Home work #5 Zachary Hightower

1. Let $*$ be an operation on the real numbers $\mathbb{R}$ defined by $x * y = x + y - xy$ for all $x, y \in \mathbb{R}$

(a) Is $*$ closed on $\mathbb{R}$?

Yes, none of the operations within the function for $*$ will take us out of the set of real numbers.

(b.) Is $*$ commutative on $\mathbb{R}$

$x * y = y * x$, which is equal to:
$x + y - xy = y + x - yx$

Changing the order of the $y + x$ component won't result in any overall difference.

Changing the order of yx won't change the result.

However, if we change the order of subtraction to $yx - y + x$, this will change the final result.

So, we can say that $*$ is not commutative on $\mathbb{R}$.

(c) Is $*$ associative on $\mathbb{R}$?

$(x \circ y) \circ z = x \circ (y \circ z)$

$(x+y-xy)+z-(x+y-xy)z = (y+z-yz)+x-x(y+z-yz)$

$x+y-xy+z-zx-zy+zxy = "$

$x+y-xy+z-zx-zy+zxy = y+z-yz+x-xy+zx+zxy$

We can see that there are several differences between the equations, which means they are not equal and thus not associative

(d.) Does $(\mathbb{R}, *)$ have an identity element

$1+y-1(y) = 1$

$0+y-0(y) = y$

So, we can see that 0 serves as our identity element.

Contin. (d) If so, does every element have an inverse?
Hint: What if $x=1$?

No, not every element has an inverse.
0 is a member of the set of real numbers and it has no inverse under $+$, $\cdot$, or $-$ operation. So not every element has an inverse.

(e.) Is $(\mathbb{R}, *)$ a group?

The set is closed under $*$

The set is not associative under $*$

The set has an identity element

Not every element of the set has an inverse.

So, it fails two of our tests. This means $(\mathbb{R}, *)$ is not a group.

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2. 101 and 103 are prime numbers and
$101 \times 103 = 10403$

(a.) How many elements are in the set $\mathbb{Z}^*_{10403}$?

We know the elements of $\mathbb{Z}^*_{10403} = \varphi(n)$
$= \varphi(10403)$

So, $\varphi(10403) = 10403 \times \left(1 - \frac{1}{101}\right)\left(1 - \frac{1}{103}\right)$
$= 10200$

(b.) How many elements are in the set $\mathbb{Z}^*_{120}$

$\varphi(120) = 120 \times \left(1 - \frac{1}{2}\right)\left(1 - \frac{1}{3}\right)\left(1 - \frac{1}{5}\right)$
$= 32$

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2. cont

(c.) How many elements are in the set $\mathbb{Z}_{p^2q^3r^4}$, where p, q, and r are distinct primes?

$$\mathbb{Z}_{p^2q^3r^4} = n \prod_{\substack{p \mid n \\ p \text{ prime}}} \left(1 - \frac{1}{p}\right) = \left(1 - \frac{1}{p}\right)\left(1 - \frac{1}{q}\right)\left(1 - \frac{1}{r}\right)$$

3. Let $\phi(n)$ be the Euler phi function. If n is an odd positive integer, explain why $\phi(2n) = \phi(n)$

If n is odd, then $n \bmod 2 = 1$

We know that when we use ϕ it is multiplicative. So it follows that $\phi(2n) = \phi(2)\,\phi(n)$

$= \phi(n)$

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4(a). Write out the elements in $\mathbb{Z}_9^*$

$\mathbb{Z}_9^* = \{1, 2, 4, 5, 7, 8\}$

(b) Is $(\mathbb{Z}_9^*, \otimes)$ a cyclic group? If so, find all its generators

$a \otimes b = ab \bmod n$

1 is not a generator, it only gives 1

$2^1 = 2$, $2^2 = 4$, $2^3 = 8$, $2^4 = 7$, $2^5 = 5$, $2^6 = 1$, 2 is a generator

$4^1 = 4$, $4^2 = 7$, $4^3 = 1$, $4^4 = 4$, $4^5 = 7$, 4 is not a generator

$5^1 = 5$, $5^2 = 7$, $5^3 = 8$, $5^4 = 4$, $5^5 = 2$, $5^6 = 1$, 5 is a generator

$7^1 = 7$, $7^2 = 4$, $7^3 = 1$, $7^4 = 7$, 7 is not a generator

$8^1 = 8$, $8^2 = 1$, $8^3 = 8$, 8 is not a generator

Generators of the cyclic group $(\mathbb{Z}_9^*, \otimes)$ = 2, 5

5. (a) Write out the elements in $\mathbb{Z}_{12}^*$

$\{1, 5, 7, 11\}$

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5 cont. (b) Is $(\mathbb{Z}_{12}^*, \otimes)$ a cyclic group? If so, find all its generators.

1, 5, 7, 11

1 is not a generator, only gives 1

$5^1 = 5$, $5^2 = 1$, $5^3 = 5$, 5 is not a generator

$7^1 = 7$, $7^2 = 1$, $7^3 = 7$, 7 is not a generator

$11^1 = 11$, $11^2 = 1$, $11^3 = 11$, 11 is not a generator

The group $(\mathbb{Z}_{12}^*, \otimes)$ has no generators so it is not a cyclic group.