Real Numbers and Limits

Name:________________________ SID:________________________

Prerequisite Concepts:

1. Basic set theory ($a \in A$, $A \subset B$, $A = B$, $A \setminus B$, $A \cup B$, $A \cap B$)
2. Arithmetic and algebra
3. Intuitive understanding of the real number line

Basic Concepts:

1. Rational numbers ($\mathbb{Q}$)
2. Real numbers ($\mathbb{R}$)
3. Intervals
4. Maximum and minimum
5. Upper and lower bounds

Advanced Concepts:

1. Infimum and supremum
2. Limit of a sequence
3. Limit of a function

1 Numbers

There are many different types of numbers. Each type of number should, strictly speaking, be considered its own type of object. We will be more lax and consider natural numbers a type of integer, integers a type of rational number, and rational numbers a type of real number. That is, we will consider $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$ to be a true statement.

1.1 Natural Numbers ($\mathbb{N}$)

The natural numbers are a fundamental building block of mathematics and their fundamental properties of are summarized in what are called the *Peano Axioms*:

1. $0 \in \mathbb{N}$.
2. If $n \in \mathbb{N}$, then n has a unique successor $S(n) \in \mathbb{N}$.
3. For any $n \in \mathbb{N}$, it is not the case that $S(n) = 0$.
4. For all $n, m \in \mathbb{N}$, if $S(n) = S(m)$, then $n = m$.
5. If there is some set A for which both statements $0 \in A$ and $n \in A \implies S(n) \in A$ are true, then $A = \mathbb{N}$.

In more plain (and less precise) language, these can be summarized as follows: 1) $\mathbb{N}$ is a set with at least one element. Call this element 0. 2) every element of $\mathbb{N}$ has exactly one next (successor) element. Let $S(n)$ represent the successor of n (assuming $n \in \mathbb{N}$). 3) Our first element 0 has no predecessor. 4) The successor function is one-to-one. 5) The principle of induction, which is useful for proving a statement is true for all elements of $\mathbb{N}$.

The set of all natural numbers is denoted by $\mathbb{N}$ and can be listed as

$$\mathbb{N} = \{0, 1, 2, 3, 4, \dots\}. \tag{1}$$

Sometimes 0 is not included in this set, but its inclusion (or lack thereof) is rarely meaningful.

Addition ($m + n$), multiplication ($m \times n$ or $m \cdot n$ or mn), exponentiation (m^n), less than ($m < n$), and greater than ($m > n$) can all be defined in terms of this successor function. These definitions in terms of the successor function are beyond our current scope, but you must be familiar with these operations and their properties. It is sufficient here to note that $S(0) = 1$ and $S(n) = n + 1$ for all $n \in \mathbb{N}$.

Note that some addition equations such as $3 + x = 10$ have solutions but others (such as $3 + x = 1$) do not (when x is restricted to $\mathbb{N}$).

A few important facts about $\mathbb{N}$:

1. If $n \in \mathbb{N}$, then its successor is $n + 1$ and is the "next largest" natural number.
2. There is no largest element of $\mathbb{N}$.
3. There is a smallest element of $\mathbb{N}$.

1.2 Integers ($\mathbb{Z}$)

The fundamental difference between $\mathbb{Z}$ and $\mathbb{N}$ is the each element of $\mathbb{Z}$ has a predecessor. That is, for each $n \in \mathbb{Z}$, there is exactly one predecessor $m \in \mathbb{Z}$ such that $S(m) = n$. This means that starting at 0, we have a chain of successors: 1, 2, 3, etc. and a chain of predecessors: -1, -2, -3, etc.

The set of all integers is denoted by $\mathbb{Z}$ and can be listed as

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}. \tag{2}$$

Addition and multiplication are effectively directly inherited from $\mathbb{N}$. Note that some multiplication equations such as $2n = 8$ have solutions but others (such as $2n = 7$) do not (when requiring $n \in \mathbb{Z}$).

A few important facts about $\mathbb{Z}$:

1. If $n \in \mathbb{Z}$, then $n + 1$ is its successor and is the "next largest" element of $\mathbb{Z}$.
2. If $n \in \mathbb{Z}$, then $n - 1$ is its predecessor and is the "next smallest" element of $\mathbb{Z}$.
3. There is no largest element of $\mathbb{Z}$.
4. There is no smallest element of $\mathbb{Z}$.
5. If $n \in \mathbb{Z}$ is a multiple of 2 (i.e., $n = 2m$ for some $m \in \mathbb{Z}$), then n is *even*.
6. If $n \in \mathbb{Z}$ is either one greater than or one less than a multiple of 2 (i.e., $n = 2m \pm 1$ for some $m \in \mathbb{Z}$), then n is *odd*.
7. Each $n \in \mathbb{Z}$ is either even or odd.

1.3 Rationals ($\mathbb{Q}$)

Rational numbers are numbers expressible as the ratio of two integers. Because rational numbers may have more than one representation (e.g., 1/2 and 2/4 are the same rational number). 1/2 and 2/4 are *different fractions* that represent the *same rational number*.

The set of rational numbers is denoted by $\mathbb{Q}$. It is possible to list out $\mathbb{Q}$, but not in an increasing order like $\mathbb{N}$ and $\mathbb{Z}$. Rather, we define $\mathbb{Q}$ as

$$\mathbb{Q} = \left\{ \frac{m}{n} \colon m, n \in \mathbb{Z} \text{ and } n \neq 0, \text{ and } m, n \text{ are co-prime} \right\}. \tag{3}$$

If $n > 0$ and m and n share no common factors, then they are said to be *co-prime* and m/n is said to be *simplified*. We still consider fractions such as 2/4 to be in $\mathbb{Q}$, but the correct representation would be the simplified 1/2 fraction. There is no difference between the notations m/n and $\frac{m}{n}$. Additionally, if the denominator is one, it is often omitted and the rational numbers $-(a/b)$, $(-a)/b$, and $a/(-b)$ are all the same and should be simply written as $-a/b$. As an aside, both $-(a/b)$ and $(-a)/b$ are both read as "the additive inverse of a over b." This is one reason why mathematics is done using a combination of standard language (e.g., English) and extra symbols.

Addition and multiplication for rationals can be defined from addition and multiplication in $\mathbb{Z}$.

Addition of a/b and c/d is defined by

$$\frac{a}{b} + \frac{c}{d} := \frac{ad + cb}{bd} \tag{4}$$

while multiplication is defined by

$$\frac{a}{b} \cdot \frac{c}{d} := \frac{ac}{bd}. \tag{5}$$

In both cases, a simplification step may be required to get the correct representation of the sum or product. The comparison of rational numbers (greater than and less than) is also inherited from $\mathbb{Z}$. If a/b and c/d are fractions with $b, d > 0$, then we say $a/b < c/d$ exactly when $ad < cb$. Note that every equation of the form $x + y = z$ has a unique solution and every equation of the form $xy = z$ also has a unique solution provided that none of x, y, z are 0. This is due to each $q \in \mathbb{Q}$ having an additive inverse $-q \in \mathbb{Q}$ such that $q + (-q) = 0$ and, if $q \neq 0$, a multiplicative inverse $q^{-1} \in \mathbb{Q}$ such that $q \cdot q^{-1} = 1$. However, polynomial equations such as $x^2 = 2$ may not have rational solutions.

In addition to having a simplified fraction representation, each rational number has (at least one) decimal expansion/representation. For example 1/2 has the (finite) decimal expansion 0.5 and the (infinite) decimal expansion $0.4\overline{9}$ while 1/7 has only the (infinite) decimal expansion $0.\overline{142857}$. The statements $1/2 = 0.5$, $1/2 = 0.50000\ldots$, and $1/2 = 0.4999\ldots$ are all true. The fact that the decimals 0.5 and $0.49999\ldots$ represent the same number is a property of the real numbers.

A few important facts about $\mathbb{Q}$:

1. Let $p, q \in \mathbb{Q}$. Then exactly one of the following statements is true: $p < q$, $q < p$, or $p = q$.
2. Let $p, q \in \mathbb{Q}$ such that $p < q$. Then there is a third rational number r such that $p < r < q$.
3. Let $p \in \mathbb{Q}$. There is no "next largest" or "next smallest" rational number relative to q.
4. Let $p \in \mathbb{Q}$. Then q has both a simplified fractional representation and at least one decimal representation. Any decimal expansion of a rational number is either finite, or ends in a finite repeating pattern.
5. Let $p, q \in \mathbb{Q}$. Then both $p + q$ and pq are also in $\mathbb{Q}$.

Remark: The distinction between a rational number and its representation can be lessened using a concept called *equivalence classes*. This equates a rational number to the *set of all equivallent fractions* and any element of that set can be chosen as a representative.

1.4 Reals ($\mathbb{R}$)

One way to view real numbers are as a special type (Cauchy) of sequence of rational numbers. For example, there is no rational number $a \in \mathbb{Q}$ such that $a^2 = 2$. However, one can construct a *sequence of rational numbers* $a_0, a_1, a_2, \ldots$ such that a_n^2 *approaches* 2. It is then this sequence $a_1, a_2, a_3, \ldots$ that represents the real number $\sqrt{2}$. *Just as different fractions may represent the same rational number, different sequences can represent the same real number.* A precise construction of the real numbers is far beyond the scope of this course, but there are several properties that must be known.

Addition and Multiplication:

1. (Commutativity) Let $a, b \in \mathbb{R}$. Then $a + b = b + a$ and $ab = ba$.
2. (Associativity) Let $a, b, c \in \mathbb{R}$. Then $(a + b) + c = a + (b + c)$ and $(ab)c = a(bc)$.
3. (Distributivity) Let $a, b, c \in \mathbb{R}$. Then $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.

The arithmetic properties above should be quite familiar and, because $\mathbb{N} \subset \mathbb{Z} \subset \mathbb{Q} \subset \mathbb{R}$, hold for each of $\mathbb{N}$, $\mathbb{Z}$, and $\mathbb{Q}$ as well. There are a few other properties that you should be aware of. The first is what allows us to view $\mathbb{R}$ as a number line.

Properties of $\mathbb{R}$:

1. Let $a, b \in \mathbb{R}$. Then exactly one of the following statements is true: $a < b$, $a = b$, or $a > b$.
2. Let $a \in \mathbb{R}$. If $a > 0$, then a is said to be *positive*. If $a \geq 0$, then a is said to be *non-negative*.
3. Let $a \in \mathbb{R}$. If $a < 0$, then a is said to be *negative*. If $a \leq 0$, then a is said to be *non-positive*.
4. Let $a, b \in \mathbb{R}$ such that $a < b$. Then there is some $c \in \mathbb{R}$ such that $a < c < b$.
5. Let $p, q \in \mathbb{Q}$ such that $p < q$, then there is some $r \in \mathbb{R} \setminus \mathbb{Q}$ such that $p < r < q$.
6. If the statement $0 \leq |a - b| \leq \varepsilon$ is true for all $\varepsilon > 0$, then $a = b$.
7. The symbol ∞ is useful, but is not an element of $\mathbb{R}$.

Note that if $a \neq b$, then there is some number c that fits between a and b. Conversely, if no number c can fit between a and b, then $a = b$. This property is why $0.49999\ldots$ and 0.5 both represent the same number.

2 Subsets of $\mathbb{R}$

There are many properties and a large theory of subsets of $\mathbb{R}$, but we will only cover the essentials.

Definition 2.1 (Maximum): Let $A \subset \mathbb{R}$. We say that $M \in A$ is the *maximum* of A (or maximal element of A) if $x \leq M$ is true for all $x \in A$.

Definition 2.2 (Bounded Above, Upper Bound): Let $A \subset \mathbb{R}$. We say that A is *bounded above* if there exists some $M \in \mathbb{R}$ such that $x \leq M$ for all $x \in A$. Any such M is called an *upper bound* of A.

Sets such as $A = \{x \in \mathbb{R} \colon x < 2\}$ and $B = \{x \in \mathbb{R} \colon x \leq 2\}$ are bounded above and has many upper bounds (e.g., 100, π, $1 + \sqrt{2}$, 2.01, 2.0001, and 2 are all upper bounds of both A and B). Note that 2 is the maximum of B, but A has no maximum, but for both A and B, 2 is the smallest possible upper bound.

Definition 2.3 (Supremum, Least Upper Bound): Let $A \subset \mathbb{R}$ be bounded above. Then A has a *least upper bound*, or *supremum*, which is denoted by $\sup A$.

Analogous definitions for minimum, bounded below, lower bound, and infimum (greatest lower bound) are left as homework to define. A subset of $\mathbb{R}$ is *bounded* if it is both bounded above and bounded below, but the conventional definition is as follows.

Definition 2.4 (Bounded): A set A is said to be *bounded* if there exists some $M > 0$ such that $|x| < M$ for all $x \in A$.

The most important type of subset of $\mathbb{R}$ is an *interval*.

Bounded Intervals: Let $a, b \in \mathbb{R}$ such that $a < b$. Then the following are all bounded intervals. Each interval has an infimum of a and a supremum of b. Not all intervals have a maximum or minimum.

1. (Open Interval) $(a, b) \coloneqq \{x \in \mathbb{R} \colon a < x < b\}$
2. (Closed Interval) $[a, b] \coloneqq \{x \in \mathbb{R} \colon a \leq x \leq b\}$
3. (Half Open Interval) $(a, b] \coloneqq \{x \in \mathbb{R} \colon a < x \leq b\}$
4. (Half Open Interval) $[a, b) \coloneqq \{x \in \mathbb{R} \colon a \leq x < b\}$

Note that we do not consider degenerate intervals where $a = b$.

Unbounded Intervals: Let $a \in \mathbb{R}$.

1. (Open Interval) $(a, \infty) \coloneqq \{x \in \mathbb{R} \colon a < x\}$
2. (Open Interval) $(-\infty, a) \coloneqq \{x \in \mathbb{R} \colon x < a\}$
3. (Closed Interval) $[a, \infty) \coloneqq \{x \in \mathbb{R} \colon a \leq x\}$
4. (Closed Interval) $(-\infty, a] \coloneqq \{x \in \mathbb{R} \colon x \leq a\}$
5. (Open and Closed Interval) $(-\infty, \infty) \coloneqq \mathbb{R}$

Note that ∞ and $-\infty$ are not real numbers themselves. They are simply symbols to represent unbounded (in the positive or negative direction). We will work with this concept extensively in this course. Note that some intervals are called *open*, some are called *closed*, and some are neither. The concept of open and closed can be extended to other types of sets, but we do not do that here. However, we do introduce two operations: given an interval I, $\overline{I}$ is the *closure* of I and $\mathring{I}$ is the *interior* of I. Rather than defining these precisely, we give several examples, but the primary concept is that $\overline{I}$ *includes* its (finite) endpoints while $\mathring{I}$ does not. If I is an interval, then $\overline{I}$ will always be a closed interval and $\mathring{I}$ will always be an open interval.

Example 2.1 (Closure and Interior (intervals)): Let $a, b \in \mathbb{R}$ such that $a < b$.

1. If $I = (a, b]$, then $\overline{I} = [a, b]$ and $\mathring{I} = (a, b)$.
2. If $I = [a, b]$, then $\overline{I} = [a, b]$ and $\mathring{I} = (a, b)$.
3. If $I = (a, b)$, then $\overline{I} = [a, b]$ and $\mathring{I} = (a, b)$.
4. If $I = (-\infty, a]$, then $\overline{I} = (-\infty, a]$ and $\mathring{I} = (-\infty, a)$.
5. If $I = (b, \infty)$, then $\overline{I} = [b, \infty)$ and $\mathring{I} = (b, \infty)$.

Properties of intervals:

1. Every interval is either bounded or unbounded. The shorthand for an unbounded interval always includes the symbol ∞ or $-\infty$.
2. Intervals may be open, closed, neither, or both. The only *interval* that is both open and closed is $(-\infty, \infty)$.
3. Given an interval I, $\overline{I}$ is a closed interval and consists of I plus its endpoints.
4. Given an interval I, $\mathring{I}$ is an open interval and consists of I minus its endpoints.

2.1 Limits of Sequences

Informally, a *sequence* is an infinite list of numbers a_n. If the terms of the sequence **get close and stay close** to some number L, then we say that the L is the limit of the sequence and may write $a_n \to L$ as $n \to \infty$ or $\lim\limits_{n \to \infty} a_n = L$. It is important to remember that, unlike a set, a sequence is ordered.

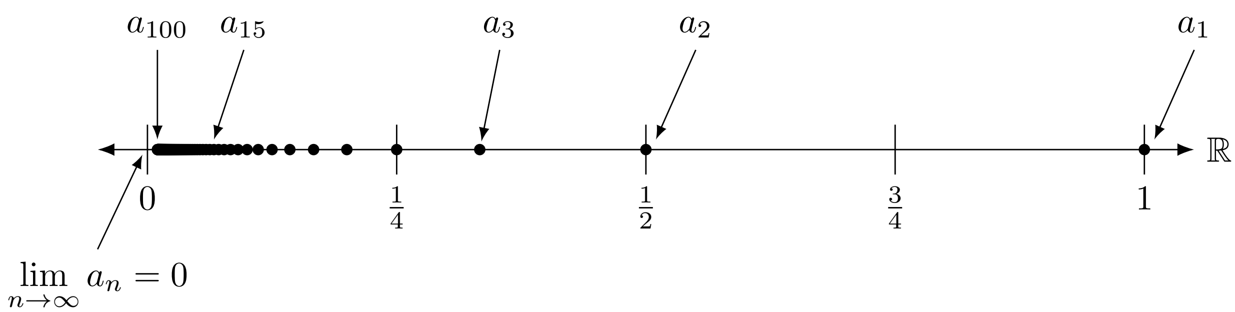

Figure 1: Illustration of the sequence $a_n = 1/n$ with $n = 1, 2, 3, \ldots$. Only the first 100 terms are plotted.

Definition 2.5 (Sequence): A real-valued *sequence* is a function from $\mathbb{N}$ (or a different infinite subset of $\mathbb{Z}$) to $\mathbb{R}$. Often times a sequence is introduced by the notation $(a_n)_{n=0}^{\infty}$ or $(a_n)_{n=1}^{\infty}$. If the starting index of the sequence is irrelevant, then (a_n) may be used as well.

For example, $(n)_{n=0}^{\infty}$ is the sequence $0, 1, 2, 3, \ldots$, and $(1/n)_{n=1}^{\infty}$ is the sequence $1, 1/2, 1/3, \ldots$.

Definition 2.6 (Limit of a sequence): Let (a_n) be a real-valued sequence. If there exists some $L \in \mathbb{R}$ such that for any $\varepsilon > 0$, there exists some $N \in \mathbb{N}$ (likely dependent on ε) such that

$$|a_n - L| < \varepsilon$$

for all $n > N$, then we say that $a_n \to L$ as $n \to \infty$ or $\lim_{n\to\infty} a_n = L$. If any such L exists, then it is necessarily unique and the sequence is said to *converge*. If no such L exists, then the sequence is said to *diverge*. Note that *it is not necessary for the sequence to reach its limit L*. For some divergent sequences, we may say a bit more. For example, the sequence $(n^2)_{n=1}^{\infty}$ clearly diverges *to infinity*.

Definition 2.7 (Infinite limit of a sequence): Let (a_n) be a real-valued sequence. If for any $M \in \mathbb{R}$, there exists an $N \in \mathbb{N}$ such that

$$a_n > M$$

for all $n > N$, then we say that (a_n) diverges to (positive) infinity and may write $a_n \to \infty$ as $n \to \infty$ or $\lim_{n\to\infty} a_n = \infty$.

There is an analogous definition for diverging to negative infinity.

Example 2.2: Sequences can converge, diverge to either positive or negative infinity, or simply diverge. It may be useful to plot the sequences below.

1. The sequence $1, \frac{1}{2}, \frac{1}{3}, \frac{1}{4}, \frac{1}{5}, \ldots$ converges to 0.
2. The sequence $-1, \frac{1}{2}, -\frac{1}{3}, \frac{1}{4}, -\frac{1}{5}, \ldots$ converges to 0.
3. The sequence $1, -1, 1, -1, 1, -1, \ldots$ diverges.
4. The sequence $(n^2)_{n=1}^{\infty}$ diverges to ∞.

5. The sequence $(-n^2)_{n=1}^{\infty}$ diverges to $-\infty$.

6. The sequence $((-1)^n n)_{n=0}^{\infty}$ diverges.

EXAMPLE 2.3: Not all sequences are given in explicit form (i.e., a_n as a function of n). For example, the Fibonacci sequence is given by the recurrence relation $a_n = a_{n-1} + a_{n-2}$ $(n \geq 2)$ and the starting terms $a_0 = 1$ and $a_1 = 1$. This information completely determines the sequence, which goes:

$$1, 1, 2, 3, 5, 8, 13, 21, \ldots$$

Important concepts for real-valued sequences:

1. Every sequence either converges or diverges.
2. A divergent sequence may or may not diverge to $\pm\infty$.
3. Informally, a sequence converges if it **gets close and stays close** to some limit value L.
4. A sequence can have at most one limit value.
5. Sequences have a specified order of terms.

3 Limits of functions

In this section, we will limit ourselves to functions of the form $f\colon I \to \mathbb{R}$ where $I \subset \mathbb{R}$ where I is an interval or finite union of disjoint intervals (e.g., $I = (-\infty, 0) \cup (0, \infty)$ is allowed, but $I = \mathbb{Q}$ is not). The closure $\overline{I}$ of the domain I will be an important concept. If $I = I_1 \cup I_2$ where I_1 and I_2 are intervals, then $\overline{I} = \overline{I_1} \cup \overline{I_2}$. For example, if $I = (-\infty, 0) \cup (0, \infty)$, then $\overline{I} = (-\infty, 0] \cup [0, \infty) = \mathbb{R}$.

Definition 3.1 (Limit of a function at a point): Let $f\colon I \to \mathbb{R}$ and $x_0 \in \overline{I}$. If there exists an $L \in \mathbb{R}$ such that for all $\varepsilon > 0$, there exists some $\delta > 0$ (likely depending on ε) such that

$$\text{for all } x \in I \text{ such that } 0 < |x - x_0| < \delta, \text{ we have that } |f(x) - L| < \epsilon,$$

then we say that $f(x) \to L$ as $x \to x_0$ or $\lim\limits_{x \to x_0} f(x) = L$.

Note that the point x_0 is not necessarily in the domain of f (it could be on the boundary of the domain). The intuition is that if $f(x)$ is close to L for all values of x that are close to x_0 (*but not including* x_0), then $f(x) \to L$ as $x \to x_0$.

Example 3.1: Consider the following limits. It may be useful to plot each function f and the point (x_0, L).

1. $\lim\limits_{x \to 0} x = 0$
2. $\lim\limits_{x \to 0} \dfrac{1}{x}$ does not exist
3. $\lim\limits_{x \to 1} \dfrac{1}{x} = 1$
4. $\lim\limits_{x \to 0} \dfrac{|x|}{x}$ does not exist
5. $\lim\limits_{x \to 5} \dfrac{x^2 - 25}{x - 5} = 10$

In this definition, if I includes points to the left and right of x_0, then both sides must be considered. However, sometimes we only wish to consider limits coming from a single direction (i.e., left or right). We use the notation $x \to x_0^+$ to denote x approaching x_0 from the right (i.e., we require $x > x_0$) and $x \to x_0^-$ to denote x approaching x_0 from the left (i.e., we require $x < x_0$).

Definition 3.2 (One sided limits): Let $f\colon I \to \mathbb{R}$ with $x_0 \in \overline{I}$ and $(x_0, b) \subset I$ for some $b > x_0$. Then we say $\lim_{x \to x_0^+} f(x) = L$ if for all $\varepsilon > 0$, there exists $\delta > 0$ such that $|f(x) - L| < \varepsilon$ whenever $0 < x - x_0 < \delta$.

Remark: The criteria that $(x_0, b) \subset I$ is so that $f(x)$ is defined immediately to the right of x_0. The inequality $0 < x - x_0 < \delta$ could be written as $x_0 < x < x_0 + \delta$. There is an analogous definition for $x \to x_0^-$.

Example 3.2: Consider the following limits. It may be useful to plot each function f and the point (x_0, L).

1. $\lim\limits_{x \to 0^+} \frac{|x|}{x} = 1$
2. $\lim\limits_{x \to 0^-} \frac{|x|}{x} = -1$
3. $\lim\limits_{x \to -1^+} (3x + 1) = -2$
4. $\lim\limits_{x \to -1^-} (3x + 1) = -2$

Theorem 3.1: Let $f \colon I \to \mathbb{R}$. If $x_0 \in \mathring{\overline{I}}$, then $\lim\limits_{x \to x_0} f(x) = L$ if and only if $\lim\limits_{x \to x_0^+} f(x) = L$ and $\lim\limits_{x \to x_0^-} f(x) = L$.

Remark: The requirement that $x_0 \in \mathring{\overline{I}}$ is so that x_0 the limit from the left and the limit from the right both make sense.

Sometimes it is usefull to "assign a value" of $\pm\infty$ (i.e., ∞ or $-\infty$) to the limit of a function (similar to the limit of a sequence). This indicates that the function values are *unbounded* near x_0 in a particular manner.

Definition 3.3 (Infinite limits): Let $f \colon I \to \mathbb{R}$ with $x_0 \in \overline{I}$. Then we say $\lim\limits_{x \to x_0} f(x) = \infty$ if for all $M > 0$, there exists a $\delta > 0$ such that $f(x) > M$ whenever $0 < |x - x_0| < \delta$.

Remark: The analogous definition for $\lim\limits_{x \to x_0} f(x) = -\infty$ replaces $f(x) > M$ with $f(x) < -M$. The analogous one-sided definitions for $\lim\limits_{x \to x_0^{\pm}} f(x) = \pm\infty$ are as expected.

Finally, just as it is sometimes desirable to allow the values of $f(x)$ to be unbounded, we sometimes wish to allow the values of x to become unbounded. That is consider $x \to \pm\infty$.

Definition 3.4 (Limits at infinity): Let $f \colon \mathbb{R} \to \mathbb{R}$. Then we say $\lim\limits_{x \to \infty} f(x) = L$ if for all $\varepsilon > 0$, there exists an $M > 0$ such that $|f(x) - L| < \varepsilon$ whenever $x > M$.

Remark: The appropriate definitions for the case when $x \to -\infty$ and/or the limit value should be clear.

Example 3.3: Consider the following limits. It may be useful to plot each function f.

1. $\lim\limits_{x \to \infty} \frac{1}{x} = 0$

2. $\lim_{x\to\infty} x = \infty$

3. $\lim_{x\to\infty} -x = -\infty$

4. $\lim_{x\to 3^+} \dfrac{x}{x-3} = \infty$

5. $\lim_{x\to -3^+} \dfrac{x}{x+3} = -\infty$

6. $\lim_{x\to -\infty} \sin(x)$ does not exist

Important concepts for limits of functions of the type $f\colon \mathbb{R} \to \mathbb{R}$:

1. The value $f(x_0)$ is irrelevant to the value of the limit $\lim_{x\to x_0} f(x)$. One may exist and the other does not. If both exist, they need not be the same.

2. The notation $x \to x_0^+$ (resp. $x \to x_0^-$) means that x is "moving" towards x_0 along X-axis *from* the positive (resp. negative) direction. The formal definition of the limit of $f(x)$ will involve $x > M$ (resp. $x < -M$) in this case.

3. The notation $x \to \infty$ (resp. $x \to -\infty$) means that x is "moving" unbounded along the X-axis in the positive (resp. negative) direction. The formal definition of the limit of $f(x)$ will involve $x > M$ (resp. $x < -M$) in this case.

4. The notation $\lim f(x) = \infty$ (resp. $\lim f(x) = -\infty$) means that the function values are increasing (resp. decreasing) without bound (regardless of what type of behavior for x is specified). The formal definition of the limit will involve $f(x) > M$ (resp. $f(x) < -M$) in this case.

5. If one considers x to be "moving" when taking a limit, the intuition of "gets close and stays close" carries over from sequences.

Theorem 3.2: Let $f\colon \mathbb{R} \to \mathbb{R}$. Then $\lim_{x\to x_0} f(x) = L$ if and only if, for *every* sequence (a_n) with $a_n \to x_0$ and $a_n \neq x_0$, it is true that $f(x_n) \to L$.

Remark: The theorem above is quite useful for showing that a particular limit does not exist. It is not often useful for showing that a limit does exist. Analogous theorems hold for each type of limit of $f(x)$.

Example 3.4: We can show that $\lim_{x\to\infty} \sin(x)$ does not exist as follows. First, consider the sequence (a_n) with $a_n = 2n\pi$ for $n \in \mathbb{N}$. Clearly $a_n \to \infty$ and $f(a_n) = \sin(2n\pi) = 0$ is constant, so $f(a_n) \to 0$. However, if we consider the second sequence (b_n) with $b_n = 2n\pi + \frac{\pi}{2}$ for $n \in \mathbb{N}$, we see that again $b_n \to \infty$ but now $f(b_n) = \sin(2n\pi + \frac{\pi}{2}) = 1$ so that $f(b_n) \to 1$. This means that there are two sequences going to ∞ with *different* limits for their respective function values. Thus the limit $\lim_{x\to\infty} \sin(x)$ cannot exist.

4 Problems

1. Use only the properties of Associativity, Commutativity, and Distributivity to show that

$$(a+b)^2 = a^2 + 2ab + b^2$$

for any $a, b \in \mathbb{R}$. For each step, specify the property that is used. [Hint: start with $(a+b)^2 = (a+b)(a+b)$ and then distribute one of the the quantities $(a+b)$ across the other.]

2. Is 0 even, odd, or neither?

3. Let **E** represent an arbitrary even integer and **O** represent an arbitrary odd integer.

   (a) Show that $\mathbf{E} + \mathbf{E} = \mathbf{E}$. [Hint: consider $(2n) + (2m)$.]

   (b) Show that $\mathbf{E} + \mathbf{O} = \mathbf{O}$. [Hint: consider $(2n) + (2m+1)$.]

   (c) Show that $\mathbf{O} + \mathbf{O} = \mathbf{E}$.

4. Let $p, q \in \mathbb{Q}$ such that $p < q$. Assume that $p = a/b$ and $q = n/m$ are the simplified fractional representations of p and q respectively.

   (a) Find a fractional representation of $r := \frac{1}{2}(p+q)$ (in terms of a, b, c, d) with a positive denominator.

   (b) Show that $p < r$.

   (c) Show that $r < q$.

   (d) Explain why there is no "smallest positive rational number." [Hint: Recall that a is positive if and only if $a > 0$ is true.]

5. We will prove that $\sqrt{2} \notin \mathbb{Q}$ in several steps. We will do this by a *proof by contradiction*, which means that we will assume that there is some $q \in \mathbb{Q}$ such that $q^2 = 2$ (i.e., $q = \sqrt{2}$) and then showing how this cannot be true.

   (a) Show that if $n \in \mathbb{Z}$, then n^2 has the same even/odd value as n. [Hint: Split this into two cases: 1) $n = 2k$ and 2) $n = 2k+1$.]

   (b) Suppose that $q = \sqrt{2}$ is a rational number with simplified fractional form n/m. Show that n^2 and m^2 are both even.

   (c) Explain why n^2 and m^2 both being even means that n/m is not simplified. This is contradicts a prior assumption (i.e., $q \in \mathbb{Q}$), which means that the prior assumption is false (i.e., $q \notin \mathbb{Q}$ must be true).

6. Define the terms *Minimum*, *Bounded Below*, *Lower Bound*, and *Infimum* analogous to "maximum", "bounded above", "upper bound", and "supremum."

7. Give an example of a subset of $\mathbb{R}$ that is bounded, has a minimum, but has no maximum.

8. Let $x \in \mathbb{R}$ be the real number whose integer part is 0 and whose decimal part consists of 1's and 0's with 1 copy of 0 separating the first two 1's, 2 copies separating the second pair, 3 copies separating the third pair, and so on.

$$x = 0.101001000100001000001\ldots$$

Is x rational or irrational? Explain.

9. A *pythagorean triple* is a triple (a, b, c) of positive integers such that $a^2 + b^2 = c^2$. Verify that $(3, 4, 5)$ is a pythagorean triple and find another. Use your second pythagorean triple to find a point on the unit circle with rational coordinates (i.e., find $p, q \in \mathbb{Q}$ such that $p^2 + q^2 = 1$).

10. Write the intervals $A = (0, 1)$, $B = (0, 1]$, $C = [-5, \infty)$, $D = (-\infty, 3)$ using set builder notation. For each set that is bounded above, give its supremum. For each set that is bounded below, give its infimum.

11. What is the primary difference between the *set* $\{1, 2, 3, 4, 5, \ldots\}$ and the *sequence* $1, 2, 3, 4, 5, \ldots$?

12. Let $A = \{1/n \in \mathbb{R} \colon n = 1, 2, 3, \ldots\}$.

    (a) Is A bounded above? If so, what is $\sup A$?

    (b) Is A bounded below? If so, what is $\inf A$?

13. Write out the first five terms of the sequence $((-1)^n)_{n=0}^{\infty}$. Describe its convergence/divergence.

14. Write out the first five terms of the sequence $((-1)^n(2n+1))_{n=4}^{\infty}$. Describe its convergence/divergence.

15. Write out the first five terms of the sequence $((-1)^n/n)_{n=1}^{\infty}$. Describe its convergence/divergence.

16. What is the infimum and supremum of the set $\{1 - \frac{1}{n+1} : n \in \mathbb{N}\}$? Does the set have a maximum? Does the set have a minimum?

17. Let $f(x) = \frac{x}{x}$. Is $f(0)$ defined? Evaluate $\lim_{x \to 0} f(x)$. Plot the graph of f over the interval $[-1, 1]$.

18. Let $f(x) = \frac{x^2}{1-x^2}$. Find all values of $a \in \mathbb{R}$ such that

    a) $\lim\limits_{x \to a^+} f(x) = \infty$

    b) $\lim\limits_{x \to a^-} f(x) = \infty$

    c) $\lim\limits_{x \to a^+} f(x) = -\infty$

    d) $\lim\limits_{x \to a^-} f(x) = -\infty$

    [Hint: find the vertical asymptotes.]

19. Graph a function such that $f(0) = 0$, $\lim\limits_{x \to 0} f(x) = 1$, $\lim\limits_{x \to -\infty} f(x) = 0$, and $\lim\limits_{x \to \infty} \dfrac{f(x)}{x} = 1$

20. While it is true that if $f(x) \to L_1$ and $g(x) \to L_2$ as $x \to x_0$, then $(f + g)(x) \to L_1 + L_2$, the converse is not always true. Find functions f and g such that $\lim\limits_{x \to 0}(f(x) + g(x))$ exists but neither $\lim\limits_{x \to 0} f(x)$ nor $\lim\limits_{x \to 0} g(x)$ exist. In this context, "exists" means that the limit value is a real number (i.e., not $\pm\infty$).