

# Assignment 2

## 1.1 Are either $\lceil \lg n \rceil!$ or $\lceil \lg \lg n \rceil!$ polynomially bounded?

Polynomially bounded means  $f_n = O(n^k)$  for some constant  $k$  (e.g., whether  $f_n \leq c \cdot n^k$  for constants  $c$  and  $k$  as  $n$  approaches  $\infty$ ).

For the first function  $\lceil \lg n \rceil!$ , without loss of generality, assume  $n = 2^a$  (where  $a \in \mathbb{N}$ ). The statement  $a! \leq c \cdot 2^{ak}$  (see below) is a contradiction, as the factorial function  $a!$  is not exponentially bounded. Therefore,  $\lceil \lg n \rceil!$  is not polynomially bounded (via proof by contradiction).

$$\lceil \lg n \rceil! \leq c \cdot n^k$$

$$\lg(2^a)! \leq c \cdot (2^a)^k$$

$$a! \leq c \cdot 2^{ak}$$

For the second function  $\lceil \lg \lg n \rceil!$ , without loss of generality, assume  $n = 2^{2^a}$  (where  $a \in \mathbb{N}$ ). The statement  $1 \cdot 2 \cdot 3 \cdots a \leq c \cdot (2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^a k})$  (see below) is obviously true. Therefore  $\lceil \lg \lg n \rceil!$  is polynomially bounded (via direct proof).

$$\lceil \lg \lg n \rceil! \leq c \cdot n^k$$

$$\lg \lg(2^{2^a})! \leq c \cdot (2^{2^a})^k$$

$$a! \leq c \cdot 2^{k \cdot 2^a}$$

$$1 \cdot 2 \cdot 3 \cdots a \leq c \cdot (2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^a \cdot k})$$

## 1.2 Use induction to prove $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$ , where $F_i = F_{i-2} + F_{i-1}$ , and $\phi$ is the golden ratio $\frac{1+\sqrt{5}}{2}$ .

To prove by induction, write out the expressions  $f_n$  and  $f_{n+1}$  (note:  $f_{n+1}$  is the same as  $f_n$ , but with  $(n+1)$  substituted everywhere in place of  $n$ ). Next, if applicable, re-write the expression  $f_{n+1}$  in terms of  $f_n$  then perform algebraic manipulations on the expression until you reach some variation of  $f_{n+1} = f_{n+1}$ . Lastly, show that the expression  $f_c$  also holds for some

constant  $c$ . The algebra is called "the inductive step", and the calculation for on the constant is called "the base case".

In this problem, the expression to prove is  $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$ , where  $\phi = \frac{1+\sqrt{5}}{2}$ . Start by demonstrating the expression holds for constants  $c = 0$  and  $c = 1$ .

$$\begin{aligned} F_0 &= \frac{\phi^0 - \hat{\phi}^0}{\sqrt{5}} = \frac{1 - 1}{\sqrt{5}} = 0 \\ F_1 &= \frac{\phi^1 - \hat{\phi}^1}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} \\ &= \frac{(1 + \sqrt{5}) - (1 - \sqrt{5})}{2\sqrt{5}} = 1 \end{aligned}$$

After showing the expression holds for some base cases  $F_0$  and  $F_1$ , the next step is algebra. Setup the expression  $F_n$  in terms of  $F_{n-1}$ , then solve (see below).

$$\begin{aligned} F_i &= \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} = F_{i-1} + F_{i-2} & F_{i-1} &= \frac{\phi^{i-1} - \hat{\phi}^{i-1}}{\sqrt{5}} \\ F_i &= F_{i-1} + F_{i-2} \\ \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} &= \frac{\phi^{i-1} - \hat{\phi}^{i-1}}{\sqrt{5}} + \frac{\phi^{i-2} - \hat{\phi}^{i-2}}{\sqrt{5}} \\ &= \frac{\phi^{i-1} + \hat{\phi}^{i-1} - \phi^{i-2} - \hat{\phi}^{i-2}}{\sqrt{5}} \\ &= \frac{\phi^{i-1} - \phi^{i-2} + \hat{\phi}^{i-1} - \hat{\phi}^{i-2}}{\sqrt{5}} \\ &= \frac{[(\phi \cdot \phi^{i-2}) + \phi^{i-2}] - [(\hat{\phi} \cdot \hat{\phi}^{i-2}) + \hat{\phi}^{i-2}]}{\sqrt{5}} \\ &= \frac{\phi^{i-2}(\phi + 1) - \hat{\phi}^{i-2}(\hat{\phi} + 1)}{\sqrt{5}} \\ &= \frac{\phi^{i-2}(\phi^2) - \hat{\phi}^{i-2}(\hat{\phi}^2)}{\sqrt{5}} \\ &= \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} \end{aligned}$$

Since we have shown  $F_i$  is obtainable via  $F_{i-1}$ , we have completed the inductive step. Since both the inductive step and base cases have been shown, the proof by induction is complete.

1.3 Show that  $k \lg k = \Theta(n)$  implies  $k = \Theta\left(\frac{n}{n \lg n}\right)$ .

1.4 Are either  $2^{n+1}$  or  $2^{2n}$  big- $O$  of  $2^n$ ?

The former is, the latter is not. Suppose  $2^{n+1} = O(2^n)$ , then  $2 \cdot 2^n \leq c \cdot 2^n$  as  $n \rightarrow \infty$ ; this is obviously true for all  $c \geq 2$ .

Regards to whether  $2^{2n} = O(2^n)$ , consider the inequality  $2^{2n} \leq c \cdot 2^n$ . This is equivalent to saying  $2^n \cdot 2^n \leq c \cdot 2^n$ . Dividing both sides by  $2^n$  gives  $2^n \leq c$ , which is obviously false (the exponential function is not constant bound).

1.5 For each pair of functions  $(A, B)$ , indicate whether  $A$  is  $O, o, \Omega, \omega$ , or  $\Theta$  of  $B$ . Assume  $k \geq 1$ ,  $\epsilon > 0$ ,  $c > 1$  are constants.

	$A$	$B$	$O$	$o$	$\Omega$	$\omega$	$\Theta$
a.	$\lg^k n$	$n^\epsilon$	yes	yes			
b.	$n^k$	$c^n$	yes	yes			
c.	$\sqrt{n}$	$n^{\sin n}$					
d.	$2^n$	$2^{n/2}$			yes	yes	
e.	$n^{\lg c}$	$c^{\lg n}$	yes		yes		yes
f.	$\lg(n!)$	$\lg(n^n)$	yes		yes		yes

In terms of growth,  $f_n = \Omega(g_n)$  is  $f_n \geq c \cdot g_n$ ,  $f_n = \omega(g_n)$  is  $f_n > c \cdot g_n$ ,  $f_n = O(g_n)$  is  $f_n \leq c \cdot g_n$ ,  $f_n = o(g_n)$  is  $f_n < c \cdot g_n$ , and  $f_n = \Theta(g_n)$  is  $f_n = c \cdot g_n$ .

To demonstrate whether  $f_n$  is something of something, isolate the constant  $c$  in the equality and observe whether it holds. Also note that big- $\Omega$  precludes little- $o$  and big- $O$  precludes little- $\omega$  (e.g., if  $f_n = O(g_n)$ , then  $f_n = \omega(g_n)$  is false, and vice-versa).

1.6 Order the following functions such that  $f_1 = \Omega(f_2)$ ,  $f_2 = \Omega(f_3)$ , ...,  $f_{29} = \Omega(f_{30})$ , and partition them into equivalence classes such that each function is big- $\Theta$  of each other.

In terms of growth,  $f_1 = \Omega(f_2)$  means  $f_1 \leq f_2$ . Therefore, the order of functions  $f_1 = \Omega(f_2), f_2 = \Omega(f_3), \dots, f_{29} = \Omega(f_{30})$  is as follows:  $2^{2^n} = \Omega((n+1)!)$ ,  $(n+1)! = \Omega(n!)$ ,  $n! = \Omega(e^n)$ ,  $e^n = \Omega(n \cdot 2^n)$ ,  $n \cdot 2^n = \Omega(2^n)$ ,  $2^n = \Omega\left(\left(\frac{3}{2}\right)^n\right)$ ,  $\left(\frac{3}{2}\right)^n = \Omega(n^{\lg \lg n})$ ,  $n^{\lg \lg n} = \Omega((\lg n)^{\lg n})$ ,  $(\lg n)^{\lg n} = \Omega((\lg n)!)$ ,  $(\lg n)! = \Omega(N^3)$ ,  $N^3 = \Omega(n^2)$ ,  $n^2 = \Omega(4^{\lg n})$ ,  $4^{\lg n} = \Omega(\lg(n!))$ ,  $\lg(n!) = \Omega(n \lg n)$ ,  $n \lg n = \Omega(2^{\lg n})$ ,  $2^{\lg n} = \Omega(n)$ ,  $n = \Omega\left((\sqrt{2})^{\lg n}\right)$ ,  $(\sqrt{2})^{\lg n} = \Omega(\sqrt{n})$ ,  $\sqrt{n} = \Omega\left(2^{\sqrt{2} \lg n}\right)$ ,  $2^{\sqrt{2} \lg n} = \Omega(\lg^2 n)$ ,  $\lg^2 n = \Omega(\ln n)$ ,  $\ln n = \Omega\left(\sqrt{\lg n}\right)$ ,  $\sqrt{\lg n} = \Omega(\ln \ln n)$ ,  $\ln \ln n = \Omega\left(2^{\lg^* n}\right)$ ,

$$2^{\lg^* n} = \Omega(\lg^* n), \lg^* n = \Omega(\lg^*(\lg n)), \lg^*(\lg n) = \Omega(\lg(\lg^* n)), \lg(\lg^* n) = \Omega\left(n^{\frac{1}{\lg n}}\right), n^{\frac{1}{\lg n}} = \Omega(1).$$

An equivalence class is a set containing elements that all adhere to some property. In this case, the elements are functions  $f$ , and the property is that each function is big- $\Theta$  of every other function in the set. The functions above can be partitioned into the following equivalence classes:  $\{n^{\lg \lg n}, (\lg n)^{\lg n}\}$ ,  $\{n^2, 4^{\lg n}\}$ ,  $\{\lg(n!), n \lg n\}$ ,  $\{2^{\lg n}, n\}$ ,  $\{(\sqrt{2})^{\lg n}, \sqrt{n}\}$ ,  $\{\lg^* n, \lg^*(\lg n)\}$ ,  $\{n^{\frac{1}{\lg n}}, 1\}$ .