Chapter 7
Pushdown Automata

Vocabulary

- Backus Naur Form (BNF)
- λ–productions
- λ–free languages
- Unit production
- s-grammars
- LL grammars

Creating a CFG from a NPDA

For simplicity, assume these requirements must be met.

1. Single final state entered if stack is empty
2. All transitions must be in the form $\delta(q_i, a, A) = \{c_1, c_2, \ldots, c_n\}$

   where

   $$c_i = (q_j, \lambda) \text{ or } c_i = (q_j, BC)$$

Each move should increase or decrease the size of the stack by a single symbol.

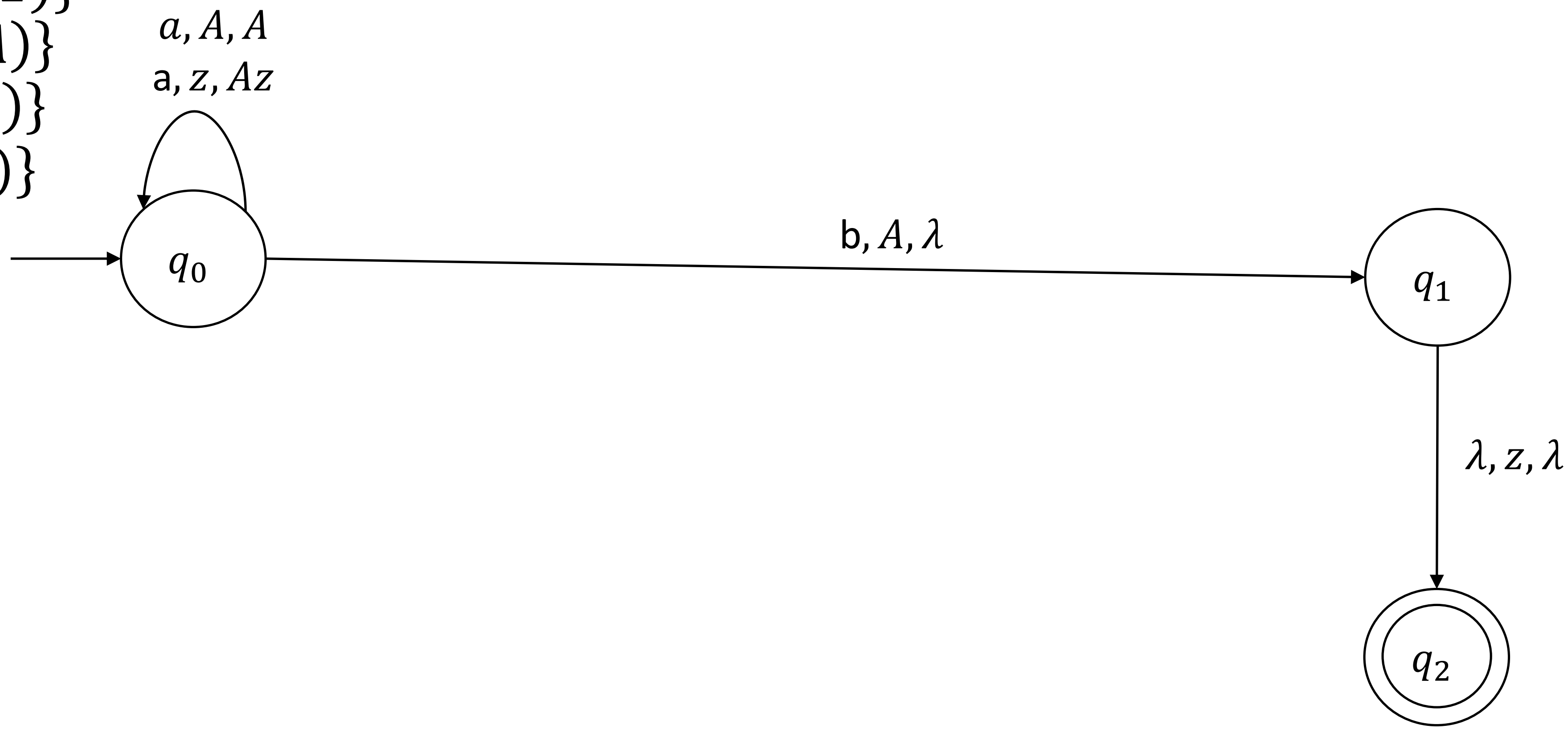
Example 5: Consider the npda with transitions

$$\delta(q_0, a, z) = \{(q_0, Az)\}$$
$$\delta(q_0, a, A) = \{(q_0, A)\}$$
$$\delta(q_0, b, A) = \{(q_1, \lambda)\}$$
$$\delta(q_1, \lambda, z) = \{(q_2, \lambda)\}$$

condition 1: Single final state entered if stack is empty (q2)

condition 2: $c_i = (q_j, \lambda)$ or $c_i = (q_j, BC)$

$$\delta(q_0, a, z) = \{(q_0, Az)\}$$
$$\delta(q_0, a, A) = \{(q_0, A)\}$$
$$\delta(q_0, b, A) = \{(q_1, \lambda)\}$$
$$\delta(q_1, \lambda, z) = \{(q_2, \lambda)\}$$



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condition 1: Single final state entered if stack is empty (q2)

condition 2: $c_i = (q_j, \lambda)$ or $c_i = (q_j, BC)$

Example 5: Consider the npda with transitions

$\delta(q_0, a, z) = \{(q_0, Az)\}$
$\delta(q_0, a, A) = \{(q_3, \lambda)\}$
$\delta(q_3, \lambda, z) = \{(q_0, Az)\}$
$\delta(q_0, b, A) = \{(q_1, \lambda)\}$
$\delta(q_1, \lambda, z) = \{(q_2, \lambda)\}$

condition 1: Single final state entered if stack is empty (q2)

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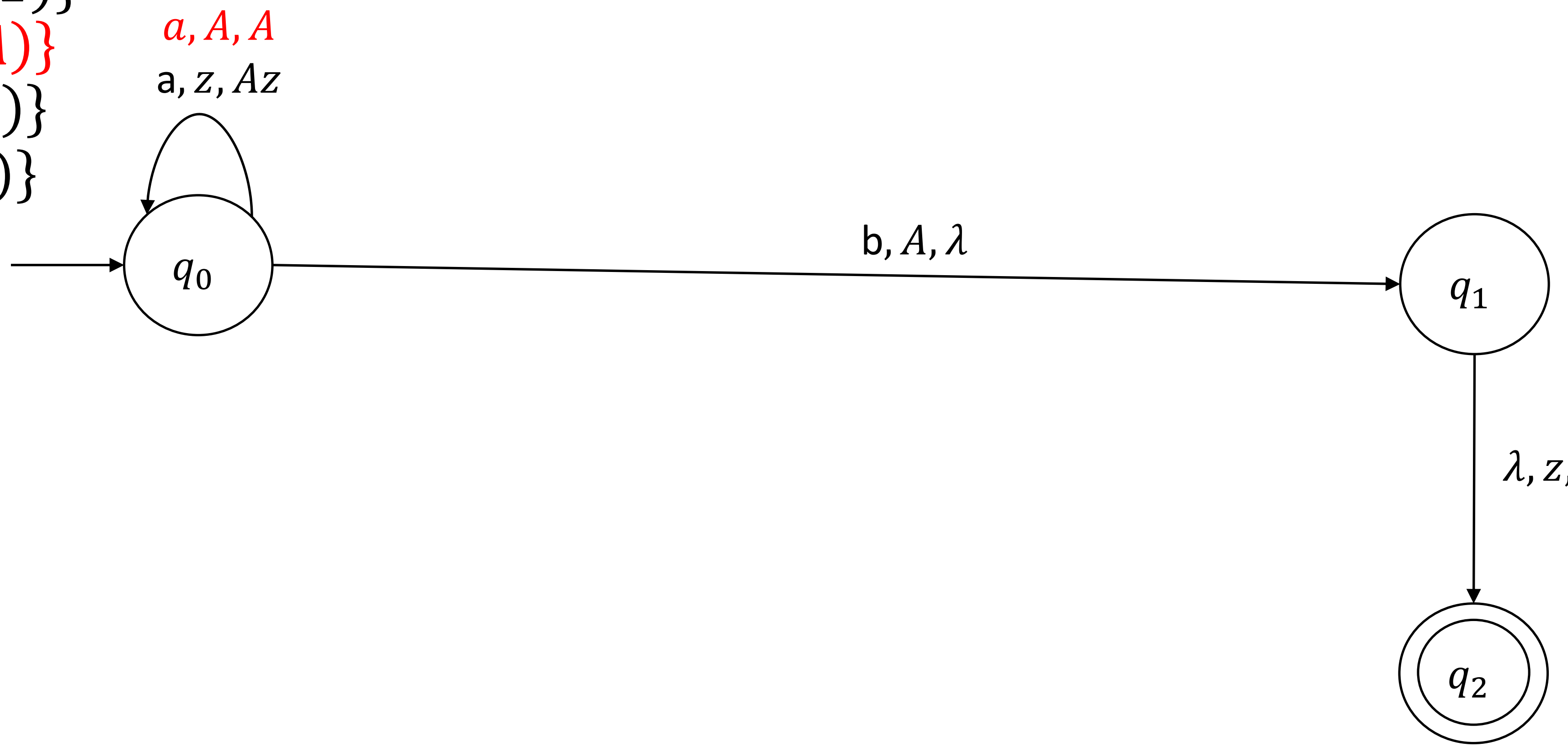

$$\delta(q_0, a, z) = \{(q_0, Az)\}$$
$$\delta(q_0, a, A) = \{(q_3, \lambda)\}$$
$$\delta(q_3, \lambda, z) = \{(q_0, Az)\}$$
$$\delta(q_0, b, A) = \{(q_1, \lambda)\}$$
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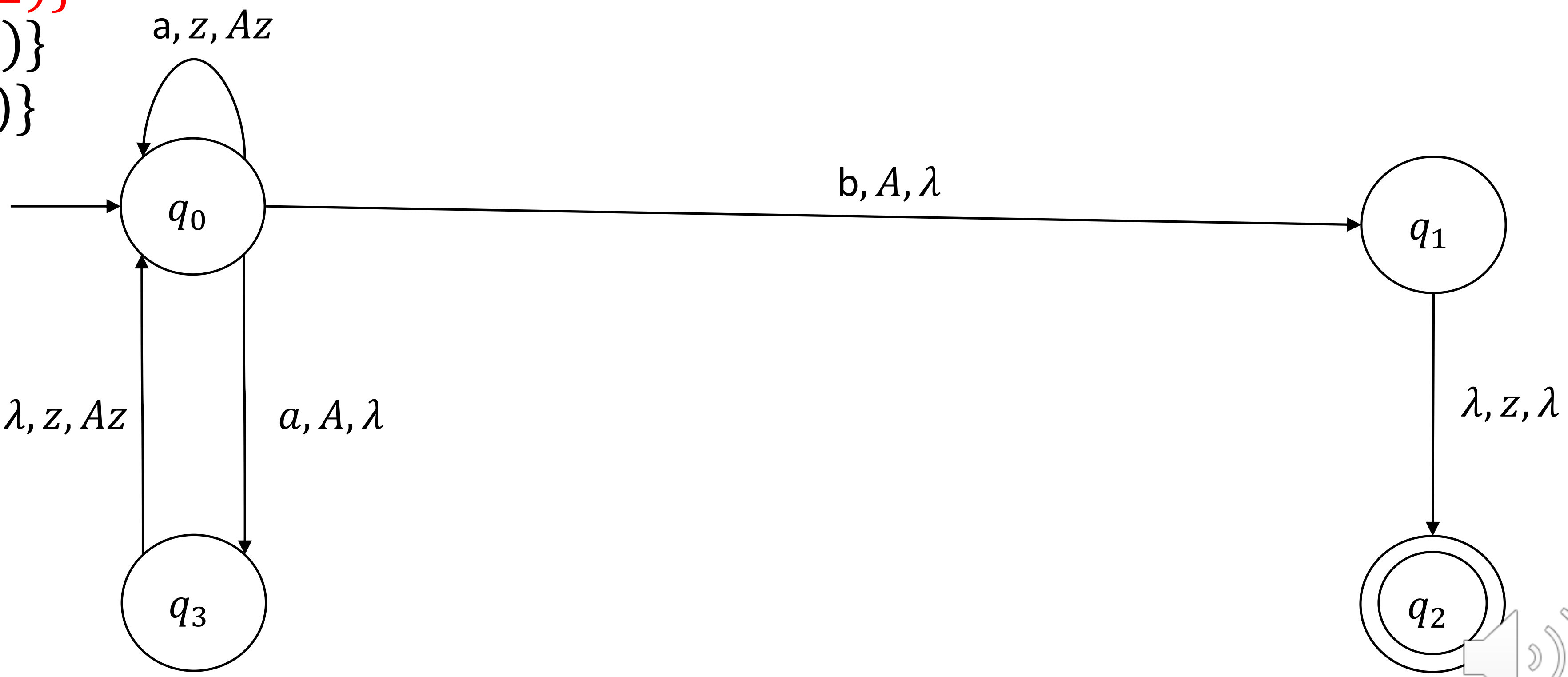

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$\delta(q_0, a, z) = \{(q_0, Az)\}$
$\delta(q_0, a, A) = \{(q_3, \lambda)\}$
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$$\delta(q_0, a, z) = \{(q_0, Az)\}$$
$$\delta(q_0, a, A) = \{(q_3, \lambda)\}$$
$$\delta(q_3, \lambda, z) = \{(q_0, Az)\}$$
$$\delta(q_0, b, A) = \{(q_1, \lambda)\}$$
$$\delta(q_1, \lambda, z) = \{(q_2, \lambda)\}$$

$$c_i = (q_j, \lambda)$$

Example 5: Consider the npda with transitions

$\delta($[illegible]$, a, z)$ [illegible] $\{$[illegible]$, A$[illegible]

$\delta(q_0, a, A) = \{(q_3, \lambda)\}$

$\delta($[illegible]$, \lambda, z)$ [illegible] $\{$[illegible]$, A$[illegible]

$\delta($[illegible]$, b, A$[illegible]$) = \{(q_1, \lambda)\}$

$\delta(q_1, \lambda, z) = \{(q_2, \lambda)\}$

$c_i = (q_j, \lambda)$

$(q_0 A q_3) \rightarrow a$

$(q_0 A q_1) \rightarrow b$

$(q_1 z q_2) \rightarrow \lambda$

Example 5: Consider the npda with transitions

$\delta(q_0, a, z) = \{(q_0, Az)\}$
$\delta(q_0, a, A) = \{(q_3, \lambda)\}$
$\delta(q_3, \lambda, z) = \{(q_0, Az)\}$
$\delta(q_0, b, A) = \{(q_1, \lambda)\}$
$\delta(q_1, \lambda, z) = \{(q_2, \lambda)\}$

$(q_0 A q_3) \rightarrow$a
$(q_0 A q_1) \rightarrow$ b
$(q_1 z q_2) \rightarrow \lambda$

$c_i = (q_j, \lambda)$

Example 5: Consider the npda with transitions

$\delta(q_0, a, z) = \{(q_0, Az)\}$
$\delta(q_0, a, A) = \{(q_3, \lambda)\}$
$\delta(q_3, \lambda, z) = \{(q_0, Az)\}$
$\delta(q_0, b, A) = \{(q_1, \lambda)\}$
$\delta(q_1, \lambda, z) = \{(q_2, \lambda)\}$

$(q_0 A q_3) \rightarrow$ a
$(q_0 A q_1) \rightarrow$ b
$(q_1 z q_2) \rightarrow \lambda$

$c_i = (q_j, BC)$

Example 5: Consider the npda with transitions

$\delta(q_0, a, \mathbf{z}) = \{(q_0, Az)\}$

$(q_0\mathbf{z}q_0) \rightarrow \mathrm{a}(q_0Aq_0)(q_0zq_0) \mid \mathrm{a}(q_0Aq_1)(q_1zq_0) \mid$
$\mathrm{a}(q_0Aq_2)(q_2zq_0) \mid \mathrm{a}(q_0Aq_3)(q_3zq_0)$

$(q_0\mathbf{z}q_1) \rightarrow \mathrm{a}(q_0Aq_0)(q_0zq_1) \mid \mathrm{a}(q_0Aq_1)(q_1zq_1) \mid$
$\mathrm{a}(q_0Aq_2)(q_2zq_1) \mid \mathrm{a}(q_0Aq_3)(q_3zq_1)$

$(q_0\mathbf{z}q_2) \rightarrow \mathrm{a}(q_0Aq_0)(q_0zq_2) \mid \mathrm{a}(q_0Aq_1)(q_1zq_2) \mid$
$\mathrm{a}(q_0Aq_2)(q_2zq_2) \mid \mathrm{a}(q_0Aq_3)(q_3zq_2)$

$(q_0\mathbf{z}q_3) \rightarrow \mathrm{a}(q_0Aq_0)(q_0zq_3) \mid \mathrm{a}(q_0Aq_1)(q_1zq_3) \mid$
$\mathrm{a}(q_0Aq_2)(q_2zq_3) \mid \mathrm{a}(q_0Aq_3)(q_3zq_3)$

Example 5: Consider the npda with transitions

$$\delta(q_3, \lambda, \boxed{z}) = \{(q_0, Az)\}$$

$$(q_3 \boxed{z} q_0) \to (q_0 A q_0)(q_0 z q_0) \mid (q_0 A q_1)(q_1 z q_0) \mid (q_0 A q_2)(q_2 z q_0) \mid (q_0 A q_3)(q_3 z q_0)$$

$$(q_3 \boxed{z} q_1) \to (q_0 A q_0)(q_0 z q_1) \mid (q_0 A q_1)(q_1 z q_1) \mid (q_0 A q_2)(q_2 z q_1) \mid (q_0 A q_3)(q_3 z q_1)$$

$$(q_3 \boxed{z} q_2) \to (q_0 A q_0)(q_0 z q_2) \mid (q_0 A q_1)(q_1 z q_2) \mid (q_0 A q_2)(q_2 z q_2) \mid (q_0 A q_3)(q_3 z q_2)$$

$$(q_3 \boxed{z} q_3) \to (q_0 A q_0)(q_0 z q_3) \mid (q_0 A q_1)(q_1 z q_3) \mid (q_0 A q_2)(q_2 z q_3) \mid (q_0 A q_3)(q_3 z q_3)$$

Example 5: Consider the npda with transitions

$\delta(q_0, a, z) = \{(q_0, Az)\}$

$(q_0Aq_3) \rightarrow \mathrm{a}$
$(q_0Aq_1) \rightarrow \mathrm{b}$
$(q_1zq_2) \rightarrow \lambda$

$(q_0zq_0) \rightarrow \cancel{\mathrm{a}(q_0Aq_0)(q_0zq_0)} \mid \cancel{\mathrm{a}(q_0Aq_1)(q_1zq_0)} \mid$
$\cancel{\mathrm{a}(q_0Aq_2)(q_2zq_0)} \mid \mathrm{a}(q_0Aq_3)(q_3zq_0)$

$(q_0zq_1) \rightarrow \cancel{\mathrm{a}(q_0Aq_0)(q_0zq_1)} \mid \cancel{\mathrm{a}(q_0Aq_1)(q_1zq_1)} \mid$
$\cancel{\mathrm{a}(q_0Aq_2)(q_2zq_1)} \mid \mathrm{a}(q_0Aq_3)(q_3zq_1)$

$(q_0zq_2) \rightarrow \cancel{\mathrm{a}(q_0Aq_0)(q_0zq_2)} \mid \mathrm{a}(q_0Aq_1)(q_1zq_2) \mid$
$\cancel{\mathrm{a}(q_0Aq_2)(q_2zq_2)} \mid \mathrm{a}(q_0Aq_3)(q_3zq_2)$

$(q_0zq_3) \rightarrow \cancel{\mathrm{a}(q_0Aq_0)(q_0zq_3)} \mid \cancel{\mathrm{a}(q_0Aq_1)(q_1zq_3)} \mid$
$\cancel{\mathrm{a}(q_0Aq_2)(q_2zq_3)} \mid \mathrm{a}(q_0Aq_3)(q_3zq_3)$

Example 5: Consider the npda with transitions

$\delta(q_3, \lambda, z) = \{(q_0, Az)\}$

$(q_0 A q_3) \rightarrow$a
$(q_0 A q_1) \rightarrow$ b
$(q_1 z q_2) \rightarrow \lambda$

$(q_3 z q_0) \rightarrow \cancel{(q_0 A q_0)(q_0 z q_0)} \mid \cancel{(q_0 A q_1)(q_1 z q_0)} \mid$
$\cancel{(q_0 A q_2)(q_2 z q_0)} \mid (q_0 A q_3)(q_3 z q_0)$

$(q_3 z q_1) \rightarrow \cancel{(q_0 A q_0)(q_0 z q_1)} \mid \cancel{(q_0 A q_1)(q_1 z q_1)} \mid$
$\cancel{(q_0 A q_2)(q_2 z q_1)} \mid (q_0 A q_3)(q_3 z q_1)$

$(q_3 z q_2) \rightarrow \cancel{(q_0 A q_0)(q_0 z q_2)} \mid (q_0 A q_1)(q_1 z q_2) \mid$
$\cancel{(q_0 A q_2)(q_2 z q_2)} \mid (q_0 A q_3)(q_3 z q_2)$

$(q_3 z q_3) \rightarrow \cancel{(q_0 A q_0)(q_0 z q_3)} \mid \cancel{(q_0 A q_1)(q_1 z q_3)} \mid$
$\cancel{(q_0 A q_2)(q_2 z q_3)} \mid (q_0 A q_3)(q_3 z q_3)$

Example 5: Consider the npda with transitions

$(q_0 A q_3) \rightarrow$a
$(q_0 A q_1) \rightarrow$ b
$(q_1 z q_2) \rightarrow \lambda$
$(q_0 z q_0) \rightarrow$ a$(q_0 A q_3)(q_3 z q_0)$
$(q_0 z q_1) \rightarrow$ a$(q_0 A q_3)(q_3 z q_1)$
$(q_0 z q_2) \rightarrow$ a$(q_0 A q_1)(q_1 z q_2)$ | a$(q_0 A q_3)(q_3 z q_2)$
$(q_0 z q_3) \rightarrow$ a$(q_0 A q_3)(q_3 z q_3)$
$(q_3 z q_0) \rightarrow (q_0 A q_3)(q_3 z q_0)$
$(q_3 z q_1) \rightarrow (q_0 A q_3)(q_3 z q_1)$
$(q_3 z q_2) \rightarrow (q_0 A q_1)(q_1 z q_2) \mid (q_0 A q_3)(q_3 z q_2)$
$(q_3 z q_3) \rightarrow (q_0 A q_3)(q_3 z q_3)$

Example 5: Final CFG

$(q_0 A q_3) \rightarrow$a
$(q_0 A q_1) \rightarrow$ b
$(q_1 z q_2) \rightarrow \lambda$
$(q_0 z q_0) \rightarrow$ a$(q_0 A q_3)(q_3 z q_0)$
$(q_0 z q_1) \rightarrow$ a$(q_0 A q_3)(q_3 z q_1)$
$(q_0 z q_2) \rightarrow$ a$(q_0 A q_1)(q_1 z q_2)$ | a$(q_0 A q_3)(q_3 z q_2)$
$(q_0 z q_3) \rightarrow$ a$(q_0 A q_3)(q_3 z q_3)$
$(q_3 z q_0) \rightarrow (q_0 A q_3)(q_3 z q_0)$
$(q_3 z q_1) \rightarrow (q_0 A q_3)(q_3 z q_1)$
$(q_3 z q_2) \rightarrow (q_0 A q_1)(q_1 z q_2)$ | $(q_0 A q_3)(q_3 z q_2)$
$(q_3 z q_3) \rightarrow (q_0 A q_3)(q_3 z q_3)$

7.3 Deterministic PDA and Deterministic CFL

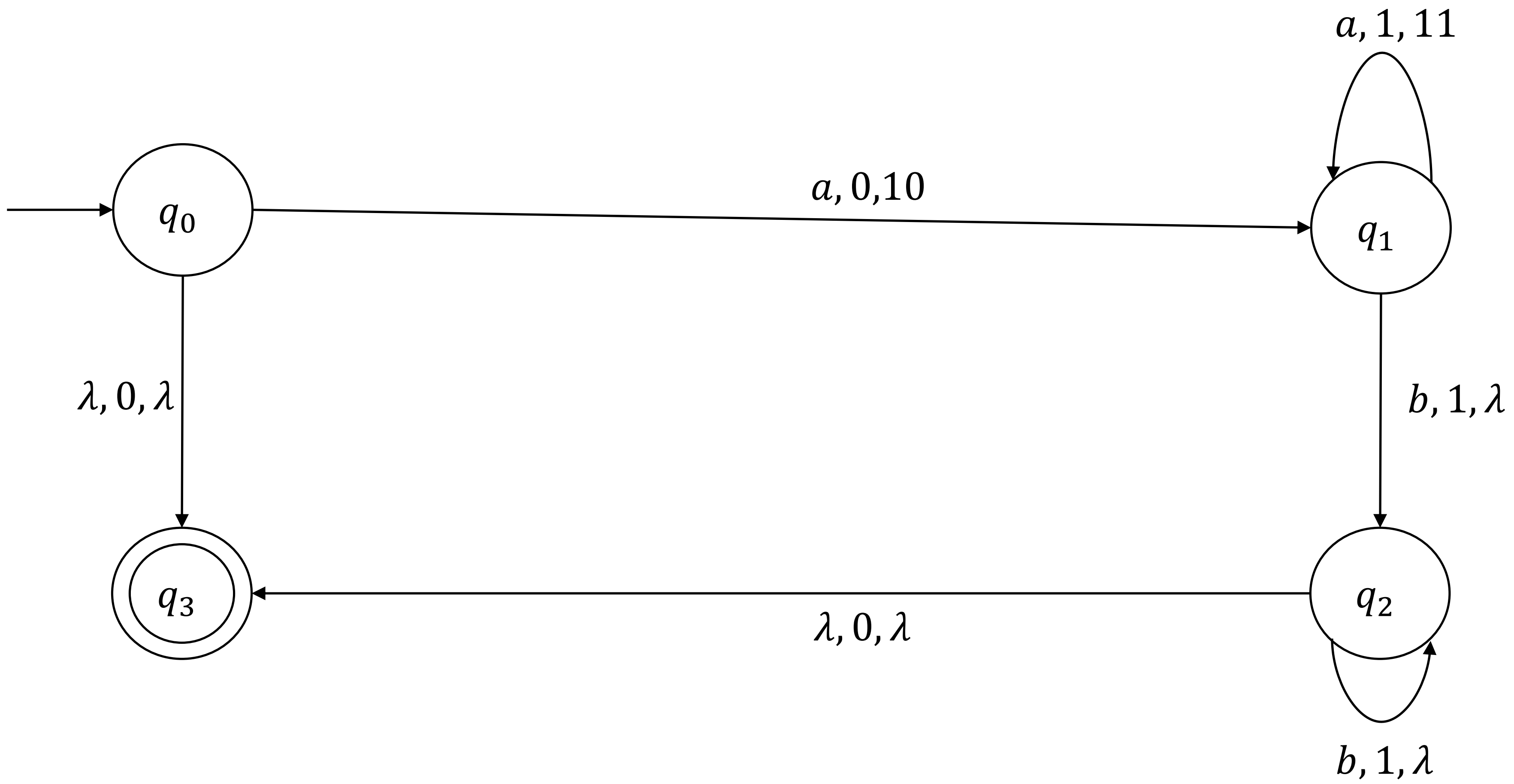
Definition of a DPDA

$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$

Restriction 1: $|\delta(q, a, b)|$<=1

Restriction 2: if $\delta(q, \lambda, b)$ is not empty then all $\delta(q, c, b)$ must be.

Example 1: $L = \{a^n b^n : n \geq 0\}$



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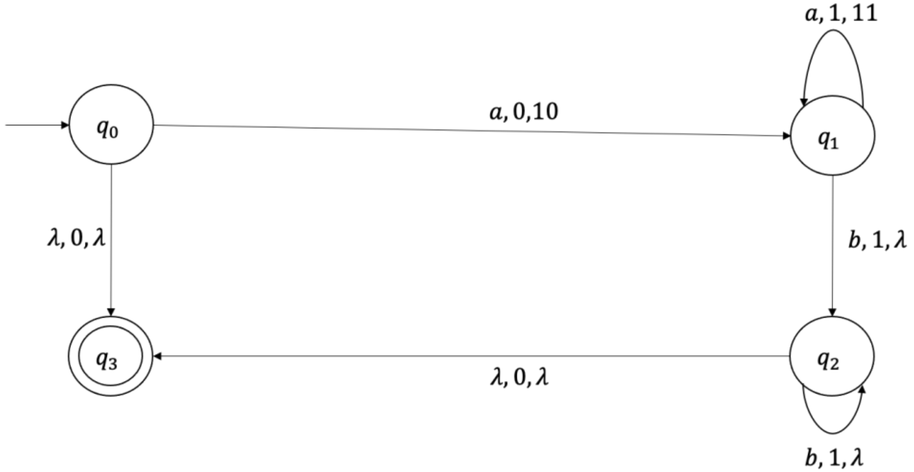
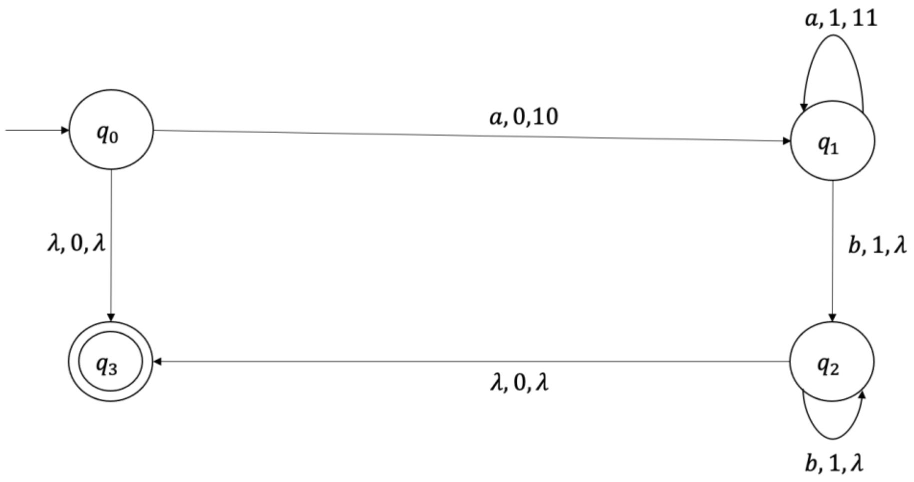


$$M = (\{q_0, q_1, q_2, q_3\},\ \{a, b\},\ \{0, 1\}, \delta, q_0, 0,\ \{q_3\})$$

$$\delta(q_0, a, 0) = \{(q_1, 10)\}$$
$$\delta(q_0, \lambda, 0) = \{(q_3, \lambda)\}$$
$$\delta(q_1, a, 1) = \{(q_1, 11)\}$$
$$\delta(q_1, b, 1) = \{(q_2, \lambda)\}$$
$$\delta(q_2, b, 1) = \{(q_2, \lambda)\}$$
$$\delta(q_2, \lambda, 0) = \{(q_3, \lambda)\}$$

Example 1: $L = \{a^n b^n : n \geq 0\}$



$$M = (\{q_0, q_1, q_2, q_3\},\ \{a, b\},\ \{0, 1\}, \delta, q_0, 0,\ \{q_3\})$$

$$\begin{aligned}
\delta(q_0, a, 0) &= \{(q_1, 10)\} \\
\delta(q_0, \lambda, 0) &= \{(q_3, \lambda)\} \\
\delta(q_1, a, 1) &= \{(q_1, 11)\} \\
\delta(q_1, b, 1) &= \{(q_2, \lambda)\} \\
\delta(q_2, b, 1) &= \{(q_2, \lambda)\} \\
\delta(q_2, \lambda, 0) &= \{(q_3, \lambda)\}
\end{aligned}$$

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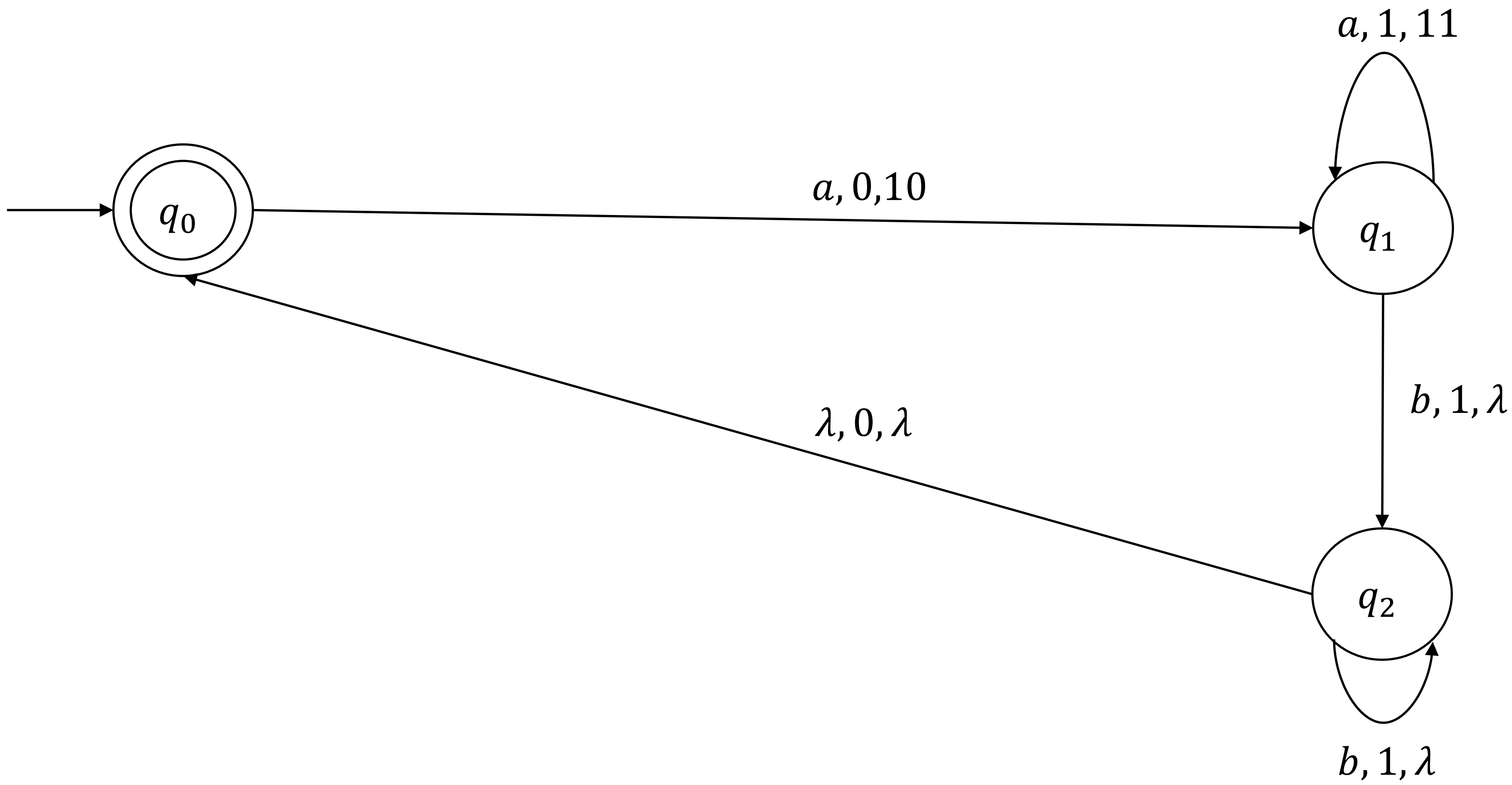

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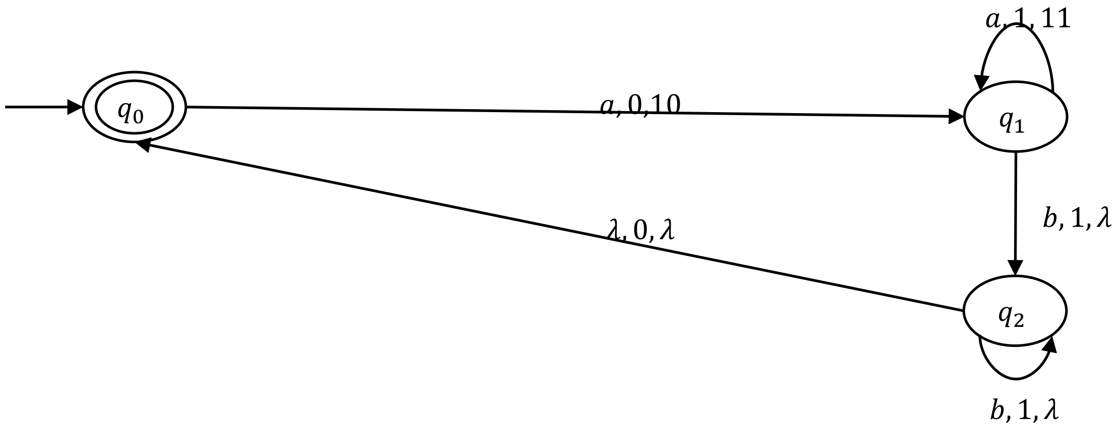

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$$\delta(q_0, a, 0) = \{(q_1, 10)\}$$
$$\delta(q_1, a, 1) = \{(q_1, 11)\}$$
$$\delta(q_1, b, 1) = \{(q_2, \lambda)\}$$
$$\delta(q_2, b, 1) = \{(q_2, \lambda)\}$$
$$\delta(q_2, \lambda, 0) = \{(q_0, \lambda)\}$$

Example 2:

$$\delta(q_0, \lambda, z) = \{(q_1, Sz)\}$$
$$\delta(q_1, a, S) = \{(q_1, A)\}$$
$$\delta(q_1, a, A) = \{(q_1, ABC), (q_1, \lambda)\}$$
$$\delta(q_1, b, A) = \{(q_1, B)\}$$
$$\delta(q_1, b, B) = \{(q_1, \lambda)\}$$

Example 2:

$$\delta(q_0, \lambda, z) = \{(q_1, Sz)\}$$
$$\delta(q_1, a, S) = \{(q_1, A)\}$$
$$\delta(q_1, a, A) = \{(q_1, ABC), (q_1, \lambda)\}$$
$$\delta(q_1, b, A) = \{(q_1, B)\}$$
$$\delta(q_1, b, B) = \{(q_1, \lambda)\}$$

Definition of a DPDA

$M = (Q, \Sigma, \Gamma, \delta, q_0, z, F)$

Restriction 1: $|\delta(q, a, b)|$<=1

Restriction 2: if $\delta(q, \lambda, b)$ is not empty then all $\delta(q, c, b)$ must be.

LL grammars

- LL Grammars are grammars are those in which the input are scanned from left to right and the derivations are leftmost derivations.
- LL(k) denotes that the ability to read the current symbol and looking ahead (k-1) symbols and then determining what production must be used to create the string being derived.
- If we cannot in k-1 lookahead then it is not and LL (k) grammar. Additionally what if we increase k and if it is not possible for all values of k, then the grammar is not an LL grammar.
- Although the grammar is not an LL grammar, doesn’t mean the language is not deterministic. There could be an equivalent grammar that is an LL grammar.
- If g is an LL grammar then the L(G) is a deterministic context-free language.