

Assignment 2

1.1 Are either $\lceil \lg n \rceil!$ or $\lceil \lg \lg n \rceil!$ polynomially bounded?

Polynomially bounded means $f_n = O(n^k)$ for some constant k (e.g., whether $f_n \leq c \cdot n^k$ for constants c and k as n approaches ∞). For the first function $\lceil \lg n \rceil!$, without loss of generality, assume $n = 2^a$ (where $a \in \mathbb{N}$).

$$\begin{aligned}\lceil \lg n \rceil! &\leq c \cdot n^k \\ \lg(2^a)! &\leq c \cdot (2^a)^k \\ a! &\leq c \cdot 2^{ak}\end{aligned}$$

The statement $a! \leq c \cdot 2^{ak}$ is a contradiction, as the factorial function $a!$ is not exponentially bounded. Therefore, $\lceil \lg n \rceil!$ is not polynomially bounded (via proof by contradiction). For the second function $\lceil \lg \lg n \rceil!$, without loss of generality, assume $n = 2^{2^a}$ (where $a \in \mathbb{N}$).

$$\begin{aligned}\lceil \lg \lg n \rceil! &\leq c \cdot n^k \\ \lg \lg(2^{2^a})! &\leq c \cdot (2^{2^a})^k \\ a! &\leq c \cdot 2^{k \cdot 2^a} \\ 1 \cdot 2 \cdot 3 \cdots a &\leq c \cdot (2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^a \cdot k})\end{aligned}$$

The statement $1 \cdot 2 \cdot 3 \cdots a \leq c \cdot (2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{2^a \cdot k})$ is obviously true. Therefore $\lceil \lg \lg n \rceil!$ is polynomially bounded (via direct proof).

- 1.2 Use induction to prove $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$; where $F_i = F_{i-2} + F_{i-1}$, and ϕ is the golden ratio $\frac{1+\sqrt{5}}{2}$.
- 1.3 Show that $k \lg k = \Theta(n)$ implies $k = \Theta\left(\frac{n}{n \lg n}\right)$.
- 1.4 Are either 2^{n+1} or 2^{2n} big- O of 2^n ?
- 1.5 For each pair of functions (A, B) , indicate whether A is O, o, Ω, ω , or Θ of B . Assume $k \geq 1$, $\epsilon > 0$, $c > 1$ are constants.

In terms of growth, $f_n = \Omega(g_n)$ means $f_n \geq c \cdot g_n$, $f_n = \omega(g_n)$ means $f_n > c \cdot g_n$, $f_n = O(g_n)$ means $f_n \leq c \cdot g_n$, $f_n = o(g_n)$ means $f_n < c \cdot g_n$, and $f_n = \Theta(g_n)$ means $f_n = c \cdot g_n$.

| | A | B | O | o | Ω | ω | Θ |
|----|-------------|--------------|-----|-----|----------|----------|----------|
| a. | $\lg^k n$ | n^ϵ | yes | yes | | | |
| b. | n^k | c^n | yes | yes | | | |
| c. | \sqrt{n} | $n^{\sin n}$ | | | | | |
| d. | 2^n | $2^{n/2}$ | | | yes | yes | |
| e. | $n^{\lg c}$ | $c^{\lg n}$ | yes | | yes | | yes |
| f. | $\lg(n!)$ | $\lg(n^n)$ | yes | | yes | | yes |

To demonstrate whether something is big-something of something, isolate the constant c in the equality and observe whether it holds. Also note that big- Ω precludes little- o and big- O precludes little- ω (e.g., if $f_n = O(g_n)$, then $f_n = \omega(g_n)$ is false, and vice-versa).

1.6 Order the following functions such that $f_1 = \Omega(f_2)$, $f_2 = \Omega(f_3)$, ..., $f_{29} = \Omega(f_{30})$, and partition them into equivalence classes such that each function is big- Θ of each other.

In terms of growth, $f_1 = \Omega(f_2)$ means $f_1 \leq f_2$. Therefore, the order of functions $f_1 = \Omega(f_2)$, $f_2 = \Omega(f_3)$, ..., $f_{29} = \Omega(f_{30})$ is as follows: $2^{2^n} = \Omega((n+1)!)$, $(n+1)! = \Omega(n!)$, $n! = \Omega(e^n)$, $e^n = \Omega(n \cdot 2^n)$, $n \cdot 2^n = \Omega(2^n)$, $2^n = \Omega\left(\left(\frac{3}{2}\right)^n\right)$, $\left(\frac{3}{2}\right)^n = \Omega(n^{\lg \lg n})$, $n^{\lg \lg n} = \Omega((\lg n)^{\lg n})$, $(\lg n)^{\lg n} = \Omega((\lg n)!)$, $(\lg n)! = \Omega(N^3)$, $N^3 = \Omega(n^2)$, $n^2 = \Omega(4^{\lg n})$, $4^{\lg n} = \Omega(\lg(n!))$, $\lg(n!) = \Omega(n \lg n)$, $n \lg n = \Omega(2^{\lg n})$, $2^{\lg n} = \Omega(n)$, $n = \Omega\left((\sqrt{2})^{\lg n}\right)$, $(\sqrt{2})^{\lg n} = \Omega(\sqrt{n})$, $\sqrt{n} = \Omega\left(2^{\sqrt{2} \lg n}\right)$, $2^{\sqrt{2} \lg n} = \Omega(\lg^2 n)$, $\lg^2 n = \Omega(\ln n)$, $\ln n = \Omega\left(\sqrt{\lg n}\right)$, $\sqrt{\lg n} = \Omega(\ln \ln n)$, $\ln \ln n = \Omega\left(2^{\lg^* n}\right)$, $2^{\lg^* n} = \Omega(\lg^* n)$, $\lg^* n = \Omega(\lg^*(\lg n))$, $\lg^*(\lg n) = \Omega(\lg(\lg^* n))$, $\lg(\lg^* n) = \Omega\left(n^{\frac{1}{\lg n}}\right)$, $n^{\frac{1}{\lg n}} = \Omega(1)$.

An equivalence class is a set containing elements that all adhere to some property. In this case, the elements are functions f , and the property is that each function is big- Θ of every other function in the set. The functions above can be partitioned into the following equivalence classes: $\{n^{\lg \lg n}, (\lg n)^{\lg n}\}$, $\{n^2, 4^{\lg n}\}$, $\{\lg(n!), n \lg n\}$, $\{2^{\lg n}, n\}$, $\{(\sqrt{2})^{\lg n}, \sqrt{n}\}$, $\{\lg^* n, \lg^*(\lg n)\}$, $\{n^{\frac{1}{\lg n}}, 1\}$.