1) $(a+b)(a+b) = (a+b)\cdot a + (a+b)\cdot b$ (dist.)

$= a^2+ba+ab+b^2$ (dist.)

$= a^2+2ab+b^2$ (comm.)

2) Even

3) $2m+2n = 2(m+n)$

$2m+(2n+1) = 2(m+n)+1$

$(2m+1)+(2n+1) = 2(m+n+1)$

4) $\frac{a}{b} < \frac{n}{m} \iff am < nb$

a) $r := \frac{1}{2}\left(\frac{a}{b}+\frac{n}{m}\right) = \frac{am+nb}{2bm}$

b) $r = \frac{am+nb}{2bm} > \frac{2am}{2bm} = \frac{a}{b} = p \Rightarrow r>p \Rightarrow p<r$

c) $r = \frac{am+nb}{2bm} < \frac{2nb}{2bm} = \frac{n}{m} = q \Rightarrow r<q$

d) Suppose $S>0$ was the smallest pos. rational number. Then $t = \frac{1}{2}S$ is rational and $0<t<S$, so t would be the smallest rational number. This is a contradiction, so no such S can exist. ∎

5) a) Let $n \in \mathbb{Z}$. Then $n^2 \in \mathbb{Z}$ as well and must be either even or odd.

Case 1) n is even $\Rightarrow n=2k \Rightarrow n^2 = 4k^2$ is also even.

Case 2) n is odd $\Rightarrow n=2k+1 \Rightarrow n^2 = 4k^2+4k+1 = 2(2k^2+2k)+1$ is also odd.

Thus n and n^2 have the same even/oddness $\forall n \in \mathbb{Z}$.

b) Let $\sqrt{2} = n/m$ with n, m co-prime and $m>0$. Then $2 = n^2/m^2 \Rightarrow n^2 = 2m^2$ so n^2 is even. Thus n is even and we have $n=2k$ for some $n \in \mathbb{Z}$. Now $2 = n^2/m^2 = 4k^2/m^2 \Rightarrow m^2 = 2k^2$ so m^2 is even as well.

c) We assumed that n/m was simplified (i.e., n & m have no common factors), but we found that if $2 = (n/m)^2$, then n^2 and m^2 must both be even. From a), this means that n & m are both even, and therefore have a common factor of 2. This is a contradiction, as n/m was assumed to be simplified, so no such fraction exists. ∎

6) Def. (Minimum)

Let $A \subset \mathbb{R}$. Then $a \in A$ is the minimum of A if $a \le x \ \forall x \in A$.

Def. (Bounded Below & lower bound)

Let $A \subset \mathbb{R}$. We say A is bounded below if $\exists M \in \mathbb{R}$ s.t. $M \le x \ \forall x \in A$.

In such a case, M is a lower bound of A.

Def. (Infimum)

Let $A \subset \mathbb{R}$. We say $a \in \mathbb{R}$ is the infimum of A (written $\inf A = a$) if

1) a is a lower bound of A and
2) if M is another lower bound of A, then $M \le a$.

7) Any interval of the form $[a, b)$. $\{1 - 1/n : n = 1, 2, 3, \dots\}$ would also work.

8) x is irrational. If x were rational, its decimal expansion would have a finite block of digits that repeat.

9) $(12, 5, 13)$ is another Pythagorean triple. To see this, observe that
$12^2 + 5^2 = 144 + 25 = 169 = 13^2$. Dividing by 13^2, we see

$$\frac{12^2}{13^2} + \frac{5^2}{13^2} = 1 \implies \left(\frac{12}{13}\right)^2 + (5/13)^2 = 1 \implies (12/13, 5/13)$$ is a rational point on the unit circle.

10)
$A = (0,1) = \{x \in \mathbb{R} : 0 < x < 1\}$ $\inf A = 0$, $\sup A = 1$

$B = (0,1] = \{x \in \mathbb{R} : 0 < x \le 1\}$ $\inf A = 0$, $\sup A = 1$

$C = [-5, \infty) = \{x \in \mathbb{R} : -5 \le x\}$ $\inf A = -5$

$D = (-\infty, 3) = \{x \in \mathbb{R} : x < 3\}$ $\sup A = 3$

11) The set $\{1, 2, 3, \dots\}$ is unordered but the sequence $1, 2, 3, \dots$ is ordered.

12) Let $A = \{1/n \in \mathbb{R} : n = 1, 2, 3, \dots\}$.

a) A is bounded above. $\sup A = 1$

b) A is bounded below. $\inf A = 0$

13) $((-1)^n)_{n=0}^{\infty} = 1, -1, 1, -1, 1, \dots$ diverges (oscillates)

14) $((-1)^n(2n+1))_{n=4}^{\infty} = 9, -11, 13, -15, 17, \dots$ diverges (oscilates to $\pm\infty$)

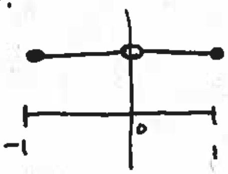
15) $((-1)^n/n)_{n=1}^{\infty} = -1, 1/2, -1/3, 1/4, -1/5, \dots$ Converges to 0

16) Let $A = \{ 1 - \frac{1}{n+1} : n \in \mathbb{N} \}$.

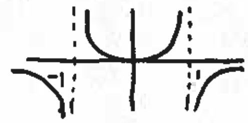
A has a supremum of 1 and an infimum of 0. [redacted]

A has no maximum but a minimum of 0.

17) $f(0)$ is not defined. $\lim_{x \to 0} \frac{x}{x} = \lim_{x\to 0} 1 = 1$.



18) Let $f(x) = \frac{x^2}{1-x^2} = \frac{x^2}{(1-x)(1+x)}$



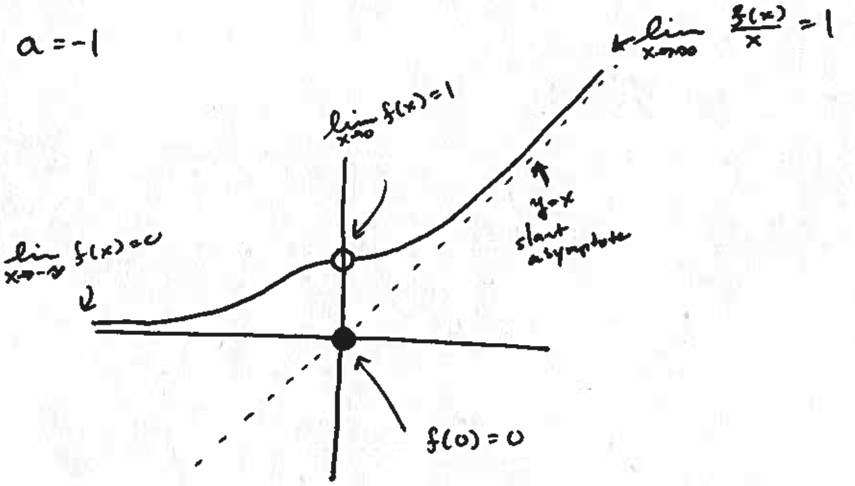
a) $a = -1$

b) $a = 1$

c) $a = 1$

d) $a = -1$

19)



20) Let $f(x) = 1/x$ and $g(x) = -1/x$. Then $\lim_{x \to 0} f(x)$ and $\lim_{x \to 0} g(x)$ do not exist but $\lim_{x \to 0} (f(x) + g(x)) = \lim_{x \to 0} \left(\frac{1}{x} - \frac{1}{x} \right) = \lim_{x \to 0} (0) = 0$.