University of Toronto
Department of Electrical and Computer Engineering
ECE557F Systems Control
Solution to Assignment 2

1. The system matrix is $A = \begin{bmatrix} -2 & 2 & -1 \\ -1 & -1 & 1 \\ -1 & -2 & -2 \end{bmatrix}$. Its eigenvalues and eigenvectors are

$$\begin{aligned} &\lambda_{1,2} = -1 \pm i2, \lambda_3 = -3 \\ &v_{1,2} = [1 \quad \pm i \quad -1]^\top, v_3 = [1 \;\; 0 \;\; 1]^\top. \end{aligned}$$

The eigenvalues are distinct and so A is diagonalizable. They all have negative real part, and so the origin is asymptotically stable. The state transition matrix is

$$\exp(At) = P \begin{bmatrix} \exp(-t)\cos(2t) & \exp(-t)\sin(2t) & 0 \\ -\exp(-t)\sin(2t) & \exp(-t)\cos(2t) & 0 \\ 0 & 0 & \exp(-3t) \end{bmatrix} P^{-1},$$

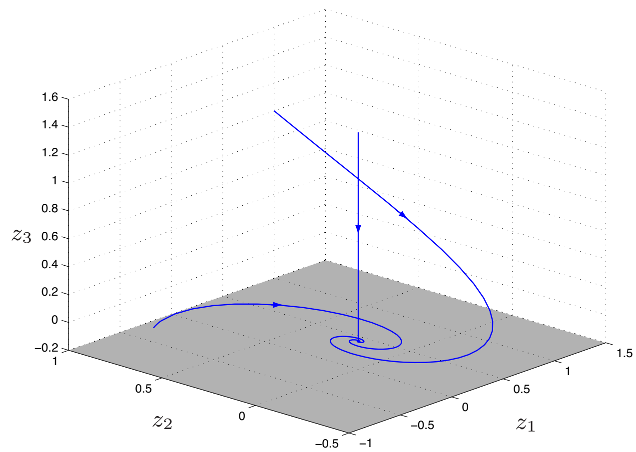
where $P = [\mathrm{Re}(v_1) \quad \mathrm{Im}(v_1) \quad v_3]$. Let's compute:

$$\begin{aligned} \exp(At) &= \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} \exp(-t)\cos(2t) & \exp(-t)\sin(2t) & 0 \\ -\exp(-t)\sin(2t) & \exp(-t)\cos(2t) & 0 \\ 0 & 0 & \exp(-3t) \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}^{-1} \\ &= \begin{bmatrix} 1/2\, e^{-t}\cos(2\,t) + 1/2\, e^{-3\,t} & e^{-t}\sin(2\,t) & -1/2\, e^{-t}\cos(2\,t) + 1/2\, e^{-3\,t} \\ -1/2\, e^{-t}\sin(2\,t) & e^{-t}\cos(2\,t) & 1/2\, e^{-t}\sin(2\,t) \\ -1/2\, e^{-t}\cos(2\,t) + 1/2\, e^{-3\,t} & -e^{-t}\sin(2\,t) & 1/2\, e^{-t}\cos(2\,t) + 1/2\, e^{-3\,t} \end{bmatrix}. \end{aligned}$$

In canonical coordinates $z = P^{-1}x$, the system takes the form

$$\dot{z} = \left[\begin{array}{cc|c} -1 & 2 & 0 \\ -2 & -1 & 0 \\ \hline 0 & 0 & -3 \end{array}\right] z.$$

The system is composed of two decoupled subsystems. The (z_1, z_2) subsystem is a stable focus, i.e., for any initial condition, $(z_1(t), z_2(t))$ spirals towards zero. Additionally, $z_3(t)$ decays to zero exponentially. In particular, then, the phase curves on the (z_1, z_2) plane remain on the plane and are spirals converging exponentially (with exponential rate 1) to the origin. Similarly, the phase curves on the z_3 axis remain on it, and converge to the origin exponentially with rate 3. To understand the behavior of solutions with initial conditions in the middle of the z space, we have to put together the two behaviors just discussed. By doing that, we deduce that all solutions converge to the origin while spiraling around the z_3 axis. We can say even more: since $z_3(t)$ converges to zero at a faster exponential rate than $(z_1(t), z_2(t))$, all solutions (except those on the z_3 axis) approach the origin tangentially to the (z_1, z_2) plane.



2. This example is purely computational, and so I just present the expression for $y(t)$,

$$y(t) = -\frac{2}{3}\exp(-3t) + \frac{1}{2}\exp(-2t) + \frac{1}{6}.$$

3. Consider the second-order system

$$\begin{aligned}\dot{x} &= \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} x + \begin{bmatrix} 1 \\ 0 \end{bmatrix} u \\ y &= [1 \;\; 0]x.\end{aligned}$$

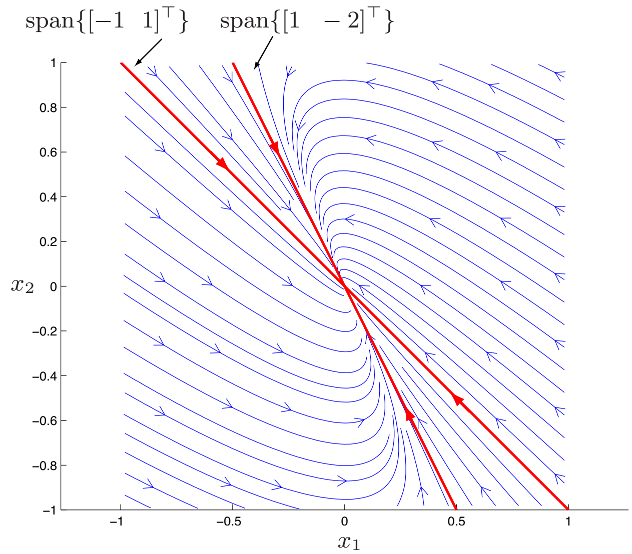
   The control system is composed of two decoupled subsystems. The x_1 subsystem is affected by the control input, while the x_2 subsystem isn't. The unforced system is clearly unstable, because A has one positive real eigenvalue at 1 corresponding to the x_2 subsystem. On the other hand, since $y = x_1$ and the x_1 dynamics are asymptotically stable, the instability of the x_2 dynamics does not affect the output signal. This observation is confirmed by the transfer function

$$Y(s)/U(s) = \frac{1}{s+1},$$

   which has a pole at -1 and hence is BIBO stable. The transfer function is first-order, while the original LTI system is second-order. This is hardly surprising, because as we have observed earlier, the x_2 subsystem is not "observable" from an input-output point of view.

4. When $\lambda_1 > \lambda_2$, the z_1 component of the solution converges to zero faster than the z_2 component. Therefore, all phase curves through initial conditions not on the z_1 axis become tangent to the z_2 axis as they approach $z = 0$. On the other hand, when $\lambda_1 < \lambda_2$, all phase curves through initial conditions not on the z_2 axis become tangent to the z_1 axis as they approach $z = 0$.

   Now consider the system with $A = \begin{bmatrix} -3 & -1 \\ 2 & 0 \end{bmatrix}$. Straightforward computations yield $A = P^{-1}\begin{bmatrix} -1 & 0 \\ 0 & -2 \end{bmatrix}P$, with $P = \begin{bmatrix} 1 & -1 \\ -2 & 1 \end{bmatrix}$. Consequently, the system is a stable node with $\lambda_1 = -1$ and $\lambda_2 = -2$. All phase curves, except those on the z_2 axis, approach the origin tangentially to the z_1 axis. In x coordinates, the z_1 and z_2 axes are mapped to the lines $\text{span}\{[1 \;\; -2]^\top\}$ and $\text{span}\{[-1 \;\; 1]^\top\}$, respectively. In conclusion, in x coordinates the lines $\text{span}\{[1 \;\; -2]^\top\}$ and $\text{span}\{[-1 \;\; 1]^\top\}$ are invariant. All phase curves not on $\text{span}\{[-1 \;\; 1]^\top\}$ approach $x = 0$ tangentially to the line $\text{span}\{[1 \;\; -2]^\top\}$. The phase portrait below confirms this prediction.



5. See the document *Solution to Practice Problems on Linear Algebra.*

6. **Part (a):** Let $\{1, x, x^2, \ldots, x^n\}$ be a basis for P^n. To compute the matrix representation of L in this basis, we apply L to each basis element:

$$L(1) = 1, \quad L(x) = x + 1, \quad L(x^2) = x^2 + 2x, \; \cdots, \; L(x^k) = x^k + kx^{k-1}.$$

The matrix representation of L is, therefore,

$$A = \begin{bmatrix} 1 & 1 & 0 & \cdots & 0 \\ 0 & 1 & 2 & \ddots & \vdots \\ 0 & 0 & 1 & \ddots & 0 \\ 0 & 0 & 0 & \cdots & n \\ 0 & 0 & \cdots & \cdots & 1 \end{bmatrix}$$

Part (b): The matrix representation of L is an invertible matrix ($\det(A) = 1$), and so its $n+1$ columns are linearly independent. Therefore, $\ker(L) = 0$ which implies that L is one-to-one.

Part (c): The rows of A are linearly independent, and so L is onto.

7. A vector $v \in \mathcal{V}$ has the form $v = [c_1 \; c_2 \; 0]^\top$. We have $Av = [2c_2 \;\; -2c_1 \; 0]^\top \in \mathcal{V}$. Therefore, $\mathcal{V}$ is A-invariant.