Continuity Solutions

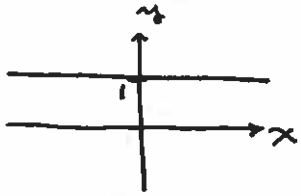
1) Solve $\sin(1/x) = 1$. $1/x$ must be an angle at the top of the unit circle, so $\frac{1}{x} = \frac{\pi}{2} + 2n\pi$ for any $n \in \mathbb{Z}$. Thus, any x of the form $x = \dfrac{1}{\frac{\pi}{2}(1+4n)} = \dfrac{2}{\pi(1+4n)}$ with $n \in \mathbb{Z}$ is a solution.

$$\boxed{x_n = \frac{2}{\pi(1+4n)}}$$

2.a) $f(x) = 1$

$D(f) = \mathbb{R}$

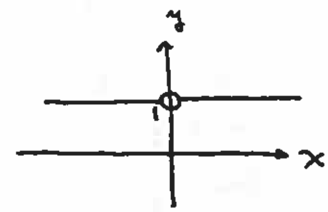
continuous



2.b) $f(x) = \frac{x}{x}$

$D(f) = \mathbb{R}\setminus\{0\}$

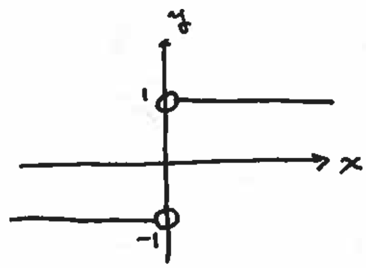
Continuous



2.c) $f(x) = \dfrac{x}{\sqrt{x^2}} = \dfrac{x}{|x|}$

$D(f) = \mathbb{R}\setminus\{0\}$

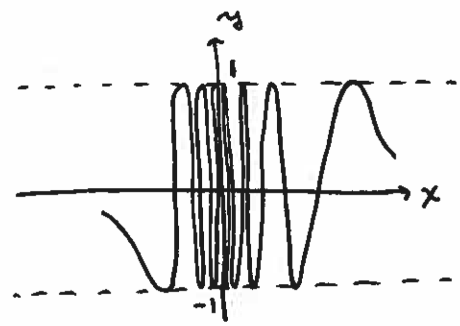
continuous



2.d) $f(x) = \sin(1/x)$

$D(f) = \mathbb{R}\setminus\{0\}$

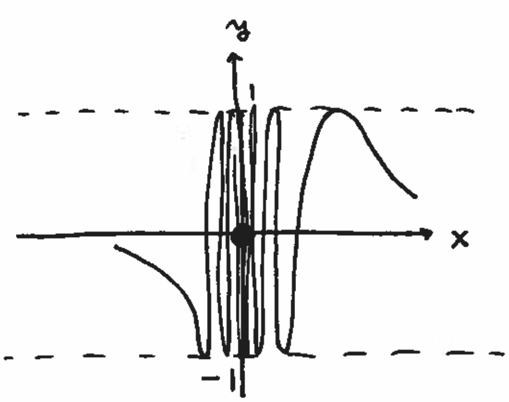
Continuous



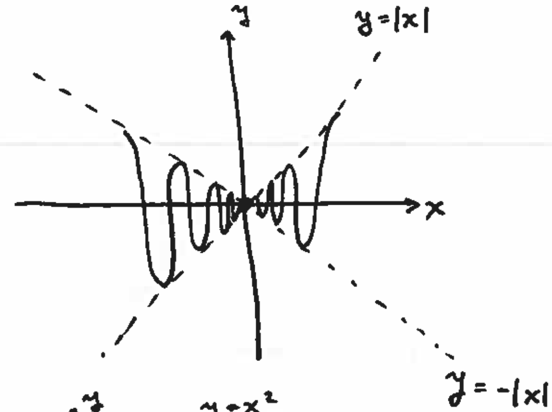
2.e) $f(x) = \begin{cases} \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$

$D(f) = \mathbb{R}$

discontinuous. oscillating discontinuity at $x_0 = 0$.



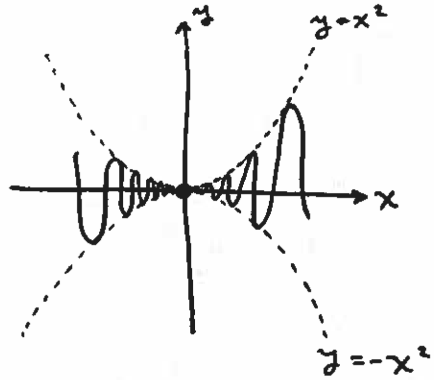
2.f) $f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$



$\mathcal{D}(f) = \mathbb{R}$

Continuous

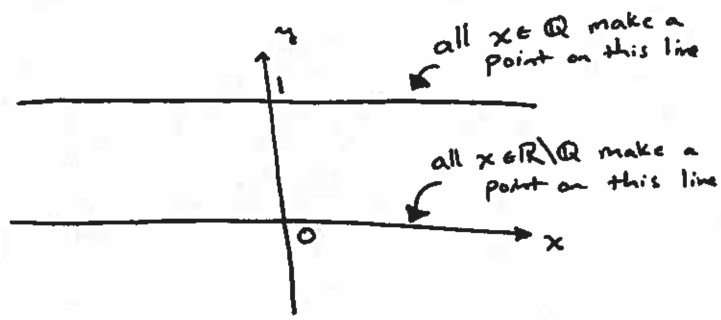
2.g) $f(x) = \begin{cases} x^2\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$



$\mathcal{D}(f) = \mathbb{R}$

Continuous

2.h) $f(x) = \begin{cases} 1, & x \in \mathbb{Q} \\ 0, & x \notin \mathbb{Q} \end{cases}$



$\mathcal{D}(f) = \mathbb{R}$

discontinuous. There is an oscillating discontinuity at each $x_0 \in \mathbb{R}$.

Because $\mathbb{Q}$ is dense in $\mathbb{R}$, the lines appear indistinguishable. However, there are "more" points on the $y=0$ line.

3) a) no discontinuities

b) removable disc. at $x_0 = 0$

c) jump disc. at $x_0 = 0$

d) oscillating disc. at $x_0 = 0$

e) oscillating disc. at $x_0 = 0$

f) no discontinuities

g) no discontinuities

h) oscillating disc. at every $x_0 \in \mathbb{R}$.

4) <u>Claim</u>: Let $f: D(f) \to \mathbb{R}$ with $D(f) \subset \mathbb{R}$ and suppose that at some $x_0 \in D(f)$, we have that $\lim_{x \to x_0} f(x) = f(x_0)$. Then f is continuous at x_0.

<u>Proof</u>: We must show that $\forall \varepsilon > 0$, $\exists \delta > 0$ s.t. $|x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon$.

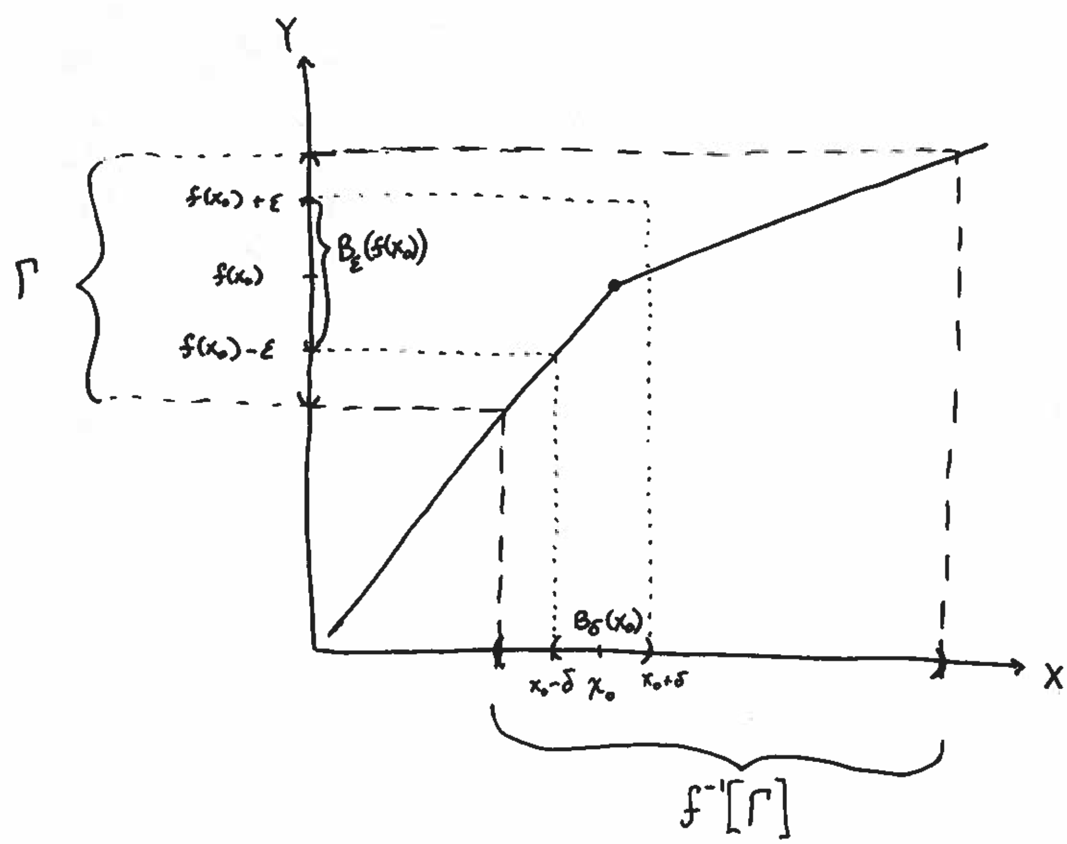
Let $\varepsilon > 0$. Because $\lim_{x \to x_0} f(x) = f(x_0)$, $\exists \delta > 0$ s.t.

$$0 < |x - x_0| < \delta \implies |f(x) - f(x_0)| < \varepsilon.$$

Thus, the only other point that we must consider is $x = x_0$. However, in this case, $|f(x_0) - f(x_0)| = 0$. Thus $|f(x) - f(x_0)| < \varepsilon$ for all x s.t. $|x - x_0| < \delta$. ■

5) $B_\delta(x_0) = \{x \in \mathbb{R} : |x - x_0| < \delta\} = \boxed{(x_0 - \delta,\ x_0 + \delta)}$

6)



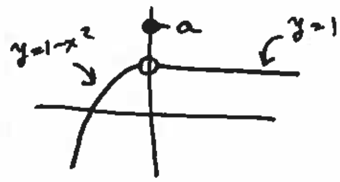
7) Consider $\mathbb{Q} \subset \mathbb{R}$.

a) $\mathbb{Q}$ does not have any interior points. If $x_0 \in \mathbb{Q}$ were an interior point, then there would exist some $\delta > 0$ s.t. $(x_0 - \delta, x_0 + \delta)$ contained only rational points. This is impossible, so $\mathbb{Q}$ has no interior point.

b) Each $x_0 \in \mathbb{R}$ is a limit point of $\mathbb{Q}$. Pick any $x_0 \in \mathbb{R}$ and any $\delta > 0$. Then the interval $(x_0 - \delta, x_0 + \delta)$ has a rational number $r \in \mathbb{Q} \cap (x_0 - \delta, x_0 + \delta)$ with $r \neq x_0$.

c) $\mathring{\mathbb{Q}} = \{\}$ and $\overline{\mathbb{Q}} = \mathbb{R}$.

8) $f(x) = \begin{cases} 1, & x > 0 \\ a, & x = 0 \\ 1 - x^2, & x < 0 \end{cases}$

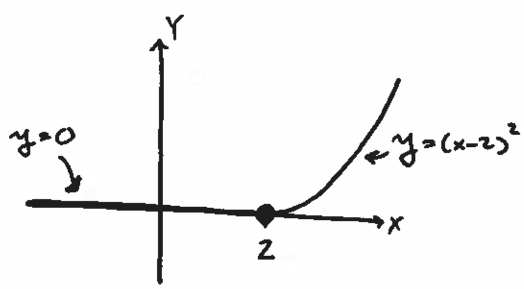


$\boxed{f \text{ is continuous when } a = 1}$

Note: $\lim\limits_{x \to 0^-} f(x) = \lim\limits_{x \to 0^-} (1 - x^2) = 1$ and $\lim\limits_{x \to 0^+} f(x) = \lim\limits_{x \to 0^+} 1 = 1$.

9) $f(x) = \begin{cases} x^2 - 4x + 4, & x > a \\ 0, & x \leq a \end{cases}$

Note:
1. $x^2 - 4x + 4 = (x-2)^2$
2. $\lim\limits_{x \to a^-} f(x) = (a-2)^2$
3. $\lim\limits_{x \to a^+} f(x) = 0$
4. $\lim\limits_{x \to a} f(x) = f(a) = 0$ when $\boxed{a = 2}$



10) Claim: $f(x)=0$ is continuous

Proof: Let $\varepsilon>0$ and pick any $\delta>0$. Then, for any $x_0 \in \mathbb{R}$ and $x \in \mathbb{R}$, we have $|f(x)-f(x_0)| = |0-0| = 0 < \varepsilon$. Because $\varepsilon>0$ was arbitrary, f is continuous at x_0. Because $x_0 \in \mathbb{R}$ was arbitrary, f is continuous on $\mathbb{R}$. $\blacksquare$

11) Claim: $f(x)=x$ is continuous.

Proof: Pick any $x_0 \in \mathbb{R}$ and $\varepsilon>0$. Set $\delta=\varepsilon$. Then, for all $x \in \mathbb{R}$ s.t. $|x-x_0|<\delta$, we have

$$|f(x)-f(x_0)| = |x-x_0| < \delta = \varepsilon.$$

Because $\varepsilon>0$ was arbitrary, f is continuous at x_0. Because $x_0 \in \mathbb{R}$ was arbitrary, f is continuous on $\mathbb{R}$. $\blacksquare$

12) Claim: $f(x)=\begin{cases} x, & x \in \mathbb{Q} \\ 0, & x \in \mathbb{R}\setminus\mathbb{Q} \end{cases}$ is continuous at $x_0=0$.

Proof: Let $\varepsilon>0$ and set $\delta=\varepsilon$. Then for all x s.t. $|x|<\delta$, we have two cases. First, if $x \in \mathbb{Q}$, then

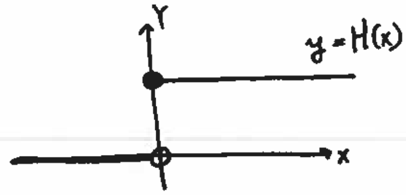
$$|f(x)-f(x_0)| = |x-0| = |x| < \delta \le \varepsilon.$$

Second, if $x \in \mathbb{R}\setminus\mathbb{Q}$, then

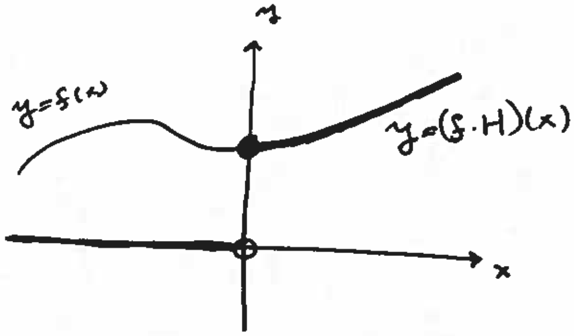
$$|f(x)-f(x_0)| = |0-0| = 0 < \varepsilon.$$

Thus $|f(x)-f(0)| < \varepsilon$ for all x s.t. $|x-0|<\delta$, and f is continuous at $x_0=0$. $\blacksquare$

13) Let $H(x) = \begin{cases} 1, & x \geq 0 \\ 0, & x < 0 \end{cases}$.



Note that $(f \cdot H)(x) = f(x) \cdot H(x) = \begin{cases} f(x), & x \geq 0 \\ 0, & x < 0 \end{cases}$. Thus, $f \cdot H$ will be continuous provided that 1) <u>f is continuous on $x \geq 0$</u> and 2) <u>$f(0) = 0$</u>.



14) Let $H(x)$ be as in 13). Note that $(f \circ H)(x) = \begin{cases} f(1), & x \geq 0 \\ f(0), & x < 0 \end{cases}$.

Thus, $f \circ H$ will be continuous if <u>$f(1) = f(0)$</u>.