Assignment 2

1.1 Are either $\lceil \lg n \rceil!$ or $\lceil \lg \lg n \rceil!$ polynomially bounded?

Polynomially bounded means $f_n = O(n^k)$ for some constant k (e.g., whether $f_n \leq c \cdot n^k$ for constants c and k as n approaches ∞). For the first function $\lceil \lg n \rceil!$, without loss of generality, assume $n = 2^a$ (where $a \in \mathbb{N}$).

$$\begin{aligned} \lceil \lg n \rceil! &\leq c \cdot n^k \\ \lg(2^a)! &\leq c \cdot (2^a)^k \\ a! &\leq c \cdot 2^{ak} \end{aligned}$$

The statement $a! \leq c \cdot 2^{ak}$ is a contradiction, as the factorial function $a!$ is not exponentially bounded. Therefore, $\lceil \lg n \rceil!$ is not polynomially bounded (via proof by contradiction). For the second function $\lceil \lg \lg n \rceil!$, without loss of generality, assume $n = 2^{2^a}$ (where $a \in \mathbb{N}$).

$$\begin{aligned} \lceil \lg \lg n \rceil! &\leq c \cdot n^k \\ \lg \lg \left(2^{2^a}\right)! &\leq c \cdot \left(2^{2^a}\right)^k \\ a! &\leq c \cdot 2^{k \cdot 2^a} \\ 1 \cdot 2 \cdot 3 \cdots a &\leq c \cdot \left(2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^a \cdot k}\right) \end{aligned}$$

The statement $1 \cdot 2 \cdot 3 \cdots a \leq c \cdot \left(2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^a k}\right)$ is obviously true. Therefore $\lceil \lg \lg n \rceil!$ is polynomially bounded (via direct proof).

1.2 Use induction to prove $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$; where $F_i = F_{i-2} + F_{i-1}$, and ϕ is the golden ratio $\frac{1+\sqrt{5}}{2}$.

1.3 Show that $k \lg k = \Theta(n)$ implies $k = \Theta\left(\frac{n}{n \ln n}\right)$.

1.4 Are either 2^{n+1} or 2^{2n} big-O of 2^n?

1.5 For each pair of functions (A, B), indicate whether A is O, o, Ω, ω, or Θ of B. Assume $k \geq 1$, $\epsilon > 0$, $c > 1$ are constants.

	A	B	O	o	Ω	ω	Θ
a.	$\lg^k n$	n^ϵ	yes	yes			
b.	n^k	c^n	yes	yes			
c.	$\sqrt{n}$	$n^{\sin n}$					
d.	2^n	$2^{n/2}$			yes	yes	
e.	$n^{\lg c}$	$c^{\lg n}$	yes		yes		yes
f.	$\lg(n!)$	$\lg(n^n)$	yes		yes		yes

1.6 Order the following functions such that $f_1 = \Omega(f_2), f_2 = \Omega(f_3), \ldots, f_{29} = \Omega(f_{30})$, and partition them into equivalence classes such that each function is big-Θ of each other.

$2^{2^{n+1}} = \Omega\left(2^{2^n}\right)$, $2^{2^n} = \Omega((n+1)!)$, $(n+1)! = \Omega(n!)$, $n! = \Omega(e^n)$, $e^n = \Omega(n \cdot 2^n)$,
$n \cdot 2^n = \Omega(2^n)$, $2^n = \Omega\left(\left(\frac{3}{2}\right)^n\right)$, $\left(\frac{3}{2}\right)^n = \Omega\left(n^{\lg\lg n}\right)$, $\{n^{\lg\lg n} = \Omega\left((\lg n)^{\lg n}\right), (\lg n)^{\lg n} = \Omega((\lg n)!),\}$
$(\lg n)! = \Omega\left(N^3\right), N^3 = \Omega\left(n^2\right), n^2 = \Omega\left(4^{\lg n}\right), 4^{\lg n} = \Omega(\lg(n!)), \lg(n!) = \Omega(n \lg n)$,
$n \lg n = \Omega\left(2^{\lg n}\right), 2^{\lg n} = \Omega(n), n = \Omega\left(\left(\sqrt{2}\right)^{\lg n}\right), \left(\sqrt{2}\right)^{\lg n} = \Omega\left(\sqrt{n}\right), \sqrt{n} = \Omega\left(2^{\sqrt{2 \lg n}}\right)$,
$2^{\sqrt{2 \lg n}} = \Omega\left(\lg^2 n\right), \lg^2 n = \Omega(\ln n), \ln n = \Omega\left(\sqrt{\lg n}\right), \sqrt{\lg n} = \Omega(\ln \ln n), \ln \ln n =$
$\Omega\left(2^{\lg^* n}\right), 2^{\lg^* n} = \Omega\left(\lg^* n\right), \lg^* n = \Omega(\lg *(\lg n)), \lg *(\lg n) = \Omega(\lg(\lg * n)), \lg(\lg * n) =$
$\Omega\left(n^{\frac{1}{\lg n}}\right), n^{\frac{1}{\lg n}}$