MATH 164 Review (B)

Problems

1. Verify the following identities.

   (a) $\tan(\alpha \pm \beta) = \dfrac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$

   (b) $\dfrac{\sin\theta + \sin 3\theta}{2\sin 2\theta} = \cos\theta$

   (c) $\dfrac{\cos\theta - \cos 3\theta}{\sin\theta + \sin 3\theta} = \tan\theta$

   (d) $\dfrac{\cos\alpha + \cos\beta}{\cos\alpha - \cos\beta} = -\cot\left(\dfrac{\alpha+\beta}{2}\right)\cot\left(\dfrac{\alpha-\beta}{2}\right)$

2. Write $\cos^4(\theta)$ as a sum of trig functions with no exponents.

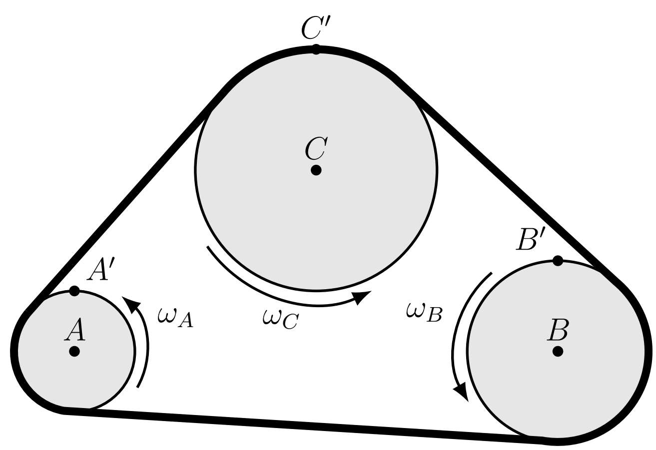
3. Consider the belt system shown below (not drawn to scale). Assume that the belt does not slip on any wheel. Let pulley A, B, and C have radii r_A, r_B, and r_C respectively and angular speed ω_A, ω_B, and ω_C respectively. Suppose $r_B = 2r_A$ and $r_C = 3r_A$. At $t = 0$ [s], points A', B', and C' are all at 12-o'clock on their respective pulleys (as shown, assume that the points are physically marked on each pulley). Let $\omega_A = 60$ [rpm] and $r_A = 6$ [in].

   (a) Find the *linear* speeds of A', B', and C'. [Hint: it is important that the belt is not slipping.]

   (b) Find ω_B and ω_C.

   (c) In the time that it takes pulley A to make one revolution, how many degrees have the pulleys B and C turned?

   (d) When is the first time that point A' returns to its starting position? Where are B' and C' when this happens? Answer with an angle measured from the center of their respective pulley (they both start at 90°).



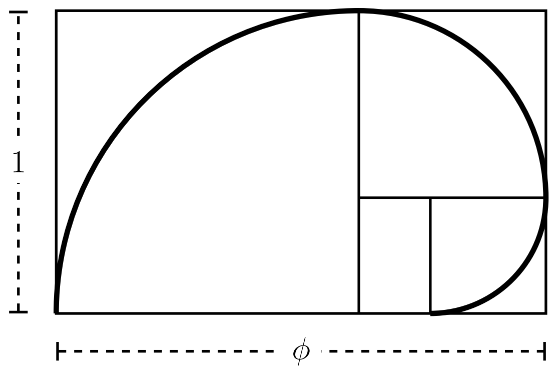
   (e) What is the first time ($t > 0$) that A' and B' are *simultaneously* at 12-o'clock? How far (in *revolutions*) have each of the three points traveled?

   (f) What is the first time ($t > 0$) that A' and C' are *simultaneously* at 12-o'clock? How far (in *revolutions*) have each of the three points traveled?

   (g) What is the first time ($t > 0$) that A', B', and C' are *all simultaneously* at 12-o'clock? How far (in *revolutions*) have each of the three points traveled?

4. Consider a rectangle with side lengths a and $a+b$ where $a, b > 0$. Suppose that the rectangle can be cut into a square (size $a \times a$) and smaller rectangle (size $(a \times b)$ such that the two rectangles are similar (i.e., the ratio of the long side to the short side is equal). What is the ratio a/b? Find an exact expression and approximate to 5 decimal places. [Hint: write an equation in terms of a and b. Set $x = a/b$ and re-arrange into a quadratic equation. Complete the square.]

5. Consider a sequence of rectangles in proportion to the golden ratio $\phi = \frac{1+\sqrt{5}}{2}$ as drawn below. Each rectangle can be split into a square and a similar rectangle and a quarter circle drawn in the square. The first three iterations of this process are drawn below. Continuing the process produces a curve called the golden spiral.



Find the length of the portion of the golden spiral drawn above. [Hint: show that $\phi - 1 = \frac{1}{\phi}$ first to simplify your calculations.] [Bonus: Find an elagent way to write your answer in terms of ϕ and make a conjecture about what the length would be if the curve was exteneded.]

6. Evaluate the following operations in $\mathbb{C}$ and simplify. Here $i = \sqrt{-1}$.
   (a) $(1+5i)+(-4+2i)$
   (b) $(-3i)-(4-5i)$
   (c) $(1+3i)(4-2i)$
   (d) $(5+8i)/i$
   (e) $(4-2i)/(1+2i)$
   (f) $e^{i\pi}+1$
   (g) $e^{i\pi/2}$
   (h) $\left(e^{i\pi/4}\right)^2$

7. Convert each complex number in cartesian $(a+bi)$ form into exponential $(re^{i\theta})$ form. Restrict θ to $-\pi < \theta \leq \pi$.
   (a) $1+i$
   (b) $\sqrt{3}/2 - i/2$
   (c) $4\cos(17\pi/12) + 4\sin(17\pi/12)i$
   (d) $-3-5i$

8. Convert each complex number from exponential form into cartesian form. Here $\exp(x) = e^x$.
   (a) $2\exp(\pi i)$
   (b) $2\exp\left(\frac{\pi}{2}i\right)$
   (c) $6\exp\left(-\frac{5\pi}{6}i\right)$
   (d) $2\exp\left(\frac{3\pi}{4}i\right)$

9. Determine which of the following statements are true, and which are false. Recall that if $z = a + bi$, then $\overline{z} = a - bi$. For every true statement provide a short proof. For every false statement, provide a counterexample.
   (a) If $z = a + bi$, then $z\overline{z} = a^2 + b^2$
   (b) If $z = a + bi$, then $z + \overline{z} = b$
   (c) If $z = a + bi$, then $(z - \overline{z})/i = b$
   (d) If $z = re^{i\theta}$, then $\overline{z} = re^{-i\theta}$

10. How are processes of rationalizing the denominator and dividing complex numbers (in cartesian form) the same? Rationalize $\frac{2}{1+\sqrt{5}}$ and divide $2/(1+\sqrt{5}i)$ to help demonstrate your explanation.

11. Graph the following relations in $\mathbb{C}$. Assume a and b are real numbers.
   (a) $\{z \in \mathbb{C} \mid |z| = 1\}$
   (b) $\{a + bi \mid a = 1\}$
   (c) $\{a + bi \mid a + b = 1\}$
   (d) $\{re^{i\theta} \mid \theta = \pi/6\}$
   (e) $\{re^{i\theta} \mid r = 4\}$

12. Solve each equation for x. Include complex answers.
   (a) $x^2 - 1 = 0$
   (b) $x^2 + 1 = 0$
   (c) $x^2 + 6x + 1 = 0$
   (d) $3x^2 - 2x + 1 = 0$

13. Solve each equation for all possible real-valued solutions. If there are no solutions, state this.
   (a) $\cos^2\theta + 5\sin^2\theta = 0$
   (b) $\cos^2\theta - \sin^2\theta = 1/2$
   (c) $\sqrt{1+\cos\theta} = \sin\theta$
   (d) $e^{\tan\theta} = 1$
   (e) $\ln\ln x = 1$
   (f) $3\cos\theta + 4\sin\theta = 0$ [Hint: combine into a single sine wave.]

14. Use trig identities and the standard unit circle to evaluate each expression exactly.
   (a) $\sin\dfrac{\pi}{12}$
   (b) $\cos\dfrac{\pi}{12}$
   (c) $\tan\dfrac{5\pi}{12}$
   (d) $\sin\dfrac{\pi}{24}$

15. Convert each trigonometric expression into an algebraic expression.
   (a) $\sin(\arccos x)$
   (b) $\cos(\arctan(1/x))$ for $x > 0$ [Bonus: make your expression work for all $x \neq 0$.]
   (c) $\sin(\arctan(1/x))$ for $x > 0$ [Bonus: make your expression work for all $x \neq 0$.]

16. Plot $y = \arccos(\cos x)$ over the interval $-\pi < x \leq \pi$.