Assignment 2

1.1 Are either $\lceil \lg n \rceil!$ or $\lceil \lg \lg n \rceil!$ polynomially bounded?

Polynomially bounded means $f_n = O(n^k)$ for some constant k (e.g., whether $f_n \le c \cdot n^k$ for constants c and k as n approaches ∞). For the first function $\lceil \lg n \rceil!$, without loss of generality, assume $n = 2^a$ (where $a \in \mathbb{N}$).

$$\lceil \lg n \rceil! \le c \cdot n^k$$
$$\lg(2^a)! \le c \cdot (2^a)^k$$
$$a! \le c \cdot 2^{ak}$$

The statement $a! \le c \cdot 2^{ak}$ is a contradiction, as the factorial function a! is not exponentially bounded. Therefore, $\lceil \lg n \rceil !$ is not polynomially bounded (via proof by contradiction). For the second function $\lceil \lg \lg n \rceil !$, without loss of generality, assume $n = 2^{2^a}$ (where $a \in \mathbb{N}$).

$$\begin{split} \lceil \lg \lg n \rceil! &\leq c \cdot n^k \\ \lg \lg \left(2^{2^a} \right)! &\leq c \cdot \left(2^{2^a} \right)^k \\ a! &\leq c \cdot 2^{k \cdot 2^a} \\ 1 \cdot 2 \cdot 3 \cdots a &\leq c \cdot \left(2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^a \cdot k} \right) \end{split}$$

The statement $1 \cdot 2 \cdot 3 \cdots a \leq c \cdot (2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^{a_k}})$ is obviously true. Therefore $\lceil \lg \lg n \rceil!$ is polynomially bounded (via direct proof).

- 1.2 Use induction to prove $F_i = \frac{\phi^i \hat{\phi}^i}{\sqrt{5}}$; where $F_i = F_{i-2} + F_{i-1}$, and ϕ is the golden ratio $\frac{1+\sqrt{5}}{2}$.
- 1.3 Show that $k \lg k = \Theta(n)$ implies $k = \Theta\left(\frac{n}{n \ln n}\right)$.
- 1.4 Are either 2^{n+1} or 2^{2n} big-O of 2^{n} ?
- 1.5 For each pair of functions (A,B), indicate whether A is O, o, Ω, ω , or Θ of B. Assume $k \ge 1$, $\epsilon > 0$, c > 1 are constants.

	\boldsymbol{A}	B	0	o	Ω	ω	Θ
a.	$\lg^k n$	n^ϵ	yes	yes			
b.	n^k	c^n	yes	yes			
c.	\sqrt{n}	$n^{\sin n}$					
d.	2^n	$2^{n/2}$			yes	yes	
e.	$n^{\lg c}$	$c^{\lg n}$	yes		yes		yes
f.	$\lg(n!)$	$\lg(n^n)$	yes		yes		yes

The main idea here is that (in terms of growth rate), $f_n = \Omega(g_n)$ means $f_n \ge c \cdot g_n$, $f_n = \omega(g_n)$ means $f_n > c \cdot g_n$, $f_n = O(g_n)$ means $f_n \le c \cdot g_n$, $f_n = o(g_n)$ means $f_n < c \cdot g_n$, and $f_n = \Theta(g_n)$ means $f_n = c \cdot g_n$.

To demonstrate whether something is big-something of something, you perform algebraic manipulations on the respective aforementioned inequalities (e.g., isolate the constant c) and observe whether the inequality holds. Also note that $\underline{\text{big-}\Omega}$ precludes little-o and $\underline{\text{big-}O}$ precludes little-o (e.g., if $f_n = O(g_n)$, then $f_n = \omega(g_n)$ is false, and vice-versa).

1.6 Order the following functions such that $f_1 = \Omega(f_2)$, $f_2 = \Omega(f_3)$, ..., $f_{29} = \Omega(f_{30})$, and partition them into equivalence classes such that each function is big- Θ of each other.

 $\begin{array}{l} \underline{\text{In terms of growth, } f_1 = \Omega(f_2) \text{ means } f_1 \leq f_2.} \text{ Therefore, the order of functions } f_1 = \Omega(f_2), f_2 = \Omega(f_3), ..., f_{29} = \Omega(f_{30}) \text{ is as follows: } 2^{2^n} = \Omega((n+1)!), \\ (n+1)! = \Omega(n!), \ n! = \Omega(e^n), \ e^n = \Omega(n \cdot 2^n), \ n \cdot 2^n = \Omega(2^n), \ 2^n = \Omega\left(\left(\frac{3}{2}\right)^n\right), \\ \left(\frac{3}{2}\right)^n = \Omega\left(n^{\lg\lg n}\right), \ n^{\lg\lg n} = \Omega\left((\lg n)^{\lg n}\right), \ (\lg n)^{\lg n} = \Omega((\lg n)!), \ (\lg n)! = \Omega(N^3), \\ N^3 = \Omega\left(n^2\right), \ n^2 = \Omega\left(4^{\lg n}\right), \ 4^{\lg n} = \Omega(\lg(n!)), \ \lg(n!) = \Omega(n\lg n), \ n\lg n = \Omega\left(2^{\lg n}\right), \\ 2^{\lg n} = \Omega(n), \ n = \Omega\left(\left(\sqrt{2}\right)^{\lg n}\right), \ \left(\sqrt{2}\right)^{\lg n} = \Omega\left(\sqrt{n}\right), \ \sqrt{n} = \Omega\left(2^{\sqrt{2\lg n}}\right), \ 2^{\sqrt{2\lg n}} = \Omega\left(\lg^2 n\right), \ \lg^2 n = \Omega(\ln n), \ \ln n = \Omega\left(\sqrt{\lg n}\right), \ \sqrt{\lg n} = \Omega(\ln \ln n), \ \ln \ln n = \Omega\left(2^{\lg^* n}\right), \\ 2^{\lg^* n} = \Omega\left(\lg^* n\right), \ \lg^* n = \Omega(\lg * (\lg n)), \ \lg * (\lg n) = \Omega(\lg(\lg * n)), \ \lg(\lg * n) = \Omega\left(n^{\frac{1}{\lg n}}\right), \ n^{\frac{1}{\lg n}} = \Omega(1). \end{array}$

An equivalence class is a set containing elements that all adhere to some property. In this case, the elements are functions f, and the property is that each function is big- Θ of every other function in the set. The functions above can be partitioned into the following equivalence classes: $\{n^{\lg\lg n},(\lg n)^{\lg n}\},\{n^2,4^{\lg n}\},\{\lg(n!),n\lg n\},\{2^{\lg n},n\},\{(\sqrt{2})^{\lg n},\sqrt{n}\},\{\lg^* n,\lg^*(\lg n)\},\{n^{\frac{1}{\lg n}},1\}.$