

Assignment 2

1.1 Are either $\lceil \lg n \rceil!$ or $\lceil \lg \lg n \rceil!$ polynomially bounded?

Polynomially bounded means $f_n = O(n^k)$ for some constant k (e.g., whether $f_n \leq c \cdot n^k$ for constants c and k as n approaches ∞).

For the first function $\lceil \lg n \rceil!$, without loss of generality, assume $n = 2^a$ (where $a \in \mathbb{N}$). The statement $a! \leq c \cdot 2^{ak}$ (see below) is a contradiction, as the factorial function $a!$ is not exponentially bounded. Therefore, $\lceil \lg n \rceil!$ is not polynomially bounded (via proof by contradiction).

$$\lceil \lg n \rceil! \leq c \cdot n^k$$

$$\lg(2^a)! \leq c \cdot (2^a)^k$$

$$a! \leq c \cdot 2^{ak}$$

For the second function $\lceil \lg \lg n \rceil!$, without loss of generality, assume $n = 2^{2^a}$ (where $a \in \mathbb{N}$). The statement $1 \cdot 2 \cdot 3 \cdots a \leq c \cdot (2^{2^k} \cdot 2^{4^k} \cdot 2^{8^k} \cdots 2^{2^{a \cdot k}})$ (see below) is obviously true. Therefore $\lceil \lg \lg n \rceil!$ is polynomially bounded (via direct proof).

$$\lceil \lg \lg n \rceil! \leq c \cdot n^k$$

$$\lg \lg(2^{2^a})! \leq c \cdot (2^{2^a})^k$$

$$a! \leq c \cdot 2^{k \cdot 2^a}$$

$$1 \cdot 2 \cdot 3 \cdots a \leq c \cdot (2^{2^k} \cdot 2^{4^k} \cdot 2^{8^k} \cdots 2^{2^{a \cdot k}})$$

1.2 Use induction to prove $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$, where $F_i = F_{i-2} + F_{i-1}$, and ϕ is the golden ratio $\frac{1+\sqrt{5}}{2}$.

To prove by induction, write out the expressions f_n and f_{n+1} (note: f_{n+1} is the same as f_n , but with $(n+1)$ substituted everywhere in place of n). Next, if applicable, re-write the expression f_{n+1} in terms of f_n then perform algebraic manipulations on the expression until you reach some variation of $f_{n+1} = f_{n+1}$. Lastly, show that the expression f_c also holds for some

constant c . The algebra is called "the inductive step", and the calculation for on the constant is called "the base case".

In this problem, the expression to prove is $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$, where $\phi = \frac{1+\sqrt{5}}{2}$. Start by demonstrating the expression holds for constants $c = 0$ and $c = 1$.

$$\begin{aligned} F_0 &= \frac{\phi^0 - \hat{\phi}^0}{\sqrt{5}} = \frac{1 - 1}{\sqrt{5}} = 0 \\ F_1 &= \frac{\phi^1 - \hat{\phi}^1}{\sqrt{5}} = \frac{\frac{1+\sqrt{5}}{2} - \frac{1-\sqrt{5}}{2}}{\sqrt{5}} \\ &= \frac{(1 + \sqrt{5}) - (1 - \sqrt{5})}{2\sqrt{5}} = 1 \end{aligned}$$

After showing the expression holds for some base cases F_0 and F_1 , the next step is algebra. Setup the expression F_n in terms of F_{n-1} , then solve (see below).

$$\begin{aligned} F_i &= \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} = F_{i-1} + F_{i-2} & F_{i-1} &= \frac{\phi^{i-1} - \hat{\phi}^{i-1}}{\sqrt{5}} \\ F_i &= F_{i-1} + F_{i-2} \\ \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} &= \frac{\phi^{i-1} - \hat{\phi}^{i-1}}{\sqrt{5}} + \frac{\phi^{i-2} - \hat{\phi}^{i-2}}{\sqrt{5}} \\ &= \frac{\phi^{i-1} + \hat{\phi}^{i-1} - \phi^{i-2} - \hat{\phi}^{i-2}}{\sqrt{5}} \\ &= \frac{\phi^{i-1} - \phi^{i-2} + \hat{\phi}^{i-1} - \hat{\phi}^{i-2}}{\sqrt{5}} \\ &= \frac{[(\phi \cdot \phi^{i-2}) + \phi^{i-2}] - [(\hat{\phi} \cdot \hat{\phi}^{i-2}) + \hat{\phi}^{i-2}]}{\sqrt{5}} \\ &= \frac{\phi^{i-2}(\phi + 1) - \hat{\phi}^{i-2}(\hat{\phi} + 1)}{\sqrt{5}} \\ &= \frac{\phi^{i-2}(\phi^2) - \hat{\phi}^{i-2}(\hat{\phi}^2)}{\sqrt{5}} \\ &= \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} \end{aligned}$$

Since we have shown F_i is obtainable via F_{i-1} , we have completed the inductive step. Since both the inductive step and base cases have been shown, the proof by induction is complete.

1.3 Show that $k \lg k = \Theta(n)$ implies $k = \Theta\left(\frac{n}{\lg n}\right)$.

If $f_n = \Theta(g_n)$, then $\Theta(f_n) = g_n$ by symmetric property of big- Θ . We can use this with some algebra to solve this problem:

$$\begin{aligned}
 k \lg k &= \Theta(n) \implies \Theta(k \lg k) = n \\
 \lg[n] &= \Theta(\lg[k \lg k]) \\
 &= \Theta(\lg k + \lg \lg k) \\
 &= \Theta(\lg k) \\
 n &= \Theta(k \lg k) \\
 \frac{n}{\lg n} &= \frac{\Theta(k \lg k)}{\Theta(\lg k)} = \Theta\left(\frac{k \lg k}{\lg k}\right) = \Theta(k) \\
 \Theta(k) &= \frac{n}{\lg n} \\
 k &= \Theta\left(\frac{n}{\lg n}\right)
 \end{aligned}$$

1.4 Are either 2^{n+1} or 2^{2n} big- O of 2^n ?

The former is, the latter is not. Suppose $2^{n+1} = O(2^n)$, then $2 \cdot 2^n \leq c \cdot 2^n$ as $n \rightarrow \infty$; this is obviously true for all $c \geq 2$.

Regards to whether $2^{2n} = O(2^n)$, consider the inequality $2^{2n} \leq c \cdot 2^n$. This is equivalent to saying $2^n \cdot 2^n \leq c \cdot 2^n$. Dividing both sides by 2^n gives $2^n \leq c$, which is obviously false (the exponential function is not constant bound).

1.5 For each pair of functions (A, B) , indicate whether A is O, o, Ω, ω , or Θ of B . Assume $k \geq 1$, $\epsilon > 0$, $c > 1$ are constants.

	A	B	O	o	Ω	ω	Θ
a.	$\lg^k n$	n^ϵ	yes	yes			
b.	n^k	c^n	yes	yes			
c.	\sqrt{n}	$n^{\sin n}$					
d.	2^n	$2^{n/2}$			yes	yes	
e.	$n^{\lg c}$	$c^{\lg n}$	yes		yes		yes
f.	$\lg(n!)$	$\lg(n^n)$	yes		yes		yes

In terms of growth, $f_n = \Omega(g_n)$ is $f_n \geq c \cdot g_n$, $f_n = \omega(g_n)$ is $f_n > c \cdot g_n$, $f_n = O(g_n)$ is $f_n \leq c \cdot g_n$, $f_n = o(g_n)$ is $f_n < c \cdot g_n$, and $f_n = \Theta(g_n)$ is $f_n = c \cdot g_n$.

To demonstrate whether f_n is something of something, isolate the constant c in the equality and observe whether it holds. Also note that big- Ω precludes little- o and big- O precludes little- ω (e.g., if $f_n = O(g_n)$, then $f_n = \omega(g_n)$ is false, and vice-versa).

1.6 Order the following functions such that $f_1 = \Omega(f_2), f_2 = \Omega(f_3), \dots, f_{29} = \Omega(f_{30})$, and partition them into equivalence classes such that each function is big- Θ of each other.

In terms of growth, $f_1 = \Omega(f_2)$ means $f_1 \leq f_2$. Therefore, the order of functions $f_1 = \Omega(f_2), f_2 = \Omega(f_3), \dots, f_{29} = \Omega(f_{30})$ is as follows: $2^{2^n} = \Omega((n+1)!)$, $(n+1)! = \Omega(n!)$, $n! = \Omega(e^n)$, $e^n = \Omega(n \cdot 2^n)$, $n \cdot 2^n = \Omega(2^n)$, $2^n = \Omega\left(\left(\frac{3}{2}\right)^n\right)$, $\left(\frac{3}{2}\right)^n = \Omega(n^{\lg \lg n})$, $n^{\lg \lg n} = \Omega((\lg n)^{\lg n})$, $(\lg n)^{\lg n} = \Omega((\lg n)!)$, $(\lg n)! = \Omega(N^3)$, $N^3 = \Omega(n^2)$, $n^2 = \Omega(4^{\lg n})$, $4^{\lg n} = \Omega(\lg(n!))$, $\lg(n!) = \Omega(n \lg n)$, $n \lg n = \Omega(2^{\lg n})$, $2^{\lg n} = \Omega(n)$, $n = \Omega\left((\sqrt{2})^{\lg n}\right)$, $(\sqrt{2})^{\lg n} = \Omega(\sqrt{n})$, $\sqrt{n} = \Omega\left(2^{\sqrt{2} \lg n}\right)$, $2^{\sqrt{2} \lg n} = \Omega(\lg^2 n)$, $\lg^2 n = \Omega(\ln n)$, $\ln n = \Omega\left(\sqrt{\lg n}\right)$, $\sqrt{\lg n} = \Omega(\ln \ln n)$, $\ln \ln n = \Omega\left(2^{\lg^* n}\right)$, $2^{\lg^* n} = \Omega(\lg^* n)$, $\lg^* n = \Omega(\lg * (\lg n))$, $\lg * (\lg n) = \Omega(\lg(\lg * n))$, $\lg(\lg * n) = \Omega\left(n^{\frac{1}{\lg n}}\right)$, $n^{\frac{1}{\lg n}} = \Omega(1)$.

An equivalence class is a set containing elements that all adhere to some property. In this case, the elements are functions f , and the property is that each function is big- Θ of every other function in the set. The functions above can be partitioned into the following equivalence classes: $\{n^{\lg \lg n}, (\lg n)^{\lg n}\}$, $\{n^2, 4^{\lg n}\}$, $\{\lg(n!), n \lg n\}$, $\{2^{\lg n}, n\}$, $\{(\sqrt{2})^{\lg n}, \sqrt{n}\}$, $\{\lg^* n, \lg^*(\lg n)\}$, $\{n^{\frac{1}{\lg n}}, 1\}$.