Integrals

Prerequisite Concepts:

1. Knowledge of functions $f\colon I \to \mathbb{R}$ with $I \subset \mathbb{R}$ an interval
2. Continuity
3. Derivatives
4. Limits
5. Area
6. Infimum and Supremum

Basic Concepts:

1. Signed area
2. Anti-derivatives
3. Summation notation

Advanced Concepts:

1. Riemann integral
2. First and second fundamental theorems of calculus

1 Motivation

Suppose we have a container of water. Over time, water is added to the container. Suppose $V(t)$ is the volume of water in the container at time t and $t = 0$ represents the time at which water was started to be added to the container. Then $V(0)$ would be the volume of water that was initially in the container. The derivative $\dot{V}(t) = \frac{\mathrm{d}}{\mathrm{d}\,t}V(t)$ measures how fast the volume of water in the container is changing at time t. If water is being added, then $V(t)$ is increasing and $\dot{V}(t) > 0$. If $V(t)$ is known over some interval $0 \le t \le T$, then we can find $\dot{V}(t)$ by taking the derivative. Although we do not *yet* have a tool to do so, it is reasonable to expect that if we know both the initial volume of water (i.e., $V(0)$) and the rate at which the volume is changing (i.e., $\dot{V}(t)$), we can recover the volume at T by tracking how much volume accumulates over the interval $[0, T]$. This technique used to accumulate $\dot{V}$ over $[0, T]$ to get $V(T)$ is *integration*.

Example 1.1: Suppose the volume of water is given by $V(t) = rt$ where r is constant. Observe the following:

1. $\dot{V}(t) = r$
2. The units of V are volume
3. The units of t are time
4. The units of $\dot{V}$ and r are both volume per time
5. If ΔV is the change in volume over a time interval of width Δt, then $\Delta V = r\Delta t$.

Example 1.2: Suppose water is being added at a constant rate r (i.e., $\dot{V}(t) = r$ with r constant). Then we still have that $\Delta V = r\Delta t$ over *any* interval of width Δt. **Note that $r\Delta t$ is the area under the graph of $\dot{V}(t) = r$ over some interval $[t_0, t_0 + \Delta t]$.** In particular, $\Delta V = rT$ where $\Delta V = V(T) - V(0)$ is the net change of volume over the interval $[0, T]$. See Figure 1.

Example 1.3: Suppose water is being added at a variable rate that decreases linearly to 0 at T (i.e., $\dot{V}(t) = r(1 - t/T)$ for r and T constants). If we make a leap of faith from the previous example and *assume* that the net change in volume, ΔV, over some interval I is given by the area under the graph of $\dot{V}(t)$ above I, then we can compute ΔV over $[0, T]$ using geometry. Using the area of a triangle, we obtain $\Delta V = rT/2$. See Figure 1.

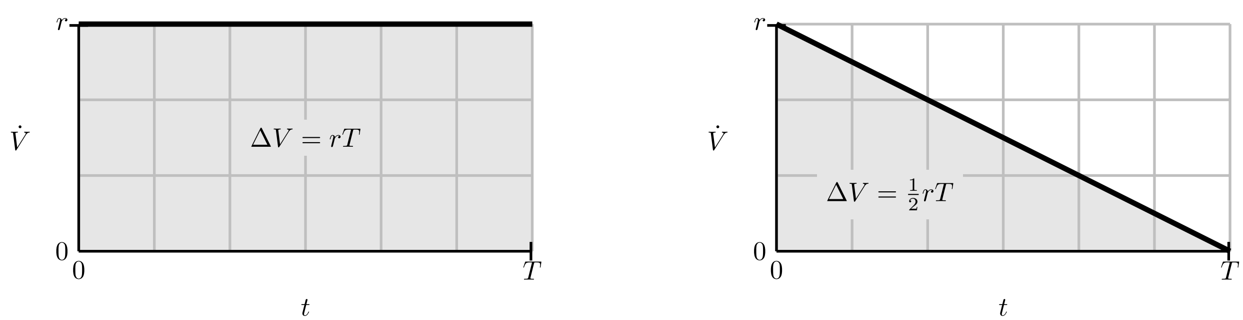


Figure 1: Left: diagram for Example 1.2. Right: diagram for Example 1.3.

2 Area Under a Continuous Curve

Let $f\colon [a,b] \to \mathbb{R}$ be a continuous function. Let us *assume* that there exists a function $\mathcal{A}\colon [a,b] \to \mathbb{R}$ such that $\mathcal{A}(x)$ is the area under f over the interval $[a,x]$, as depicted in Figure 2.

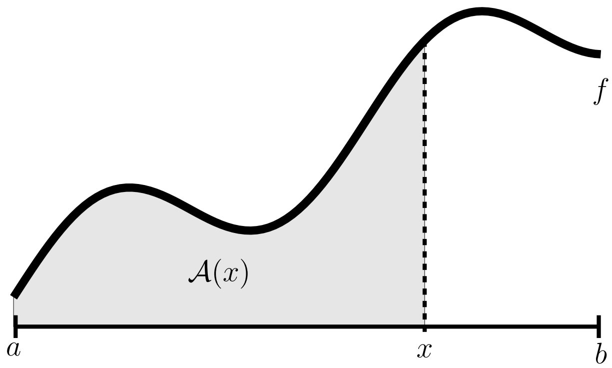


Figure 2: An area function.

Note that we allow f to have negative values, so $\mathcal{A}$ represents the *signed area* under f. Using our intuition, we should expect the following properties.

Properties of the area function:

1. $\mathcal{A}(a) = 0$
2. If $f(x) > 0$, then $\mathcal{A}$ accumulates positive area and is increasing
3. If $f(x) < 0$, then $\mathcal{A}$ accumulates negative area and is decreasing

Example 2.1: Let $f\colon [a,b] \to \mathbb{R}$ be given by $f(x) = c$. Then $\mathcal{A}(x) = c(x-a)$ is the area of the rectangle with height c and width $x - a$.

Example 2.2: Let $f\colon [0,10] \to \mathbb{R}$ be given by $f(x) = x$. Then $\mathcal{A}(x) = \frac{1}{2}x^2$ is the area of the right triangle with length and height both equal to x.

Remark: It is not strictly necessary that the domain of f is a closed and bounded in order to construct the area function $\mathcal{A}$.

The properties posited above can be proved with the following theorem.

Theorem 2.1: *Let $f\colon [a,b] \to \mathbb{R}$ be continuous and $\mathcal{A}\colon [a,b] \to \mathbb{R}$ its area function. Then $\mathcal{A}$ is differentiable and $\mathcal{A}'(x) = f(x)$.*

Proof. Fix any $x \in [a,b]$. Let $\Delta x \neq 0$ such that $x + \Delta x \in [a,b]$ and let I be the closed interval between x and $x + \Delta x$ (i.e., either $I = [x, x+\Delta x]$ or $I = [x + \Delta x, x]$, depending on the sign of Δx). Because f is continuous and I is closed and bounded, f attains both its maximum and minimum values over I. Thus there exist values $c, C \in I$ such that

$$f(c) = \min_{t \in I} f(t)$$

and

$$f(C) = \max_{t \in I} f(t),$$

as illustrated in Figure 3. Observe that c and C depend on Δx and that both $c \to x$ and $C \to x$ as $\Delta x \to 0$. Setting $\Delta\mathcal{A} = \mathcal{A}(x + \Delta x) - \mathcal{A}(x)$, we must have that

$$f(c)\Delta x \leq \Delta\mathcal{A} \leq f(C)\Delta x$$

so that

$$f(c) \leq \frac{\Delta\mathcal{A}}{\Delta x} \leq f(C).$$

Taking $\Delta x \to 0$ and using the continuity of f as well as the squeeze theorem, we see that

$$\lim_{\Delta x \to 0} \frac{\Delta\mathcal{A}}{\Delta x} = f(x),$$

which of course is the definition of $\mathcal{A}'(x) = f(x)$. $\square$

COROLLARY 2.1: *If f has an antiderivative F on $[a, b]$, then $\mathcal{A}(x) = F(x) - F(a)$.*

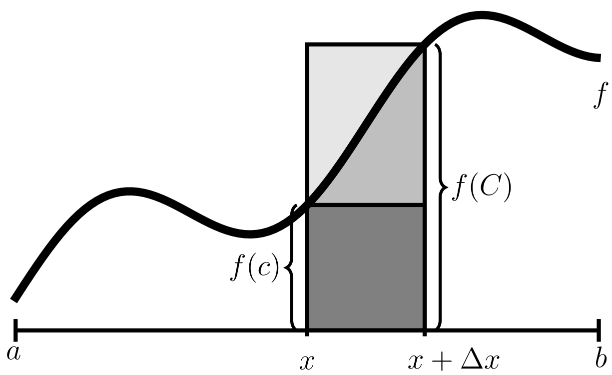


Figure 3: Upper and lower estimates for the area difference $\Delta\mathcal{A}$ over an interval $[x, x + \Delta x]$.

> REMARK: In the language of differentials, we have $\mathrm{d}\,\mathcal{A} = f(x)\,\mathrm{d}\,x$ where $\mathrm{d}\,\mathcal{A}$ is the differential area and $\mathrm{d}\,x$ is the differential width. See Figure 4. It is often useful to visualize the area function $\mathcal{A}(x)$ as the area swept out by the differential $\mathrm{d}\,\mathcal{A}$ as a dummy variable t increases from a to x, or as an (uncountably infinite) sum of differential areas over the interval $[a, x]$.

It turns out that requiring f to be continuous is more strict than is truly necessary in order for the area function $\mathcal{A}$ to exist. For example, if f has finitely many jump discontinuities in the interval $[a, b]$, then we could construct $\mathcal{A}$ as above on each of the sub-intervals on which f is continuous. In some cases, f is allowed to have infinitely many discontinuities, and the area function will still exist. This extension requires the definition of the Riemann integral, which in turn requires us to understand Sigma notation.

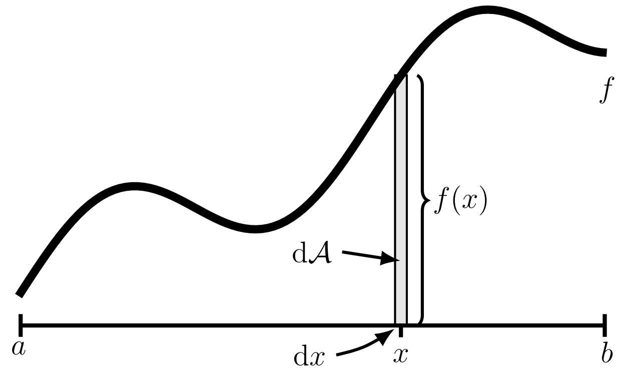


Figure 4: Depiction of the area differential $\mathrm{d}\,\mathcal{A} = f(x)\,\mathrm{d}\,x$.

3 Sigma Notation

Sigma notation (or summation notation) is a compact method for writing down the sum of many numbers. For example, if we wished to write down the sum of all the integers squared from 1 to 100, this would take a large amount of space. Because there is an obvious pattern, we can get away with writing

$$1^2 + 2^2 + 3^2 + \ldots + 99^2 + 100^2.$$

If the pattern was less obvious, or we wish to be more precise, we can write this sum using sigma notation. To use sigma notation we define a pattern (k^2 in this example), a starting place/index ($k = 1$ in this example), and an ending place/index ($k = 100$ in this example). The Greek capital letter Σ (the lowercase is σ and is the analogue for the Latin letter "S" for sum) is used to put everything together. The index variable and starting value goes under the symbol, the ending index goes above, and the pattern goes to the right. Our example sum is then written

$$\sum_{k=1}^{100} k^2.$$

As an aside, the same notation is used for products with the capital Greek letter Π (the lowercase is π and is the analogue for the Latin letter "P" for product) replacing Σ.

Definition 3.1 (Sigma Notation): *Let $a_m, a_{m+1}, \ldots, a_n$ be real numbers. Then*

$$\sum_{k=m}^{n} f_k = f_m + f_{m+1} + \ldots + f_n.$$

Here k is the index *of the sum and ranges from its starting value of $m \in \mathbb{Z}$ to its ending value of $n \in \mathbb{Z}$.* The index is always integer-valued. *If the summation bounds are understood (by context), then the notation $\sum a_k$ may be used as a shorthand as the sum of a_k over all valid indices. By convention, if $n < m$, then $\sum_{k=m}^{n} a_k = 0$.*

Example 3.1: Let $a_k = 1/2^k$. Then

$$\sum_{k=0}^{3} a_k = 1 + 1/2 + 1/4 + 1/8 = (8 + 4 + 2 + 1)/8 = 15/8$$

and

$$\sum_{k=1}^{3} a_k = 1/2 + 1/4 + 1/8 = 7/8.$$

Example 3.2: Let $a_k = 2k+1$. Then

$$\sum_{k=0}^{4} a_k = 1+3+5+7+9 = 25.$$

Example 3.3:

$$\sum_{k=-3}^{3} k = -3-2-1+0+1+2+3 = 0$$

Example 3.4:

$$\sum_{k=100}^{100} k = 100$$

Because we are only concerned with sums of a finite number of terms, we have the following observations.

Important Observations:

1. $\sum a_k \pm \sum b_k = \sum (a_k \pm b_k)$ **if each sum has the same index range**
2. $\sum (ca_k) = c \sum a_k$
3. $\displaystyle\sum_{k=p}^{q} a_k + \sum_{k=q+1}^{r} a_k = \sum_{k=p}^{r} a_k$ where $p \le q \le r$ are integers.

Given a sequence a_k for $k \in \mathbb{N}$, we often wish to define a cumulative sum function $f(n) = \sum_{k=0}^{n} a_k$.

Theorem 3.1 (Special Sums): *The following sums can be written as an explicit function of their upper index.*

1) $\displaystyle\sum_{k=1}^{n} c = nc$ *for any constant* $c \in \mathbb{R}$

2) $\displaystyle\sum_{k=1}^{n} k = \frac{n(n+1)}{2}$

3) $\displaystyle\sum_{k=1}^{n} (2k-1) = n^2$

4) $\displaystyle\sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a}$ *for any* $a \notin \{0,1\}$

Claim 1) is trivially true, so we omit its proof.

Proof of Claim 2). Fix any $n \in \mathbb{N}$ and set $S = \sum_{k=1}^{n} k = 1+2+3+\ldots+n$. Note that we could reverse the order of addition to write $S = n+(n-1)+\ldots+3+2+1$. Adding both the forward and reverse orders together results in

$$\begin{aligned} S+S &= 1+2+3+\ldots+(n-1)+n \\ &\quad +n+(n-1)+(n-2)+\ldots+2+1 \end{aligned}$$

so that

$$\begin{aligned} 2S &= (n+1) + (n+1) + \ldots + (n+1) + (n+1) \\ &= \sum_{k=1}^{n} (n+1) \\ &= n(n+1), \end{aligned}$$

which finally gives us $S = \dfrac{n(n+1)}{2}$. $\square$

Proof of Claim 3). We will prove this claim by induction. First, note that the base case when $n = 1$ is true. Next, assume that our claim is true for each $n \leq N$ where $N \geq 1$ is an integer. We now show that our claim must also be true for $n = N + 1$. Observe that

$$\begin{aligned} \sum_{k=1}^{N+1} (2k-1) &= \sum_{k=1}^{N} (2k-1) + (2N+1) && \text{(split the sum)} \\ &= N^2 + 2N + 1 && \text{(claim for } n = N \text{ is true)} \\ &= (N+1)^2 && \text{(factor),} \end{aligned}$$

so our claim being true for $n = N$ implies our claim is also true for $n = N + 1$. Because our claim is true for $n = 1$, it must also be true for $n = 2, 3, 4, \ldots$, and in fact true for all positive integers n. $\square$

Proof of Claim 4). Fix any valid a, $n \in \mathbb{N}$, and set $S = \sum_{k=0}^{n} a^k = 1 + a + a^2 + \ldots + a^n$. Observe that

$$aS = a + a^2 + a^3 + \ldots + a^n + a^{n+1} = S - 1 + a^{n+1}.$$

Re-arranging $aS = S - 1 + a^{n+1}$ to solve for S, we see that $S = (1 - a^{n+1})/(1 - a)$. $\square$

Sometimes the index for the sum is called a *dummy variable* as it only has an intermediate role (e.g., the sum $\sum_{k=1}^{5} k$ does not depend on k). It is sometimes helpful to *re-index* a sum to a more simplified form. When the sum is re-indexed, it is essential that both the number of terms and the value of the terms do not change.

Example 3.5: Observe that $\displaystyle\sum_{k=0}^{5} (k+3)$ and $\displaystyle\sum_{k=3}^{8} k$ both evaluate to $3 + 4 + 5 + 6 + 7 + 8 = 33$.

> Theorem 3.2 (Re-indexing): *Let $j \in \mathbb{Z}$. Then the sums $\displaystyle\sum_{k=0}^{n} a_k$ and $\displaystyle\sum_{k=j}^{n+j} a_{k-j}$ are the same.*

Proof. Both sums evaluate to $a_0 + a_1 + a_2 + \ldots + a_n$. $\square$

4 The Riemann Integral

Let us return to the problem of finding the area under some function $f : [a, b] \to \mathbb{R}$. Before we proceed, we need to clarify some terminology.

4.1 Partitions and Sums

Definition 4.1 (Partition): *A* partition*, P, of the interval $[a, b]$ is a finite collection of points*

$$\{x_0, x_1, x_2, \ldots, x_{n-1}, x_n\} \subset [a, b]$$

such that

$$a = x_0 < x_1 < x_2 < \ldots < x_{n-1} < x_n = b.$$

A partition with $n + 1$ points naturally creates n sub-intervals $I_k = [x_{k-1}, x_k]$ with width $\Delta x_k = x_k - x_{k-1}$ for $k = 1, 2, 3, \ldots, n$ with disjoint interiors and whose union is all of $[a, b]$. We measure the size of P by $||P|| = \max\limits_n \Delta x_n$. If all Δx_k are equal, then we say that P is a uniform *partition.*

Definition 4.2 (Tagged Partition): *A* tagged partition*, $(P, \{c_k\})$ of $[a, b]$ is a partition of $[a, b]$ with each sub-interval has a sample point $c_k \in I_k$ chosen.*

With this terminology, we may now introduce three different sums to approximate the area under f.

Definition 4.3 (Lower and Upper Sums): *Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Given a partition P of $[a, b]$, set*

$$m_k = \inf_{x \in I_k} f(x)$$

and

$$M_k = \sup_{x \in I_k} f(x).$$

Then

$$L(f, P) = \sum_{k=1}^{n} m_k \Delta x_k \tag{1}$$

is the lower sum *of f with respect to P, and*

$$U(f, P) = \sum_{k=1}^{n} M_k \Delta x_k \tag{2}$$

is the upper sum *of f with respect to P. See Figure 5.*

Remark: Recall that $f : [a, b] \to \mathbb{R}$ is *bounded* if there exist numbers $m, M \in \mathbb{R}$ such that

$$m \le f(x) \le M$$

for all $x \in [a, b]$. This is required to ensure that the values m_k and M_k exist in the upper and lower sums. However, if f is continuous we can say

$$m_k = \min_{x \in I_k} f(x)$$

and

$$M_k = \max_{x \in I_k} f(x).$$

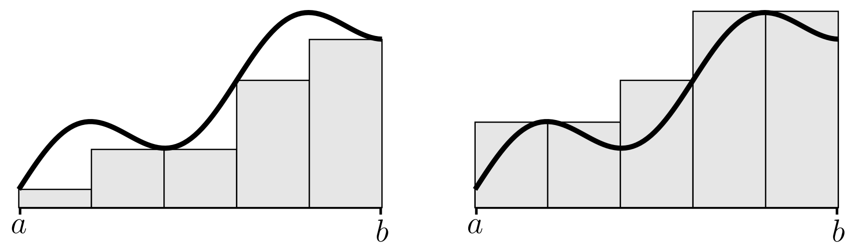


Figure 5: A partition with $n = 5$ sub-intervals of equal width. Left: the lower sum of f. Right: the upper sum of f.

Definition 4.4 (Riemann Sum): *Let* $f\colon [a, b] \to \mathbb{R}$. *Given a tagged partition* $(P, \{c_k\})$ *of* $[a, b]$ *the sum*

$$R(f, P) = \sum_{k=1}^{n} f(c_k)\Delta x_k \tag{3}$$

is the Riemann sum *of* f *with respect to* $(P, \{c_k\})$.

Remark: If the points c_k in a tagged partition of $[a, b]$ are chosen such that $c_k = x_{k-1}$, then $R(f, P)$ is known as the *left Riemann sum*, because f is evaluated at the left endpoint of each interval. Similarly, if $c_k = x_k$, then $R(f, P)$ is known as the *right Riemann sum*, because f is evaluated at the right endpoint of each interval. If $c_k = (x_{k-1} + x_k)/2$, then $R(f, P)$ is known as the *midpoint rule* or *midpoint approximation*, because f is evaluated at the midpoint of each interval. See Figure 6 and Figure 7.

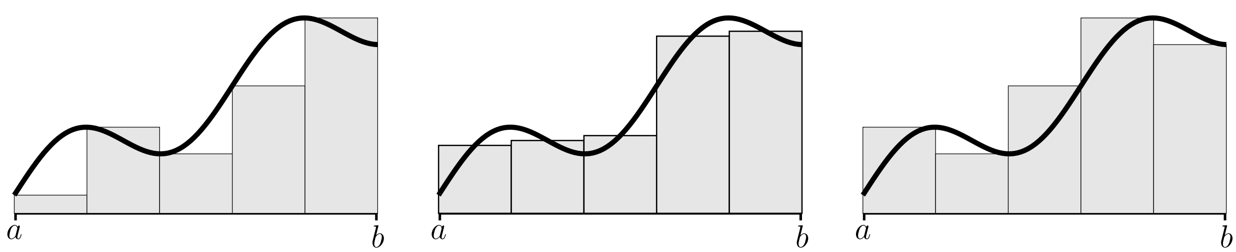


Figure 6: Riemann sums over a uniform partition with $n = 5$ sub-intervals. From left to right: left Riemann sum, midpoint approximation, right Riemann sum.

Figure 7: Riemann sums over a uniform partition with $n = 20$ sub-intervals. From left to right: left Riemann sum, midpoint approximation, right Riemann sum.

Note that each of these sums are approximations to the area under f. In particular, $L(f, P)$ is an under estimate, $U(f, P)$ is an over estimate, and $R(f, P)$ is easy to compute (as we know ahead of time where we will evaluate f) but could either over-estimate or under-estimate. In each of these cases, we expect each of these values to get close to the true area as the partition is refined (i.e., as we take $||P|| \to 0$). A function for which this is true is said to be Riemann integrable, although there are a few (equivalent) definitions for this.

4.2 Riemann Integral

There are (at least) two common definitions for the Riemann integral.

DEFINITION 4.5 (RIEMANN INTEGRAL (MODERN)): *Let* $f\colon [a,b] \to \mathbb{R}$ *be a bounded function and let* $\mathcal{P}$ *be the class of all partitions of* $[a,b]$. *Define*

$$L(f) = \sup_{P \in \mathcal{P}} L(f,P)$$

and

$$U(f) = \inf_{P \in \mathcal{P}} U(f,P)$$

as the greatest lower sum *and* least upper sum *respectively. If* $L(f) = U(f)$, *then we say that* f *is* Riemann integrable *over* $[a,b]$ *and say that*

$$\int_a^b f$$

is the Riemann integral *of* f *from* a *to* b. *The value of the integral is given by* $\int_a^b f = L(P) = U(P)$.

DEFINITION 4.6 (RIEMANN INTEGRAL (HISTORIC)): *Let* $f\colon [a,b] \to \mathbb{R}$ *be a bounded function. We say that* f *is* Riemann integrable *with Riemann integral* $\int_a^b f = A$ *if for all* $\varepsilon > 0$ *there exists a* $\delta > 0$ *such that*

$$|R(f,P) - A| < \varepsilon$$

for all tagged partitions $(P, \{c_k\})$ *of* $[a,b]$ *with* $||P|| < \delta$.

REMARK: For the sake of clarity, it is often insisted upon that $\int_a^b f$ be written as $\int_a^b f(x)\,\mathrm{d}\,x$. This is of more importance when multiple variables are involved. In this case x plays the role of a dummy variable. For example, $\int_a^b f(t)\,\mathrm{d}\,t$ is equivalent to the prior expression. This is similar to the role of the index k in $\sum_{k=1}^n a_k$.

REMARK: By convention, we set $\int_a^a f = 0$ and $\int_b^a f = -\int_a^b f$.

THEOREM 4.1 (EQUIVALENCE OF RIEMANN INTEGRAL): *Let* $f\colon [a,b] \to \mathbb{R}$. *Then* f *is Riemann integrable by Definition 4.5 if and only if it is Riemann integrable by Definition 4.6. In this case, the value of the integral* $\int_a^b f$ *in both definitions are equal.*

Proof. Beyond the scope of this course. □

Theorem 4.2 (Properties of Riemann Integrals): *Let $f, g\colon [a,b] \to \mathbb{R}$ be Riemann integrable. Then*

1) $\int_a^b (f \pm g) = \int_a^b f \pm \int_a^b g$
2) $\int_a^b kf = k\int_a^b f$ *for any **constant** $k \in \mathbb{R}$*
3) $\int_a^b f = \int_a^c f + \int_c^b f$ *for any $c \in (a,b)$*
4) *If $m \le f(x) \le M$ on $[a,b]$, then $m(b-a) \le \int_a^b f \le M(b-a)$*
5) *If $g(x) \le f(x)$ on $[a,b]$, then $\int_a^b g \le \int_a^b f$*
6) *$|f|$ is integrable and $|\int_a^b f| \le \int_a^b |f|$*
7) *$F\colon [a,b] \to \mathbb{R}$ given by $F(x) = \int_a^x f$ is continuous.*
8) *If f is continuous, there exists a $c \in [a,b]$ such that $f(c) = \frac{1}{b-a}\int_a^b f$*

Proof of Claim 1). It is sufficient to prove the $+$ case. Let $\varepsilon > 0$. Choose $\delta > 0$ such that for any tagged partition $(P, \{c_k\})$ of $[a,b]$ with $||P|| < \delta$, we have $|R(f,P) - A_f| < \varepsilon/2$ and $|R(g,P) - A_g| < \varepsilon/2$, where we set $A_f = \int_a^b f$ and $A_g = \int_a^b f$ for convenience. Now we have

$$\begin{aligned}
|R(f+g,P) - (A_f + A_g)| &= |\sum (f(c_k) + g(c_k))\Delta x_k - A_f - A_g| && \text{definition of } R\\
&= |\sum f(c_k)\Delta x_k - A_f + \sum g(c_k)\Delta x_k - A_g| && \text{split the sum}\\
&\le |\sum f(c_k)\Delta x_k - A_f| + |\sum g(c_k)\Delta x_k - A_g| && \text{triangle inequality}\\
&< \frac{\varepsilon}{2} + \frac{\varepsilon}{2}\\
&= \varepsilon.
\end{aligned}$$

Thus $\int_a^b (f+g) = A_f + A_g = \int_a^b f + \int_a^b g$ by Definition 4.6. □

Proof of Claim 2). Left as an exercise. □

Proof of Claim 3). Let $\varepsilon > 0$. Let P be any partition of $[a,b]$ such that $U(f,P) - L(f,P) < \varepsilon$ and $c = x_k$ is some point in the partition. Set $P_a = P \cap [a,c]$ and $P_b = P \cap [c,b]$ as partitions of $[a,c]$ and $[c,b]$ respectively. Then

$$U(f,P_a) - L(f,P_a) \le U(f,P) - L(f,P) < \varepsilon$$

and similar for P_b so that f is integrable on both $[a,c]$ and $[c,b]$. Because of the method by which P_a and P_b were constructed, we have $U(f,P) = U(f,P_a) + U(f,P_b)$ and $L(f,P) = L(f,P_a) + L(f,P_b)$ so that we have

$$U(f,P) \ge \int_a^c f + \int_c^b f$$

and

$$L(f,P) \le \int_a^c f + \int_c^b f.$$

Thus,

$$\int_a^b f \ge \int_a^c f + \int_c^b f$$

and

$$\int_a^b f \leq \int_a^c f + \int_c^b f$$

so that $\int_a^b f = \int_a^c f + \int_c^b f$. $\square$

Proof of Claim 4). Let P be any partition of $[a, b]$. Clearly $m \leq m_k$ and $M \geq M_k$ so that

$$L(f, P) = \sum m_k \Delta x_k \geq m \sum \Delta x_k = m(b-a).$$

A similar observation for $U(f, P)$ yields

$$m(b-a) \leq L(f, P) \leq U(f, P) \leq M(b-a),$$

from which our claim follows. $\square$

Proof of Claim 5). Let P be any partition of $[a, b]$. Clearly $U(g, P) \leq U(f, P)$ and $L(g, P) \leq L(f, P)$ due to $\sup_{I_k} g \leq \sup_{I_k} f$ and $\inf_{I_k} g \leq \inf_{I_k} f$. Our claim immediately follows from this observation. $\square$

Proof of Claim 6). Let P be any partition of $[a, b]$ and set

$$m_k = \inf_{I_k} f,$$

$$m'_k = \inf_{I_k} |f|,$$

$$M_k = \sup_{I_k} f,$$

and

$$M'_k = \sup_{I_k} |f|$$

(note that the prime notation does not refer to a derivative). Note that

$$\begin{aligned}
M_k - m_k &= \sup_{I_k} f - \inf_{I_k} f \\
&\geq ||\sup_{I_k} f| - |\inf_{I_k} f|| && \text{(reverse triangle inequality)} \\
&\geq |\sup_{I_k} |f| - \inf_{I_k} |f|| && \text{(see comment below)} \\
&= |M'_k - m'_k| \\
&= M'_k - m'_k && (M'_k \geq m'_k),
\end{aligned}$$

where we used the fact that $\sup_{I_k} |f| \geq |\sup_{I_k} f|$ and a similar inequality for $\inf_{I_k} |f|$. Multiplying by Δx_k and summing over k, we know that

$$U(f, P) - L(f, P) \geq U(|f|, P) - L(|f|, P) \geq 0,$$

so that $|f|$ is integrable over $[a, b]$. Observing that

$$\begin{aligned}
|U(f, P)| &= |\sum M_k \Delta x_k| \\
&\leq \sum |M_k| \Delta x_k \\
&\leq \sum M'_k \Delta x_k \\
&= U(|f|, P)
\end{aligned}$$

finally shows that $|\int_a^b f| \leq \int_a^b |f|$. $\square$

Proof of Claim 7). From Claim 3), we know that F is well-defined. To see that F is continuous, observe that for any $x, x_0 \in [a,b]$ we have

$$\begin{aligned}
|F(x) - F(x_0)| &= |\int_a^x f - \int_a^{x_0} f| \\
&= |\int_x^{x_0} f| \\
&\leq \int_x^{x_0} |f| \\
&\leq M|x - x_0|
\end{aligned}$$

where $|f(t)| \leq M$ on $[a,b]$, so that F must be continuous. □

Proof of Claim 8). Let $m = \min_{[a,b]} f$ and $M = \max_{[a,b]} f$. From Claim 4), we see that

$$m \leq \frac{1}{b-a} \int_a^b f \leq M.$$

Because f is continuous, by the Intermediate Value Theorem, there must be some $c \in [a,b]$ such that $f(c) = \frac{1}{b-a} \int_a^b f$. □

4.3 Riemann Integration and Area

The cornerstone to connect Riemann integration to area functions is due the following theorems.

Theorem 4.3 (Continuity implies integrability): *Let $f\colon [a,b] \to \mathbb{R}$ be continuous. Then f is Riemann integrable.*

Proof. Recall that because f is continuous over a compact domain, f is uniformly continuous. That is, for every $\varepsilon > 0$ there exists a $\delta > 0$ such that for all $x, y \in [a,b]$we have that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon.$$

Let $\varepsilon > 0$ and choose $\delta > 0$ as above. Let P be any partition of $[a,b]$ such that $||P|| < \delta$. Using Definition 4.5, we have

$$\begin{aligned}
0 &\leq U(f) - L(f) \\
&\leq U(f,P) - L(f,P) \\
&= \sum_{k=1}^{n} (M_k \Delta x_k) - \sum_{k=1}^{n} (m_k \Delta x_k) \\
&= \sum_{k=1}^{n} (M_k - m_k)\Delta x_k,
\end{aligned}$$

where M_k and m_k are the maximum and minimum of f over $I_k = [x_{k-1}, x_k]$ respectively. We have assumed without proof that $L(f) \leq U(f)$. Because $M_k = f(C_k)$ and $m_k = f(c_k)$ with $|C_k - c_k| < \delta$, we must have $M_k - m_k < \varepsilon$. Continuing our line of reasoning, we see that

$$\begin{aligned}
\sum_{k=1}^{n} (M_k - m_k)\Delta x_k &< \sum_{k=1}^{n} \varepsilon \Delta x_k \\
&= \varepsilon \sum_{k=1}^{n} \Delta x_k \\
&= \varepsilon(b - a).
\end{aligned}$$

Thus altogether, we have that $0 \leq U(f) - L(f) < \varepsilon(b-a)$. Because $\varepsilon > 0$ was arbitrary, we must have that $U(f) = L(f)$ so that f is Riemann integrable on $[a, b]$. $\square$

Theorem 4.4 (Existence of the antiderivative (Fundamental Theorem of Calculus I)): *Let $f\colon [a,b] \to \mathbb{R}$ be continuous. $F\colon [a,b] \to \mathbb{R}$ given by $F(x) = \int_a^x f$ is differentiable with $F'(x) = f(x)$.*

Proof. Let $x \in [a, b]$ and let $h \in \mathbb{R}$ be such that $x + h \in [a, b]$. Then

$$\begin{aligned} F(x+h) - F(x) &= \int_a^{x+h} f - \int_a^x f \\ &= \int_x^{x+h} f \\ &= f(c)h \end{aligned}$$

for some c between x and $x + h$. Now

$$F'(x) = \lim_{h \to 0} \frac{F(x+h) - F(x)}{h} = \lim_{h \to 0} f(c) = f(x),$$

where we used the fact that $c \to x$ as $h \to 0$ and f is continuous. $\square$

Remark: The theorem above is often written as

$$\frac{\mathrm{d}}{\mathrm{d}\,x} \int_a^x f = f(x), \tag{4}$$

which holds whenever f is integrable over $[a, x]$ and continuous at x.

Example 4.1: Because $f(x) = x$ is continuous, the value $\int_0^1 f$ exists. Because this value exists, then for any sequence of tagged partitions $(P^{(n)}, \{c_k^{(n)}\})$ such that $||P^{(n)}|| \to 0$, we have that $R(f, P^{(n)}) \to \int_0^1 f$. Let us take $P^{(n)}$ to be the partition of $[0, 1]$ into n sub-intervals of equal width and $c_k^{(n)}$ to be the left endpoint of the n-th sub interval I_k, so that $R(f, P^{(n)})$ is the left Riemann sum over the partition of $[0, 1]$ with every sub-interval having width $\Delta x_k^{(n)} = 1/n$.

Because the width of every sub-interval of $P^{(n)}$ is $1/n$, we must have that $c_k^{(n)} = (k-1)/n$ and

$$\begin{aligned}
R(f, P^{(n)}) &= \sum_{k=1}^{n} f(c_k^{(n)})\Delta x_k^{(n)} && \text{(definition of Riemann sum)} \\
&= \sum_{k=1}^{n} f\left(\frac{k-1}{n}\right) \cdot \frac{1}{n} && \text{(using tagged partition choice)} \\
&= \sum_{k=1}^{n} \frac{k-1}{n^2} && \text{(using } f\text{)} \\
&= \frac{1}{n^2} \sum_{k=1}^{n} (k-1) && \text{(factor a constant)} \\
&= \frac{1}{n^2} \sum_{k=0}^{n-1} k && \text{(re-indexing)} \\
&= \frac{1}{n^2} \sum_{k=1}^{n-1} k && \text{(first term of sum is 0)} \\
&= \frac{1}{n^2} \cdot \frac{(n-1)(n)}{2} && \text{(formula to evaluate the sum)} \\
&= \frac{n^2 - n}{2n^2} && \text{(simplify),}
\end{aligned}$$

so that, by taking $n \to \infty$, we evaluate

$$\int_0^1 f = \lim_{n\to\infty} \frac{n^2-n}{2n^2} = \frac{1}{2}.$$

REMARK: If f is continuous, then the integral $\int_a^b f$ is sometimes defined as

$$\int_a^b f(x)\,\mathrm{d}\,x = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_{k-1}) \frac{b-a}{n}$$

or

$$\int_a^b f(x)\,\mathrm{d}\,x = \lim_{n\to\infty} \sum_{k=1}^{n} f(x_k) \frac{b-a}{n}$$

where $x_k = a + (b-a)k/n$ for $k = 0, 1, 2, \ldots, n$. This is simply the limit of the left Riemann sum (top) or right Riemann sum (bottom) over an equal-width partition of $[a, b]$ with n sub-intervals.

Observe the connection between $\int$ and $\sum$, $\mathrm{d}\,x$ and Δx_k, and finally x and k.

THEOREM 4.5 (AREA UNDER A CURVE (FUNDAMENTAL THEOREM OF CALCULUS II)): *Let $f\colon [a, b] \to \mathbb{R}$ be integrable. If $F\colon [a, b] \to \mathbb{R}$ is continuous and $F'(x) = f(x)$ on (a, b), then*

$$\int_a^b f = F(b) - F(a).$$

Proof. Let P be any partition of $[a, b]$ and, observing that F satisfies the assumptions for the Mean Value Theorem on $I_k = [x_{k-1}, x_k]$, choose tags $c_k \in (x_{k-1}, x_k)$ such that

$$f(c_k) = F'(c_k) = \frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}}$$

for each $k = 1, 2, 3, \ldots, n$. Now

$$\begin{aligned}
R(f, P) &= \sum_{k=1}^{n} f(c_k)\Delta x_k \\
&= \sum_{k=1}^{n} \frac{F(x_k) - F(x_{k-1})}{x_k - x_{k-1}}\Delta x_k \\
&= \sum_{k=1}^{n} (F(x_k) - F(x_{k-1})) \\
&= \sum_{k=1}^{n} F(x_k) - \sum_{k=1}^{n} F(x_{k-1}) \\
&= \sum_{k=1}^{n} F(x_k) - \sum_{k=0}^{n-1} F(x_k) \\
&= F(x_n) + \sum_{k=1}^{n-1} F(x_k) - \sum_{k=1}^{n-1} F(x_K) - F(x_0) \\
&= F(x_n) - F(x_0) \\
&= F(b) - F(a).
\end{aligned}$$

Because P was arbitrary and f is Riemann integrable, we see that we must have $\int_a^b f = F(b) - F(a)$. □

Remark: The theorem above is often written as

$$\int_a^b F' = F(b) - F(a). \tag{5}$$

However, this assumes that F' is integrable. See Example 4.6.

Connection to the Area function:

1. $\mathcal{A}(x) = \int_a^x f$ is the *signed area* under f over $[a, x]$.
2. $\int_a^x |f|$ is the *total area* under f over $[a, x]$.
3. Use $\int_a^b F' = F(b) - F(a)$ to evaluate the area function. The most difficult part is finding a suitable F given F'.
4. When setting up an integral to compute the area, it is often useful to first find the differential area $\mathrm{d}\,\mathcal{A}$ first. This usually supplies both the function f to be integrated as well as the variable of integration (from the differential length in $\mathrm{d}\,\mathcal{A} = f(x)\,\mathrm{d}\,x$).

Notation and Terminology:

1. The shorthand $F(x)|_a^b$ and $[F(x)]_a^b$ are both used to indicate the difference $F(b) - F(a)$.
2. $\int_a^b f$ is called a *definite integral*
3. $\int f$ is called an *indefinite integral* and is used to represent an arbitrary anti-derivative of f
4. If $F' = f$, then F is an antiderivative of f. The term *primitive of* f is also used.
5. If the domain of f is connected (i.e., an interval in $\mathbb{R}$), then the antiderivative of f is unique up to an additive constant.
6. When using the notation $\int_a^b f(x)\,\mathrm{d}\,x$, the variable indicated by the differential $\mathrm{d}\,x$ is the variable that the function is being integrated with respect to and is a dummy variable.

Example 4.2: Note that if $F'(x) = f(x)$, then $\frac{\mathrm{d}}{\mathrm{d}\,x}[F(x) + C] = f(x)$ as well. However, either the choice $F(x)$ or $F(x) + C$ as the antiderivative of f will result in the same evaluation of $\int_a^b f$. This can be seen by noting

$$\begin{aligned}[F(x) + C]_a^b &= (F(b) - C) - (F(a) - C) \\ &= F(b) - C - F(a) + C \\ &= F(b) - F(a) \\ &= [F(x)]_a^b.\end{aligned}$$

Example 4.3: Evaluate the following definite integrals.

1) $\displaystyle\int_0^1 x\,\mathrm{d}\,x = \frac{1}{2}x^2\Big|_0^1 = \frac{1}{2}\cdot 1^2 - \frac{1}{2}\cdot 0^2 = \frac{1}{2}$

2) $\displaystyle\int_1^2 x\,\mathrm{d}\,x = \frac{1}{2}x^2\Big|_1^2 = \frac{1}{2}(2^2 - 1^2) = \frac{3}{2}$

3) $\displaystyle\int_{-1}^1 x^3\,\mathrm{d}\,x = \frac{1}{4}x^4\Big|_{-1}^1 = \frac{1}{4}[1^4 - (-1)^4] = 0$

4) $\displaystyle\int_0^1 x^n\,\mathrm{d}\,x = \frac{1}{n}x^{n+1}\Big|_0^1 = \frac{1}{n}$ where $n > 0$.

5) $\displaystyle\int_0^\pi \sin t\,\mathrm{d}\,t = -\cos t\Big|_0^\pi = \cos t\Big|_\pi^0 = \cos 0 - \cos\pi = 1 - (-1) = 2$ (Note that $F(x)|_a^b = -F(x)|_b^a$.)

6) $\displaystyle\int_0^2 xy^2\,\mathrm{d}\,x = \frac{1}{2}x^2y^2\Big|_{x=0}^2 = \frac{1}{2}y^2(4-0) = 2y^2$ (Note that the variable of integration is x, so y is treated as a constant. Also, when there are multiple variables involved, it is good practice to explicitly state the variable which is used for the substitution of the integral bounds.)

7) $\displaystyle\int_0^2 xy^2\,\mathrm{d}\,y = \frac{1}{3}xy^3\Big|_{y=0}^2 = \frac{1}{3}x(8-0) = \frac{8}{3}x$

8) $\displaystyle\int_0^{2\pi} \cos^2\theta\,\mathrm{d}\,\theta = \int_0^{2\pi}\frac{1}{2}(1+\cos 2\theta)\,\mathrm{d}\,\theta = \frac{1}{2}\left(\theta + \frac{1}{2}\sin 2\theta\right)\Big|_0^{2\pi} = \frac{1}{2}(2\pi - 0) = \pi$

9) $\displaystyle\int_0^1 (x - x^2)\,\mathrm{d}\,x = \left(\frac{1}{2}x^2 - \frac{1}{3}x^3\right)\Big|_0^1 = \frac{1}{2} - \frac{1}{3} = \frac{1}{6}$

10) $\displaystyle\int_{-a}^{a} x^2 \,\mathrm{d}\,x = \frac{1}{3}x^3\Big|_{-a}^{a} = \frac{1}{3}\left(a^3 - (-a)^3\right) = \frac{2}{3}a^3$

Example 4.4 (A discontinuous integrable function): Let

$$f(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$$

and suppose we wish to evaluate $\int_a^b f$ where $a < 0$ and $b > 0$. Graphing f and appealing to our geometric intuition of area, the value $\int_a^b f = b$ is a likely candidate. Let $\varepsilon > 0$ and $(P, \{c_k\})$ be a tagged partition of $[a, b]$. We must find $\delta > 0$ such that if $||P|| < \delta$ then $|R(f, P) - b| < \varepsilon$. We compute

$$\begin{aligned} R(f, P) - b &= \sum_{k=1}^{n} f(c_k)\Delta x_k - b \\ &= \sum_{k=m}^{n} f(c_k)\Delta x_k - b \\ &= f(c_m)\Delta x_m + \sum_{k=m+1}^{n} \Delta x_k - b \\ &= f(c_m)\Delta x_m + (b - x_m) - b \\ &= f(c_m)\Delta x_m - x_m \end{aligned}$$

where m is chosen to be the largest index such that $I_{m-1} = [x_{m-2}, x_{m-1}] \subset [a, 0)$ so that $f(c_k) = 0$ if $k < m$. Note that if $||P|| < \delta$, then we must have $\Delta x_m < \delta$ and $|x_m| < \delta$ so that

$$\begin{aligned} |R(f, P) - b| &= |f(c_m)\Delta x_m - x_m| \\ &\leq |f(c_m)\Delta x_m| + |x_m| \\ &\leq 2\delta \end{aligned}$$

so that choosing $\delta = \varepsilon/2$ shows that $\int_a^b f = b$ by Definition 4.6.

Example 4.5 (A non-integrable bounded function): Let

$$f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$$

and $[a, b] \subset \mathbb{R}$. Let P be any partition of $[a, b]$. Then, as each sub-interval I_k contains both rational and irrational points, we must have that $M_k = \sup_{I_k} f = 1$ and $m_k = \inf_{I_k} f = 0$. Thus, $L(f, P) = 0$ and $U(f, P) = b - a$ for any partition P so that $L(f) = \sup_P L(f, P) = 0$ and $U(f) = \inf_P U(f, P) = b - a$. Because $L(f) \neq U(f)$, f is not integrable over any interval $[a, b]$.

Example 4.6 (A non-integrable derivative): Let

$$F(x) = \frac{1}{2}x^2 \sin\left(\frac{1}{x^2}\right)$$

when $x \neq 0$ and $F(0) = 0$. Then,

$$F'(x) = x \sin\left(\frac{1}{x^2}\right) + \frac{1}{x}\cos\left(\frac{1}{x^2}\right)$$

when $x \neq 0$ and

$$F'(0) = \lim_{h \to 0} \frac{F(h) - F(0)}{h} = \lim_{h \to 0} \frac{h}{2} \sin\left(\frac{1}{h^2}\right) = 0$$

so that F' exists on all of $\mathbb{R}$. However, F' is not bounded on any interval containing 0 so that $\int_a^b F'$ does not exist if $0 \in [a, b]$.

5 Simplification Techniques

Although more functions are integrable than differentiable, given an expression $f(x)$, finding an expression for $\int f$ is often much more difficult than finding an expression for f'. Here, we present several common tricks that are used to make this problem easier: both for computing indefinite and definite integrals.

5.1 Symmetry

The evaluation of the integral $\int_a^b f$ can be simplified greatly if the function f is symmetric about the center of the interval $(a+b)/2$. For simplicity, we will assume that the center of the interval is the origin.

Definition 5.1 (Even): *Let $f\colon [-a,a] \to \mathbb{R}$ for some $a > 0$. If*

$$f(-x) = f(x)$$

for all $x \in [-a,a]$, then f is said to be even.

Definition 5.2 (Odd): *Let $f\colon [-a,a] \to \mathbb{R}$ for some $a > 0$. If*

$$f(-x) = -f(x)$$

for all $x \in [-a,a]$, then f is said to be odd.

Theorem 5.1 (Even-Odd Symmetry): *Let $f\colon [-a,a] \to \mathbb{R}$ for some $a > 0$ be Riemann integrable. If f is even, then $\int_{-a}^{a} f = 2\int_0^a f$. If f is odd, then $\int_{-a}^{a} f = 0$.*

Proof. Exercise. ☐

Example 5.1: Because $\sin x$ is odd, we know immediately that $\int_{-1}^{1} \sin x \,\mathrm{d}\, x = 0$. However, we can verify this by observing

$$\int_{-1}^{1} \sin x \,\mathrm{d}\, x = -\cos x\Big|_{-1}^{1} = \cos x\Big|_{-1}^{1} = \cos 1 - \cos(-1) = \cos 1 - \cos 1 = 0,$$

where we used the fact that $\cos x$ is even.

Using even/odd symmetry can be especially helpful when integrating the absolute value of an even or odd function.

Example 5.2: Because $|\sin x|$ is even, we know that

$$\int_{-\pi}^{\pi} |\sin x| \,\mathrm{d}\, x = 2\int_0^{\pi} |\sin x| \,\mathrm{d}\, x = 2\int_0^{\pi} \sin x \,\mathrm{d}\, x = 2(-\cos x)\Big|_0^{\pi} = -2(\cos\pi - \cos 0) = -2(-1-1) = 4.$$

Note that we used the fact that $\sin x \geq 0$ on $[0,\pi]$ so that (on this interval) $|\sin x| = \sin x$.

5.2 Substitution

Recall the chain rule $(f \circ g)' = (f' \circ g) \cdot g'$. If $F = \int f$, we can re-write this as $(F \circ g)' = (f \circ g) \cdot g'$. Integrating both sides, we have that

$$\int (f \circ g) \cdot g' = F \circ g. \tag{6}$$

Example 5.3: Compute the following indefinite integrals.

1) $\displaystyle\int (1+x^2)^{10} \cdot 2x \,\mathrm{d}\,x = \frac{1}{11}(1+x^2)^{11} + C$. Here $f(x) = x^{10}$, $F(x) = \frac{1}{11}x^{11} + C$, and $g(x) = 1 + x^2$.

2) $\displaystyle\int \cos(\sin x) \cos x \,\mathrm{d}\,x = \sin(\sin x) + C$. Here $f(x) = \cos x$ and $g(x) = \sin x$.

3) $\displaystyle\int \sqrt{\sin x} \cos x \,\mathrm{d}\,x = \frac{2}{3}(\sin x)^{3/2} = C$. Here $f(x) = \sqrt{x}$ and $g(x) = \sin x$.

This method is called u-substitution and is more often presented as follows:

$$\int f(g(x)) \cdot g'(x) \,\mathrm{d}\,x = \int f(u) \,\mathrm{d}\,u, \tag{7}$$

where $u = g(x)$. When using this technique, it is often the case that one proceeds along the following lines.

substitution steps:

1. Define $u = g(x)$
2. Compute/define the differential $\mathrm{d}\,u = g'(x)\,\mathrm{d}\,x$. Sometimes, this is re-arraged as $\mathrm{d}\,x = \frac{1}{g'(x)}\,\mathrm{d}\,u$ or some similar re-arrangement that is convenient for the specific problem.
3. Solve $\int f(u)\,\mathrm{d}\,u$
4. Back substitute to get your final solution in terms of the original variable x

Note that there is nothing special about using the variable u. Any variable that is free in the context of the problem can be used, but u and w are common choices.

Example 5.4: Let us evaluate

$$\int x\sqrt{1-x^2}\,\mathrm{d}\,x.$$

Setting $u = 1 - x^2$, we have that $\mathrm{d}\,u = -2x\,\mathrm{d}\,x$, which we can re-arrange to get $x\,\mathrm{d}\,x = -\frac{1}{2}\,\mathrm{d}\,u$. Now we integrate

$$\int \sqrt{u} \cdot -\frac{1}{2}\,\mathrm{d}\,u = -\frac{1}{2}\int u^{1/2}\,\mathrm{d}\,u = -\frac{1}{2} \cdot \frac{2}{3}u^{3/2} + C.$$

After back-substitution and simplifying, we conclude that

$$\int x\sqrt{1-x^2}\,\mathrm{d}\,x = -\frac{1}{2}\left(1-x^2\right)^{3/2} + C.$$

Example 5.5: Let us evaluate

$$\int x^3\sqrt{1+x^2}\,\mathrm{d}\,x.$$

Setting $u = 1 + x^2$, we have that $\mathrm{d}\,u = 2x\,\mathrm{d}\,x$, or $x\,\mathrm{d}\,x = \frac{1}{2}\,\mathrm{d}\,u$. Multiplying both sides by x^2 and using $x^2 = u - 1$, we have that $x^3\,\mathrm{d}\,x = \frac{1}{2}(u-1)\,\mathrm{d}\,u$. Now, we have

$$\begin{aligned}
\int x^3\sqrt{1+x^2}\,\mathrm{d}\,x &= \frac{1}{2}\int (u-1)\sqrt{u}\,\mathrm{d}\,u \\
&= \frac{1}{2}\int \left(u^{3/2} - u^{1/2}\right)\mathrm{d}\,u \\
&= \frac{1}{2}\left(\frac{2}{5}u^{5/2} - \frac{2}{3}u^{3/2}\right) + C \\
&= \frac{1}{5}u^{5/2} - \frac{1}{3}u^{3/2} + C \\
&= \frac{1}{5}(1+x^2)^{5/2} - \frac{1}{3}(1+x^2)^{3/2} + C.
\end{aligned}$$

Note that the re-arranging of the differentials amounts to re-arranging the original integral to

$$\int x^3\sqrt{1+x^2}\,\mathrm{d}\,x = \int \left[(1+x^2)\sqrt{1+x^2} - \sqrt{1+x^2}\right] x\,\mathrm{d}\,x$$

and using $u = 1 + x^2$.

There are two methods to use this technique to evaluate a definite integral. The first method is to compute the full anti-derivative (including back-substitution) and then evaluate using the original bounds. This takes the form

$$\int_a^b f(g(x))\cdot g'(x)\,\mathrm{d}\,x = f(g(x))\Big|_a^b = f(g(b)) - f(g(a)).$$

However, this can be re-written as

$$\int_a^b f(g(x))\cdot g'(x)\,\mathrm{d}\,x = \int_{g(a)}^{g(b)} f(u)\,\mathrm{d}\,u, \tag{8}$$

which is our second technique. Note that the back-substitution step is not necessary here.

Example 5.6: Let us evaluate the definite integral $\int_0^1 x\sqrt{1-x^2}\,\mathrm{d}\,x$. Setting $u = g(x) = 1-x^2$, we re-arrange our integral to get

$$\begin{aligned}
\int_0^1 x\sqrt{1-x^2}\,\mathrm{d}\,x &= -\frac{1}{2}\int_0^1 \sqrt{1-x^2}\cdot(-2x)\,\mathrm{d}\,x \\
&= -\frac{1}{2}\int_{g(0)}^{g(1)} \sqrt{u}\,\mathrm{d}\,u \\
&= -\frac{1}{2}\int_1^0 u^{1/2}\,\mathrm{d}\,u \\
&= \frac{1}{2}\int_0^1 u^{1/2}\,\mathrm{d}\,u & \left(\text{recall } \int_b^a f = -\int_a^b f\right) \\
&= \frac{1}{2}\cdot\frac{2}{3}u^{3/2}\Big|_0^1 \\
&= \frac{1}{3}.
\end{aligned}$$

Note that the formal manipulation of symbols can sometimes be misleading and only valid on certain intervals. Errors are especially common when dealing with absolute values and problems involving “trig substitution.”

5.2.1 Geometric Interpretation

When $g'(x) > 0$, Equation (8) has an important geometric meaning. Assume that $a < b$. Because $g'(x) > 0$ on $[a, b]$, we know that g is increasing so that $g(a) < g(b)$ and that g^{-1} exists. Let $A = g(a)$ and $B = g(b)$ so that we may re-write our equation as

$$\int_a^b f(g(x)) \cdot g'(x) \,\mathrm{d}\, x = \int_A^B f(u) \,\mathrm{d}\, u.$$

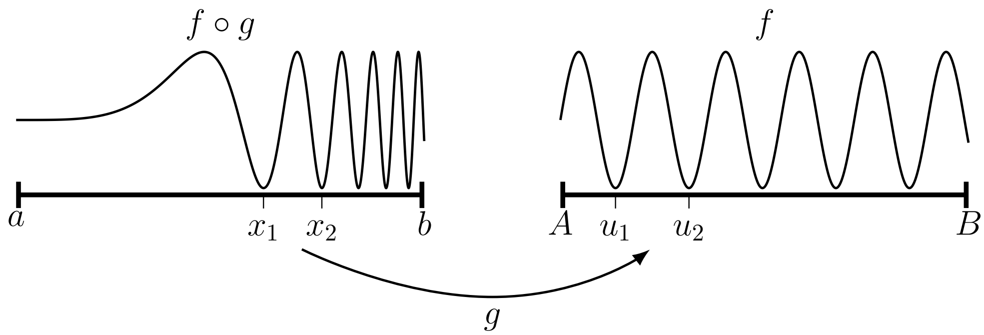


Figure 8: Diagram of u-substitution. Here $u_1 = g(x_1)$ and $u_2 = g(x_2)$.

Examine Figure 8 closely. Note that we **do not** have that $\int_a^b f \circ g = \int_A^B f$. This is because *the interval* $[a, b]$ *is deformed in a non-uniform way to get the interval* $[A, B]$. The manner of this deformation is determined by $g'(x)$. In regions where $g'(x) > 1$, values are "spread out" and in regions where $0 < g'(x) < 1$, values are "compressed." To be more precise, if $g'(x)$ is not constant and $P = \{x_0, x_1, \ldots, x_n\}$ is a uniform partition of $[a, b]$, then $g[P] = \{u_0, u_1, \ldots, u_n\}$ is not a uniform partition of $[A, B]$, where $u_k = g(x_k)$ for $k = 0, 1, 2, \ldots, n$. From the Mean Value Theorem, for each $k = 1, 2, \ldots, n$, there is some $c_k \in (x_{k-1}, x_k)$ such that $\Delta u_k = g'(c_k)\Delta x_k$. Letting $(P, \{c_k\})$ be a tagged partition of $[a, b]$ (with P arbitrary and tags $\{c_k\}$ as defined above), we have that $(g[P], \{C_k\})$ is a tagged partition of $[A, B]$, where $C_k = g(c_k)$. Now the corresponding Riemann sums are

$$R(f \circ g, P) = \sum f(g(c_k)) \cdot g'(c_k)\Delta x_k$$

and

$$R(f, g[P]) = \sum f(C_k) \cdot \Delta u_k = \sum f(g(c_k)) \cdot g'(c_k)\Delta x_k$$

so that $R(f \circ g, P) = R(f, g[P])$. Taking $||P|| \to 0$, we see that $\int_a^b f(g(x)) \cdot g'(x) \,\mathrm{d}\, x = \int_A^B f(u) \,\mathrm{d}\, u$ as before, but with a more geometric intuition. This technique of changing variables via $u = g(x)$ is a technique of great importance in calculus.

6 Application Examples

When using integration to solve a practical problem, one usually has some quantity of interest in mind (e.g., volume, energy, position, area, etc.). When setting up your integral, it is often easiest to do so by first computing the differential of the quantity of interest. This is best illustrated by several examples.

Much of Calculus II is devoted to a more precise and thorough survey of integral applications.

Example 6.1 (Area between two curves): Let $f, g\colon [a,b] \to \mathbb{R}$ such that f and g are both continuous and $g(x) \leq f(x)$ on $[a,b]$. We wish to determine the area between the two functions over $[a,b]$. Picking an arbitrary $x \in [a,b]$ and drawing a rectangle centered at x with width $\mathrm{d}\,x$ and height $f(x) - g(x)$ gives us a differential area of $\mathrm{d}\,\mathcal{A} = (f(x) - g(x))\,\mathrm{d}\,x$. See Figure 9. Letting $\mathcal{A}$ be the area function, we have that the area between f and g over $[a,b]$ is given by

$$\mathcal{A}(b) - \mathcal{A}(a) = \int_a^b \mathrm{d}\,\mathcal{A} = \int_a^b (f(x) - g(x))\,\mathrm{d}\,x = \int_a^b (f - g).$$

More formally, this can be seen by using $f(c_k) - g(c_k)$ as the height in a Riemann sum. However, the same result will be obtained.

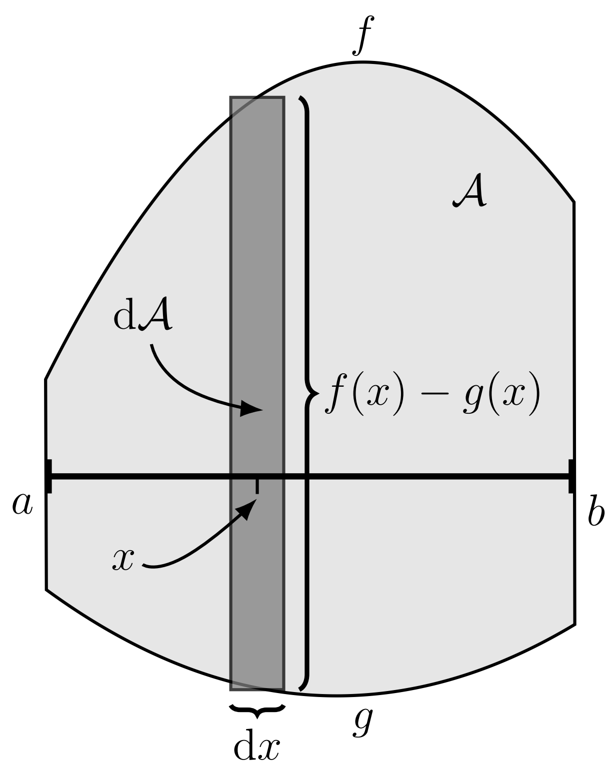


Figure 9: A differential area between two curves.

Example 6.2 (Area of a circle): Consider a circle of radius R. At a smaller radius r, construct a differential area $\mathrm{d}\,\mathcal{A}$ as an annulus with average radius r and width $\mathrm{d}\,r$ so that

$$\mathrm{d}\,\mathcal{A} = 2\pi r\,\mathrm{d}\,r.$$

Summing (integrating) $\mathrm{d}\,\mathcal{A}$ from $r = 0$ to $r = R$ gives us

$$\begin{aligned}\mathcal{A} &= \int_0^R 2\pi r\,\mathrm{d}\,r \\ &= \pi r^2\Big|_0^R \\ &= \pi R^2\end{aligned}$$

as the area of the circle. See Figure 10.

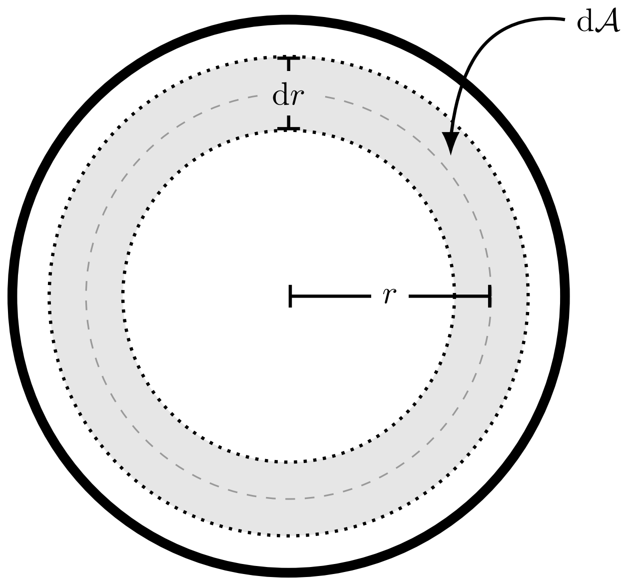


Figure 10: A differential area of a cricle.

Example 6.3 (Volume of a cone): Consider a right circular cone with base radius R and height H. At a distance h from the apex of the cone construct a differential volume as a disk of radius r and height $\mathrm{d}\,h$. That is, $\mathrm{d}\,V = \pi r^2 \,\mathrm{d}\,h$. From similar triangles, we know that $r/h = R/H$ so that $r = hR/H$ and

$$\mathrm{d}\,V = \frac{\pi R^2}{H^2} h^2 \,\mathrm{d}\,h.$$

Summing (integrating) $\mathrm{d}\,V$ over $0 \le h \le H$ gives us

$$\begin{aligned} V &= \frac{\pi R^2}{H^2} \int_0^H h^2 \,\mathrm{d}\,h \\ &= \frac{\pi R^2}{H^2} \cdot \frac{1}{3} H^3 \\ &= \frac{\pi R^2 H}{3} \\ &= \frac{\pi}{3} R^2 H \end{aligned}$$

as the volume of the cone. See Figure 11.

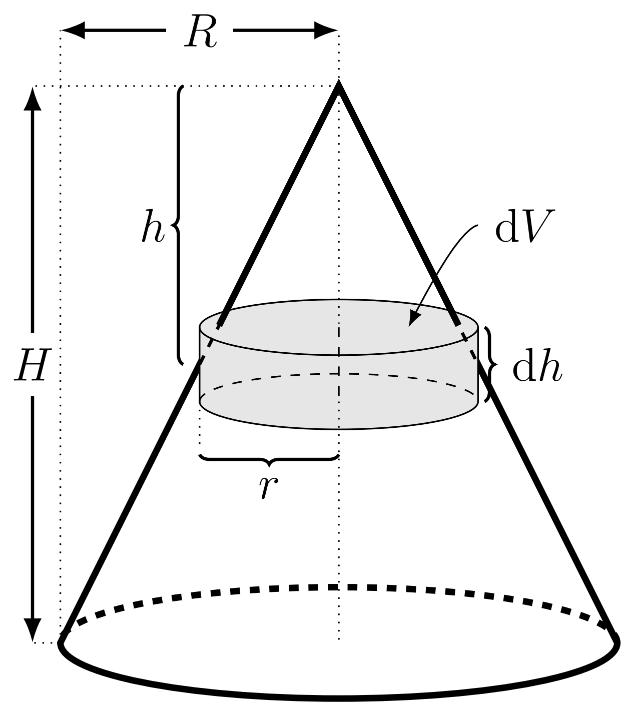


Figure 11: A differential volume of a cone.

Example 6.4 (Distance): Suppose an object moves in the plane with coordinates given as a function of time as

$$x(t) = t^{3/2}$$

and

$$y(t) = \frac{3}{2}t$$

and we wish to know how far the object moves (total distance, not displacement) over the interval $0 \le t \le T$. To solve this, we informally reason that if we set $\Delta x = x(t + \Delta t) - x(t)$ and $\Delta y = y(t + \Delta t) - y(t)$ that the distance that the object travels between times t and $t + \Delta t$ can be approximated as

$$\Delta s \approx \sqrt{(\Delta x)^2 + (\Delta y)^2}.$$

Dividing by Δt gives us

$$\frac{\Delta s}{\Delta t} \approx \frac{1}{\Delta t}\sqrt{(\Delta x)^2 + (\Delta y)^2} = \sqrt{\left(\frac{\Delta x}{\Delta t}\right)^2 + \left(\frac{\Delta y}{\Delta t}\right)^2}$$

and taking $\Delta t \to 0$ gives us $\dot{s} = \sqrt{\dot{x}^2 + \dot{y}^2}$, after taking some liberties with the approximation. Here $\dot{s}$ is the *speed* of the object while $\dot{x} = (3/2)t^{1/2}$ and $\dot{y}(t) = 3/2$ are the x and y components of the velocity of the object. Integrating speed yields distance so we compute

$$\begin{aligned} s(T) &= \int_0^T \dot{s}\,\mathrm{d}\,t \\ &= \int_0^T \sqrt{\left(\frac{3}{2}t^{1/2}\right)^2 + \left(\frac{3}{2}\right)^2}\,\mathrm{d}\,t \\ &= \frac{3}{2}\int_0^T \sqrt{t+1}\,\mathrm{d}\,t \\ &= \frac{3}{2}\cdot\frac{2}{3}(t+1)^{3/2}\Big|_0^T \\ &= (T+1)^{3/2} - 1 \end{aligned}$$

as the distance traveled over the interval $[0, T]$. See Figure 12.

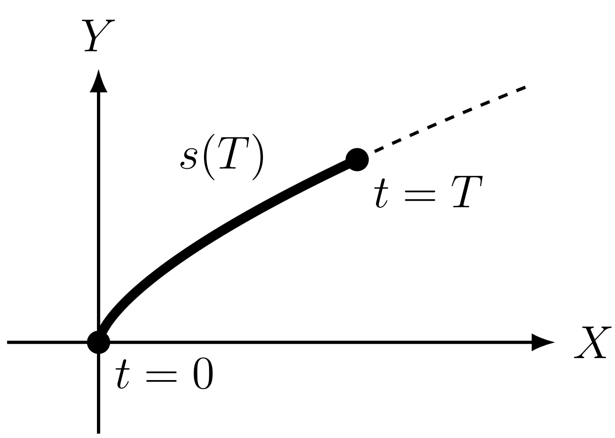


Figure 12: A trajectory (dashed) and a distance $s(T)$ (solid) along the trajectory.

7 Problems

1) Give an example of a function $f\colon [0,1] \to \mathbb{R}$ that is unbounded and piecewise algebraic. Provide the graph of your function. [Hint: define f piecewise, using algebraic expressions for each piece of its domain. The function F' in Example 4.6 is not piecewise algebraic.]

2) Use Definition 4.6 to prove Claim 2) of Theorem 4.2.

3) Let

$$f(x) = \begin{cases} -1, & x \le 0 \\ 1, & x > 0 \end{cases}.$$

a) Find an expression for $F(x) = \int_0^x f$ for any $x \in \mathbb{R}$.

b) Does the specific value $f(0)$ affect your solution to the previous part?

c) For what values of x is $F'(x) = f(x)$ true?

4) Let $f(x) = x^2$. We will compute $\int_0^1 f$ using Riemann sums in the following steps.

a) Review the proof of Theorem 3.1 Claim 3). Use induction to prove that

$$\sum_{k=1}^{n} k^2 = \frac{n(n+1)(2n+1)}{6}.$$

b) Review Example 4.1. Let $P^{(n)}$ be the uniform partition of $[0,1]$ with n sub-intervals and let $c_k^{(n)} = x_k^{(n)}$ be the tags for the *right* Riemann sum. Draw a diagram to illustrate the sum $R(f, P^{(3)})$ and compute this value.

c) Is $R(f, P^{(3)})$ an over-estimate or under-estimate to $\int_0^1 f$? [Hint: the answer should be clear from your diagram in the previous part.]

d) Find an expression for $R(f, P^{(n)})$ and simplify as much as possible. [Hint: find a simple expression for $x_k^{(n)}$ first.]

e) Evaluate $\lim_{n\to\infty} R(f, P^{(n)})$.

5) Let $f(x) = x$. Use the midpoint rule to approximate $\int_0^1 f$ using a uniform partition with 2 sub-intervals. List the partition points x_k and the tags c_k used for evaluation.

6) Evaluate each indefinite integral. Assume all symbols other than the variable of integration represent constants.

a) $\displaystyle\int 5x^2 \,\mathrm{d}\,x$

b) $\displaystyle\int ax^2 \,\mathrm{d}\,x$

c) $\displaystyle\int \sin(3x) \,\mathrm{d}\,x$

d) $\displaystyle\int \sin(yx) \,\mathrm{d}\,x$

e) $\displaystyle\int x\sin(x^2 + a^2) \,\mathrm{d}\,x$

f) $\displaystyle\int \left(\sqrt{x} - \sqrt{2x} + \frac{x}{4}\right) \mathrm{d}\,x$

g) $\displaystyle\int \frac{\cos\sqrt{x}}{\sqrt{x}} \,\mathrm{d}\,x$

h) $\displaystyle\int \cos^4(2t)\sin(2t) \,\mathrm{d}\,t$

7) Evaluate each definite integral. Assume all symbols other than the variable of integration represent constants.

a) $\displaystyle\int_0^1 \left(1 + x^2\right)^2 \mathrm{d}\,x$

b) $\displaystyle\int_0^1 x\left(1 + x^2\right)^{10} \mathrm{d}\,x$

c) $\displaystyle\int_0^1 t\sqrt{1 - t^2} \,\mathrm{d}\,t$

d) $\displaystyle\int_0^R 4\pi r^2 \,\mathrm{d}\,r$

e) $\displaystyle\int_{-1}^1 |x| \,\mathrm{d}\,x$

f) $\displaystyle\int_{-1}^1 \left(\sin(x^3) + x\right) \mathrm{d}\,x$

g) $\displaystyle\int_{-\pi}^{\pi} \cos|\theta| \,\mathrm{d}\,\theta$

h) $\displaystyle\int_{-1}^1 \left(\sin(-x^7) + \sqrt{|x|}\right) \mathrm{d}\,x$

8) Review Example 6.4. Suppose that an object moves in the plane according to the equations

$$x(t) = R\cos(\omega t)$$

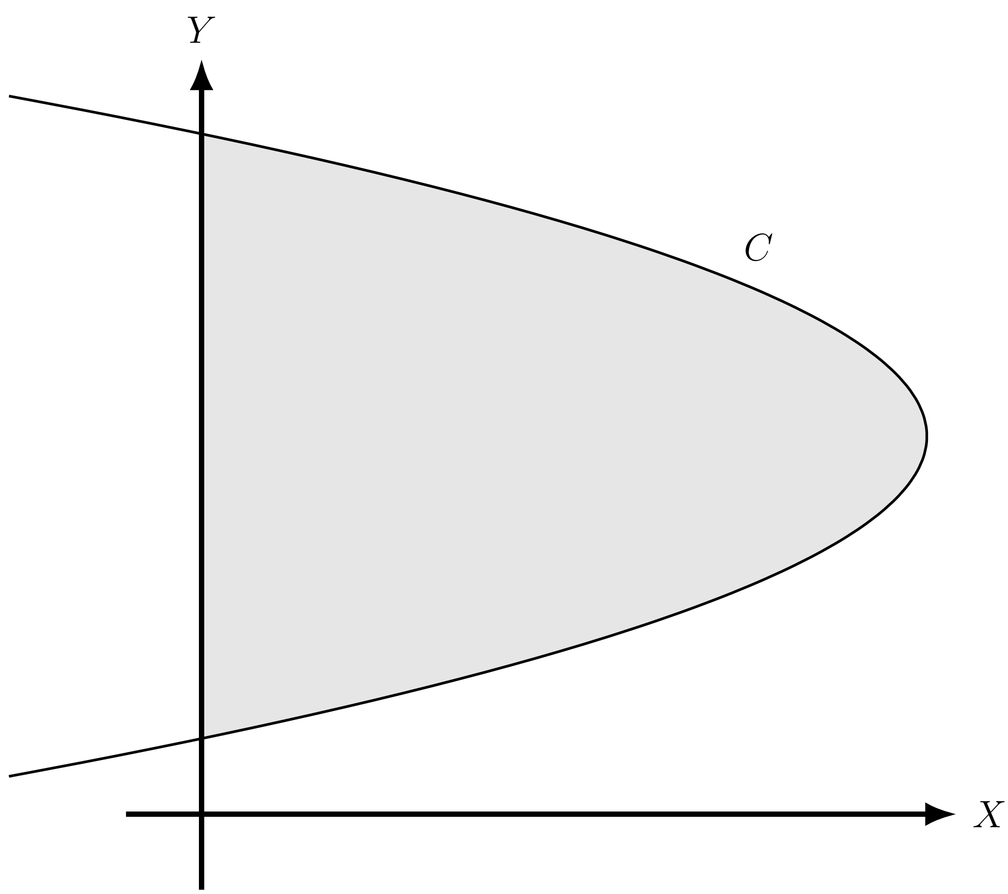
and

$$y(t) = R\sin(\omega t)$$

where $R > 0$ and $\omega > 0$ are constant.

- *a)* Evaluate $x^2 + y^2$. What shape does the trajectory of the object lie on?
- *b)* Let $\dot{s} = \sqrt{\dot{x}^2 + \dot{y}^2}$ as in the example. Compute $\dot{s}(t)$.
- *c)* At $t = 0$, the object is at the coordinate $(R, 0)$. Let $T > 0$ be the first time that the object returns to that coordinate. What is T? [Hint: x and y are both periodic.]
- *d)* How far does the travel (i.e., total distance) over the time interval $[0, T]$? Compute this distance using an integral. Is your answer consistent with your answer from part *a)*?

9) What is the area of the shaded region? The curve C is given by $-y^2 + 5y - 4 = x$.



10) Decide if each statement is true or false. If the statement is false, provide a counter-example. Assume that $f\colon [a, b] \to \mathbb{R}$ and $x \in (a, b)$.

- *a)* If f is continuous, it is integrable.
- *b)* If f is integrable, it is continuous.
- *c)* If f' exists, then $\int f$ exists.
- *d)* If $\int f$ exists, then f' exists.
- *e)* It is always true that $\frac{\mathrm{d}}{\mathrm{d}\,x} \int_a^x f = f(x)$.
- *f)* If f' exists, then it is true that $\int_a^b f' = f(b) - f(a)$.