Assignment 2

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1.1 Are either $\lceil \lg n \rceil!$ or $\lceil \lg \lg n \rceil!$ polynomially bounded?

Polynomially bounded means $f_n = O(n^k)$ for some constant k (e.g., whether $f_n \leq c \cdot n^k$ for constants c and k as n approaches ∞).

For the first function $\lceil \lg n \rceil!$, without loss of generality, assume $n = 2^a$ (where $a \in \mathbb{N}$). The statement $a! \leq c \cdot 2^{ak}$ (see below) is a contradiction, as the factorial function $a!$ is not exponentially bounded. Therefore, $\lceil \lg n \rceil!$ is not polynomially bounded (via proof by contradiction).

$$\lceil \lg n \rceil! \leq c \cdot n^k$$

$$\lg(2^a)! \leq c \cdot (2^a)^k$$

$$a! \leq c \cdot 2^{ak}$$

For the second function $\lceil \lg \lg n \rceil!$, without loss of generality, assume $n = 2^{2^a}$ (where $a \in \mathbb{N}$). The statement $1 \cdot 2 \cdot 3 \cdots a \leq c \cdot \left(2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^a k}\right)$ (see below) is obviously true. Therefore $\lceil \lg \lg n \rceil!$ is polynomially bounded (via direct proof).

$$\lceil \lg \lg n \rceil! \leq c \cdot n^k$$

$$\lg \lg \left(2^{2^a}\right)! \leq c \cdot \left(2^{2^a}\right)^k$$

$$a! \leq c \cdot 2^{k \cdot 2^a}$$

$$1 \cdot 2 \cdot 3 \cdots a \leq c \cdot \left(2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^a \cdot k}\right)$$

1.2 Use induction to prove $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$, where $F_i = F_{i-2} + F_{i-1}$, and ϕ is the golden ratio $\frac{1+\sqrt{5}}{2}$.

To prove by induction, write out the expressions f_n and f_{n+1} (note: f_{n+1} is the same as f_n, but with $(n+1)$ substituted everywhere in place of n).

Next, if applicable, re-write the expression f_{n+1} in terms of f_n then perform algebraic manipulations on the expression until you reach some variation of $f_{n+1} = f_{n+1}$. Lastly, show that the expression f_c also holds for some constant c. The algebra is called "the inductive step", and the calculation for on the constant is called "the base case".

In this problem, the expression to prove is $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$, where $\phi = \frac{1+\sqrt{5}}{\sqrt{5}}$. Start by demonstrating the expression holds for constants $c = 0$ and $c = 1$.

$$F_0 = \frac{\phi^0 - \hat{\phi}^0}{\sqrt{5}} = \frac{1-1}{\sqrt{5}} = 0$$

$$\begin{aligned}
F_1 = \frac{\phi^1 - \hat{\phi}^1}{\sqrt{5}} &= \frac{\frac{(1+\sqrt{5})}{\sqrt{5}} + \frac{2}{(1+\sqrt{5})}}{\sqrt{5}} \\
&= \frac{\left(1+\sqrt{5}\right) - \left(1-\sqrt{5}\right)}{2\sqrt{5}} = 1
\end{aligned}$$

After showing the expression holds for some base cases F_0 and F_1, the next step is algebra. Setup the expression F_n in terms of F_{n-1}, then solve (see below).

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} = F_{i-1} + F_{i-2} \qquad F_{i-1} = \frac{\phi^{i-1} - \hat{\phi}^{i-1}}{\sqrt{5}}$$

$$\begin{aligned}
F_i &= F_{i-1} + F_{i-2} \\
\frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} &= \frac{\phi^{i-1} - \hat{\phi}^{i-1}}{\sqrt{5}} + \frac{\phi^{i-2} - \hat{\phi}^{i-2}}{\sqrt{5}} \\
&= \frac{\phi^{i-1} + \hat{\phi}^{i-1} - \phi^{i-2} - \hat{\phi}^{i-2}}{\sqrt{5}} \\
&= \frac{\phi^{i-1} - \phi^{i-2} + \hat{\phi}^{i-1} - \hat{\phi}^{i-2}}{\sqrt{5}} \\
&= \frac{\left[(\phi \cdot \phi^{i-2}) + \phi^{i-2}\right] - \left[(\hat{\phi} \cdot \hat{\phi}^{i-2}) + \hat{\phi}^{i-2}\right]}{\sqrt{5}} \\
&= \frac{\phi^{i-2}\left(\phi + 1\right) - \hat{\phi}^{i-2}\left(\hat{\phi} + 1\right)}{\sqrt{5}} \\
&= \frac{\phi^{i-2}\left(\phi^2\right) - \hat{\phi}^{i-2}\left(\hat{\phi}^2\right)}{\sqrt{5}} \\
&= \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}
\end{aligned}$$

Since we have shown F_i is obtainable via F_{i-1}, we have completed the inductive step. Since both the inductive step and base cases have been shown, the proof by induction is complete.

1.3 Show that $k \lg k = \Theta(n)$ implies $k = \Theta\left(\frac{n}{\ln n}\right)$.

If $f_n = \Theta(g_n)$, then $\Theta(f_n) = g_n$ by symmetric property of big-Θ. We can use this with some algebra to solve this problem:

$$k \ln k = \Theta(n) \Longrightarrow \Theta(k \ln k) = n$$

$$\begin{aligned}
\ln[n] &= \Theta(\ln[k \ln k]) \\
&= \Theta(\ln k + \ln \ln k) \\
&= \Theta(\ln k) \\
n &= \Theta(k \ln k) \\
\frac{n}{\ln n} &= \frac{\Theta(k \ln k)}{\Theta(\ln k)} = \Theta\left(\frac{k \ln k}{\ln k}\right) = \Theta(k) \\
\Theta(k) &= \frac{n}{\ln n} \\
k &= \Theta\left(\frac{n}{\ln n}\right)
\end{aligned}$$

1.4 Are either 2^{n+1} or 2^{2n} big-O of 2^n?

The former is, the latter is not. Suppose $2^{n+1} = O(2^n)$, then $2 \cdot 2^n \le c \cdot 2^n$ as $n \to \infty$; this is obviously true for all $c \ge 2$. Regards to whether $2^{2n} = O(2^n)$, consider the inequality $2^{2n} \le c \cdot 2^n$. This is equivalent to saying $2^n \cdot 2^n \le c \cdot 2^n$. Dividing both sides by 2^n gives $2^n \le c$, which is obviously false (the exponential function is not constant bound).

1.5 For each pair of functions (A, B), indicate whether A is O, o, Ω, ω, or Θ of B. Assume $k \ge 1$, $\epsilon > 0$, $c > 1$ are constants.

	A	B	O	o	Ω	ω	Θ
a.	$\lg^k n$	n^ϵ	yes	yes			
b.	n^k	c^n	yes	yes			
c.	$\sqrt{n}$	$n^{\sin n}$					
d.	2^n	$2^{n/2}$			yes	yes	
e.	$n^{\lg c}$	$c^{\lg n}$	yes		yes		yes
f.	$\lg(n!)$	$\lg(n^n)$	yes		yes		yes

In terms of growth, $f_n = \Omega(g_n)$ is $f_n \geq c \cdot g_n$, $f_n = \omega(g_n)$ is $f_n > c \cdot g_n$, $f_n = O(g_n)$ is $f_n \leq c \cdot g_n$, $f_n = o(g_n)$ is $f_n < c \cdot g_n$, and $f_n = \Theta(g_n)$ is $f_n == c \cdot g_n$.

To demonstrate whether f_n is something of something, isolate the constant c in the equality and observe whether it holds. Also note that big-Ω precludes little-o and big-O precludes little-ω (e.g., if $f_n = O(g_n)$, then $f_n = \omega(g_n)$ is false, and vice-versa).

1.6 Order the following functions such that $f_1 = \Omega(f_2), f_2 = \Omega(f_3), \ldots, f_{29} = \Omega(f_{30})$, and partition them into equivalence classes such that each function is big-Θ of each other.

In terms of growth, $f_1 = \Omega(f_2)$ means $f_1 \leq f_2$. Therefore, the order of functions $f_1 = \Omega(f_2), f_2 = \Omega(f_3), \ldots, f_{29} = \Omega(f_{30})$ is as follows: $2^{2^n} = \Omega((n+1)!)$, $(n+1)! = \Omega(n!)$, $n! = \Omega(e^n)$, $e^n = \Omega(n \cdot 2^n)$, $n \cdot 2^n = \Omega(2^n)$, $2^n = \Omega\left(\left(\frac{3}{2}\right)^n\right)$, $\left(\frac{3}{2}\right)^n = \Omega\left(n^{\lg\lg n}\right)$, $n^{\lg\lg n} = \Omega\left((\lg n)^{\lg n}\right)$, $(\lg n)^{\lg n} = \Omega((\lg n)!)$, $(\lg n)! = \Omega\left(N^3\right)$, $N^3 = \Omega\left(n^2\right)$, $n^2 = \Omega\left(4^{\lg n}\right)$, $4^{\lg n} = \Omega(\lg(n!))$, $\lg(n!) = \Omega(n \lg n)$, $n \lg n = \Omega\left(2^{\lg n}\right)$, $2^{\lg n} = \Omega(n)$, $n = \Omega\left(\left(\sqrt{2}\right)^{\lg n}\right)$, $\left(\sqrt{2}\right)^{\lg n} = \Omega\left(\sqrt{n}\right)$, $\sqrt{n} = \Omega\left(2^{\sqrt{2 \lg n}}\right)$, $2^{\sqrt{2\lg n}} = \Omega\left(\lg^2 n\right)$, $\lg^2 n = \Omega(\ln n)$, $\ln n = \Omega\left(\sqrt{\lg n}\right)$, $\sqrt{\lg n} = \Omega(\ln \ln n)$, $\ln \ln n = \Omega\left(2^{\lg^* n}\right)$, $2^{\lg^* n} = \Omega\left(\lg^* n\right)$, $\lg^* n = \Omega(\lg *(\lg n))$, $\lg *(\lg n) = \Omega(\lg(\lg *n))$, $\lg(\lg *n) = \Omega\left(n^{\frac{1}{\lg n}}\right)$, $n^{\frac{1}{\lg n}} = \Omega(1)$.

An equivalence class is a set containing elements that all adhere to some property. In this case, the elements are functions f, and the property is that each function is big-Θ of every other function in the set. The functions above can be partitioned into the following equivalence classes: $\left\{n^{\lg\lg n}, (\lg n)^{\lg n}\right\}$, $\left\{n^2, 4^{\lg n}\right\}$, $\{\lg(n!), n \lg n\}$, $\left\{2^{\lg n}, n\right\}$, $\left\{\left(\sqrt{2}\right)^{\lg n}, \sqrt{n}\right\}$, $\left\{\lg^* n, \lg^*(\lg n)\right\}$, $\left\{n^{\frac{1}{\lg n}}, 1\right\}$.