

## Assignment 2

- 1.1 Use induction to prove  $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$ ; where  $F_i = F_{i-2} + F_{i-1}$ , and  $\phi$  is the golden ratio  $\frac{1+\sqrt{5}}{2}$ .

To prove by induction, write out the expressions  $f_n$  and  $f_{n+1}$  (note:  $f_{n+1}$  is the same as  $f_n$ , but with  $(n+1)$  substituted everywhere in place of  $n$ ). Next, if applicable, re-write the expression  $f_{n+1}$  in terms of  $f_n$  then perform algebraic manipulations on the expression until you reach some variation of  $f_{n+1} = f_n$ . Lastly, show that the expression  $f_c$  also holds for some constant  $c$ . The algebra is called "the inductive step", and the calculation for on the constant is called "the base case".

In this problem, the expression to prove is  $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$ , where  $\phi = \frac{1+\sqrt{5}}{2}$ . Start by demonstrating the expression holds for constants  $c = 0, c = 1$ .

$$F_0 = \frac{\phi^0 - \hat{\phi}^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} \quad (1.1)$$

$$= 0 \quad (1.2)$$

$$F_1 = \frac{\phi^1 - \hat{\phi}^1}{\sqrt{5}} = \frac{\frac{(1+\sqrt{5})}{2} + \frac{2}{(1+\sqrt{5})}}{\sqrt{5}} \quad (1.1)$$

$$= \frac{(1+\sqrt{5}) - (1-\sqrt{5})}{2\sqrt{5}} \quad (1.2)$$

$$= 1 \quad (1.3)$$

After showing the expression holds for some base cases  $F_0$  and  $F_1$ , the next step is algebra. Setup the expression  $F_n$  in terms of  $F_{n-1}$ , then solve (see below).

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} = F_{i-1} + F_{i-2} \quad F_{i-1} = \frac{\phi^{i-1} - \hat{\phi}^{i-1}}{\sqrt{5}}$$

$$F_{i+1} = F_i + F_{i-1} \quad (1.1)$$

$$\frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} = \frac{\phi^{i-1} - \hat{\phi}^{i-1}}{\sqrt{5}} + \frac{\phi^{i-2} - \hat{\phi}^{i-2}}{\sqrt{5}} \quad (1.2)$$

$$= \frac{\phi^{i-1} + \hat{\phi}^{i-1} - \phi^{i-2} - \hat{\phi}^{i-2}}{\sqrt{5}} \quad (1.3)$$

$$= \frac{\phi^{i-1} - \phi^{i-2} + \hat{\phi}^{i-1} - \hat{\phi}^{i-2}}{\sqrt{5}} \quad (1.4)$$

$$= \frac{[(\phi \cdot \phi^{i-2}) + \phi^{i-2}] - [(\hat{\phi} \cdot \hat{\phi}^{i-2}) + \hat{\phi}^{i-2}]}{\sqrt{5}} \quad (1.5)$$

$$= \frac{\phi^{i-2}(\phi + 1) - \hat{\phi}^{i-2}(\hat{\phi} + 1)}{\sqrt{5}} \quad (1.6)$$

$$= \frac{\phi^{i-2}(\phi^2) - \hat{\phi}^{i-2}(\hat{\phi}^2)}{\sqrt{5}} \quad (1.7)$$

$$= \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} \quad (1.8)$$

Since we have shown  $F_{i+1}$  is obtainable via  $F_i$ , we have completed the inductive step. Since both the inductive step and base cases have been shown, the proof by induction is complete.