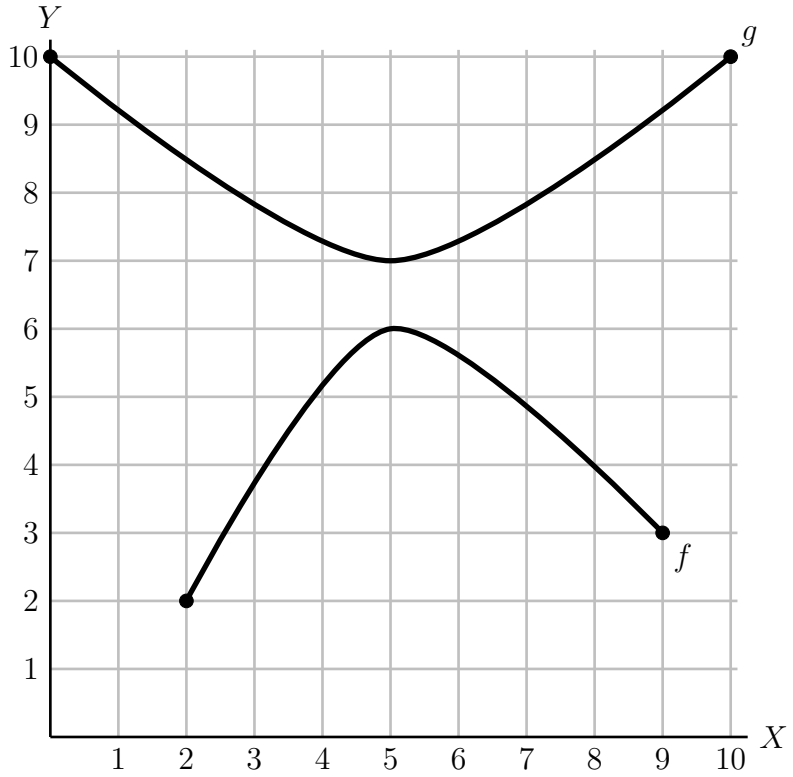
MATH 164 Exam 2 Review

Problems

1. Let X be a non-empty set and let $I_X \colon X \to X$ be the *identity function* on X (i.e., $I_X(x) = x$ for all $x \in X$).
   (a) Is I_X injective? Prove or provide a counterexample.
   (b) Is I_X surjective? Prove or provide a counterexample.
   (c) If I_X is bijective, what is its inverse?
2. Let $f \colon X \to Y$ and suppose that f is bijective. Answer the following questions about this function or state that not enough information is given.
   (a) What is the name of the function?
   (b) What is the domain of the function?
   (c) What is the co-domain of the function?
   (d) What is the range of the function?
   (e) What is the name of its inverse function?
   (f) What is the domain of its inverse function?
   (g) What is the co-domain of its inverse function?
   (h) What is the range of its inverse function?
   (i) Draw a cartoon to illustrate each of these concepts.
3. Explain why, if one uses implied domain and range $f \colon \mathcal{D}(f) \to \mathcal{R}(f)$, any one-to-one function is invertible.
4. Suppose that $f \colon \mathbb{R} \to \mathbb{R}$ is increasing and invertible. Is f^{-1} increasing or decreasing? Provide an example that demonstrates your claim and then prove.
5. Let $f(x) = \log_3 x$, $g(x) = 3^x$, and $h(x) = x^3$.
   (a) What are the (implied) domains and ranges of f, g, and h?
   (b) What is the domain and range of $f \circ g$? Write a simplified expression for $(f \circ g)(x)$.
   (c) What is the domain and range of $g \circ f$? Write a simplified expression for $(g \circ f)(x)$.
   (d) What is the domain and range of $h \circ g$? Write a simplified expression for $(h \circ g)(x)$.
   (e) What is the domain and range of $g \circ h$? Write a simplified expression for $(g \circ h)(x)$.
   (f) What is the domain and range of $h \circ f$? Write a simplified expression for $(h \circ f)(x)$.
   (g) What is the domain and range of $f \circ h$? Write a simplified expression for $(f \circ h)(x)$.

6. Consider the graphs $y = f(x)$ and $y = g(x)$ of two functions shown below.



(a) If possible, evaluate f at the points $x \in \{0, 5, 10\}$. Round to the nearest integer.

(b) If possible, evaluate g at the points $x \in \{0, 5, 10\}$. Round to the nearest integer.

(c) If possible, evaluate $f \circ g$ at the points $x \in \{0, 5, 10\}$. Round to the nearest quarter (i.e., your answers may be decimals that end in .0, .25, .5, or .75).

(d) If possible, evaluate $g \circ f$ at the points $x \in \{0, 5, 10\}$. Round to the nearest quarter (i.e., your answers may be decimals that end in .0, .25, .5, or .75).

(e) Does the equation $f(x) = g(x)$ have any solutions? If so, list them.

(f) What is the domain and range of f?

(g) What is the domain and range of g?

(h) What is the domain and range of $f \circ g$? [Hint: use images to find the range.]

(i) What is the domain and range of $g \circ f$?

(j) What is the domain and range of $f \circ f$?

(k) What is the domain and range of $g \circ g$?

(l) List and classify the global extrema of f.

(m) List and classify the global extrema of g.

(n) List and classify the local extrema of f.

(o) List and classify the local extrema of g.

(p) Solve the inequality $f(x) \leq g(x)$.

(q) Solve the inequality $f(x) \geq x$. [Hint: plot $y = x$.]

7. Let function $f\colon \mathbb{R} \to \mathbb{R}$ have the property that for every pair $x_1, x_2 \in \mathbb{R}$,

$$x_1 < x_2 \implies f(x_1) < f(x_2).$$

This property is called *strictly increasing*. Prove the following statements about f or provide a counterexample (i.e., provide a function f that demonstrates the statement is false).

(a) f is one-to-one.

(b) f is onto.

8. Re-arrange each expression to solve for y. Simplify.

(a) $\log_2 x + \log_3 y + \log_4 z = 1$

(b) $\log_2 x + \log_2 y + \log_2 z = 1$

(c) $\ln(x^y) + yx^2 = 1$

9. Solve for x. Express your solution simplified but exactly. There may be multiple solutions, extraneous solutions, or no solutions.

(a) $\log x = 50$

(b) $\log_4 x = 16$

(c) $e^{x^2} = 2$

(d) $e^{-x^2} = 2$

(e) $\log_x 10 = 1$

(f) $\log_x 10 = 2$

(g) $(\log_3 x)^2 + \log_3 x^2 + 1 = 0$

(h) $e^{2x} + 2e^x + 1 = 4$

(i) $x^2 = 10$

(j) $\log_3 x = \log_9 x$

(k) $\log_3 x = \log_9 x^2$

10. Let

$$f(x) = \frac{a}{1 + be^x}$$

with $a > 0$ and $b < 0$.

(a) As $x \to \infty$, how does $1 + be^x$ behave?

(b) As $x \to -\infty$, how does $1 + be^x$ behave?

(c) What are the horizontal asymptotes of the graph $y = f(x)$?

(d) Plot $y = f(x)$.

11. Consider the equilateral triangle in $\mathbb{R}^2$ with one vertex at $(0, 0)$, one vertex at $(1, 0)$, and another vertex on the unit circle at (x_0, y_0). Find (x_0, y_0). [Hint: draw the circle and triangle.]

(a) Find (x_0, y_0). For simplicity, assume that $x_0, y_0 > 0$.

(b) Find the length, s, of the arc on the unit circle from $(0, 0)$ to (x_0, y_0) in the positive direction.

(c) Interpret the length, as an angle, θ, in radians. What is its equivalent measure in degrees?

(d) What is the area of the triangle? What is the area of the sector of the circle that is swept out by θ?

(e) Suppose an object traverses this arc in 1 second. Provide the following speeds, including units. Take the radius of the unit circle to be 1 centimeter here.

i) What is the average linear speed of the object?

ii) What is the average angular speed of the object (in rad/s)?

iii) What is the average angular speed of the object (in rev/min)?

12. What is the average angular speed of the minute hand of an analogue clock measured in rpm? What is its average angular velocity?