Problem Set 1.1, page 8

1 The combinations give (a) a line in $\mathbf{R}^3$ (b) a plane in $\mathbf{R}^3$ (c) all of $\mathbf{R}^3$.

2 $\boldsymbol{v} + \boldsymbol{w} = (2, 3)$ and $\boldsymbol{v} - \boldsymbol{w} = (6, -1)$ will be the diagonals of the parallelogram with $\boldsymbol{v}$ and $\boldsymbol{w}$ as two sides going out from $(0, 0)$.

3 This problem gives the diagonals $\boldsymbol{v} + \boldsymbol{w}$ and $\boldsymbol{v} - \boldsymbol{w}$ of the parallelogram and asks for the sides: The opposite of Problem 2. In this example $\boldsymbol{v} = (3, 3)$ and $\boldsymbol{w} = (2, -2)$.

4 $3\boldsymbol{v} + \boldsymbol{w} = (7, 5)$ and $c\boldsymbol{v} + d\boldsymbol{w} = (2c + d, c + 2d)$.

5 $\boldsymbol{u}+\boldsymbol{v} = (-2, 3, 1)$ and $\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w} = (0, 0, 0)$ and $2\boldsymbol{u}+2\boldsymbol{v}+\boldsymbol{w} =$ (add first answers) $=$ $(-2, 3, 1)$. The vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are in the same plane because a combination gives $(0, 0, 0)$. Stated another way: $\boldsymbol{u} = -\boldsymbol{v} - \boldsymbol{w}$ is in the plane of $\boldsymbol{v}$ and $\boldsymbol{w}$.

6 The components of every $c\boldsymbol{v} + d\boldsymbol{w}$ add to zero because the components of $\boldsymbol{v}$ and of $\boldsymbol{w}$ add to zero. $c = 3$ and $d = 9$ give $(3, 3, -6)$. There is no solution to $c\boldsymbol{v}+d\boldsymbol{w} = (3, 3, 6)$ because $3 + 3 + 6$ is not zero.

7 The nine combinations $c(2, 1) + d(0, 1)$ with $c = 0, 1, 2$ and $d = (0, 1, 2)$ will lie on a lattice. If we took all whole numbers c and d, the lattice would lie over the whole plane.

8 The other diagonal is $\boldsymbol{v} - \boldsymbol{w}$ (or else $\boldsymbol{w} - \boldsymbol{v}$). Adding diagonals gives $2\boldsymbol{v}$ (or $2\boldsymbol{w}$).

9 The fourth corner can be $(4, 4)$ or $(4, 0)$ or $(-2, 2)$. Three possible parallelograms!

10 $\boldsymbol{i} - \boldsymbol{j} = (1, 1, 0)$ is in the base (x-y plane). $\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k} = (1, 1, 1)$ is the opposite corner from $(0, 0, 0)$. Points in the cube have $0 \le x \le 1$, $0 \le y \le 1$, $0 \le z \le 1$.

11 Four more corners $(1, 1, 0), (1, 0, 1), (0, 1, 1), (1, 1, 1)$. The center point is $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2})$. Centers of faces are $(\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, \frac{1}{2}, 1)$ and $(0, \frac{1}{2}, \frac{1}{2}), (1, \frac{1}{2}, \frac{1}{2})$ and $(\frac{1}{2}, 0, \frac{1}{2}), (\frac{1}{2}, 1, \frac{1}{2})$.

12 The combinations of $\boldsymbol{i} = (1, 0, 0)$ and $\boldsymbol{i} + \boldsymbol{j} = (1, 1, 0)$ fill the xy plane in xyz space.

13 Sum $=$ zero vector. Sum $= -2{:}00$ vector $= 8{:}00$ vector. 2:00 is $30°$ from horizontal $= (\cos\frac{\pi}{6}, \sin\frac{\pi}{6}) = (\sqrt{3}/2, 1/2)$.

14 Moving the origin to 6:00 adds $\boldsymbol{j} = (0, 1)$ to every vector. So the sum of twelve vectors changes from $\mathbf{0}$ to $12\boldsymbol{j} = (0, 12)$.

15 The point $\frac{3}{4}\boldsymbol{v}+\frac{1}{4}\boldsymbol{w}$ is three-fourths of the way to $\boldsymbol{v}$ starting from $\boldsymbol{w}$. The vector $\frac{1}{4}\boldsymbol{v}+\frac{1}{4}\boldsymbol{w}$ is halfway to $\boldsymbol{u}=\frac{1}{2}\boldsymbol{v}+\frac{1}{2}\boldsymbol{w}$. The vector $\boldsymbol{v}+\boldsymbol{w}$ is $2\boldsymbol{u}$ (the far corner of the parallelogram).

16 All combinations with $c+d=1$ are on the line that passes through $\boldsymbol{v}$ and $\boldsymbol{w}$. The point $\boldsymbol{V}=-\boldsymbol{v}+2\boldsymbol{w}$ is on that line but it is beyond $\boldsymbol{w}$.

17 All vectors $c\boldsymbol{v}+c\boldsymbol{w}$ are on the line passing through $(0,0)$ and $\boldsymbol{u}=\frac{1}{2}\boldsymbol{v}+\frac{1}{2}\boldsymbol{w}$. That line continues out beyond $\boldsymbol{v}+\boldsymbol{w}$ and back beyond $(0,0)$. With $c\geq 0$, half of this line is removed, leaving a *ray* that starts at $(0,0)$.

18 The combinations $c\boldsymbol{v}+d\boldsymbol{w}$ with $0\leq c\leq 1$ and $0\leq d\leq 1$ ***fill the parallelogram*** with sides $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v}=(1,0)$ and $\boldsymbol{w}=(0,1)$ then $c\boldsymbol{v}+d\boldsymbol{w}$ fills the unit square. But when $\boldsymbol{v}=(a,0)$ and $\boldsymbol{w}=(b,0)$ these combinations only fill a segment of a line.

19 With $c\geq 0$ and $d\geq 0$ we get the infinite "cone" or "wedge" between $\boldsymbol{v}$ and $\boldsymbol{w}$. For example, if $\boldsymbol{v}=(1,0)$ and $\boldsymbol{w}=(0,1)$, then the cone is the whole quadrant $x\geq 0$, $y\geq 0$. ***Question***: What if $\boldsymbol{w}=-\boldsymbol{v}$? The cone opens to a half-space. But the combinations of $\boldsymbol{v}=(1,0)$ and $\boldsymbol{w}=(-1,0)$ only fill a line.

20 (a) $\frac{1}{3}\boldsymbol{u}+\frac{1}{3}\boldsymbol{v}+\frac{1}{3}\boldsymbol{w}$ is the center of the triangle between $\boldsymbol{u},\boldsymbol{v}$ and $\boldsymbol{w}$; $\frac{1}{2}\boldsymbol{u}+\frac{1}{2}\boldsymbol{w}$ lies between $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) To fill the triangle keep $c\geq 0$, $d\geq 0$, $e\geq 0$, and $c+d+e=\mathbf{1}$.

21 The sum is $(\boldsymbol{v}-\boldsymbol{u})+(\boldsymbol{w}-\boldsymbol{v})+(\boldsymbol{u}-\boldsymbol{w})=$ **zero vector**. Those three sides of a triangle are in the same plane!

22 The vector $\frac{1}{2}(\boldsymbol{u}+\boldsymbol{v}+\boldsymbol{w})$ is *outside* the pyramid because $c+d+e=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}>1$.

23 *All* vectors are combinations of $\boldsymbol{u},\boldsymbol{v},\boldsymbol{w}$ as drawn (not in the same plane). Start by seeing that $c\boldsymbol{u}+d\boldsymbol{v}$ fills a plane, then adding $e\boldsymbol{w}$ fills all of $\mathbf{R}^3$.

24 The combinations of $\boldsymbol{u}$ and $\boldsymbol{v}$ fill one plane. The combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill another plane. Those planes meet in a *line*: *only the vectors* $c\boldsymbol{v}$ are in both planes.

25 (a) For a line, choose $\boldsymbol{u}=\boldsymbol{v}=\boldsymbol{w}=$ any nonzero vector (b) For a plane, choose $\boldsymbol{u}$ and $\boldsymbol{v}$ in different directions. A combination like $\boldsymbol{w}=\boldsymbol{u}+\boldsymbol{v}$ is in the same plane.

26 Two equations come from the two components: $c + 3d = 14$ and $2c + d = 8$. ***The solution is*** $c = 2$ ***and*** $d = 4$. Then $2(1, 2) + 4(3, 1) = (14, 8)$.

27 A four-dimensional cube has $2^4 = 16$ corners and $2 \cdot 4 = 8$ three-dimensional faces and 24 two-dimensional faces and 32 edges in Worked Example **2.4 A**.

28 There are **6** unknown numbers $v_1, v_2, v_3, w_1, w_2, w_3$. The six equations come from the components of $\boldsymbol{v} + \boldsymbol{w} = (4, 5, 6)$ and $\boldsymbol{v} - \boldsymbol{w} = (2, 5, 8)$. Add to find $2\boldsymbol{v} = (6, 10, 14)$ so $\boldsymbol{v} = (3, 5, 7)$ and $\boldsymbol{w} = (1, 0, -1)$.

29 Fact: For any three vectors $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in the plane, some combination $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$ is the zero vector (beyond the obvious $c = d = e = 0$). So if there is one combination $C\boldsymbol{u} + D\boldsymbol{v} + E\boldsymbol{w}$ that produces $\boldsymbol{b}$, there will be many more—just add c, d, e or $2c, 2d, 2e$ to the particular solution C, D, E.

The example has $3\boldsymbol{u} - 2\boldsymbol{v} + \boldsymbol{w} = 3(1, 3) - 2(2, 7) + 1(1, 5) = (0, 0)$. It also has $-2\boldsymbol{u} + 1\boldsymbol{v} + 0\boldsymbol{w} = \boldsymbol{b} = (0, 1)$. Adding gives $\boldsymbol{u} - \boldsymbol{v} + \boldsymbol{w} = (0, 1)$. In this case c, d, e equal $3, -2, 1$ and $C, D, E = -2, 1, 0$.

Could another example have $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ that could NOT combine to produce $\boldsymbol{b}$? Yes. The vectors $(1, 1), (2, 2), (3, 3)$ are on a line and no combination produces $\boldsymbol{b}$. We can easily solve $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = 0$ but not $C\boldsymbol{u} + D\boldsymbol{v} + E\boldsymbol{w} = \boldsymbol{b}$.

30 The combinations of $\boldsymbol{v}$ and $\boldsymbol{w}$ fill the plane *unless* $\boldsymbol{v}$ *and* $\boldsymbol{w}$ *lie on the same line through* $(0, 0)$. Four vectors whose combinations fill 4-dimensional space: one example is the "standard basis" $(1, 0, 0, 0), (0, 1, 0, 0), (0, 0, 1, 0)$, and $(0, 0, 0, 1)$.

31 The equations $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w} = \boldsymbol{b}$ are

$$\begin{array}{rrrl}
2c & -d & & = 1 \\
-c & +2d & -e & = 0 \\
 & -d & +2e & = 0
\end{array}
\qquad
\begin{array}{l}
\text{So } d = 2e \\
\text{then } c = 3e \\
\text{then } 4e = 1
\end{array}
\qquad
\begin{array}{l}
c = 3/4 \\
d = 2/4 \\
e = 1/4
\end{array}$$