# Assignment 2

## 1.1 Are either $\lceil \lg n \rceil!$ or $\lceil \lg \lg n \rceil!$ polynomially bounded?

Polynomially bounded means $f_n = O(n^k)$ for some constant $k$ (e.g., whether $f_n \leq c \cdot n^k$ for constants $c$ and $k$ as $n$ approaches $\infty$). For the first function $\lceil \lg n \rceil!$, without loss of generality, assume $n = 2^a$ (where $a \in \mathbb{N}$).

$$\begin{aligned} \lceil \lg n \rceil! &\leq c \cdot n^k \\ \lg(2^a)! &\leq c \cdot (2^a)^k \\ a! &\leq c \cdot 2^{ak} \end{aligned}$$

The statement $a! \leq c \cdot 2^{ak}$ is a contradiction, as the factorial function $a!$ is not exponentially bounded. Therefore, $\lceil \lg n \rceil!$ is not polynomially bounded (via proof by contradiction). For the second function $\lceil \lg \lg n \rceil!$, without loss of generality, assume $n = 2^{2^a}$ (where $a \in \mathbb{N}$).

$$\begin{aligned} \lceil \lg \lg n \rceil! &\leq c \cdot n^k \\ \lg \lg \left(2^{2^a}\right)! &\leq c \cdot \left(2^{2^a}\right)^k \\ a! &\leq c \cdot 2^{k \cdot 2^a} \\ 1 \cdot 2 \cdot 3 \cdots a &\leq c \cdot \left(2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^a \cdot k}\right) \end{aligned}$$

The statement $1 \cdot 2 \cdot 3 \cdots a \leq c \cdot \left(2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^a k}\right)$ is obviously true. Therefore $\lceil \lg \lg n \rceil!$ is polynomially bounded (via direct proof).

1.2 Use induction to prove $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$; where $F_i = F_{i-2} + F_{i-1}$, and $\phi$ is the golden ratio $\frac{1+\sqrt{5}}{2}$.

1.3 Show that $k \lg k = \Theta(n)$ implies $k = \Theta\left(\frac{n}{n \ln n}\right)$.

1.4 Are either $2^{n+1}$ or $2^{2n}$ big-$O$ of $2^n$?

1.5 For each pair of functions $(A, B)$, indicate whether $A$ is $O, o, \Omega, \omega$, or $\Theta$ of $B$. Assume $k \geq 1$, $\epsilon > 0$, $c > 1$ are constants.

| | $A$ | $B$ | $O$ | $o$ | $\Omega$ | $\omega$ | $\Theta$ |
|---|---|---|---|---|---|---|---|
| a. | $\lg^k n$ | $n^\epsilon$ | yes | yes | | | |
| b. | $n^k$ | $c^n$ | yes | yes | | | |
| c. | $\sqrt{n}$ | $n^{\sin n}$ | | | | | |
| d. | $2^n$ | $2^{n/2}$ | | | yes | yes | |
| e. | $n^{\lg c}$ | $c^{\lg n}$ | yes | | yes | | yes |
| f. | $\lg(n!)$ | $\lg(n^n)$ | yes | | yes | | yes |

The main idea here is that (in terms of growth rate), $f_n = \Omega(g_n)$ means $f_n \geq c \cdot g_n$, $f_n = \omega(g_n)$ means $f_n > c \cdot g_n$, $f_n = O(g_n)$ means $f_n \leq c \cdot g_n$, $f_n = o(g_n)$ means $f_n < c \cdot g_n$, and $f_n = \Theta(g_n)$ means $f_n == c \cdot g_n$.

To demonstrate whether something is big-something of something, you perform algebraic manipulations on the respective aforementioned inequalities (e.g., isolate the constant $c$) and observe whether the inequality holds. Also note that big-$\Omega$ precludes little-$o$ and big-$O$ precludes little-$\omega$ (e.g., if $f_n = O(g_n)$, then $f_n = \omega(g_n)$ is false, and vice-versa).

## 1.6 Order the following functions such that $f_1 = \Omega(f_2), f_2 = \Omega(f_3), \ldots, f_{29} = \Omega(f_{30})$, and partition them into equivalence classes such that each function is big-$\Theta$ of each other.

In terms of growth, $f_1 = \Omega(f_2)$ means $f_1 \leq f_2$. Therefore, the order of functions $f_1 = \Omega(f_2), f_2 = \Omega(f_3), \ldots, f_{29} = \Omega(f_{30})$ is as follows: $2^{2^n} = \Omega((n+1)!)$, $(n+1)! = \Omega(n!)$, $n! = \Omega(e^n)$, $e^n = \Omega(n \cdot 2^n)$, $n \cdot 2^n = \Omega(2^n)$, $2^n = \Omega\left(\left(\frac{3}{2}\right)^n\right)$, $\left(\frac{3}{2}\right)^n = \Omega\left(n^{\lg\lg n}\right)$, $n^{\lg\lg n} = \Omega\left((\lg n)^{\lg n}\right)$, $(\lg n)^{\lg n} = \Omega((\lg n)!)$, $(\lg n)! = \Omega\left(N^3\right)$, $N^3 = \Omega\left(n^2\right)$, $n^2 = \Omega\left(4^{\lg n}\right)$, $4^{\lg n} = \Omega(\lg(n!))$, $\lg(n!) = \Omega(n \lg n)$, $n \lg n = \Omega\left(2^{\lg n}\right)$, $2^{\lg n} = \Omega(n)$, $n = \Omega\left(\left(\sqrt{2}\right)^{\lg n}\right)$, $\left(\sqrt{2}\right)^{\lg n} = \Omega\left(\sqrt{n}\right)$, $\sqrt{n} = \Omega\left(2^{\sqrt{2\lg n}}\right)$, $2^{\sqrt{2\lg n}} = \Omega\left(\lg^2 n\right)$, $\lg^2 n = \Omega(\ln n)$, $\ln n = \Omega\left(\sqrt{\lg n}\right)$, $\sqrt{\lg n} = \Omega(\ln\ln n)$, $\ln\ln n = \Omega\left(2^{\lg^* n}\right)$, $2^{\lg^* n} = \Omega\left(\lg^* n\right)$, $\lg^* n = \Omega(\lg *(\lg n))$, $\lg *(\lg n) = \Omega(\lg(\lg *n))$, $\lg(\lg *n) = \Omega\left(n^{\frac{1}{\lg n}}\right)$, $n^{\frac{1}{\lg n}} = \Omega(1)$.

An equivalence class is a set containing elements that all adhere to some property. In this case, the elements are functions $f$, and the property is that each function is big-$\Theta$ of every other function in the set. The functions above can be partitioned into the following equivalence classes: $\left\{n^{\lg\lg n}, (\lg n)^{\lg n}\right\}$, $\left\{n^2, 4^{\lg n}\right\}$, $\{\lg(n!), n\lg n\}$, $\left\{2^{\lg n}, n\right\}$, $\left\{\left(\sqrt{2}\right)^{\lg n}, \sqrt{n}\right\}$, $\left\{\lg^* n, \lg^*(\lg n)\right\}$, $\left\{n^{\frac{1}{\lg n}}, 1\right\}$.