Assignment 2

1.1 Are either $\lceil \lg n \rceil!$ or $\lceil \lg \lg n \rceil!$ polynomially bounded?

Polynomially bounded means $f_n = O(n^k)$ for some constant k (e.g., whether $f_n \le c \cdot n^k$ for constants c and k as n approaches ∞). For the first function $\lceil \lg n \rceil!$, without loss of generality, assume $n = 2^a$ (where $a \in \mathbb{N}$).

$$
\begin{aligned}
\lceil \lg n \rceil! &\le c \cdot n^k \\
\lg(2^a)! &\le c \cdot (2^a)^k \\
a! &\le c \cdot 2^{ak}
\end{aligned}
$$

The statement $a! \le c \cdot 2^{ak}$ is a contradiction, as the factorial function $a!$ is not exponentially bounded. Therefore, $\lceil \lg n \rceil!$ is not polynomially bounded (via proof by contradiction). For the second function $\lceil \lg \lg n \rceil!$, without loss of generality, assume $n = 2^{2^a}$ (where $a \in \mathbb{N}$).

$$
\begin{aligned}
\lceil \lg \lg n \rceil! &\le c \cdot n^k \\
\lg \lg \left(2^{2^a}\right)! &\le c \cdot \left(2^{2^a}\right)^k \\
a! &\le c \cdot 2^{k \cdot 2^a} \\
1 \cdot 2 \cdot 3 \cdots a &\le c \cdot \left(2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^a \cdot k}\right)
\end{aligned}
$$

The statement $1 \cdot 2 \cdot 3 \cdots a \le c \cdot \left(2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^a k}\right)$ is obviously true. Therefore $\lceil \lg \lg n \rceil!$ is polynomially bounded (via direct proof).

1.2 Use induction to prove $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$; where $F_i = F_{i-2} + F_{i-1}$, and ϕ is the golden ratio $\frac{1+\sqrt{5}}{2}$.

1.3 Show that $k \lg k = \Theta(n)$ implies $k = \Theta\left(\frac{n}{n \ln n}\right)$.

1.4 Are either 2^{n+1} or 2^{2n} big-O of 2^n?

1.5 For each pair of functions (A, B), indicate whether A is O, o, Ω, ω, or Θ of B. Assume $k \geq 1$, $\epsilon > 0$, $c > 1$ are constants.

	A	B	O	o	Ω	ω	Θ
a.	$\lg^k n$	n^ϵ	yes	yes			
b.	n^k	c^n	yes	yes			
c.	$\sqrt{n}$	$n^{\sin n}$					
d.	2^n	$2^{n/2}$			yes	yes	
e.	$n^{\lg c}$	$c^{\lg n}$	yes		yes		yes
f.	$\lg(n!)$	$\lg(n^n)$	yes		yes		yes

The main idea here is that (in terms of growth rate), $f_n = \Omega(g_n)$ means $f_n \geq c \cdot g_n$, $f_n = \omega(g_n)$ means $f_n > c \cdot g_n$, $f_n = O(g_n)$ means $f_n \leq c \cdot g_n$, $f_n = o(g_n)$ means $f_n < c \cdot g_n$, and $f_n = \Theta(g_n)$ means $f_n == c \cdot g_n$.

To demonstrate whether something is big-something of something, you perform algebraic manipulations on the respective aforementioned inequalities (e.g., isolate the constant c) and observe whether the inequality holds.

Also note that big-Ω precludes little-o and big-O precludes little-ω (e.g., if $f_n = O(g_n)$, then $f_n = \omega(g_n)$ is false, and vice-versa).

1.6 Order the following functions such that $f_1 = \Omega(f_2), f_2 = \Omega(f_3), \ldots, f_{29} = \Omega(f_{30})$, and partition them into equivalence classes such that each function is big-Θ of each other.

In terms of growth, $f_1 = \Omega(f_2)$ means $f_1 \le f_2$. Therefore, the order of functions $f_1 = \Omega(f_2), f_2 = \Omega(f_3), \ldots, f_{29} = \Omega(f_{30})$ is as follows: $2^{2^n} = \Omega((n+1)!)$, $(n+1)! = \Omega(n!)$, $n! = \Omega(e^n)$, $e^n = \Omega(n \cdot 2^n)$, $n \cdot 2^n = \Omega(2^n)$, $2^n = \Omega\left(\left(\frac{3}{2}\right)^n\right)$, $\left(\frac{3}{2}\right)^n = \Omega\left(n^{\lg\lg n}\right)$, $n^{\lg\lg n} = \Omega\left((\lg n)^{\lg n}\right)$, $(\lg n)^{\lg n} = \Omega((\lg n)!)$, $(\lg n)! = \Omega\left(N^3\right)$, $N^3 = \Omega\left(n^2\right)$, $n^2 = \Omega\left(4^{\lg n}\right)$, $4^{\lg n} = \Omega(\lg(n!))$, $\lg(n!) = \Omega(n \lg n)$, $n \lg n = \Omega\left(2^{\lg n}\right)$, $2^{\lg n} = \Omega(n)$, $n = \Omega\left(\left(\sqrt{2}\right)^{\lg n}\right)$, $\left(\sqrt{2}\right)^{\lg n} = \Omega\left(\sqrt{n}\right)$, $\sqrt{n} = \Omega\left(2^{\sqrt{2 \lg n}}\right)$, $2^{\sqrt{2 \lg n}} = \Omega\left(\lg^2 n\right)$, $\lg^2 n = \Omega(\ln n)$, $\ln n = \Omega\left(\sqrt{\lg n}\right)$, $\sqrt{\lg n} = \Omega(\ln \ln n)$, $\ln \ln n = \Omega\left(2^{\lg^* n}\right)$, $2^{\lg^* n} = \Omega\left(\lg^* n\right)$, $\lg^* n = \Omega(\lg *(\lg n))$, $\lg *(\lg n) = \Omega(\lg(\lg * n))$, $\lg(\lg * n) = \Omega\left(n^{\frac{1}{\lg n}}\right)$, $n^{\frac{1}{\lg n}} = \Omega(1)$.

An **equivalence class** is a set containing elements that all adhere to some property. In this case, the elements are functions f, and the property is that each function is big-Θ of every other function in the set. The functions above can be partitioned into the following equivalence classes: $\left\{n^{\lg\lg n}, (\lg n)^{\lg n}\right\}$, $\left\{n^2, 4^{\lg n}\right\}$, $\{\lg(n!), n \lg n\}$, $\left\{2^{\lg n}, n\right\}$, $\left\{\left(\sqrt{2}\right)^{\lg n}, \sqrt{n}\right\}$, $\left\{\lg^* n, \lg^*(\lg n)\right\}$, $\left\{n^{\frac{1}{\lg n}}, 1\right\}$.