MATH 2418 Linear Algebra. Week 8

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Summary of this Week's Goals

This week we will review Sections 3.1 through 3.2 and cover Section 3.4 (Independence, Basis and Dimension). By the end of the week, you should have a clear understanding of the following topics: vector space; closure with resepct to vector addition and scalar multiplication; the column space, nullspace and row space of a matrix; reduced row echelon form; pivot columns versus free columns; the rank of a matrix; how to solve $A\mathbf{x} = \mathbf{0}$ for vectors in the nullspace of A; how to describe the complete solution to $A\mathbf{x} = \mathbf{b}$ as the sum of a paricular solution and nullspace solution; the definitions of independent vectors, basis and dimension. You should be able to find bases for the columns space and row space of a matrix and identify the dimension of those spaces.

Announcements

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3.1 Spaces of Vectors

Vector Spaces and Subspaces

- A vector space is a closed set of mathematical objects for which vector addition and scalar multiplication are defined. Here are the keys parts of a vector space:
  - Vectors.

    These are the mathematical objects that make up the set to which the vector space refers. They may be vectors like the ones we have discussed already (n-tuples of real numbers interpreted as arrows with magnitude and direction), or they may be more other mathematical objects like matrices and functions.
  - Vector Addition. A vector space requires that addition be defined so that any vector in the vector space may be added to any other vector in the vector space.
  - Scalar Multiplication. A vector space requires that scalar multiplication be defined so that any vector in the vector space may be multiplied by a scalar. For our purposes, scalars are usually real numbers, but they could be other sets of numbers such as rational numbers or complex numbers.
  - Closure. A vector space must be closed with respect to vector addition and scalar multiplication. This means that the sum of any two vectors in the vector space must also be a vector in the space and any scalar multiple of a vector in the vector space must also be a vector in the space. This implies that every possible linear combination of vectors in the space must also be an element of the space.
- Examples: Single-point Space, The Real Numbers $\mathbb{R}$, Euclidean Coordinates Spaces ($\mathbb{R}^n$), Matrices ($\mathbb{R}^{m\times n}$), Polynomials, Function Spaces.
- Subspaces are subsets of larger vector spaces.

- Subspaces inherit their definitions of vector addition and scalar multiplication from the vector spaces they are contained in. The critical point to showing that a subset is actually a vector space itself is to demonstrate closure.
- If a subspace contains the vectors $\mathbf{u}$ and $\mathbf{v}$, it must also contain $\mathbf{u} + \mathbf{v}$ and $c\mathbf{u}$ (for any scalar c).
- Every vector space (including a subspace) must contain a zero vector ($\{\mathbf{0}\}$).

The Column Space of a Matrix

- The column space of an $m \times n$ matrix A, denoted $C(A)$, is the set of all possible linear combinations of the columns of A. This is the set of all vectors $\mathbf{y}$ which can be written as the product $\mathbf{y} = A\mathbf{x}$ for some choice of $\mathbf{x}$.
- The column space of A contains vectors in $\mathbb{R}^m$. It is a set of "all possible results" from the operation $A\mathbf{x}$.

3.2 The Nullspace of A: Solving $A\mathbf{x} = \mathbf{0}$ and $R\mathbf{x} = \mathbf{0}$

The Nullspace of a Matrix

- The nullspace of an $m \times n$ matrix A, denoted $N(A)$, is the set of all vectors $\mathbf{x}$ which are solutions to the equation $A\mathbf{x} = \mathbf{0}$.
- The nullspace of A contains vectors in $\mathbb{R}^n$. It is a set of "all possible inputs" into the operation $A\mathbf{x}$ which result in a zero vector output.
- To find the nullspace of an $m \times n$ matrix A, we first perform row operations on the matrix to transform it to reduced row echelon form, which we call R.
- The rank r of A is the number of pivots.
- When the reduced row echelon form of the matrix A contains no free columns, the only solution to $A\mathbf{x} = \mathbf{0}$ is the zero vector $\mathbf{x} = \mathbf{0}$.
- When the reduced row echelon form of the matrix A contains at least one free column, a non-zero "special solution" to $A\mathbf{x} = \mathbf{0}$ can be found by setting one free variable equal to 1 and all other free variables equal to 0.
- A non-zero special solution can be created for each free column in the reduced row echelon form matrix derived from A. The equation $A\mathbf{x} = \mathbf{0}$ will have $n - r$ non-zero special solutions.
- The complete set of solutions to $A\mathbf{x} = \mathbf{0}$ will be the set of all possible linear combinations of special solutions derived from the free variables in the reduced row echelon form.

3.3 The Complete Solution to $A\mathbf{x} = \mathbf{b}$

Overview of the Complete Solution

- The equation $A\mathbf{x} = \mathbf{b}$ may have no solutions, a unique solution, or an infinite number of solutions.
- To solve the equation, perform row operations on $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ to put it in the reduced row echelon form $\begin{bmatrix} R & \mathbf{d} \end{bmatrix}$.
  - If there are rows in R containing all zeros and the vector $\mathbf{d}$ does not also have zeros on the corresponding rows, there will be **no solution** to the system. The vector $\mathbf{b}$ is not in the column space of A.
  - If a solution exists and R contains only pivot columns, **a unique solution** exists.

- If a solution exists and R contains both pivot columns and free columns, **an infinite number of solutions** exist.

- If a solution exists, the general form of the solution is $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$, where $\mathbf{x}_p$ is a particular solution and $\mathbf{x}_n$ is any vector in the nullspace of A.
  - The particular solution $\mathbf{x}_p$ may be picked out from $\begin{bmatrix} R & \mathbf{d} \end{bmatrix}$ by setting free variables, if there are any, to zero.
  - Special solutions in the nullspace of A may be determined by setting each of the free variables in R equal to one while holding the others fixed at zero, as we did in the previous section.
  - The vectors $\mathbf{x}_n$ in the nullspace of A are linear combinations of the special solutions in $N(A)$.

Conclusions

- Properties of matrices A having full column rank ($r = n$).
  - All columns of A are pivot columns.
  - There are no free variables or special solutions.
  - The nullspace $N(A)$ contains only the zero vector $\mathbf{x} = \mathbf{0}$.
  - If $A\mathbf{x} = \mathbf{b}$ has a solution, it has only one solution.
- Properties of matrices A having full row rank ($r = m$).
  - All rows have pivots and R has no zero rows.
  - The equation $A\mathbf{x} = \mathbf{b}$ will have a solution for any choice of $\mathbf{b}$.
  - The column space $C(A)$ is all of $\mathbb{R}^m$.
  - There are $n - r = n - m$ special solutions in the nullspace of A.
- Four possibilities.
  - Full rank, square and invertible ($r = m = n$). $A\mathbf{x} = \mathbf{b}$ will have one solution ($A^{-1}\mathbf{b}$).
  - Full row rank, short and wide ($r = m < n$). $A\mathbf{x} = \mathbf{b}$ will have infinite solutions.
  - Full column rank, tall and thin ($r = n < m$). $A\mathbf{x} = \mathbf{b}$ will have no solutions or one solution, depending on whether $\mathbf{b}$ is in the column space of A.
  - Not full rank ($r < m$ and $r < n$). $A\mathbf{x} = \mathbf{b}$ will have no solutions or infinite solutions, depending on whether $\mathbf{b}$ is in the column space of A.
    Change the last component of $\mathbf{b}$ ($= 8$) for an example with no solution.

3.4 Independence, Basis and Dimension

Independent Vectors

- The matrix A is said to have linearly independent columns if the only solution to $A\mathbf{x} = \mathbf{0}$ is the vector $\mathbf{x} = \mathbf{0}$.
- More generally, vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ... $\mathbf{v}_k$ are said to be linearly independent if the only linear combination of these vectors which produces a zero vector result is the linear combination for which every coefficient is zero. This definition applies to "vectors" which are not Euclidean Column Vectors, like the columns of a matrix A.
- This definition is equivalent to one we presented earlier, which said that vectors are linearly independent if no vector can be expressed as a linear combination of other vectors. This definition is preferred now because it does not pick out any particular vector to be expressed as a linear combination of others.

- The "span" of a set of vectors is the set of all possible linear combinations of the vectors. We also say that a space is "spanned" by a set of vectors if linear combinations of those vectors fills the space.
- One way of understanding independence is that every vector in a set is necessary in producing the span of the set. There are no *extra* vectors. Removing any vector from the set would reduce the size of the space spanned by the set of vectors.
- Linear dependence in a set of vectors, on the other hand, means that there are vectors in the set that are not necessary in producing the span of the set. One or more vectors could be removed and the span of the set would be the same.

The Basis and Dimension of a Vector Space

- A basis of a vector space S is a set of independent vectors in S which spans the space S.
  - The set of vectors in a basis is large enough to span the entire space
  - The set of vectors in a basis is no larger than it needs to be to span the space
- Every vector in S can be written as a unique linear combination of basis vectors.
- The dimension of a vector space S is the number of vectors in any basis for the space.
- Examples
  - The vectors $\mathbf{i} = (1,0)$ and $\mathbf{j} = (0,1)$ are a basis for $\mathbb{R}_2$. The vectors $2\mathbf{i} - \mathbf{j} = (2,-1)$ and $\mathbf{i} + 3\mathbf{j} = (1,3)$ are another basis for $\mathbb{R}_2$. (There are many bases for any vector space.) The dimension of $\mathbb{R}_2$ is two, so every basis for $\mathbb{R}_2$ will contain two vectors.
  - The vectors $(1,1,1)$ and $(0,0,1)$ are a basis for the column space of the following matrix:

    $$A = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 0 & 1 \\ 1 & 2 & 3 \end{bmatrix}$$

    Note that the third colummn is a linear combination of the first two, so only the first two columns are needed to span $C(A)$. The vector $(0,0,1)$ works just as well as $(0,0,2)$ appearing in the second column. In fact, any two independent linear combinations of $(1,1,1)$ and $(0,0,1)$ would work as a basis for $C(A)$.
  - The polynomials $p_0(x) = 1$, $p_1(x) = x$, and $p_2(x) = x^2$ are a basis for the vector space of polynomials having degree two or less.
  - The functions $y_1(x) = e^x$, and $y_2(x) = e^{-x}$ are a basis for the vector space of solutions to the differential equation $\mathrm{d}^2y/\mathrm{d}x^2 = y$.

Basis and Dimension of the Column Space $C(A)$

- The column space $C(A)$ is the space spanned by the columns of a matrix A.
- The column space $C(A)$ contains vectors in $\mathbb{R}^m$.
- A natural basis for $C(A)$ is the set of pivot columns in A. Choose the pivot columns from A, not R, because $C(A)$ is not the same space as $C(R)$.
- Example:

  $$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = R$$

  The first and second columns of A and R are pivot columns. A natural basis for $C(A)$ is the set of vectors containing the first and second columns of A: $(1,2,0)$ and $(1,2,1)$. Note that the first two columns of R don't span $C(A)$ since any linear combination of those columns would have a zero in the third component.

- The columns of A are linearly independent exactly when the rank of A is equal to n (A has full column rank).
- The columns of A cannot be linearly independent if $n > m$;
- The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ... $\mathbf{v}_n$ are a basis for $\mathbb{R}^n$ exactly when they are the columns of an $n \times n$ invertible matrix.

Basis and Dimension of the Row Space $C(A^T)$

- The row space $C(A^T)$ is the space spanned by the rows of a matrix A. The row space of A is the column space of A^T.
- The row space $C(A^T)$ contains vectors in $\mathbb{R}^n$.
- A natural basis for $C(A^T)$ is the set of pivot rows in R. Choose the pivot rows from R, not A, because $C(A^T)$ is the same as $C(R^T)$, and row swaps may disturb the order of the independent rows in A.
- Example:

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 2 & 0 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 2 \\ 0 & 0 & 0 \end{bmatrix} = R$$

  The first and second rows of R are pivot rows. A natural basis for $C(A^T)$ is the set of vectors containing the first and second rows of R: $(1, 0, -2)$ and $(0, 1, 2)$. Note that the first two rows of A don't span $C(A^T)$ since they are linearly dependent. The row swap changed the order of independent rows in A.

- The rows of A are linearly independent exactly when the rank of A is equal to m (A has full row rank).
- The rows of A cannot be linearly independent if $m > n$;
- The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, ... $\mathbf{v}_n$ are a basis for $\mathbb{R}^n$ exactly when they are the rows of an $n \times n$ invertible matrix.