

# Assignment 2

## 1.1 Are either $\lceil \lg n \rceil!$ or $\lceil \lg \lg n \rceil!$ polynomially bounded?

Polynomially bounded means  $f_n = O(n^k)$  for some constant  $k$  (e.g., whether  $f_n \leq c \cdot n^k$  for constants  $c$  and  $k$  as  $n$  approaches  $\infty$ ). For the first function  $\lceil \lg n \rceil!$ , without loss of generality, assume  $n = 2^a$  (where  $a \in \mathbb{N}$ ).

$$\begin{aligned}\lceil \lg n \rceil! &\leq c \cdot n^k \\ \lg(2^a)! &\leq c \cdot (2^a)^k \\ a! &\leq c \cdot 2^{ak}\end{aligned}$$

The statement  $a! \leq c \cdot 2^{ak}$  is a contradiction, as the factorial function  $a!$  is not exponentially bounded. Therefore,  $\lceil \lg n \rceil!$  is not polynomially bounded (via proof by contradiction). For the second function  $\lceil \lg \lg n \rceil!$ , without loss of generality, assume  $n = 2^{2^a}$  (where  $a \in \mathbb{N}$ ).

$$\begin{aligned}\lceil \lg \lg n \rceil! &\leq c \cdot n^k \\ \lg \lg(2^{2^a})! &\leq c \cdot (2^{2^a})^k \\ a! &\leq c \cdot 2^{k \cdot 2^a} \\ 1 \cdot 2 \cdot 3 \cdots a &\leq c \cdot (2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^a \cdot k})\end{aligned}$$

The statement  $1 \cdot 2 \cdot 3 \cdots a \leq c \cdot (2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^a \cdot k})$  is obviously true. Therefore  $\lceil \lg \lg n \rceil!$  is polynomially bounded.

- 1.2 Use induction to prove  $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$ ; where  $F_i = F_{i-2} + F_{i-1}$ , and  $\phi$  is the golden ratio  $\frac{1+\sqrt{5}}{2}$ .
- 1.3 Show that  $k \lg k = \Theta(n)$  implies  $k = \Theta\left(\frac{n}{n \lg n}\right)$ .
- 1.4 Are either  $2^{n+1}$  or  $2^{2n}$  big- $O$  of  $2^n$ ?
- 1.5 For each pair of functions  $(A, B)$ , indicate whether  $A$  is  $O, o, \Omega, \omega$ , or  $\Theta$  of  $B$ . Assume  $k \geq 1$ ,  $\epsilon > 0$ ,  $c > 1$  are constants.

$A$	$B$	$O$	$o$	$\Omega$	$\omega$	$\Theta$
$\lg^k n$	$n^\epsilon$	yes	yes	yes	yes	yes
$n^k$	$c^n$	yes	yes	yes	yes	yes
$\sqrt{n}$	$n^{\sin n}$	yes	yes	yes	yes	yes
$2^n$	$2^{n/2}$	yes	yes	yes	yes	yes
$n^{\lg c}$	$c^{\lg n}$	yes	yes	yes	yes	yes
$\lg(n!)$	$\lg(n^n)$	yes	yes	yes	yes	yes
$A$	$B$	yes	yes	yes	yes	yes

- 1.6 Order the following functions such that  $f_1 = \Omega(f_2)$ ,  $f_2 = \Omega(f_3)$ , ...,  $f_{29} = \Omega(f_{30})$ , and partition them into equivalence classes such that each function is big- $\Theta$  of each other.