TWO PROOFS OF THE GERRITZEN-GRAUERT THEOREM

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Abstract. The goal of this note is to present a more streamlined version of BGR's proof of Gerritzen-Grauert and compare it with Temkin's new proof of the theorem.

Contents

Introduction

The Gerritzen-Grauert theorem is a fundamental theorem in rigid analytic geometry whose most important corollary is that affinoid subdomains are finite unions of rational domains.

Theorem 1 (Gerritzen-Grauert)**.** *Let $\varphi : Y \longrightarrow X$ be a locally closed immersion. Then there exists a covering*

$$\bigcup_{i=1}^{n} X_i = X$$

consisting of finitely many rational domains $X_i \subset X$ such that φ induces Runge immersions $\varphi_i : \varphi^{-1}(X_i) \longrightarrow X_i$.

The theorem's importance comes from its use in proving Tate's Acyclicity Theorem: the presheaf $\mathcal{O}_X$, which assigns to an affinoid subdomain $U = \mathrm{Sp}(A') \subset X = \mathrm{Sp}(A)$ the algebra A', is a sheaf for rational domains, and hence a sheaf for finite affinoid coverings (which are coverings for the G-topology). This expository note will go through both the classical proof of the theorem as given in [BGR] and a more recent proof given by Temkin in [Tem].

Preliminaries

Fix a complete non-archimedean valued field k with non-trivial absolute value. Let A and B be (strictly) k-affinoid algebras. Let $X = \mathrm{Sp}(A)$, $Y = \mathrm{Sp}(B)$ be the corresponding affinoid varieties.

Definition 2. Let $f : X \longrightarrow Y$ be a morphism of affinoid varieties. f is a *closed immersion* if the corresponding map on k-affinoid algebras $B \longrightarrow A$ is surjective. If f is injective, we say it is an *open immersion* if it is an isomorphism on stalks $f_x : \mathcal{O}_{Y,y} \xrightarrow{\sim} \mathcal{O}_{X,x}$ for all x such that $\varphi(x) = y$. We call it a *locally closed immersion* if it is a surjection on stalks.

Definition 3. A morphism of affinoid varieties $f : X \longrightarrow Y$ is a *Runge Immersion* if it factors as

$$\begin{tikzcd} X \arrow[rr, "f"] \arrow[dr, hook, "g"'] & & Y \\ & W \arrow[ur, hook, "h"'] & \end{tikzcd}$$

where g is a closed immersion and W is a Weierstraß domain of Y via h.

Notice that open immersions of Weierstraß domains are trivially Runge because Weierstraß domains are closed in themselves (note that closed immersions are Zariski-closed by definition).

Example. All open immersions that are also closed immersions are Runge immersions. To see this, we will show that an open and closed immersion $\varphi : X \longrightarrow Y$ expresses X as a Weierstraß subdomain of Y where $X = \mathrm{Sp}(A)$, $Y = \mathrm{Sp}(B)$. For $\varphi : X \longrightarrow Y$ an open and closed immersion, the image of X in Y is Zariski-closed because φ is a closed immersion. Let $Y_1 = \varphi(X)$ and $Y_2 = Y \setminus X_1$. So $Y_1 \cap Y_2 = \varnothing$ and $Y_1 \sqcup Y_2 = Y$. Since it is a closed immersion, φ corresponds to a surjection $\varphi^* : B \longrightarrow A$ which extends to a surjection on localizations $B_{\mathfrak{m}_{\varphi(x)}} \longrightarrow A_{\mathfrak{m}_x}$ for all $x \in X$. Since φ is an open immersion, it is bijective on stalks. Localizations inject into their stalks because, by [BGR, Proposition 7.3.2/3], stalks are the localization up to completion and the Krull Intersection Theorem tells you that the $\mathfrak{m}_x$-adic topology on $A_{\mathfrak{m}_x}$ is Hausdorff. Thus, since $\mathcal{O}_{X,x} \cong \mathcal{O}_{Y,\varphi(x)}$, we get an injection $B_{\mathfrak{m}_{\varphi(x)}} \longrightarrow A_{\mathfrak{m}_x}$. So the map $B_{\mathfrak{m}_{\varphi(x)}} \longrightarrow A_{\mathfrak{m}_x}$ is a bijection for all $x \in X$. Therefore, for any $x \in X$, there exists an $f \in B$ with $f(\varphi(x)) \neq 0$ such that φ^* induces a bijection $B[f^{-1}] \longrightarrow A[f^{-1}]$. Thus $\varphi(X)$ is Zariski-open in Y. So Y_2 is also Zariski-closed in Y. Thus by the Chinese Remainder Theorem

$$B/\operatorname{rad}(B) \cong B/\operatorname{id}(Y_1) \times B/\operatorname{id}(Y_2)$$

where $\operatorname{id}(Y_i)$ denotes the ideal of vanishing of Y_i, $i = 1, 2$. So we may find a function $f \in B$ such that $f \equiv 0 \pmod{\operatorname{id}(Y_1)}$ and $f \equiv 1 \pmod{\operatorname{id}(Y_2)}$. Take $c \in k$ with $|c| > 1$. Then Y_1 is equal to the Weierstraß domain

$$Y(cf) = \{y \in Y : |cf(y)| \leq 1\} = \{y \in Y : |f(y)| \leq |c|^{-1} < 1\}$$

Thus φ is a Runge immersion because it is an embedding of a Weierstraß subdomain into Y: it factors as

$$\begin{tikzcd}
\mathrm{Sp}(A) \cong \mathrm{Sp}(B/I) \arrow[rr] \arrow[dr, "\sim"'] & & \mathrm{Sp}(B) \\
& \mathrm{Sp}(B\langle cf\rangle) \arrow[ur, hook] &
\end{tikzcd}$$

Gerritzen-Grauert à la BGR

Theorem 4 (Gerritzen-Grauert)**.** *Let $\varphi : Y \longrightarrow X$ be a locally closed immersion. Then there exists a covering*

$$\bigcup_{i=1}^{n} X_i = X$$

consisting of finitely many rational domains $X_i \subset X$ such that φ induces Runge immersions $\varphi_i : \varphi^{-1}(X_i) \longrightarrow X_i$.

The major important corollary of this theorem is the special case of when φ is also an open immersion (note that open immersions are locally closed immersions). In particular, we get

Corollary 4.1. *If $\varphi : Y \longrightarrow X$ is an open immersion, then the maps $\varphi_i : \varphi^{-1}(X_i) \longrightarrow X_i$ define $\varphi^{-1}(X_i)$ as a Weierstraß subdomain of X_i for all i. In particular, if $X' \subset X$ is an affinoid subdomain of X then there exists a covering $X = \bigcup_{i=1}^{r} X_i$ with X_i rational such that $X_i \cap X'$ is Weierstraß in X_i for all i. Importantly, X' is a finite union of rational domains in X.*

Proof. First we will show that open immersions that are also Runge are embeddings of Weierstraß domains. Since the φ_i are Runge, we get a factorization

$$\begin{tikzcd}
\varphi^{-1}(X_i) \arrow[rr, "\varphi_i"] \arrow[dr, hook, "f_i"'] & & X_i \\
& W_i \arrow[ur, hook, "g_i"'] &
\end{tikzcd}$$

with f_i closed and g_i expressing W_i as a Weierstraß subdomain of X_i. g_i must be an open immersion because it is a Weierstraß domain embedding. The φ_i are also open immersions hence f_i must also be an open immersion. Therefore f_i is open and closed and so it expresses $\varphi^{-1}(X_i)$ as a Weierstraß subdomain of W_i by the example following Definition 3. Thus by the transitivity of Weierstraß domains, $\varphi^{-1}(X_i)$ is Weierstraß in X_i for all i.

Now if X' is an affinoid subdomain of X we get an open immersion $\varphi : X' \longrightarrow X$. This means that $\varphi^{-1}(X_i) = X_i \cap X'$ is Weierstraß in X for all i. Moreover, $X_i \cap X'$ is rational for all i by the transitivity of rational domains [BGR, Theorem 7.2.4/2]. □

Before going into [BGR]'s proof of Gerrizten-Grauert in sufficiently gory detail, we will give an outline:

(1) For $\lambda : A \longrightarrow B$ corresponding to φ, we define $\langle A : B\rangle$ to be the smallest n (called the embedding dimension) such that there is an affinoid generating system of B over A consisting of n elements $f_1, \dots, f_n$ for which we can find a surjection

$$\Lambda : A\langle X_1, \dots, X_n\rangle \longrightarrow B$$
$$X_i \longmapsto f_i$$

We would like to perform (strong) induction on $n = \langle A : B\rangle$. We induce on n because such a surjection Λ can allow us to create rational domains out of "distinguished" elements in its kernel. Being distinguished is an $\mathfrak{m}_x$-local condition on the coefficients of a series for $\mathfrak{m}_x \in \mathrm{Sp}(A)$. The base case is trivial because closed immersions are Runge.

(2) We can find a surjection $\Lambda : A\langle X_1, \dots, X_n\rangle \longrightarrow B$ extending λ such that there exists some $f \in \ker(\Lambda)$ that is "X_n-distinguished of degree $\le s$." We can then find a rational domain $X' = \mathrm{Sp}(A')$ of $X = \mathrm{Sp}(A)$ consisting of the points at which f is X_n-distinguished of degree s.

(3) From this rational domain, we can show that we get a surjection

$$A'\langle X_1, \dots, X_{n-1}\rangle \longrightarrow B\widehat{\otimes}_A A'$$

which notably has embedding dimension $\langle B\widehat{\otimes}_A A' : A'\rangle \le n-1$ so we may apply the induction hypothesis and get a finite covering of X' by rational domains $X_i, 1 \le i \le m$ such that the induced maps $\varphi^{-1}(X_i) \longrightarrow X_i$ are Runge immersions.

(4) The members of this covering family are also rational in X because of the transitivity of rational domains. Thus, we may pick functions out of A that express the X_i's as rational domains of A and then thicken them to rational domains X_ε^i that still cover X' and for which $\varphi : \varphi^{-1}(X_\varepsilon^i) \longrightarrow X_\varepsilon^i$ is a Runge immersion for all i.

(5) We may then find rational domains $V_1, \dots, V_\ell$ such that $\bigcup_{i=1}^m X_\varepsilon^i \cup \bigcup_{j=1}^\ell V_j = X$ and $\bigcup_j V_j \subset X \smallsetminus X'$. Geometrically, the thickening in 4 is necessary because we would otherwise have a disjoint covering of X which would contradict X being connected (in the weak G-topology). To prove the theorem, it is enough to show that we have Runge immersions $\varphi^{-1}(V_j) \longrightarrow V_j$. f will be X_n-distinguished of degree $\le s-1$ on these rational domains because the V_j's don't meet X'. Thus we may repeat what we did before using $A_j\langle X_1, \dots, X_n\rangle \longrightarrow B$ with $\mathrm{Sp}(A_j) = V_j$ until we get down to $s = 0$. Similarly, we may induct on s because, if the theorem holds when $f|_{V_j}$ is X_n-distinguished of degree $\le s-1$ it holds for $\le s$ by the above considerations. At this point, we will have shown that the theorem holds for $\langle A : B\rangle = n$ and we are done.

We will now go through the preparations needed to prove the theorem. First we define what it means to be a distinguished element. Take $x \in X = \mathrm{Sp}(A)$ and denote its associated maximal ideal by $\mathfrak{m}_x$. Define

$$\pi_x : A\langle X_1, \dots, X_n\rangle \longrightarrow (A/\mathfrak{m}_x)\langle X_1, \dots, X_n\rangle$$
$$\sum_{\nu=0}^{\infty} a_{\nu_1\dots\nu_n} X_1^{\nu_1} \dots X_n^{\nu_n} \longmapsto \sum_{\nu=0}^{\infty} a_{\nu_1\dots\nu_n}(x) X_1^{\nu_1} \dots X_n^{\nu_n}$$

where $a_{\nu_1\dots\nu_n}(x)$ is the image of $a_{\nu_1\dots\nu_n}$ in $A/\mathfrak{m}_x$. For simplicity, we will denote with X the collection of indeterminates $X_1, \dots, X_n$ and write such a series as given above as $\sum_{\nu=0}^\infty a_\nu X^\nu$ from now on.

Definition 5. A series $f \in A\langle X_1, \dots, X_n\rangle$ is called X_n-*distinguished of degree* s *at* $x \in X$ if $\pi_x(f) \in (A/\mathfrak{m}_x)\langle X_1, \dots, X_n\rangle$ given by $f = \sum_{\nu=0}^\infty f_\nu(X_1, \dots, X_{n-1})X_n^\nu$ satisfies[1]:

(1) f_s is a unit in $(A/\mathfrak{m}_x)\langle X_1, \dots, X_{n-1}\rangle$
(2) $|f_s| = |f|$ and $|f_s| > |f_\nu|$ for all $\nu > s$ (in lexicographic order)

[1] NB: one can write any series $f \in A\langle X_1, \dots, X_n\rangle$ in this form.

We say that f is *X_n-distinguished of degree s on* $\operatorname{Sp} A$ if f is X_n-distinguished of degree s at every point $x \in \operatorname{Sp} A$. We will say that f is *X_n-distinguished of degree $\leq s$ on* $\operatorname{Sp} A$ if the evident condition holds.

Note that, since $(A/\mathfrak{m}_x)\langle X_1, \ldots, X_{n-1}\rangle$ is a Tate algebra, $|f_s| = |f|$ is equivalent to saying that $|f_s| = \max_\nu |a_\nu|$ where $a_\nu \in A/\mathfrak{m}_x$ is the coefficient of f_ν. In particular, we will demand that $|f_s| \geq |f_v|$ for $v \leq s$ (in lexicographic order).

Now we will go through and prove a number of key propositions that we will need for the proof. We will prove them in roughly the order in which they are used in the proof.

Proposition 6. *Let $\varphi : \operatorname{Sp}(B) \longrightarrow \operatorname{Sp}(A)$ be an injection corresponding to $\lambda : A \longrightarrow B$. Assume that $n = \langle A : B\rangle \geq 1$ and $A \neq 0 \neq B$. Then there exists a surjection $\Lambda : A\langle X_1, \ldots, X_n\rangle \longrightarrow B$ extending λ such that* $\ker(\Lambda)$ *contains an element f that is X_n-distinguished of degree s on* $\operatorname{Sp}(A)$ *for some $s \geq 0$.*

To show that such an element exists, we need to show that we may find a series in $A\langle X_1, \ldots, X_n\rangle$ whose coefficients do not share a common 0 because of the following lemma:

Lemma 7. *Let $f = \sum a_\nu X^\nu \in A\langle X_1, \ldots, X_n\rangle$ such that the $a_\nu \in A$ have no common* 0. *Then there exists an A-algebra automorphism $\tau : A\langle X_1, \ldots, X_n\rangle \longrightarrow A\langle X_1, \ldots, X_n\rangle$ such that $\tau(f)$ is X_n-distinguished of degree $\leq s$ on* $\operatorname{Sp}(A)$.

Proof. Since the coefficients have no common 0, they generate the unit ideal in A. Thus there are finitely many indices $\nu_1, \ldots, \nu_m$ such that the a_{ν_i} generate the unit ideal. Taking the max over i we can bound $|a_{\nu_i}(x)|$ from below by some α for all $x \in \operatorname{Sp}(A)$. The coefficients a_ν will be less than α for almost all indices ν. This shows that $t := \sup_{x \in \operatorname{Sp}(A)} t_x$ is finite for $t_x :=$ the least upper bound of all natural numbers occuring in multi-indices ν satsifying $|a_\nu(x)| = |\pi_x(f)|$. Define $c_n := 1$ and $c_{n-j} := 1 + t\sum_{d=0}^{j-1} c_{n-d}$. Define the automorphism

$$\begin{aligned} \tau : A\langle X_1, \ldots, X_n\rangle &\longrightarrow A\langle X_1, \ldots, X_n\rangle \\ X_i &\longmapsto X_i + X_n^{c_i} \qquad j = 1, \ldots, n-1 \\ X_n &\longmapsto X_n \end{aligned}$$

Since $t \geq t_x$ for all x, [BGR, Prop 5.2.4/2] implies that $\pi_x(\tau(f))$ is X_n-distinguished of some degree s_x such that

$$s_x \leq t_x \sum_{i=1}^{n} c_i \leq t \sum_{i=1}^{n} c_i$$

thus $\tau(f)$ is X_n-distinguished of degree $\leq s = t\sum_{i=1}^n c_i$ on $\operatorname{Sp}(A)$. □

We can ensure that we have series with coefficients generating the unit ideal:

Lemma 8. *Let $\lambda : A \longrightarrow B$ give rise to an injection $\varphi : \operatorname{Sp}(B) \longrightarrow \operatorname{Sp}(A)$ and let $\Lambda : A\langle X_1, \ldots, X_n\rangle \longrightarrow B$ be a surjection extending λ. Then:*

(1) If $A \neq 0$, all surjections $\Lambda_x : (A/\mathfrak{m}_x)\langle X_1, \ldots, X_n\rangle \longrightarrow B/\mathfrak{m}_x B$ have a non-trivial kernel.
(2) There exists an element $f \in \ker(\Lambda)$ that has coefficients generating the unit ideal in A.

Proof. (1) Since φ is injective, we know that for all $x \in \operatorname{Sp}(A)$, $\mathfrak{m}_x B = \Lambda(\mathfrak{m}_x)B$ is contained in at most one maximal ideal of B. Thus $B/\mathfrak{m}_x B$ is local if it is non-zero. Therefore Λ_x, which is surjective by assumption, cannot be injective hence an isomorphism because $(A/\mathfrak{m}_x)\langle X_1, \ldots, X_n\rangle$ is clearly not local.

(2) Consider the following commutative diagram[2]:

$$\begin{array}{ccc} A\langle X_1, \ldots, X_n\rangle & \xrightarrow{\Lambda} & B \\ {\scriptstyle \pi_x}\downarrow & & \downarrow{\scriptstyle \pi_x} \\ (A/\mathfrak{m}_x)\langle X_1, \ldots, X_n\rangle & \xrightarrow[\Lambda_x]{} & B/\mathfrak{m}_x B \end{array}$$

Given any $x \in \operatorname{Sp}(A)$ there exists a $g \in \ker(\Lambda)$ such that $\pi_x(g) \neq 0$. Indeed, since Λ_x is not injective from (1), we can find a series $g_1 \in A\langle X_1, \ldots, X_n\rangle$ satisfying $0 \neq \pi_x(g_1) \in \ker(\Lambda_x)$. Then $\Lambda(g_1) \in \mathfrak{m}_x B$ because

[2]We abuse notation and denote both reduction maps by π_x

it is 0 in $B/\mathfrak{m}_x B$ and we can find an inverse image g_2 of $\Lambda(g_1)$ in $\mathfrak{m}_x A\langle X_1, \ldots, X_n\rangle$ because the diagram commutes. Let $g = g_1 - g_2$. Then

$$\Lambda(g) = \Lambda(g_1) - \Lambda(g_2) = \Lambda(g_1) - \Lambda(g_1) = 0$$

and

$$\pi_x(g) = \pi_x(g_1) - \pi_x(g_2) = \pi_x(g_1) \neq 0$$

□

So now we may prove Proposition 6.

Proof of Proposition 6. We are assuming that $A \neq 0 \neq B$. Pick a surjection $\Lambda : A\langle X_1, \ldots, X_n\rangle \longrightarrow B$ extending $\lambda : A \longrightarrow B$. Then Lemma 9 implies that there exists some $f \in \ker(\Lambda)$ such that $\pi_x(f) \neq 0$ for all $x \in \mathrm{Sp}(A)$. Then by Lemma 8, we may apply an automorphism to send f to an X_n-distinguished series of degree $\leq s$. □

Next we will construct a rational domain out of points at which f is X_n-distinguished of degree strictly s.

Proposition 9. *Let $f \in A\langle X_1, \ldots, X_n\rangle$ be X_n-distinguished of degree $\leq s$ on $\mathrm{Sp}(A)$. Then the collection of points $U = \{x \in \mathrm{Sp}(A) \mid f$ is X_n-distinguished of degree s at $x\}$ is a rational domain in $\mathrm{Sp}(A)$.*

Proof. Let $f = \sum f_\nu X_n^\nu$ for $f_\nu \in A\langle X_1, \ldots, X_{n-1}\rangle$ and let a_ν denote the constant term of f_ν. Then f being X_n-distinguished of degree $d \leq s$ at $x \in \mathrm{Sp}(A)$ implies that

1. $|\pi_x(f_\nu)| \leq |\pi_x(f_d)|$ for $\nu \leq d$
2. $|\pi_x(f_\nu)| < |\pi_x(f_d)|$ for $\nu > d$
3. $\pi_x(f_d)$ is a unit in $(A/\mathfrak{m}_x)\langle X_1, \ldots, X_{n-1}\rangle$

1 comes from the fact that $|\cdot|$ is the Gauß norm on $(A/\mathfrak{m}_x)\langle X_1, \ldots, X_{n-1}\rangle$ while 2 and 3 come from the definition of distinguished elements. Since $a_\nu(x) = \pi_x(a_\nu)$ the constant term of $\pi_x(f_\nu)$ we get that $|a_\nu(x)| \leq |\pi_x(f_\nu)|$ which is an equality for $\nu = d$ by the definition of distinguished element. Thus

$$|\pi_x(f_\nu)| \leq |\pi_x(f_d)| \quad \text{for } \nu \leq d$$
$$|\pi_x(f_\nu)| < |\pi_x(f_d)| \quad \text{for } \nu > d$$

So we notice that $a_d(x) \neq 0$. Since f is X_n-distinguished of degree $d \leq s$ for all $x \in \mathrm{Sp}(A)$ we see that $a_0, \ldots, a_s \in A$ have no common 0 in $\mathrm{Sp}(A)$. Therefore

$$U = \{x \in \mathrm{Sp}(A) : |a_\nu(x)| \leq |a_s(x)|,\ \nu = 0, \ldots, s\}$$

is a rational domain.

□

Next we need the result that will allow us to use the induction hypothesis.

Proposition 10. *Let $f \in A\langle X_1, \ldots, X_n\rangle$ be X_n-distinguished of degree s for all $x \in \mathrm{Sp}(A)$. Then the canonical map $\iota : A\langle X_1, \ldots, X_{n-1}\rangle \longrightarrow A\langle X_1, \ldots, X_n\rangle/(f)$ is finite.*

Proof Sketch. The idea here is that every power series whose coefficients generate the unit ideal is a distinguished/Weierstraß polynomial up to multiplication by a unit.

Let $f = \sum f_\nu X_n^\nu$ for $f_\nu \in A\langle X_1, \ldots, X_{n-1}\rangle$. Then $\pi_x(f_\nu)$ is a unit and hence has no 0 for all $x \in \mathrm{Sp}(A)$ (an element in a ring is a unit if and only if it is non-zero modulo every maximal ideal). Thus f_s has no zeroes for all $x \in \mathrm{Sp}(A)$ and hence is a unit in $A\langle X_1, \ldots, X_{n-1}\rangle$. Replacing f by $f_s^{-1}f$ we may assume that $f_s = 1$ thus

$$|f_\nu|_{\sup} \leq 1 \quad \nu \leq s$$
$$|f_\nu|_{\sup} < 1 \quad \nu > s$$

Therefore, denoting by b the residue class of X_n in $A\langle X_1, \ldots, X_n\rangle/(f)$ we get that

$$|b^s + \iota(f_{s-1})b^{s-1} + \ldots + \iota(f_0)|_{\sup} < 1$$

Since b is an affinoid generating system of $A\langle X_1, \ldots, X_n\rangle/(f)$ over $A\langle X_1, \ldots, X_{n-1}\rangle$, ι is finite by [BGR, Theorem 6.3.2/2]. □

We define the rational domain
$$X_\varepsilon := X\left(\varepsilon^{-1}\frac{f_1}{g},\ldots,\varepsilon^{-1}\frac{f_r}{g}\right)$$
for $f_1,\ldots,f_r,g\in A$ generating the unit ideal and $\varepsilon\in\sqrt{|k^*|}$. We note that $X_\varepsilon\subset X_{\varepsilon'}$ if $\varepsilon\le\varepsilon'$. We can think of X_ε as a thickening of X_1. We would like to be able to enlarge our rational domains (for reasons we will discuss in a moment) that we get from the induction hypothesis and still have Runge immersions. The following lemma comes in handy.

Lemma 11 (Extension Lemma). *Let $\varphi_{\varepsilon_0}: Y_{\varepsilon_0}\longrightarrow X_{\varepsilon_0}$ be a Runge immersion for some $\varepsilon_0\in\sqrt{|k^*|}$. Then there exists an $\varepsilon>\varepsilon_0$ such that $\varphi_\varepsilon: Y_\varepsilon\longrightarrow X_\varepsilon$ is also Runge immersion.*

Proof Sketch. We may replace X and Y by $X_{\varepsilon'}$ and $Y_{\varepsilon'}$ to assume that X_{ε_0} and Y_{ε_0} are Weierstraß subdomains (rational subdomains of rational domains are Weierstraß in those rational domains). Because φ_{ε_0} is Runge, the image of A_{ε_0} in B_{ε_0} is dense in B_{ε_0}. We thus get that A is dense in B_{ε_0}.[3] We make a Weierstraß subdomain of X out of the inverse image of elements arbitrarily close (in the supremum norm) to affinoid generators of B_{ε_0} (we may do so because of density). In particular, we may make the image of these elements power bounded to force them to factor through the Weierstraß subdomain. We would like this map into the Weierstraß subdomain to be a closed immersion. We can show that the map is finite. The maximum modulus principle allows us to thicken in such a way so that for $X_{\varepsilon_0}\cap Y=\varnothing$ we may find an $\varepsilon>\varepsilon_0$ such that $X_\varepsilon\cap Y=\varnothing$. Then [BGR, Proposition 7.3.3/8] tells us that we still have a locally closed immersion which will be closed by finiteness and so φ_ε is Runge. □

Take some $\varepsilon\in\sqrt{|k^*|}$. We define
$$X_\varepsilon^i := X\left(\varepsilon^{-1}\frac{g_1^i}{g_{n_i}^i},\ldots,\varepsilon^{-1}\frac{g_{n_i-1}^i}{g_{n_i}^i}\right)$$
$$X_\varepsilon := \bigcup_{i=1}^r X_\varepsilon^i$$
We then have the following easy lemma.

Lemma 12. *For all $\varepsilon\in\sqrt{|K^*|}$ with $\varepsilon>1$, there exist finitely many rational subdomains $V_1,\ldots V_\ell\subset X$ such that*
$$\bigcup_{j=1}^{\ell} V_j\subset X\smallsetminus X_1$$
and
$$X = X_\varepsilon\cup\bigcup_{j=1}^{\ell}V_j$$

Proof. Let $r=1$ i.e., assume we are only dealing with one of the X_ε^i's. For $j=1,\ldots,\ell$ define
$$V_j := X\left(\frac{g_1}{g_j},\ldots,\frac{g_{n-1}}{g_j},\frac{\varepsilon g_n}{g_j}\right)$$
This works. It is evident we may do this for any r. □

We will note that a similar fact is used in Temkin's proof of Gerritzen-Grauert, but it follows from the compactness of $\mathcal{M}(A)$.

A quick discussion of what's going on here geometrically: we need the thickening lemma because otherwise, we'd have a disjoint rational covering $X=X_1\sqcup\bigcup V_j$ which would constitute an admissible covering of X in the weak G-topology. But then X would be disconnected. In particular, this thickening is necessary from the point of view that Gerritzen-Grauert is necessary for Tate Acyclicity. If X is disconnected, you get non-trivial idempotents! See [BGR, Proposition 9.1.4/8] and the discussion following it for more details. We also note that the connection to compactness here involves the fact that we use non-strict inequalities.

[3] Algebras are dense in their Weierstraß subdomains: the completion of A with respect to the norm given by $|f|=|\sum a_\nu g^\nu|=\max|a_\nu|$ is going to be the functions on the Weierstraß domain $\mathrm{Sp}(A\langle g\rangle)$

Proposition 13. *Let $\varphi^* : A \longrightarrow A'$ define $\mathrm{Sp}(A')$ as an affinoid subdomain of $\mathrm{Sp}(A)$ via $\varphi : \mathrm{Sp}(A') \longrightarrow \mathrm{Sp}(A)$. Then $f \in A\langle X_1, \ldots, X_n\rangle$ is X_n-distinguished of degree s at $x \in \mathrm{Sp}(A')$ if and only if $f' \in A'\langle X_1, \ldots, X_n\rangle$ is X_n-distinguished of degree s at x where f' is the image of f along $A\langle X_1, \ldots, X_n\rangle \longrightarrow A'\langle X_1, \ldots, X_n\rangle$.*

Proof. For $x \in \mathrm{Sp}(A')$ we have $\mathfrak{m}_x = \varphi^*(\mathfrak{m}_{\varphi(x)})A'$. [BGR, Proposition 7.2.2/1] implies that $A'/\mathfrak{m}_x \cong A/\mathfrak{m}_{\varphi(x)}$ (isometric and preserves units so preserves distinguishedness). □

Proposition 14. *$f \in A\langle X_1, \ldots, X_n\rangle$ is a unit if and only if f is X_n-distinguished of degree $s = 0$ for all $x \in \mathrm{Sp}(A)$.*

Proof. Since f is a unit in $A\langle X_1, \ldots, X_n\rangle$, $\pi_x(f)$ is a unit in $(A/\mathfrak{m}_x)\langle X_1, \ldots, X_n\rangle$ for all $x \in \mathrm{Sp}(A)$. Thus by [BGR, Proposition 5.1.3/1][4] it is equivalent to say that $\pi_x(f)$ is X_n-distinguished of degree $s = 0$. Indeed, if $f_0 = f(0)$ is a unit in $(A/\mathfrak{m}_x)\langle X_1, \ldots, X_{n-1}\rangle$ and $|f_0| = |f|$ with $|f_0| > |f_\nu|$ for all $\nu > 0$, we may scale to 1 which means f is a unit if and only if it is constant in $\widetilde{A/\mathfrak{m}_x}[X_1, \ldots, X_{n-1}]$ which is true if and only if $\tilde{f}$ has degree 0 which is true if and only if it is X_n-distinguished of degree 0. Conversely, if f is X_n-distinguished of degree $s = 0$ for all x, then all of the $\pi_x(f)$ are units. Hence, f cannot have any 0 on $\mathrm{Sp}(A) \times \mathbb{B}^n = \mathrm{Sp}(A\langle X_1, \ldots, X_n\rangle)$, hence f must be a unit. □

Now we are ready to do [BGR]'s proof of Gerritzen-Grauert:

Proof of Gerritzen-Grauert. Let $\varphi : Y \longrightarrow X$ be a locally closed immersion for $Y = \mathrm{Sp}(B)$ and $X = \mathrm{Sp}(A)$ and take $\lambda : A \longrightarrow B$ to be the associated map on algebras. Extend to a surjection $\Lambda : A\langle X_1, \ldots, X_n\rangle \longrightarrow B$. This can always be done because affinoid generating systems always exist: you can take n such that B is isomorphic to a quotient of the Tate Algebra T_n. Use Proposition 6 to find $f \in \ker(\Lambda)$: an X_n-distinguished series of degree $\leq s$ on $\mathrm{Sp}(A)$.

Let X' be the rational domain of X corresponding to the points at which f is X_n-distinguished of degree s (Proposition 9). Let A' be the corresponding algebra i.e., the algebra such that $\mathrm{Sp}(A') = X' \subset X$. We have the corresponding map on algebras $A \longrightarrow A'$. By tensoring Λ with A' over A we get a surjection[5]

$$\Lambda' : A'\langle X_1, \ldots, X_n\rangle \longrightarrow B\widehat{\otimes}_A A'$$

Since f is X_n-distinguished of degree s on $\mathrm{Sp}(A')$, we can use Proposition 10 to get a finite map

$$\iota : A'\langle X_1, \ldots, X_{n-1}\rangle \longrightarrow A'\langle X_1, \ldots, X_n\rangle/(f)$$

Thus we get a finite map $\Lambda'' : A'\langle X_1, \ldots, X_{n-1}\rangle \longrightarrow B\widehat{\otimes}_A A'$ extending $\lambda' : A' \longrightarrow B\widehat{\otimes}_A A'$ which is obtained by tensoring over λ with A'. Since λ corresponds to a locally closed immersion, it is an epimorphism because locally closed immersions are monomorphisms in the category of affinoid varieties (we will prove this in the section on Temkin's proof of Gerritzen-Grauert). Completed tensor product preserves epimorphisms and so λ' is also a locally closed immersion hence Λ'' is as well. One could also see this by realizing that a surjection on stalks will be retained by the completed tensor product. Thus by [BGR, Proposition 7.3.3/8 (ii)] Λ'' is a closed immersion because it is finite and locally closed and thus we have $\langle A' : B\widehat{\otimes}_A A'\rangle \leq n - 1$ and we may apply the induction hypothesis to $\varphi' : \varphi^{-1}(X') \longrightarrow X'$.

Hence there exists a covering $\bigcup_{i=1}^r X^i = X'$ with X^i rational in X' and $\varphi^{-1}(X^i) \longrightarrow X^i$ a Runge immersion for all i. The X^i are rational in X by the transitivity of rational domains (X' is rational in X). Thus, we may take functions on X describing the X^i. Define rational domains X^i_ε for $\varepsilon \in \sqrt{|k^*|}$ such that $X^i = X^i_1$. Then Lemma 11 tells us that there exists an $\varepsilon > 1$ such that the $\varphi^{-1}(X^i_\varepsilon) \longrightarrow X^i_\varepsilon$ are still Runge immersions. By Lemma 12, we may find rational domains $V_1, \ldots, V_\ell \subset X$ not meeting $X' = \bigcup X^i = \bigcup X^i_1$ such that $X = \bigcup X^i_\varepsilon \cup \bigcup V_j$.

Let $V_j = \mathrm{Sp}(A_j)$. Because we already know the theorem holds for the X^i_ε's, it is enough to show that the theorem holds for the locally closed immersions $\varphi^{-1}(V_j) \longrightarrow V_j$ corresponding to $\lambda_j : A_j \longrightarrow B\widehat{\otimes}_A A_j$ induced by λ (because they cover X along with the X^i_ε's). Tensoring with A_j we get epimorphisms Λ_j :

[4] If $g \in T_n$ is a unit with norm 1 if and only if $|g(0) - g| < 1$. So g is a unit if and only if $\tilde{g}$ is a unit i.e., a constant in $\tilde{T}_n = k[X_1, \ldots, X_n]$. This is the same as being X_n-distinguished of degree 0 and of norm 1. We may norm to 1.

[5] Surjections are epimoprhims, but not all epimorphisms are surjections because then locally closed would imply closed. I make this note because BGR says "epimorphism" in places where it should really say "surjection."

$A_j\langle X_1,\ldots,X_n\rangle \longrightarrow B\widehat{\otimes}_A A_j$. Thus $\langle A_j : B\widehat{\otimes}_A A_j\rangle \le n$ for all j. Since none of the V_j meet X', Proposition 13 implies that the image f_j of f in $A_j\langle X_1,\ldots,X_n\rangle$ ($f_j = f|_{\mathrm{Sp}(A_j)}$) is X_n-distinguished of degree $\le s-1$ on $V_j = \mathrm{Sp}(A_j)$ for all j (because f was taken to be distinguished of degree $\le s$ and the f_j are outside of the points at which f is distinguished of degree s). $f \in \ker(\Lambda)$ so $f_j \in \ker(\Lambda_j)$ thus we may proceed with the maps $\varphi^{-1}(V_j) \longrightarrow V_j$ using the same method as before with φ. We see that, since f is X_n-distinguished of degree $\le s-1$ on the V_j's, the theorem would be true by an induction argument on s because if we can show it to be true for $s-1$ then it becomes true for s. So it suffices to show that the theorem holds for $s=0$. If $s=0$, all f_j's are units by Proposition 14 and so $B\widehat{\otimes}_A A_j = 0$ because, if a unit is in the kernel of the surjection Λ_j, the codomain will be isomorphic to 0. Thus $\varphi^{-1}(V_j) = \varnothing$ and so the $\varphi^{-1}(V_j) \longrightarrow V_j$ are Runge immersions and we're done. $\square$

Gerritzen-Grauert à la Temkin

Temkin's proof makes use of basic topological properties of the full spectrum $\mathcal{M}(A)$ of an affinoid algebra instead of just the max spectrum $\mathrm{Max}(A) = \mathrm{Sp}(A)$. We will define this soon. Finiteness plays a big role in this proof, allowing us to get an isomorphism between the regular tensor product and the completed tensor product. This, along with the fact that φ is a locally closed immersion (hence a monomorphism as we will see), gives us a closed immersion. So we are effectively using the same fact as [BGR, Proposition 7.3.3/8 (ii)]. Moreover Lemma 12 comes up as a result of the compactness of the spectrum. We begin by proving a categorical result that characterizes monomorphisms.

Lemma 15. *For a monomorphism $f : X \longrightarrow Y$ in any category with fiber products, the diagonal map $\triangle : X \longrightarrow X \times_Y X$ is an isomorphism*

Proof. Consider $Z \rightrightarrows X \longrightarrow Y$ where the two arrows from Z to X commute with f. They must factor through $X \times_Y X$ by the universal property of the fiber product. If f is a monomorphism then the two maps from Z to X must be equal which means that Z factors through the diagonal in $X \times_Y X$. The result follows. $\square$

Equivalently the codiagonal map is an epimorphism in the dual category. Now we prove a fact about monomorphisms and locally closed immersions that was mentioned in footnote 6. This will prove quite useful. Let $Y_0 = \mathrm{Max}(B)$ and $X_0 = \mathrm{Max}(A)$. Denote by $\kappa(x)$ the residue field at the point $\mathfrak{m}_x \in \mathrm{Max}(A)$ i.e., the field $A/\mathfrak{m}_x$. We will reinterpret these maximal ideals as multiplicative semi-norms on A in a little bit.

Proposition 16. *The map of affinoid varieties $\varphi : Y_0 \longrightarrow X_0$ is a locally closed immersion if and only if it is a monomorphism in the category of affinoid varieties.*

Proof. ($\Rightarrow$): For all $y \in Y_0$, the induced map $\mathcal{O}_{X_0,\varphi(y)} \longrightarrow \mathcal{O}_{Y_0,y}$ is a surjection. The canonical homomorphism from B to $\prod_{y\in Y_0}\mathcal{O}_{Y_0,y}$ is a injection because $B \hookrightarrow B_{\mathfrak{m}_y} \hookrightarrow \mathcal{O}_{Y_0,y}$ for all y (the Krull Intersection Theorem implies that the maximal-adic topology is Hausdorff). Thus φ is a monomorphism.

($\Leftarrow$): The codiagonal map $B\widehat{\otimes}_A B \longrightarrow B$ is an isomorphism. Therefore, for all $x \in X_0$ with $\varphi(y) = x$ for some $y \in Y_0$ we get an isomorphism $B_x\widehat{\otimes}_{\kappa(x)}B_x \longrightarrow B_x$ where $B_x = B \otimes_A \kappa(x)$. This isomorphism is acquired by successive base changes. By [Gru, 3.2.1(4)], the canonical map from a tensor product of Banach spaces over a non-archimedean field to the completed tensor product is injective. Thus we have a faithfully flat map $(Y_0)_x \longrightarrow \mathrm{Max}(\kappa(x))$ where $(Y_0)_x = \mathrm{Max}(B_x)$. So we have a fiber square

$$\begin{tikzcd}
(Y_0)_x \times_{\kappa(x)} (Y_0)_x \arrow[r, "\sim"] \arrow[d, "\sim"'] & (Y_0)_x \arrow[d, "\text{Faithfully Flat}"] \\
(Y_0)_x \arrow[r] & \mathrm{Max}(\kappa(x))
\end{tikzcd}$$

So the map $(Y_0)_x \longrightarrow \mathrm{Max}(\kappa(x))$ is an isomorphism so we get an isomorphism $\kappa(x) \cong B_x$. This implies that φ is injective and [BGR, Lemma 7.2.5/2 (b)] (an epimorphism of residue fields gives an epimorphism on some affinoid subdomain) implies that φ is a locally closed immersion. $\square$

Definition 17. Let A be a commutative Banach ring. The *spectrum* of A denoted $\mathcal{M}(A)$ is the collection of all non-zero bounded multiplicative seminorms $|\cdot| : A \longrightarrow \mathbb{R}_+$. We give it the topology for which a neighborhood base of opens around $|\cdot|_x \in \mathcal{M}(A)$ is given, for $f = f_1, \dots, f_n \in A$ and $\varepsilon = \varepsilon_1, \dots, \varepsilon_n \in \mathbb{R}$ by the sets

$$U_{f,\varepsilon} = \{|\cdot| \in \mathcal{M}(A) : ||f_i|_x - |f_i|| < \varepsilon_i, \text{ for all } 1 \le i \le n\}$$

Which is equivalently the weakest topology with respect to which the map $\mathcal{M}(A) \longrightarrow \mathbb{R}$ given by $|\cdot| \longmapsto |f|$ is continuous.

Fact: the spectrum is compact. For each $x \in \mathcal{M}(A) = X$ we get a norm on $A/\ker(|\cdot|_x)$. We may extend this norm to the fraction field $\mathcal{H}(x)$ in the obvious way. We then get a character $\chi_x : A \longrightarrow \mathcal{H}(x)$ and we define $f(x) := \chi_x(f)$. $X_0 = \text{Max}(A)$ is going to be equal to $x \in X$ such that $[\mathcal{H}(x) : k] < \infty$, the finite k-algebraic extensions we work with for $\text{Sp}(A) = \text{Max}(A)$ and for such points $\kappa(x) = \mathcal{H}(x)$.

Notice that every point of X has a fundamental system of closed neighborhoods consisting of Laurent domains $\{x \in \mathcal{M}(A) : |f_i(x)| \le r_i, |g_j(x)| \ge s_j, \text{ for all } i,j\}$ (non-strict bounded below is going to be the complement of strict bounded above). Notice that now with the full spectrum, we may take $r_i, g_j \in \mathbb{R}$ because the seminorms take values in $\mathbb{R}$ unlike before where we were restricted to values in $\sqrt{|k^*|}$. Notice that $|g_j(x)| \ge s_j$ is a closed condition. We will use this when we show that the reduction map (to be defined) is anti-continuous.

Proposition 18. *Let $\varphi : Y \longrightarrow X$ be a monomorphism of affinoid varieties (where we now understand affinoid varieties as the opposite category of commutative Banach rings via the spectrum). Then for all $y \in Y$ such that $x = \varphi(y)$ we have that $\varphi^{-1}(x) = \{y\}$ and $\mathcal{H}(x) \cong \mathcal{H}(y)$*

Proof. Let $X = \mathcal{M}(A)$ and $Y = \mathcal{M}(B)$. It is enough to enough that $\mathcal{H}(x) \cong B\widehat{\otimes}_A\mathcal{H}(x)$. φ is a monomorphism so $B\widehat{\otimes}_A B \cong B$. So, by tracing through a number of base changes, we get an isomorphism $B_{\mathcal{H}(x)}\widehat{\otimes}_{\mathcal{H}(x)}B_{\mathcal{H}(x)} \xrightarrow{\sim} B_{\mathcal{H}(x)} = B\widehat{\otimes}_A\mathcal{H}(x)$ which is the base change along $\chi_x : A \longrightarrow \mathcal{H}(x)$. Using the same fact from Proposition 14, we have an injection $B_{\mathcal{H}(x)} \otimes B_{\mathcal{H}(x)} \hookrightarrow B_{\mathcal{H}(x)}\widehat{\otimes}_{\mathcal{H}(x)}B_{\mathcal{H}(x)} \cong B_{\mathcal{H}(x)}$. Therefore by a similar argument as was used in Proposition 14, we have an isomorphism $\mathcal{H}(x) \cong B\widehat{\otimes}_A\mathcal{H}(x)$. □

Next we will prove a key finiteness lemma that will be used in the proof of Gerritzen-Grauert. Beforehand we note that [BGR, Theorem 6.3.5/1] is very important for our considerations. It says that a map between affinoid algebras is finite if and only if the corresponding map on reductions is finite.

The character $\chi_x : A \longrightarrow \mathcal{H}(x)$ gives rise to a map $\widetilde{\chi}_x : \widetilde{A} \longrightarrow \widetilde{\mathcal{H}(x)}$ in the obvious way. We then get a map $x \longmapsto \ker(\widetilde{\chi}_x)$ which induces a map $\pi : X \longrightarrow \widetilde{X} = \text{Spec}(\widetilde{A})$ (note that $\ker(|\cdot|_x)$ is prime because if $ab \in \ker(|\cdot|_x)$, $|ab|_x = |a|_x|b|_x = 0$). This "reduction map" is anticontinuous (open sets pull back to closed sets). To see this, we only need to check on the distinguished opens $D(f)$ of $\text{Spec}(\widetilde{A})$:

$$\begin{aligned}\ker(\widetilde{\chi}_x) \in D(\tilde{f}) &\Leftrightarrow \tilde{f} \notin \ker(\widetilde{\chi}_x)\\ &\Leftrightarrow \widetilde{\chi}_x(\tilde{f}) \neq 0\\ &\Leftrightarrow \chi_x(f) \notin A^{\circ\circ}\\ &\Leftrightarrow |f(x)| \ge 1\end{aligned}$$

thus the inverse image of any open set is closed (as noted before $|f(x)| \ge 1$ is a closed condition). Similarly if we consider complements of the distinguished opens (standard vanishing loci), we have

$$\begin{aligned}\ker(\widetilde{\chi}_x) \notin D(\tilde{f}) &\Leftrightarrow \tilde{f} \in \ker(\widetilde{\chi}_x)\\ &\Leftrightarrow \widetilde{\chi}_x(\tilde{f}) = 0\\ &\Leftrightarrow \chi_x(f) \in A^{\circ\circ}\\ &\Leftrightarrow |f(x)| < 1\end{aligned}$$

which is an open condition, as is evident by the neighborhood base of opens given in the definition of the spectrum.

Lemma 19. *Let $X' = \mathcal{M}(A')$ be a Laurent subdomain of $X = \mathcal{M}(A)$ given by $X' = X(f, g^{-1})$ for $f = f_1, \dots, f_n$, $g = g_1, \dots, g_m \in A$. Then:*

(i). $\widetilde{A'}$ is finite over the subalgebra generated by $\tilde{f}_1 \dots, \tilde{f}_n, \tilde{g}_1^{-1}, \dots, \tilde{g}_m^{-1}$ over the image of $\widetilde{A}$.

(ii). Let X' be a neighborhood of $x \in X$. Then $\widetilde{\chi}_x(\widetilde{A'})$ is finite over $\chi_x(A)$ i.e., it is finitely-generated as a $\widetilde{\chi}_x(\widetilde{A})$-module.

Proof. (i) Let $T = T_1, \dots, T_n$ and $S = S_1, \dots, S_m$. Consider the surjection $A\langle T, S\rangle \longrightarrow A'$ mapping $T_i \longmapsto f_i$ and $S_j \longmapsto g_j^{-1}$. Then by [BGR, Theorem 6.3.5/1], the corresponding map on reductions $\widetilde{A}[T, S] \longrightarrow \widetilde{A'}$ is finite.

(ii) We may replace X' by a smaller Laurent neighborhood of x of the same form but with different f_i's and g_j's such that $|f_i(x)| < 1$ and $|g_j(x)| > 1$ for all i, j. Then the fact follows from (i). $\square$

Now we will prove Gerritzen-Grauert (for strictly k-affinoid algebras) as Temkin does. First, we will show that another condition is sufficient to prove Gerritzen-Grauert. Namely

Proposition 20. *If for all $y \in Y$ there exists a rational U containing $x = \varphi(y)$ such that $V = \varphi^{-1}(U)$ is a neighborhood of y in Y and the induced map $V \longrightarrow U$ is a closed immersion, then Gerritzen-Grauert holds for the locally closed immersion $\varphi : Y \longrightarrow X$.*

Proof. Let $U = X\left(\frac{f}{g}\right)$ where $f = f_1, \dots, f_n$ and $f, g \in A$ generate the unit ideal. g is invertible on U so we may find a sufficiently small rational neighborhood W of x in X such that g is invertible on W and $\varphi^{-1}(W) \subset \varphi^{-1}(U)$. Then $U \cap W$ is a Weierstraß subdomain of W and therefore $\varphi^{-1}(W) \longrightarrow W$ is a Runge immersion (a closed immersion of a Weierstraß domain is Runge). Using this method along with the fact that Y is compact, we can show that there exist rational subdomains $X_1, \dots, X_m$ of X such that each $\varphi^{-1}(X_i) \longrightarrow X_i$ is Runge. Furthermore $\bigcup_{i=1}^m X_i$ contains an open neighborhood $\mathcal{U}$ of $\varphi(Y)$. X is compact so there exist rational domains $X_{m+1}, \dots, X_n$ not meeting $\varphi(Y)$ such that $X \setminus \mathcal{U} \subset \bigcup_{m+1}^n X_i$ (this is just like Lemma 12). Then $\bigcup_{i=1}^n X_i = X$ and $\varphi^{-1}(X_i) \longrightarrow X_i$ are Runge. $\square$

Thus the question stands, how do you construct such a U? We will shrink X to a rational domain X' containing x such that $Y' = \varphi^{-1}(X')$ is a neighborhood of y in Y. We note that the base change $Y' \longrightarrow X'$ along φ is still a monomorphism (limits preserve monomorphisms). Temkin proceeds by first shrinking X multiple times until $\widetilde{\chi}_y(\widetilde{B})$ is finite over $\widetilde{\chi}_x(\widetilde{A})$:

By shrinking X, one may assume $\widetilde{\chi}_y(\widetilde{B}$ is finite over $\widetilde{\chi}_x(\widetilde{A})$. Take $h_1, \dots, h_n \in B^\circ$ such that $\widetilde{\chi}_y(\widetilde{B})$ is finite over the $\widetilde{\chi}_x(\widetilde{A})$-subalgebra generated by $\tilde{h}_1, \dots, \tilde{h}_n$. Proposition 18 implies that $\mathcal{H}(x) \cong \mathcal{H}(y)$ and thus there exist $f_1, \dots, f_n, g \in A$ with $|g(x)| = 1$ and $|\frac{f_i}{g} - h_i| < 1$ for all $1 \le i \le n$. We seek to conclude that $\tilde{f}_i = \tilde{h}_i$. Let $X' = X\left(\frac{a}{g}\right)$ for some $a \in k^*$ with $|a| < 1$. Let $Y' = \varphi^{-1}(X')$. By lemma 19 (ii), $\widetilde{\chi}_y(\widetilde{B})$ is finite over $\widetilde{\chi}_y(\widetilde{B})$ and therefore we can replace X and Y by X' and Y' respectively and assume that g is invertible in A. Replacing f_i by $\frac{f_i}{g}$ we may assume $g = 1$. Furthermore, if $X' = X(f_1, \dots, f_n)$ then $Y' = \varphi^{-1}(X')$ is a neighborhood of y and therefore we can replace X and Y by X' and Y' and assume that $f_i \in A^\circ$ for all i. But then $\tilde{h}_i = \tilde{f}_i$ and thus $\widetilde{\chi}_y(\widetilde{B})$ is finite over $\widetilde{\chi}_x(\widetilde{A})$. $\square$

To show that we actually get a closed immersion, we need to show the result above extends to x' and y' for all $y' \in Y$ such that $\varphi(y') = x'$:

Shrinking X, one may assume $\widetilde{\chi}_{y'}(\widetilde{B})$ is finite over $\widetilde{\chi}_{x'}(\widetilde{A})$ for all $y' \in Y$ such that $x' = \varphi(y')$. Denote by $\mathcal{Y}$ the Zariski-closure of $\pi(y)$ in $\widetilde{Y}$ where $\pi : Y \longrightarrow \widetilde{Y}$ is the reduction map. Then for all $y' \in Y$ with $\pi(y') \in \mathcal{Y}$ we have $\ker(\widetilde{\chi}_y) \subset \ker(\widetilde{\chi}_{y'})$. Therefore $\widetilde{\chi}_{y'}(\widetilde{B})$ is finite over $\widetilde{\chi}_{x'}(\widetilde{A})$. Since $\pi^{-1}(\mathcal{Y})$ is open in Y, there exists a Laurent neighborhood $X' = X(f, g^{-1})$ of x with $Y' = \varphi^{-1}(X') \subset \pi^{-1}(\mathcal{Y})$. Let $X' = \mathcal{M}(A')$ and $Y' = \mathcal{M}(B')$. Then by Lemma 19 (ii), $\widetilde{\chi}_{y'}(\widetilde{B'})$ and $\widetilde{\chi}_{x'}(\widetilde{A'})$ are finite over the subalgebras generated by $\tilde{f}_1, \dots, \tilde{f}_n, \tilde{g}_1^{-1}, \dots, \tilde{g}_m^{-1}$ over $\widetilde{\chi}_{y'}(\widetilde{B})$ and $\widetilde{\chi}_{x'}(\widetilde{A})$. Thus for all $y' \in Y'$, $\widetilde{\chi}_{y'}(\widetilde{B'})$ is finite over $\widetilde{\chi}_{x'}(\widetilde{A'})$. $\square$

We finish by showing that we actually have a closed immersion, thus Gerritzen-Grauert follows from Proposition 20:

In the situation above, φ is a closed immersion. For all minimal primes $\mathfrak{p}$ of $\widetilde{B}$, there exists a $y' \in Y$ with $\ker(\widetilde{\chi}_{y'}) = \mathfrak{p}$. In our case, $\widetilde{B}/\mathfrak{p}$ is finite over $\widetilde{A}$. If $\mathfrak{p}_1, \dots, \mathfrak{p}_n$ is the set of minimal primes of $\widetilde{B}$ (note that the reductions of affinoid algebras are noetherian because they are just $\tilde{k}$-polynomial algebras in finitely many indeterminates), the homomorphism $\widetilde{B} \longrightarrow C = \bigoplus_{i=1}^{n} \widetilde{B}/\mathfrak{p}_{\mathfrak{i}}$ is injective because $\widetilde{B}$ has no nilpotents. Since $\widetilde{A}$ is noetherian and C is is a finite $\widetilde{A}$-module, its submodule $\widetilde{B}$ is also finite. [BGR, Proposition 6.3.5/1] (finite morphism iff reduction map finite) implies that B is finite over A. In particular, $B \otimes_A B \cong B \widehat{\otimes}_A B$. φ is a monomorphism and so $B \otimes_A B \cong B$ i.e., the finite morphism $\mathrm{Spec}(B) \longrightarrow \mathrm{Spec}(A)$ is a monomorphism. Thus we need to show that for a ring A and a finite A-algebra B such that $B \otimes_A B \cong B$, the canonical map $A \longrightarrow B$ is a surjection. We may localize A at a prime and assume it to be local (localization is exact). Nakayama's lemma implies we may replace A with its residue field. In this case the required fact is trivial i.e., for a field k, the map $k \longrightarrow B$ must be a surjection in the diagram

$$\begin{tikzcd} B \otimes_k B \cong B & B \arrow[l] \\ B \arrow[u] & k \arrow[l] \arrow[u] \end{tikzcd}$$

This last part is actually an alternate proof of [BGR, Proposition 7.3.3/8 (ii)]. $\square$

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