MATH 164 Exam 2 Review

1) **Claim:** I_X is injective

a) **Proof:** Let not. Then $\exists\, x_1, x_2 \in X$ s.t. $I_X(x_1) = I_X(x_2)$ but $x_1 \neq x_2$. However $I_X(x_1) = x_1$ and $I_X(x_2) = x_2$, so we must have $x_1 = I_X(x_1) = I_X(x_2) = x_2$. This is a contradiction, so I_X must be injective. ∎

Proof by contradiction

Proof: Because $I_X(x) = x \;\forall\, x \in X$, we have that $I_X(x_1) = I_X(x_2)$ iff $x_1 = x_2$. Thus I_X is injective. ∎

Direct proof

b) **Claim:** I_X is surjective.

Proof: Pick any $x \in X$. Then $I_X(x) = x$ so $x \in \mathcal{R}(I_X)$. Thus $X \subset \mathcal{R}(I_X)$. ~~We always have~~ Because X is the co-domain of I_X, we also have $\mathcal{R}(I_X) \subset X$. Thus $X = \mathcal{R}(I_X)$ and we see that I_X is surjective. ∎

*c) I_X is bijective with $I_X^{-1} = I_X$.

$\ast$ 2) Let $f: X \to Y$ be bijective.

a) Function name is f

b) $\mathcal{D}(f) = X$

c) co-domain is Y

d) $\mathcal{R}(f) = Y$ (from surjective)

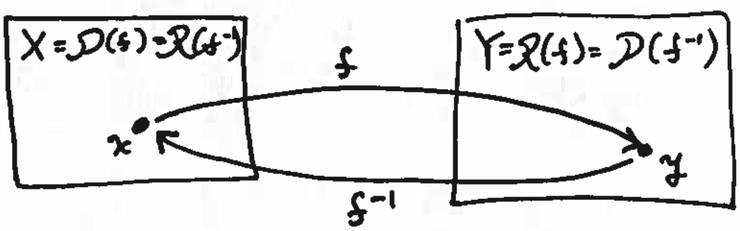
e) Inverse function is f^{-1} (from bijective)

f) $\mathcal{D}(f^{-1}) = Y$

g) co-domain of f^{-1} is X

h) $\mathcal{R}(f^{-1}) = X$

i) $f: X \to Y$, $f^{-1}: Y \to X$



3) Every function that uses the implied domain and range $f: \mathcal{D}(f) \to \mathcal{R}(f)$ uses the range as its co-domain. Thus the function is automatically onto/surjective and one-to-one/injective is all that is needed to check to verify that f is invertible/bijective.

4) Let $f : \mathbb{R} \to \mathbb{R}$ be increasing and invertible.

* Claim: f^{-1} is increasing.

* Example: Let $f(x) = 2x$. Then $f^{-1}(x) = \frac{1}{2}x$. Clearly both f and f^{-1} are increasing as they are lines with positive slope.

Proof: Pick any $y_1, y_2 \in \mathbb{R}$ such that $y_1 < y_2$. Then we must show that $f^{-1}(y_1) \leq f^{-1}(y_2)$.

Because f is invertible, $\exists$ unique $x_1, x_2 \in \mathbb{R}$ such that $f(x_1) = y_1$ and $f(x_2) = y_2$. Note that $f(x_1) = y_1 < y_2 = f(x_2)$.

Because f is increasing and $f(x_1) < f(x_2)$, we must have $x_1 < x_2$ (clearly $x_1 \neq x_2$ as $f(x_1) \neq f(x_2)$ and $x_2 < x_1$ would imply $f(x_2) \leq f(x_1)$, which is false).

Thus $f^{-1}(y_1) = x_1 < x_2 = f^{-1}(y_2)$, and we may conclude that f^{-1} is increasing. ■

* 5) Let $f(x) = \log_3 x$, $g(x) = 3^x$, $h(x) = x^3$

a) $\mathcal{D}(f) = (0,\infty)$ $\quad \mathcal{R}(f) = \mathbb{R}$

$\mathcal{D}(g) = \mathbb{R}$ $\quad \mathcal{R}(g) = (0,\infty)$

$\mathcal{D}(h) = \mathbb{R}$ $\quad \mathcal{R}(h) = \mathbb{R}$

b) $(f\circ g)(x) = f(3^x) = \log_3 3^x = x$

$\mathcal{D}(f\circ g) = \mathcal{R}(f\circ g) = \mathbb{R}$

$\Big\}\ f\circ g = I_{\mathbb{R}}$

identity function on $\mathbb{R}$ or $(0,\infty)$

c) $(g\circ f)(x) = g(\log_3 x) = 3^{\log_3 x} = x$

$\mathcal{D}(g\circ f) = \mathcal{R}(g\circ f) = (0,\infty)$

$\Big\}\ g\circ f = I_{(0,\infty)}$

d) $(h\circ g)(x) = h(3^x) = (3^x)^3 = 3^{3x} = (3^3)^x = 27^x$

$\mathcal{D}(h\circ g) = \mathbb{R}$ $\quad \mathcal{R}(h\circ g) = (0,\infty)$

e) $(g\circ h)(x) = g(x^3) = 3^{(x^3)}$

$\mathcal{D}(g\circ h) = \mathbb{R}$ $\quad \mathcal{R}(g\circ h) = (0,\infty)$

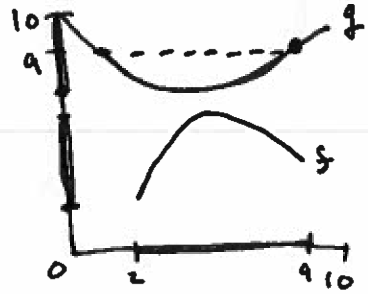
f) $(h\circ f)(x) = h(\log_3 x) = (\log_3 x)^3$

$\mathcal{D}(h\circ f) = (0,\infty)$ $\quad \mathcal{R}(h\circ f) = \mathbb{R}$

g) $(f\circ h)(x) = f(x^3) = \log_3(x^3) = 3\log_3 x$

$\mathcal{D}(f\circ h) = (0,\infty)$ $\quad \mathcal{R}(f\circ h) = \mathbb{R}$

* 6)



ND = Not Defined

a) $f(0) = ND$ $\quad f(5) = 6$ $\quad f(10) = ND$

b) $g(0) = 10$ $\quad g(5) = 7$ $\quad g(10) = 10$

c) $f(g(0)) = f(10) = ND$
$f(g(5)) = f(7) \approx 4.75$
$f(g(10)) = f(10) = ND$

d) $g(f(0)) = ND$
$g(f(5)) = g(6) \approx 7.25$
$g(f(10)) = ND$

e) No. Otherwise the graphs $y = f(x)$ and $y = g(x)$ would intersect at (x_0, y_0) where x_0 is a solution to $f(x) = g(x)$.

f) $\mathcal{D}(f) = [2, 9]$ $\quad \mathcal{R}(f) = [2, 6]$

g) $\mathcal{D}(g) = [0, 10]$ $\quad \mathcal{R}(g) = [7, 10]$

h) $\mathcal{D}(f \circ g) = \{x \in \mathcal{D}(g) \mid g(x) \in \mathcal{D}(f)\} = \{x \mid 0 \le x \le 10 \text{ and } 2 \le g(x) \le 9\}$
$= [1.25, 8.75]$

$\mathcal{R}(f \circ g) = (f \circ g)([1.25, 8.75]) = f(g([1.25, 8.75]))$ ← images
$= f([7, 9]) = [3, 4.75]$

i) $\mathcal{D}(g \circ f) = \{x \mid x \in \mathcal{D}(f) \text{ and } f(x) \in \mathcal{D}(g)\} = \{x \mid 2 \le x \le 9 \text{ and } 0 \le f(x) \le 10\}$
$= [2, 9]$

$\mathcal{R}(g \circ f) = g(f([2, 9])) = g([2, 6]) = [7, 8.5]$

* 6) cont.

j) $D(f \circ f) = \{x \mid x \in D(f) \text{ and } f(x) \in D(x)\} = \{x \mid 2 \le x \le 9 \text{ and } 2 \le f(x) \le 9\}$
$= [2, 9]$

$\mathcal{R}(f \circ f) = f(f([2,9])) = f([2,6]) = [2,6]$

k) $D(g \circ g) = \{x \mid x \in D(g) \text{ and } g(x) \in D(g)\} = [0,10]$

$\mathcal{R}(g \circ g) = g(g([0,10])) = g([7,10]) = [7.75, 10]$

l) f has a global min at $x=2$ and a global max at $x=5$

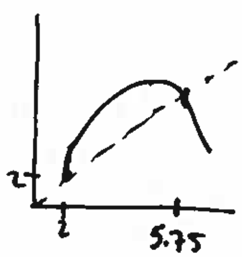
m) g has a global min at $x=5$ and two global [redacted] maxima at $x=0$ and $x=10$.

n) f has one local max at $x=5$ and no local min.

o) g has one local min at $x=5$ and no local max.

p) $f(x) \le g(x)$ when $2 \le x \le 9$

q)



$f(x) \ge x$ when $2 \le x \le 5.75$

7) Let $f:\mathbb{R}\to\mathbb{R}$ be strictly increasing.

a) *<u>Claim:</u> f is one-to-one.

<u>Proof:</u> Let not. Then $\exists\, x_1 \neq x_2$ such that $f(x_1)=f(x_2)$. Let x_1 be the smaller so that $x_1 < x_2$. However, because f is increasing, we must have $f(x_1) < f(x_2)$. Thus we cannot have $f(x_1)=f(x_2)$ and so f must be one-to-one. ■

*b) <u>Claim:</u> f need not be onto.

<u>Proof:</u> Consider $f(x)=2^x$. Then f is strictly increasing but $\mathcal{R}(f)=(0,\infty)\neq\mathbb{R}$, so f is not onto.

*8) a) $\log_2 x + \log_3 y + \log_4 z = 1 \to \log_3 y = 1-\log_2 x-\log_4 z$

$$\to y = 3^{1-\log_2 x-\log_4 z} = 3^1\cdot 3^{-\log_2 x}\cdot 3^{-\log_4 z}$$

$$\to y = \frac{3}{3^{\log_2 x}\, 3^{\log_4 z}} = \frac{3}{3^{\frac{\log_3 x}{\log_3 2}}\cdot 3^{\log_3 z/\log_3 4}}$$

$$\to \boxed{y = \frac{3}{x^{1/\log_3 2}\cdot z^{1/\log_3 4}}} \qquad (x,z>0)$$

b) $\log_2 x + \log_2 y + \log_2 z = 1 \to \log_2(xyz)=1 \to xyz=2$

$$\to \boxed{y = 2/xz} \qquad (x,z>0)$$

c) $\ln(x^y) + yx^2 = 1 \to y\ln x + yx^2 = 1 \to y(\ln x + x^2) = 1 \to \boxed{y = \frac{1}{x^2+\ln x}}$

* 9) a) $\log_{10} x = 50 \to \boxed{x = 10^{50}}$

b) $\log_4 x = 16 \to \boxed{x = 4^{16}}$

c) $e^{x^2} = 2 \to x^2 = \ln 2 \to \boxed{x = \pm\sqrt{\ln 2}}$ (note $\ln 2 > 0$ as $2 > 1$)

d) $e^{-x^2} = 2 \to -x^2 = \ln 2 \to \boxed{\text{no solution}}$

e) $\log_x 10 = 1 \to x^1 = 10 \to \boxed{x = 10}$

f) $\log_x 10 = 2 \to x^2 = 10 \to \boxed{x = \sqrt{10}}$

or use change of base $\log_x 10 = \dfrac{\ln 10}{\ln x}$

g) $(\log_3 x)^2 + 2\log_3 x + 1 = 0 \to (\log_3 x + 1)^2 = 0$

$\to \log_3 x = -1 \to x = 3^{-1} \to \boxed{x = 1/3}$

h) $(e^x)^2 + 2e^x + 1 = 4 \to (e^x + 1)^2 = 4 \to e^x + 1 = \pm 2 \to e^x = -1 \pm 2$

$\to e^x = 1 \to \boxed{x = 0}$

i) $x^2 = 10 \to \boxed{x = \pm\sqrt{10}}$

j) $\log_3 x = \log_9 x \to$ $\log_3 x = \dfrac{\log_3 x}{\log_3 9} \to \log_3 x = \dfrac{\log_3 x}{2}$

$\to \log_3 x = 0 \to \boxed{x = 1}$

k) $\log_3 x = \log_9 x^2 \to \log_3 x = 2\log_9 x = 2\dfrac{\log_3 x}{\log_3 9} = \log_3 x$

All $x > 0$ solve the equation.

10) $f(x) = \dfrac{a}{1+be^x}$, $a>0$, $b<0$

a) as $x \to \infty$, $e^x \to \infty$, so $\underline{1+be^x \to -\infty}$ as $b<0$

b) as $x \to -\infty$, $e^x \to 0$, so $\underline{1+be^x \to 1}$

c) as $x \to \infty$, $\underline{\dfrac{a}{1+be^x} \to 0}$ as $1+be^x \to -\infty$

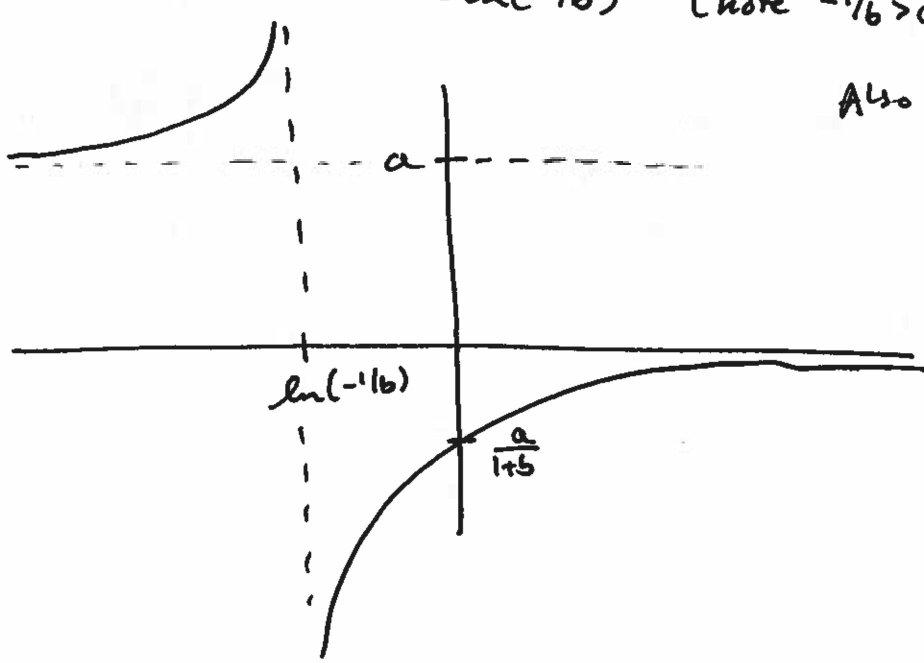
as $x \to -\infty$, $\underline{\dfrac{a}{1+be^x} \to a}$ as $1+be^x \to 1$

The two horizontal asymptotes are at $y=0$ and $y=a$

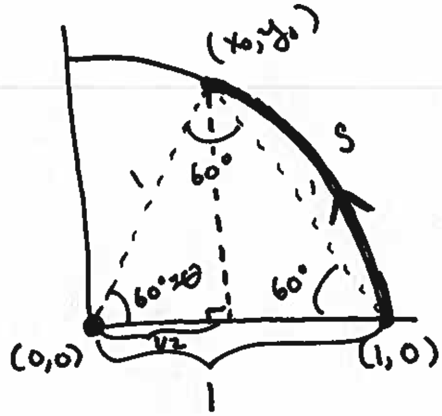
d) Note there is a vertical asymptote when $1+be^x = 0 \to e^x = -1/b$

$\to x = \ln(-1/b)$ (note $-1/b > 0$ as $b<0$).

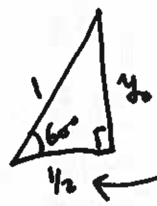
Also $f(0) = \dfrac{a}{1+b}$



11)



a)



$x_0 = 1/2$ from splitting the equilateral triangle into two isosceles triangles.

$$(1/2)^2 + y_0^2 = 1^2 \to y_0 = \sqrt{1 - 1/4} = \sqrt{3/4} = \sqrt{3}/2 \to \boxed{(1/2, \sqrt{3}/2)}$$

b) $\frac{60°}{180°} = \frac{\theta}{\pi} \to \frac{1}{3} = \frac{\theta}{\pi} \to \theta = \pi/3 \to \boxed{S = 1 \cdot \theta = \pi/3}$

c) 60°

d)

$\frac{1}{2}A = \frac{(\frac{1}{2})(\sqrt{3}/2)}{2} \to \boxed{A_{\text{Triangle}} = \sqrt{3}/4}$

$\to \boxed{A_{\text{Sector}} = \frac{1}{2} r^2 \theta = \pi/6}$

e) i) $\boxed{\pi/3 \text{ cm/s}}$ ii) $\boxed{\pi/3 \text{ rad/s}}$ ← or $\pi/3$ 1/s

iii) $\frac{\pi}{3} \frac{\text{rad}}{\text{s}} \cdot \frac{\text{rev}}{2\pi \text{ rad}} \cdot \frac{60\text{s}}{\text{min}} = \frac{60}{6}$ rev/min = 10 rpm

12) 1 rpm speed, −1 rpm velocity → ccw is the positive direction