Derivatives

Prerequisite Concepts:

1. Limit points of a set $X \subset \mathbb{R}$ (esp. when X is a finite union of intervals)
2. Continuity and limits
3. Compositions of functions, multiplication of functions, etc.

Basic Concepts:

1. Limit definition of the derivative
2. Tangent lines
3. Derivative notations
4. Derivative rules (product, quotient, etc.)

Advanced Concepts:

1. Linearity
2. Derivatives as linear approximations
3. Slopes of implicitly defined curves
4. Related rates
5. The derivative operator

1 Slope and Tangent Lines

Recall that the slope of a line given by $y = y_0 + m(x - x_0)$ is m. With the exception of a vertical line, every line in the plane can be written in this "point-slope" form and has a single slope $m \in \mathbb{R}$. We would like to extend the notion of slope to more general functions.

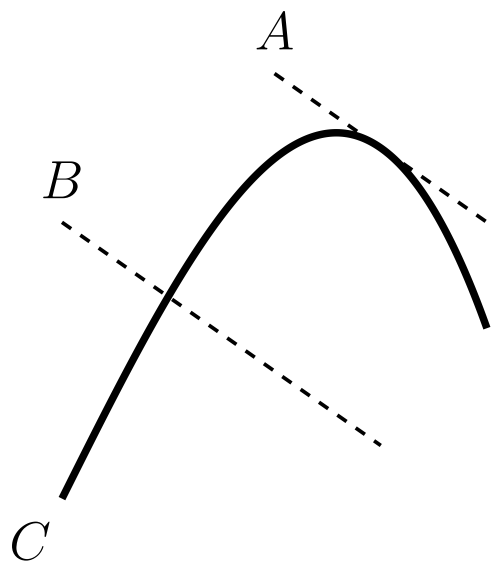

Figure 1: A curve C (solid). The line A is tangent to C at their point of intersection. The line B is not tangent to C at their point of intersection.

Definition 1.1 (Tangent): *We say that two curves are* tangent at a point *if they intersect at the point and their slopes are the same at that point.*

We would like to define the slope of a curve at a point *as* the slope of the tangent line at the point (assuming it exists and is uniquely defined), but this would clearly be circular. Instead, we will assume that our curve is the graph of a function (i.e., $y = f(x)$ for some function f) and define the slope via its function f. The case where the curve is not the such a graph will be handled later.

Definition 1.2 (Slope, Derivative, Tangent Line): *Let $f\colon \mathcal{D}(f) \to \mathbb{R}$ with $\mathcal{D}(f) \subset \mathbb{R}$ and $x_0 \in \mathcal{D}(f)$ be a limit point of $\mathcal{D}(f)$. Let the curve C be the graph of f. We say that C (or equivalently f) has* slope $f'(x_0) \in \mathbb{R}$ *if the limit*

$$f'(x_0) := \lim_{h \to 0} \frac{f(x_0 + h) - f(x_0)}{h} \tag{1}$$

exists and is finite. The value $f'(x_0)$ is the derivative of f at x_0. *The* tangent line to C at the point $(x_0, f(x_0))$ *is given by the equation*

$$y = f(x_0) + f'(x_0)(x - x_0). \tag{2}$$

Definition 1.3 (Differentiable): *Let $f\colon \mathcal{D}(f) \to \mathbb{R}$. If $f'(x_0)$ exists for some $x_0 \in \mathcal{D}(f)$, then f is said to be* differentiable at x_0. *If $f'(x_0)$ exists for all $x_0 \in \mathcal{D}(f)$, then f is said to be* differentiable.

In Figure 1, we see that the curve (line) A is tangent to the curve C *at their point of intersection* while B is not. The ratio $(f(x_0+h)-f(x_0))/h$ is called the *Newton Quotient* and is the slope of the secant line between the two points $(x_0, f(x_0))$ and $(x_0+h, f(x_0+h))$ on the graph of f.

Example 1.1: Consider the function $f(x) = x^2$ and suppose we wish to know the slope of f at $x_0 = 3$ as well as the equation for the tangent line at that point. Using (1), we can compute $f'(3)$ as follows:

$$\begin{aligned} f'(3) &= \lim_{h\to 0} \frac{f(3+h)-f(3)}{h} \\ &= \lim_{h\to 0} \frac{(3+h)^2-3^2}{h} \\ &= \lim_{h\to 0} \frac{9+6h+h^2-9}{h} \\ &= \lim_{h\to 0} (6+h) \\ &= 6. \end{aligned}$$

Thus the slope of f at $x_0 = 3$ is 6, and our equation for the tangent line at $x_0 = 6$ is $y = 9 + 6(x-3)$. Plot $y = f(x)$, the tangent line, and the point $(3, 9)$ on a graphing utility.

Example 1.2: Let $f(x) = |x|$ and suppose we wish to know the slope of f at $x_0 = 0$. Using (1), we see that

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0} \frac{|h|}{h}. \end{aligned}$$

Taking the limit from the left, we observe that $\lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0^-} \frac{-h}{h} = -1$. However, taking the limit from the right, we observe that $\lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0^+} \frac{h}{h} = 1$. Thus the overall limit does not exist, and $f'(0)$ is not defined.

Remark: In the example above, one sees that the *left derivative* ($h \to 0^-$ case) and the *right derivative* ($h \to 0^+$ case) both exist, but the derivative does not.

Example 1.3: Let

$$f(x) = \begin{cases} x^2 \sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}$$

we wish to find $f'(0)$. Using (1), we see that

$$\begin{aligned} f'(0) &= \lim_{h\to 0} \frac{f(0+h)-f(0)}{h} \\ &= \lim_{h\to 0} \frac{h^2 \sin(1/h)}{h} \\ &= \lim_{h\to 0} h\sin(1/h) \\ &= 0, \end{aligned}$$

where we used the Squeeze Theorem for the final evaluation of the limit (note $|h\sin(1/h)| \leq |h|$). Plot $y = x^2\sin(1/x)$, $y = x^2$, and $y = -x^2$ in a graphing utility.

Example 1.4: Let $f(x) = \sqrt{x}$. Let us find an algebraic expression for $f'(x)$ when $x > 0$. Using (1), we see that

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h} \\
&= \lim_{h\to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) \\
&= \lim_{h\to 0} \frac{x+h-x}{h(\sqrt{x+h} + \sqrt{x})} \\
&= \lim_{h\to 0} \frac{1}{\sqrt{x+h} + \sqrt{x}} \\
&= \frac{1}{2\sqrt{x}},
\end{aligned}$$

where we used the fact that algebraic functions are continuous on their domains in the last step. Thus, we have that $f'(x) = \frac{1}{2}x^{-1/2}$ for all $x > 0$.

Example 1.5: Let $f(x) = 1/x$. Let us find an algebraic expression for $f'(x)$ when $x \neq 0$. Using (1), we see that

$$\begin{aligned}
f'(x) &= \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} \\
&= \lim_{h\to 0} \frac{1/(x+h) - 1/x}{h} \\
&= \lim_{h\to 0} \frac{1}{h} \cdot \frac{x - (x+h)}{x(x+h)} \\
&= \lim_{h\to 0} \frac{-1}{x(x+h)} \\
&= \frac{-1}{x^2} \\
&= -x^{-2}
\end{aligned}$$

Theorem 1.1: *Consider a function $f \colon \mathcal{D}(f) \to \mathbb{R}$ with $\mathcal{D}(f) \subset \mathbb{R}$. If $f'(x_0)$ exists, then f is continuous at x_0.*

Proof. Because $f'(x_0)$ exists, we know that $x_0 \in \mathcal{D}(f)$ and x_0 is a limit point of $\mathcal{D}(f)$. Thus, it is sufficient to show that $\lim_{x\to x_0} f(x) = f(x_0)$, or (equivalently) that $\lim_{h\to 0} f(x_0 + h) = f(x_0)$.

Consider the function

$$g(h) = \frac{f(x_0+h) - f(x_0)}{h} - f'(x_0).$$

From (1), we know that $\lim_{h\to 0} g(h) = 0$. Re-arranging, we see that

$$f(x_0 + h) = f(x_0) + hf'(x_0) + hg(h) \to f(x_0)$$

as $h \to 0$. □

A few important facts about derivatives:

1. If $f'(x_0)$ exists, then f is continuous at x_0
2. $\lim_{h\to 0^-}(f(x_0+h)-f(x_0))/h$ is the *left derivative of* f *at* x_0
3. $\lim_{h\to 0^+}(f(x_0+h)-f(x_0))/h$ is the *right derivative of* f *at* x_0
4. The left and right derivatives of f may exist at x_0 without $f'(x_0)$ existing
5. If $f'(x_0)$ exists, then both the left and right derivatives exist at x_0 and all three are equal

2 Derivative Rules

In Example 1.4, we saw that it is possible to find an expression for $f'(x)$. That is, given a function f, it is possible to find the function f'. Because using (1) is cumbersome, we wish to find rules that will allow us to construct the function f' from known, more simple functions. In order to do this, there are several tools that will be required.

THEOREM 2.1 (SCALAR MULTIPLE RULE): *Let $f\colon \mathcal{D}(f)\to\mathbb{R}$ with $\mathcal{D}(f)\subset\mathbb{R}$ and suppose that $f'(x_0)$ exists for some $x_0\in\mathcal{D}(f)$. Then for any $k\in\mathbb{R}$, we have*

$$(kf)'(x_0) = kf'(x_0).$$

REMARK: Recall that $(kf)(x) = k\cdot f(x)$ for any constant $k\in\mathbb{R}$.

Proof. See Problem *8*. □

THEOREM 2.2 (SUM RULE): *Let $f\colon\mathcal{D}(f)\to\mathbb{R}$ and $g\colon\mathcal{D}(g)\to\mathbb{R}$ with $\mathcal{D}(f)\subset\mathbb{R}$ and $\mathcal{D}(g)\subset\mathbb{R}$. Suppose there exists $x_0\in\mathcal{D}(f)\cap\mathcal{D}(g)$ with x_0 a limit point of $\mathcal{D}(f)\cap\mathcal{D}(g)$ such that $f'(x_0)$ and $g'(x_0)$ both exist. Then*

$$(f+g)'(x_0) = f'(x_0)+g'(x_0).$$

Proof. See Problem *9*. □

THEOREM 2.3 (PRODUCT RULE): *Let $f\colon\mathcal{D}(f)\to\mathbb{R}$ and $g\colon\mathcal{D}(g)\to\mathbb{R}$ with $\mathcal{D}(f)\subset\mathbb{R}$ and $\mathcal{D}(g)\subset\mathbb{R}$. Suppose there exists $x_0\in\mathcal{D}(f)\cap\mathcal{D}(g)$ such that $f'(x_0)$ and $g'(x_0)$ both exist. Then*

$$(fg)'(x_0) = f'(x_0)g(x_0)+f(x_0)g'(x_0).$$

Proof. Using (1), we have

$$\begin{aligned}
(fg)'(x_0) &= \lim_{h\to 0}\frac{f(x_0+h)g(x_0+h)-f(x_0)g(x_0)}{h}\\
&= \lim_{h\to 0}\frac{f(x_0+h)g(x_0+h)-f(x_0+h)g(x_0)+f(x_0+h)g(x_0)-f(x_0)g(x_0)}{h}\\
&= \lim_{h\to 0}\left(\frac{f(x_0+h)g(x_0+h)-f(x_0+h)g(x_0)}{h}+\frac{f(x_0+h)g(x_0)-f(x_0)g(x_0)}{h}\right)\\
&= \lim_{h\to 0}\left(f(x_0+h)\frac{g(x_0+h)-g(x_0)}{h}+g(x_0)\frac{f(x_0+h)-f(x_0)}{h}\right)\\
&= f(x_0)g'(x_0)+g(x_0)f'(x_0),
\end{aligned}$$

where we used the fact that each of the three limits exist individually (as f is continuous at x_0 and $f'(x_0)$ and $g'(x_0)$ both exist) in the last step. Re-arranging yields the desired expression. $\square$

Example 2.1: Let $f(x) = x^2$ and $g(x) = x^3$. Using results from the Problems section, we know that $f'(x) = 2x$ and $g'(x) = 3x^2$. Setting $h = fg$ (i.e., $h(x) = x^5$), we see that

$$h'(x) = f'(x)g(x) + f(x)g'(x) = 2x \cdot x^3 + x^2 \cdot 3x^2 = 5x^4.$$

In order to prove the next theorem, we must introduce *little-o* notation.

Definition 2.1 (Little-o): *Given a function g, we say that a function f is $o(g)$, alternatively $f = o(g)$, if for all $\varepsilon > 0$ there exists a $\delta > 0$ such that $|h| < \delta \implies |f(h)| < \varepsilon|g(h)|$. This is equivalent to the criteria that*

$$\lim_{h\to 0} \frac{f(h)}{g(h)} = 0$$

assuming that $g(h) \neq 0$ for all h sufficiently close to 0 (with the possible exception of $h = 0$).

In common use, if we know that some function f is $o(h)$, we often do not care about the other properties of f and replace its usage with simply the symbol $o(h)$. Here, the function g is simply $g(h) = h$.

Lemma 2.1: *Let f be a function and suppose that $f'(x_0)$ exists. Then there is another function g of class $o(h)$ such that*

$$f(x_0 + h) = f(x_0) + hf'(x_0) + g(h)$$

for all h such that $x_0 + h \in \mathcal{D}(f)$.

Remark: We may re-write this claim without the use of g as $f(x_0 + h) = f(x_0) + hf'(x_0) + o(h)$, as the exact form of g is often irrelevant.

Proof. Define $g(h) \coloneqq f(x_0 + h) - f(x_0) - hf'(x_0)$. Then

$$\begin{aligned}
\lim_{h\to 0} \frac{g(h)}{h} &= \lim_{h\to 0} \frac{f(x_0 + h) - f(x_0) - hf'(x_0)}{h} \\
&= \lim_{h\to 0} \left(\frac{f(x_0 + h) - f(x_0)}{h} - f'(x_0) \right) \\
&= f'(x_0) - f'(x_0) \\
&= 0.
\end{aligned}$$

Thus g both exists and is of class $o(h)$. $\square$

Theorem 2.4 (Chain Rule): *Let $f\colon \mathcal{D}(f) \to \mathbb{R}$ and $g\colon \mathcal{D}(g) \to \mathbb{R}$ with $\mathcal{D}(f) \subset \mathbb{R}$ and $\mathcal{D}(g) \subset \mathbb{R}$. Suppose there exists $x_0 \in \mathcal{D}(g)$ such that $g(x_0) \in \mathcal{D}(f)$ and that both $g'(x_0)$ and $f'(g(x_0))$ both exist. Then*

$$(f \circ g)'(x_0) = f'(g(x_0)) \cdot g'(x_0).$$

Proof. Without taking the limit, let us examine the appropriate Newton Quotient. We see that

$$\frac{f(g(x_0 + h)) - f(g(x_0))}{h} = \frac{f(g(x_0) + hg'(x_0) + o(h)) - f(g(x_0))}{h}$$

by using Lemma 2.1 on the expression $g(x_0+h)$. Setting $t = hg'(x_0) + o(h)$ and applying the lemma again, we see that

$$\frac{f(g(x_0)+t) - f(g(x_0))}{h} = \frac{f(g(x_0)) + tf'(g(x_0)) + o(t) - f(g(x_0))}{h} = \frac{tf'(g(x_0)) + o(t)}{h}.$$

Substituting for t and noting that any function that is $o(t)$ is also $o(h)$, we see

$$\frac{tf'(g(x_0)) + o(t)}{h} = f'(g(x_0))\frac{hg'(x_0) + o(h)}{h} + \frac{o(h)}{h}$$

Taking $h \to 0$, we see that

$$\begin{aligned}
(f \circ g)'(x_0) &= \lim_{h\to 0} \frac{f(g(x_0+h)) - f(g(x_0))}{h} \\
&= \lim_{h\to 0} f'(g(x_0))\frac{hg'(x_0) + o(h)}{h} + \lim_{h\to 0} \frac{o(h)}{h} \\
&= f'(g(x_0))\left(g'(x_0) + \lim_{h\to 0} \frac{o(h)}{h}\right) + \lim_{h\to 0} \frac{o(h)}{h} \\
&= f'(g(x_0)) \cdot g'(x_0),
\end{aligned}$$

where we used the fact that $o(h)/h \to 0$. $\square$

Theorem 2.5 (Quotient Rule): *Let $f\colon \mathcal{D}(f) \to \mathbb{R}$ and $g\colon \mathcal{D}(g) \to \mathbb{R}$ with $\mathcal{D}(f) \subset \mathbb{R}$ and $\mathcal{D}(g) \subset \mathbb{R}$. Suppose there exists $x_0 \in \mathcal{D}(f) \cap \mathcal{D}(g)$ such that $f'(x_0)$ and $g'(x_0)$ both exist and that $g(x_0) \neq 0$. Then*

$$\left(\frac{f}{g}\right)'(x_0) = \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.$$

Proof. Let $h(x) = x^{-1}$ so that $h'(x) = -x^{-2}$ from Example 1.5. Note that $f/g = f \cdot (h \circ g)$. Using the product rule and then the chain rule, we see that

$$\begin{aligned}
\left(\frac{f}{g}\right)'(x_0) &= f'(x_0) \cdot (h \circ g)(x_0) + f(x_0) \cdot (h \circ g)'(x_0) \\
&= \frac{f'(x_0)}{g(x_0)} + f(x_0) \cdot (h'(g(x_0)) \cdot g'(x_0)) \\
&= \frac{f'(x_0)}{g(x_0)} + f(x_0) \cdot \left(\frac{-1}{g(x_0)^2} \cdot g'(x_0)\right) \\
&= \frac{f'(x_0)g(x_0) - f(x_0)g'(x_0)}{g(x_0)^2}.
\end{aligned}$$

$\square$

Theorem 2.6 (Inverse Function Rule): *Let $f\colon X \to X$ with $X \subset \mathbb{R}$ be invertible and both f and f^{-1} differentiable. Then*

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))}$$

for all $x \in X$ such that the denominator is non-zero.

Proof. Because f is invertible, we know that $(f \circ f^{-1})(x) = x$ for all $x \in X$. Taking the derivative on both sides and applying the chain rule, we see that $f'(f^{-1}(x)) \cdot (f^{-1})'(x) = 1$. Dividing by $f'(f^{-1}(x))$ yields the desired result. $\square$

EXAMPLE 2.2: Consider $f\colon (0,\infty) \to (0,\infty)$ given by $f(x) = x^2$. Then $f^{-1}(x) = \sqrt{x}$. Using the rule above and the fact that $f'(x) = 2x$, we see that

$$(f^{-1})'(x) = \frac{1}{f'(f^{-1}(x))} = \frac{1}{2f^{-1}(x)} = \frac{1}{2\sqrt{x}},$$

which is consistent with the results in Example 1.4.

3 The Derivative as a Linear Operator

The term "operator" is similar to function. The derivative operator has an input (some function f) and an output (the function f'). It is often useful to express the derivative operator in Leibniz notation.

DEFINITION 3.1 (LEIBNIZ NOTATION): *Consider a function f that has a derivative f'. We use the notation*

$$\frac{d}{dx}f = f'$$

to denote the derivative of a function or expression with respect to the variable x.

This notation can be used in a variety of ways. For example $\frac{dy}{dx}$ reads "the derivative of y with respect to x." The symbols dx and dy are called "differentials" and are related to tiny changes in the variables x and y. If one sets $y = f(x)$ for some function f and $\Delta x = x - x_0$ and $\Delta y = f(x) - f(x_0)$, then $\frac{dy}{dx}|_{x_0} = \lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = f'(x_0)$. The vertical bar $|_{x_0}$ is instructions to evaluate the expression immediately to the left at x_0. It is crucial to note that $\frac{dy}{dx}$ **is not** a fraction.

EXAMPLE 3.1: From the previous examples, we know that

1. $\frac{d}{dx}\sqrt{x} = -\frac{1}{2\sqrt{x}}$ and
2. $\frac{d}{dx}x^{-1} = -\frac{1}{x^2}$

Additionally, we can use other letters. For example, $\frac{d}{du}\sqrt{u} = -\frac{1}{2\sqrt{u}}$.

EXAMPLE 3.2: Suppose y is independent of x. Then

$$\frac{d}{dx}\left(x^2 y\right) = 2xy.$$

EXAMPLE 3.3: Suppose y is depends on x (i.e., $y = f(x)$ for some unknown function f). Assuming the relationship is differentiable (i.e., that f' exists), then

$$\frac{d}{dx}\left(x^2 y\right) = 2xy + x^2 y',$$

where $y' = f'(x)$. Note that the product rule was used.

In order to appreciate linearity, one must understand what a vector space is.

DEFINITION 3.2 (VECTOR SPACE): *Consider a set V with the operation $\mathbf{u} + \mathbf{v}$ (vector addition) defined for every $\mathbf{u}, \mathbf{v} \in V$ and the operation $k\mathbf{v}$ (scalar multiplication) defined for every $k \in \mathbb{R}$ and $\mathbf{v} \in V$. We call V a* vector space *(or* linear space*) if:*

1. $\mathbf{u} + (\mathbf{v} + \mathbf{w}) = (\mathbf{u} + \mathbf{v}) + \mathbf{w}$ *for all* $\mathbf{u}, \mathbf{v}, \mathbf{w} \in V$ *(vector addition is associative)*
2. $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ *for all* $\mathbf{u}, \mathbf{v} \in V$ *(vector addition is commutative)*
3. *There exists* $\mathbf{0} \in V$ *such that* $\mathbf{0} + \mathbf{v} = \mathbf{v}$ *for all* $\mathbf{v} \in V$ *(the zero vector is the additive identity)*
4. *For every* $\mathbf{v} \in V$ *there is another element* $-\mathbf{v} \in V$ *such that* $\mathbf{v} + (-\mathbf{v}) = \mathbf{0}$ *(additive inverse)*
5. *For all* $a, b \in \mathbb{R}$ *and* $\mathbf{v} \in V$, $a(b\mathbf{v}) = (ab)\mathbf{v}$ *(associativity of scalar multiplication)*
6. *For all* $\mathbf{v} \in V$, $1\mathbf{v} = \mathbf{v}$ *(the scalar* $1 \in \mathbb{R}$ *is the multiplicative identity)*
7. *For all* $a \in \mathbb{R}$ *and* $\mathbf{u}, \mathbf{v} \in V$, $a(\mathbf{u} + \mathbf{v}) = a\mathbf{u} + a\mathbf{v}$ *(distributivity of scalar multiplication with vector addition)*
8. *For all* $a, b \in \mathbb{R}$ *and* $\mathbf{u} \in V$, $(a + b)\mathbf{u} = a\mathbf{u} + b\mathbf{u}$ *(distributivity of scalar multiplication with scalar addition)*

In particular, $\mathbf{u} + \mathbf{v} \in V$ *for all* $\mathbf{u}, \mathbf{v} \in V$ *(V is closed under vector addition), and* $a\mathbf{v} \in V$ *for all* $a \in \mathbb{R}$ *and* $\mathbf{v} \in V$ *(V is closed under scalar multiplication).*

Example 3.4: $\mathbb{R}$ is a vector space.

Example 3.5: $\mathbb{R}^2$ is a vector space. Define $\mathbf{u} = (u_1, u_2) \in \mathbb{R}^2$ with vector addition $\mathbf{u}+\mathbf{v} = (u_1+v_1, u_2+v_2) \in \mathbb{R}^2$ and scalar multiplication $k\mathbf{v} = (kv_1, kv_2) \in \mathbb{R}^2$. In order to verify that $\mathbb{R}^2$ is a vector space, we must verify each of the 8 criteria above. 1 and 2 are immediate, 3 holds with $\mathbf{0} = (0, 0)$, 4 holds with $-\mathbf{v} = (-v_1, -v_2)$, and 5–8 are all immediate.

Example 3.6: Let $\mathcal{F}(X) = \{f : X \to \mathbb{R}\}$ where X is a non-empty set (i.e., $\mathcal{F}(X)$ is the set of all functions from a common domain X to $\mathbb{R}$). Then $\mathcal{F}(X)$ is a vector space with vector addition $f + g$ for all $f, g \in \mathcal{F}(X)$ and scalar multiplication kf for all $k \in \mathbb{R}$ and $f \in \mathcal{F}(X)$ defined as usual (i.e., $(f + g)(x) = f(x) + g(x)$ and $(kf)(x) = kf(x)$ for all $x \in X$). The verification of all 8 properties is straightforward with the constant 0 function playing the role of the zero vector.

Theorem 3.1: *Let* $C(X) = \{f : X \to \mathbb{R} \mid f \text{ is continuous}\}$. *Then* $C(X)$ *is a vector space.*

Proof. Because $C(X) \subset \mathcal{F}(X)$ and $\mathcal{F}(X)$ is a vector space, it is sufficient to prove closure under vector addition and scalar multiplication. That is if both f and g are continuous at x_0, then $f + g$ is continuous at x_0 and if $k \in \mathbb{R}$ then kf is also continuous at x_0. These follow immediately from the ε–δ definition of continuity. $\square$

Theorem 3.2: *Let* $D(X) = \{f : X \to \mathbb{R} \mid f' \text{ exists}\}$. *Then* $D(X)$ *is a vector space.*

Proof. Because $D(X) \subset C(X)$ and $C(X)$ is a vector space, it is sufficient to prove closure under vector addition and scalar multiplication. That is if both f and g are differentiable at x_0, then $f+g$ is differentiable at x_0 and if $k \in \mathbb{R}$ then kf is also differentiable at x_0. These are exactly the results of Theorems 2.2 and 2.1. $\square$

Definition 3.3 (Linear): *Let* U *and* V *be a vector spaces and* T *some function (or operator)* $T\colon U \to V$*. We say that* T *is* linear *if*

1 $T(\mathbf{u}_1 + \mathbf{u}_2) = T(\mathbf{u}_1) + T(\mathbf{u}_2)$ *for all* $\mathbf{u}_1, \mathbf{u}_2 \in U$ *and*

2 $T(a\mathbf{u}) = aT(\mathbf{u})$ *for all* $a \in \mathbb{R}$ *and* $\mathbf{u} \in U$*.*

Example 3.7: The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = 2x$ is linear. To see this, note that

$$f(x_1 + x_2) = 2 \cdot (x_1 + x + 2) = 2 \cdot x_1 + 2 \cdot x_2 = f(x_1) + f(x_2)$$

for all $x_1, x_2 \in \mathbb{R}$. Similarly,

$$f(kx) = 2 \cdot (kx) = k \cdot 2x = kf(x)$$

for all $k \in \mathbb{R}$ and all $x \in \mathbb{R}$.

Example 3.8: The function $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = x + 1$ is not linear. To see this, it is sufficient to note that

$$f(x_1 + x_2) = (x_1 + x_2) + 1 = (x_1 + 1) + (x_2 + 1) - 1 = f(x_1) + f(x_2) - 1$$

for all $x_1, x_2 \in \mathbb{R}$.

Theorem 3.3 (Linearity of the Derivative): *Let* $X \subset \mathbb{R}$*,* $\mathcal{F}(X)$*, and* $D(X)$ *be as in the examples above. Then* $\frac{d}{dx}\colon D(X) \to \mathcal{F}(X)$ *is linear. In particular, if* f *and* g *are differentiable functions on a common domain* X *and* $k \in \mathbb{R}$*, then* $f + g$ *and* kf *are both differentiable on* X *and their derivatives are given by*

$$\frac{d}{dx}(f + g)(x) = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

and

$$\frac{d}{dx}kf(x) = k\frac{d}{dx}f(x).$$

Proof. This is exactly the content of Theorem 3.2. □

As a final note, consider a function $f\colon X \to Y$ where X and Y are vector spaces. Sometimes this function f has a derivative at a point (say $f'(x_0)$). This derivative at the point x_0 is the best *linear* approximation to f (treating the point $(x_0, f(x_0)$ as the origin in $X \times Y$). That is, we can view $f'(x_0)$ as a linear function from X to Y. This is the generalization of the tangent line when $f\colon \mathbb{R} \to \mathbb{R}$. For example if $f\colon \mathbb{R}^n \to \mathbb{R}$, $f'(x_0)$ is represented by what is called a *gradient vector*, while if $f\colon \mathbb{R}^n \to \mathbb{R}^m$, $f'(x_0)$ is represented by what is called a *Jacobean matrix*. These are topics that are covered in multi-variable calculus.

4 A Library of Derivatives

Theorem 4.1 (Power Rule for Positive Integers): *Let* $n \in \{1, 2, 3, \ldots\}$*. Then*

$$\frac{d}{dx}x^n = nx^{n-1}$$

for $x \in \mathbb{R}$*.*

Proof. We will prove this by *induction*. Let us first verify the case when $n = 1$.

$$\frac{d}{dx}x = \lim_{h \to 0} \frac{(x+h)-(x)}{h} = 1.$$

Now let us suppose that our claim is true for $n = 1, \ldots, m$ for some $m \geq 1$. We will show that our claim must be true for $n = m+1$ as well.

$$\begin{aligned}
\frac{d}{dx}x^{m+1} &= \frac{d}{dx}x \cdot x^m && \text{(re-write)} \\
&= x^m \frac{d}{dx}x + x\frac{d}{dx}x^m && \text{(product rule)} \\
&= x^m + x \cdot mx^{m-1} && \text{(our claim is true for } n = m\text{)} \\
&= (m+1)x^m && \text{(simplify)}.
\end{aligned}$$

Thus, our claim being true for $n = 1$ implies that our claim must be true for $n = 2$ (by taking $m = 1$), which in turn implies that our claim must be true for $n = 3$ (by taking $m = 2$) and so on. □

Theorem 4.2 (Power Rule for Negative Integers): *Let* $n \in \{1, 2, 3, \ldots\}$. *Then*

$$\frac{d}{dx}x^{-n} = -nx^{-n-1}$$

for $x \in \mathbb{R} \setminus \{0\}$.

Proof. Note that $x^{-n}x^n = 1$ for all $x \neq 0$. Taking the derivative on both sides and applying the product rule yields

$$\begin{aligned}
\frac{d}{dx}1 &= x^n \frac{d}{dx}x^{-n} + x^{-n}\frac{d}{dx}x^n \\
&= x^n \frac{d}{dx}x^{-n} + x^{-n} \cdot nx^{n-1} \\
&= x^n \frac{d}{dx}x^{-n} + nx^{-1}
\end{aligned}$$

Re-arranging yields

$$-nx^{-1} = x^n \frac{d}{dx}x^{-n} \implies -nx^{-n-1} = \frac{d}{dx}x^{-n}.$$

□

Remark: Note that the two theorems above give us $\frac{d}{dx}x^n = nx^{n-1}$ for all $n \in \mathbb{Z}$.

Theorem 4.3 (Power Rule for Reciprocals of Integers): *Let* $n \in \mathbb{Z} \setminus \{0\}$. *Then*

$$\frac{d}{dx}x^{1/n} = \frac{1}{n}x^{1/n-1}$$

for all $x \neq 0$ *if* n *is odd and all* $x > 0$ *if* n *is even.*

Remark: Note that the exponent $1/n - 1$ is the same as $(1/n) - 1$ and not $1/(n-1)$.

Proof. Set $f(x) = x^n$ so that $f^{-1}(x) = x^{1/n}$. From the theorems above, we have that $f'(x) = nx^{n-1}$. Using Theorem 2.6, we see that

$$\begin{aligned} \frac{d}{dx} f^{-1}(x) &= \frac{1}{n(x^{1/n})^{n-1}} \\ &= \frac{1}{n} \cdot \frac{1}{x^{(n-1)/n}} \\ &= \frac{1}{n} \cdot \frac{1}{x^{1-1/n}} \\ &= \frac{1}{n} x^{1/n-1}. \end{aligned}$$

□

THEOREM 4.4 (POWER RULE): *Let $q \in \mathbb{Q}$. Then*

$$\frac{d}{dx} x^q = q x^{q-1}$$

for all x in the natural domain of x^q.

REMARK: If $q \in \mathbb{Q}$ has a simplified form of $q = n/m$ (i.e., n and m have no common factors and $m > 0$), then the natural domain of x^q is as follows. If $n < 0$, then $0 \notin \mathcal{D}(x^q)$. Additionally, if m is even, then $\mathcal{D}(x^q)$ does not contain any negative numbers. With the exception of those two restrictions, $\mathcal{D}(x^q)$ is the set of all real numbers.

Proof. Let n/m be the simplified form of q. Using the previous theorems and the chain rule, we see that

$$\begin{aligned} \frac{d}{dx} x^q = \frac{d}{dx} x^{n/m} = \frac{d}{dx} \left(x^n\right)^{1/m} &= \frac{1}{m} \left(x^n\right)^{1/m-1} \cdot n x^{n-1} \\ = \frac{n}{m} \cdot x^{n/m-n} \cdot x^{n-1} &= \frac{n}{m} x^{n/m-1} = q x^{q-1}. \end{aligned}$$

□

REMARK: Although we do not prove it, Theorem 4.4 holds with $q \in \mathbb{R}$.

Next, we will look at the derivatives of $\sin x$ and $\cos x$, but we will first need to prove a few lemmas.

THEOREM 4.5: *The* sin *and* cos *functions are both continuous on all of* $\mathbb{R}$*.*

Proof. Examine Figure 2. From the geometric definitions of $\sin\theta$ and $\cos\theta$, we clearly must have $\sin\theta \to 0$ and $\cos\theta \to 1$ as $\theta \to 0$. Thus sin and cos are continuous at $\theta_0 = 0$.

Next, pick any $\theta_0 \in \mathbb{R}$. Then

$$\begin{aligned} \lim_{\theta \to \theta_0} \sin(\theta) &= \lim_{\theta \to \theta_0} \sin(\theta - \theta_0 + \theta_0) \\ &= \lim_{\theta \to \theta_0} \big(\sin(\theta - \theta_0)\cos(\theta_0) + \cos(\theta - \theta_0)\sin(\theta_0)\big) \\ &= 0 \cdot \cos\theta_0 + 1 \cdot \sin\theta_0 \\ &= \sin\theta_0, \end{aligned}$$

where we used the trig identity $\sin(u + v) = \sin(u)\cos(v) + \cos(u)\sin(v)$ and the continuity of sin and cos at 0. Thus sin is continuous on $\mathbb{R}$. Finally, because $\cos(\theta) = \sin(\pi/2 - \theta)$ and compositions of continuous functions are continuous, we have that cos is also continuous on $\mathbb{R}$. □

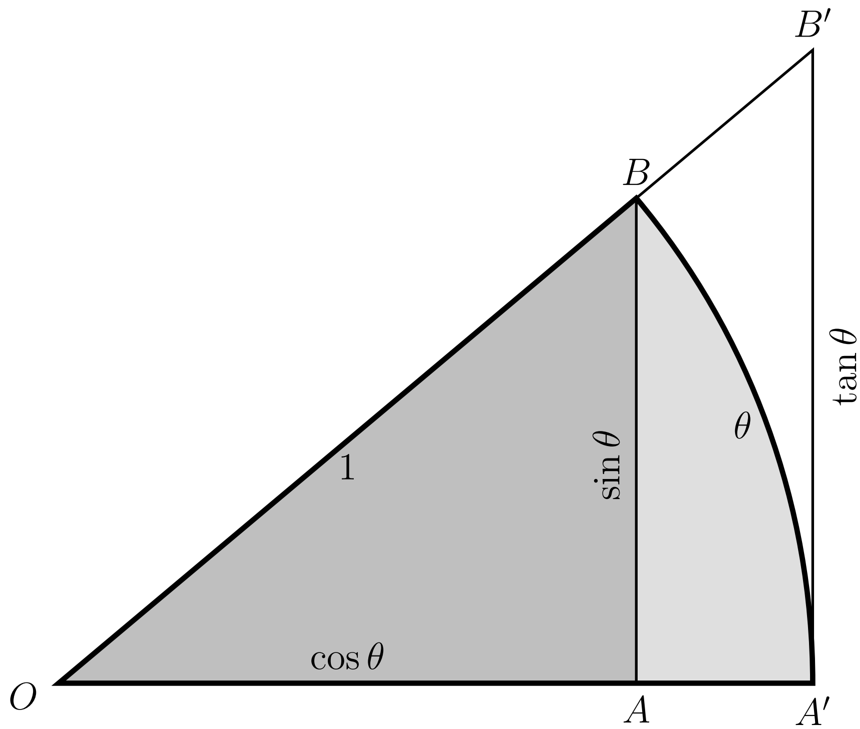

Figure 2: Geometric relationships between trig functions on the unit circle.

Lemma 4.1:

$$\lim_{\theta\to 0}\frac{\sin\theta}{\theta} = 1$$

Proof. Let $f(\theta) = \sin(\theta)/\theta$. Noting that $f(-\theta) = f(\theta)$, without loss of generality, we may assume that $\theta > 0$ in our limit. Examining Figure 2, we observe that the area of $\triangle OAB$ is $\frac{1}{2}\sin\theta\cos\theta$, the area of the sector $OA'B$ is $\theta/2$, and the area of $\triangle OA'B'$ is $\frac{1}{2}\tan\theta$. Thus, we have

$$\frac{1}{2}\sin\theta\cos\theta < \frac{1}{2}\theta < \frac{1}{2}\tan\theta.$$

Dividing by $\frac{1}{2}\sin\theta$ (note $\sin\theta > 0$) yields

$$\cos\theta < \frac{\theta}{\sin\theta} < \frac{1}{\cos\theta}.$$

Taking the reciprocal and re-arranging yields

$$\cos\theta < \frac{\sin\theta}{\theta} < \frac{1}{\cos\theta}.$$

Taking $\theta \to 0$ gives

$$1 \leq \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \leq 1.$$

$\square$

THEOREM 4.6 (DERIVATIVES OF TRIG FUNCTIONS): *The function* sin *and* cos *are differentiable on* $\mathbb{R}$ *with*

$$\frac{d}{d\theta}\sin\theta = \cos\theta$$

and

$$\frac{d}{d\theta}\cos\theta = -\sin\theta.$$

Proof. Pick any $\theta \in \mathbb{R}$. Then, recalling that $\sin u - \sin v = 2\cos((u+v)/2)\sin((u-v)/2)$, we have

$$\begin{aligned}\frac{d}{d\theta}\sin\theta &= \lim_{h\to 0}\frac{\sin(\theta+h)-\sin(\theta)}{h}\\ &= \lim_{h\to 0}\frac{2\cos(\theta-h/2)\sin(h/2)}{h}\\ &= \lim_{h\to 0}\cos(\theta-h/2)\frac{\sin(h/2)}{h/2}\\ &= \cos\theta.\end{aligned}$$

Thus $\frac{d}{d\theta}\sin\theta = \cos\theta$. Using the identity $\cos\theta = \sin(\pi/2-\theta)$ and the chain rule, we see that $\frac{d}{d\theta}\cos\theta = \cos(\pi/2-\theta)\cdot -1 = -\sin\theta$. ☐

ESSENTIAL FACTS:

1. Let $n \in \mathbb{R}$. Then $\dfrac{d}{dx}x^n = nx^{n-1}$ for all $x \in \mathbb{R}$ such that x^n and x^{n-1} are both defined.
2. For any $\theta \in \mathbb{R}$, $\dfrac{d}{d\theta}\sin\theta = \cos\theta$ and $\dfrac{d}{d\theta}\cos\theta = -\sin\theta$.
3. If f and g are both differentiable on appropriate domains and $k \in \mathbb{R}$, then:
   (a) $(f+g)' = f'+g'$ and $(kf)' = kf'$ (the derivative is linear)
   (b) $(fg)' = f'g + fg'$ (product rule)
   (c) $(f\circ g)' = (f'\circ g)\cdot g'$ (chain rule)
   (d) $(f/g)' = (f'g - fg')/g^2$ (quotient rule)

5 Tangent Lines

Let us now return to the topic of lines that are tangent to a curve $C \subset \mathbb{R}^2$. There are (at least) three ways of representing such a curve.

1. As the graph of a function: $C = \{(x,y) \in \mathbb{R}^2 : y = f(x)\}$ for some $f\colon \mathcal{D}(f) \to \mathbb{R}$. Ex: $y = \sqrt{1-x^2}$ with $-1 \le x \le 1$.
2. As the the solution set to a relation in x and y: $C = \{(x,y) \in \mathbb{R}^2 : h(x,y) = 0\}$ for some $h\colon \mathbb{R}^2 \to \mathbb{R}$. Ex: $h(x,y) = x^2 + y^2 - 1$.
3. As the image of a function $\mathbf{g}\colon \mathcal{D}(\mathbf{g}) \to \mathbb{R}^2$: $C = \{\mathbf{g}(t) : t \in \mathcal{D}(\mathbf{g})\}$. Ex: $\mathbf{g}(t) = (\cos t, \sin t)$ with $0 \le t < 2\pi$.

The first represents C as the graph of an *explicitly* defined function f, the second represents C as a relation or (locally, sometimes) the graph of an *implicitly* defined function, and the third represents C *parametrically* by giving the coordinates x and y in terms of some parameter t (in the example above, $x = \cos t$ and $y = \sin t$).

5.1 Explicit Form

This is the first case that was discussed. If C is the graph $y = f(x)$, then (if $f'(x_0)$ exists), the equation for the line tangent to C at the point $(x_0, f(x_0))$ is given by

$$y = f(x_0) + f'(x_0)(x - x_0),$$

which we could re-write to $y = L(x)$ with $L(x) = f(x_0) + f'(x_0)(x - x_0)$ the *linearization* of f at x_0.

EXAMPLE 5.1: Let C be the graph of $f(x) = \sqrt{1-x^2}$ over the interval $[-1, 1]$, then $f'(x) = -x/\sqrt{1-x^2}$. The tangent line at $x_0 = -1/\sqrt{2}$ is given by the graph of $L(x) = f(x_0) + f'(x_0)(x + 1/\sqrt{2})$. Evaluating f and f' at x_0 gives us $f(x_0) = 1/\sqrt{2}$ and $f'(x_0) = 1$, so $L(x) = 1/\sqrt{2} + 1 \cdot (x + 1/\sqrt{2}) = \sqrt{2} + x$. Plot $y = f(x)$ and $y = L(x)$ in a graphing utility.

5.2 Implicit Form

In this case, we will treat y as an (unknown) function of x. We could treat the general case where $h(x, y) = 0$ using a technique called *partial differentiation*. However, we will skip this for now and only look at specific examples. Here, it may be more convenient to write the tangent line in the form $y - y_0 = m(x - x_0)$ where m is the slope and (x_0, y_0) is the point of tangency on C.

EXAMPLE 5.2: Let C be the solution to the relation $x^2 + y^2 = 1$ in $\mathbb{R}^2$. We will find the equation of the line tangent to C at the point $\left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Treating y as an unknown function of x, we differentiate both sides of the equation to find:

$$\frac{d}{dx}1 = \frac{d}{dx}\left(x^2 + y^2\right) = \frac{d}{dx}x^2 + \frac{d}{dx}y^2 = 2x + 2y\frac{d}{dx}y = 2x + 2yy'.$$

This means that $0 = 2x + 2yy'$ where y' is the derivative of the unknown function of x. *This expression holds for all $(x, y) \in C$ where y' exists.* At the point of interest, we have $x_0 = -1/\sqrt{2}$ and $y_0 = 1/\sqrt{2}$ so that $0 = -2/\sqrt{2} + 2y'/\sqrt{2}$. Solving for y' gives us $y'|_{(x_0,y_0)} = 1$. This is the slope of our tangent line, and our equation of the tangent line is $y - 1/\sqrt{2} = x + 1/\sqrt{2}$, which simplifies to $y = x + \sqrt{2}$.

REMARK: Note that the expression $2x + 2yy' = 0$ can be solved for $y' = -x/y$. This is undefined when $y = 0$, which occurs when $x = \pm 1$ on C in the example above.

EXAMPLE 5.3: Let C be the solution to the relation $2\sin(xy) = 1$. Observe that the point $(x_0, y_0) = (\pi/3, 5/2)$ is on C because $\sin(5\pi/6) = 1/2$. Differentiating the relation, we find

$$0 = 2\frac{d}{dx}\sin(xy) = 2\cos(xy)\frac{d}{dx}(xy) = 2\cos(xy)(y + xy').$$

Solving for y' at (x_0, y_0) gives us $y'|_{(x_0,y_0)} = -15/2\pi$, so the equation of the tangent line is

$$y - \frac{5}{2} = -\frac{15}{2\pi}\left(x - \frac{\pi}{3}\right).$$

Plot C, (x_0, y_0), and the tangent line using a graphing utility.

This technique is of great use when looking many applied problems called *related rates* problems.

Example 5.4: Consider a rectangle of width x and height y, both measured in [cm]. Suppose that x is increasing at 1 [cm/s] and y is decreasing at 2 [cm/s]. If at some time t_0, the rectangle has width $x_0 = 10$ [cm] and height $y_0 = 30$ [cm], find the rate of change of both the area and perimeter of the rectangle.

1) Let $P = 2x + 2y$ be the perimeter. Clearly P, x, and y are all functions of t. Differentiating both sides with respect to t, we have
$$P' = 2x' + 2y'.$$
At the time in question $x' = 1$ [cm/s] and $y' = -2$ [cm/2], so $P' = -2$ [cm/2]. Thus, the perimeter is decreasing at a rate of 2 [cm/2].

2) Let $A = xy$ be the area. Differentiating both sides with respect to t give us $A' = x'y + xy'$. Evaluating gives us $A' = (1\ [\text{cm/s}])(30\ [\text{cm}]) + (10\ [\text{cm}])(-2\ [\text{cm/s}]) = 10\ \left[\text{cm}^2/\text{s}\right]$. Thus, the area is increasing at a rate of $10\ \left[\text{cm}^2/\text{s}\right]$.

5.3 Parametric Form

We will discuss this case in detail later. In short, if $\mathbf{g} \colon \mathbb{R} \to \mathbb{R}^2$ has the form $\mathbf{g}(t) = (g_1(t), g_2(t))$ where $g_1, g_2 \colon \mathbb{R} \to \mathbb{R}$ are differentiable, then then the line tangent to C at the point $(x_0, y_0) = (g)(t_0) \equiv (g_1(t_0), g_2(t_0))$ in parametric form is the graph of

$$\mathbf{l}(t) = \mathbf{g}(t_0) + \mathbf{g}'(t_0)(t - t_0).$$

To write the line in standard form, one can manipulate the equation to get

$$\frac{y - y_0}{x - x_0} = \frac{g_1'(t_0)}{g_2'(t_0)},$$

assuming that $g_2'(t_0) \neq 0$. Here (x, y) is any point on the tangent line other than (x_0, y_0), so the slope of the tangent line is given by $g_1'(t_0)/g_2'(t_0)$ when it is defined.

Example 5.5: Let $\mathbf{g}(t) = (\cos t, \sin t)$. At $t_0 = 3\pi/4$, we have $\mathbf{g}(t_0) = (x_0, y_0) = \left(-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$. Here $g_1(t) = \cos t$ and $g_2(t) = \sin t$, so at t_0, $g_2'(t_0)/g_1'(t_0) = \sin t_0 / \cos t_0 = 1$. Then the equation of the tangent line is given by $y = \sqrt{2} + x$ after simplifying. In parametric form, the line is given by the graph of

$$\mathbf{l}(t) = \begin{bmatrix} \cos t_0 \\ \sin t_0 \end{bmatrix} + (t - t_0) \begin{bmatrix} -\sin t_0 \\ \cos t_0 \end{bmatrix} = \begin{bmatrix} -1/\sqrt{2} - (t - 3\pi/4)/\sqrt{2} \\ 1/\sqrt{2} - (t - 3\pi/4)/\sqrt{2} \end{bmatrix}.$$

You may wish to plot $\mathbf{g}$ and $\mathbf{l}$ using a graphing utility.

Helpful hints:

1. You can take the derivative of some expression $h(x, y)$ with respect to x, y, or some other variable t. Treat x and y as functions and apply the various differentiation rules as appropriate.
2. If t is time, then $\frac{d}{dt}x$ is the rate of change of x. Sometimes this is written as $\dot{x}$.

6 Problems

1) Let $f(x) = x^2$. Use (1) to find an algebraic expression for $f'(x)$.

2) Let $f(x) = x^3$. Use (1) to find an algebraic expression for $f'(x)$.

3) Let

$$f(x) = \begin{cases} x\sin(1/x), & x \neq 0 \\ 0, & x = 0 \end{cases}.$$

Plot $y = f(x)$, $y = |x|$, and $y = -|x|$. Use (1) to show that $f'(0)$ does not exist.

4) Let $f(x) = \frac{1}{4}(1-x^2)^2$. Find the equation of the line tangent to $y = f(x)$ at $x_0 = \frac{1}{2}$. Plot $y = f(x)$ and the tangent line over the interval $-\sqrt{2} \leq x \leq \sqrt{2}$.

5) Take the derivative of $f(x) = (1 + x + x^2)^2$ by expanding and applying the power rule.

6) Take the derivative of $f(x) = (1 + x + x^2)^2$ by applying the chain rule.

7) Take each specified derivative.

a) $\dfrac{d}{dx}\left(\dfrac{1+x+x^2}{1-x+x^2}\right)$

b) $\dfrac{d}{dx}\left(10^{10} + \pi x + \dfrac{1}{x} + \dfrac{2}{\sqrt{x}}\right)$

c) $\dfrac{d}{dx}\left(xy^2 - x^2y + y^{-\pi}\right)$ where y is independent of x

d) $\dfrac{d}{dx}\left(xy^2 - x^2y + y^{-\pi}\right)$ where y is dependent on x [Hint: use the notation y' for $\frac{d}{dx}y$]

e) $\dfrac{d}{dy}\left(xy^2 - x^2y + y^{-\pi}\right)$ where x is independent of y

f) $\dfrac{d}{dt}\left(x^2 + y^2\right)$ where x and y are differentiable functions of t.

g) $\dfrac{d}{dt}(xy)$ where x and y are differentiable functions of t.

h) $\dfrac{d}{dt}\left(\sqrt{3t} + (t-1)^{-2}\right)$

i) $\dfrac{d}{dx}(\sin(x)\cos(2x))$

j) $\dfrac{d}{dx}\sin(1/x)$

k) $\dfrac{d}{dx}x^2\sin(1/x)$

l) $\dfrac{d}{d\theta}\tan\theta$

m) $\dfrac{d}{d\theta}\sec\theta$

n) $\dfrac{d}{d\phi}\left(\cos^2\phi + \sin^2\phi\right)$

o) $\dfrac{d}{d\phi}\cos^2(2\phi+1)$

p) $\dfrac{d}{dx}\sin(\sin^2(x))$

8) Prove Theorem 2.1. You may assume that $\mathcal{D}(f) = \mathcal{D}(g) = \mathbb{R}$.

9) Prove Theorem 2.2. You may assume that $\mathcal{D}(f) = \mathcal{D}(g) = \mathbb{R}$.

10) Prove that $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = mx$ with $m \in \mathbb{R}$ constant is linear.

11) Prove that $f\colon \mathbb{R} \to \mathbb{R}$ given by $f(x) = mx + b$ with $m, b \in \mathbb{R}$ constant is not linear. This is an example of an *affine* function, but it is often called linear for convenience and because its graph is a line in $\mathbb{R}^2$.

12) Suppose

$$\lim_{x \to x_0} \frac{f(x)}{x} = L.$$

Evaluate (with justification)

$$\lim_{x \to x_0} \frac{f(3x)}{x}.$$

13) Consider the function

$$f(x) = \begin{cases} \frac{1}{2}x^2, & x < 2 \\ y_0 + m(x-2), & x \geq 2 \end{cases},$$

where $y_0, m \in \mathbb{R}$ are constants that will be determined.

a) Choose y_0 to make f continuous. [Hint: find the left and right limits of f as $x \to 2^{\pm}$.]

b) Choose m to make f differentiable. [Hint: find the left and right derivatives of f at $x_0 = 2$.]

c) Plot $y = f(x)$.

14) Consider the curve C given by the relation $x^2 + y^2 = 25$. Given a point (x_0, y_0) in this relation, this *implicitly* defines a function f such that $y = f(x)$ for all x in some interval $x_0 - \delta < x < x_0 + \delta$. The values of $x_0 = \pm 1$ are exceptions. Observe that the relation has a vertical tangent line at the corresponding points.

a) Use implicit differentiation to find y' in terms of x and y.

b) Use your result from above to write the equation of the line tangent to C at the point $(3, 4)$.

c) Use your result from above to write the equation of the line tangent to C at the point $(3, -4)$.

d) Plot C and both lines on the same coordinate system.

15) Let C be the solution to $\cos(xy) = 1/\sqrt{2}$ in $\mathbb{R}^2$. Observe that $(x_0, y_0) = (\pi, 1/4)$ is on C. Find the equation of the line tangent to C at (x_0, y_0).

16) Recall that the volume of a sphere is given by $V = \frac{4\pi}{3}r^3$. If r is increasing at a rate of 1 [cm/s], how fast is V increasing when:

a) $r = 1$ [cm]

b) $r = 2$ [cm]

[Hint: treat V and r as functions of t (time). Then differentiate both sides of the volume equation with respect to t.]

17) Recall that the area of a disk is given by $A = \pi r^2$. If A is increasing at a rate of 1 $\left[\text{cm}^2/\text{s}\right]$, how fast is r increasing when:

a) $r = 1$ [cm]

b) $r = 2$ [cm]

[Hint: treat A and r as functions of t (time). Then differentiate both sides of the area equation with respect to t.]