32 A is singular when its third column $\boldsymbol{w}$ is a combination $c\boldsymbol{u} + d\boldsymbol{v}$ of the first columns. A typical column picture has $\boldsymbol{b}$ outside the plane of $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$. A typical row picture has the intersection line of two planes parallel to the third plane. *Then no solution.*

33 $\boldsymbol{w} = (5, 7)$ is $5\boldsymbol{u} + 7\boldsymbol{v}$. Then $A\boldsymbol{w}$ equals 5 times $A\boldsymbol{u}$ plus 7 times $A\boldsymbol{v}$. **Linearity** means : When w is a combination of $\boldsymbol{u}$ and $\boldsymbol{v}$, then $A\boldsymbol{w}$ is the same combination of $A\boldsymbol{u}$ and $A\boldsymbol{v}$.

34 $\begin{bmatrix} 2 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ has the solution $\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 4 \\ 7 \\ 8 \\ 6 \end{bmatrix}$.

35 $\boldsymbol{x} = (1, \ldots, 1)$ gives $S\boldsymbol{x} =$ sum of each row $= 1 + \cdots + 9 = 45$ for Sudoku matrices. 6 row orders $(1, 2, 3)$, $(1, 3, 2)$, $(2, 1, 3)$, $(2, 3, 1)$, $(3, 1, 2)$, $(3, 2, 1)$ are in Section 2.7. The same 6 permutations of ***blocks*** of rows produce Sudoku matrices, so $6^4 = 1296$ orders of the 9 rows all stay Sudoku. (And also 1296 permutations of the 9 columns.)

Problem Set 2.2, page 53

1 Multiply equation 1 by $\ell_{21} = \frac{10}{2} = \mathbf{5}$ and subtract from equation 2 to find $2x + 3y = 1$ (unchanged) and $-6y = 6$. The pivots to circle are 2 and -6.

2 $-6y = 6$ gives $y = -1$. Then $2x + 3y = 1$ gives $x = 2$. Multiplying the right side $(1, 11)$ by 4 will multiply the solution by 4 to give the new solution $(x, y) = (8, -4)$.

3 Subtract $-\frac{1}{2}$ (or add $\frac{1}{2}$) times equation 1. The new second equation is $\boldsymbol{3y = 3}$. Then $y = 1$ and $x = 5$. If the right side changes sign, so does the solution: $(x, y) = (-5, -1)$.

4 Subtract $\ell = \frac{c}{a}$ times equation 1 from equation 2. The new second pivot multiplying y is $d - (cb/a)$ or $(ad - bc)/a$. Then $y = (ag - cf)/(ad - bc)$. Notice the "determinant of A" $= ad - bc$. It must be nonzero for this division.

5 $6x + 4y$ is 2 times $3x + 2y$. There is no solution unless the right side is $2 \cdot 10 = 20$. Then all the points on the line $3x+2y = 10$ are solutions, including $(0, 5)$ and $(4, -1)$. The two lines in the row picture are the same line, containing all solutions.

6 Singular system if $b = 4$, because $4x + 8y$ is 2 times $2x + 4y$. Then $g = 32$ makes the lines $2x + 4y = 16$ and $4x + 8y = 32$ become the *same*: infinitely many solutions like $(8, 0)$ and $(0, 4)$.

7 If $a = 2$ elimination must fail (two parallel lines in the row picture). The equations have no solution. With $a = 0$, elimination will stop for a row exchange. Then $3y = -3$ gives $y = -1$ and $4x + 6y = 6$ gives $x = 3$.

8 If $k = 3$ elimination must fail: no solution. If $k = -3$, elimination gives $0 = 0$ in equation 2: infinitely many solutions. If $k = 0$ a row exchange is needed: one solution.

9 On the left side, $6x - 4y$ is 2 times $(3x - 2y)$. Therefore we need $b_2 = 2b_1$ on the right side. Then there will be infinitely many solutions (two parallel lines become one single line in the row picture). The column picture has both columns along the same line.

10 The equation $y = 1$ comes from elimination (subtract $x + y = 5$ from $x + 2y = 6$). Then $x = 4$ and $5x - 4y = 20 - 4 = c = 16$.

11 (a) Another solution is $\frac{1}{2}(x+X, y+Y, z+Z)$. (b) If 25 planes meet at two points, they meet along the whole line through those two points.

12 Elimination leads to this upper triangular system; then comes back substitution.

$$\begin{aligned} 2x + 3y + z &= 8 \\ y + 3z &= 4 \\ 8z &= 8 \end{aligned} \quad \text{gives} \quad \begin{aligned} x &= 2 \\ y &= 1 \\ z &= 1 \end{aligned}$$

If a zero is at the start of row 2 or row 3, that avoids a row operation.

13
$$\begin{aligned} 2x - 3y \phantom{{}+z} &= 3 \\ 4x - 5y + z &= 7 \\ 2x - y - 3z &= 5 \end{aligned} \quad \text{gives} \quad \begin{aligned} 2x - 3y &= 3 \\ y + z &= 1 \\ 2y + 3z &= 2 \end{aligned} \quad \text{and} \quad \begin{aligned} 2x - 3y &= 3 \\ y + z &= 1 \\ -5z &= 0 \end{aligned} \quad \text{and} \quad \begin{aligned} x &= 3 \\ y &= 1 \\ z &= 0 \end{aligned}$$

Here are steps $1, 2, 3$: Subtract $2 \times$ row 1 from row 2, subtract $1 \times$ row 1 from row 3, subtract $2 \times$ row 2 from row 3

14 Subtract 2 times row 1 from row 2 to reach $(d-10)y-z=2$. Equation (3) is $y-z=3$. If $d=10$ exchange rows 2 and 3. If $d=11$ the system becomes singular.

15 The second pivot position will contain $-2-b$. If $b=-2$ we exchange with row 3. If $b=-1$ (singular case) the second equation is $-y-z=0$. But equation (3) is the same so there is a *line of solutions* $(x,y,z)=(1,1,-1)$.

16 (a) **Example of 2 exchanges**
$$\begin{aligned} 0x+0y+2z&=4\\ x+2y+2z&=5\\ 0x+3y+4z&=6 \end{aligned}$$
(exchange 1 and 2, then 2 and 3)

(b) **Exchange but then breakdown**
$$\begin{aligned} 0x+3y+4z&=4\\ x+2y+2z&=5\\ 0x+3y+4z&=6 \end{aligned}$$
(rows 1 and 3 are not consistent)

17 If row $1=$ row 2, then row 2 is zero after the first step; exchange the zero row with row 3 and row 3 has no pivot. If column $2=$ column 1, then column 2 has no pivot.

18 *Example* $x+2y+3z=0$, $4x+8y+12z=0$, $5x+10y+15z=0$ has 9 different coefficients but rows 2 and 3 become $0=0$: infinitely many solutions to $A\boldsymbol{x}=\mathbf{0}$ but almost surely no solution to $A\boldsymbol{x}=\boldsymbol{b}$ for a random $\boldsymbol{b}$.

19 Row 2 becomes $3y-4z=5$, then row 3 becomes $(q+4)z=t-5$. If $q=-4$ the system is singular—no third pivot. Then if $t=5$ the third equation is $0=0$ which allows infinitely many solutions. Choosing $z=1$ the equation $3y-4z=5$ gives $y=3$ and equation 1 gives $x=-9$.

20 Singular if row 3 is a combination of rows 1 and 2. From the end view, the three planes form a triangle. This happens if rows 1+2=row 3 on the left side but not the right side: $x+y+z=0$, $x-2y-z=1$, $2x-y=4$. No parallel planes but still no solution. The three planes in the row picture form a triangular tunnel.

21 (a) Pivots $2, \frac{3}{2}, \frac{4}{3}, \frac{5}{4}$ in the equations $2x+y=0, \frac{3}{2}y+z=0, \frac{4}{3}z+t=0, \frac{5}{4}t=5$ after elimination. Back substitution gives $t=4, z=-3, y=2, x=-1$. (b) If the off-diagonal entries change from $+1$ to -1, the pivots are the same. The solution is $(1,2,3,4)$ instead of $(-1,2,-3,4)$.

22 The fifth pivot is $\frac{6}{5}$ for both matrices (1's or -1's off the diagonal). The nth pivot is $\frac{n+1}{n}$.

23 If ordinary elimination leads to $x + y = 1$ and $2y = 3$, the original second equation could be $2y + \ell(x+y) = 3 + \ell$ for any ℓ. Then ℓ will be the multiplier to reach $2y = 3$, by subtracting ℓ times equation 1 from equation 2.

24 Elimination fails on $\begin{bmatrix} a & 2 \\ a & a \end{bmatrix}$ if $a = 2$ or $a = 0$. (You could notice that the determinant $a^2 - 2a$ is zero for $a = 2$ and $a = 0$.)

25 $a = 2$ (equal columns), $a = 4$ (equal rows), $a = 0$ (zero column).

26 Solvable for $s = 10$ (add the two pairs of equations to get $a+b+c+d$ on the left sides, 12 and $2 + s$ on the right sides). So 12 must agree with $2 + s$, which makes $s = 10$. The four equations for a, b, c, d are **singular**! Two solutions are $\begin{bmatrix} 1 & 3 \\ 1 & 7 \end{bmatrix}$ and $\begin{bmatrix} 0 & 4 \\ 2 & 6 \end{bmatrix}$,

$$A = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 \end{bmatrix} \text{ and } U = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}.$$

27 Elimination leaves the diagonal matrix $\text{diag}(3, 2, 1)$ in $3x = 3, 2y = 2, z = 2$. Then $x = 1, y = 1, z = 2$.

28 $A(2, :) = A(2, :) - 3 * A(1, :)$ subtracts 3 times row 1 from row 2.

29 The average pivots for rand(3) *without* row exchanges were $\frac{1}{2}, 5, 10$ in one experiment—but pivots 2 and 3 can be arbitrarily large. Their averages are actually infinite ! ***With row exchanges*** in MATLAB's **lu** code, the averages $.75$ and $.50$ and $.365$ are much more stable (and should be predictable, also for **randn** with normal instead of uniform probability distribution for the numbers in A).

30 If $A(5, 5)$ is 7 not 11, then the last pivot will be 0 not 4.

31 Row j of U is a combination of rows $1, \ldots, j$ of A (when there are no row exchanges). If $A\boldsymbol{x} = \mathbf{0}$ then $U\boldsymbol{x} = \mathbf{0}$ (not true if $\boldsymbol{b}$ replaces $\mathbf{0}$). U just keeps the diagonal of A when A is *lower triangular*.

32 The question deals with 100 equations $A\boldsymbol{x} = \mathbf{0}$ when A is singular.

(a) Some linear combination of the 100 rows is **the row of 100 zeros**.

(b) Some linear combination of the 100 **columns** is **the column of zeros**.

(c) A very singular matrix has all ones: $A = \textbf{ones}(100)$. A better example has 99 random rows (or the numbers $1^i, \ldots, 100^i$ in those rows). The 100th row could be the sum of the first 99 rows (or any other combination of those rows with no zeros).

(d) The row picture has 100 planes **meeting along a common line through 0**. The column picture has 100 vectors all in the same 99-dimensional hyperplane.

Problem Set 2.3, page 66

1 $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

2 $E_{32}E_{21}\boldsymbol{b} = (1, -5, -35)$ but $E_{21}E_{32}\boldsymbol{b} = (1, -5, 0)$. When E_{32} comes first, row 3 feels no effect from row 1.

3 $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$ $M = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$.

Those E's are in the right order to give $MA = U$.

4 Elimination on column 4: $\boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{E_{21}} \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} \xrightarrow{E_{31}} \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \xrightarrow{E_{32}} \begin{bmatrix} 1 \\ -4 \\ 10 \end{bmatrix}$. The original $A\boldsymbol{x} = \boldsymbol{b}$ has become $U\boldsymbol{x} = \boldsymbol{c} = (1, -4, 10)$. Then back substitution gives $\boldsymbol{z} = -5, y = \frac{1}{2}, x = \frac{1}{2}$. This solves $A\boldsymbol{x} = (1, 0, 0)$.

5 Changing a_{33} from 7 to 11 will change the third pivot from 5 to 9. Changing a_{33} from 7 to 2 will change the pivot from 5 to *no pivot*.