Project Euler Theorems

Zach Taylor

April 18, 2019

1. $\sum_{k=1}^{n} k = \frac{n*(n+1)}{2}$ for all $n \geq 1$.

Proof. The formula clearly holds when $n = 1$ and $n = 2$. Now suppose the formula holds when $n = m \geq 2$. Then

$$\sum_{k=1}^{m+1} k = \sum_{k=1}^{m} k + m + 1 = \frac{m*(m+1)}{2} + m + 1 = \frac{(m+1)*(m+2)}{2}$$

Thus the formula holds for $n = m + 1$, so by the principle of induction it holds for all $n \geq 1$. $\square$

2. Let $a, b, c \in \mathbb{Z}$. Then $b \mid a$ and $c \mid a$ if and only if $\mathrm{lcm}(b, c) \mid a$.

Proof. Set $n = \mathrm{lcm}(b, c)$, and suppose $b \mid a$ and $c \mid a$. By the division algorithm, $a = pn + r$ for some $p, r \in \mathbb{Z}$, where $0 \leq r < n$. Thus, since $b \mid a$ and $b \mid n$, we have $b \mid r$. By the same reasoning, we also have $c \mid r$. Since r is a common multiple of b and c and $r < n$, r must be 0. Thus $a = pn$, that is, $n \mid a$. Conversely, let $k, j \in \mathbb{Z}$ such that $n = kb$ and $n = jc$. Then if $n \mid a$, we have, for some $m \in \mathbb{Z}$, $a = mn = (mk)b = (mj)c$, and thus $c \mid a$ and $b \mid a$. $\square$

3. Let $a, b \in \mathbb{Z}$. The following hold:

- If a is odd and b is even, $a + b$ is odd.
- If a and b are odd, $a + b$ is even.

Proof. Note that if $a, b \in \mathbb{Z}$ where a is odd and b is even, then $a = 2k + 1$ and $b = 2l$ for $k, l \in \mathbb{Z}$, so $a + b = 2(k + l) + 1$ is odd. On the other hand, if both a and b are odd, then we can write $b = 2l + 1$ and $a + b = 2(k + l + 1)$ is even. $\square$

4. Let $\{F_n\}_0^\infty$ be the sequence of Fibonacci numbers. If F_k is even, then the next even Fibonacci number is F_{k+3}.

Proof. The first 5 Fibonacci numbers are 1, 2, 3, 5, and 8. Clearly the result holds here, since $F_2 = 2$ is even and $F_{2+3} = F_5 = 8$ is also even. Now suppose that the pattern even, odd, odd, even holds for F_k through F_{k+3}. By theorem 3, $F_{k+4} = F_{k+3} + F_{k+2}$ is odd, which implies that $F_{k+5} = F_{k+4} + F_{k+3}$ is odd, which implies that $F_{k+6} = F_{k+5} + F_{k+4}$ is even. Thus the pattern holds for F_{k+3} through F_{k+6}, so by the priniciple of induction the pattern holds for the entire sequence. □

5. Let $\{E_n\}_1^\infty$ be the sequence of even Fibonacci numbers. Then $E_n = 4E_{n-1} + E_{n-2}$.

Proof. If $\{F_n\}_0^\infty$ is the Fibonacci sequence,

$$\begin{aligned} F_n &= F_{n-1} + F_{n-2} \\ &= F_{n-2} + F_{n-3} + F_{n-3} + F_{n-4} \\ &= 2F_{n-3} + F_{n-2} + F_{n-4} \\ &= 2F_{n-3} + F_{n-3} + F_{n-4} + F_{n-5} + F_{n-6} \\ &= 3F_{n-3} + F_{n-4} + F_{n-5} + F_{n-6} \\ &= 4F_{n-3} + F_{n-6} \end{aligned}$$

Thus, by theorem 4, $E_n = 4E_{n-1} + E_{n-2}$. □