Combinatorics
Chapter 2

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Wednesday, September 6, 2022

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Introduction

Definition: Combinatorics

The area of mathematics concerned with *counting* in a finite space is called combinatorics.

- Goal: Determine the total number of ways that objects can be arranged/selected or the number of ways events can occur.
- Two particular types of arrangements are:
  1. Permutation: when order matters
     - Ex: Passwords
  2. Combination: when order does not matter
     - Ex: Choosing some people at random from an entire group.

Examples

Permutation (order) or Combination (no order)?

Example	Perm. (P) or Comb. (C)?
1. The number of ways to finish a race in 1st, 2nd, or 3rd place.	P
2. The number of ways to get exactly 3 heads when I flip a coin ten times.	C
3. The winning numbers to a 3-digit lottery.	C
4. Selecting 5 people for a group where everyone has the same role.	C
5. Selecting 5 people for a group where the first person selected is president, then vice president, etc.	P

Definition: The Fundamental Counting Principle (FCP)

A method to determine the number of ways multiple independent events can occur.

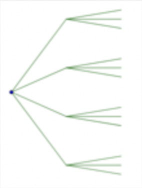
Event 1: a_1 ways
Event 2: a_2 ways
.
.
Event n: a_n ways

Total number of ways is $a_1 \bullet a_2 \ldots \bullet a_n$

Example:

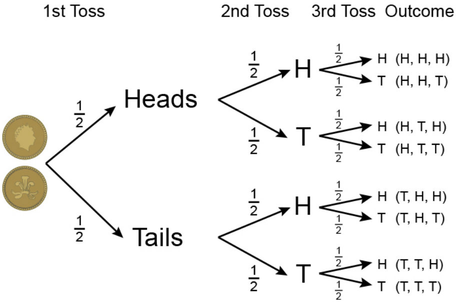
If event 1 can occur in 4 ways and event 2 can occur in 3 ways then the total number of ways that events 1 and 2 can occur is $3 \times 4 = 12$.

Tree Diagram



Example: FCP

The number of outcomes possible after flipping a coin 3 times.



Toss	Outcomes Possible
1st	2
2nd	2
3rd	2

The total number of ways that the three tosses can occur is $2 \times 2 \times 2 = 8$ ways.

Factorial Numbers

Definition: $n!$

n factorial (denoted $n!$) is defined as

$$n! = n(n-1)(n-2)\dots(3)(2)(1)$$

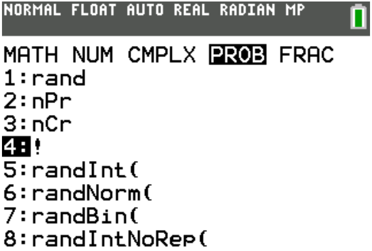
and computes the number of ordered arrangements of n objects (all of them!) without replacement.

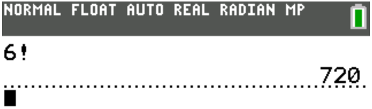
Examples:

1. $5! = 5 \times 4 \times 3 \times 2 \times 1 = 120$
   There are 120 ordered arrangements 5 objects without replacement.
2. One special number: $0! = 1$
   - There is one way to arrange no objects: do nothing! (It's a choice.)

Factorial Numbers - on the TI84

- Example: Calculate 6!
- Steps: Type in 6 $\rightarrow$ $\boxed{MATH}$ $\rightarrow$ $\boxed{PROB}$ $\rightarrow$ $\boxed{4:!}$





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Permutation

Definition/Formula: Permutation

A permutation is the number of ways to select r objects without replacement from n total objects in which order matters. The number of permutations of r objects selected from n objects is given by

$$\boxed{{}_nP_r = \frac{n!}{(n-r)!}}$$

- Ex: ${}_8P_4 = \frac{8!}{(8-4)!} = \frac{8!}{4!} = \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{4\cdot3\cdot2\cdot1} = 8 \cdot 7 \cdot 6 \cdot 5 = 1680$

  There are 1680 possible ordered arrangements of 4 objects selected from 8.

Permutation Examples

- Suppose 20 people are in a race. In how many ways can they finish in first, second, and third place?

  Solution: ${}_{20}P_3 = \frac{20!}{17!} = 20 \cdot 19 \cdot 18 = 6840$

- What permutation is represented by the product $100 \cdot 99 \cdot 98 \cdot 97 \cdot 96$?

  Solution: ${}_{100}P_5$

Permutation Examples

- If there are n objects, how many ways can I order all of them?
  Solution: Let $r = n$:

$$
\begin{aligned}
{}_nP_n &= \frac{n!}{(n-n)!} \\
&= \frac{n!}{0!} \\
&= n!
\end{aligned}
$$

  Answer: $n!$

Factorial numbers are permutations.

Birthday Example

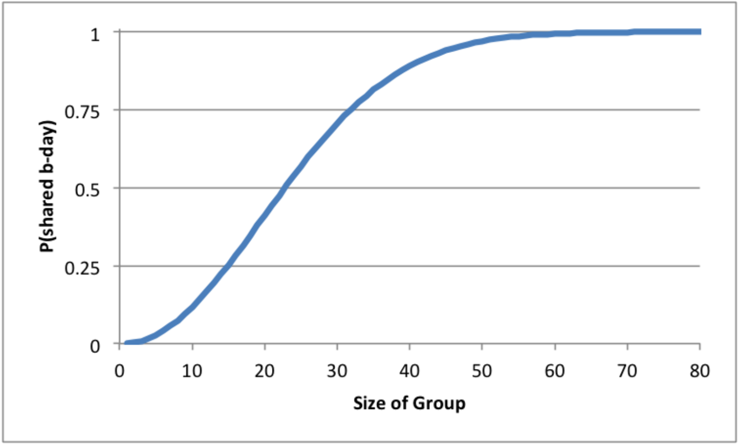
Suppose there are 25 people in a room. What is the probability that two or more people share the same birthday?

Solution:

$$\begin{aligned} P(\text{at least one birthday}) &= 1 - P(\text{no shared birthdays}) \\ &= 1 - \frac{365}{365} \cdot \frac{364}{365} \cdot \frac{363}{365} \cdots \frac{341}{365} \\ &= 1 - \frac{{}_{365}P_{25}}{365^{25}} \\ &= \boxed{0.569} \end{aligned}$$

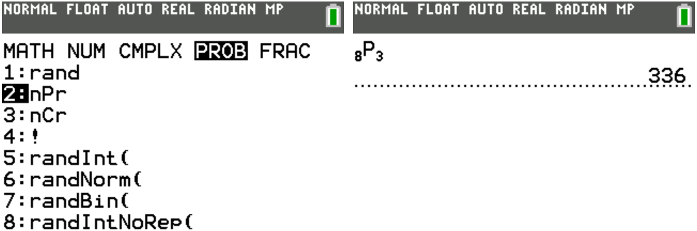
Birthday Example (Continued)

The plot shows the probability of at least one shared birthday versus group size, n, where $P(\text{shared b-day}) = 1 - \frac{{}_{365}P_n}{365^n}$:



Permutations - on the TI84

- Example: Calculate ${}_8P_3$:
- Steps: $\boxed{MATH} \rightarrow \boxed{PROB} \rightarrow \boxed{2: {}_nP_r}$



- Older Calculators: $8\ {}_nP_r\ 3$

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Combinations

Definition/Formula: Combination

A combination is the number of ways to choose r objects from n total objects without regard to order. The number of combinations of r objects selected from n objects is given by

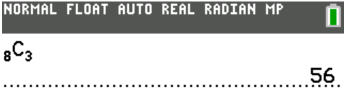
$$\boxed{{}_nC_r = \frac{n!}{(n-r)!r!}}$$

- Ex: ${}_8C_3 = \frac{8!}{(8-3)!3!} = \frac{8!}{5!3!} = \frac{8\cdot7\cdot6\cdot5\cdot4\cdot3\cdot2\cdot1}{(5\cdot4\cdot3\cdot2\cdot1)(3\cdot2\cdot1)} = \frac{8\cdot7\cdot6}{3\cdot2\cdot1} = 56$

  There are 56 ways to choose 3 objects selected from 8 without regard to order.

Combinations - on the TI84

- Example: Calculate ${}_8C_3$:
- Steps: $\boxed{MATH} \rightarrow \boxed{PROB} \rightarrow \boxed{3 : {}_nC_r}$



- Older Calculators: $8 \; {}_nC_r \; 3$