

Assignment 2

- 1.1 Use induction to prove $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$; where $F_i = F_{i-2} + F_{i-1}$, and ϕ is the golden ratio $\frac{1+\sqrt{5}}{2}$.

To prove by induction, write out the expressions f_n and f_{n+1} (note: f_{n+1} is the same as f_n , but with $(n+1)$ substituted everywhere in place of n). Next, if applicable, re-write the expression f_{n+1} in terms of f_n then perform algebraic manipulations on the expression until you reach some variation of $f_{n+1} = f_n$. Lastly, show that the expression f_c also holds for some constant c . The algebra is called "the inductive step", and the calculation for on the constant is called "the base case".

In this problem, the expression to prove is $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$, where $\phi = \frac{1+\sqrt{5}}{2}$. Start by demonstrating the expression holds for constants $c = 0$ and $c = 1$.

$$\begin{aligned} F_0 &= \frac{\phi^0 - \hat{\phi}^0}{\sqrt{5}} = \frac{1 - 1}{\sqrt{5}} \\ &= 0 \\ F_1 &= \frac{\phi^1 - \hat{\phi}^1}{\sqrt{5}} = \frac{\frac{(1+\sqrt{5})}{2} - \frac{2}{(1+\sqrt{5})}}{\sqrt{5}} \\ &= \frac{(1+\sqrt{5}) - (1-\sqrt{5})}{2\sqrt{5}} \\ &= 1 \end{aligned}$$

After showing the expression holds for some base cases F_0 and F_1 , the next step is algebra. Setup the expression F_n in terms of F_{n-1} , then solve (see below).

$$F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} = F_{i-1} + F_{i-2} \qquad F_{i-1} = \frac{\phi^{i-1} - \hat{\phi}^{i-1}}{\sqrt{5}}$$

$$F_{i+1} = F_i + F_{i-1} \quad (1.1)$$

$$\frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} = \frac{\phi^{i-1} - \hat{\phi}^{i-1}}{\sqrt{5}} + \frac{\phi^{i-2} - \hat{\phi}^{i-2}}{\sqrt{5}} \quad (1.2)$$

$$= \frac{\phi^{i-1} + \hat{\phi}^{i-1} - \phi^{i-2} - \hat{\phi}^{i-2}}{\sqrt{5}} \quad (1.3)$$

$$= \frac{\phi^{i-1} - \phi^{i-2} + \hat{\phi}^{i-1} - \hat{\phi}^{i-2}}{\sqrt{5}} \quad (1.4)$$

$$= \frac{[(\phi \cdot \phi^{i-2}) + \phi^{i-2}] - [(\hat{\phi} \cdot \hat{\phi}^{i-2}) + \hat{\phi}^{i-2}]}{\sqrt{5}} \quad (1.5)$$

$$= \frac{\phi^{i-2}(\phi + 1) - \hat{\phi}^{i-2}(\hat{\phi} + 1)}{\sqrt{5}} \quad (1.6)$$

$$= \frac{\phi^{i-2}(\phi^2) - \hat{\phi}^{i-2}(\hat{\phi}^2)}{\sqrt{5}} \quad (1.7)$$

$$= \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}} \quad (1.8)$$

Since we have shown F_{i+1} is obtainable via F_i , we have completed the inductive step. Since both the inductive step and base cases have been shown, the proof by induction is complete.