Continuity

Prerequisite Concepts:

1. Knowledge of functions of the general type $f : X \to Y$
2. Compositions of functions
3. Limits of functions of the type $f : \mathcal{D}(f) \to \mathbb{R}$ with $\mathcal{D}(f) \subset \mathbb{R}$
4. Understanding of $\mathbb{R}$, $\mathbb{Q}$, and the relation between them
5. The natural domain of a function $f : \mathcal{D}(f) \to \mathbb{R}$ with a given evaluation rule $f(x)$ provided

Basic Concepts:

1. Intuitive understanding of continuity
2. Distance in $\mathbb{R}$
3. Limit points

Advanced Concepts:

1. Precise understanding of continuity
2. Open and closed sets
3. Limit point
4. Image and pre-image

1 Intuition

The first definition of continuity that students most often hear is *f is continuous if its graph can be drawn without picking up your pen.* As written, this definition is insufficient and (due to its lack of precision) incorrect. First, as we will see, the function $f(x) = 1/x$ is continuous on the interval $(0, 1]$, but its graph has infinite length and thus cannot be drawn. This is of course not in the spirit of the definition as it could be drawn *in concept*. A more fundamental problem is that $f(x) = 1/x$ is continuous on the domain $[-1, 0) \cup (0, 1]$ as well. The definition makes more sense if we split it into two notions.

Definition 1.1 (Intuitive continuity): *A* continuous curve *in the plane $\mathbb{R}^2$ is a curve that could be drawn (possibly in an infinite amount of time) without picking up your pen. A function $f\colon \mathcal{D}(f) \to \mathbb{R}$ where $\mathcal{D}(f) \subset \mathbb{R}$ is continuous if, for every interval $I \subset \mathcal{D}(f)$, the graph of f above I is a continuous curve in $I \times \mathbb{R}$.*

The fact that $f(x) = 1/x$ on $\mathbb{R}\backslash\{0\}$ is consistent with this "definition". In fact, using this definition, we expect to see that every polynomial and rational function is continuous.

Example 1.1: Consider the functions $f\colon \mathcal{D}(f) \to \mathbb{R}$ where $\mathcal{D}(f) \subset \mathbb{R}$ is the largest subset of $\mathbb{R}$ on which its rule for evaluation is valid (i.e., their natural domains).

(1.1.1) $f(x) = \sqrt{x}$ is continuous

(1.1.2) $f(x) = 1/x$ is continuous

(1.1.3) $f(x) = \begin{cases} 1/x, & x \neq 0 \\ 0, & x = 0 \end{cases}$ is not continuous.

(1.1.4) $f(x) = \begin{cases} |x|, & x \neq 0 \\ 1, & x = 0 \end{cases}$ is not continuous.

Graph each of the functions in Example 1.1. You will notice that the manner in which (1.1.3) and (1.1.4) are discontinuous are fundamentally different. The first is an example of an *infinite discontinuity* while the second is an example of a *removable discontinuity*.

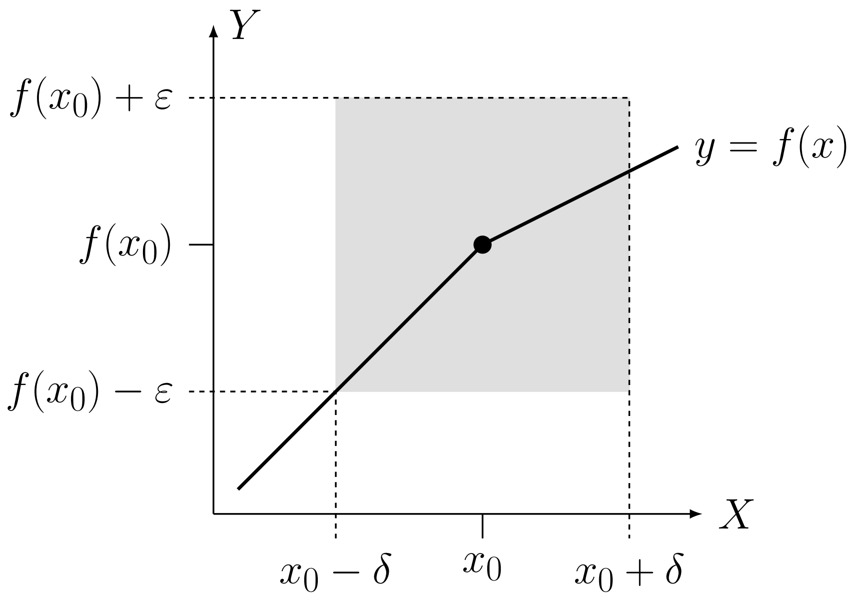

Figure 1: A function that is continuous at x_0.

2 Continuity

Now, let us precisely define what it means for a function $f\colon \mathcal{D}(f) \to \mathbb{R}$ to be continuous.

Definition 2.1 (Continuity at a point $x_0 \in \mathbb{R}$): *Let $f\colon \mathcal{D}(f) \to \mathbb{R}$ with $\mathcal{D}(f) \subset \mathbb{R}$. We say that f is* continuous at $x_0 \in \mathcal{D}(f)$ *exactly when the following criteria is met:*

> *For all $\varepsilon > 0$, there exists a $\delta > 0$ such that if $x \in \mathcal{D}(f)$ and $|x - x_0| < \delta$, then $|f(x) - f(x_0)| < \varepsilon$.*

If f is not continuous at $x_0 \in \mathcal{D}(f)$, then it is discontinuous at x_0.

Remark: The criteria that $x \in \mathcal{D}(f)$ is often not explicitly stated, but it is always implied due to the evaluation of $f(x)$. The same is true regarding the definition of a limit.

Theorem 2.1 (Continuity implies the limit): *Let $f\colon \mathcal{D}(f) \to \mathbb{R}$ with $\mathcal{D}(f) \subset \mathbb{R}$ be continuous at $x_0 \in \mathcal{D}(f)$. Then $\lim_{x\to x_0} f(x) = f(x_0)$.*

Proof. Let $\varepsilon > 0$. Because f is continuous at x_0, there exists a $\delta > 0$ such that

$$|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$

Setting $L = f(x_0)$, we see that

$$
\begin{aligned}
0 < |x - x_0| < \delta &\implies |x - x_0| < \delta \\
&\implies |f(x) - f(x_0)| < \varepsilon && \text{(continuity)} \\
&\implies |f(x) - L| < \varepsilon && (f(x_0) = L.
\end{aligned}
$$

This is precisely the definition of continuity, so we must have $\lim_{x \to x_0} f(x) = L$ with $L = f(x_0)$.

$\square$

Theorem 2.2 (Limit implies continuity): *Let $f\colon \mathcal{D}(f) \to \mathbb{R}$ with $\mathcal{D}(f) \subset \mathbb{R}$ and suppose that at some $x_0 \in \mathcal{D}(f)$, we have that $\lim_{x \to x_0} f(x) = f(x_0)$. Then f is continuous at x_0.*

Proof. Problem *4* $\square$

Definition 2.2 (Continuous function): *Let f be a function. If f is continuous at every $x_0 \in \mathcal{D}(f)$, then f is said to be* continuous*. A function that is not continuous is* discontinuous*.*

Remark: It is a common mistake to say that if a function $f\colon \mathcal{D}(f) \to \mathbb{R}$ is discontinuous if it is not continuous for all $x_0 \in \mathbb{R}$. This would then imply functions such as $f(x) = 1/x$ with $\mathcal{D}(f) = \{x \in \mathbb{R} \colon x \neq 0\}$ are discontinuous. This is not true.

Although we must have $x_0 \in \mathcal{D}(f)$ for f to be continuous at x_0, it is useful to allow us to talk about discontinuities at any limit point of $\mathcal{D}(f)$. The precise notion of a limit point will be discussed in the next section. For now, it is sufficient to note that if $\Omega = (a, b) \cup (b, c)$, then the set of limit points of Ω is the closed interval $[a, c]$ (note that the hole at b and both endpoints are filled).

Definition 2.3 (Removable discontinuity): *Let $f\colon \mathcal{D}(f) \to \mathbb{R}$ with $\mathcal{D}(f) \subset \mathbb{R}$ and x_0 a limit point of $\mathcal{D}(f)$. If f is not continuous at x_0, but there exists some $a \in \mathbb{R}$ such that the function*

$$
g(x) = \begin{cases} f(x), & x \in \mathcal{D}(f) \setminus \{x_0\} \\ a, & x = x_0 \end{cases}
$$

is continuous, then f is said to have a removable discontinuity *at x_0.*

Remark: The function g is the same as f, just re-defined at x_0 to a. Graphs of functions with removable discontinuities have holes in them.

Definition 2.4 (Jump discontinuity): *Let $f\colon \mathcal{D}(f) \to \mathbb{R}$ with $\mathcal{D}(f) \subset \mathbb{R}$ and $x_0 \in \mathbb{R}$ such that $\lim_{x \to x_0^-} f(x) = L_1$ and $\lim_{x \to x_0^+} f(x) = L_2$ with $L_1 \neq L_2$. Then f is said to have a* jump discontinuity *at x_0.*

Remark: Graphs of functions with jump discontinuities have gaps with a *positive finite* vertical distance.

Definition 2.5 (Oscillating discontinuity): *Let $f\colon \mathcal{D}(f) \to \mathbb{R}$ with $\mathcal{D}(f) \subset \mathbb{R}$ and let $x_0 \in \mathbb{R}$ be a limit point of $\mathcal{D}(f)$. If f is bounded on some interval $(x_0 - \delta, x_0 + \delta)$ with $\delta > 0$ but either $\lim_{x\to x_0^+} f(x)$ or $\lim_{x\to x_0^-} f(x)$ does not exist, then f is said to have an* oscillating discontinuity *at x_0.*

Remark: Graphs of functions with oscillating discontinuities are impossible to draw. They have a top "envelope" and bottom "envelope" that do not meet at x_0. For example, if $f(x) = \sin(1/x)$, take $g(x) = 1$ as the top envelope and $h(x) = -1$ as the bottom. The graph of f intersects the graphs of g and h infinitely many times over the interval $(-\delta, \delta)$ for any $\delta > 0$.

Remark: It is possible to drop the requirement of boundedness for the definition of an oscillating discontinuity. However, doing so requires the notion of lim inf and lim sup (limits combined with infima and suprema), so we do not discuss those here.

Definition 2.6 (Infinite discontinuity): *Let $f\colon \mathcal{D}(f) \to \mathbb{R}$ with $\mathcal{D}(f) \subset \mathbb{R}$ and let $x_0 \in \mathbb{R}$ be a limit point of $\mathcal{D}(f)$. If either $\lim_{x\to x_0^-} f(x) = \pm\infty$ or $\lim_{x\to x_0^+} f(x) = \pm\infty$, then f is said to have an infinite discontinuity at x_0.*

Remark: The graph of a function with an infinite discontinuity has a vertical asymptote.

Example 2.1: Consider the functions $f\colon \mathcal{D}(f) \to \mathbb{R}$ with their natural domain $\mathcal{D}(f) \subset \mathbb{R}$.

(2.1.1) $f(x) = (x^2 - 1)/(x + 1)$ has a removable discontinuity at $x_0 = -1$.

(2.1.2) $f(x) = (x^2 - 1)/(x + 2)$ has an infinite discontinuity at $x_0 = -2$.

(2.1.3) $f(x) = \tan(x)$ is continuous, but has infinitely many infinite discontinuities at each of the points $\frac{\pi}{2} + n\pi$ for $n \in \mathbb{Z}$.

(2.1.4) $f(x) = 1/\tan(x)$ is continuous but has infinitely many removable discontinuities at each of the points $\frac{\pi}{2} + n\pi$ and infinite discontinuities at the points $n\pi$ for $n \in \mathbb{Z}$.

(2.1.5) $f(x) = \tan(1/x)$ has infinitely many infinite discontinuities for any interval $(-\delta, \delta)$ with $\delta > 0$.

(2.1.6) $f(x) = |x|/x$ has a jump discontinuity at $x_0 = 0$.

A few important facts about continuity:

1. For a function f to be continuous at x_0, we must have $x_0 \in \mathcal{D}(f)$.
2. For a function f to be discontinuous at x_0, x_0 must be a limit point of $\mathcal{D}(f)$.
3. A continuous function may have discontinuities at points *outside* of its domain but is continuous *on its domain*.
4. **f is continuous at a limit point of its domain exactly when** $\lim_{x\to x_0} f(x) = f(x_0)$
5. The statements "f is continuous" and "f is continuous on its domain" are equivalent.

3 Topology

In the previous section, the notion of limit points of a domain $\mathcal{D}(f) \subset \mathbb{R}$ was introduced. Here we will define what this means. Intuitively, these points are "anywhere it makes sense to talk about a limit of $f(x)$," but we will make this precise and discuss continuity in terms of *open* sets.

Definition 3.1 (Open ball): *Let $\delta > 0$. We say that $B_\delta(x_0)$ is the* open ball of radius δ centered at x_0. *This $B_\delta(x_0)$ is defined as*

$$B_\delta(x_0) \coloneqq \{x \colon |x - x_0| < \delta\}.$$

Definition 3.2 (Interior point): *Let $\Omega \subset \mathbb{R}$. We say that $x_0 \in \Omega$ is an* interior point *(to Ω) if there exists some $\delta > 0$ such that $B_\delta(x_0) \subset \Omega$.*

Definition 3.3 (Open set): *Let $\Omega \subset \mathbb{R}$. We say that $\Omega \subset \mathbb{R}$ is* open *if every $x_0 \in \Omega$ is an interior point.*

Definition 3.4 (Limit point): *Let $\Omega \subset \mathbb{R}$. We say that $x_0 \in \mathbb{R}$ is a* limit point *(of Ω) if for all $\delta > 0$, there exists some $x \in \Omega \cap B_\delta(x_0)$ such that $x \neq x_0$.*

Remark: Intuitively, a limit point of Ω is any point x_0 that can be approached by some sequence that is contained in Ω and does not contain the value x_0 as any of its terms.

Definition 3.5 (Closed set): *Let $\Omega \subset \mathbb{R}$. We say that Ω is* closed *if Ω contains all of its own limit points.*

Definition 3.6 (Isolated point): *Let $\Omega \subset \mathbb{R}$. We say the $x_0 \in \mathbb{R}$ is an* isolated point *(of Ω) if there exists some $\delta > 0$ such that $B_\delta(x_0) \cap \Omega = \{x_0\}$ (i.e., $B_\delta(x_0)$ contains no points in Ω except for x_0 itself).*

Remark: It is important that we are considering subsets of $\mathbb{R}$. For example, if $\Omega = \{x \colon x^2 < 2\} \subset \mathbb{R}$, then $\sqrt{2}$ is a limit point of Ω. However, if we considered $\Omega = \{x \colon x^2 < 2\} \subset \mathbb{Q}$, then $\sqrt{2}$ would not be a limit point because $\sqrt{2} \notin \mathbb{Q}$. We will only consider sets embedded (i.e., subsets of) in $\mathbb{R}$.

Example 3.1: Consider $\mathbb{Z} \subset \mathbb{R}$. Then each $n \in \mathbb{Z}$ is an isolated point because $B_{1/2}(n) \cap \mathbb{Z} = \{n\}$ for each integer n.

Example 3.2: Consider $\mathbb{Q} \subset \mathbb{R}$. Then each $q \in \mathbb{Q}$ is a limit point because for any $\delta > 0$, there is a rational number $r \in \mathbb{Q}$ such that $q < r < q + \delta$. If we assume $\delta \in \mathbb{Q}$, we can choose the point $r = q + \frac{1}{2}\delta$.

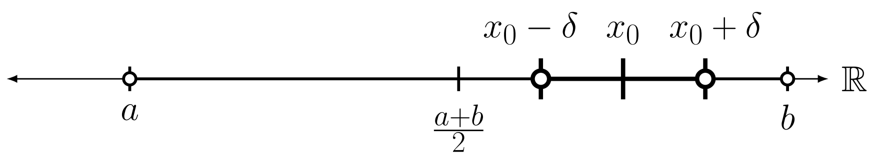

Figure 2: Illustration of the interval (a, b) being open.

EXAMPLE 3.3: Consider $\mathbb{Q} \subset \mathbb{R}$. Then each $q \in \mathbb{Q}$ is not an interior point because for any $\delta > 0$, there is an irrational number $x \in \mathbb{R} \setminus \mathbb{Q}$ such that $q < r < q + \delta$. If we assume $\delta \in \mathbb{R} \setminus \mathbb{Q}$, we can choose the point $r = q + \frac{1}{2}\delta$.

EXAMPLE 3.4: Let $\Omega = \{x \in \mathbb{R} \colon x = 1/n, n = 1, 2, 3, \ldots\}$. Then $0 \notin \Omega$, but 0 is a limit point of Ω.

EXAMPLE 3.5: Let $\Omega = (a, b)$ with $a < b$. Then Ω is open. To see this, choose any $x_0 \in \Omega$ and set $\delta = \frac{1}{2}\min(x_0 - a, b - x_0)$. For simplicity, assume that x_0 is in the right-half of the interval so that $\delta = \frac{1}{2}(b - x_0)$. Then $B_\delta(x_0) = \{x \in \mathbb{R} \colon |x - x_0| < \delta\} = (x_0 - \delta, x_0 + \delta) \subset (a, b)$ as $\delta > 0$ was chosen so that $x_0 + \delta < b$ and $x_0 - \delta > a$. See Figure 2.

DEFINITION 3.7 (INTERIOR): *Let* $\Omega \subset \mathbb{R}$. *Then* $\mathring{\Omega} = \{x \in \Omega \colon x \textit{ is an interior point}\}$ *is the* interior *of* Ω.

DEFINITION 3.8 (CLOSURE): *Let* $\Omega \subset \mathbb{R}$. *Then* $\overline{\Omega} = \Omega \cup \{\textit{limit points of } \Omega\}$ *is the* closure *of* Ω.

DEFINITION 3.9 (BOUNDARY): *Let* $\Omega \subset \mathbb{R}$. *Then* $\partial\Omega = \overline{\Omega} \setminus \mathring{\Omega}$ *is the* boundary *of* Ω.

REMARK: Note that $\mathring{\Omega} \subset \Omega \subset \overline{\Omega}$. Also, Ω is closed exactly when $\Omega = \overline{\Omega}$ and is open exactly when $\Omega = \mathring{\Omega}$.

REMARK: Re-defining $B_\delta(x_0)$ to the appropriate 2-dimensional distance, these definitions hold for $\Omega \subset \mathbb{R}^2$.

EXAMPLE 3.6: Let $\Omega = (-\infty, 0) \cup (0, 1] \cup \{2\} \cup (3, \infty) \subset \mathbb{R}$. Then

$$\mathring{\Omega} = (-\infty, 0) \cup (0, 1) \cup (3, \infty),$$

$$\overline{\Omega} = (-\infty, 0] \cup \{2\} \cup [3, \infty),$$

$$\partial\Omega = \{0, 1, 2, 3\}.$$

Ω has exactly one isolated point at 2. Ω is neither open (as not every point in Ω is interior) nor closed (as it does not contain limit points at 0 or 1).

Note that the definition of continuity can be written in terms of open balls, but it will be useful to have two notions first.

Definition 3.10 (Image): *Let* $f\colon X \to Y$ *and* $\Omega \subset X$. *Then the set* $f[X] \coloneqq \{y \in Y : y = f(x) \text{ for some } x \in \Omega\}$ *is the* image *of* Ω *under* f.

Definition 3.11 (Pre-Image): *Let* $f\colon X \to Y$ *and* $\Gamma \subset Y$. *Then the set* $f^{-1}[\Gamma] \coloneqq \{x \in X : y = f(x) \text{ for some } y \in \Gamma\}$ *is the* pre-image *of* Γ *under* f.

We state the topological definition of continuity as a theorem and then prove its equivalence to Definition 2.2. For simplicity, we consider $\mathcal{D}(f) = \mathbb{R}$.

Theorem 3.1 (Topological Continuity): *Let* $f\colon \mathbb{R} \to \mathbb{R}$. *Then* f *is continuous if and only if the pre-image of every open set* $\Gamma \subset \mathbb{R}$ *is open.*

Proof. ($\Longleftarrow$) Let f be continuous. Choose any open set $\Gamma \subset \mathbb{R}$. We must show that $f^{-1}[\Gamma]$ is open. Choose any $x_0 \in f^{-1}[\Gamma]$. Then $f(x_0) \in \Gamma$. Because Γ is open there is some $\varepsilon > 0$ such that $B_\varepsilon(f(x_0)) \subset \Gamma$ (i.e., all y-values such that $|y - f(x_0)| < \varepsilon$ is true are in Γ). Because f is continuous there is a $\delta > 0$ such that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$. This is exactly the same as $f[B_\delta(x_0)] \subset B_\varepsilon(f(x_0))$. Because $B_\varepsilon(f(x_0)) \subset \Gamma$, we must have $B_\delta(x_0) \subset f^{-1}[\Gamma]$. Thus x_0 is an interior point of $f^{-1}[\Gamma]$. Because Γ was arbitrary, the pre-image of any open set must also be open.

($\Longrightarrow$) Let f have the property that the pre-image of every open set is open. Choose any $x_0 \in \mathcal{D}(f) = \mathbb{R}$ and any $\varepsilon > 0$. Let $\Gamma = B_\varepsilon(f(x_0))$. Because Γ is open, the pre-image $f^{-1}[\Gamma] = \{x \in \mathbb{R} : |f(x) - f(x_0)| < \varepsilon\}$ is open. Because $x_0 \in f^{-1}[\Gamma]$ and $f^{-1}[\Gamma]$ is open, there exists some $\delta > 0$ such that $B_\delta(x_0) \subset f^{-1}[\Gamma]$. Then we must have $f[B_\delta(x_0)] \subset \Gamma = B_\varepsilon(f(x_0))$, which is equivalent to the statement that $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$. Because $\varepsilon > 0$ was arbitrary, f is continuous at x_0. Because x_0 was arbitrary, f is continuous. $\square$

Remark: Go to Figure 1. Note that the intervals $B_\delta(x_0)$ and $B_\varepsilon(f(x_0))$ are marked. Draw an open interval Γ on the Y-axis that contains $B_\varepsilon(f(x_0))$. Draw $f^{-1}[\Gamma]$ on the X-axis. It should contain $B_\delta(x_0)$.

A few important facts about topology:

1. A set Ω has four options: open, closed, neither, both. The sets $\emptyset$ and $\mathbb{R}$ are the only subsets of $\mathbb{R}$ that are both open and closed. A half-open interval such as $(a, b]$ (with $a < b$) is neither.
2. The boundary of an *interval* is the set of its endpoints.
3. The set of limit points *of an interval* is the closure of the interval.
4. The set $B_\delta(x_0)$ is the set of points at a distance *strictly* less than δ from x_0. This notion makes sense in any set in which you can measure distance (i.e., $\mathbb{R}$, $\mathbb{R}^2$, $\mathbb{R}^3$, $\mathbb{R}^d$, $\mathbb{C}$, and much more).
5. $\Omega \subset \mathbb{R}$ is open if and only if $\Omega^c \coloneqq \mathbb{R} \setminus \Omega$ is closed. (simple proof.)

4 A Few Properties

There are a few properties of limits that are especially important.

THEOREM 4.1 (CONTINUITY PROPERTIES): *Let $f\colon \mathcal{D}(f) \to \mathbb{R}$ and $g\colon \mathcal{D}(g) \to \mathbb{R}$ be continuous functions. Then the following functions are also continuous on their appropriate domains.*

1. $f + g$
2. $f - g$
3. $f \cdot g$
4. f/g
5. kf *for any* $k \in \mathbb{R}$
6. f^n, *for any* $n = 1, 2, 3, \ldots$
7. $f^{1/n}$, *for any* $n = 1, 2, 3, \ldots$
8. $f \circ g$

Proof. Items 1–7 follow directly from the corresponding properties of limits. Property 8 is effectively two-applications of Theorem 3.1, and a proof can be found in the text. □

REMARK: A result of these properties are that polynomials, rational functions, and algebraic functions are all continuous on their domain.

A few important facts:

1. Most functions for which a closed-form representation exists are continuous.
2. Every polynomial is continuous and does not have any discontinuities.
3. Every rational function is continuous but may have removable or infinite discontinuities (but only finitely many of either).
4. Algebraic functions are continuous but may have removable, jump, or infinite discontinuities (but only finitely many of either).

5 Problems

1) Find *all* of the solutions $x \in \mathbb{R}$ to the equation $\sin(1/x) = 1$. Express your answer in the form $x_n = g(n)$ for $n \in \mathbb{Z}$ such that $\sin(1/g(n)) = 1$ for all $n \in \mathbb{Z}$.

2) Consider the functions $f\colon \mathcal{D}(f) \to \mathbb{R}$ below where $\mathcal{D}(f) \subset \mathbb{R}$ is the largest subset such that the provided evaluation rule is valid. Provide $\mathcal{D}(f)$, plot each function, and state if they are continuous or discontinuous. For any function that is discontinuous, provide all points $x_0 \in \mathcal{D}(f)$ where f is not continuous. Graphing utilities are allowed, but the graph must still be drawn as part of your turned in work. You may assume without proof that $g(x) = \sin(x)$ is continuous on $\mathbb{R}$. Some graphs may be impossible to draw. In such a case, do your best and provide a short description of its behavior.

2.a) $f(x) = 1$

2.b) $f(x) = x/x$

2.c) $f(x) = x/\sqrt{x^2}$

2.d) $f(x) = \sin(1/x)$

2.e) $f(x) = \begin{cases} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

2.f) $f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

2.g) $f(x) = \begin{cases} x^2\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

2.h) $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$

3) For each of the functions in Problem *2*, find every discontinuity and classify each one. If there are no discontinuities, state this.

4) Prove Theorem 2.2. [Hint: reverse the line of reasoning from the preceding proof.]

5) Let $\delta > 0$ and $x_0 \in \mathbb{R}$. Write $B_\delta(x_0)$ in interval notation.

6) Read the remark at the end of the proof of Theorem 3.1. Follow its instructions and recreate the drawing. Label Γ, $f^{-1}[\Gamma]$, $B_\varepsilon(f(x_0))$, and $B_\delta(x_0)$ as *distinct* open intervals on the appropriate X or Y-axis.

7) Consider $\mathbb{Q} \subset \mathbb{R}$. Does $\mathbb{Q}$ have any interior points? Does $\mathbb{Q}$ have any limit points? Find $\mathring{\mathbb{Q}}$ and $\overline{\mathbb{Q}}$.

8) Consider the function

$$f(x) = \begin{cases} 1, & x > 0 \\ a, & x = 0 \\ 1 - x^2 & x < 0 \end{cases}.$$

Is it possible to choose $a \in \mathbb{R}$ such that f is continuous? If so, what is it? If not, why?

9) Consider the function

$$f(x) = \begin{cases} x^2 - 4x + 4, & x > a \\ 0, & x \leq a \end{cases}$$

where $a \in \mathbb{R}$. Find a so that f is continuous. Then plot f.

10) Prove that the constant function $f(x) = 0$ is continuous.

11) Prove that the identity function $f(x) = x$ is continuous.

12) Prove that the function

$$f(x) = \begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R} \setminus \mathbb{Q} \end{cases}$$

is continuous at $x_0 = 0$. Use the definition of continuity provided in Definition 2.2.

13) The Heaviside function is defined as

$$H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}.$$

Under what conditions on $f\colon \mathbb{R} \to \mathbb{R}$ will the function $f \cdot H$ be continuous? Explain. [Hint: it may be helpful to graph $f \cdot H$ by first graphing a representative f and then graphing $f \cdot H$.]

14) Let H be the Heaviside function and $f\colon \mathbb{R} \to \mathbb{R}$. Under what conditions will the function $f \circ H$ be continuous? Explain.