Assignment 2

1.1 Are either $\lceil \lg n \rceil !$ or $\lceil \lg \lg n \rceil !$ polynomially bounded?

Polynomially bounded means $f_n = O(n^k)$ for some constant k (e.g., whether $f_n \leq c \cdot n^k$ for constants c and k as n approaches ∞). For the first function $\lceil \lg n \rceil !$, without loss of generality, assume $n = 2^a$ (where $a \in \mathbb{N}$).

$$\begin{aligned}
\lceil \lg n \rceil ! &\leq c \cdot n^k \\
\lg(2^a)! &\leq c \cdot (2^a)^k \\
a! &\leq c \cdot 2^{ak}
\end{aligned}$$

The statement $a! \leq c \cdot 2^{ak}$ is a contradiction, as the factorial function $a!$ is not exponentially bounded. Therefore, $\lceil \lg n \rceil !$ is not polynomially bounded (via proof by contradiction). For the second function $\lceil \lg \lg n \rceil !$, without loss of generality, assume $n = 2^{2^a}$ (where $a \in \mathbb{N}$).

$$\begin{aligned}
\lceil \lg \lg n \rceil ! &\leq c \cdot n^k \\
\lg \lg \left(2^{2^a}\right)! &\leq c \cdot \left(2^{2^a}\right)^k \\
a! &\leq c \cdot 2^{k \cdot 2^a} \\
1 \cdot 2 \cdots (a-1) \cdot a &\leq c \cdot \left(2^{2k} \cdot 2^{4k} \cdot 2^{8k} \cdots 2^{k \cdot 2^{(a-1)}}) \cdot 2^{k \cdot 2^a}\right)
\end{aligned}$$

1.2 Use induction to prove $F_i = \frac{\phi^i - \hat{\phi}^i}{\sqrt{5}}$; where $F_i = F_{i-2} + F_{i-1}$, and ϕ is the golden ratio $\frac{1+\sqrt{5}}{2}$.

1.3 Show that $k \lg k = \Theta(n)$ implies $k = \Theta\left(\frac{n}{n \ln n}\right)$.

1.4 Are either 2^{n+1} or 2^{2n} big-O of 2^n?

1.5 For each pair of functions (A, B), indicate whether A is O, o, Ω, ω, or Θ of B. Assume $k \geq 1$, $\epsilon > 0$, $c > 1$ are constants.

A	B	O	o	Ω	ω	Θ
$\lg^k n$	n^ϵ	yes	yes	yes	yes	yes
n^k	c^n	yes	yes	yes	yes	yes
$\sqrt{n}$	$n^{\sin n}$	yes	yes	yes	yes	yes
2^n	$2^{n/2}$	yes	yes	yes	yes	yes
$n^{\lg c}$	$c^{\lg n}$	yes	yes	yes	yes	yes
$\lg(n!)$	$\lg(n^n)$	yes	yes	yes	yes	yes
A	B	yes	yes	yes	yes	yes

1.6 Order the following functions such that $f_1 = \Omega(f_2), f_2 = \Omega(f_3), \ldots, f_{29} = \Omega(f_{30})$, and partition them into equivalence classes such that each function is big-Θ of each other.