Assignment 2

1.1 Are either $\lceil \lg n \rceil!$ or $\lceil \lg \lg n \rceil!$ polynomially bounded?

Polynomially bounded means $f_n = O(n^k)$ for some constant k (e.g., whether $f_n \leq c \cdot n^k$ for constants c and k as n approaches ∞). For the first function $\lceil \lg n \rceil!$, without loss of generality, assume $n = 2^a$ (where $a \in \mathbb{N}$).

$$\begin{aligned} \lceil \lg n \rceil! &\leq c \cdot n^k \\ \lg(2^a)! &\leq c \cdot (2^a)^k \\ a! &\leq c \cdot 2^{ak} \end{aligned}$$

The statement $a! \leq c \cdot 2^{ak}$ is a contradiction, as the factorial function $a!$ is not exponentially bounded. Therefore, $\lceil \lg n \rceil!$ is not polynomially bounded (via proof by contradiction). For the second function $\lceil \lg \lg n \rceil!$, without loss of generality, assume $n = 2^{2^a}$ (where $a \in \mathbb{N}$).

$$\begin{aligned} \lceil \lg \lg n \rceil! &\leq c \cdot n^k \\ \lg \lg \left(2^{2^a}\right)! &\leq c \cdot \left(2^{2^a}\right)^k \\ a! &\leq c \cdot 2^{k \cdot 2^a} \\ 1 \cdot 2 \cdot 3 \cdots a &\leq c \cdot \left(2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^a \cdot k}\right) \end{aligned}$$

The statement $1 \cdot 2 \cdot 3 \cdots a \leq c \cdot \left(2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^a k}\right)$ is obviously true. Therefore $\lceil \lg \lg n \rceil!$ is polynomially bounded (via direct proof).

1.2 Use induction to prove $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$; where $F_i = F_{i-2} + F_{i-1}$, and ϕ is the golden ratio $\frac{1+\sqrt{5}}{2}$.

1.3 Show that $k \lg k = \Theta(n)$ implies $k = \Theta\left(\frac{n}{n \ln n}\right)$.

1.4 Are either 2^{n+1} or 2^{2n} big-O of 2^n?

1.5 For each pair of functions (A, B), indicate whether A is O, o, Ω, ω, or Θ of B. Assume $k \geq 1$, $\epsilon > 0$, $c > 1$ are constants.

	A	B	O	o	Ω	ω	Θ
a.	$\lg^k n$	n^ϵ	yes	yes			
b.	n^k	c^n	yes	yes			
c.	$\sqrt{n}$	$n^{\sin n}$					
d.	2^n	$2^{n/2}$			yes	yes	
e.	$n^{\lg c}$	$c^{\lg n}$	yes		yes		yes
f.	$\lg(n!)$	$\lg(n^n)$	yes		yes		yes

1.6 Order the following functions such that $f_1 = \Omega(f_2), f_2 = \Omega(f_3), \ldots, f_{29} = \Omega(f_{30})$, and partition them into equivalence classes such that each function is big-Θ of each other.

$$
\begin{aligned}
2^{2^{n+1}} &= \Omega\left(2^{2^n}\right), \\
2^{2^n} &= \Omega\left((n+1)!\right), \\
(n+1)! &= \Omega\left(n!\right), \\
n! &= \Omega\left(e^n\right), \\
e^n &= \Omega\left(n \cdot 2^n\right), \\
n \cdot 2^n &= \Omega\left(2^n\right), \\
2^n &= \Omega\left(\left(\frac{3}{2}\right)^n\right), \\
\left(\frac{3}{2}\right)^n &= \Omega\left(n^{\lg \lg n}\right), \\
n^{\lg \lg n} &= \Omega\left((\lg n)^{\lg n}\right), \\
(\lg n)^{\lg n} &= \Omega\left((\lg n)!\right), \\
(\lg n)! &= \Omega\left(N^3\right), \\
N^3 &= \Omega\left(n^2\right), \\
n^2 &= \Omega\left(4^{\lg n}\right), \\
4^{\lg n} &= \Omega\left(\lg(n!)\right),
\end{aligned}
$$

$$
\begin{aligned}
\lg(n!) &= \Omega(n \lg n), \\
n \lg n &= \Omega\left(2^{\lg n}\right), \\
2^{\lg n} &= \Omega(n), \\
n &= \Omega\left(\left(\sqrt{2}\right)^{\lg n}\right), \\
\left(\sqrt{2}\right)^{\lg n} &= \Omega\left(\sqrt{n}\right), \\
\sqrt{n} &= \Omega\left(2^{\sqrt{2 \lg n}}\right), \\
2^{\sqrt{2 \lg n}} &= \Omega\left(\lg^2 n\right), \\
\lg^2 n &= \Omega(\ln n), \\
\ln n &= \Omega\left(\sqrt{\lg n}\right), \\
\sqrt{\lg n} &= \Omega(\ln \ln n), \\
\ln \ln n &= \Omega\left(2^{\lg^* n}\right), \\
2^{\lg^* n} &= \Omega\left(\lg^* n\right), \\
\lg^* n &= \Omega(\lg *(\lg n)), \\
\lg *(\lg n) &= \Omega(\lg(\lg * n)), \\
\lg(\lg * n) &= \Omega\left(n^{\frac{1}{\lg n}}\right), \\
n^{\frac{1}{\lg n}}
\end{aligned}
$$