

Assignment 2

1.1 Are either $\lceil \lg n \rceil!$ or $\lceil \lg \lg n \rceil!$ polynomially bounded?

Polynomially bounded means $f_n = O(n^k)$ for some constant k (e.g., whether $f_n \leq c \cdot n^k$ for constants c and k as n approaches ∞). For the first function $\lceil \lg n \rceil!$, without loss of generality, assume $n = 2^a$ (where $a \in \mathbb{N}$).

$$\begin{aligned}\lceil \lg n \rceil! &\leq c \cdot n^k \\ \lg(2^a)! &\leq c \cdot (2^a)^k \\ a! &\leq c \cdot 2^{ak}\end{aligned}$$

Which is a contradiction, as the factorial function $a!$ is not exponentially bounded. Therefore, $\lceil \lg n \rceil!$ is not polynomially bounded (via proof by contradiction). For the second function $\lceil \lg \lg n \rceil!$, without loss of generality, assume $n = 2^{2^a}$ (where $a \in \mathbb{N}$).

$$\begin{aligned}\lceil \lg \lg n \rceil! &\leq c \cdot n^k \\ \lg \lg(2^{2^a})! &\leq c \cdot (2^{2^a})^k \\ a! &\leq c \cdot 2^{k \cdot 2^a} \\ 1 \cdot 2 \cdot 3 \cdots a &\leq c \cdot (2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^a k})\end{aligned}$$

Which is obviously true for $c > 0$. Therefore $\lceil \lg \lg n \rceil!$ is polynomially bounded.

- 1.2 Use induction to prove $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$; where $F_i = F_{i-2} + F_{i-1}$, and ϕ is the golden ratio $\frac{1+\sqrt{5}}{2}$.
- 1.3 Show that $k \lg k = \Theta(n)$ implies $k = \Theta\left(\frac{n}{n \lg n}\right)$.
- 1.4 Are either 2^{n+1} or 2^{2n} big- O of 2^n ?
- 1.5 For each pair of functions (A, B) , indicate whether A is O, o, Ω, ω , or Θ of B . Assume $k \geq 1, \epsilon > 0, c > 1$ are constants.

| A | B | O | o | Ω | ω | Θ |
|-------------|--------------|-----|-----|----------|----------|----------|
| $\lg^k n$ | n^ϵ | yes | yes | yes | yes | yes |
| n^k | c^n | yes | yes | yes | yes | yes |
| \sqrt{n} | $n^{\sin n}$ | yes | yes | yes | yes | yes |
| 2^n | $2^{n/2}$ | yes | yes | yes | yes | yes |
| $n^{\lg c}$ | $c^{\lg n}$ | yes | yes | yes | yes | yes |
| $\lg(n!)$ | $\lg(n^n)$ | yes | yes | yes | yes | yes |
| A | B | yes | yes | yes | yes | yes |

- 1.6 Order the following functions such that $f_1 = \Omega(f_2), f_2 = \Omega(f_3), \dots, f_{29} = \Omega(f_{30})$, and partition them into equivalence classes such that each function is big- Θ of each other.