap \operatorname{Im}\chi$ and $\operatorname{Im} f_{\chi_{\mathrm{top}}(\varphi)} \subset \operatorname{Im} i$. By Proposition 3.6, the latter condition is satisfied. Hence, $\chi_{\mathrm{top}}(\varphi)$ comes from a unit if $\operatorname{Im}\chi_{\mathrm{top}} \subset \operatorname{Im}\chi$. Consider the short exact sequence

$$1 \longrightarrow \widehat{\mathscr{O}}^*(\overline{X}) \xrightarrow{\chi} H^1_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}}(1)) \longrightarrow \widehat{T}(\operatorname{Pic}(\overline{X})) \longrightarrow 1$$

Taking Galois-fixed points gives

$$1 \longrightarrow \widehat{\mathscr{O}}^*(\overline{X})^{\Gamma_K} \xrightarrow{\chi} H^1_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}}(1))^{\Gamma_K} \longrightarrow \widehat{T}(\operatorname{Pic}(X))$$

We have $\operatorname{Pic}(X) \cong \operatorname{Pic}(X_0)[1/p]$ by Lemma 3.3. So we have $\widehat{T}(\operatorname{Pic}(X)) \cong \widehat{T}(\operatorname{Pic}(X_0)[1/p])$. But Theorem 3.9 tells us that $\widehat{T}(\operatorname{Pic}(X_0)[1/p]) = 0$. Hence,

$$\operatorname{Im} \chi \supset \chi(\widehat{\mathscr{O}}^*(\overline{X})^{\Gamma_K}) \cong H^1_{\text{ét}}(\overline{X}, \widehat{\mathbb{Z}}(1))^{\Gamma_K} \supset \operatorname{Im} \chi_{\text{top}}$$

Let $\widetilde{\varphi}$ be the unit that maps to $\chi_{\text{top}}(\varphi)$. So $\chi_{\text{top}}(\widetilde{\varphi}_{\text{ét}}) = \chi_{\text{top}}(\varphi)$ by Lemma 3.4. Therefore we have $f_{\chi_{\text{top}}(\widetilde{\varphi}_{\text{ét}})}(\overline{x}) = f_{\chi_{\text{top}}(\varphi)}(\overline{x})$ for all $\overline{x} \in X(\overline{K})$. Hence, by Proposition 3.6, we have $i(\overline{\widetilde{\varphi}_{\text{ét}}(\overline{x})}) = i(\overline{\varphi}(\overline{x}))$ for all $\overline{x} \in X(\overline{K})$. Thus, since i is injective (i.e., K has no infinitely divisible elements), $\overline{\widetilde{\varphi}_{\text{ét}}}(\overline{x}) = \widetilde{\varphi}(\overline{x}) = \overline{\varphi}(\overline{x})$ for all $\overline{x} \in X(\overline{K})$. In particular, $\widetilde{\varphi}$ coincides with φ on geometric points. $\square$

Corollary 3.11. *Let X be the perfection of a K-scheme of finite type. Let $Y = Y_0^{\text{perf}}$ be the perfection of a K-scheme of finite type. Then the natural map*

$$\operatorname{Mor}_{K^{p^{-\infty}}}(X, Y) \longrightarrow \operatorname{Mor}^\bullet_{K_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}})$$

is bijective.

Proof. We have

$$\begin{aligned}\operatorname{Mor}_{K^{p^{-\infty}}}(X, Y) &\cong \lim_{\Phi} \operatorname{Mor}_{K^{p^{-\infty}}}(X, Y_{0,K^{p^{-\infty}}}) \\ &\cong \lim_{\Phi_{\text{ét}}} \operatorname{Mor}^\bullet_{K_{\text{ét}}}(X_{\text{ét}}, (Y_{0,K^{p^{-\infty}}})_{\text{ét}}) && \text{(Theorem 3.10)} \\ &\cong \operatorname{Mor}^\bullet_{K_{\text{ét}}}(X_{\text{ét}}, Y_{\text{ét}}) && \text{(Topological Invariance)}\end{aligned}$$

$\square$

Corollary 3.12. *Let X and Y be perfections of schemes of finite type over K. Then $X \cong Y$ if and only if $X_{\text{ét}} \simeq Y_{\text{ét}}$ (over $K^{p^{-\infty}}$ and $\operatorname{Spec}(K)_{\text{ét}}$ respectively).* $\square$

Sanity Check 3.13 (Multiplication by p on $\mathbb{G}_m$). The p-th power map $p : \mathbb{G}_m \xrightarrow{x \mapsto x^p} \mathbb{G}_m$ gives rise to an equivalence on étale sites $p^* : \mathbb{G}_{m,\text{ét}} \xrightarrow{\sim} \mathbb{G}_{m,\text{ét}}$ whose inverse is not realized by a morphism of schemes. We can understand this concretely by looking at classes.

Let $\mathbb{G}_m = \operatorname{Spec}(K^{p^{-\infty}}[t^{\pm 1}])$. The p-th power map is the unit t^p. So the corresponding class at finite level n (prime-to-p) is the étale μ_n-torsor $\operatorname{Spec}(\overline{K}[t^{\pm 1}, X]/(X^n - t^p))$. As a class in $H^1_{\text{ét}}(\mathbb{G}_{m,\overline{K}}, \mu_n) \cong \mathbb{Z}/n\mathbb{Z}$, this corresponds to $p \pmod n$. The corresponding element in $H^1_{\text{ét}}(\mathbb{G}_{m,\overline{K}}, \widehat{\mathbb{Z}}_{\neq p}) \cong \widehat{\mathbb{Z}}_{\neq p}$ is p. The torsor $\lim_{p \nmid n} \operatorname{Spec}(\overline{K}[t^{\pm 1}, X]/(X^n - t))$ corresponds to $1 \in \widehat{\mathbb{Z}}_{\neq p}$. And raising t to a power on the torsor side corresponds to multiplication on the class side.

The inverse of p^* should give a map on étale fundamental groups that is the inverse of the map on étale fundamental groups induced by p^*. In particular, the class associated to the inverse of p^* should be the inverse of the class associated to p^*. This class is $1/p \in \widehat{\mathbb{Z}}_{\neq p}$. It lies both in the image of χ and the image of χ_{top}, but it does not come from a unit. We conclude that decompletion (Proposition 3.8) cannot go through for generic imperfect schemes. To see the failure of the decompletion argument in the case of $\mathbb{G}_m$, we note that $\mathscr{O}^*_1(\mathbb{G}_{m,\overline{K}}) \cong \mathbb{Z}$ is not a free $\mathbb{Z}[1/p]$-module. In particular, we would need to show that $\varepsilon_i \in \mathbb{Z}$, not $\mathbb{Z}[1/p]$. But there is no analogue of [Voe, Lemma 3.1]. Indeed, the completed unit with image $1/p \in H^1(\mathbb{G}_{m,\overline{K}}, \widehat{\mathbb{Z}}_{\neq p})$ is given by $\hat{u} = t^{1/p}$ (where $1/p$ is understood as a profinite prime-to$-p$ integer), but $1/p \notin \mathbb{Z}$.

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