Assignment 2

1.1 Are either $\lceil \lg n \rceil!$ or $\lceil \lg \lg n \rceil!$ polynomially bounded?

Polynomially bounded means $f_n = O(n^k)$ for some constant k (e.g., whether $f_n \le c \cdot n^k$ for constants c and k as n approaches ∞). For the first function $\lceil \lg n \rceil!$, without loss of generality, assume $n = 2^a$ (where $a \in \mathbb{N}$).

$$\begin{aligned}
\lceil \lg n \rceil! &\le c \cdot n^k \\
\lg(2^a)! &\le c \cdot (2^a)^k \\
a! &\le c \cdot 2^{ak}
\end{aligned}$$

The statement $a! \le c \cdot 2^{ak}$ is a contradiction, as the factorial function $a!$ is not exponentially bounded. Therefore, $\lceil \lg n \rceil!$ is not polynomially bounded (via proof by contradiction). For the second function $\lceil \lg \lg n \rceil!$, without loss of generality, assume $n = 2^{2^a}$ (where $a \in \mathbb{N}$).

$$\begin{aligned}
\lceil \lg \lg n \rceil! &\le c \cdot n^k \\
\lg \lg \left(2^{2^a}\right)! &\le c \cdot \left(2^{2^a}\right)^k \\
a! &\le c \cdot 2^{k \cdot 2^a} \\
1 \cdot 2 \cdot 3 \cdots a &\le c \cdot \left(2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^a \cdot k}\right)
\end{aligned}$$

The statement $1 \cdot 2 \cdot 3 \cdots a \le c \cdot \left(2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^a k}\right)$ is obviously true. Therefore $\lceil \lg \lg n \rceil!$ is polynomially bounded (via direct proof).

1.2 Use induction to prove $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$; where $F_i = F_{i-2} + F_{i-1}$, and ϕ is the golden ratio $\frac{1+\sqrt{5}}{2}$.

1.3 Show that $k \lg k = \Theta(n)$ implies $k = \Theta\left(\frac{n}{n \ln n}\right)$.

1.4 Are either 2^{n+1} or 2^{2n} big-O of 2^n?

1.5 For each pair of functions (A,B), indicate whether A is O, o, Ω, ω, or Θ of B. Assume $k \geq 1$, $\epsilon > 0$, $c > 1$ are constants.

	A	B	O	o	Ω	ω	Θ
a.	$\lg^k n$	n^ϵ	yes	yes			
b.	n^k	c^n	yes	yes			
c.	$\sqrt{n}$	$n^{\sin n}$					
d.	2^n	$2^{n/2}$			yes	yes	
e.	$n^{\lg c}$	$c^{\lg n}$	yes		yes		yes
f.	$\lg(n!)$	$\lg(n^n)$	yes		yes		yes

1.6 Order the following functions such that $f_1 = \Omega(f_2), f_2 = \Omega(f_3), \ldots, f_{29} = \Omega(f_{30})$, and partition them into equivalence classes such that each function is big-Θ of each other.

$$\begin{aligned}
2^{2^{n+1}} &= \Omega(2^{2^n}),\\
2^{2^n} &= \Omega((n+1)!),\\
(n+1)! &= \Omega(n!),\\
n! &= \Omega(e^n),\\
e^n &= \Omega(n \cdot 2^n),\\
n \cdot 2^n &= \Omega(2^n),\\
2^n &= \Omega(\left(\frac{3}{2}\right)^n),\\
\left(\frac{3}{2}\right)^n &= \Omega(n^{\lg\lg n}), n^{\lg\lg n} = \Omega((\lg n)^{\lg n}),\\
(\lg n)^{\lg n} &= \Omega((\lg n)!),\\
(\lg n)! &= \Omega(N^3),\\
N^3 &= \Omega(n^2),\\
n^2 &= \Omega(4^{\lg n}),\\
4^{\lg n} &= \Omega(\lg(n!)),\\
\lg(n!) &= \Omega(n \lg n),\\
n \lg n &= \Omega(2^{\lg n}),\\
2^{\lg n} &= \Omega(n),\\
n &= \Omega(\left(\sqrt{2}\right)^{\lg n}),\\
\left(\sqrt{2}\right)^{\lg n} &= \Omega(\sqrt{n}),\\
\sqrt{n} &= \Omega(2^{\sqrt{2\lg n}}),\\
2^{\sqrt{2\lg n}} &= \Omega(\lg^2 n),\\
\lg^2 n &= \Omega(\ln n),\\
\ln n &= \Omega(\sqrt{\lg n}),\\
\sqrt{\lg n} &= \Omega(\ln\ln n),\\
\ln\ln n &= \Omega(2^{\lg^* n}),\\
2^{\lg^* n} &= \Omega(\lg^* n),\\
\lg^* n &= \Omega(\lg * (\lg n)),\\
\lg * (\lg n) &= \Omega(\lg(\lg * n)),\\
\lg(\lg * n) &= \Omega(n^{\frac{1}{\lg n}}),\\
n^{\frac{1}{\lg n}} &
\end{aligned}$$