Contents

Chapter 1

Fundamentals of Vectors

1.1 Vectors in Euclidean Space

Let's establish some notation first. We will use $\mathbb{R}^n$ to denote the set of all ordered lists of n real numbers. For instance, $\mathbb{R}^2$ denotes the set of all ordered pairs (x, y) of real numbers, which we can visualize as being the xy-plane. $\mathbb{R}^3$ denotes the set of all ordered triplets (x, y, z) of real numbers. We refer to $\mathbb{R}^n$ as being n-dimensional Euclidean space.

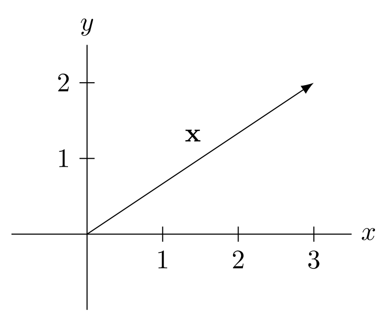
Elements of $\mathbb{R}^n$ are called vectors. Thus, a vector $\mathbf{x}$ in $\mathbb{R}^n$ is an ordered list of n real numbers, so $\mathbf{x}$ can be expressed as $\mathbf{x} = (x_1, \ldots, x_n)$. In linear algebra however, we will typically write

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

instead. The notation $(x_1, \ldots, x_n)$ will often be used to save vertical space, but it's best to view vectors as columns in this class, for reasons to be apparent later.

For the special case when $n = 1$, we will write $\mathbb{R}$ instead of $\mathbb{R}^1$. Vectors in $\mathbb{R}$ are just real numbers. We usually call them scalars instead of vectors.

As mentioned previously, $\mathbb{R}^2$ can be visualized as the familiar xy-plane. Vectors in $\mathbb{R}^2$ thus correspond to points in the plane. It is also useful, however, to visualize vectors as arrows or directed line segments emanating from the origin. For instance, the vector $\mathbf{x} = (3, 2)$ is depicted below as an arrow:



When viewed as a directed line segment, we see that vectors have two properties associated with them: magnitude (i.e., length) and direction. When viewed as directed line segments, two vectors are considered equal to each other if and only if they have the same magnitude and the same direction. Many physical quantities have magnitude and direction and are thus modeled as vectors. The prototypical example would be velocity, which has magnitude (speed) and direction.

When viewing vectors as directed line segments, we can also choose to view them emanating from points other than the origin. And, although we can't visualize $\mathbb{R}^n$ in general for n larger than three, it is still helpful to visualize vectors as arrows in these cases.

1.1.1 Vector Addition

Given two vectors

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_n \end{bmatrix}$$

in $\mathbb{R}^n$, we define their sum $\mathbf{x}+\mathbf{y}$ by simply adding component-wise:

$$\mathbf{x}+\mathbf{y} = \begin{bmatrix} x_1+y_1 \\ x_2+y_2 \\ \vdots \\ x_n+y_n \end{bmatrix}.$$

Example. Let

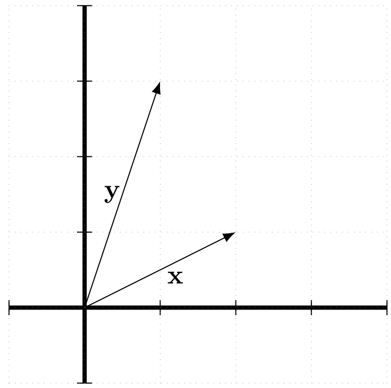
$$\mathbf{x} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{y} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}.$$

Then

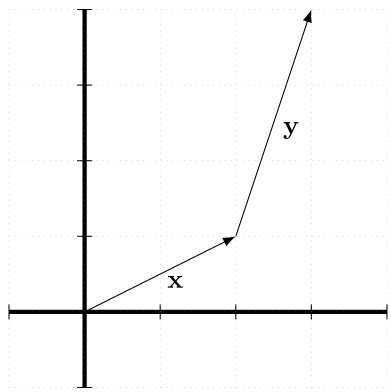
$$\mathbf{x}+\mathbf{y} = \begin{bmatrix} 3 \\ 4 \end{bmatrix}. \qquad \diamond$$

Now, relying on computations with coordinates can sometimes complicate a problem, and you will likely encounter a few instances of this in this class. Thus, it may be useful to have a way of adding vectors without relying on their coordinates. This can be done by visualizing vectors as directed line segments. Here's how it works: if we connect the tail of $\mathbf{y}$ to the arrowhead of $\mathbf{x}$, then $\mathbf{x}+\mathbf{y}$ will be the arrow that starts from the tail of $\mathbf{x}$ and goes out to the head of $\mathbf{y}$. We illustrate with some examples on the next few pages.

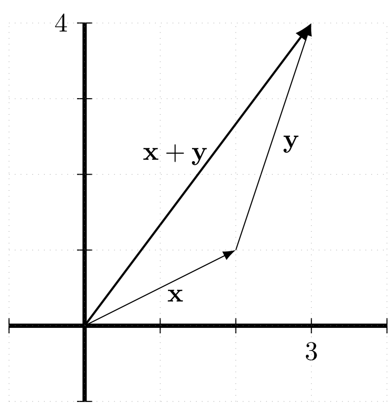
Example. Let's use the vectors $\mathbf{x} = (2, 1)$ and $\mathbf{y} = (1, 3)$ again:



To draw the vector $\mathbf{x} + \mathbf{y}$, move the vector $\mathbf{y}$ so that its tail is connected to the arrowhead of $\mathbf{x}$:

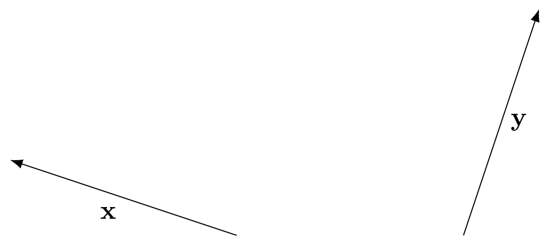


Then $\mathbf{x} + \mathbf{y}$ is the vector that starts from the tail of $\mathbf{x}$ and goes out to the arrowhead of $\mathbf{y}$:

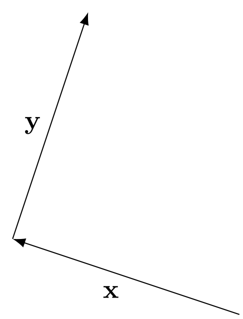


Notice that we get $\mathbf{x} + \mathbf{y} = (3, 4)$, as expected. $\diamond$

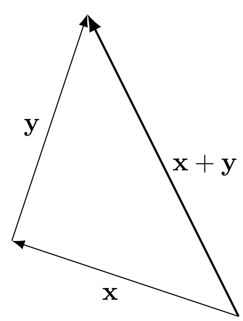
Example. Here is a coordinate-free example. Consider the vectors $\mathbf{x}$ and $\mathbf{y}$ depicted below:



To draw the vector $\mathbf{x}+\mathbf{y}$, move the vectors so that the tail of $\mathbf{y}$ is connected to the arrowhead of $\mathbf{x}$:



Then $\mathbf{x} + \mathbf{y}$ is the vector that starts from the tail of $\mathbf{x}$ and goes out to the arrowhead of $\mathbf{y}$:



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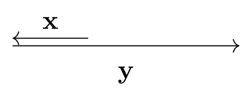
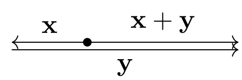
As you can see from these two examples, when we connect $\mathbf{x}$ and $\mathbf{y}$ head-to-tail, their sum forms the third edge of a triangle (again, starting from the tail of $\mathbf{x}$ to the head of $\mathbf{y}$). There is one somewhat exceptional situation where this is not the case though, as the next example illustrates.

Example. Consider the vectors $\mathbf{x}$ and $\mathbf{y}$ depicted below:

To draw the vector $\mathbf{x}+\mathbf{y}$, move the vectors so that the tail of $\mathbf{y}$ is connected to the arrowhead of $\mathbf{x}$. To make this easier to see, I have drawn the vector $\mathbf{y}$ slightly below $\mathbf{x}$ instead. But notice that the tail of $\mathbf{y}$ and the head of $\mathbf{x}$ are aligned, as if the tail of $\mathbf{y}$ were starting at the head of $\mathbf{x}$.



Then $\mathbf{x}+\mathbf{y}$ is the vector that starts from the tail of $\mathbf{x}$ and goes out to the arrowhead of $\mathbf{y}$. I have placed a dot where the tail of $\mathbf{x}$ is located.

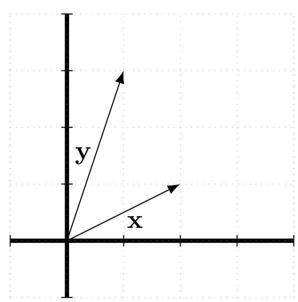


$\diamond$

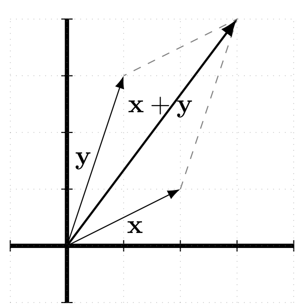
We say that two vectors are parallel if they point in the same direction, and anti-parallel if they point in opposite directions. In the previous example, the vectors $\mathbf{x}$ and $\mathbf{y}$ were anti-parallel.

There is another useful way of visualizing vector addition, but it only works when the two vectors are neither parallel nor anti-parallel. Suppose we have two such vectors $\mathbf{x}$ and $\mathbf{y}$. If we connect the vectors tail-to-tail, then the vector $\mathbf{x}+\mathbf{y}$ is the diagonal of the parallelogram formed by the vectors $\mathbf{x}$ and $\mathbf{y}$, with its tail connected to the tail of $\mathbf{x}$ and $\mathbf{y}$. Again, we'll illustrate with some examples.

Example. Let's use the vectors $\mathbf{x} = (2, 1)$ and $\mathbf{y} = (1, 3)$ again:

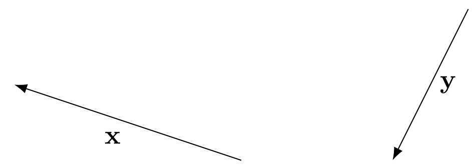


If we form the parallelogram determined by $\mathbf{x}$ and $\mathbf{y}$, then $\mathbf{x}+\mathbf{y}$ will be the diagonal of the parallelogram, with its tail at the same point as the tail of $\mathbf{x}$ and $\mathbf{y}$.

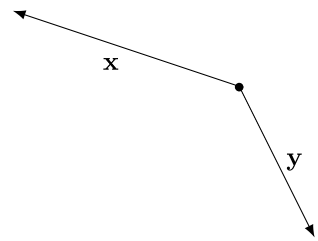


Once again, we get that $\mathbf{x}+\mathbf{y} = (3, 4)$. $\diamond$

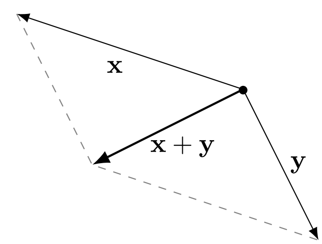
Example. Here is a coordinate-free example. Consider the vectors $\mathbf{x}$ and $\mathbf{y}$ depicted below:



To draw the vector $\mathbf{x} + \mathbf{y}$, move the vectors so that their tails are connected to each other:



If we form the parallelogram determined by $\mathbf{x}$ and $\mathbf{y}$, then $\mathbf{x} + \mathbf{y}$ will be the diagonal of the parallelogram, with its tail at the same point as the tail of $\mathbf{x}$ and $\mathbf{y}$.



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1.1.2 Scalar Multiplication

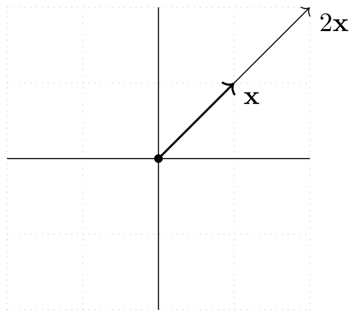
Given a scalar c and a vector

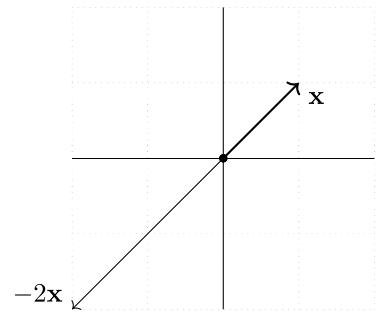
$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$$

in $\mathbb{R}^n$, we define their product $c\mathbf{x}$ by simply multiplying each component of $\mathbf{x}$ by c:

$$c\mathbf{x} = \begin{bmatrix} cx_1 \\ cx_2 \\ \vdots \\ cx_n \end{bmatrix}$$

As an example, if $\mathbf{x} = (1, 1)$, then $2\mathbf{x} = (2, 2)$ and $-2\mathbf{x} = (-2, -2)$. Below are some illustrations of this:





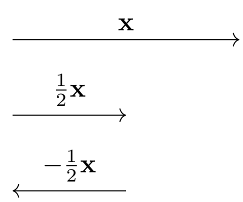
In the above figures, notice that:

- both $2\mathbf{x}$ and $-2\mathbf{x}$ are twice as long as $\mathbf{x}$;
- $2\mathbf{x}$ points in the same direction as $\mathbf{x}$; and
- $-2\mathbf{x}$ points in the opposite direction.

In general, the vector $c\mathbf{x}$:

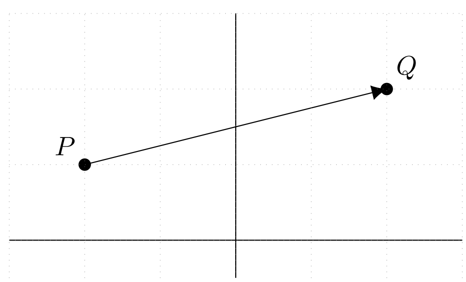
- will be $|c|$ times as long as $\mathbf{x}$;
- will point in the same direction as $\mathbf{x}$ if $c > 0$; and
- will point in the opposite direction as $\mathbf{x}$ if $c < 0$.

Here are some illustrations without a coordinate system cluttering the picture:



1.1.3 Visualizing Vector Subtraction

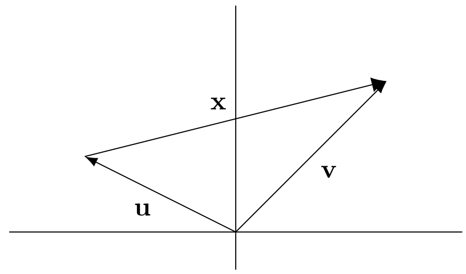
There is one last thing I want to talk about in this section. I mentioned earlier that it is often useful to view vectors starting somewhere other than the origin. Consider the following picture with two vectors $P = (-2, 1)$ and $Q = (2, 2)$ labeled. I have also drawn a vector starting at P and ending at Q.



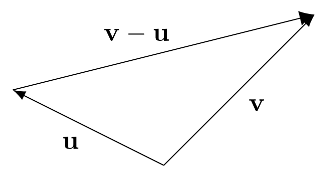
What are the coordinates of this vector? From the picture, we see that the vector goes to the right 4 units, and goes up 1 unit. Thus, the coordinates of this vector are $(4, 1)$. Notice that we get the same result by simply subtracting P from Q:

$$Q - P = \begin{bmatrix} 2 \\ 2 \end{bmatrix} - \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix}.$$

To see why this works, let's add more vectors into the picture now.



Notice that **u** and **x** are connected head-to-tail, and connects from the tail of **u** to the head of **x**. From our discussion of vector addition earlier, this means that $\mathbf{v} = \mathbf{u} + \mathbf{x}$, or equivalently, that $\mathbf{x} = \mathbf{v} - \mathbf{u}$:



In general, if **u** and **v** are connected tail-to-tail, then $\mathbf{u} - \mathbf{v}$ is the vector starting from the head of **u** to the head of **v**, and usually forms the third edge of a triangle.

Here is a simple example of doing some vector algebra.

Example. Solve the following equation for the vector **v**:

$$3\begin{bmatrix} 2 \\ -1 \\ 3 \end{bmatrix} - 2\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}.$$

Solution. Multiply the 3 into the vector $(2, -1, 3)$:

$$\begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix} - 2\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix}.$$

Subtract both sides of the equation by the vector $(6, -3, 9)$:

$$-2\mathbf{v} = \begin{bmatrix} 2 \\ 1 \\ 4 \end{bmatrix} - \begin{bmatrix} 6 \\ -3 \\ 9 \end{bmatrix} = \begin{bmatrix} -4 \\ 4 \\ -5 \end{bmatrix}.$$

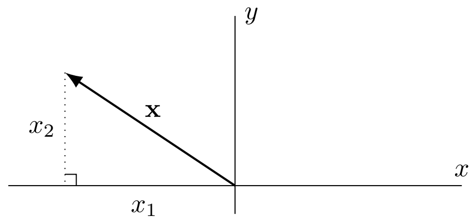
Divide both sides by -2 to get

$$\mathbf{v} = \begin{bmatrix} 2 \\ -2 \\ \frac{5}{2} \end{bmatrix}. \qquad \diamond$$

1.2 Norm/Length of a Vector

Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$. The norm of $\mathbf{x}$, denoted $\|\mathbf{x}\|$, is defined to be the length of $\mathbf{x}$. We would like to come up with a formula for $\|\mathbf{x}\|$.

Suppose first that $\mathbf{x} = (x_1, x_2)$ is a vector in $\mathbb{R}^2$:



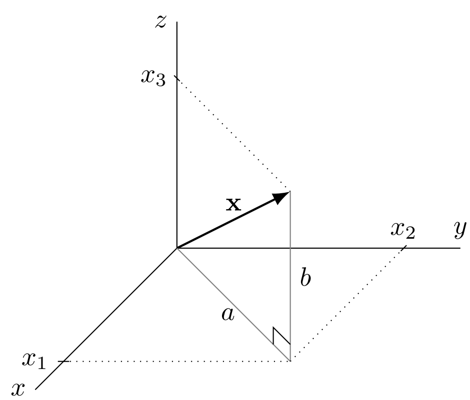
Notice that $\mathbf{x}$ is the hypotenuse of a right triangle, whose height is $|x_2|$, and whose base has a length of $|x_1|$. We are using absolute values here because in general x_1 and x_2 could be negative, but length should be positive. From the Pythagorean Theorem, we find that

$$\|\mathbf{x}\|^2 = x_1^2 + x_2^2,$$

or equivalently,

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2}.$$

We'll look at the three-dimensional case on the next page. Suppose $\mathbf{x} = (x_1, x_2, x_3)$ is a vector in $\mathbb{R}^3$:



Once again, the vector $\mathbf{x}$ is the hypotenuse of a right triangle. We've labeled the lengths of the other edges as a and b. From the Pythagorean Theorem,

$$\|\mathbf{x}\|^2 = a^2 + b^2.$$

We can see from the picture that $b = |x_3|$. As for a, notice that a is just the length of the vector (x_1, x_2) in $\mathbb{R}^2$. Thus,

$$a = \sqrt{x_1^2 + x_2^2}.$$

It follows that

$$a^2 = x_1^2 + x_2^2 \qquad \text{and} \qquad b^2 = x_3^2,$$

and thus,

$$\|\mathbf{x}\|^2 = x_1^2 + x_2^2 + x_3^2,$$

or equivalently,

$$\|\mathbf{x}\| = \sqrt{x_1^2 + x_2^2 + x_3^2}.$$

Generalizing to higher dimensions leads to the following:

Definition. Let $\mathbf{x}$ be a vector in $\mathbb{R}^n$ given by

$$\mathbf{x} = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}.$$

The ***norm*** of $\mathbf{x}$, denoted $\|\mathbf{x}\|$, is defined by the following formula:

$$\|\mathbf{x}\| = \sqrt{x_1^2 + \ldots + x_n^2}.$$

It represents the length of the vector $\mathbf{x}$.

Example. Compute the length of the vector $\mathbf{x}$ in $\mathbb{R}^4$ given by

$$\mathbf{x} = \begin{bmatrix} 5 \\ 2 \\ -4 \\ 2 \end{bmatrix}.$$

Solution.

$$\|\mathbf{x}\| = \sqrt{5^2 + 2^2 + (-4)^2 + 2^2} = \sqrt{25 + 4 + 16 + 4} = \sqrt{49} = 7.$$

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1.2.1 Unit Vectors, Normalization

One property of the norm is that for any scalar c, and for any vector $\mathbf{x}$,

$$\|c\mathbf{x}\| = |c|\,\|\mathbf{x}\|.$$

In words, the length of $c\mathbf{x}$ is $|c|$ times the length of $\mathbf{x}$. We saw this already when we were talking about scalar multiplication. In particular, if we choose c to be $1/|\mathbf{x}|$, then the length of $c\mathbf{x}$ would be equal to 1.

Definition. A *unit vector* is any vector whose norm (or length) is equal to 1.

As mentioned above, given a nonzero vector $\mathbf{x}$, we can always obtain a unit vector pointing in the same direction as $\mathbf{x}$ by simply dividing $\mathbf{x}$ by its norm. Unit vectors are often more convenient to work with, and in some problems it is necessary to be using unit vectors. They are important enough that this process of dividing a vector by its norm has a name: it is called normalization.

Example. We saw earlier that the norm of

$$\mathbf{x} = \begin{bmatrix} 5 \\ 2 \\ -4 \\ 2 \end{bmatrix}$$

is 7. To get a unit vector pointing in the same direction as $\mathbf{x}$, we just need to normalize $\mathbf{x}$ by dividing it by its norm. Thus,

$$\frac{1}{7}\begin{bmatrix} 5 \\ 2 \\ -4 \\ 2 \end{bmatrix}$$

is a unit vector, and it points in the same direction as $\mathbf{x}$. ◇

1.3 The Dot Product

We've talked about how to add vectors, and how to multiply a scalar with a vector. What about multiplying a vector with another vector? Well, it turns out that there are quite a few interesting ways do that. In this section, we are going to discuss the dot product (a.k.a. scalar product) of two vectors.

Definition. Let $\mathbf{u}, \mathbf{v}$ be vectors in $\mathbb{R}^n$ given by

$$\mathbf{u} = \begin{bmatrix} u_1 \\ \vdots \\ u_n \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ \vdots \\ v_n \end{bmatrix}.$$

The dot product of $\mathbf{u}$ and $\mathbf{v}$, denoted $\mathbf{u} \cdot \mathbf{v}$, is defined by the following formula:

$$\mathbf{u} \cdot \mathbf{v} = u_1 v_1 + u_2 v_2 + \ldots + u_n v_n.$$

Note that the dot product of two vectors results in a scalar.

Example. Compute the dot product of the following vectors:

$$\mathbf{u} = \begin{bmatrix} 2 \\ 3 \\ -2 \\ 1 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} -1 \\ 4 \\ 5 \\ 2 \end{bmatrix}.$$

Solution.

$$\mathbf{u} \cdot \mathbf{v} = 2(-1) + 3(4) + (-2)(5) + 1(2) = -2 + 12 - 10 + 2 = 2.$$ ◇

The following proposition lists some of the properties of the dot product:

Proposition. For any three vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^n$, and for any scalar c,

1. $(\mathbf{u} + \mathbf{v}) \cdot \mathbf{w} = (\mathbf{u} \cdot \mathbf{w}) + (\mathbf{v} \cdot \mathbf{w})$.
2. $\mathbf{u} \cdot (\mathbf{v} + \mathbf{w}) = (\mathbf{u} \cdot \mathbf{v}) + (\mathbf{u} \cdot \mathbf{w})$.
3. $(c\mathbf{u}) \cdot \mathbf{v} = c(\mathbf{u} \cdot \mathbf{v})$.
4. $\mathbf{u} \cdot (c\mathbf{v}) = c(\mathbf{u} \cdot \mathbf{v})$.
5. $\mathbf{u} \cdot \mathbf{v} = \mathbf{v} \cdot \mathbf{u}$.
6. $\mathbf{u} \cdot \mathbf{u} \geq 0$, and $\mathbf{u} \cdot \mathbf{u} = 0$ only if $\mathbf{u}$ is the zero vector.

The first two properties basically say that the distributive laws hold. The next two basically say that constants can be pulled to the front.

So, why do we care about the dot product? It turns out that the dot product between two vectors carries a lot of geometric information about them. We proceed to illustrate what we mean by this statement.

1.3.1 The Connection between the Dot Product and the Norm

If $\mathbf{v} = (v_1, \ldots, v_n)$, then

$$\mathbf{v} \cdot \mathbf{v} = v_1^2 + \ldots + v_n^2.$$

Since

$$\|\mathbf{v}\| = \sqrt{v_1^2 + \ldots + v_n^2},$$

it follows that

$$\mathbf{v} \cdot \mathbf{v} = \|\mathbf{v}\|^2.$$

This means that we can compute the norm of a vector using the dot product. In this sense, the dot product encodes information about lengths of vectors.

Example. Suppose $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^n$ such that $\|\mathbf{u}\| = 2$, $\|\mathbf{v}\| = 3$, and $\mathbf{u} \cdot \mathbf{v} = -4$. Compute $\|\mathbf{u} + \mathbf{v}\|$.

Solution. Since the magnitude of a vector, squared, is equal to the dot product of the vector with itself, we have

$$\|\mathbf{u} + \mathbf{v}\|^2 = (\mathbf{u} + \mathbf{v}) \cdot (\mathbf{u} + \mathbf{v}).$$

Using properties of the dot product, we can "foil" out the right side of this equation to get

$$\begin{aligned} \|\mathbf{u} + \mathbf{v}\|^2 &= \mathbf{u} \cdot \mathbf{u} + \mathbf{u} \cdot \mathbf{v} + \mathbf{v} \cdot \mathbf{u} + \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{u}\|^2 + 2\,\mathbf{u} \cdot \mathbf{v} + \|\mathbf{v}\|^2. \end{aligned}$$

Plugging in the given information, we find that

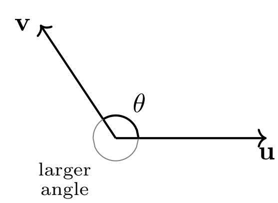
$$\|\mathbf{u} + \mathbf{v}\|^2 = 4 - 8 + 9 = 5.$$

Thus,

$$\|\mathbf{u} + \mathbf{v}\| = \sqrt{5}.$$ ◇

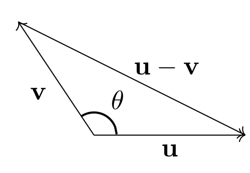
1.3.2 The Angle Between Two Vectors

Suppose we have two nonzero vectors $\mathbf{u}$ and $\mathbf{v}$ connected tail-to-tail. There are two angles that could be considered the angle between $\mathbf{u}$ and $\mathbf{v}$; a smaller angle, which we denote by θ, and a larger angle.



When we refer to the angle between two vectors, we are always referring to θ. In particular then, the angle between two vectors is always between 0 and π radians.

Recall that if two vectors $\mathbf{u}$ and $\mathbf{v}$ are connected tail-to-tail, then the vector connecting the head of $\mathbf{v}$ to the head of $\mathbf{u}$ is $\mathbf{u} - \mathbf{v}$:



If we apply the Law of Cosines to the above picture, we find that

$$\|\mathbf{u} - \mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 - 2\,\|\mathbf{u}\|\,\|\mathbf{v}\|\,\cos\theta.$$

We saw earlier that the norm of a vector, squared, is equal to the vector dot product with itself. We can thus rewrite the above equation as

$$(\mathbf{u} - \mathbf{v}) \cdot (\mathbf{u} - \mathbf{v}) = (\mathbf{u} \cdot \mathbf{u}) + (\mathbf{v} \cdot \mathbf{v}) - 2\,\|\mathbf{u}\|\,\|\mathbf{v}\|\,\cos\theta.$$

Using properties of the dot product, we can "foil" out the left side of this equation to get

$$(\mathbf{u} \cdot \mathbf{u}) - 2(\mathbf{u} \cdot \mathbf{v}) + (\mathbf{v} \cdot \mathbf{v}) = (\mathbf{u} \cdot \mathbf{u}) + (\mathbf{v} \cdot \mathbf{v}) - 2\,\|\mathbf{u}\|\,\|\mathbf{v}\|\,\cos\theta.$$

We can cancel out the $\mathbf{u} \cdot \mathbf{u}$ and $\mathbf{v} \cdot \mathbf{v}$ terms from both sides of this equation. This leaves us with

$$-2(\mathbf{u} \cdot \mathbf{v}) = -2\,\|\mathbf{u}\|\,\|\mathbf{v}\|\,\cos\theta.$$

Next, we can cancel the -2 from both sides to get the following formula:

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\cos\theta.$$

This formula allows us to compute the angle between two vectors using the dot product. Thus, the dot product encodes information about the angle between two vectors as well.

Example. Compute the angle between the following two vectors:

$$\mathbf{u} = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -2 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} 3 \\ 0 \\ 4 \\ 0 \end{bmatrix}$$

Solution. We just need to compute $\mathbf{u} \cdot \mathbf{v}$, $\|\mathbf{u}\|$, $\|\mathbf{v}\|$, and then plug everything into the formula

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\cos\theta,$$

to solve for θ.

$$\mathbf{u} \cdot \mathbf{v} = 3 \qquad \|\mathbf{u}\| = 3 \qquad \|\mathbf{v}\| = \sqrt{2}.$$

Plugging these into the above formula gives us

$$3 = (3)(5)\cos\theta \iff \tfrac{1}{5} = \cos\theta.$$

Thus, we see that the angle between $\mathbf{u}$ and $\mathbf{v}$ is $\cos^{-1}\left(\frac{1}{5}\right)$. ◇

1.3.3 Orthogonality

The word orthogonal is synonymous with the word perpendicular. Thus, to say that two vectors are orthogonal to each other is the same as saying that the angle between the two vectors is $\frac{\pi}{2}$ radians. Since $\cos(\frac{\pi}{2}) = 0$, it follows from the formula

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\,\|\mathbf{v}\|\,\cos\theta$$

that two nonzero vectors are orthogonal to each other if and only if their dot product is equal to zero. By convention, the zero vector is orthogonal to every vector.

Example. Find all values of k for which the following two vectors are orthogonal to each other.

$$\mathbf{u} = \begin{bmatrix} 2 \\ -3 \\ 1 \\ 4 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} k^2 \\ 1 \\ 3 \\ -2 \end{bmatrix}.$$

Solution. For $\mathbf{u}$ and $\mathbf{v}$ to be orthogonal to each other, we need their dot product to be zero. So we will compute $\mathbf{u} \cdot \mathbf{v}$, set it equal to zero, and solve for k.

$$\mathbf{u} \cdot \mathbf{v} = 2k^2 - 8 = 0 \implies k = \pm 2.$$ ◇

1.3.4 More On the Relation Between Dot Products and Angles

In the last section, we proved that

$$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta,$$

where θ is the angle between $\mathbf{u}$ and $\mathbf{v}$ that is between 0 and π radians. As a consequence of this, we saw that two nonzero vectors are orthogonal to each other if and only if their dot product equals 0.

Assuming $\mathbf{u}$ and $\mathbf{v}$ are nonzero, notice from this formula that $\mathbf{u} \cdot \mathbf{v}$ will be positive when $\cos\theta$ is positive, and likewise for negative. Now, $\cos\theta$ will be positive when θ is an acute angle (i.e. $0 \leq \theta < \frac{\pi}{2}$), and negative when θ is an obtuse angle (i.e. $\frac{\pi}{2} < \theta \leq \pi$).

$\mathbf{u} \cdot \mathbf{v} > 0$ $\qquad\qquad$ $\mathbf{u} \cdot \mathbf{v} < 0$

When the angle is acute, $\mathbf{u}$ and $\mathbf{v}$ tend to point more towards the same direction, and their dot product will be positive. When the angle is obtuse, $\mathbf{u}$ and $\mathbf{v}$ tend to point more in opposite directions, and their dot product will be negative. Hence, the sign of the dot product gives us a sense for whether the vectors point more towards the same direction or more in opposite directions.

Cauchy-Bunyakovsky-Schwarz Inequality

Again, considering the formula $\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos\theta$, using the fact that $|\cos\theta| \leq 1$, we find that

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|.$$

This is the Cauchy-Bunyakovsky-Schwarz inequality, or CBS inequality for short. We don't need to use this inequality much in this class, but it is very important in mathematics and thus worth mentioning.

Note that equality holds, i.e. $|\mathbf{u} \cdot \mathbf{v}| = \|\mathbf{u}\|\|\mathbf{v}\|$, only when $\theta = 0$ or π. If $\theta = 0$, then $\mathbf{u}, \mathbf{v}$ point in the same direction, and if $\theta = \pi$, then $\mathbf{u}, \mathbf{v}$ point in opposite directions. In both cases, $\mathbf{u}$ and $\mathbf{v}$ lie on the same line, meaning they are linearly dependent. Thus, equality holds if and only if $\mathbf{u}, \mathbf{v}$ are linearly dependent.

In an attempt to give the CBS inequality the attention it deserves, we summarize our discussion of it in the fancy box below.

Cauchy-Bunyakovsky-Schwarz Inequality

For any two vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$,

$$|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\|\|\mathbf{v}\|.$$

Equality holds if and only if $\mathbf{u}, \mathbf{v}$ are linearly dependent.

Triangle Inequality

In the last section, we saw that by writing $\|\mathbf{u}+\mathbf{v}\|^2$ as $(\mathbf{u}+\mathbf{v})\cdot(\mathbf{u}+\mathbf{v})$ and foiling everything out,

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2(\mathbf{u}\cdot\mathbf{v}).$$

Replacing $\mathbf{u}\cdot\mathbf{v}$ with $\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta$ gives us

$$\|\mathbf{u}+\mathbf{v}\|^2 = \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\|\cos\theta.$$

Since $\cos\theta \le 1$, it follows that

$$\|\mathbf{u}+\mathbf{v}\|^2 \le \|\mathbf{u}\|^2 + \|\mathbf{v}\|^2 + 2\|\mathbf{u}\|\|\mathbf{v}\|\cdot 1.$$

Notice that the right hand side can be factored as $(\|\mathbf{u}\|+\|\mathbf{v}\|)^2$, so that ultimately, we find that

$$\|\mathbf{u}+\mathbf{v}\|^2 \le (\|\mathbf{u}\|+\|\mathbf{v}\|)^2,$$

or, square rooting both sides,

$$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\|+\|\mathbf{v}\|.$$

This last inequality is known as the triangle inequality. It is quite possibly the most widely used inequality in all of mathematics.

Looking back at our work above, notice that equality holds in the triangle inequality precisely when $\cos\theta = 1$, which would be when $\theta = 0$. In this case, $\mathbf{u}$ and $\mathbf{v}$ point in the same direction, meaning they are parallel to one another.

As with the CBS inequality, we attempt to give the triangle inequality the attention it deserves by summarizing our discussion of it so far in the fancy box below.

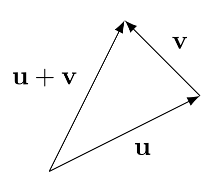
Triangle Inequality

For any two vectors $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$,

$$\|\mathbf{u}+\mathbf{v}\| \le \|\mathbf{u}\|+\|\mathbf{v}\|.$$

Equality holds only when $\mathbf{u}, \mathbf{v}$ are parallel to each other (i.e. one is a nonnegative multiple of the other).

As for why we call it the triangle inequality, recall that the vectors $\mathbf{u}, \mathbf{v}$, and $\mathbf{u}+\mathbf{v}$ typically form the edges of a triangle.



The triangle inequality says that the length of one of the edges of this triangle is less than the sum of the lengths of the other two edges, which reflects a well known geometric fact about triangles that the length of any edge is less than the sum of the lengths of the other two edges.

Example. The main application of the triangle inequality is in establishing other inequalities. We'll illustrate this by proving the following results using the triangle inequality.

1. For any $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in $\mathbb{R}^n$, $\|\mathbf{x}+\mathbf{y}+\mathbf{z}\| \leq \|\mathbf{x}\|+\|\mathbf{y}\|+\|\mathbf{z}\|$.
2. For any $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$, $\|\mathbf{u}-\mathbf{v}\| \leq \|\mathbf{u}\|+\|\mathbf{v}\|$.
3. For any $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$, $\big|\,\|\mathbf{u}\|-\|\mathbf{v}\|\,\big| \leq \|\mathbf{u} \pm \mathbf{v}\|$.
4. For any $\mathbf{x}, \mathbf{y}, \mathbf{z}$ in $\mathbb{R}^n$, $\|\mathbf{x}-\mathbf{z}\| \leq \|\mathbf{x}-\mathbf{y}\|+\|\mathbf{y}-\mathbf{z}\|$.

Proof.

1. Treating $\mathbf{x}+\mathbf{y}$ as a single vector and $\mathbf{z}$ as another vector, we can apply the triangle inequality to conclude that
$$\|(\mathbf{x}+\mathbf{y})+\mathbf{z}\| \leq \|\mathbf{x}+\mathbf{y}\|+\|\mathbf{z}\|.$$
The triangle inequality also tells us that
$$\|\mathbf{x}+\mathbf{y}\| \leq \|\mathbf{x}\|+\|\mathbf{y}\|.$$
Combining these two inequalities proves the desired result.

2. According to the triangle inequality,
$$\|\mathbf{u}+(-\mathbf{v})\| \leq \|\mathbf{u}\|+\|-\mathbf{v}\|.$$
This is easily seen to be equivalent to the desired result.

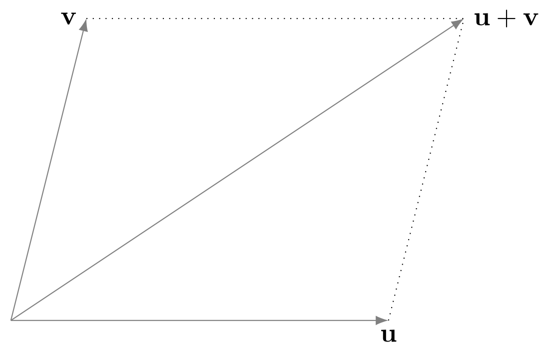
3. Notice that
$$\|\mathbf{u}\| = \|(\mathbf{u}+\mathbf{v})-\mathbf{v}\| \leq \|\mathbf{u}+\mathbf{v}\|+\|\mathbf{v}\|.$$
Ignoring the $\|(\mathbf{u}+\mathbf{v})-\mathbf{v}\|$ part of the above expression now gives
$$\|\mathbf{u}\| \leq \|\mathbf{u}+\mathbf{v}\|+\|\mathbf{v}\|.$$
Subtracting $\|\mathbf{v}\|$ from both sides of this inequality gives us
$$\|\mathbf{u}\|-\|\mathbf{v}\| \leq \|\mathbf{u}+\mathbf{v}\|.$$
Repeating the same argument but with $\mathbf{u}$ and $\mathbf{v}$ swapped gives us
$$\|\mathbf{v}\|-\|\mathbf{u}\| \leq \|\mathbf{u}+\mathbf{v}\|.$$
These last two inequalities together are equivalent to saying
$$\big|\,\|\mathbf{u}\|-\|\mathbf{v}\|\,\big| \leq \|\mathbf{u}+\mathbf{v}\|.$$
A similar argument proves the result with $\|\mathbf{u}-\mathbf{v}\|$ at the end rather than $\|\mathbf{u}+\mathbf{v}\|$.

4. Notice that
$$\|\mathbf{x}-\mathbf{z}\| = \|(\mathbf{x}-\mathbf{y})+(\mathbf{y}-\mathbf{z})\| \leq \|\mathbf{x}-\mathbf{y}\|+\|\mathbf{y}-\mathbf{z}\|.$$

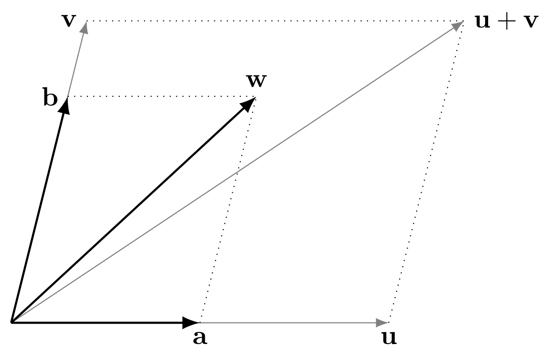
$\diamond$

1.4 Parallelograms, Linear Combinations, and Span

Last time, we saw that if we have two vectors $\mathbf{u}$ and $\mathbf{v}$ which are connected tail-to-tail and are neither parallel nor anti-parallel, then their sum $\mathbf{u}+\mathbf{v}$ is equal to the diagonal of the parallelogram formed by $\mathbf{u}$ and $\mathbf{v}$:



What if we choose a vector $\mathbf{w}$ inside of the parallelogram now? How is the vector $\mathbf{w}$ related to $\mathbf{u}$ and $\mathbf{v}$? Consider the following illustration.



The vector $\mathbf{w}$ is inside of the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$. We've drawn two vectors $\mathbf{a}$ and $\mathbf{b}$ that point in the same direction as $\mathbf{u}$ and $\mathbf{v}$, and for which $\mathbf{w}$ is the diagonal of the parallelogram formed by $\mathbf{a}$ and $\mathbf{b}$, i.e., for which $\mathbf{w} = \mathbf{a} + \mathbf{b}$.

Because $\mathbf{a}$ points in the same direction as $\mathbf{u}$, it must be that $\mathbf{a} = c\mathbf{u}$ for some scalar $c > 0$. And since the length of $\mathbf{a}$ is less than the length of $\mathbf{u}$, we can say that $0 < c < 1$. Similarly, we can conclude that $\mathbf{b} = d\mathbf{v}$ for some scalar d between 0 and 1. Thus, we have $\mathbf{w} = c\mathbf{u} + d\mathbf{v}$.

One can make this same argument for every vector inside of the parallelogram. Namely, every vector inside the parallelogram can be expressed as $c\mathbf{u}+d\mathbf{v}$ for some scalars c and d between 0 and 1. The reader should convince themselves that the reverse is true as well, i.e., that every vector of the form $c\mathbf{u} + d\mathbf{v}$ where $0 \leq c \leq 1$ and $0 \leq d \leq 1$ is inside or on the parallelogram. Thus, we conclude that the set of all vectors inside or on the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$ consists precisely of all vectors of the form $c\mathbf{u} + d\mathbf{v}$, where $0 \leq c \leq 1$ and $0 \leq d \leq 1$.

The expression $c\mathbf{u}+d\mathbf{v}$ is an example of what we call a linear combination of the vectors $\mathbf{u}$ and $\mathbf{v}$. More generally, we have the following

Definition. Let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. A ***linear combination*** of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is any vector of the form

$$c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k,$$

where $c_1, c_2, \ldots, c_k$ are scalars.

Example. Let $\mathbf{u} = (1,1)$ and $\mathbf{v} = (2,3)$. Then the vector $(4,5)$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, since

$$\begin{bmatrix}4\\5\end{bmatrix} = 2\begin{bmatrix}1\\1\end{bmatrix} + 1\begin{bmatrix}2\\3\end{bmatrix}.$$

The vector $(1, -1)$ is also a linear combination of $\mathbf{u}$ and $\mathbf{v}$, since

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 5\begin{bmatrix} 1 \\ 1 \end{bmatrix} - 2\begin{bmatrix} 2 \\ 3 \end{bmatrix}. \qquad \diamond$$

Example. By definition, a linear combination of a single vector $\mathbf{v}$ is any vector of the form $c\mathbf{v}$, where c is any scalar. Thus, a linear combination of a single vector is simply a scalar multiple of that vector. $\diamond$

Example. Every vector (x, y, z) in $\mathbb{R}^3$ is a linear combination of the vectors $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$, since

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} = x\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} + y\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + z\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \qquad \diamond$$

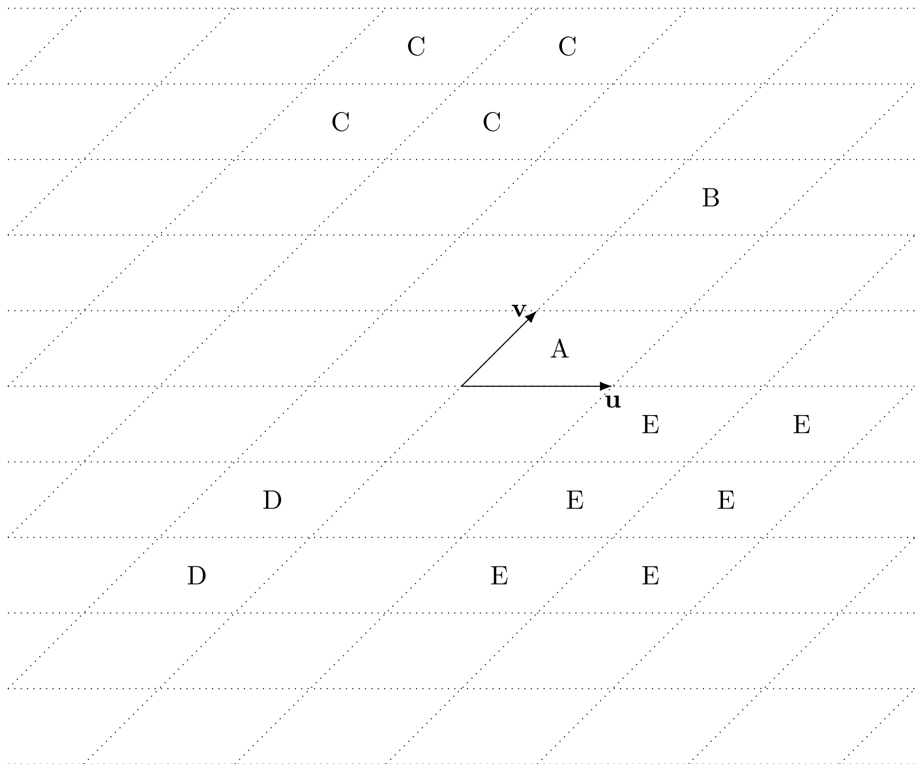
We can rephrase the results from the previous page by saying that the set of all linear combinations $c\mathbf{u} + d\mathbf{v}$, where c and d are restricted to be between 0 and 1, coincides with the set of all points inside and on the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$.

Now consider the set of all linear combinations of $\mathbf{u}$ and $\mathbf{v}$ of the form $c\mathbf{u} + d\mathbf{v}$, where:

A. $0 \le c \le 1$ and $0 \le d \le 1$.
B. $0 \le c \le 1$ and $2 \le d \le 3$.
C. $-3 \le c \le -1$ and $3 \le d \le 5$.
D. $-1 \le c \le 0$ and $-3 \le d \le -1$.
E. $1 \le c \le 3$ and $-3 \le d \le 0$.

Again, we know that the region A corresponds to the set of all points inside and on the parallelogram determined by $\mathbf{u}$ and $\mathbf{v}$. Could you sketch the regions corresponding to B, C, D, and E? Think about it for a moment! The answer appears on the next page.

Here is an illustration indicating what region each of these sets of linear combinations is describing:



Can you see it?

Now, what if we consider the set of all possible linear combinations of $\mathbf{u}$ and $\mathbf{v}$, without any restrictions on the coefficients c and d? Hopefully, you can see that the set of all linear combinations of $\mathbf{u}$ and $\mathbf{v}$ would give us the entire infinite plane determined by $\mathbf{u}$ and $\mathbf{v}$. This motivates the following definition:

Definition. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. The ***span*** of $\mathbf{v}_1, \ldots, \mathbf{v}_k$, denoted $\operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$, is defined to be the set of all linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

Using set notation, we can write

$$\operatorname{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\} = \{\mathbf{b} \in \mathbb{R}^n \ ; \ \mathbf{b} = c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k \quad \substack{\text{for some scalars} \\ c_1,\ldots,c_k}\}.$$

In the above picture, $\operatorname{span}\{\mathbf{u}, \mathbf{v}\}$ is equal to the (infinite) plane determined by $\mathbf{u}$ and $\mathbf{v}$. *This can be rephrased by saying that a vector* $\mathbf{w}$ *is a linear combination of* $\mathbf{u}$ *and* $\mathbf{v}$ *if and only if* $\mathbf{w}$ *lies in the plane determined by* $\mathbf{u}$ *and* $\mathbf{v}$.

Example. Is the vector $\mathbf{w} = (1, -1)$ a linear combination of the vectors $\mathbf{u} = (2, 1)$ and $\mathbf{v} = (1, 1)$?

Solution. We need to check if there exists scalars x_1 and x_2 such that

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = x_1 \begin{bmatrix} 2 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

Reading off the components of both sides of this equation gives us

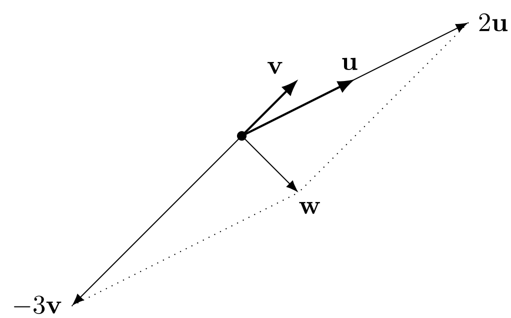
$$\begin{aligned} 1 &= 2x_1 + x_2 \\ -1 &= \ \ x_1 + x_2. \end{aligned}$$

Subtracting the second equation from the first equation gives us $x_1 = 2$. Plugging $x_1 = 2$ into either equation then gives us $x_2 = -3$. Thus, we see that

$$\begin{bmatrix} 1 \\ -1 \end{bmatrix} = 2 \begin{bmatrix} 2 \\ 1 \end{bmatrix} - 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

It follows that $(1, -1)$ is a linear combination of $(2, 1)$ and $(1, 1)$. ◇

Here is a small illustration of the above example. All vectors have their tail located at the same point, which is marked with a dot.



The vector $\mathbf{w}$ is the diagonal of the parallelogram determined by the vectors $2\mathbf{u}$ and $-3\mathbf{v}$. In particular, the vector $\mathbf{w}$ is a linear combination of the vectors $\mathbf{u}$ and $\mathbf{v}$ because it lies in the plane determined by $\mathbf{u}$ and $\mathbf{v}$.

Example. Is the vector $(3, 3)$ a linear combination of the vectors $(1, 2)$ and $(2, 4)$?

Solution. We need to check if there exists scalars x_1 and x_2 such that

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

Reading off the components of both sides of this equation gives us

$$\begin{aligned} 3 &= x_1 + 2x_2 \\ 3 &= 2x_1 + 4x_2. \end{aligned}$$

If we multiply the first equation by two, we get

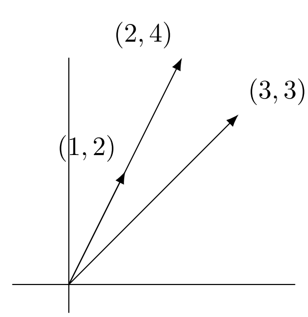
$$\begin{aligned} 6 &= 2x_1 + 4x_2 \\ 3 &= 2x_1 + 4x_2. \end{aligned}$$

It is not possible for $2x_1 + 4x_2$ to be equal to 6 and 3 at the same time, so we see that there is no solution to this system of equations. This means that there does *not* exist scalars x_1 and x_2 for which

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 2 \\ 4 \end{bmatrix}.$$

It follows that $(3, 3)$ is *not* a linear combination of the vectors $(1, 2)$ and $(2, 4)$. ◇

The following picture illustrates what is happening:



The vector $(3, 3)$ is not a linear combination of $(1, 2)$ and $(2, 4)$, because the vectors $(1, 2)$ and $(2, 4)$ only span a line, and the vector $(3, 3)$ does not lie on that line.

Example. Consider the following question: Does the vector $\mathbf{w} = (5, 1, 5)$ lie in the plane determined by the vectors $\mathbf{u} = (1, 0, 2)$ and $\mathbf{v} = (3, 1, 1)$?

This question may seem difficult at first, as you probably are not able to visualize these vectors very well in your head, so you probably can't see what the answer is. Remember though that for $\mathbf{w}$ to be in the plane determined by $\mathbf{u}$ and $\mathbf{v}$, it just needs to be a linear combination of these vectors. In other words, this question is equivalent to the following question:

Is the vector $\mathbf{w} = (5, 1, 5)$ a linear combination of the vectors $\mathbf{u} = (1, 0, 2)$ and $\mathbf{v} = (3, 1, 1)$?

We can answer this the same way we did before. We need to check if there are scalars x_1 and x_2 such that

$$\begin{bmatrix} 5 \\ 1 \\ 5 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} 3 \\ 1 \\ 1 \end{bmatrix}.$$

To cut to the chase, I will just note that $x_1 = 2$ and $x_2 = 1$ satisfies the above equation. Thus, we can say that $\mathbf{w}$ is a linear combination of $\mathbf{u}$ and $\mathbf{v}$, and therefore that $\mathbf{w}$ lies in the plane determined by $\mathbf{u}$ and $\mathbf{v}$. $\diamond$

You need to be able to understand the geometry associated with algebraic concepts such as linear combinations, if only because you may be given a seemingly geometric question, as in the previous example, and you must be able to translate it into an algebraic question in order to answer it.

Closing Remarks

In the first example, we ended up with the equations

$$\begin{aligned} 1 &= 2x_1 + x_2 \\ -1 &= x_1 + x_2, \end{aligned}$$

while in the second example, we ended up with the equations

$$\begin{aligned} 3 &= x_1 + 2x_2 \\ 3 &= 2x_1 + 4x_2. \end{aligned}$$

These are examples of what we call a linear system of equations. They are linear because all of the variables appear by themselves (they're not multiplying each other or anything like that) and all exponents of the variables are equal to one.

We'll end these notes by pointing out to the reader that in order to check if a vector is a linear combination of some other vectors, we end up needing to solve a linear system of equations. Keep this in mind as you continue.

1.5 Linear Independence

1.5.1 The Inutuition

Let's start by recalling the definitions of linear combination and span. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$.

- A linear combination of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is any vector of the form $c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k$, where $c_1, \ldots, c_k$ are any scalars. In particular, we noted that a linear combination of a single vector is just a scalar multiple of that vector.
- The span of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$, denoted $\text{span}\{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$, is defined to be the set of all linear combinations of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$.

Previously, we had a picture of two vectors $\mathbf{u}$ and $\mathbf{v}$, and we saw that their span was equal to the plane that they determined. As a consequence of this, we noted that a vector $\mathbf{w}$ was a linear combination of $\mathbf{u}$ and $\mathbf{v}$ if and only if $\mathbf{w}$ was in the plane determined by $\mathbf{u}$ and $\mathbf{v}$. Also, in a previous example we did, we had the vectors $(1, 2)$ and $(2, 4)$ which only span a line (b/c they are multiples of each other).

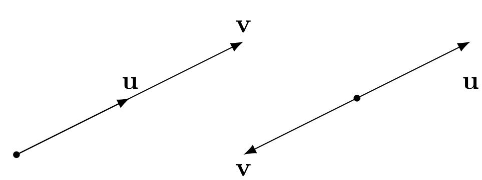
It will be good to have some intuition about dimension at this point. A plane is considered to be a 2-dimensional object, since it has area. Similarly, a line is considered to be a 1-dimensional object, since it has length, and a 3-dimensional object would have volume instead. A 0-dimensional object is just a point.

Now suppose we have a single vector $\mathbf{u}$. There are two possibilities for what its span can be.

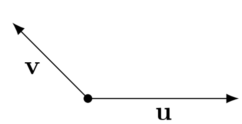
1. If $\mathbf{u} = \mathbf{0}$, then the span of $\mathbf{u}$ is just the zero vector, i.e. it is just a point. In this case, the dimension of $\text{span}\{\mathbf{u}\}$ is zero.
2. If $\mathbf{u} \neq 0$, then the span of $\mathbf{u}$ will be the line determined by $\mathbf{u}$. In this case, the dimension of $\text{span}\{\mathbf{u}\}$ is one.

Next, let's look at all of the possibilities when we have two vectors $\mathbf{u}$ and $\mathbf{v}$.

1. If $\mathbf{u} = \mathbf{v} = \mathbf{0}$, then the span of $\mathbf{u}$ and $\mathbf{v}$ is just the zero vector. In this case, $\text{span}\{\mathbf{u}, \mathbf{v}\}$ is just a point, so it is 0-dimensional.
2. The span of $\mathbf{u}$ and $\mathbf{v}$ could be 1-dimensional. This would happen if $\mathbf{u}$ and $\mathbf{v}$ are parallel or anti-parallel, as in the left and right pictures below, respectively. The vectors would only determine a line, and this line would be the span of $\mathbf{u}$ and $\mathbf{v}$.



3. Otherwise, if $\mathbf{u}$ and $\mathbf{v}$ are both nonzero and neither parallel nor anti-parallel, then their span will be the plane that they determine. This is the "nice" scenario, where the span of two vectors is two-dimensional.



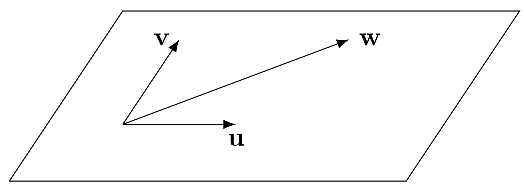
In the second case, where the span of $\mathbf{u}$ and $\mathbf{v}$ is 1-dimensional, this is happening because $\mathbf{u}$ or $\mathbf{v}$ is a scalar multiple of the other. We can rephrase this by saying that one of the vectors is a linear combination of the other. The same could be said for the first case, where $\mathbf{u} = \mathbf{v} = \mathbf{0}$. Technically, in this case, $\mathbf{u}$ and $\mathbf{v}$ are also scalar multiples of each other, so they are also linear combinations of each other. *Thus, two vectors fail to span a 2-dimensional space if and only if one of them is a linear combination of the other.*

Suppose now that we have three vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{w}$. Again, let's take a look at all of the possibilities for their span.

1. If $\mathbf{u} = \mathbf{v} = \mathbf{w} = \mathbf{0}$, then their span is just the origin, which is just a point. So their span is 0-dimensional. We can say that in this case, at least one of the vectors is a linear combination of the others.

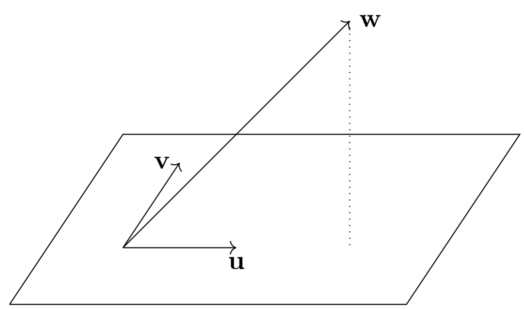
2. Their span could be 1-dimensional. This will happen if the three vectors are not all $\mathbf{0}$ but lie along the same line, in which case, their span would be the line that they determine. Again, we can say in this case that at least one of the vectors is a linear combination of the others.

3. Their span could be 2-dimensional. This will happen if the vectors do not lie along some line, but they lie witin some plane. In this case, their span would be the plane that they lie in.

   Since one of the vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$ lies in the plane determined by the other two, we see once again that one of them is a linear combination of the others.

4. Otherwise, their span is 3-dimensional. This will happen if the three vectors do not lie in a plane.

Note that *three vectors fail to span a 3-dimensional space when at least one of them is a linear combination of the others*, similar to how two vectors fail to span a 2-dimensional space when one of them is a linear combination of the other.

Although we can't quite visualize higher dimensions, we can still imagine that the same phenomena occurs. Namely, suppose we have k vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in $\mathbb{R}^n$. Suppose one of these vectors is a linear combination of the others. For definiteness, let's suppose that $\mathbf{v}_k$ is a linear combination of the others. Then $\mathbf{v}_k$ lies in the span of $\mathbf{v}_1, \ldots, \mathbf{v}_{k-1}$, which, intuitively, is at most $(k-1)$-dimensional. So the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ can span at most $k-1$ dimensions if one of them is a linear combination of the others.

All of this discussion was to try and motivate the following:

Intuitive Definition. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$.

- We say that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are ***linearly independent*** if none of them is a linear combination of the others.

- We say that the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are ***linearly dependent*** if at least one of them is a linear combination of the others.

The geometric interpretation of linear independence is the following: k vectors are linearly independent if and only if they span a k-dimensional space. If k vectors are linearly dependent, then the dimension of their span will be less than k.

You'll notice that we labeled the above definition as an "intuitive definition". It is actually not the official definition that we want to use. Here is the reason why: suppose we are given $\mathbf{a}_1, \ldots, \mathbf{a}_k$ and are asked to check if they are linearly independent or not. We would probably start with the first vector and check to see if it is a linear combination of the others; if it's not, then we would continue down the list until we get to a vector that is a linear combination of the others, or until we have exhausted the list. This can be very time-consuming!

Therefore, we want another definition of linear independence that is easier to use. The definition that we will arrive at is, in my opinion, less intuitive than the one given above. I think it is very important to have some intuition in this class, however; hence, I chose to present this intuitive definition.

1.5.2 The Formal Definition

Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. Consider the equation

$$x_1\mathbf{v}_1 + \ldots + x_k\mathbf{v}_k = \mathbf{0},$$

where the scalars $x_1, \ldots, x_k$ are the unknowns. This equation is called the ***homogeneous equation***; the word homogeneous signifies one side of the equation is zero. The homogeneous equation always has at least one solution: namely, the solution where $x_1 = \ldots = x_k = 0$. This is called the trivial solution. Any nonzero solutions to the homogeneous equation, if there are any, are called nontrivial solutions.

To arrive at the definition of linear independence that we officially want to use, let's suppose that we have vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Let's further assume that they are linearly dependent. For definiteness, suppose that $\mathbf{v}_1$ is a linear combination of the others, so that

$$\mathbf{v}_1 = c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k$$

for some scalars $c_2, \ldots, c_k$. We can move $\mathbf{v}_1$ to the other side of the equation to get

$$\mathbf{0} = -\mathbf{v}_1 + c_2\mathbf{v}_2 + \ldots + c_k\mathbf{v}_k.$$

This says that

$$x_1 = -1 \qquad x_2 = c_2 \qquad \ldots \qquad x_k = c_k$$

is a solution to the homogeneous equation

$$\mathbf{0} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_k\mathbf{v}_k.$$

In particular, since $x_1 \neq 0$, we have a nontrivial solution to the homogeneous equation.

Let's now reverse the process. Suppose first that we have a nontrivial solution to the homogeneous equation

$$x_1\mathbf{v}_1 + \ldots + x_k\mathbf{v}_k = \mathbf{0}.$$

This means that one of the x's is nonzero. For definiteness, let's suppose x_1 is nonzero. Then we can move the second through last terms to the other side of the equation to get

$$x_1\mathbf{v}_1 = -x_2\mathbf{v}_2 - \ldots - x_k\mathbf{v}_k.$$

Since x_1 is nonzero, we can divide both sides by x_1 to get

$$\mathbf{v}_1 = -\tfrac{x_2}{x_1}\mathbf{v}_2 - \ldots - \tfrac{x_k}{x_1}\mathbf{v}_k.$$

This tells us that $\mathbf{v}_1$ is a linear combination of $\mathbf{v}_2, \ldots, \mathbf{v}_k$. Thus, the vectors $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_k$ are linearly dependent.

What we saw just now is that, if $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly dependent, then there exists a nontrivial solution to the homogeneous equation. Conversely, if there exists a nontrivial solution to the homogeneous equation, then the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly dependent. In other words, being linearly dependent is equivalent to having nontrivial solutions to the homogeneous equation. This leads us at last to the formal definition of linear independence.

Definition. Let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors in $\mathbb{R}^n$. The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are ***linearly independent*** if the only solution to the homogeneous equation

$$x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_k\mathbf{v}_k = \mathbf{0}$$

is the trivial solution where $x_1 = x_2 = \ldots = x_k = 0$.

The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are ***linearly dependent*** if they are not linearly independent, i.e., they are linearly dependent if there exists a nontrivial (i.e., nonzero) solution to the homogeneous equation.

Typically, this definition is the one that you will apply to problems. However, keep in mind the geometric interpretation and intuitive definition given earlier as well.

Example. Are the vectors

$$\begin{bmatrix} 1 \\ 4 \\ 0 \\ -2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2 \\ -1 \\ 3 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 5 \\ 2 \end{bmatrix}$$

linearly independent?

Solution. We need to check if there are any nontrivial (i.e., nonzero) solutions to the equation

$$x_1 \begin{bmatrix} 1 \\ 4 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ -1 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

Looking at the four components, we obtain the following four equations:

$$\begin{aligned} x_1 + \quad\; - \; x_3 &= 0 \\ 4x_1 + 2x_2 \qquad &= 0 \\ - \; x_2 + 5x_3 &= 0 \\ -2x_1 + 3x_3 + 2x_3 &= 0. \end{aligned}$$

The first equation gives us $x_1 = x_3$. Substitute into the second equation to get

$$4x_3 + 2x_2 = 0.$$

This gives us $x_2 = -2x_3$. Substitute into the third equation to get

$$2x_3 + 5x_3 = 0.$$

This gives us $x_3 = 0$. From $x_2 = -2x_3$, it follows that $x_2 = 0$ as well. And from $x_1 = x_3$, it follows that $x_1 = 0$.

Since the only solution to the equation

$$x_1 \begin{bmatrix} 1 \\ 4 \\ 0 \\ -2 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 2 \\ -1 \\ 3 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 5 \\ 2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

is the trivial solution where $x_1 = x_2 = x_3 = 0$, it follows that the given vectors are linearly independent. $\diamond$

Closing Remarks

Notice that in order to check if vectors are linearly independent, we end up needing to solve a linear system of equations, just like when we were checking if a given vector is a linear combination of some other vectors. This should not be surprising, since checking if vectors are linearly independent is, according to our intuitive definition, the same as checking to see if vectors are linear combinations of other vectors.

Since these linear systems of equations keep showing up, it seems like we should study them a bit, don't you think? That is what we will do next. Don't forget everything that we've covered so far!

Chapter 2

Linear Systems of Equations

A linear system of equations is a set of equations of the form

$$\begin{aligned}
a_{11}x_1 + \ a_{12}x_2 + \ \ldots + \ a_{1n}x_n &= b_1 \\
a_{21}x_1 + \ a_{22}x_2 + \ \ldots + \ a_{2n}x_n &= b_2 \\
&\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \ \ldots + a_{mn}x_n &= b_m.
\end{aligned}$$

There are m equations in this system. The variables $x_1, x_2, \ldots, x_n$ are the unknowns, i.e., what we'd want to solve for. A solution to this LSE is a set of numbers $x_1, x_2, \ldots, x_n$ that satisfies all m equations simultaneously.

We've seen some examples of linear systems of equations (LSEs) already. For instance,

$$\begin{aligned}
x_1 + \qquad - \ \ x_3 &= 0 \\
4x_1 + 2x_2 \qquad\quad &= 0 \\
- \ \ x_2 + 5x_3 &= 0 \\
-2x_1 + 3x_3 + 2x_3 &= 0
\end{aligned}$$

appeared in a previous example.

We'll begin our study of LSEs by looking at some of the geometry associated with them. After that, we will learn a standard technique known as ***elimination*** for solving LSEs.

2.1 The Row and Column Pictures

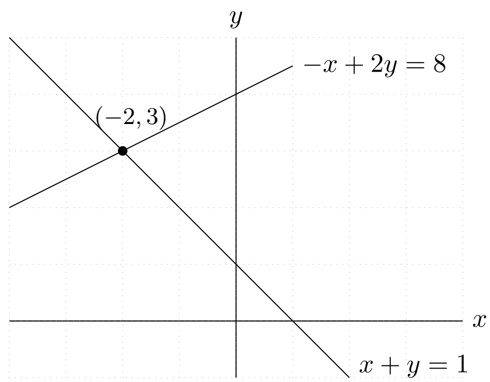
2.1.1 The "Row Picture"

Let me be immediately clear that the phrase "row picture" is not standard, but it is terminology that Gilbert Strang uses in his linear algebra textbook.

Consider the following linear system of equations:

$$\begin{aligned}
-x + 2y &= 8 \\
x + \ \ y &= 1.
\end{aligned}$$

Each of these equations is an equation of a line. A solution (x, y) to this system has to satisfy both equations, meaning that it has to lie on both lines. Thus, the solutions are where the two lines intersect. Below is the row picture for this system of equations:



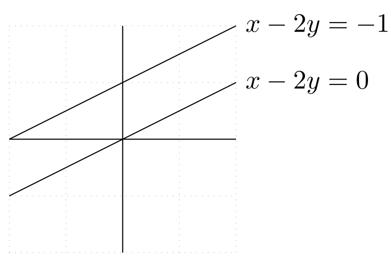
The two lines given by the system of equations are depicted. Their intersection point, $(-2,3)$ is also in the picture. Thus, we see that $(x,y)=(-2,3)$ should be the solution to the LSE. You can double check that this is the case by plugging in $x=-2$ and $y=3$ into the system of equations.

So, to recap, in a linear system with two equations and two unknowns, each equation represents a line, and the intersection of the two lines represents the set of solutions to the linear system. The row picture is simply graphing lines and where they intersect.

What are the possibilities when two lines intersect? We describe them below.

1. The two lines never intersect. This would happen if they are parallel and do not coincide. This would be the case for the following LSE:

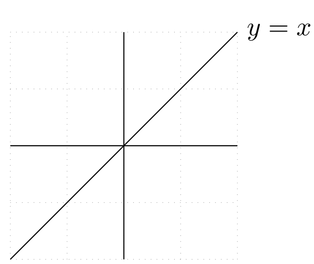
$$\begin{aligned} x-2y &= 0 \\ x-2y &= -1 \end{aligned}$$



In this case, there would be no solution to the corresponding LSE, because the lines do not intersect.

2. The two lines coincide. This would be the case for the following LSE, since the two equations are equivalent:

$$\begin{aligned} x-y &= 0 \\ 2x-2y &= 0 \end{aligned}$$



Since the two lines coincide, there are infinitely many intersection points. Thus, there would be infinitely many solutions to the corresponding LSE.

3. The two lines intersect at one point. In this case, there would be a unique solution to the corresponding LSE.

It turns out that this phenomenon persists no matter how many equations and unknowns there are. Namely, a linear system of equations will have either no solution, infinitely many solutions, or a unique solution. There are no other possibilities! For instance, you'll never find exactly two solutions; as soon as you have more than one solution, you'll have infinitely many. This fact is important enough to restate below.

Proposition. A linear system of equations will have either no solution, a unique solution, or infinitely many solutions.

2.1.2 The "Column Picture"

Again, the phrase "column picture" is not standard, but it is terminology that Gilbert Strang uses in his linear algebra textbook. Between the row and column pictures, the column picture is more important for us!

If you've been going through this book from the start, then you've actually already seen examples of column pictures for a LSEs! But don't worry, we'll provide another example here as well.

Let's use the same example as we did for the row picture.

$$\begin{aligned} -x + 2y &= 8 \\ x + y &= 1. \end{aligned}$$

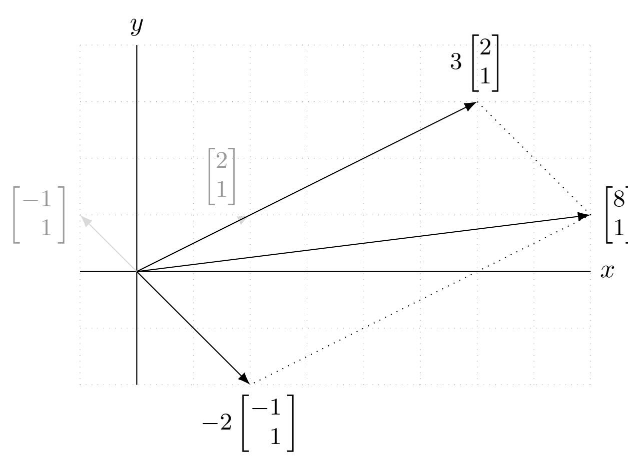
We found the solution to be $(x, y) = (-2, 3)$. Notice that we can rewrite the above system of equations as the following vector equation:

$$x \begin{bmatrix} -1 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}.$$

What does this tell us? It tells us that solving the above system of equations is the same as checking whether the vector $(8, 1)$ is a linear combination of the vectors $(-1, 1)$ and $(2, 1)$! Since we found a solution, we know that the answer is yes, the vector $(8, 1)$ is indeed a linear combination of the vectors $(-1, 1)$ and $(2, 1)$. In fact, since the solution was $(x, y) = (-2, 3)$, we know that

$$-2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} + 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 8 \\ 1 \end{bmatrix}.$$

Here is the column picture for this LSE:



In the column picture, we are basically illustrating how the sum of the vectors

$$-2 \begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad 3 \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

is equal to $(8, 1)$, by showing that $(8, 1)$ is the diagonal of the parallelogram determined by these two vectors.

Closing Remarks

Consider now the general linear system of m equations with n unknowns:

$$\begin{aligned} a_{11}x_1 + \ a_{12}x_2 + \ \dots + \ a_{1n}x_n &= b_1 \\ a_{21}x_1 + \ a_{22}x_2 + \ \dots + \ a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ \dots + a_{mn}x_n &= b_m. \end{aligned}$$

For now, let's denote this system by $(*)$. The idea of the column picture was to rewrite this as the following vector equation:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \dots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

If we let

$$\mathbf{a}_1 = \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} \qquad \mathbf{a}_2 = \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} \qquad \cdots \qquad \mathbf{a}_n = \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix},$$

then we have the following observation, which I think is important enough to be stated as a theorem.

Theorem. The linear system of equations $(*)$ has a solution if and only if the vector $\mathbf{b}$ is a linear combination of the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$. In other words, the linear system of equations has a solution if and only $\mathbf{b}$ is in the span of the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$.

Notice that the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$, and $\mathbf{b}$ are vectors in $\mathbb{R}^m$, since they have m components. In particular then, if the vectors $\mathbf{a}_1, \mathbf{a}_2, \dots, \mathbf{a}_n$ span all of $\mathbb{R}^m$, then the equation $(*)$ will always have a solution.

2.2 Using Elimination to Solve a Linear System of Equations

We've encountered a few linear systems of equations (LSEs) so far in some examples, and we kind of just messed around with the equations in order to find the solution. In these notes, and the two that follow, we are going to learn a standard procedure for solving LSEs which is called Gaussian elimination, or simply just elimination.

Let's consider the following LSE as an example:

$$\begin{aligned} x + 2y &= 1 \\ 2x + 7y &= -4 \end{aligned}$$

What we are going to do is eliminate the variable x from the second equation. We will do this with the following operation: (Equation 2) $-$ 2(Equation 1). Doing this leaves us with

$$\begin{aligned} x + 2y &= 1 \\ 3y &= -6 \end{aligned}$$

Now that we have eliminated x from the second equation, we can solve for y. We get $y = -2$. Then we substitute this back into the first equation to solve for x. We get $x = 5$.

Notice that the variables x and y are not too important when solving the LSE. For instance, we could label the variables as c and d instead, or anything else that we'd like. Thus, we can ignore the variables and only focus on the numbers. To do that, we rewrite the LSE in the following form:

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 7 & -4 \end{array}\right]$$

We call this the ***augmented matrix*** corresponding to the LSE. The matrix

$$\begin{bmatrix} 1 & 2 \\ 2 & 7 \end{bmatrix}$$

is called the ***coefficient matrix*** corresponding to the LSE. The operation that we performed earlier to eliminate x from the second equation now corresponds to the following operation on the rows of the augmented matrix: (Row 2) $-$ 2(Row 1). Doing this leaves us with

$$\left[\begin{array}{cc|c} 1 & 2 & 1 \\ 2 & 7 & -4 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 2 & 1 \\ 0 & 3 & -6 \end{array}\right]$$

Notice that the coefficient matrix was reduced to an "upper triangular" matrix, i.e., a matrix with zeroes below the diagonal of the matrix. Roughly speaking, this is the goal of elimination: to take the augmented matrix, and perform row operations until the the coefficient matrix has been reduced to an upper triangular form. Once the coefficient matrix is in upper triangular form, we can then solve for the last variable, and then substitute back into the previous equations to solve for the remaining variables.

We should formally define everything now before we continue.

Definition. For the linear system of equations

$$\begin{aligned} a_{11}x_1 + \ a_{12}x_2 + \ldots + \ a_{1n}x_n &= b_1 \\ a_{21}x_1 + \ a_{22}x_2 + \ldots + \ a_{2n}x_n &= b_2 \\ &\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m, \end{aligned}$$

the *matrix of coefficients* is

$$\begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix},$$

and the corresponding *augmented matrix* is

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right].$$

2.2.1 Allowable Row Operations

You'll probably get the hang of elimination after seeing some examples. But before that, we need to discuss the procedure a bit more.

First, what kind of operations are we allowed to perform? In the previous example, we performed the operation $R_2 - 2R_1$. So, we are allowed to add/subtract a multiple of a row from another row. This is the row operation that you'll use the most, but there are two more types of row operations that we can perform.

When doing elimination, we are allowed to swap rows in the augmented matrix. This is because

$$\begin{aligned} x + 2y &= 1 \\ 2x + 7y &= -4 \end{aligned}$$

and

$$\begin{aligned} 2x + 7y &= -4 \\ x + 2y &= 1 \end{aligned}$$

are essentially the same LSE. Swapping the order of the equations does not affect the solution.

We can also multiply any row by a nonzero number when doing elimination. This is because if we have an equation

$$x + 2y = 1,$$

and we multiply both sides by a nonzero number, we're not really changing anything.

So in summary, when doing elimination, there are three types of row operations that we can perform:

The Three Types of Row Operations

1. We can add/subtract a multiple of a row from another row.
2. We can swap rows.
3. We can multiply any row by a nonzero number.

2.2.2 Row Echelon Form

As we mentioned earlier, the goal of elimination is to take the augmented matrix of a linear system of equations, and to perform row operations on it until it has been reduced to an "upper triangular" form. We now need to be more specific about what we mean by "upper triangular". Specifically, we want to perform row operations until the augmented matrix is in what we call row echelon form.

Definition. A matrix is said to be in ***row echelon form*** (REF) if:

- All rows of zeroes, if there are any, appear on the bottom of the matrix, below all of the nonzero rows.
- The first nonzero entry in a row, if there are any, appears to the right of the first nonzero entry in the previous row.
- For each column containing the first nonzero entry of a row, the entries below that first nonzero entry are zeros.

If a matrix is in REF, the first nonzero entry in each row is called a ***pivot***.

Example. The matrix

$$\begin{bmatrix} 1 & 2 & -3 & 2 \\ 0 & -2 & 3 & 4 \\ 0 & 0 & 5 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

is in row echelon form. All rows of zeros are at the bottom. The first nonzero entry in a row appears to the right of the first nonzero entry in the previous row (the -2 in the second row appears to the right of the 1 in the first row, and the 5 in the third row appears to the right of the -2 in the second row). The pivots are the 1, -2, and 5, an all entries beneath them are 0. $\diamond$

Example. The matrix

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & -5 & 3 & 2 & -4 \\ 0 & 0 & 0 & 0 & 2 \end{bmatrix}$$

is in row echelon form. There are no rows of zeros. The pivots are the 1, -5, and 2 in the first, second, and third rows. They appear to the right of the previous pivot, and all entries beneath them are 0. $\diamond$

Example. The matrix

$$\begin{bmatrix} 1 & 2 & 0 & 1 & 3 \\ 0 & -5 & 3 & 2 & -4 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 2 & 0 & 0 \end{bmatrix}$$

is not in row echelon form. The second condition is not satisfied, because the first nonzero entry in the fourth row appears to the left of the first nonzero entry in the third row. To be in REF, it needs to appear to the right instead.

If we swapped the third and fourth rows, then it would be in REF. $\diamond$

Comments

Unfortunately, it is difficult to give an exact step-by-step procedure to do elimination, since different problems will work out differently. Here are a few general guidelines:

- Use the pivots to produce zeros below them. I'll try to point out what I mean by this in the examples. Generally speaking, you don't HAVE to do this, but it's a "safe" way to do elimination. Also, you HAVE to do it this way when you're finding an LU factorization of a matrix (we'll talk about that later).
- Do it one column at a time, i.e., get all of the zeros in the first column, then get all of the zeros in the next column, etc.

My hope is that you will get the hang of elimination from examples. We will present numerous examples in the next section.

2.3 Examples of Performing Elimination

2.3.1 Case 1: Unique Solution

Example. Solve the following linear system of equations using elimination:

$$\begin{aligned} x + 2y + z &= 3 \\ 2x + 5y + z &= 2 \\ -3x - 4y + 2z &= 4. \end{aligned}$$

Solution. First, set up the augmented matrix:

$$\left[\begin{array}{rrr|r} \boxed{1} & 2 & 1 & 3 \\ 2 & 5 & 1 & 2 \\ -3 & -4 & 2 & 4 \end{array}\right]$$

I've already highlighted the first pivot. We want to use the pivot to produce zeros below it. This means we want to add/subtract multiples of the first row to the other rows.

$$\left[\begin{array}{rrr|r} \boxed{1} & 2 & 1 & 3 \\ 2 & 5 & 1 & 2 \\ -3 & -4 & 2 & 4 \end{array}\right] \begin{array}{c} \xrightarrow{R_2 - 2R_1} \\ \xrightarrow{R_3 + 3R_1} \end{array} \left[\begin{array}{rrr|r} \boxed{1} & 2 & 1 & 3 \\ 0 & \boxed{1} & -1 & -4 \\ 0 & 2 & 5 & 13 \end{array}\right].$$

The second pivot is now highlighted as well. We want to use the second pivot to produce zeros below it. This means we want to add/subtract multiples of the second row to the other rows. We will do $R_3 - 2R_2$ to get a zero below the second pivot. If you are wondering what's wrong with doing $R_3 - R_1$ instead to get a zero below the second pivot, allow me to illustrate real quick:

$$\left[\begin{array}{rrr|r} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -4 \\ 0 & 2 & 5 & 13 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{rrr|r} 1 & 2 & 1 & 3 \\ 0 & 1 & -1 & -4 \\ -1 & 0 & 4 & 10 \end{array}\right].$$

Notice that we lost a zero in the first column, so we basically undid what we did in the previous step. On the other hand, if we go back and do $R_3 - 2R_2$, we get:

$$\left[\begin{array}{rrr|r} \boxed{1} & 2 & 1 & 3 \\ 0 & \boxed{1} & -1 & -4 \\ 0 & 2 & 5 & 13 \end{array}\right] \xrightarrow{R_3 - 2R_2} \left[\begin{array}{rrr|r} \boxed{1} & 2 & 1 & 3 \\ 0 & \boxed{1} & -1 & -4 \\ 0 & 0 & \boxed{7} & 21 \end{array}\right].$$

Now the augmented matrix is in row echelon form. We write out the corresponding system of equations now:

$$\begin{aligned} x + 2y + z &= 3 \\ y - z &= -4 \\ 7z &= 21. \end{aligned}$$

Solving for z in the third equation gives us $z = 3$. Substituting $z = 3$ into the second equation and solving for y gives us $y = -1$. Substituting $y = -1$ and $z = 3$ into the first equation and solving for x gives us $x = 2$. ◇

Example. Solve the following linear system of equations using elimination:

$$\begin{aligned} -2x + 4y - 8z &= 8 \\ 3x + 2y + 3z &= 13 \\ x \qquad + 2z &= 2 \end{aligned}$$

Solution. Again, start by setting up the augmented matrix:

$$\left[\begin{array}{rrr|r} -2 & 4 & -8 & 8 \\ 3 & 2 & 3 & 13 \\ 1 & 0 & 2 & 2 \end{array}\right].$$

We could use the -2 as a pivot to produce zeros in the first column. However, to get rid of the 3 below the -2, we'd have to do $R_2 + \frac{3}{2}R_1$, and an operation like this has the potential to introduce fractions everywhere (although it wouldn't in this particular example). Elimination tends to be easiest when the pivots are equal to 1, so I want to illustrate some ways of getting a pivot to equal 1. In this example, we can simply swap Row 1 and Row 3:

$$\left[\begin{array}{rrr|r} -2 & 4 & -8 & 8 \\ 3 & 2 & 3 & 13 \\ 1 & 0 & 2 & 2 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{rrr|r} 1 & 0 & 2 & 2 \\ 3 & 2 & 3 & 13 \\ -2 & 4 & -8 & 8 \end{array}\right].$$

Next, use the pivot to make everything below it zero:

$$\left[\begin{array}{rrr|r} 1 & 0 & 2 & 2 \\ 3 & 2 & 3 & 13 \\ -2 & 4 & -8 & 8 \end{array}\right] \begin{array}{c} \xrightarrow{R_2 - 3R_1} \\ \xrightarrow{R_3 + 2R_1} \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 2 & 2 \\ 0 & 2 & -3 & 7 \\ 0 & 4 & -4 & 12 \end{array}\right].$$

Use the second pivot to produce zeros below it:

$$\left[\begin{array}{rrr|r} 1 & 0 & 2 & 2 \\ 0 & 2 & -3 & 7 \\ 0 & 4 & -4 & 12 \end{array}\right] \xrightarrow{R_3 - 2R_2} \left[\begin{array}{rrr|r} 1 & 0 & 2 & 2 \\ 0 & 2 & -3 & 7 \\ 0 & 0 & 2 & -2 \end{array}\right].$$

Now that we've reduced the augmented matrix to row echelon form, we write out the corresponding system of equations and solve via back-substitution:

$$\begin{aligned} x \qquad + 2z &= 2 \\ 2y - 3z &= 7 \\ 2z &= -2. \end{aligned}$$

The third equation gives us $z = -1$. Substitute this into the second equation to get $y = 2$. Substitute everything into the first equation to get $x = 4$. ◇

Example. Solve the following linear system of equations using elimination:

$$\begin{aligned} 2x \phantom{{}-y} - z &= 3 \\ 3x - y + z &= 2 \\ 2x + 2y + z &= 1. \end{aligned}$$

Solution. Set up the augmented matrix:

$$\left[\begin{array}{rrr|r} 2 & 0 & -1 & 3 \\ 3 & -1 & 1 & 2 \\ 2 & 2 & 1 & 1 \end{array}\right].$$

Again, we could use the 2 as a pivot, but we'd have to do $R_2 - \frac{3}{2}R_1$ to produce a zero below it, and in general, this has the potential to introduce fractions everywhere. In order to avoid fractions, let's do $R_2 - R_1$ to get a 1 in the first column, and then we'll swap rows so we can use the 1 as a pivot:

$$\left[\begin{array}{rrr|r} 2 & 0 & -1 & 3 \\ 3 & -1 & 1 & 2 \\ 2 & 2 & 1 & 1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{rrr|r} 2 & 0 & -1 & 3 \\ 1 & -1 & 2 & -1 \\ 2 & 2 & 1 & 1 \end{array}\right] \xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|r} 1 & -1 & 2 & -1 \\ 2 & 0 & -1 & 3 \\ 2 & 2 & 1 & 1 \end{array}\right].$$

Next, use the pivot to produce zeros below it.

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & -1 \\ 2 & 0 & -1 & 3 \\ 2 & 2 & 1 & 1 \end{array}\right] \begin{array}{c} \xrightarrow{R_2 - 2R_1} \\ \xrightarrow{R_3 - 2R_1} \end{array} \left[\begin{array}{rrr|r} 1 & -1 & 2 & -1 \\ 0 & 2 & -5 & 5 \\ 0 & 4 & -3 & 3 \end{array}\right].$$

Next, use the second pivot to produce zeros below it.

$$\left[\begin{array}{rrr|r} 1 & -1 & 2 & -1 \\ 0 & 2 & -5 & 5 \\ 0 & 4 & -3 & 3 \end{array}\right] \xrightarrow{R_3 - 2R_2} \left[\begin{array}{rrr|r} 1 & -1 & 2 & -1 \\ 0 & 2 & -5 & 5 \\ 0 & 0 & 7 & -7 \end{array}\right].$$

Now that we have reduced the augmented matrix to row echelon form, we write out the corresponding system of equations and back-substitute to solve:

$$\begin{aligned} x - y + 2z &= -1 \\ 2y - 5z &= 5 \\ 7z &= -7. \end{aligned}$$

The third equation gives us $z = -1$. Substitute this into the second equation to get $y = 0$. Substitute everything into the first equation to get $x = 1$. ◇

Example. Is the vector

$$\begin{bmatrix} -7 \\ -2 \\ 0 \end{bmatrix}$$

a linear combination of the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}?$$

Solution. We need to check and see if there exists scalars x, y, z such that

$$x \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} + y \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} + z \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \\ 0 \end{bmatrix}.$$

Equivalently, we need to check and see if there exists a solution to the following system of equations:

$$\begin{aligned} x + 2y - 2z &= -7 \\ 2x + 4y \phantom{{}-2z} &= -2 \\ x + 3y + z &= 0. \end{aligned}$$

We'll solve this using elimination. Set up the augmented matrix:

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & -7 \\ 2 & 4 & 0 & -2 \\ 1 & 3 & 1 & 0 \end{array}\right].$$

Use the pivot to produce zeros below it.

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & -7 \\ 2 & 4 & 0 & -2 \\ 1 & 3 & 1 & 0 \end{array}\right] \xrightarrow[R_3 - R_1]{R_2 - 2R_1} \left[\begin{array}{ccc|c} 1 & 2 & -2 & -7 \\ 0 & 0 & 4 & 12 \\ 0 & 1 & 3 & 7 \end{array}\right].$$

We can't do anything with a zero pivot, so we'll swap rows to get a nonzero pivot:

$$\left[\begin{array}{ccc|c} 1 & 2 & -2 & -7 \\ 0 & 0 & 4 & 12 \\ 0 & 1 & 3 & 7 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 2 & -2 & -7 \\ 0 & 1 & 3 & 7 \\ 0 & 0 & 4 & 12 \end{array}\right].$$

Now that we've reduced the augmented matrix to row echelon form, we write the corresponding system of equations and back-substitute to solve:

$$\begin{aligned} x + 2y - 2z &= -7 \\ y + 3z &= 7 \\ 4z &= 12. \end{aligned}$$

The third equation gives us $z = 3$. Substitute this into the second equation to get $y = -2$. Substitute everything into the first equation to get $x = 3$.

Since $x = 3$, $y = -2$, and $z = 3$, it follows that

$$3 \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ 4 \\ 3 \end{bmatrix} + 3 \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -7 \\ -2 \\ 0 \end{bmatrix}.$$

Thus, the vector $(-7, -2, 0)$ is a linear combination of the vectors $(1, 2, 1)$, $(2, 4, 3)$, and $(-2, 0, 1)$. $\diamond$

Example. Are the vectors
$$\begin{bmatrix} 2 \\ 3 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ -2 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 1 \\ 2 \\ -5 \\ 8 \end{bmatrix}$$
linearly independent?

Solution. The vectors are linearly independent if and only if the only solution to the homogeneous equation
$$x_1 \begin{bmatrix} 2 \\ 3 \\ -2 \\ 4 \end{bmatrix} + x_2 \begin{bmatrix} 1 \\ 1 \\ 3 \\ -2 \end{bmatrix} + x_3 \begin{bmatrix} 1 \\ 2 \\ -5 \\ 8 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$
is the trivial solution where $x_1 = x_2 = x_3 = 0$. We rewrite the above vector equation as the following linear system of equations:
$$\begin{aligned} 2x_1 + x_2 + x_3 &= 0 \\ 3x_1 + x_2 + 2x_2 &= 7 \\ -2x_1 + 3x_2 - 5x_3 &= 0 \\ 4x_1 - 2x_2 + 8x_3 &= 0. \end{aligned}$$
We'll solve this system using elimination. Set up the augmented matrix:
$$\left[\begin{array}{rrr|r} 2 & 1 & 1 & 0 \\ 3 & 1 & 2 & 0 \\ -2 & 3 & -5 & 0 \\ 4 & -2 & 8 & 0 \end{array}\right].$$
In order to avoid fractions, let's do $R_2 - R_1$ to get a 1 in the first column, and then we'll swap rows so that we can use the 1 as a pivot.
$$\left[\begin{array}{rrr|r} 2 & 1 & 1 & 0 \\ 3 & 1 & 2 & 0 \\ -2 & 3 & -5 & 0 \\ 4 & -2 & 8 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{rrr|r} 2 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ -2 & 3 & -5 & 0 \\ 4 & -2 & 8 & 0 \end{array}\right]$$
$$\xrightarrow{R_1 \leftrightarrow R_2} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ -2 & 3 & -5 & 0 \\ 4 & -2 & 8 & 0 \end{array}\right].$$
Now that we have a nice pivot, we'll use it to make everything below it zero:
$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 2 & 1 & 1 & 0 \\ -2 & 3 & -5 & 0 \\ 4 & -2 & 8 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{R_2 - 2R_1} \\ \xrightarrow{R_3 + 2R_1} \\ \xrightarrow{R_4 - 4R_1} \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -2 & 4 & 0 \end{array}\right].$$
Use the second pivot to make everything below it zero:
$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 3 & -3 & 0 \\ 0 & -2 & 4 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{R_3 - 3R_2} \\ \xrightarrow{R_4 + 2R_2} \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right].$$
If want our matrix in row echelon form, then all rows of zeros should appear on the bottom. So we should swap rows 3 and 4:
$$\left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 2 & 0 \end{array}\right] \xrightarrow{R_3 \leftrightarrow R_4} \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Now that we've reduced the augmented matrix to row echelon form, we write out the corresponding system of equations and do the usual back-substitution to solve.

$$\begin{aligned} x_1 \qquad + \ x_3 &= 0 \\ x_2 - \ x_3 &= 0 \\ 2x_3 &= 0. \end{aligned}$$

Note that the fourth equation was just $0 = 0$, so we didn't bother to write it. Solving for x_3 in the third equation gives us $x_3 = 0$. Substituting this into the second equation gives us $x_2 = 0$. Substituting $x_2 = 0$ and $x_3 = 0$ into the first equation gives us $x_1 = 0$.

Thus, the only solution to the homogeneous equation was the trivial solution. It follows that the vectors

$$\begin{bmatrix} 2 \\ 3 \\ -2 \\ 4 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 3 \\ -2 \end{bmatrix}, \text{ and } \begin{bmatrix} 1 \\ 2 \\ -5 \\ 8 \end{bmatrix}$$

are linearly independent. ◇

2.3.2 Case 2: No solution

Example. Is the vector

$$\begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}$$

a linear combination of the vectors

$$\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix}, \quad \text{and} \quad \begin{bmatrix} 3 \\ -2 \\ -7 \end{bmatrix}?$$

Solution. We need to check if there exists a solution to the following equation:

$$x_1 \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 4 \end{bmatrix} + x_3 \begin{bmatrix} 3 \\ -2 \\ -7 \end{bmatrix} = \begin{bmatrix} 2 \\ 3 \\ 6 \end{bmatrix}.$$

Equivalently, we need to check if there exists a solution to the following linear system of equations:

$$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 2 \\ 2x_1 + x_2 - 2x_3 &= 3 \\ 3x_1 + 4x_2 - 7x_3 &= 6. \end{aligned}$$

We'll set up the augmented matrix and perform elimination:

$$\left[\begin{array}{rrr|r} 1 & -2 & 3 & 2 \\ 2 & 1 & -2 & 3 \\ 3 & 4 & -7 & 6 \end{array}\right] \xrightarrow[R_3 - 3R_1]{R_2 - 2R_1} \left[\begin{array}{rrr|r} 1 & -2 & 3 & 2 \\ 0 & 5 & -8 & -1 \\ 0 & 10 & -16 & 0 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{rrr|r} 1 & -2 & 3 & 2 \\ 0 & 5 & -8 & -1 \\ 0 & 0 & 0 & 2 \end{array}\right].$$

Looking at the row echelon form, notice that the third row gives us the equation $0x_1+0x_2+0x_3 = 2$. Clearly there is no solutions to this equation, and thus, this system of equations has no solution. It follows that the vector $(2, 3, 6)$ is not a linear combination of the vectors $(1, 2, 3)$, $(-2, 1, 4)$, and $(3, -2, -7)$. ◇

Basically, if a linear system of equations has no solution, then after reducing the augmented matrix to row echelon form, you will always end up with an equation of the form "zero equals something nonzero". In particular then, there must be at least one row of zeros in the row echelon form of the coefficient matrix. It follows that if there are no rows of zeros in the row echelon form of the coefficient matrix, then there will be at least one solution.

2.3.3 Case 3: Infinitely Many Solutions

It will be important to know what to do when there are infinitely many solutions. This will be the situation that you'll encounter the most in the latter half of the semester. We'll take a look at some examples first.

Example. Consider the following linear system of equations:

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ x_1 \phantom{{}- x_2} - 2x_3 &= -1. \end{aligned}$$

Let's set up the augmented matrix and row reduce:

$$\left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 1 & 0 & -2 & -1 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & -1 & 2 & 1 \\ 0 & 1 & -4 & -2 \end{array}\right].$$

Now that the augmented matrix has been reduced to row echelon form, we write out the corresponding system of equations:

$$\begin{aligned} x_1 - x_2 + 2x_3 &= 1 \\ x_2 - 4x_3 &= -2. \end{aligned}$$

What's happening is that we have two equations, so we can only solve for two of the three variables. It is customary to solve for the variables corresponding to the pivots in the row echelon form; meaning, since the pivots are located in the first and second column, we will choose to solve for x_1 and x_2. Solving for x_2 in the second equation gives us

$$x_2 = -2 + 4x_3.$$

Substitute this into the first equation to get

$$x_1 + 2 - 2x_3 = 1 \implies x_1 = -1 + 2x_3.$$

What about x_3? Well, we don't have an equation telling us what x_3 must be equal to, so what is happening is that there is no restriction on what x_3 can be. We express this by saying that x_3 is a *free variable*. It is customary to introduce a new variable, called a parameter, to represent each free variable. Let's put $x_3 = t$. Then plugging $x_3 = t$ into the above expressions for x_1 and x_2 gives us the following solution:

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 + 2t \\ -2 + 4t \\ t \end{bmatrix}.$$

The parameter t can be any number. Hence, we are getting infinitely many solutions, one for each value of t. You can check this by plugging $x_1 = -1 + 2t$, $x_2 = -2 + 4t$, and $x_3 = t$ in the the original system of equations to see that the equations are satisfied, regardless of what t is. ◇

Example. Consider the following linear system of equations:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ 2x_1 + 4x_2 - x_3 + x_4 &= 3 \\ 3x_1 + 6x_2 - x_3 + 2x_4 &= 4. \end{aligned}$$

We'll set up the augmented matrix and row reduce:

$$\left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 2 & 4 & -1 & 1 & 3 \\ 3 & 6 & -1 & 2 & 4 \end{array}\right] \begin{array}{c} \xrightarrow{R_2 - 2R_1} \\ \xrightarrow{R_3 - 3R_1} \end{array} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 2 & 2 & -2 \end{array}\right]$$

$$\xrightarrow{R_3 - 2R_2} \left[\begin{array}{cccc|c} 1 & 2 & -1 & 0 & 2 \\ 0 & 0 & 1 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

Now that we've reduced to row echelon form, let's write out the corresponding system of equations:

$$\begin{aligned} x_1 + 2x_2 - x_3 &= 2 \\ x_3 + x_4 &= -1. \end{aligned}$$

We end up with only two equations, which means we can only solve for two of the four variables; the other two variables will be free variables. Once again, we will follow the convention of solving for the variables corresponding to the pivots. In this case, the pivots are in Columns 1 and 3, so we will solve for x_1 and x_3. This makes x_2 and x_4 our free variables. From the second equation, we get

$$x_3 = -1 - x_4.$$

Substituting this into the first equation and solving for x_1 gives us

$$x_1 = 1 - 2x_2 - x_4.$$

Let's also parametrize our free variables by setting

$$x_2 = s \qquad x_4 = t.$$

Then plugging these in to the above expressions for x_1 and x_3 gives us

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 - 2s - t \\ s \\ -1 - t \\ t \end{bmatrix}.$$

This represents the general solution to the original linear system of equations. In particular, we are getting infinitely many solutions, since the parameters s and t can be any number.

By the way, notice that we can rewrite the general solution as

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix} + s \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ -1 \\ 1 \end{bmatrix}.$$

Rewriting it in this way helps us to visualize what the set of solutions looks like; namely, this tells us that the set of solutions to the original system of equations is the plane spanned by the vectors $(-2, 1, 0, 0)$ and $(-1, 0, -1, 1)$, shifted to pass through the point $(1, 0, -1, 0)$. ◇

Let's summarize the procedure and make a few comments regarding when there are infinitely many solutions to a linear system of equations.

The procedure is to set up the augmented matrix and row reduce to row echelon form, as usual. Then, write out the system of equations corresponding to the row echelon form. Solve for the variables corresponding to the pivots; the other variables are free variables. Assign a parameter to each free variable, and express the solution in terms of the parameters.

Now for some comments. In order to have infinitely many solutions, there must be at least one free variable. The variables that we solve for correspond to the columns with pivots, and the free variables correspond to the columns without pivots. Thus, we have the following:

Theorem.

- If a linear system of equations has a solution, then it will have infinitely many solutions if and only if the coefficient matrix has at least one column without a pivot, since then there will be at least one free variable.
- If a linear system of equations has a solution, then the solution is unique if and only if every column of the coefficient matrix has a pivot, since then there will be no free variables.

Example. Are the vectors

$$\begin{bmatrix}1\\2\\0\\1\end{bmatrix}, \begin{bmatrix}0\\1\\2\\0\end{bmatrix}, \quad \text{and} \quad \begin{bmatrix}2\\5\\2\\2\end{bmatrix}$$

linearly independent?

Solution. The vectors are linearly independent if and only if the only solution to the homogeneous equation

$$\begin{aligned} x_1 \qquad\quad + 2x_3 &= 0 \\ 2x_1 + \ \ x_2 + 5x_3 &= 0 \\ + 2x_2 + 2x_3 &= 0 \\ x_1 + \qquad + 2x_3 &= 0. \end{aligned}$$

is the trivial solution where $x_1 = x_2 = x_0$. Let's take the coefficient matrix and row reduce it:

$$\begin{bmatrix}1 & 0 & 2\\2 & 1 & 5\\0 & 2 & 2\\1 & 0 & 2\end{bmatrix} \begin{array}{c}\xrightarrow{R_2-2R_1} \\ \\ \xrightarrow{R_4-R_1}\end{array} \begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 2 & 2\\0 & 0 & 0\end{bmatrix} \xrightarrow{R_3-2R_2} \begin{bmatrix}1 & 0 & 2\\0 & 1 & 1\\0 & 0 & 0\\0 & 0 & 0\end{bmatrix}.$$

Notice that the third column is missing a pivot, which means we will have a free variable. This means we will have infinitely many solutions. In particular, there are nontrivial solutions to the homogeneous equation, which means that the vectors are not linearly independent. ◇

Generalizing from this example, we have the following

Theorem. Let $\mathbf{a}_1, \ldots, \mathbf{a}_k$ be vectors in $\mathbb{R}^n$, and let A be the $n \times k$ matrix whose first column is $\mathbf{a}_1$, second column is $\mathbf{a}_2$, etc. Then the vectors $\mathbf{a}_1, \ldots, \mathbf{a}_k$ are linearly independent if and only if every column has a pivot. They will be linearly dependent otherwise.

2.4 Matrix-Vector Multiplication

We've seen that the linear system of equations

$$\begin{aligned} a_{11}x_1 + \; a_{12}x_2 + \ldots + \; a_{1n}x_n &= b_1 \\ a_{21}x_1 + \; a_{22}x_2 + \ldots + \; a_{2n}x_n &= b_2 \\ \vdots \qquad \vdots \qquad \ddots \qquad \vdots \quad &\;\; \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ldots + a_{mn}x_n &= b_m \end{aligned} \tag{2.1}$$

can be represented by the following vector equation:

$$x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}, \tag{2.2}$$

and also by the following augmented matrix:

$$\left[\begin{array}{cccc|c} a_{11} & a_{12} & \ldots & a_{1n} & b_1 \\ a_{21} & a_{22} & \ldots & a_{2n} & b_2 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} & b_m \end{array}\right].$$

We are now going to consider yet another way of representing (2.1). Let

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

We are going to define matrix-vector multiplication so that (2.1) can be represented as:

$$A\mathbf{x} = \mathbf{b}.$$

Note the sizes of all of the matrices right now. A is size $m \times n$ since it has m rows and n columns, $\mathbf{x}$ is size $n \times 1$, and $\mathbf{b}$ is size $m \times 1$.

$$\underbrace{A}_{m\times n} \underbrace{\mathbf{x}}_{n\times 1} = \underbrace{\mathbf{b}}_{m\times 1}.$$

Since we want $A\mathbf{x} = \mathbf{b}$ to be equivalent (2.2), it looks like we should have

$$A\mathbf{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

Let's make all of this discussion official now.

Definition (Matrix-Vector Multiplication)**.** Given an $m \times n$ matrix

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix}$$

and an $n \times 1$ vector

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

we define the product $A\mathbf{x}$ to be the $m \times 1$ vector given by

$$A\mathbf{x} = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + \ldots + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}.$$

In words, $A\mathbf{x}$ is equal to x_1 times the first column of A, plus x_2 times the second column of A, plus etc.

One remark before the examples. Matrix-vector multiplication is only defined when A is $m \times n$ and $\mathbf{x}$ is $n \times 1$, i.e., it is only defined when the number of columns in A is equal to the number of entries in $\mathbf{x}$.

Example. If

$$A = \begin{bmatrix} 1 & 2 & -1 \\ 2 & -3 & 0 \\ -4 & 3 & 1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ 5 \end{bmatrix},$$

then

$$A\mathbf{x} = 3 \begin{bmatrix} 1 \\ 2 \\ -4 \end{bmatrix} - 2 \begin{bmatrix} 2 \\ -3 \\ 3 \end{bmatrix} + 5 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} -6 \\ 12 \\ -13 \end{bmatrix}.$$

◇

Example. If

$$A = \begin{bmatrix} 1 & 0 & -1 & 3 \\ 5 & -2 & 4 & 2 \\ 2 & 1 & -3 & 0 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

then the product $A\mathbf{x}$ is undefined, because the number of columns of A does not match the number of entries in $\mathbf{x}$. ◇

Example. If

$$A = \begin{bmatrix} 1 & 0 & -1 & 3 & 6 \\ 5 & -2 & 4 & 2 & 0 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ -1 \\ -3 \\ 2 \end{bmatrix},$$

then

$$A\mathbf{x} = 2 \begin{bmatrix} 1 \\ 5 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ -2 \end{bmatrix} - \begin{bmatrix} -1 \\ 4 \end{bmatrix} - 3 \begin{bmatrix} 3 \\ 2 \end{bmatrix} + 2 \begin{bmatrix} 6 \\ 0 \end{bmatrix} = \begin{bmatrix} 6 \\ -6 \end{bmatrix}.$$

◇

Before looking at a few more examples, let's take a moment to make an observation. According to our definition of matrix-vector multiplication, if

$$A = \overbrace{[\mathbf{a}_1 \ \mathbf{a}_2 \ \ldots \ \mathbf{a}_n]}^{\text{columns of } A} \qquad \text{and} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

then

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n.$$

Recall that the j^{th} standard basis vector of $\mathbb{R}^n$, denoted $\mathbf{e}_j$, is the vector whose j^{th} coordinate is 1, with the remaining coordinates equal to 0. It follows that

$$A\mathbf{e}_j = 0\mathbf{a}_1 + \ldots + 1\mathbf{a}_j + \ldots + 0\mathbf{a}_n \qquad \Longleftrightarrow \qquad A\mathbf{e}_j = \mathbf{a}_j.$$

In words, A times the j^{th} standard basis vector returns the j^{th} column of A. This is a very useful fact.

Another way to compute $A\mathbf{x}$ is the following: the i^{th} entry of the product is equal to the i^{th} row of A, dot product with the vector $\mathbf{x}$.

Example. Using the previous example where

$$A = \begin{bmatrix} 1 & 0 & -1 & 3 & 6 \\ 5 & -2 & 4 & 2 & 0 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} 2 \\ 3 \\ -1 \\ -3 \\ 2 \end{bmatrix},$$

the first entry of the product $A\mathbf{x}$ is equal to the first row of A, dot product with $\mathbf{x}$:

$$(1^{\text{st}} \text{ Row of } A) \cdot \mathbf{x} = 1(2) + 0(3) - 1(-1) + 3(-3) + 6(2) = 6.$$

The second entry of the product $A\mathbf{x}$ is equal to the second row of A, dot product with $\mathbf{x}$:

$$(2^{\text{nd}} \text{ Row of } A) \cdot \mathbf{x} = 5(2) - 2(3) + 4(-1) + 2(-3) + 0(2) = -6.$$

Thus, we find that

$$A\mathbf{x} = \begin{bmatrix} 6 \\ -6 \end{bmatrix},$$

which is the same as what we found in the previous example. ◇

Example. Consider the following linear system of equations.

$$\begin{aligned} 2x_1 - 3x_2 + x_3 \phantom{{}+6x_4} - 7x_5 &= 4 \\ 5x_1 + 4x_2 - 2x_3 + 6x_4 - x_5 &= -1 \\ - x_2 + 4x_3 - 7x_4 \phantom{{}-7x_5} &= 0 \\ -8x_1 + 5x_2 - 7x_3 + x_4 \phantom{{}-7x_5} &= 3 \\ x_1 + - x_3 - 9x_4 + 6x_5 &= -8. \end{aligned}$$

The matrix of coefficients is given by

$$\begin{bmatrix} 2 & -3 & 1 & 0 & -7 \\ 5 & 4 & -2 & 6 & -1 \\ 0 & -1 & 4 & -7 & 0 \\ -8 & 5 & -7 & 1 & 0 \\ 1 & 0 & -1 & -9 & 6 \end{bmatrix}.$$

So the above linear system of equations can be rewritten as the following matrix equation:

$$\begin{bmatrix} 2 & -3 & 1 & 0 & -7 \\ 5 & 4 & -2 & 6 & -1 \\ 0 & -1 & 4 & -7 & 0 \\ -8 & 5 & -7 & 1 & 0 \\ 1 & 0 & -1 & -9 & 6 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} 4 \\ -1 \\ 0 \\ 3 \\ -8 \end{bmatrix}.$$ ◇

Example. The following linear system of equations

$$\begin{aligned} 2x_1 + x_2 - 4x_3 &= 2 \\ -3x_1 + 2x_2 + 5x_3 &= 5 \\ -3x_2 + x_3 &= -4 \\ 5x_1 - 8x_2 - 7x_3 &= 1 \end{aligned}$$

can be rewritten as the following matrix equation:

$$\begin{bmatrix} 2 & 1 & -4 \\ -3 & 2 & 5 \\ 0 & -3 & 1 \\ 5 & -8 & -7 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 5 \\ -4 \\ 1 \end{bmatrix}. \qquad \diamond$$

We'll end this section by mentioning how matrix-vector multiplication behaves with vector addition and scalar multiplication. These two properties are very important to keep in mind!

Proposition. Let A be an $m \times n$ matrix.

- For any two vectors $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$,
$$A(\mathbf{x} + \mathbf{y}) = A\mathbf{x} + A\mathbf{y}.$$
- For any vector $\mathbf{x}$ in $\mathbb{R}^n$, and for any scalar c,
$$A(c\mathbf{x}) = c(A\mathbf{x}).$$

Closing Remarks

For us, linear algebra is primarily going to be the study of three objects: vectors, linear systems of equations, and matrices. At this point, we've introduced all three of these objects. Most of the rest of the semester will pretty much involve studying the connections between these three objects.

In the next couple of notes, we will study matrices a bit more in depth. In particular, we will learn how to multiply two matrices together.

2.5 Review

We have seen that the general linear system of equations

$$\begin{aligned} a_{11}x_1 + \ a_{12}x_2 + \ \ldots + \ a_{1n}x_n &= b_1 \\ a_{21}x_1 + \ a_{22}x_2 + \ \ldots + \ a_{2n}x_n &= b_2 \\ \vdots \qquad\quad \vdots \qquad \ddots \qquad\quad \vdots \ \ &\ \ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \ \ldots + a_{mn}x_n &= b_m \end{aligned}$$

can be represented as the matrix equation

$$A\mathbf{x} = \mathbf{b},$$

where

$$A = \begin{bmatrix} a_{11} & a_{12} & \ldots & a_{1n} \\ a_{21} & a_{22} & \ldots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \ldots & a_{mn} \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{bmatrix}.$$

We have also seen how to solve a linear system of equations using elimination. Remember that there are three possibilities: we'll either have no solution, infinitely many solutions, or a unique solution.

In this section, we will take a look at how the size of A relates to: whether or not the equation $A\mathbf{x} = \mathbf{b}$ has a solution for all possible $\mathbf{b}$; and, if it is solvable, whether it will have a unique solution or infinitely many solutions.

2.5.1 When is $A\mathbf{x} = \mathbf{b}$ NOT Always Solvable?

Let's take the linear system of equations

$$\begin{aligned} x + \ \ y &= b_1 \\ 2x + 2y &= b_2 \end{aligned}$$

and perform elimination:

$$\left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 2 & 2 & b_2 \end{array}\right] \xrightarrow{R_2 - 2R_1} \left[\begin{array}{cc|c} 1 & 1 & b_1 \\ 0 & 0 & b_2 - 2b_1 \end{array}\right].$$

From the second row, we end up with the equation

$$0 = b_2 - 2b_1.$$

This equation is only solvable if $b_2 - 2b_1 = 0$. Otherwise, the system of equations is inconsistent (i.e. has no solution). In particular then, the equation $A\mathbf{x} = \mathbf{b}$ is not always solvable when

$$A = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix}.$$

What happened in the above example is that the row echelon form of A had a row of zeroes in it. More generally, if any row echelon form of A has a zero in it, then the equation $A\mathbf{x} = \mathbf{b}$ will not always be solvable, i.e., it will be inconsistent for some values of $\mathbf{b}$. On the other hand, if there are no rows of zeros in any row echelon form of A, then the equation $A\mathbf{x} = \mathbf{b}$ will always be consistent. We state this result formally below.

Theorem. The equation $A\mathbf{x} = \mathbf{b}$ is not always solvable (in the sense that there will be no solution for some values of $\mathbf{b}$) if and only if there is at least one row of zeros in any row echelon form of A.

Theorem. The equation $A\mathbf{x} = \mathbf{b}$ is always solvable (in the sense that there is always a solution no matter what $\mathbf{b}$ is) if and only if there are no rows of zeros in any row echelon form of A.

So, when is a row echelon form of A guaranteed to have at least one row of zeros? Consider the following examples of 5×4 matrices in REF:

$$\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \qquad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Since there can be at most one pivot in each column, a 5×4 matrix can have at most 4 pivots. This means that there will always be at least one row without a pivot. Notice that in any REF of any matrix, a row without a pivot is going to be a row of zeros. Hence, any REF of a 5×4 matrix will always have at least one row of zeros.

More generally, we can apply the same logic to conclude that if A is $m \times n$ and has more rows than columns (i.e. $m > n$), then there will be at least one row of zeros in its REF. Hence, if $m > n$, we can conclude that the equation $A\mathbf{x} = \mathbf{b}$ is not always solvable. We'll state this result formally below.

Theorem. If A is an $m \times n$ matrix with $m > n$, then the general equation $A\mathbf{x} = \mathbf{b}$ is not always solvable, in the sense that there are values of $\mathbf{b}$ for which the equation has no solution.

Note: We are NOT saying that if $m \leq n$, that the equation $A\mathbf{x} = \mathbf{b}$ will always be solvable. This is not true, as it is still possible that you will get a row of zeros when row reducing.

2.5.2 When Does $A\mathbf{x} = \mathbf{b}$ Have Infinitely Many Solutions?

This was already addressed in Section ???, when we were working through examples with infinitely many solutions. Let us briefly summarize that discussion here; you may want to look back at those notes though if you don't remember. Basically, there will be infinitely many solutions if and only if we have at least one free variable, and each column without a pivot produces a free variable. This led us the the following results.

Theorem. If the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then it will have infinitely many solutions if and only if at least one column is missing a pivot in any row echelon form of A.

Theorem. If the equation $A\mathbf{x} = \mathbf{b}$ is consistent, then it will have a unique solution if and only if every column has a pivot in any row echelon form of A.

So, when is a matrix A guaranteed to have at least one column without a pivot? Consider the following example of a 4×5 matrix in REF:

$$\begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{bmatrix}.$$

Notice that there can be at most one pivot in each row in any REF of a matrix. Thus, a 4×5 matrix can have at most 4 pivots, which means there will always be at least one column without a pivot.

More generally, we can apply the same logic to conclude that if A is $m \times n$ with more columns than rows (i.e. $n > m$), then there will always be at least one column without a pivot. Hence, if $n > m$, then if the equation $A\mathbf{x} = \mathbf{b}$ is consistent, it will have infinitely many solutions. We state this result formally below.

Theorem. If A is an $m \times n$ matrix with $n > m$, then the equation $A\mathbf{x} = \mathbf{b}$ will have infinitely many solutions whenever it is consistent.

Again, this is not saying that if $n \leq m$, that there will be a unique solution. That will depend on if every column of A ends up having a pivot or not.

Chapter 3

Linear Transformations

3.1 Introduction

The Signature of a Function

Consider the function $f(x) = x^2$. The input, x, is a real number, and the output, x^2, is also a real number. We express this with the following notation:

$$f : \mathbb{R} \to \mathbb{R}.$$

As another example, consider the function

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} y \\ x+y \\ x \end{bmatrix}.$$

The input, (x, y), is a vector in $\mathbb{R}^2$, and the output, $(y, x+y, x)$, is a vector in $\mathbb{R}^3$. We express this with the following notation:

$$f : \mathbb{R}^2 \to \mathbb{R}^3.$$

In general, the expression $f : \mathbb{R}^n \to \mathbb{R}^m$ is called the signature of the function f. It specifies that the domain of f, which is the set of all allowable inputs, is $\mathbb{R}^n$, while the codomain of f, which is the space where the outputs live, is $\mathbb{R}^m$.

Linear Transformations

In calculus, one is primarily interested in studying differentiable functions. Similarly, in linear algebra, we are primarily interested in studying a certain class of functions, which are called linear transformations (or linear maps).

Definition. A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is said to be a ***linear transformation***, or ***linear map***, if the following two conditions are satisfied:

- For every $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$, $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.
- For every $\mathbf{v}$ in $\mathbb{R}^n$, and for every scalar α, $T(\alpha\mathbf{v}) = \alpha\, T(\mathbf{v})$.

Example. The function

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} y \\ x+y \\ x \end{bmatrix}$$

given above is a linear map. We will prove this by showing that f satisfies the two defining conditions of a linear map.

The key is to notice that we can rewrite the formula for f as

$$f\begin{pmatrix} x \\ y \end{pmatrix} = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}, \quad \text{since} \quad \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ x+y \\ x \end{bmatrix}.$$

If we let

$$A = \begin{bmatrix} 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} x \\ y \end{bmatrix},$$

then the function f is given by $f(\mathbf{v}) = A\mathbf{v}$. Then, using the properties of matrix-vector multiplication given at the end of the previous notes,

$$f(\mathbf{u}+\mathbf{v}) = A(\mathbf{u}+\mathbf{v}) = A\mathbf{u} + A\mathbf{v} = f(\mathbf{u}) + f(\mathbf{v}),$$

and so the first condition is satisfied, and

$$f(c\mathbf{v}) = A(c\mathbf{v}) = c\,A\mathbf{v} = \alpha\, f(\mathbf{v}),$$

and so the second condition is satisfied as well. $\diamond$

Example. We can generalize the previous example as follows: if A is an $m \times n$ matrix, then the function $T : \mathbb{R}^n \to \mathbb{R}^m$ defined by $T(\mathbf{x}) = A\mathbf{x}$ is a linear map. Note that the sizes make sense: A is $m \times n$, the input $\mathbf{x}$ is $n \times 1$, and the output $A\mathbf{x}$ is $m \times 1$. $\diamond$

If you would like to see a different example, then I have to apologize, because that is actually not possible! It turns out that *there are no other examples*. More precisely, the example states that if A is an $m \times n$ matrix, then $T(\mathbf{x}) = A\mathbf{x}$ defines a linear map $T : \mathbb{R}^n \to \mathbb{R}^m$. We are saying that the converse is also true: namely, if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then there exists an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$. This will be the first theorem that we prove in these notes!

Theorem. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear transformation. Then there exists an $m \times n$ matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

Proof. The key to the proof is the following formula:

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} + \ldots + x_n \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}. \tag{3.1}$$

Let $\mathbf{e}_1, \mathbf{e}_2, \ldots, \mathbf{e}_n$ be the vectors in $\mathbb{R}^n$ given by

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix} \quad , \ldots, \quad \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

These vectors are called the standard basis vectors of $\mathbb{R}^n$. Then formula (3.1) can be rewritten as

$$\mathbf{x} = x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_n\mathbf{e}_n.$$

Thus, we can write

$$T(\mathbf{x}) = T(x_1\mathbf{e}_1 + x_2\mathbf{e}_2 + \ldots + x_n\mathbf{e}_n).$$

Using the linearity of T, we can distribute it to each term in the sum and factor out the scalars to get

$$T(\mathbf{x}) = x_1T(\mathbf{e}_1) + x_2T(\mathbf{e}_2) + \ldots + x_nT(\mathbf{e}_n).$$

Since $T : \mathbb{R}^n \to \mathbb{R}^m$, $T(\mathbf{e}_j)$ is a vector in $\mathbb{R}^m$. If we denote

$$T(\mathbf{e}_j) = \begin{bmatrix} a_{1j} \\ a_{2j} \\ \vdots \\ a_{mj} \end{bmatrix}$$

for $j = 1, 2, \ldots, n$, then equivalently, we can write

$$T(\mathbf{x}) = x_1 \begin{bmatrix} a_{11} \\ a_{21} \\ \vdots \\ a_{m1} \end{bmatrix} + x_2 \begin{bmatrix} a_{12} \\ a_{22} \\ \vdots \\ a_{m2} \end{bmatrix} + x_n \begin{bmatrix} a_{1n} \\ a_{2n} \\ \vdots \\ a_{mn} \end{bmatrix}. \tag{3.2}$$

The expression on the right is exactly how we defined matrix-vector multiplication! Namely, if we let

$$A = \begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix},$$

then (3.2) is equivalent to $T(\mathbf{x}) = A\mathbf{x}$. That completes the proof! $\square$

From the proof, notice that the first column of A is the vector $T(\mathbf{e}_1)$, the second column of A is the vector $T(\mathbf{e}_2)$, and so forth. This gives us an explicit formula for finding the matrix A from the transformation T.

Corollary. Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map. Then $T(\mathbf{x}) = A\mathbf{x}$, where A is the $m \times n$ matrix given by

$$A = [\, T(\mathbf{e}_1) \,|\, T(\mathbf{e}_2) \,|\, \cdots \,|\, T(\mathbf{e}_n) \,],$$

i.e., the j^{th} column of A is the vector $T(\mathbf{e}_j)$. This matrix is called the ***standard matrix*** of T.

By the way, the fact that the j^{th} column of A is $T(\mathbf{e}_j)$ shouldn't be surprising. Since $T(\mathbf{x}) = A\mathbf{x}$, $T(\mathbf{e}_j) = A\mathbf{e}_j$, and we actually mentioned in the section on matrix-vector multiplication how a matrix multliplied with $\mathbf{e}_j$ returns its j^{th} column.

In the next sections, we will use this formula to find the matrices of specific types of linear transformations.

Closing Remarks

Every linear transformation can be represented by some matrix. Thus, the study of linear transformations can be reduced to the study of matrices. In this class, we will almost exclusively be working with matrices rather than linear transformations. That's a bit unfortunate, in my opinion, but oh well.

Let me make one comment. A linear transformation T with signature $T : \mathbb{R}^n \to \mathbb{R}^n$ (note how both the domain and codomain are $\mathbb{R}^n$) is often called a linear *operator* on $\mathbb{R}^n$. Basically, the word operator is reserved for the special case where both the domain and codomain are the same. Not everyone uses this terminology, but many do.

3.2 Visualizing A Linear Transformation

Last time, we saw that any linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ has the form $T(\mathbf{x}) = A\mathbf{x}$ for some $m \times n$ matrix A.

Consider the function $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by

$$T(x, y) = \begin{bmatrix} x + 2y \\ y \end{bmatrix}.$$

This can be rewritten as

$$T(x, y) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}.$$

Thus, T has the form $T(\mathbf{x}) = A\mathbf{x}$, where

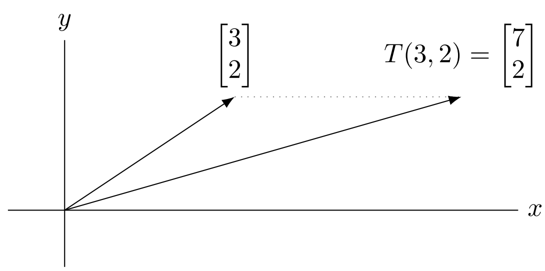
$$A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}.$$

Hence, it follows that T is a linear transformation.

Let's try to visualize what the transformation T is doing. Consider the following:

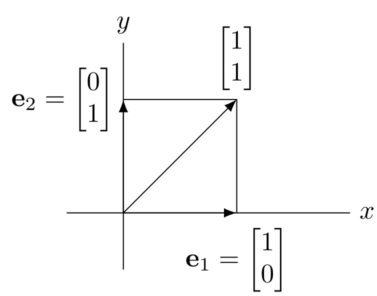
$$T(3, 2) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \end{bmatrix}.$$

Let's draw a picture of the input and the output:



We imagine that T is transforming the vector $(3, 2)$ into the vector $(7, 2)$. You can try plugging in other inputs besides $(3, 2)$ to see the effect of the transformation T.

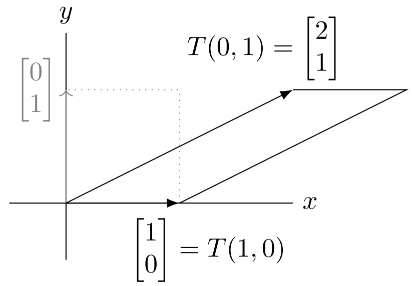
We can also try to see how the transformation T transforms areas. For example, consider the unit square determined by the vectors $\mathbf{e}_1$ and $\mathbf{e}_2$:



To see how the unit square gets transformed by T, we compute:

$$T(1, 0) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \qquad T(0, 1) = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}.$$

Let's plot the results:



We see that the unit square gets transformed into the parallelogram determined by the vectors $(1, 0)$ and $(2, 1)$.

Another Perspective On Matrix-Vector Multiplication

Since $T(\mathbf{x}) = A\mathbf{x}$, instead of thinking that the transformation T is transforming the vector $\mathbf{x}$, we can also think instead that the matrix A is transforming the vector $\mathbf{x}$. This gives us another way of looking at matrix-vector multiplication: multiplying A with $\mathbf{x}$ has the effect of transforming the vector $\mathbf{x}$ into the vector $A\mathbf{x}$.

Usually, when I look at a matrix, I don't just see a rectangular array of numbers. I usually think of matrices as operators that operate or act on vectors. Multiplying a vector by a matrix has the effect of transforming it in some way.

3.3 Rotations in $\mathbb{R}^2$

Review

The standard basis vectors of $\mathbb{R}^n$ are defined to be the vectors

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \mathbf{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \mathbf{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

In words, $\mathbf{e}_i$ is the vector whose coordinates are all 0, except the i^{th} coordinate which is 1.

A function $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map if it satisfies the following two conditions:

- For any two vector $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$, $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$.
- $T(c\mathbf{v}) = cT(\mathbf{v})$ for any scalar c and $\mathbf{v}$ in $\mathbb{R}^n$.

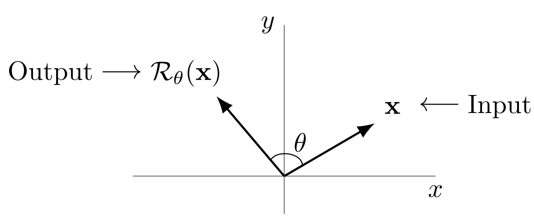
We proved in Section 3.1 that if $T : \mathbb{R}^n \to \mathbb{R}^m$ is a linear map, then there exists an $m \times n$ matrix A such that

$$T(\mathbf{x}) = A\mathbf{x}.$$

The matrix A is called the standard matrix of T. We also saw how to compute the matrix A if we know the transformation T: the first column of A is $T(\mathbf{e}_1)$; the second column of A is $T(\mathbf{e}_2)$; and so forth.

Rotations in $\mathbb{R}^2$

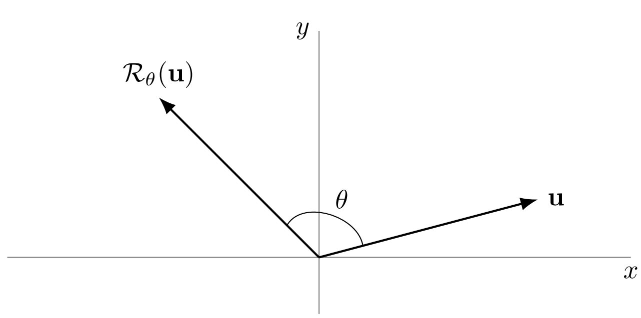
Let $\mathcal{R}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ be the function that takes a vector $\mathbf{x}$ in $\mathbb{R}^2$ as an input, and whose output $\mathcal{R}_\theta(\mathbf{x})$ is the vector $\mathbf{x}$ rotated counterclockwise by an angle of θ.



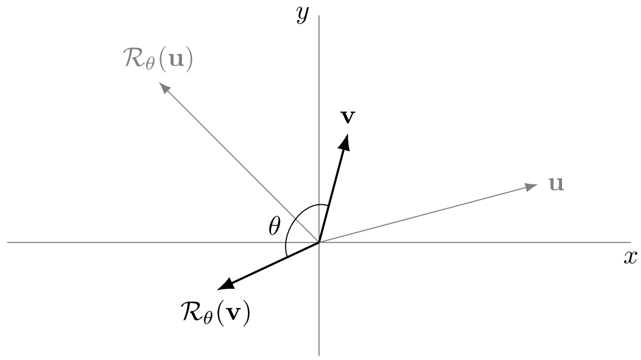
As you might have guessed (since we are talking about it), $\mathcal{R}_\theta$ is an example of a linear transformation. To see why, we will show using pictures that $\mathcal{R}_\theta$ satisfies the two defining conditions of a linear map:

$$\mathcal{R}_\theta(\mathbf{u} + \mathbf{v}) = \mathcal{R}_\theta(\mathbf{u}) + \mathcal{R}_\theta(\mathbf{v}) \qquad \text{and} \qquad \mathcal{R}_\theta(c\mathbf{v}) = c\,\mathcal{R}_\theta(\mathbf{v}).$$

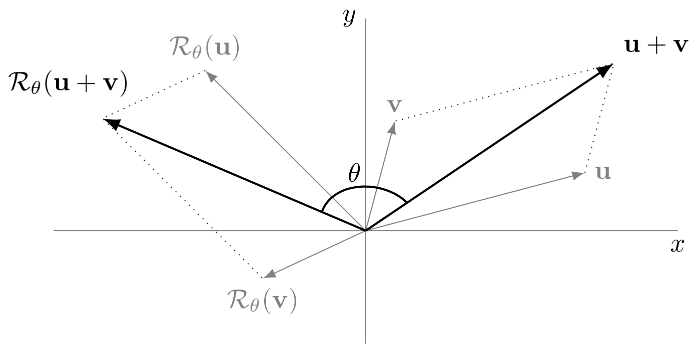
To illustrate how $\mathcal{R}_\theta(\mathbf{u} + \mathbf{v}) = \mathcal{R}_\theta(\mathbf{u}) + \mathcal{R}_\theta(\mathbf{v})$, consider the following series of pictures. First, we have a vector $\mathbf{u}$, as well as $\mathcal{R}_\theta(\mathbf{u})$, which is just the vector $\mathbf{u}$ rotated counterclockwise by the angle θ:



Next, we add a vector $\mathbf{v}$ to the picture, as well as the vector $\mathcal{R}_\theta(\mathbf{v})$, which is just $\mathbf{v}$ rotated counterclockwise by the same angle θ:



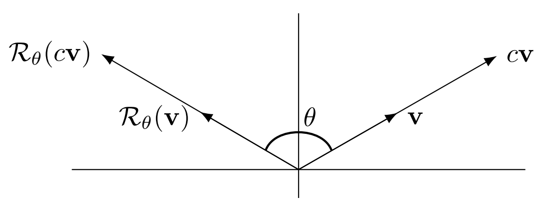
Now we add the vector $\mathbf{u}+\mathbf{v}$ to the picture, as well as the vector $\mathcal{R}_\theta(\mathbf{u}+\mathbf{v})$, which is just $\mathbf{u}+\mathbf{v}$ rotated counterclockwise by the same angle θ:



As one can see, the parallelogram created by $\mathbf{u}$ and $\mathbf{v}$ gets rotated into the parallelogram created by $\mathcal{R}_\theta(\mathbf{u})$ and $\mathcal{R}_\theta(\mathbf{v})$. In particular, since $\mathcal{R}_\theta(\mathbf{u}+\mathbf{v})$ is the diagonal of the parallelogram formed by $\mathcal{R}_\theta(\mathbf{u})$ and $\mathcal{R}_\theta(\mathbf{v})$, it follows that

$$\mathcal{R}_\theta(\mathbf{u}+\mathbf{v}) = \mathcal{R}_\theta(\mathbf{u}) + \mathcal{R}_\theta(\mathbf{v}).$$

Next, we show that $\mathcal{R}_\theta(c\mathbf{v}) = c \cdot \mathcal{R}_\theta(\mathbf{v})$ by showing that the two vectors have the same length and point in the same direction. We'll draw the case where $c > 0$, and leave it to the reader to verify that the $c < 0$ case holds as well. (The case $c = 0$ is obvious.)



Since $c > 0$, the vector $c\mathbf{v}$ points in the same direction as $\mathbf{v}$, so after rotating counterclockwise by θ, $\mathcal{R}_\theta(c\mathbf{v})$ and $\mathcal{R}_\theta(\mathbf{v})$ point in the same direction. Likewise, since $c > 0$, $c \cdot \mathcal{R}_\theta(\mathbf{v})$ points in the same direction as $\mathcal{R}_\theta(\mathbf{v})$. Thus, both $\mathcal{R}_\theta(c\mathbf{v})$ and $c \cdot \mathcal{R}_\theta(\mathbf{v})$ point in the same direction.

To see why they have the same length, keep in mind that rotating a vector isn't going to change its length. Thus, $\mathcal{R}_\theta(\mathbf{v})$ and $\mathbf{v}$ have the same length, and similarly, $\mathcal{R}_\theta(c\mathbf{v})$ and $c\mathbf{v}$ have the same length. Since $\mathcal{R}_\theta(\mathbf{v})$ and $\mathbf{v}$ have the same length, the vectors $c\mathcal{R}_\theta(\mathbf{v})$ and $c\mathbf{v}$ also have the same length. Thus, both vectors $\mathcal{R}_\theta(c\mathbf{v})$ and $c \cdot \mathcal{R}_\theta(\mathbf{v})$ have the same length as $c\mathbf{v}$, and thus they have the same lengths.

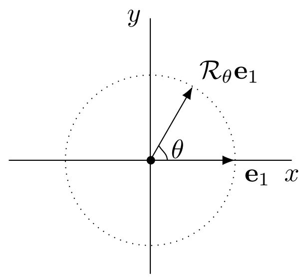
Since $\mathcal{R}_\theta(c\mathbf{v})$ and $c \cdot \mathcal{R}_\theta(\mathbf{v})$ have the same length and point in the same direction, they must be the equal to each other.

Now that we have verified that the rotation $\mathcal{R}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear map, it follows that there exists a 2×2 matrix R_θ such that

$$\mathcal{R}_\theta(\mathbf{x}) = R_\theta \mathbf{x}.$$

To find the matrix R_θ, we know that its first column is $\mathcal{R}_\theta(\mathbf{e}_1)$, where $\mathbf{e}_1 = (1, 0)$; and its second column is $\mathcal{R}_\theta(\mathbf{e}_2)$, where $\mathbf{e}_2 = (0, 1)$.

We'll start by finding the first column of R_θ, which is just $\mathcal{R}_\theta(\mathbf{e}_1)$.



If you remember how cosine and sine are defined in terms of the unit circle, then you'll recognize that the coordinates of $\mathcal{R}_\theta \mathbf{e}_1$ are $(\cos\theta, \sin\theta)$. Thus, the first column of R_θ is

$$\begin{bmatrix} \cos\theta \\ \sin\theta \end{bmatrix}.$$

Similarly, the second column of R_θ is $\mathcal{R}_\theta(\mathbf{e}_2)$, and since $\mathbf{e}_2$ starts $\pi/2$ radians ahead of $\mathbf{e}_1$, it follows that

$$\mathcal{R}_\theta(\mathbf{e}_2) = \begin{bmatrix} \cos(\theta + \frac{\pi}{2}) \\ \sin(\theta + \frac{\pi}{2}) \end{bmatrix}.$$

Using geometry or a trig identity, this can be simplified as

$$\mathcal{R}_\theta(\mathbf{e}_2) = \begin{bmatrix} -\sin\theta \\ \cos\theta \end{bmatrix}.$$

This gives us the second column of R_θ.

Summary

Putting everything together, we see that the map $\mathcal{R}_\theta : \mathbb{R}^2 \to \mathbb{R}^2$ which rotates vectors counterclockwise by an angle of θ is a linear operator. Furthermore, we have $\mathcal{R}_\theta(\mathbf{x}) = R_\theta \mathbf{x}$, where R_θ is the standard matrix of $\mathcal{R}_\theta$ and is given by

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Clockwise Rotations

What about clockwise rotations? Well, rotating clockwise by an angle of θ is the same as rotating counterclockwise by an angle of $-\theta$. So the standard matrix for a clockwise rotation can be obtained by taking the standard matrix for a counterclockwise rotation and replacing θ with $-\theta$. Doing so, and using the facts that

$$\cos(-\theta) = \cos\theta \qquad \text{and} \qquad \sin(-\theta) = -\sin\theta,$$

we find that the standard matrix for a clockwise rotation by angle θ is given by

$$\begin{bmatrix} \cos\theta & \sin\theta \\ -\sin\theta & \cos\theta \end{bmatrix}.$$

Example. Find the coordinates of the vector obtained by rotating the vector $(28, -34)$ counterclockwise by an angle of $210°$.

Solution. Let

$$\mathbf{x} = \begin{bmatrix} 28 \\ -34 \end{bmatrix}.$$

and $\theta = 210°$. Then the question is asking us to find $\mathcal{R}_\theta(\mathbf{x})$. We know that

$$\mathcal{R}_\theta(\mathbf{x}) = R_\theta \mathbf{x},$$

where

$$R_\theta = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

Plugging in $\theta = 210°$ gives us

$$R_\theta = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix}.$$

Then

$$\mathcal{R}_\theta(\mathbf{x}) = R_\theta \mathbf{x} = \begin{bmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} \\ -\frac{1}{2} & -\frac{\sqrt{3}}{2} \end{bmatrix} \begin{bmatrix} 28 \\ -34 \end{bmatrix} = \begin{bmatrix} -14\sqrt{3} - 17 \\ -14 + 17\sqrt{3} \end{bmatrix}.$$

Thus, rotating the vector $(28, -34)$ counterclockwise by $210°$ gives us the vector

$$\begin{bmatrix} -14\sqrt{3} - 17 \\ -14 + 17\sqrt{3} \end{bmatrix}.$$

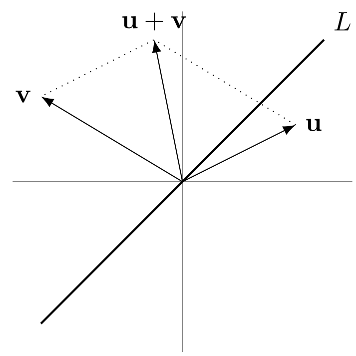
◇

3.4 Reflections in $\mathbb{R}^2$

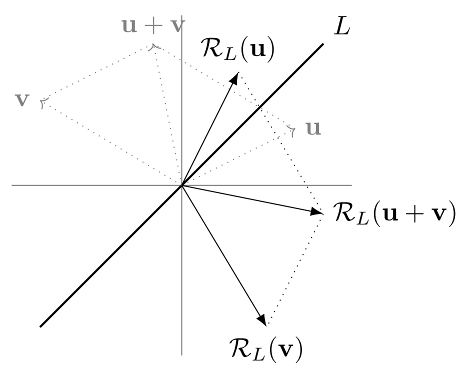
Let L be a line passing through the origin in $\mathbb{R}^2$. Let $\mathcal{R}_L : \mathbb{R}^2 \to \mathbb{R}^2$ denote the function that takes a vector $\mathbf{x}$ as input and whose output $\mathcal{R}_L(\mathbf{x})$ is the reflection of $\mathbf{x}$ about the line L. Like rotations, reflections are also linear transformations. Again, we provide some illustrations on why $\mathcal{R}_L$ satisfies

$$\mathcal{R}_L(\mathbf{u}+\mathbf{v}) = \mathcal{R}_L(\mathbf{u}) + \mathcal{R}_L(\mathbf{v}) \qquad \text{and} \qquad \mathcal{R}_L(c\mathbf{v}) = c\,\mathcal{R}_L(\mathbf{v}).$$

To see that $\mathcal{R}_L(\mathbf{u}+\mathbf{v}) = \mathcal{R}_L(\mathbf{u}) + \mathcal{R}_L(\mathbf{v})$, let's start with the following picture with the line L, and vectors $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$:

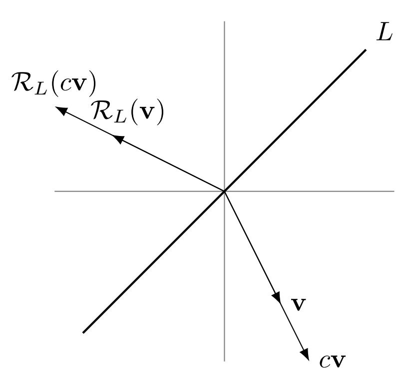


Next, we have the vectors $R_L(\mathbf{u})$, $R_L(\mathbf{v})$, and $R_L(\mathbf{u}+\mathbf{v})$, which are obtained by reflecting $\mathbf{u}$, $\mathbf{v}$, and $\mathbf{u}+\mathbf{v}$ about the line L, respectively:



We see that the parallelogram created by $\mathbf{u}$ and $\mathbf{v}$ gets reflected into the parallelogram created by $\mathcal{R}_L(\mathbf{u})$ and $\mathcal{R}_L(\mathbf{v})$. In particular then, $\mathcal{R}_L(\mathbf{u}+\mathbf{v})$ is the diagonal of the parallelogram formed by $\mathcal{R}_L(\mathbf{u})$ and $\mathcal{R}_L(\mathbf{v})$. Thus, $\mathcal{R}_L(\mathbf{u}+\mathbf{v}) = \mathcal{R}_L(\mathbf{u}) + \mathcal{R}_L(\mathbf{v})$.

Next, we will illustrate how $\mathcal{R}_L(c\mathbf{v}) = c \cdot \mathcal{R}_L(\mathbf{v})$. We will only illustrate the case where $c > 0$, and leave it to the reader to verify the case $c < 0$. The argument is very similar to the one given for rotations. Consider the following picture:



Since $c > 0$, the vector $c\mathbf{v}$ points in the same direction as $\mathbf{v}$. Hence, after reflection, the vector $\mathcal{R}_L(c\mathbf{v})$ points in the same direction as $\mathcal{R}_L(\mathbf{v})$. On the other hand, $c \cdot \mathcal{R}_L(\mathbf{v})$ also points in the same direction as $\mathcal{R}_L(\mathbf{v})$, since $c > 0$. So the vectors $\mathcal{R}_L(c\mathbf{v})$ and $c \cdot \mathcal{R}_L(\mathbf{v})$ point in the same direction.

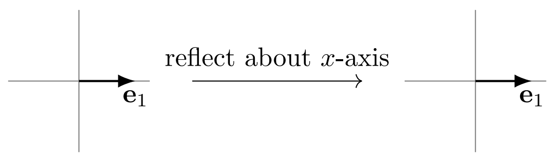
To see why the vectors $\mathcal{R}_L(c\mathbf{v})$ and $c \cdot \mathcal{R}_L(\mathbf{v})$ have the same length, the key is recognizing that, as with rotations, reflecting a vector isn't going to change its length. Thus, $\mathcal{R}_L(\mathbf{v})$ and $\mathbf{v}$ have the same length, which implies that $c \cdot \mathcal{R}_L(\mathbf{v})$ has the same length as $c\mathbf{v}$. On the other hand, $\mathcal{R}_L(c\mathbf{v})$ also has the same length as $c\mathbf{v}$. Hence, the vectors $\mathcal{R}_L(c\mathbf{v})$ and $c \cdot \mathcal{R}_L(\mathbf{v})$ have the same length.

Now that we have verified that the reflection $\mathcal{R}_L : \mathbb{R}^2 \to \mathbb{R}^2$ is a linear operator, it follows that there exists a 2×2 matrix R_L such that

$$\mathcal{R}_L(\mathbf{x}) = R_L\mathbf{x}.$$

To find the matrix R_L, we know that its first column is $\mathcal{R}_L(\mathbf{e}_1)$, where $\mathbf{e}_1 = (1, 0)$; and its second column is $\mathcal{R}_L(\mathbf{e}_2)$, where $\mathbf{e}_2 = (0, 1)$. Let's consider a few special cases first. Not because it will help with the general case, but because it's nice to do things at a leisurely pace. ☺

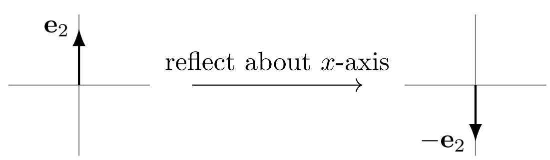
Let's start by finding the standard matrix for a reflection about the x-axis. To do this, we need to reflect the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$ about the x-axis. Consider the following illustration.



Since the vector $\mathbf{e}_1$ already lies along the x-axis, its reflection about the x-axis is itself. Thus, the first column of the standard matrix for the reflection about the x-axis is

$$\mathbf{e}_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}.$$

Now consider this next illustration.



We see that the reflection of $\mathbf{e}_2$ about the x-axis is $-\mathbf{e}_2$. Thus, the second column of the standard matrix for the reflection about the x-axis is

$$-\mathbf{e}_2 = \begin{bmatrix} 0 \\ -1 \end{bmatrix}.$$

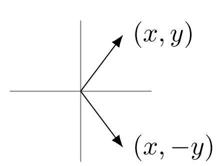
Putting it all together, we find that the standard matrix for the reflection about the x-axis in $\mathbb{R}^2$ is

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

Thus, for any vector (x, y), its reflection about the x-axis is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}.$$

Notice that this is exactly what we'd expect:

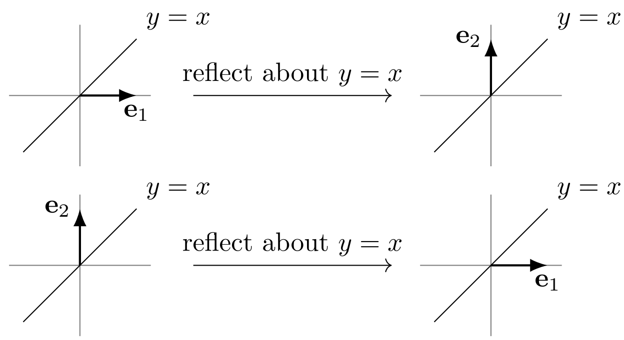


Similarly, one finds that the standard matrix for the reflection about the y-axis in $\mathbb{R}^2$ is

$$\begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}.$$

You should try to derive this yourself!

Finally, let's compute the standard matrix for the reflection about the line $y = x$. Again, we can find the standard matrix by reflecting the standard basis vectors $\mathbf{e}_1$ and $\mathbf{e}_2$.



We see from the above illustrations that $\mathbf{e}_1$ gets reflected to $\mathbf{e}_2$; hence, $\mathbf{e}_2$ is the first column of the standard matrix for the reflection about $y = x$. Similarly, $\mathbf{e}_2$ gets reflected to $\mathbf{e}_1$, so $\mathbf{e}_1$ is the second column. Thus, the standard matrix for the reflection about the line $y = x$ is given by

$$\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

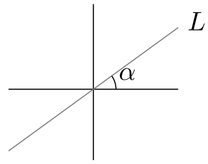
Similarly, the standard matrix for the reflection about the line $y = -x$ is given by

$$\begin{bmatrix} 0 & -1 \\ -1 & 0 \end{bmatrix}.$$

Again, you should try and derive this yourself!

The General Formula for the Matrix of a Reflection Operator on $\mathbb{R}^2$

Let's now consider the case of an arbitrary line L. For our present purposes, we will need to reference the angle that L makes with the positive x-axis; we choose to denote this angle by α.

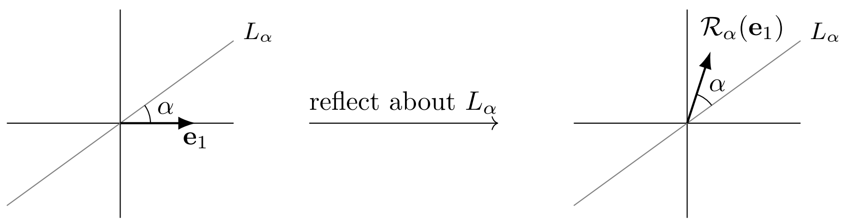


At this point, we will also refine our notation a bit. To indicate how the line L depends on the angle α, we will now denote the line by L_α instead of just L. Furthermore, we now denote the operator that reflects about the line L_α by $\mathcal{R}_\alpha$. Thus, for any vector $\mathbf{x}$ in $\mathbb{R}^2$, $\mathcal{R}_\alpha(\mathbf{x})$ represents the reflection of $\mathbf{x}$ about the line L_α. Correspondingly, we now use R_α to denote the standard matrix of $\mathcal{R}_\alpha$, so that

$$\mathcal{R}_\alpha(\mathbf{x}) = R_\alpha \mathbf{x}.$$

The key to deriving the formula for R_α is to use the fact that reflections preserve length and angles, as we shall see below.

The first column of R_α is equal to $\mathcal{R}_\alpha(\mathbf{e}_1)$, which is the reflection of $\mathbf{e}_1$ about the line L_α. Consider the illustration below.

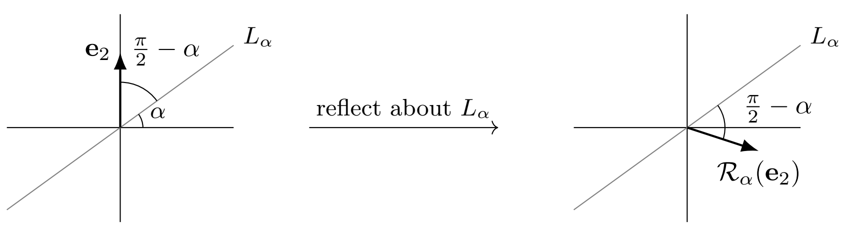


Since the angle between $\mathbf{e}_1$ and L_α is α, so too is the angle between $\mathcal{R}_\alpha(\mathbf{e}_1)$ and L_α (because reflections preserve angles). In particular then, the angle between $\mathcal{R}_\alpha(\mathbf{e}_1)$ and the positive x-axis is 2α. Furthermore, since $\mathbf{e}_1$ is a unit vector, so too is $\mathcal{R}_\alpha(\mathbf{e}_1)$ (because reflections preserve length). As a unit vector that makes an angle of 2α with the positive x-axis, we conclude that

$$\mathcal{R}_\alpha(\mathbf{e}_1) = \begin{bmatrix} \cos(2\alpha) \\ \sin(2\alpha) \end{bmatrix}.$$

This is the first column of R_α.

Similarly, the second column of R_α is obtained by reflecting $\mathbf{e}_2$ about the line L_α. Here is the illustration:



Since the angle between $\mathbf{e}_2$ and L_α is $\frac{\pi}{2} - \alpha$, so too is the angle between $\mathcal{R}_\alpha(\mathbf{e}_2)$ and L_α. In particular though, the angle between $\mathcal{R}_\alpha(\mathbf{e}_2)$ and the positive x-axis is now

$$\left(\tfrac{\pi}{2} - \alpha\right) - \alpha = \tfrac{\pi}{2} - 2\alpha.$$

However, notice that this is measured *clockwise* from the positive x-axis, so we need to negate this angle in order to express $\mathcal{R}_\alpha(\mathbf{e}_2)$ in terms of cos and sin. Thus,

$$\mathcal{R}_\alpha(\mathbf{e}_2) = \begin{bmatrix} \cos\left(2\alpha - \frac{\pi}{2}\right) \\ \sin\left(2\alpha - \frac{\pi}{2}\right) \end{bmatrix}.$$

By using trig identities or some geometry, we can rewrite this as

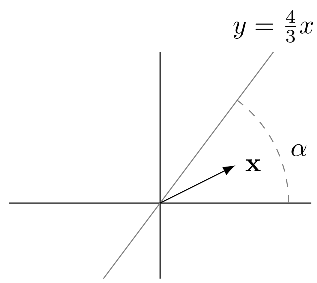
$$\mathcal{R}_\alpha(\mathbf{e}_2) = \begin{bmatrix} \sin(2\alpha) \\ -\cos(2\alpha) \end{bmatrix}.$$

This is the second column of R_α.

In summary, the standard matrix for the operator $\mathcal{R}_\alpha$ in $\mathbb{R}^2$ that reflects about the line L_α is

$$R_\alpha = \begin{bmatrix} \cos(2\alpha) & \sin(2\alpha) \\ \sin(2\alpha) & -\cos(2\alpha) \end{bmatrix}.$$

Example. Let $\mathbf{x} = (2, 1)$. We want to find the reflection of $\mathbf{x}$ about the line $y = \frac{4}{3}x$.



Using some trig, one finds that

$$\alpha = \arctan(\tfrac{4}{3}) \;\approx\; .9273 \text{ rad} \;\approx\; 53.13^\circ.$$

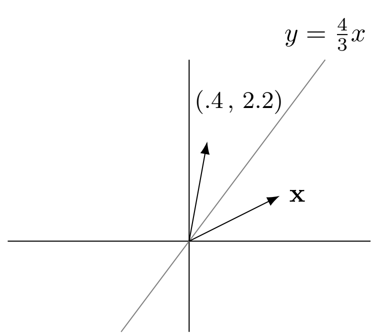
Plugging this into the formula for R_α, we find that

$$R_\alpha \approx \begin{bmatrix} -.28 & .96 \\ .96 & .28 \end{bmatrix}.$$

Now we compute:

$$R_\alpha \mathbf{x} \approx \begin{bmatrix} -.28 & .96 \\ .96 & .28 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} .4 \\ 2.2 \end{bmatrix}$$

Thus, the reflection of the vector $(2, 1)$ about the line $y = \frac{4}{3}x$ should approximately be the vector $(.4\,,\,2.2)$. Below are the two vectors plotted. What do you think? Does it look like $(.4\,,\,2.2)$ is the reflection of $(2, 1)$ about the line $y = \frac{4}{3}x$?



◇

3.5 Matrix Addition

Addition of matrices is very similar to addition of vectors. Suppose A and B are two matrices of the same size. We define their sum, $A+B$, to be the matrix obtained by adding the entries of A and B component-wise. Note that $A+B$ is undefined if A and B are different sizes.

Example. Let

$$A = \begin{bmatrix} 2 & -3 & 0 \\ 1 & 4 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 3 & -1 & 2 \\ -5 & 3 & 0 \end{bmatrix}.$$

Then

$$A+B = \begin{bmatrix} 5 & -4 & 2 \\ -4 & 7 & -1 \end{bmatrix}.$$

◇

Example. Let

$$A = \begin{bmatrix} 2 & -3 & 0 \\ 1 & 4 & -1 \end{bmatrix} \qquad B = \begin{bmatrix} 1 & 2 \\ 3 & 4 \end{bmatrix}.$$

Then $A+B$ is undefined, since A and B are different sizes (A is 2×3, B is 2×2). ◇

There is not much else to say about matrix addition. The rest of this section is optional, i.e., you'll be fine if you don't read the rest. We discuss why matrix addition is defined the way it is.

3.5.1 Addition of Linear Maps

Let S and T be linear maps with signature $\mathbb{R}^n \to \mathbb{R}^m$. We define their sum, $S+T$, to be the function $\mathbb{R}^n \to \mathbb{R}^m$ given by the following rule:

$$(S+T)(\mathbf{x}) = S(\mathbf{x}) + T(\mathbf{x}).$$

It turns out that the sum of two linear transformations results in a linear transformation. To see this, we will show that $S+T$ satisfies the two defining conditions of a linear transformation:

1. $(S+T)(\mathbf{x}+\mathbf{y}) = (S+T)(\mathbf{x}) + (S+T)(\mathbf{y})$ for any $\mathbf{x}, \mathbf{y}$ in $\mathbb{R}^n$.
2. $(S+T)(c\mathbf{x}) = c(S+T)(\mathbf{x})$ for any scalar c and $\mathbf{x}$ in $\mathbb{R}^n$.

Proof. We start by proving the first statement. By definition of $S+T$,

$$(S+T)(\mathbf{x}+\mathbf{y}) = S(\mathbf{x}+\mathbf{y}) + T(\mathbf{x}+\mathbf{y}).$$

Since S and T are linear,

$$S(\mathbf{x}+\mathbf{y}) = S(\mathbf{x}) + S(\mathbf{y}) \qquad \text{and} \qquad T(\mathbf{x}+\mathbf{y}) = T(\mathbf{x}) + T(\mathbf{y}).$$

Plugging these into the previous equation gives us

$$(S+T)(\mathbf{x}+\mathbf{y}) = S(\mathbf{x}) + S(\mathbf{y}) + T(\mathbf{x}) + T(\mathbf{y}).$$

Rearranging the middle two terms gives us

$$(S+T)(\mathbf{x}+\mathbf{y}) = S(\mathbf{x}) + T(\mathbf{x}) + S(\mathbf{y}) + T(\mathbf{y}).$$

Since

$$S(\mathbf{x}) + T(\mathbf{x}) = (S+T)(\mathbf{x}) \qquad \text{and} \qquad S(\mathbf{y}) + T(\mathbf{y}) = (S+T)(\mathbf{y}),$$

it follows that

$$(S+T)(\mathbf{x}+\mathbf{y}) = (S+T)(\mathbf{x}) + (S+T)(\mathbf{y}).$$

This proves the first statement.

Next, by definition of $S+T$,

$$(S+T)(c\mathbf{x}) = S(c\mathbf{x}) + T(c\mathbf{x}).$$

Since S and T are linear, we can factor the c out. Thus,

$$\begin{aligned}(S+T)(c\mathbf{x}) &= cS(\mathbf{x}) + cT(\mathbf{x}) \\ &= c\big[S(\mathbf{x}) + T(\mathbf{x})\big] \\ &= c(S+T)(\mathbf{x}).\end{aligned}$$

This proves the second statement. $\square$

Now, since S and T are linear, it follows that there exists $m \times n$ matrices A and B, namely their standard matrices, such that

$$S(\mathbf{x}) = A\mathbf{x} \qquad \text{and} \qquad T(\mathbf{x}) = B\mathbf{x}.$$

Furthermore, we know that the j^{th} columns of A and B are $S(\mathbf{e}_j)$ and $T(\mathbf{e}_j)$, respectively.

Similarly, since $S+T$ is linear, it follows that there exists an $m \times n$ matrix C, namely its standard matrix, such that

$$(S+T)(\mathbf{x}) = C\mathbf{x}.$$

And we know that the j^{th} column of C is equal to $(S+T)(\mathbf{e}_j)$.

Now, by definition,

$$(S+T)(\mathbf{e}_j) = S(\mathbf{e}_j) + T(\mathbf{e}_j).$$

This says that the j^{th} column of C should be equal to the j^{th} column of A plus the j^{th} column of B. In other words, C is just the matrix $A+B$!

Essentially, what this tells us is that addition of matrices corresponds to addition of the associated linear transformations. That is to say, the standard matrix of a sum of linear transformations is the sum of their standard matrices. This is, of course, no coincidence: ***matrix addition is defined the way it is precisely so that this is true.***

3.6 Matrix Multiplication

Matrix multiplication is not as natural or simple as matrix addition. As we did with matrix addition, we will first define how matrix multiplication works and look at some examples. Afterwards, if you are interested, there is an optional section explaining why matrix multiplication is defined the way it is.

3.6.1 Definition of Matrix Multiplication

Definition. Given an $m \times k$ matrix A and a $k \times n$ matrix B, we define the product AB to be the $m \times n$ matrix whose j^{th} column is equal to the matrix A times the j^{th} column of B:

$$j^{\text{th}} \text{ column of } AB = A \begin{pmatrix} j^{\text{th}} \text{ column} \\ \text{of } B \end{pmatrix}.$$

Note how the sizes of A and B need to match up in order to multiply them. Specifically, the number of columns of A must be equal to the number of rows of B in order for the product AB to be defined. If A is $m \times k$ and B is $k \times n$, then AB is $m \times n$.

Example. If

$$A = \overbrace{\begin{bmatrix} 2 & 1 & -1 \\ 0 & -3 & 2 \end{bmatrix}}^{2\times 3} \qquad \text{and} \qquad B = \overbrace{\begin{bmatrix} -1 & 2 \\ 2 & 1 \\ -3 & 0 \end{bmatrix}}^{3\times 2},$$

then AB and BA are both defined. AB is of size 2×2, while BA is of size 3×3.

To compute AB, multiply A to each column of B:

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & -3 & 2 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \\ -3 \end{bmatrix} = \begin{bmatrix} 3 \\ -12 \end{bmatrix} \qquad \begin{bmatrix} 2 & 1 & -1 \\ 0 & -3 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \end{bmatrix}$$

Thus,

$$AB = \begin{bmatrix} 3 & 5 \\ -12 & -3 \end{bmatrix}.$$

Similarly, to compute BA, multiply B to each column of A:

$$\begin{bmatrix} -1 & 2 \\ 2 & 1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 2 \\ 0 \end{bmatrix} = \begin{bmatrix} -2 \\ 4 \\ -6 \end{bmatrix} \qquad \begin{bmatrix} -1 & 2 \\ 2 & 1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -3 \end{bmatrix} = \begin{bmatrix} -7 \\ -1 \\ -3 \end{bmatrix} \qquad \begin{bmatrix} -1 & 2 \\ 2 & 1 \\ -3 & 0 \end{bmatrix} \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 0 \\ 3 \end{bmatrix}.$$

Thus,

$$BA = \begin{bmatrix} -2 & -7 & 5 \\ 4 & -1 & 0 \\ -6 & -3 & 3 \end{bmatrix}.$$

◇

Example. If

$$A = \overbrace{\begin{bmatrix} 2 & -1 & 5 \end{bmatrix}}^{1\times 3} \qquad \text{and} \qquad B = \overbrace{\begin{bmatrix} 1 & 2 \\ 3 & 1 \\ 0 & 2 \end{bmatrix}}^{3\times 2},$$

then AB is defined and of size 1×2, while BA is not defined.

To compute AB, multiply A to each column of B:

$$\begin{bmatrix} 2 & -1 & 5 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 0 \end{bmatrix} = -1 \qquad \begin{bmatrix} 2 & -1 & 5 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 2 \end{bmatrix} = 13.$$

Thus,

$$AB = \begin{bmatrix} -1 & 13 \end{bmatrix}.$$

◇

An alternative method for computing AB: the ij-entry of AB is equal to the dot product between the i^{th} row of A and the j^{th} column of B:

$$ij\text{-entry of } AB = \left(i^{\text{th}} \text{ row of } A\right) \cdot \left(j^{\text{th}} \text{ column of } B\right).$$

We'll refer to this method of computing a matrix product as the row-dot-column method. This method is probably more practical for computations, but the original definition should still be kept in mind.

Example. Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 3 & -2 \\ -1 & 2 & 1 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 2 & -1 \\ 1 & 0 \\ 3 & 2 \end{bmatrix}.$$

Then AB is defined and of size 3×2, while BA is not defined. We will compute AB using the row-dot-column method. We'll start by taking the first row of A, and multiplying it with each column of B:

$$\begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 7 \qquad \begin{bmatrix} 1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = 1.$$

This gives us the first row of the product AB:

$$AB = \begin{bmatrix} 7 & 1 \\ & \\ & \end{bmatrix}.$$

Next, take the second row of A and multiply it with each column of B:

$$\begin{bmatrix} 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = -3 \qquad \begin{bmatrix} 0 & 3 & -2 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = -4.$$

This gives us the second row of AB:

$$AB = \begin{bmatrix} 7 & 1 \\ -3 & -4 \\ & \end{bmatrix}.$$

Finally, take the third row of A and multiply it with each column of B:

$$\begin{bmatrix} -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 3 \qquad \begin{bmatrix} -1 & 2 & 1 \end{bmatrix} \begin{bmatrix} -1 \\ 0 \\ 2 \end{bmatrix} = 3.$$

This gives us the third and final row of AB:

$$AB = \begin{bmatrix} 7 & 1 \\ -3 & -4 \\ 3 & 3 \end{bmatrix}.$$ ◇

Matrix multiplication might seem weird at first, but if you practice a few examples, you'll get the hang of it fairly quickly. You need to practice though!

We list some of the properties of matrix multiplication below. It should be assumed in the statement of these properties that the sizes of all matrices match up so that everything is defined.

1. Matrix multiplication is associative, i.e., $(AB)C = A(BC)$.
2. Matrix multiplication distributes over matrix addition, i.e., $A(B + C) = AB + AC$ and $(A + B)C = AC + BC$.

3. Matrix multiplication is not commutative, i.e, in general $AB \neq BA$. The above examples illustrate this.

Finally, we summarize a few facts about matrix multiplication (the last one is new; you should verify it yourself!):

1. The j^{th} column of AB is equal to A times the j^{th} column of B.
2. The ij-entry of AB is equal to the i^{th} row of A times the j^{th} column of B.
3. The i^{th} row of AB is equal to the i^{th} row of A times B.

3.6.2 Composition of Linear Maps

This subsection is optional. The motivation for defining matrix multiplication comes from the perspective of linear transformations (as was the case with matrix addition). First, let $f : \mathbb{R}^k \to \mathbb{R}^m$ and $g : \mathbb{R}^n \to \mathbb{R}^k$ be two functions (not necessarily linear). Then we can form their composition $f \circ g : \mathbb{R}^n \to \mathbb{R}^m$. For instance, if $f : \mathbb{R}^3 \to \mathbb{R}$ is given by

$$f(y_1, y_2, y_3) = y_1 + 5y_2y_3,$$

and $g : \mathbb{R}^2 \to \mathbb{R}^3$ is given by

$$g(x_1, x_2) = \begin{bmatrix} x_1x_2 \\ x_1 + x_2 \\ x_1e^{x_2} \end{bmatrix},$$

then $f \circ g : \mathbb{R}^2 \to \mathbb{R}$ is the function given by

$$(f \circ g)(x_1, x_2) = f\big(g(x_1, x_2)\big) = x_1x_2 + 5(x_1 + x_2)(x_1e^{x_2}).$$

Now we specialize to the case where we have linear maps. The following proposition is the key to defining matrix multiplication:

Proposition. The composition of two linear maps is a linear map. More precisely, if $S : \mathbb{R}^k \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^k$ are linear, then their composition $S \circ T : \mathbb{R}^n \to \mathbb{R}^m$ is also linear.

Proof. First, let $\mathbf{u}, \mathbf{v}$ be two vectors in $\mathbb{R}^n$. We will show that

$$(S \circ T)(\mathbf{u} + \mathbf{v}) = (S \circ T)(\mathbf{u}) + (S \circ T)(\mathbf{v}).$$

For now, we have

$$(S \circ T)(\mathbf{u} + \mathbf{v}) = S\big(T(\mathbf{u} + \mathbf{v})\big).$$

Since T is linear, $T(\mathbf{u} + \mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$. Thus, we have

$$(S \circ T)(\mathbf{u} + \mathbf{v}) = S\big(T(\mathbf{u}) + T(\mathbf{v})\big).$$

But S is also linear, and so $S\big(T(\mathbf{u}) + T(\mathbf{v})\big) = S\big(T(\mathbf{u})\big) + S\big(T(\mathbf{v})\big)$. Thus,

$$(S \circ T)(\mathbf{u} + \mathbf{v}) = S\big(T(\mathbf{u})\big) + S\big(T(\mathbf{v})\big).$$

This is equivalent to

$$(S \circ T)(\mathbf{u} + \mathbf{v}) = (S \circ T)(\mathbf{u}) + (S \circ T)(\mathbf{v}),$$

which is what we wanted to show.

Next, let c be a scalar, and let $\mathbf{v}$ a vector in $\mathbb{R}^n$. We need to show that

$$(S \circ T)(c\mathbf{v}) = c(S \circ T)(\mathbf{v}).$$

For now, we have

$$(S \circ T)(c\mathbf{v}) = S\big(T(c\mathbf{v})\big).$$

Since T is linear, $T(c\mathbf{v}) = cT(\mathbf{v})$. Thus, we have

$$(S \circ T)(c\mathbf{v}) = S\big(cT(\mathbf{v})\big).$$

But S is also linear, and so $S\big(cT(\mathbf{v})\big) = cS\big(T(\mathbf{v})\big)$. Thus,

$$(S \circ T)(c\mathbf{v}) = cS\big(T(\mathbf{v})\big).$$

This is equivalent to

$$(S \circ T)(c\mathbf{v}) = c(S \circ T)(\mathbf{v}),$$

which is what we wanted to show.

Thus, $S \circ T$ satisfies the two defining conditions of a linear map. $\square$

Once again, suppose we have linear maps $S : \mathbb{R}^k \to \mathbb{R}^m$ and $T : \mathbb{R}^n \to \mathbb{R}^k$. Let A and B be the standard matrices of S and T, respectively, so that A is $m \times k$, B is $k \times n$, and

$$S(\mathbf{y}) = A\mathbf{y} \qquad \text{and} \qquad T(\mathbf{x}) = B\mathbf{x}.$$

We've just proven that $S \circ T : \mathbb{R}^n \to \mathbb{R}^m$ is linear, which means there exists an $m \times n$ matrix C such that

$$(S \circ T)(\mathbf{x}) = C\mathbf{x}.$$

On the other hand, we know that

$$(S \circ T)(\mathbf{x}) = S\big(T(\mathbf{x})\big).$$

Since $T(\mathbf{x}) = B\mathbf{x}$ for any $\mathbf{x}$, and $S(\mathbf{y}) = A\mathbf{y}$ for any $\mathbf{y}$, it follows $S\big(T(\mathbf{x})\big) = A(B\mathbf{x})$. Thus,

$$(S \circ T)(\mathbf{x}) = A(B\mathbf{x}).$$

Comparing this with $(S \circ T)(\mathbf{x}) = C\mathbf{x}$, the idea is that we should define matrix multiplication so that $AB = C$.

So, what is C? Well, remember that in order to find the standard matrix of a linear map, we just need to apply the linear map to the standard basis vectors. More specifically, the j^{th} column of C is given by

$$(S \circ T)(\mathbf{e}_j).$$

From the previous paragraph,

$$(S \circ T)(\mathbf{e}_j) = A(B\mathbf{e}_j) = A \left(\substack{j^{\text{th}} \text{ column} \\ \text{of } B} \right).$$

Thus, we define the product of an $m \times k$ matrix A with a $k \times n$ matrix B to be the $m \times n$ matrix whose j^{th} column is equal to A times the j^{th} column of B.

In conclusion, matrix multiplication corresponds to the composition of the associated linear transformations, and matrix multiplication is defined the way it is precisely so that this is true.

Chapter 4

Inverses, Elementary Matrices, LU Decomposition & Its Variations

4.1 Matrix Inverses

4.1.1 Identity Matrices

The $n \times n$ identity matrix is defined to be the $n \times n$ matrix with ones on the diagonal and zeros off the diagonal. Thus, the 2×2 identity matrix is

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and the 3×3 identity matrix is

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The 1×1 identity matrix is just the number 1.

We will use I_n to denote the $n \times n$ identity matrix. Often, however, the size of the matrix is not important or can be inferred from the context, so we may simply use I to denote an identity matrix as well.

What is special about identity matrices is that they satisfy the following property: for any matrix A,

$$AI = A \qquad \text{and} \qquad IA = A.$$

In particular,

$$I\mathbf{x} = \mathbf{x}$$

for any vector $\mathbf{x}$ as well. Essentially then, identity matrices are the matrix analog of the number 1, in the sense that for any number a,

$$a1 = 1a = a.$$

4.1.2 Definition of Matrix Inverse

Recall that any linear system of equations can represented as a matrix equation of the form $A\mathbf{x} = \mathbf{b}$. The simplest scenario would be when we have only one equation with one unknown. Such a system would look like

$$ax = b,$$

where a, x, and b are all scalars, with x being the unknown. To solve for x, we would divide both sides by a, or equivalently, we would multiply both sides by a^{-1}:

$$a^{-1}ax = a^{-1}b.$$

We know that $a^{-1}a = 1$, and so we get

$$1x = a^{-1}b.$$

And finally, we know that $1x = x$, and so we find that

$$x = a^{-1}b.$$

The idea of matrix inverses is to try and apply the above procedure to the general case. So, starting with

$$A\mathbf{x} = \mathbf{b},$$

we would like to have available an inverse of A, denoted A^{-1}, that we can multiply both sides of this equation with:

$$A^{-1}A\mathbf{x} = A^{-1}\mathbf{b}.$$

The inverse should satisfy $A^{-1}A = I$, analogous to how we had $a^{-1}a = 1$ in the previous paragraph. Then we would have

$$I\mathbf{x} = A^{-1}\mathbf{b},$$

and because $I\mathbf{x} = \mathbf{x}$, we find the solution to the linear system of equations is given by

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

Now, let's take a look at what it means for a matrix A to have an inverse A^{-1}. Let's assume that A is $m \times n$. Recall that, in general, the equation $A\mathbf{x} = \mathbf{b}$ can have either no solution, infinitely many solutions, or a unique solution. If A is invertible, we see that the equation

$$A\mathbf{x} = \mathbf{b}$$

will always have a solution, namely

$$\mathbf{x} = A^{-1}\mathbf{b}.$$

So we can rule out the possibility of having no solution. It turns out that if A is invertible, we can also rule out the possibility of having infinitely many solutions. To see why, suppose that we have two solutions $\mathbf{x}_1$ and $\mathbf{x}_2$, so that

$$A\mathbf{x}_1 = \mathbf{b} \qquad \text{and} \qquad A\mathbf{x}_2 = \mathbf{b}.$$

Since $A\mathbf{x}_1$ and $A\mathbf{x}_2$ are both equal to $\mathbf{b}$, they are also equal to each other:

$$A\mathbf{x}_1 = A\mathbf{x}_2.$$

But now if we multiply both sides by A^{-1},

$$A^{-1}A\mathbf{x}_1 = A^{-1}A\mathbf{x}_2,$$

we would be left with

$$I\mathbf{x}_1 = I\mathbf{x}_2,$$

or equivalently,

$$\mathbf{x}_1 = \mathbf{x}_2.$$

Thus, if A is invertible, then any two solutions to the equation $A\mathbf{x} = \mathbf{b}$ must be equal to each other, meaning that there can only be one, unique solution. We thus conclude that if A is invertible, then the general equation $A\mathbf{x} = \mathbf{b}$ always has a unique solution, which is given by $\mathbf{x} = A^{-1}\mathbf{b}$. It cannot have no solution, and it cannot have infinitely many solutions.

In Section 2.5, we saw that if there are more equations than unknowns, that is, if $m > n$, then the general equation $A\mathbf{x} = \mathbf{b}$ will not always have a solution. Thus, A cannot be invertible if $m > n$. On the other hand, we saw that if there are more unknowns than equations, that is, if $n > m$, then the equation $A\mathbf{x} = \mathbf{b}$ will always have infinitely many solutions, if it is consistent. Thus, A cannot be invertible if $n > m$ either. We therefore conclude that for a matrix A to be invertible, it must be of size $n \times n$, i.e., it must be a square matrix. Thus, we are led to only consider square matrices in the following

Definition. Let A be a square matrix. A is said to be ***invertible*** if there exists a matrix, denoted A^{-1}, such that

$$A^{-1}A = AA^{-1} = I.$$

If it exists, the matrix A^{-1} is called the inverse of A.

Remarks

- Note that if A is invertible, then its inverse is unique. To see why, suppose B and C are inverses, meaning

  $$BA = AB = I \qquad \text{and} \qquad CA = AC = I.$$

  Then we have

  $$BAC = (BA)C = IC = C \qquad \text{and} \qquad BAC = B(AC) = BI = B.$$

  Since B and C both equal BAC, they are equal to each other. This shows that any two inverses of A must equal each other, and hence that the inverse of A is unique.

- As an immediate consequence of the definition, if A is invertible, then so too is its inverse, with $(A^{-1})^{-1} = A$.

- Not every square matrix is invertible. As an example, consider the following matrix:

  $$A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$$

  Then

  $$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix}.$$

  From this, we see that for any matrix B, BA will have a column of zeros, and thus can never equal the identity. It follows that A has no inverse. In general, any matrix with a column or row of zeroes will not be invertible. Note, though, that it is possible for a matrix to not have a row or column of zeros and still be not invertible.

There is one more remark we want to mention, but the explanation of this one is a little bit more involved, so we have separated it into the following:

Proposition. Let A be an $n \times n$ matrix. If B satisfies $BA = I$, then it also satisfies $AB = I$ and therefore $B = A^{-1}$. Similarly, if C satisfies $AC = I$, then it also satisfies $CA = I$ and therefore $C = A^{-1}$.

Proof. We will prove the first part of the proposition, as the second part is somewhat similar. Assume that $BA = I$. In order to show that $AB = I$ as well, we first note that the general equation $A\mathbf{x} = \mathbf{b}$ always has a solution. To see why, just multiply both sides on the left by B to get $BA\mathbf{x} = B\mathbf{b}$. Since $BA = I$, this becomes $\mathbf{x} = B\mathbf{b}$. Hence, we managed to find one solution for $\mathbf{x}$, namely $\mathbf{x} = B\mathbf{b}$. If we plug this solution back into the original $A\mathbf{x} = \mathbf{b}$ equation, we get

$$A(B\mathbf{b}) = \mathbf{b} \qquad \Longleftrightarrow \qquad (AB)\mathbf{b} = \mathbf{b}.$$

Notice that $\mathbf{b}$ is arbitrary, so in fact we are seeing that $(AB)\mathbf{b} = \mathbf{b}$ for all $\mathbf{b}$. In particular then, we have

$$(AB)\mathbf{e}_j = \mathbf{e}_j \quad \text{for } j = 1, \ldots, n,$$

where $\mathbf{e}_j$ is the j^{th} standard basis vector of $\mathbb{R}^n$. Recalling how multiplying a matrix by $\mathbf{e}_j$ returns the j^{th} column of the matrix, this last equation is saying that

$$\substack{j^{\text{th}} \text{ column} \\ \text{of } AB} = \mathbf{e}_j, \quad j = 1, \ldots, n.$$

Since the j^{th} column of AB is $\mathbf{e}_j$ for all j, it follows that $AB = I$. Then, since $AB = I$ and $BA = I$, it follows that $B = A^{-1}$. $\square$

In the mean time, the question that must be on your mind right now is: given a matrix A, how can we find A^{-1}? We will address this shortly.

4.1.3 The 2×2 Formula

Let us dispose of a special case right now. It turns out that 2×2 matrices are small enough as to where we can simply memorize a formula for their inverse. Suppose

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

It turns out that A is invertible if and only if the quantity $ad - bc$, called the ***determinant*** of A, is nonzero. If the determinant of A is nonzero, then A^{-1} is given by the following formula:

$$A^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}.$$

We will discuss determinants and a generalization of this formula for $n \times n$ matrices much later in this course. For now, feel free to use it!

Example. Consider the following linear system of equations:

$$\begin{aligned} 2x - 2y &= 4 \\ 3x - 5y &= -3. \end{aligned}$$

We can rewrite this as

$$\begin{bmatrix} 2 & -2 \\ 3 & -5 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}.$$

The matrix

$$\begin{bmatrix} 2 & -2 \\ 3 & -5 \end{bmatrix}$$

is invertible, since its determinant is

$$2(-5) - (-2)(3) = -4$$

is nonzero. Thus, we can solve the system of equations as

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 & -2 \\ 3 & -5 \end{bmatrix}^{-1} \begin{bmatrix} 4 \\ -3 \end{bmatrix}.$$

Using the formula for the inverse of a 2×2, we find that

$$\begin{bmatrix} x \\ y \end{bmatrix} = -\frac{1}{4} \begin{bmatrix} -5 & 2 \\ -3 & 2 \end{bmatrix} \begin{bmatrix} 4 \\ -3 \end{bmatrix}.$$

If multiply everything out, we find that

$$\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} \frac{13}{2} \\ \frac{9}{2} \end{bmatrix}.$$

You can plug these into the original equations to double check that this is correct. ◇

4.1.4 The Dressing-Undressing Principle

We'll end these notes with a property that is useful to know. Suppose A and B are $n \times n$ invertible matrices. Notice that

$$(B^{-1}A^{-1})(AB) = I.$$

Thus, it follows that

$$(AB)^{-1} = B^{-1}A^{-1}.$$

This formula is sometimes referred to as the dressing-undressing principle. When you dress yourself, you put your underwear on first, and then some pants (or whatever). When you invert the process and undress yourself, you take your pants off first, and then your underwear. Notice how the order goes from: underwear, pants; to: pants, underwear. So, when inverting a process, you invert each step in reverse order.

We can extend this to any finite number of invertible matrices.

Proposition. Let $A_1, A_2, \ldots, A_k$ be $n \times n$ invertible matrices. Then their product $A_1 A_2 \ldots A_k$ is invertible, and

$$(A_1 A_2 \ldots A_k)^{-1} = A_k^{-1} \ldots A_2^{-1} A_1^{-1}.$$

4.2 Algorithm for Computing A^{-1}

4.2.1 Gauss-Jordan Elimination

We need to cover one thing before we talk about the general algorithm for finding the inverse of a matrix. Consider the following linear system of equations:

$$\begin{aligned} x + \ \ y \qquad &= 1 \\ 2x + 3y + 2z &= 3 \\ 3x + 5y + \ \ z &= 2. \end{aligned}$$

Let's set up the augmented matrix and start performing row operations in order to solve:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 2 & 3 & 2 & 3 \\ 3 & 5 & 1 & 2 \end{array}\right] \xrightarrow[R_3-3R_1]{R_2-2R_1} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 1 & -1 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -3 & -3 \end{array}\right].$$

Now that we've reduced everything to row echelon form, normally what we would do is back-substitute to solve for x, y, and z. We're going to do something else this time. Instead, we are going to continue row-reducing until we have the identity matrix on the left. This requires making all of the pivots equal to 1, and it requires making everything above the pivots equal to 0 as well. Let's start by dividing Row 3 by -3 to get a 1 as a pivot. Then, we'll perform a few more row operations to get zeros above the pivots:

$$\left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -3 & -3 \end{array}\right] \xrightarrow{-\frac{1}{3}R_3} \left[\begin{array}{ccc|c} 1 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right] \xrightarrow{R_1-R_2} \left[\begin{array}{ccc|c} 1 & 0 & -2 & 0 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & 1 & 1 \end{array}\right]$$

$$\xrightarrow[R_2-2R_3]{R_1+2R_3} \left[\begin{array}{ccc|c} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 1 \end{array}\right].$$

Now that we have reduced the left side to the identity matrix, if we write out the corresponding system of equations from the last matrix, we get

$$\begin{aligned} x &= \ \ 2 \\ y &= -1 \\ z &= \ \ 1. \end{aligned}$$

This is the solution! The point we want to make here is that if we reduce the left side of the augmented matrix to the identity matrix, then what is on the right side will be the solution to our system of equations.

This method of solving a linear system of equations, where reduce our matrix to row echelon form, make all the pivots equal to 1, and then make everything above the pivots equal to 0, is called Gauss-Jordan Elimination.

4.2.2 The Algorithm

Given a matrix A, we want to try and find A^{-1}, if possible. The idea at first is to try and find A^{-1} one column at a time. For the sake of clarity, let's suppose that our matrices are of size 3×3. Suppose

$$A^{-1} = \begin{bmatrix} a & d & g \\ b & e & h \\ c & f & i \end{bmatrix}.$$

We know that multiplying a matrix by the j^{th} standard basis vector returns its j^{th} column, so

$$\begin{bmatrix} a \\ b \\ c \end{bmatrix} = A^{-1} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} d \\ e \\ f \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \begin{bmatrix} g \\ h \\ i \end{bmatrix} = A^{-1} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

If we multiply both sides of each of these equations by A on the left, we get

$$A\begin{bmatrix} a \\ b \\ c \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \qquad A\begin{bmatrix} d \\ e \\ f \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad A\begin{bmatrix} g \\ h \\ i \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}.$$

Let's focus on the left most equation right now. If we want to solve for (a, b, c), which represents the first column of A^{-1}, we could set up the augmented matrix

$$\left[\begin{array}{c|c} A & \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \end{array}\right]$$

and perform Gauss-Jordan elimination. We would row-reduce the left side of the augmented matrix to an identity matrix, and then the solutions, which are the values of a, b, c, will appear on the right side:

$$\left[\begin{array}{c|c} A & \begin{matrix} 1 \\ 0 \\ 0 \end{matrix} \end{array}\right] \longrightarrow \left[\begin{array}{c|c} I & \begin{matrix} a \\ b \\ c \end{matrix} \end{array}\right].$$

We can solve for (d, e, f) and (g, h, i), the second and third columns of A^{-1}, by doing the same thing:

$$\left[\begin{array}{c|c} A & \begin{matrix} 0 \\ 1 \\ 0 \end{matrix} \end{array}\right] \longrightarrow \left[\begin{array}{c|c} I & \begin{matrix} d \\ e \\ f \end{matrix} \end{array}\right] \quad \text{and} \quad \left[\begin{array}{c|c} A & \begin{matrix} 0 \\ 0 \\ 1 \end{matrix} \end{array}\right] \longrightarrow \left[\begin{array}{c|c} I & \begin{matrix} g \\ h \\ i \end{matrix} \end{array}\right].$$

Now here is the key idea: instead of doing this three separate times, just do everything at once! That is to say: instead of augmenting A with $(1, 0, 0)$ and row reducing to find the first column of A^{-1}, and then augmenting A with $(0, 1, 0)$ and row reducing to find the second column of A^{-1}, and then augmenting A with $(0, 0, 1)$ and row reducing to find the third column of A^{-1}, we should augment A with $(1, 0, 0)$, $(0, 1, 0)$, and $(0, 0, 1)$ all at once, and row reduce everything to find all three columns of A^{-1} at once! Written symbolically, we are suggesting to do the following:

$$\left[\begin{array}{c|ccc} & 1 & 0 & 0 \\ A & 0 & 1 & 0 \\ & 0 & 0 & 1 \end{array}\right] \longrightarrow \left[\begin{array}{c|ccc} & a & d & g \\ I & b & e & h \\ & c & f & i \end{array}\right].$$

This generalizes for any $n \times n$ matrix. The general algorithm for finding A^{-1} is the following: take the matrix A and augment it with an identity matrix on the right. Then row reduce until you have the identity on the left; whatever is on the right will be A^{-1}!

$$\left[\begin{array}{c|c} A & I \end{array}\right] \longrightarrow \left[\begin{array}{c|c} I & A^{-1} \end{array}\right]$$

Example. Suppose we want to find the inverse of the following matrix:

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 5 \\ 1 & -2 & 2 \end{bmatrix}.$$

Start by augmenting it with a 3×3 identity matrix:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -1 & 5 & 0 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 1 \end{array}\right].$$

Now we start row reducing:

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -1 & 5 & 0 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 1 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-2R_1} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$

$$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 2 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{-1R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 2 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow[R_3-2R_2]{R_1+R_2}$$

$$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 & 1 & 2 \end{array}\right] \xrightarrow{R_1-2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & -2 & -5 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 & 1 & 2 \end{array}\right].$$

Thus,

$$A^{-1} = \begin{bmatrix} 8 & -2 & -5 \\ 1 & 0 & -1 \\ -3 & 1 & 2 \end{bmatrix}.$$

You can confirm that multiplying this with A will result in an identity matrix. $\diamond$

Remark. This algorithm will fail if we are not able to reduce A to an identity matrix. If this happens, it means that A is not invertible. As a consequence of this, we have the following

Proposition. A square matrix is invertible if and only if it can be row-reduced to an identity matrix.

4.3 Elementary Matrices

An elementary matrix is any matrix that can be obtained from an identity matrix by performing exactly one row operation. Let's start by performing some row operations on an identity matrix to get some elementary matrices to play with.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3+5R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix}.$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_1 \leftrightarrow R_2} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The matrices

$$E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \quad \text{and} \quad F = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

are elementary matrices, since they were obtained by performing exactly one row operation on an identity matrix.

Let's take our first elementary matrix E and multiply it with an arbitrary 3×3 matrix to see what happens.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} a & b & c \\ d & e & f \\ g+5a & h+5b & i+5c \end{bmatrix}.$$

Notice the result was to add 5 of Row 1 to Row 3, the same operation that was performed on the identity matrix to find E in the first place!

Similarly, if we take the elementary matrix F and multiply it with an arbitrary 3×3 matrix:

$$\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix} = \begin{bmatrix} d & e & f \\ a & b & c \\ g & h & i \end{bmatrix},$$

we see that the result is to swap Row 1 and Row 2; again, this is the same row operation that we performed on the identity matrix to obtain F!

These observations generalize to the following

Proposition. Multiplying a matrix on the left by an elementary matrix has the same effect as performing the corresponding row operation.

Thus, we can imagine that every time we perform a row operation, we are really multiplying our matrix on the left by an elementary matrix. This perspective yields some interesting and sometimes useful results. We present one of them further below.

Remark. What if we multiply on the right by an elementary matrix? Notice that E can also be obtained from the identity matrix by adding 5 of Column 3 to Column 1. In general, every elementary matrix has a corresponding *column* operation as well, and multiplying a matrix on the right by an elementary matrix has the effect of performing the corresponding column operation. Try taking an arbitrary 3×3 matrix and multiplying it by E on the right to confirm this result!

4.3.1 The Inverse of an Elementary Matrix

Every row operation has an inverse row operation, i.e., a row operation that will undo it. For instance, consider how we obtained the matrix E:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3+5R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} = E.$$

Notice that we can undo this and go back to the identity by performing the following:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 5 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 - 5R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

So $R_3 - 5R_1$ is the inverse row operation to $R_3 + 5R_1$. The elementary matrix corresponding to this inverse row operation turns out to be the inverse of E. Thus, if we compute:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3 - 5R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix},$$

then we find that

$$E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix}.$$

This discussion generalizes to the following:

Proposition. Every elementary matrix is invertible, and the inverse of any elementary matrix is also an elementary matrix. More specifically, the inverse of an elementary matrix is the elementary matrix corresponding to the inverse row operation.

4.3.2 Factoring an Invertible Matrix into a Product of Elementary Matrices

In Subsection 4.2.2 we found the inverse of the matrix

$$A = \begin{bmatrix} 1 & -1 & 2 \\ 2 & -1 & 5 \\ 1 & -2 & 2 \end{bmatrix}$$

Here is the work, copy-pasted from before:

> Start by augmenting it with a 3×3 identity matrix:
>
> $$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -1 & 5 & 0 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 1 \end{array}\right].$$
>
> Now we start row reducing:
>
> $$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 2 & -1 & 5 & 0 & 1 & 0 \\ 1 & -2 & 2 & 0 & 0 & 1 \end{array}\right] \underset{\xrightarrow{R_3 - R_1}}{\xrightarrow{R_2 - 2R_1}} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 2 & 1 & -1 & 1 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3}$$
>
> $$\left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & -1 & 0 & -1 & 0 & 1 \\ 0 & 2 & 1 & -1 & 1 & 0 \end{array}\right] \xrightarrow{-1R_2} \left[\begin{array}{ccc|ccc} 1 & -1 & 2 & 1 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 2 & 1 & -1 & 1 & 0 \end{array}\right] \underset{\xrightarrow{R_3 - 2R_2}}{\xrightarrow{R_1 + R_2}}$$
>
> $$\left[\begin{array}{ccc|ccc} 1 & 0 & 2 & 2 & 0 & -1 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 & 1 & 2 \end{array}\right] \xrightarrow{R_1 - 2R_3} \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 8 & -2 & -5 \\ 0 & 1 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 & 1 & 2 \end{array}\right].$$
>
> Thus,
>
> $$A^{-1} = \begin{bmatrix} 8 & -2 & -5 \\ 1 & 0 & -1 \\ -3 & 1 & 2 \end{bmatrix}.$$

Let's take each of the row operations performed, and find the corresponding elementary matrices.

$$R_2 - 2R_1 \qquad E_1 = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 - R_1 \qquad E_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix}$$

$$R_2 \leftrightarrow R_3 \qquad E_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

$$-1R_2 \qquad E_4 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_1 + R_2 \qquad E_5 = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$R_3 - 2R_2 \qquad E_6 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$$

$$R_1 - 2R_3 \qquad E_7 = \begin{bmatrix} 1 & 0 & -2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Note that the elementary matrices are numbered according to the order in which the row operations were performed. Performing these 7 row operations reduced A to an identity matrix. Thus, multiplying A by these 7 elementary matrices results in the identity matrix:

$$E_7E_6E_5E_4E_3E_2E_1A = I.$$

Note the order the elementary matrices were written in: since E_1 corresponds to the first row operation performed, it should be the first elementary matrix that is multiplied to A; hence, why E_1 is closest to A; etc. The above equation implies that

$$A^{-1} = E_7E_6E_5E_4E_3E_2E_1.$$

By the dressing-undressing principle, it follows that

$$A = E_1^{-1}E_2^{-1}E_3^{-1}E_4^{-1}E_5^{-1}E_6^{-1}E_7^{-1}.$$

Thus, we see that A and A^{-1} can be factored into a product of elementary matrices. We can apply this procedure to any invertible matrix. Thus, we have the following:

Proposition. Every invertible matrix is a product of elementary matrices.

4.4 LU Decomposition

4.4.1 The Idea & Definition

Let's just do an example first. Consider the following matrix:

$$A = \begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -3 \\ 3 & 4 & 4 \end{bmatrix}.$$

Let's reduce A to an upper triangular matrix:

$$\begin{bmatrix} 1 & 0 & 1 \\ -2 & 1 & -3 \\ 3 & 4 & 4 \end{bmatrix} \xrightarrow[R_3-3R_1]{R_2+2R_1} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 4 & 1 \end{bmatrix} \xrightarrow{R_3-4R_2} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix}.$$

Call the upper triangular matrix that we ended up with U:

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 5 \end{bmatrix}.$$

Next, let's get the elementary matrices corresponding to the row operations performed:

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_2+2R_1} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = E_1$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3-3R_1} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix} = E_2$$

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{R_3-4R_2} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -4 & 1 \end{bmatrix} = E_3.$$

Since performing row operations is equivalent to multiplication on the left by the corresponding elementary matrices, it follows that

$$E_3E_2E_1A = U,$$

or equivalently,

$$A = E_1^{-1}E_2^{-1}E_3^{-1}U.$$

Let $L = E_1^{-1}E_2^{-1}E_3^{-1}$, so that $A = LU$. We compute:

$$L = \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{E_1^{-1}} \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}}^{E_2^{-1}} \overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 4 & 1 \end{bmatrix}}^{E_3^{-1}} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 3 & 4 & 1 \end{bmatrix}.$$

Notice that L is a lower triangular matrix with 1's on the diagonal. Thus, we have factored A into the product of a lower triangular matrix and an upper triangular matrix. This is the LU decomposition of a matrix A.

Definition. The LU decomposition of a matrix A consists of factoring A into a product of two matrices, $A = LU$. The matrix L is lower triangular with 1's on the diagonal, and the matrix U is upper triangular.

Let's take a closer look at the above example. The upper triangular matrix U is obtained by row reducing the matrix A. Notice the type of row operation that we did: we only added and subtracted multiples of the pivot rows to produce the zeros below the diagonal. It is important that you do the same when trying to find an LU factorization of a matrix. In particular, you should never: swap rows, multiply a row by a nonzero number, or use a non-pivot row to produce zeros. Sometimes, it is not possible to row reduce a matrix into upper triangular form without performing a row swap; thus, not every matrix has an LU decomposition.

The lower triangular matrix L with 1's on the diagonal was obtained by finding the elementary matrices corresponding to the row operations performed, and multiplying their inverses together appropriately. But, we don't really need to do all that. Instead, take a look at the row operations performed, and the end result:

Row Operations Performed: $R_2 \boxed{+2} R_1 \qquad R_3 \boxed{-3} R_1 \qquad R_3 \boxed{-4} R_2.$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ \boxed{-2} & 1 & 0 \\ \boxed{3} & \boxed{4} & 1 \end{bmatrix}.$$

The observation is that the numbers used in the row operations go into the corresponding entries of L with opposite sign. In particular,

- the operation $R_2 + 2R_1$ puts a -2 into the $(2,1)$-entry of L;
- the operation $R_3 - 3R_1$ puts a $+3$ into the $(3,1)$-entry of L; and
- the operation $R_3 - 4R_2$ puts a $+4$ into the $(3,2)$-entry of L.

We summarize the procedure for finding an LU decomposition below.

1. Find U by reducing A to an upper triangular matrix. You must reduce by adding or subtracting multiples of the pivot rows to produce zeros below the diagonal. If this is not possible (i.e., because a row swap is necessary), then A does not have an LU decomposition.

2. Find L by taking the numbers used in the row operations and putting them into the corresponding entries below the diagonal of L with opposite sign. The diagonal entries are all 1's.

4.4.2 Example & Application

Example. Find an LU decomposition for the following matrix, if possible:

$$A = \begin{bmatrix} 2 & 0 & 2 \\ -3 & 1 & -4 \\ 4 & 5 & 2 \end{bmatrix}.$$

Solution. Start by reducing A to find U:

$$A = \begin{bmatrix} 2 & 0 & 2 \\ -3 & 1 & -4 \\ 4 & 5 & 2 \end{bmatrix} \xrightarrow[R_3 - 2R_1]{R_2 + \frac{3}{2}R_1} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 5 & -2 \end{bmatrix} \xrightarrow{R_3 - 5R_2} \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix} = U.$$

To find L, we know it is lower triangular with 1's on the diagonal.

$$L = \begin{bmatrix} 1 & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{bmatrix}.$$

The remaining entries are found by taking the numbers used in the row operations and changing their sign:

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix}.$$

You can check by multiplying L and U that $A = LU$. ◇

Example. Use the results from the previous problem to solve the following linear system of equations.

$$\begin{aligned} 2x_1 + \phantom{x_2 - {}} 2x_3 &= -2 \\ -3x_1 + x_2 - 4x_3 &= 5 \\ 4x_1 + 5x_2 + 2x_3 &= -6. \end{aligned}$$

Solution. We can rewrite this system of equations as

$$A\mathbf{x} = \mathbf{b},$$

where A is the matrix from the previous example,

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \qquad \text{and} \qquad \mathbf{b} = \begin{bmatrix} -2 \\ 5 \\ -6 \end{bmatrix}.$$

Since $A = LU$, this is equivalent to solving

$$LU\mathbf{x} = \mathbf{b}.$$

Here is the strategy for solving for $\mathbf{x}$. Let $U\mathbf{x} = \mathbf{y}$. So now the equation becomes

$$L\mathbf{y} = \mathbf{b}.$$

We'll solve this equation for $\mathbf{y}$, and once we solve for $\mathbf{y}$, we can go back to the equation $U\mathbf{x} = \mathbf{y}$ to solve for $\mathbf{x}$.

Let's start with the equation $L\mathbf{y} = \mathbf{b}$. We have

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} -2 \\ 5 \\ -6 \end{bmatrix}.$$

So $L\mathbf{y} = \mathbf{b}$ is equivalent to

$$\begin{aligned} y_1 \phantom{{}+5y_2+y_3} &= -2 \\ -\tfrac{3}{2}y_1 + y_2 \phantom{{}+y_3} &= 5 \\ 2y_1 + 5y_2 + y_3 &= -6. \end{aligned}$$

We already have $y_1 = -2$. Substituting this into the second equation gives us $y_2 = 2$. Substituting these into the third equation gives us $y_3 = -12$. Thus,

$$\mathbf{y} = \begin{bmatrix} -2 \\ 2 \\ -12 \end{bmatrix}.$$

Now that we have we have $\mathbf{y}$, we go back to the equation $U\mathbf{x} = \mathbf{y}$ to solve for $\mathbf{x}$. Since

$$U = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix},$$

the equation $U\mathbf{x} = \mathbf{y}$ is equivalent to

$$\begin{aligned} 2x_1 + \phantom{x_2 - {}} 2x_3 &= -2 \\ x_2 - x_3 &= 2 \\ 3x_3 &= -12. \end{aligned}$$

The third equation gives us $x_3 = -4$. Substituting this into the second equation gives us $x_2 = -2$. Substituting these into the first equation gives us $x_1 = 3$. Thus,

$$\mathbf{x} = \begin{bmatrix} 3 \\ -2 \\ -4 \end{bmatrix}. \qquad \diamond$$

Summary

In order to solve $A\mathbf{x} = \mathbf{b}$ using $A = LU$:

1. Let $\mathbf{y} = U\mathbf{x}$ and solve $L\mathbf{y} = \mathbf{b}$ for $\mathbf{y}$. Since L is lower triangular, you can solve for $\mathbf{y}$ using forward-substitution.

2. Once you have $\mathbf{y}$, go back and solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$. Since U is upper triangular, you can solve for $\mathbf{x}$ using backward-substitution.

4.5 LDU Decomposition

A slight modification to finding an LU decomposition leads to an LDU decomposition. In an LDU decomposition, L is lower triangular with 1's on the diagonal, D is diagonal, and U is upper triangular with 1's on the diagonal. It's easiest to illustrate with an example.

Example. Previously, we found an LU decomposition of the following matrix:

$$A = \begin{bmatrix} 2 & 0 & 2 \\ -3 & 1 & -4 \\ 4 & 5 & 2 \end{bmatrix}.$$

In particular, the lower and upper triangular matrices that we found were

$$L = \begin{bmatrix} 1 & 0 & 0 \\ -\frac{3}{2} & 1 & 0 \\ 2 & 5 & 1 \end{bmatrix} \qquad U_0 = \begin{bmatrix} 2 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 3 \end{bmatrix}.$$

Note that I am calling the U in the LU decomposition U_0 now, so as not to confuse with the U in the LDU decompostion that we will find below. The lower triangular matrix L that we found in the LU decomposition will also be the lower triangular matrix that we use in the LDU decomposition.

To find D in LDU, take the diagonal entries of the upper triangular matrix U_0 from the LU decomposition and place them into a diagonal matrix:

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 3 \end{bmatrix}.$$

To find U in LDU, divide each row of U_0 by its corresponding diagonal entry.

$$U = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{bmatrix}.$$

You can check that $A = LDU$ by checking that $DU = U_0$. $\diamond$

We summarize the procedure for finding an $A = LDU$ decomposition below:

- Start by finding an $A = LU$ decomposition. Call the lower and upper triangular matrices found L and U_0, respectively. We already have L from this first step.
- Take the diagonal entries of U_0 and put them into a diagonal matrix. This gives us D.
- Finally, divide each row of U_0 by its corresponding diagonal entry to get U.

Note, the step where we divide each row of U_0 by its diagonal entry is not possible if one of the diagonal entries is zero. In such a case, an LDU decomposition with 1's along the diagonal of U is not possible. However, if we drop the requirement that the diagonal entries of U are all 1's, then it is sometimes possible to get $A = LDU$ even if U_0 has a diagonal entry of zero.

4.6 LDL^T Decomposition for Symmetric Matrices

4.6.1 The Transpose of a Matrix

Let A be an $m \times n$ matrix. Its transpose, denoted A^T and read as "A transpose", is the $n \times m$ matrix whose first column is the first row of A; whose second column is the second row of A; etc.

Example. If

$$A = \begin{bmatrix} 1 & 5 & 0 & 3 \\ 3 & 4 & 2 & 1 \\ 2 & 0 & 5 & 7 \\ 6 & 2 & 1 & 4 \\ 3 & 7 & 8 & 2 \end{bmatrix},$$

then

$$A^T = \begin{bmatrix} 1 & 3 & 2 & 6 & 3 \\ 5 & 4 & 0 & 2 & 7 \\ 0 & 2 & 5 & 1 & 8 \\ 3 & 1 & 7 & 4 & 2 \end{bmatrix}.$$

Notice how the first row of A is the first column of A^T; the second row of A is the second column of A^T; etc. ◇

Proposition. The following is a list of properties of transposition.

1. Assuming A and B are the same size (so they can be added),
$$(A + B)^T = A^T + B^T.$$
2. For any scalar k,
$$(kA)^T = kA^T.$$
3. Assuming AB is defined,
$$(AB)^T = B^T A^T.$$
Thus, transposition follows the "dressing-undressing principle".
4. If A is invertible, then so is A^T, and
$$(A^T)^{-1} = (A^{-1})^T,$$
i.e., the inverse of A^T is the transpose of A^{-1}.

Let's prove the fourth property. If A is invertible, then we have $AA^{-1} = I$. Notice from Property 3 that $(AA^{-1})^T = (A^{-1})^T A^T$, while the transpose of an identity matrix is itself. Thus, transposing both sides of $AA^{-1} = I$ gives us $(A^{-1})^T A^T = I$. Since multiplying A^T by $(A^{-1})^T$ gives an identity matrix, this proves that A^T is invertible, and that $(A^{-1})^T$ is its inverse.

The importance of matrix transposition is related to the following fact: if $\mathbf{u}$, $\mathbf{v}$ are vectors in $\mathbb{R}^n$, then $\mathbf{u}^T\mathbf{v}$ is equal to the dot product of $\mathbf{u}$ and $\mathbf{v}$. To see this, let

$$\mathbf{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix} \qquad \mathbf{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}.$$

Then

$$\begin{aligned} \mathbf{u}^T\mathbf{v} &= \begin{bmatrix} u_1 & u_2 & \cdots & u_n \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix} \\ &= u_1 v_1 + u_2 v_2 + \ldots + u_n v_n \\ &= \mathbf{u} \cdot \mathbf{v}. \end{aligned}$$

4.6.2 Symmetric Matrices

A matrix A is said to be symmetric if it satisfies $A = A^T$. In words, A is symmetric if it is equal to its own transpose. It is easy to see that if A is symmetric then it must be square. The reasoning for the terminology is most apparent by looking at an example.

A typical 3×3 symmetric matrix looks like this:

$$A = \begin{bmatrix} a & d & e \\ d & b & f \\ e & f & c \end{bmatrix}.$$

You should verify that $A = A^T$. In general, a symmetric matrix is symmetric, in the colloquial sense, about the diagonal.

4.6.3 The LDL^T Decomposition

Symmetric matrices have some particularly nice properties. One of the main results in this class is the Spectral Theorem for real symmetric matrices, which leads to an important factorization of symmetric matrices called the spectral decomposition. Until then, we discuss another factorization of symmetric matrices, called the LDL^T decomposition.

An LDL^T decomposition of a symmetric matrix is really just an LDU factorization. What happens is, if A is symmetric, then the U in an LDU factorization will turn out to be the transpose of L. We illustrate this with an example.

Example. Consider the following symmetric matrix:

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 3 & -1 \\ -2 & -1 & -1 \end{bmatrix}.$$

Let's find an LDU decomposition of A. Start by row reducing A to an upper triangular matrix:

$$\begin{bmatrix} 1 & 2 & -2 \\ 2 & 3 & -1 \\ -2 & -1 & -1 \end{bmatrix} \xrightarrow[R_3+2R_1]{R_2-2R_1} \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 3 \\ 0 & 3 & -5 \end{bmatrix} \xrightarrow{R_3+3R_2} \begin{bmatrix} 1 & 2 & -2 \\ 0 & -1 & 3 \\ 0 & 0 & 4 \end{bmatrix}.$$

Then

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -3 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

To find U, divide each row of the above upper triangular matrix by its pivot:

$$U = \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}.$$

Notice that U is the transpose of L, as promised. Thus, $A = LDU = LDL^T$. ◇

More generally, to find an LDL^T decomposition of a symmetric matrix A,

1. Start by finding an LU decomposition of A. The L obtained in this step is the L needed for the LDL^T decomposition.
2. Find the matrix D by taking the diagonal entries of U and putting them into an diagonal matrix.
3. Then $A = LDL^T$.

Note that this procedure works even if the U in the LU decomposition has zeros along its diagonal; just omit dividing that row by 0. ☺

4.7 $PA = LU$ Decomposition

Let A be a square matrix. All of the decompositions that we have discussed so far require that one can row reduce A to an upper triangular matrix without swapping rows. So what happens if we have to swap rows in order to row reduce A to an upper triangular matrix? In such a case, it is possible to obtain what people call a $PA = LU$ decomposition instead. The idea is to perform any necessary row swaps first. P will be the product of all the elementary matrices corresponding to the row swaps performed. Then, since we have done all the necessary row swaps, we can now find an LU decomposition of PA. We will illustrate with an example.

Example. Let

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 7 & 5 \end{bmatrix}.$$

If we start trying to find an LU decomposition of A, we get

$$\begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 7 & 5 \end{bmatrix} \begin{matrix} \xrightarrow{R_2 - 2R_1} \\ \xrightarrow{R_3 - 3R_1} \end{matrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 0 & 1 \\ 0 & 1 & 2 \end{bmatrix}.$$

In order to row reduce A to upper triangular form, we see at this point that we will have to swap rows 3 and 2. In particular, A does not admit an LU decomposition.

Let's go back to A and swap rows first.

$$A = \begin{bmatrix} 1 & 2 & 1 \\ 2 & 4 & 3 \\ 3 & 7 & 5 \end{bmatrix} \xrightarrow{R_2 \leftrightarrow R_3} \begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 5 \\ 2 & 4 & 3 \end{bmatrix}.$$

The elementary matrix P corresponding to this row swap is

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix},$$

so

$$PA = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 5 \\ 2 & 4 & 3 \end{bmatrix}.$$

Now we proceed to find an LU decomposition of PA.

$$PA = \begin{bmatrix} 1 & 2 & 1 \\ 3 & 7 & 5 \\ 2 & 4 & 3 \end{bmatrix} \begin{matrix} \xrightarrow{R_2 - 3R_1} \\ \xrightarrow{R_3 - 2R_1} \end{matrix} \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus,

$$U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}.$$

You can multiply everything out to verify that $PA = LU$. ◇

Given a $PA = LU$ decomposition, we can use it solve a system of equations $A\mathbf{x} = \mathbf{b}$ as follows:

1. Multiply both sides of $A\mathbf{x} = \mathbf{b}$ on the left by P to get $PA\mathbf{x} = P\mathbf{b}$.
2. Since $PA = LU$, this becomes $LU\mathbf{x} = P\mathbf{b}$.
3. Let $\mathbf{y} = U\mathbf{x}$, and solve $L\mathbf{y} = P\mathbf{b}$ for $\mathbf{y}$, using forward substitution.

4. Once we have solved for $\mathbf{y}$, we can then solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$, using backward substitution.

Example. Using the matrix A from the previous example, and the $PA = LU$ decomposition, solve the equation $A\mathbf{x} = \mathbf{b}$, where $\mathbf{b} = (1, 3, 4)$.

Solution. First, we compute:

$$P\mathbf{b} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}.$$

Next, we need to solve $L\mathbf{y} = P\mathbf{b}$ for $\mathbf{y}$. With

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix} \qquad \mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \qquad P\mathbf{b} = \begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix},$$

we get the following system of equations:

$$\begin{aligned} y_1 \qquad\quad &= 1 \\ 3y_1 + y_2 \qquad &= 4 \\ 2y_1 + \qquad y_3 &= 3. \end{aligned}$$

Solving this with forward substitution yields $y_1 = 1$, $y_2 = 1$, and $y_3 = 1$.

Finally, we need to solve $U\mathbf{x} = \mathbf{y}$ for $\mathbf{x}$. Since

$$U = \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 1 \end{bmatrix}$$

and $\mathbf{y} = (1, 1, 1)$, we get the following system of equations:

$$\begin{aligned} x_1 + 2x_2 + \ x_3 &= 1 \\ x_2 + 2x_3 &= 1 \\ x_3 &= 1. \end{aligned}$$

Solving this with backward substitution gives us $x_3 = 1$, $x_2 = -1$, and $x_1 = 2$. Thus, the solution to $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

◇

Chapter 5

Vector Spaces, Subspaces, Basis & Dimension, and The Fundamental Subspaces of a Matrix

5.1 General Vector Spaces

So far in this class, we've been working a lot with vectors in $\mathbb{R}^n$, as well as with matrices. For both vectors and matrices, we are able to add them as well as multiply them by scalars. You have encountered other objects in other classes that can be added and multiplied by scalars. The most notable example would be functions.

A general vector space is essentially any collection of objects that can be added and multiplied by scalars. The usual definition of a vector space that is presented to students in an introductory linear algebra class is something like this:

Definition. A vector space is a set V, equipped with a notion of addition and scalar multiplication that satisfies:

1. V is closed under addition; i.e., for any $\mathbf{u}$, $\mathbf{v}$ in V, $\mathbf{u}+\mathbf{v}$ is in V.
2. Addition is associative; i.e., for any $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ in V, $(\mathbf{u}+\mathbf{v})+\mathbf{w}=\mathbf{u}+(\mathbf{v}+\mathbf{w})$.
3. Addition is commutative; i.e., for any $\mathbf{u}$, $\mathbf{v}$ in V, $\mathbf{u}+\mathbf{v}=\mathbf{v}+\mathbf{u}$.
4. There exists an additive identity, denoted $\mathbf{0}$, in V; i.e., there exists an element in V, denoted $\mathbf{0}$, such that for any $\mathbf{v}$ in V, $\mathbf{0}+\mathbf{v}=\mathbf{v}$.
5. Every element in V has an additive inverse; i.e., for any $\mathbf{v}$ in V, there exists an element in V, denoted $-\mathbf{v}$, such that $(-\mathbf{v})+\mathbf{v}=\mathbf{0}$.
6. V is closed under scalar multiplication; i.e., for any scalar c, and for any $\mathbf{v}$ in V, $c\mathbf{v}$ is in V.
7. The following properties hold for any scalars c, d, and any vectors $\mathbf{u}$, $\mathbf{v}$:
   - $c(\mathbf{u}+\mathbf{v})=c\mathbf{u}+c\mathbf{v}$.
   - $(c+d)\mathbf{v}=c\mathbf{v}+d\mathbf{v}$.
   - $(cd)\mathbf{v}=c(d\mathbf{v})$.
8. For any $\mathbf{v}$ in V, $1\mathbf{v}=\mathbf{v}$.

At the time I am writing this, in MATH 2418 at UTD, you will almost exclusively consider scalars to be real numbers. However, you might occasionally need to consider complex scalars as well. In general, scalars could come from other fields as well, but you don't need to worry about that.

And honestly, it's not too important for you to memorize the above axioms. They basically say that anything that seems reasonable will be true in a vector space. For instance, suppose we have a vector space V. Then for any vector $\mathbf{v}$ in V, it is true that $0 \cdot \mathbf{v} = \mathbf{0}$. This isn't mentioned anywhere in the definition, but it actually follows as a consequence of it.

We list some important examples of vector spaces below:

1. $\mathbb{R}^n$, with the usual notions of addition and scalar multiplication, is a vector space.

2. $M_{m\times n}(\mathbb{R})$ denotes the set of all $m \times n$ matrices with entries in $\mathbb{R}$. With the usual notions of matrix addition and scalar multiplication, $M_{m\times n}(\mathbb{R})$ is a vector space. For the special case where $m = n$, we use the notation $M_n(\mathbb{R})$ instead of the more cumbersome $M_{n\times n}(\mathbb{R})$. As a shortcut, we will often drop the $\mathbb{R}$ from the notation and write $M_{m\times n}$ and M_n.

3. $P_n(\mathbb{R})$ denotes the set of all polynomial function with coefficients in $\mathbb{R}$ of degree less than or equal to n. With the usual notions of adding two polynomials and multiplying a polynomial by a scalar, $P_n(\mathbb{R})$ is a vector space. Again, as a shortcut, we will often drop the $\mathbb{R}$ from the notation and write P_n.

4. $C^k([a, b])$ denotes the set of all functions that are k-times continuously differentiable on the interval $[a, b]$. For the special case where $k = 0$, $C^0([a, b])$ denotes the set of all continuous functions on the interval $[a, b]$. With the usual notions of adding two functions and multiplying a function by a scalar, $C^k([a, b])$ is a vector space. We won't be dealing with this guy much in this class, but it is important enough to be mentioned.

5.2 Subspaces

Last time, we introduced the notion of a vector space. Essentially, a vector space is a set of objects that can be added and multiplied by scalars. We provided, as examples of vectors spaces, the following:

- $\mathbb{R}^n$
- $M_{m\times n}$ and M_n - the vector spaces of all $m \times n$ matrices and all $n \times n$ matrices, respectively (
- P_n - the vector space of all polynomial functions of degree less than or equal to n

Pretty much all of the vector spaces that we will work with in this class turn out to be subspaces of the three classes of vector spaces listed above, and mostly $\mathbb{R}^n$ at that. A ***subspace*** of a vector space V is a subset W of V that is a vector space in and of itself. It turns out that W is a subspace if and only if the following two conditions hold:

1. W is closed under addition, i.e. for any $\mathbf{u}, \mathbf{v}$ in W, $\mathbf{u} + \mathbf{v}$ is in W.
2. W is closed under scalar multiplication, i.e. for any scalar k, and for any $\mathbf{v}$ in W, $k\mathbf{v}$ is in W.

To prove that something is a subspace, one must show that it satisfies these two conditions. These types of problems are a bit more abstract and often nothing quite like what students have encountered before, and because of this, I find that students often struggle a bit with them. In an effort to help out, my plan for this section is to inundate you with a plethora of examples.

5.2.1 Examples of Subspaces in $\mathbb{R}^n$

Example. Show that the set W of all vectors of the form $(a, 2a - 3b, b)$, $a, b \in \mathbb{R}$, is a subspace of $\mathbb{R}^3$.

Solution. First, we check that W is closed under addition. Let $\mathbf{u}, \mathbf{v}$ be two vectors in W, so

$$\mathbf{u} = \begin{bmatrix} a_1 \\ 2a_1 - 3b_1 \\ b_1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} a_2 \\ 2a_2 - 3b_2 \\ b_2 \end{bmatrix}$$

for some a_1, b_1, a_2, b_2 in $\mathbb{R}$. Then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} a_1 + a_2 \\ 2(a_1 + a_2) - 3(b_1 + b_2) \\ b_1 + b_2 \end{bmatrix}$$

If we let $a = a_1 + a_2$ and $b = b_1 + b_2$, then we see that $\mathbf{u} + \mathbf{v}$ has the form $(a, 2a - 3b, b)$. Thus, $\mathbf{u} + \mathbf{v}$ is in W. This proves that the sum of any two vectors in W is also in W, so we conclude that W is closed under addition.

Next, we check that W is closed under scalar multiplication. For any real number k,

$$k\mathbf{u} = \begin{bmatrix} ka_1 \\ 2ka_1 - 3kb_1 \\ kb_1 \end{bmatrix}$$

If we let $a = ka_1$ and $b = kb_1$, then we see that $k\mathbf{u}$ has the form $(a, 2a - 3b, b)$. Thus, $k\mathbf{u}$ is in W. This shows that any scalar multiple of a vector in W is also in W, so we conclude that W is closed under scalar multiplication.

Since W is closed under addition and scalar multiplication, it follows that W is a subspace of $\mathbb{R}^3$. $\diamond$

Example. Let's modify the previous question ever so slightly. Suppose W is the set of all vectors of the form $(a+1, 2a-3b, b)$, $a, b \in \mathbb{R}$. (The modification is the extra $+1$ in the x-component.) Let's check to see if this new W is a subspace.

To check closure under addition, let $\mathbf{u}, \mathbf{v}$ be vectors in W, so

$$\mathbf{u} = \begin{bmatrix} a_1+1 \\ 2a_1 - 3b_1 \\ b_1 \end{bmatrix} \qquad \text{and} \qquad \mathbf{v} = \begin{bmatrix} a_2+1 \\ 2a_2 - 3b_2 \\ b_2 \end{bmatrix}$$

for some a_1, b_1, a_2, b_2 in $\mathbb{R}$. Then

$$\mathbf{u} + \mathbf{v} = \begin{bmatrix} a_1 + a_2 + 2 \\ 2(a_1 + a_2) - 3(b_1 + b_2) \\ b_1 + b_2 \end{bmatrix}$$

Notice that $\mathbf{u} + \mathbf{v}$ is of the form $(a+2, 2a-3b, b)$, with $a = a_1 + a_2$ and $b = b_1 + b_2$, rather than of the form $(a+1, 2a-3b, b)$. Thus, $\mathbf{u} + \mathbf{v}$ is NOT in W. Since the sum of two vectors in W is no longer in W, it follows that W is NOT closed under addition, and therefore NOT a subspace. ◇

Example. Show that the set W of all vectors (x, y, z) satisfying the equation $x - 2y + 3z = 0$ is a subspace of $\mathbb{R}^3$.

Solution. Again, we'll start by checking closure under addition. Let $\mathbf{u}, \mathbf{v}$ be two vectors in W. If we write $\mathbf{u} = (x_1, y_1, z_1)$ and $\mathbf{v} = (x_2, y_2, z_2)$, then being in W means that

$$x_1 - 2y_1 + 3z_1 = 0 \qquad \text{and} \qquad x_2 - 2y_2 + 3z_2 = 0. \tag{$*$}$$

To show that W is closed under addition, we need to show that $\mathbf{u} + \mathbf{v}$ is in W, so we compute:

$$\mathbf{u} + \mathbf{v} = (\overbrace{x_1 + x_2}^{x}, \overbrace{y_1 + y_2}^{y}, \overbrace{z_1 + z_2}^{z}).$$

To show that $\mathbf{u} + \mathbf{v}$ is in W, we need to verify that $x - 2y + 3z = 0$.

$$\begin{aligned} x - 2y + 3z &= (x_1 + x_2) - 2(y_1 + y_2) + 3(z_1 + z_2) \\ &= (x_1 - 2y_1 + 3z_1) + (x_2 - 2y_2 + 3z_2) \\ &= 0 + 0 \qquad \text{b/c of } (*) \\ &= 0. \end{aligned}$$

This proves that $\mathbf{u} + \mathbf{v}$ is in W. Thus, we see that the sum of any two vectors in W is also in W, and hence we conclude that W is closed under addition.

Next, we'll check that W is closed under scalar multiplication. Let c be a scalar. Then

$$c\mathbf{u} = (\overbrace{cx_1}^{x}, \overbrace{cy_1}^{y}, \overbrace{cz_1}^{z}).$$

To show that $c\mathbf{u}$ is in W, we again need to verify that $x - 2y + 3z = 0$.

$$\begin{aligned} x - 2y + 3z &= cx_1 - 2(cy_1) + 3(cz_1) \\ &= c(x_1 - 2y_1 + 3z_1) \\ &= c \cdot 0 \qquad \text{b/c of } (*) \\ &= 0. \end{aligned}$$

This proves that $c\mathbf{u}$ is in W. Thus, we see that any scalar multiple of a vector in W is also in W, and hence we conclude that W is closed under scalar multiplication.

Since W is both closed under addition and scalar multiplication, it follows that W is a subspace of $\mathbb{R}^3$. ◇

Example. Again, let's consider a slight modification of the previous example. Suppose W is the set of all vectors (x, y, z) satisfying the equation $x - 2y + 3z = 1$. (Note it is now $= 1$ rather than $= 0$.)

Let's check closure under addition. Let $\mathbf{u} = (x_1, y_1, z_1)$ and $\mathbf{v} = (x_2, y_2, z_2)$ be vectors in W, so that

$$x_1 - 2y_1 + 3z_1 = 1 \qquad \text{and} \qquad x_2 - 2y_2 + 3z_2 = 1. \tag{$*$}$$

Then

$$\mathbf{u} + \mathbf{v} = (\overbrace{x_1 + x_2}^{x}, \overbrace{y_1 + y_2}^{y}, \overbrace{z_1 + z_2}^{z}),$$

and

$$\begin{aligned} x - 2y + 3z &= (x_1 + x_2) - 2(y_1 + y_2) + 3(z_1 + z_2) \\ &= (x_1 - 2y_1 + 3z_1) + (x_2 - 2y_2 + 3z_2) \\ &= 1 + 1 \qquad \text{b/c of } (*) \\ &= 2. \end{aligned}$$

In particular, we see that $\mathbf{u} + \mathbf{v}$ satisfies the equation $x - 2y + 3z = 2$, rather than the equation $x - 2y + 3z = 1$. Since it does not satisfy the equation $x - 2y + 3z = 1$, it is not in W. This shows that the sum of two vectors in W is no longer in W, meaning that W is not closed under addition. Hence, W is not a subspace. ◇

5.2.2 The Nullspace of a Matrix

Going back to the example before this last one, where W was the set of all (x, y, z) satisfying $x - 2y + 3z = 0$, notice that this equation can be rewritten as the matrix equation

$$\begin{bmatrix} 1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = 0.$$

Generalizing this example, let A be an $m \times n$ matrix. The set of all $\mathbf{x}$ satisfying the equation $A\mathbf{x} = \mathbf{0}$ is called the nullspace of A, denoted $N(A)$. Notice that since A is $m \times n$, $\mathbf{x}$ must be $n \times 1$, i.e. $\mathbf{x}$ must be in $\mathbb{R}^n$. Hence, the nullspace of A is a subset of $\mathbb{R}^n$. Let us show that $N(A)$ is a subspace of $\mathbb{R}^n$.

We start by showing closure under addition. Suppose $\mathbf{u}, \mathbf{v}$ are in $N(A)$, so that $A\mathbf{u} = \mathbf{0}$ and $A\mathbf{v} = \mathbf{0}$. Then

$$A(\mathbf{u} + \mathbf{v}) = A\mathbf{u} + A\mathbf{v} = \mathbf{0} + \mathbf{0} = \mathbf{0}.$$

We see that $\mathbf{u} + \mathbf{v}$ is a solution to the equation $A\mathbf{x} = \mathbf{0}$. Hence, $\mathbf{u} + \mathbf{v}$ is in $N(A)$. This shows that the sum of any two vectors in $N(A)$ is also in $N(A)$, so $N(A)$ is closed under addition.

Checking closure under scalar multiplication next, suppose once again that $\mathbf{u}$ is in $N(A)$, so that $A\mathbf{u} = \mathbf{0}$. Then for any scalar c,

$$A(c\mathbf{u}) = c \cdot A\mathbf{u} = c \cdot \mathbf{0} = \mathbf{0}.$$

This shows that $c\mathbf{u}$ is a solution to the equation $A\mathbf{x} = \mathbf{0}$. Hence, any scalar multiple of a vector in $N(A)$ is also in $N(A)$, so $N(A)$ is closed under scalar multiplication.

Since $N(A)$ is closed under addition and scalar multiplication, it follows that $N(A)$ is a subspace of $\mathbb{R}^n$. We summarize our discussion below.

Proposition. Let A be an $m \times n$ matrix. Then $N(A)$, the nullspace of A, which is defined as the set of all solutions to the equation $A\mathbf{x} = \mathbf{0}$, is a subspace of $\mathbb{R}^n$.

5.2.3 Orthogonal Complements

Definition. Let W be a subspace of $\mathbb{R}^n$. The ***orthogonal complement*** of W, denoted $W^\perp$, is defined to be the set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in W. Note: $W^\perp$ is often read as "W perp".

Example. If W is the x-axis in $\mathbb{R}^2$, then $W^\perp$ is the y-axis. If W is the x-axis in $\mathbb{R}^3$, then $W^\perp$ is the yz-plane. ◇

We will be talking about orthogonal complements a bit when we get to the fundamental subspaces of a matrix (of which the nullspace is one of), and again when we talk about projections. We introduce them here however so that we can practice working with subspaces in the following proposition.

Proposition. For any subspace W of $\mathbb{R}^n$, its orthogonal complement $W^\perp$ is also a subspace of $\mathbb{R}^n$.

Proof. We start by showing closure under addition. Suppose $\mathbf{u}$, $\mathbf{v}$ are two vectors in $W^\perp$. Then $\mathbf{u}$ and $\mathbf{v}$ are orthogonal to every vector in W, meaning that for every $\mathbf{w}$ in W,

$$\mathbf{u}\cdot\mathbf{w} = 0 \qquad \text{and} \qquad \mathbf{v}\cdot\mathbf{w} = 0.$$

We need to show that $\mathbf{u}+\mathbf{v}$ is also in $W^\perp$, meaning, we need to show that $\mathbf{u}+\mathbf{v}$ is orthogonal to every $\mathbf{w}$ in W. Since $\mathbf{u}\cdot\mathbf{w} = 0 = \mathbf{v}\cdot\mathbf{w}$ for every $\mathbf{w}$ in W, it follows that for any $\mathbf{w}$ in W,

$$(\mathbf{u}+\mathbf{v})\cdot\mathbf{w} = \mathbf{u}\cdot\mathbf{w} + \mathbf{v}\cdot\mathbf{w} = 0 + 0 = 0.$$

Thus, $\mathbf{u}+\mathbf{v}$ is in $W^\perp$, and so $W^\perp$ is closed under addition.

Next, we check closure under scalar multiplication. Suppose that $\mathbf{u}$ is in $W^\perp$, so that for every $\mathbf{w}$ in W,

$$\mathbf{u}\cdot\mathbf{w} = 0.$$

Let k be any scalar. Then for any $\mathbf{w}$ in W,

$$(k\mathbf{u})\cdot\mathbf{w} = k(\mathbf{u}\cdot\mathbf{w}) = 0.$$

Thus, $k\mathbf{u}$ is orthogonal to every $\mathbf{w}$ in W and is thus in $W^\perp$. It follows that $W^\perp$ is also closed under scalar multiplication.

Since $W^\perp$ is closed under addition and scalar multiplication, this proves that $W^\perp$ is a subspace of $\mathbb{R}^n$. □

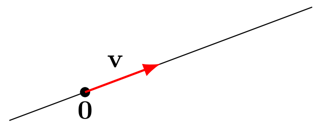
Proposition. For any subspace W of $\mathbb{R}^n$, the only vector that is common to both W and $W^\perp$ is the zero vector.

Proof. Suppose $\mathbf{v}$ is in W and $W^\perp$. We need to show that $\mathbf{v}$ must be $\mathbf{0}$. Since $\mathbf{v}$ is in $W^\perp$, it is orthogonal to every vector in W, and since $\mathbf{v}$ is in W, this means $\mathbf{v}$ is orthogonal to itself. This is equivalent to saying that $\mathbf{v}\cdot\mathbf{v} = 0$, and since $\mathbf{v}\cdot\mathbf{v} = \|\mathbf{v}\|^2$, we see that $\|\mathbf{v}\| = 0$. The only vector whose length is 0 is the zero vector, so $\mathbf{v}$ must be equal to $\mathbf{0}$. □

It is a fact that for any subspace W of $\mathbb{R}^n$, $(W^\perp)^\perp$ is equal to W. We don't quite have the tools to prove this yet though.

5.2.4 More on Subspaces of $\mathbb{R}^n$

Let us show that every line L passing through the origin in $\mathbb{R}^n$ is a subspace of $\mathbb{R}^n$. Let $\mathbf{v}$ be any nonzero vector in L.



Then every vector in L is a scalar multiple of $\mathbf{v}$. The sum of any two scalar multiples of $\mathbf{v}$ is also a scalar multiple of $\mathbf{v}$:

$$c\mathbf{v} + d\mathbf{v} = (c+d)\mathbf{v}.$$

This proves that L is closed under addition. Similarly, a scalar multiple of any scalar multiple of $\mathbf{v}$ is also a scalar multiple of $\mathbf{v}$:

$$c(d\mathbf{v}) = (cd)\mathbf{v}.$$

This proves that L is closed under scalar multiplication. Since L is both closed under addition and scalar multiplication, it follows that L is a subspace of $\mathbb{R}^n$.

Note that a line not passing through the origin would not be a subspace. In fact, any subspace of any vector space *must* contain the zero vector. Indeed, if W is a subspace, and $\mathbf{v}$ is a vector in W, then $0\mathbf{v} = \mathbf{0}$ must be in W, since W is closed under scalar multiplication.

We haven't rigorously defined dimension yet, but we have worked with the notion intuitively. Lines passing through the origin constitute the 1-dimensional subspaces of $\mathbb{R}^n$. Planes passing through the origin happen to constitute the 2-dimensional subspaces of $\mathbb{R}^n$. The set consisting only of the origin is a 0-dimensional subspace.

In $\mathbb{R}^2$, the only subspaces are:

- $\{(0,0)\}$ - the one 0-dimensional subspace
- lines passing through the origin - the 1-dimensional subspaces
- $\mathbb{R}^2$ - the one 2-dimesional subspace

All other subsets of $\mathbb{R}^2$ fail to be subspaces.

In $\mathbb{R}^3$, the only subspaces are:

- $\{(0,0,0)\}$ - the one 0-dimensional subspace
- lines passing through the origin - the 1-dimensional subspaces
- planes passing through the origin - the 2-dimensional subspaces
- $\mathbb{R}^3$ - the one 3-dimesional subspace

All other subsets of $\mathbb{R}^3$ fail to be subspaces.

Unfortunately, we don't seem to have the terminology for it, but whatever the higher dimensional analogs of lines and planes are, the ones passing through the origin would make up the higher dimensional subspaces of $\mathbb{R}^n$ for $n > 3$.

Example. Let W be the upper half-plane in $\mathbb{R}^2$, which consists of all vectors (x, y) satisfying $y \geq 0$. Since W is not the origin, a line through the origin, or $\mathbb{R}^2$ itself, it should not be a subspace of $\mathbb{R}^2$. To prove this definitively, we can show that it is not closed under scalar multiplication. For instance, the vector $(0, 1)$ lies in the upper half-plane, but if we multiply it by -1, we get the vector $(0, -1)$, which is not in the upper half-plane. $\diamond$

5.2.5 Subspaces of Matrices

Example. Let U_2 be the set of all 2×2 upper triangular matrices. Show that U_2 is a subspace of M_2 (the space of all 2×2 matrices).

Solution. We start by checking closure under addition. Let A, B be in U_2. Then A, B are 2×2 upper triangular, so we can write

$$A = \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} d & e \\ 0 & f \end{bmatrix}.$$

Then

$$A + B = \begin{bmatrix} a+d & b+e \\ 0 & c+f \end{bmatrix}.$$

We see that $A + B$ is also upper triangular and thus in U_2. Since the sum of any two matrices in U_2 is also in U_2, this shows that U_2 is closed under addition.

Next, we check closure under scalar multiplication. Let A be as above, and let k be any scalar. Then

$$kA = \begin{bmatrix} ka & kb \\ 0 & kc \end{bmatrix}.$$

We see that kA is also upper triangular and thus in U_2. This shows that any scalar multiple of a matrix in U_2 is also in U_2, and hence that U_2 is closed under scalar multiplication.

Since U_2 is both closed under addition and scalar multiplication, it follows that U_2 is a subspace of M_2. ◇

In general, the set U_n of all $n \times n$ upper triangular matrices is a subspace of M_n. It is closed under addition since the sum of two upper triangular matrices is upper triangular, and it is closed under scalar multiplication since any scalar multiple of an upper triangular matrix is upper triangular. Likewise, the set L_n of all $n \times n$ lower triangular matrices, and the set D_n of all $n \times n$ diagonal matrices, are also subspaces of M_n.

Example. Let S_n denote the set of all $n \times n$ symmetric matrices. Show that S_n is a subspace of M_n.

Solution. We start by checking closure under addition. Let A, B be two $n \times n$ symmetric matrices. Being symmetric means that

$$A^T = A \qquad \text{and} \qquad B^T = B.$$

Then

$$(A + B)^T = A^T + B^T = A + B.$$

Since $(A+B)^T = A+B$, it follows that $A+B$ is also symmetric. In short, the sum of two symmetric matrices remains symmetric, so S_n is closed under addition.

Next, we check closure under scalar multiplication. For any scalar k, and for any symmetric matrix A,

$$(kA)^T = k(A^T) = kA.$$

Since $(kA)^T = kA$, it follows that kA is symmetric. Thus, any scalar multiple of a symmetric matrix remains symmetric, so S_n is closed under scalar multiplication.

Since S_n is both closed under addition and scalar multiplication, we conclude it is a subspace of M_n. ◇

A matrix A is said to be anti-symmetric (or skew-symmetric) if it satisfies $A^T = -A$. One can show that the set of all $n \times n$ anti-symmetric matrices is a subspace of M_n, in a similar manner as we did for the symmetric case.

Example. The set of all $n \times n$ invertible matrices is not a subspace of M_n. One reason why is because it does not contain the zero matrix, as the zero matrix is not invertible. (Recall that all subspaces *must* contain the zero element.) $\diamond$

Example. The set of all $n \times n$ singular (i.e. non-invertible) matrices is not a subspace of M_n either. In particular, it fails to be closed under addition. As an example, for $n = 2$, the matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

are not invertible, but their sum is the identity matrix, which is invertible. $\diamond$

Example. The set of all $n \times n$ triangular matrices (including BOTH upper and lower) is not a subspace of $M_n(\mathbb{R})$. It is not closed under addition. For instance, when $n = 2$, the matrices

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$$

are triangular, but their sum is not. $\diamond$

5.2.6 Subspaces of Polynomials

Example. Recall that P_n is the space of all polynomial functions of degree less than or equal to n. Let W be the set of all polynomials in P_n whose graph passes through the point $(1, 0)$. Show that W is a subspace of P_n.

Solution. First, we check closure under addition. Let f and g be two polynomials in W. Since their graph passes through the point $(1, 0)$, we know that

$$f(1) = 0 \qquad \text{and} \qquad g(1) = 0.$$

Then

$$(f + g)(1) = f(1) + g(1) = 0 + 0 = 0,$$

and so the graph of $f + g$ also passes through the point $(1, 0)$. Thus, W is closed under addition.

Next, we check closure under scalar multiplication. Let f be as above, and let c be any scalar. Then

$$(cf)(1) = c \cdot f(1) = c \cdot 0 = 0,$$

and so the graph of the polynomial cf also passes through the point $(1, 0)$. Thus, W is closed under scalar multiplication.

Since W is closed under addition and scalar multiplication, we conclude that the set of all polynomials in P_n whose graph passes through the point $(1, 0)$ is a subspace of P_n. $\diamond$

Example. Let's modify this last example slightly, by having W be the set of all polynomials in P_n whose graph passes through $(1, 1)$ rather than $(1, 0)$. When checking closure under addition, we would have $f(1) = 1$ and $g(1) = 1$, and then we would have

$$(f + g)(1) = f(1) + g(1) = 1 + 1 = 2.$$

This time, we see that the graph of $f + g$ passes through the point $(1, 2)$ rather than the point $(1, 1)$. Hence, $f + g$ would no longer be in W, proving that W is not closed under addition. Hence, W would not be a subspace.

$\diamond$

5.3 Linear Combinations and Span

Way back in Section 1.4, we talked about linear combinations and span. However, back then we were doing everything strictly in $\mathbb{R}^n$. Now that we have introduced the notion of vector spaces, and we have seen examples of vector spaces other than $\mathbb{R}^n$, it is time to revisit these topics.

Definition. Let V be a vector space, and let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors in V. A ***linear combination*** of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is any vector of the form

$$c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k,$$

where $c_1, \ldots, c_k$ are scalars. The ***span*** of the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is the set of all their linear combinations.

5.3.1 Examples

Example. Is the vector $\mathbf{b} = (2, -1, 1)$ a linear combination of the vectors $\mathbf{v}_1 = (1, 0, 1)$, $\mathbf{v}_2 = (2, 1, 3)$, and $\mathbf{v}_3 = (1, 2, 3)$? Do the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span all of $\mathbb{R}^3$?

Solution. Let's start with the first question. We need to see if there exists scalars c_1, c_2, c_3 such that

$$c_1 \overbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}^{\mathbf{v}_1} + c_2 \overbrace{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}}^{\mathbf{v}_2} + c_3 \overbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}^{\mathbf{v}_3} = \overbrace{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}^{\mathbf{b}}.$$

Analyzing the x, y, z components individually, we find that this is equivalent to seeing if there exists a solution to the following system of equations:

$$\begin{aligned} c_1 + 2c_2 + \ c_3 &= \ \ 2 \\ c_2 + 2c_3 &= -1 \\ c_1 + 3c_2 + 3c_3 &= \ \ 1 \end{aligned}$$

Let's set up the augmented matrix and row reduce.

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 1 & 3 & 3 & 1 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 1 & 2 & -1 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

At this point, we can see that there are infinitely many solutions to this system of equations. The fact there are solutions to this system of equations tells us that there does exist scalars c_1, c_2, c_3 such that $c_1\mathbf{v}_1 + c_2\mathbf{v}_2 + c_3\mathbf{v}_3 = \mathbf{b}$. Hence, we can conclude that $\mathbf{b}$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$.

If we wanted to find actual numbers for c_1, c_2, c_3, then we could continue by solving this system of equations. This wasn't part of the question, but let's go ahead and do that. From the reduced matrix, we have the following:

$$\begin{aligned} c_1 + 2c_2 + \ c_3 &= \ \ 2 \\ c_2 + 2c_3 &= -1 \end{aligned}$$

Since the third column is missing a pivot, we can choose c_3 to be a free variable. Let's parametrize c_3 by setting $c_3 = t$. Plugging this into the second equation and solving for c_2 gives us $c_2 = -1 - 2t$. Plugging these into the first equation and solving for c_1 gives us $c_1 = 4 + 3t$. To get explicit numbers, let's plug in $t = 0$. Then we get $c_1 = 4$, $c_2 = -1$, and $c_3 = 0$. This tells us that

$$4 \overbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}^{\mathbf{v}_1} + (-1) \overbrace{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}}^{\mathbf{v}_2} + 0 \overbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}^{\mathbf{v}_3} = \overbrace{\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}}^{\mathbf{b}}.$$

You could plug in any other number for t to get a different set of coefficients that would also work.

Next, let's check to see if $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ span all of $\mathbb{R}^3$. To do this, we now need to check if *every* vector in $\mathbb{R}^3$ is a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. This amounts to checking if for any $\mathbf{b} = (b_1, b_2, b_3)$, there exists scalars c_1, c_2, c_3 such that

$$c_1 \overbrace{\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}}^{\mathbf{v}_1} + c_2 \overbrace{\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix}}^{\mathbf{v}_2} + c_3 \overbrace{\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}}^{\mathbf{v}_3} = \overbrace{\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}}^{\mathbf{b}}.$$

This is the same as what we did earlier, except instead of using $\mathbf{b} = (2, -1, 1)$ specifically, we are keeping $\mathbf{b}$ general and writing it as (b_1, b_2, b_3). So let's just repeat what we did earlier, with (b_1, b_2, b_3) in place of $(2, -1, 1)$:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 1 & 2 & b_2 \\ 1 & 3 & 3 & b_3 \end{array}\right] \xrightarrow{R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 1 & 2 & b_2 \\ 0 & 1 & 2 & -b_1 + b_3 \end{array}\right] \xrightarrow{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & b_1 \\ 0 & 1 & 2 & b_2 \\ 0 & 0 & 0 & -b_1 - b_2 + b_3 \end{array}\right].$$

Focusing on the last row of the reduced matrix, we have the equation

$$0c_1 + 0c_2 + 0c_3 = -b_1 - b_2 + b_3.$$

Clearly, there is no solution to the system of equations whenever $-b_1 - b_2 + b_3 \neq 0$. This tells us that there are some vectors $\mathbf{b} = (b_1, b_2, b_3)$ in $\mathbb{R}^3$ which are NOT linear combinations of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, and therefore the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ do NOT span all of $\mathbb{R}^3$.

As a tiny bonus, we do see that $\mathbf{b} = (b_1, b_2, b_3)$ is in the span of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ if and only if $-b_1 - b_2 + b_3 = 0$, since only then would we be able to find solutions for c_1, c_2, c_3. $\diamond$

Example. Let's do an example in a vector space of matrices. Specifically, let's work in L_2, the vector space of all 2×2 lower triangular matrices. Is the matrix

$$B = \begin{bmatrix} 3 & 0 \\ 4 & 2 \end{bmatrix}$$

a linear combination of

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}?$$

Do the lower triangular matrices A_1, A_2, A_3 span all of L_2?

Solution. To answer the first question, we need to see if there exists scalars c_1, c_2, c_3 such that

$$c_1 \overbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}^{A_1} + c_2 \overbrace{\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}}^{A_2} + c_3 \overbrace{\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}}^{A_3} = \overbrace{\begin{bmatrix} 3 & 0 \\ 4 & 2 \end{bmatrix}}^{B}.$$

Combining everything on the left gives us

$$\begin{bmatrix} c_1 + c_3 & 0 \\ c_1 + c_2 + 3c_3 & c_1 + 2c_2 + 2c_3 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 4 & 2 \end{bmatrix}.$$

This yields the following system of equations:

$$\begin{aligned} c_1 \phantom{{}+2c_2} + c_3 &= 3 \\ c_1 + c_2 + 3c_3 &= 4 \\ c_1 + 2c_2 + 2c_3 &= 2 \end{aligned}$$

We'll solve this in the usual manner:

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 1 & 1 & 3 & 4 \\ 1 & 2 & 2 & 2 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 2 & 1 & -1 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & 3 \\ 0 & 1 & 2 & 1 \\ 0 & 0 & -3 & -3 \end{array}\right].$$

This leaves us with

$$\begin{aligned} c_1 \quad + c_3 &= \ \ 3 \\ c_2 + 2c_3 &= \ \ 1 \\ - 3c_3 &= -3 \end{aligned}$$

Solving by back-substitution gives us $c_1 = 2$, $c_2 = -1$, and $c_3 = 1$. Hence, we find that B is a linear combination of A_1, A_2, A_3, and specifically,

$$2\overbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}^{A_1} + (-1)\overbrace{\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}}^{A_2} + 1\overbrace{\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}}^{A_3} = \overbrace{\begin{bmatrix} 3 & 0 \\ 4 & 2 \end{bmatrix}}^{B}.$$

Next, we want to see if A_1, A_2, A_3 span all of L_2, by checking to see if *every* 2×2 lower triangular matrix is a linear combination of A_1, A_2, A_3. This amounts to checking if for any lower triangular

$$B = \begin{bmatrix} b_1 & 0 \\ b_2 & b_3 \end{bmatrix},$$

there exists scalars c_1, c_2, c_3 such that

$$c_1\overbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}^{A_1} + c_2\overbrace{\begin{bmatrix} 0 & 0 \\ 1 & 2 \end{bmatrix}}^{A_2} + c_3\overbrace{\begin{bmatrix} 1 & 0 \\ 3 & 2 \end{bmatrix}}^{A_3} = \overbrace{\begin{bmatrix} b_1 & 0 \\ b_2 & b_3 \end{bmatrix}}^{B}.$$

This is the same as what we did earlier, except we are using

$$B = \begin{bmatrix} b_1 & 0 \\ b_2 & b_3 \end{bmatrix} \qquad \text{instead of} \qquad B = \begin{bmatrix} 3 & 0 \\ 4 & 2 \end{bmatrix}.$$

Let's repeat what we did earlier then, but with b_1, b_2, b_3 in place of $3, 4, 2$.

$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & b_1 \\ 1 & 1 & 3 & b_2 \\ 1 & 2 & 2 & b_3 \end{array}\right] \xrightarrow[R_3-R_1]{R_2-R_1} \left[\begin{array}{ccc|c} 1 & 0 & 1 & b_1 \\ 0 & 1 & 2 & -b_1+b_2 \\ 0 & 2 & 1 & -b_1+b_3 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{ccc|c} 1 & 0 & 1 & b_1 \\ 0 & 1 & 2 & -b_1+b_2 \\ 0 & 0 & -3 & b_1-2b_2+b_3 \end{array}\right].$$

This leaves us with

$$\begin{aligned} c_1 \quad + c_3 &= \ \ b_1 \\ c_2 + 2c_3 &= -b_1 + b_2 \\ - 3c_3 &= \ \ b_1 - 2b_2 + b_3. \end{aligned}$$

At this point, it would be a bit messy, but we could solve for c_1, c_2, c_3 via back-substitution. This shows that for any 2×2 lower triangular matrix B, we are able to solve for c_1, c_2, c_3 such that $c_1A_1 + c_2A_2 + c_3A_3$ would equal B, and hence that the lower triangular matrices A_1, A_2, A_3 span all of L_2. ◇

Before we do another example, which will involve polynomials, it will be worthwhile to reflect a bit on the two examples just now. In each example, we had a vector space V and some number of vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ in V, and we had to check if those vectors spanned all of V. In the first example,

$V = \mathbb{R}^3$ and we had the vectors $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. In the second example, $V = L_2$, and we had the vectors (a.k.a 2×2 lower triangular matrices) A_1, A_2, A_3. In each example, to check if they spanned, we took an arbitrary $\mathbf{b}$ in V and checked to see if it was a linear combination of the given vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$:

$$c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k = \mathbf{b}.$$

This led us to a system of equations to see if we could solve for the coefficients $c_1, \ldots, c_k$.

And here is what I want to highlight: the reason why $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ failed to span $\mathbb{R}^3$ in the first example *is because we ended up with a row of zeros after row reducing.* That row of zeros led to the equation

$$0c_1 + 0c_2 + 0c_3 = -b_1 - b_2 + b_3,$$

which shows that we would not be able to solve for c_1, c_2, c_3 for all vectors $\mathbf{b}$. In contrast, the reason why A_1, A_2, A_3 ended up spanning L_2 *is because we did not end up with a row of zeros after row reducing.* Because we did not get a row of zeros, we could back-substitute to solve for c_1, c_2, c_3, no matter what the 2×2 lower triangular matrix B was.

Example. Do the polynomials

$$f_1(x) = 1 + x, \qquad f_2(x) = 3 + x^2, \qquad f_3(x) = 5 + 2x + x^2$$

span P_2 (the space of all polynomials of degree ≤ 2)?

Solution. We need to check if every polynomial of degree at most two is a linear combination of f_1, f_2, f_3. To do this, let's set $b(x) = b_0 + b_1x + b_2x^2$ and see if we can solve for scalars c_1, c_2, c_3 such that

$$c_1 \overbrace{(1+x)}^{f_1(x)} + c_2 \overbrace{(3+x^2)}^{f_2(x)} + c_3 \overbrace{(5+2x+x^2)}^{f_3(x)} = \overbrace{b_0 + b_1x + b_2x^2}^{b(x)}.$$

Combining like terms on the left gives us

$$(c_1 + 3c_2 + 5c_3) + (c_1 + 2c_3)x + (c_2 + c_3)x^2 = b_0 + b_1x + b_2x^2.$$

For the two sides to equal each other, the coefficients of like powers of x must coincide. This gives us the following system of equations:

$$\begin{aligned} c_1 + 3c_2 + 5c_3 &= b_0 \\ c_1 \qquad + 2c_3 &= b_1 \\ c_2 + c_3 &= b_2. \end{aligned}$$

Again, we'll try and solve this the usual way:

$$\left[\begin{array}{ccc|c} 1 & 3 & 5 & b_0 \\ 1 & 0 & 2 & b_1 \\ 0 & 1 & 1 & b_2 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 3 & 5 & b_0 \\ 0 & -3 & -3 & -b_0 + b_1 \\ 0 & 1 & 1 & b_2 \end{array}\right] \xrightarrow{R_2 \leftrightarrow R_3} \left[\begin{array}{ccc|c} 1 & 3 & 2 & b_0 \\ 0 & 1 & 1 & b_2 \\ 0 & -3 & -3 & -b_0 + b_1 \end{array}\right]$$

$$\xrightarrow{R_3 + 3R_2} \left[\begin{array}{ccc|c} 1 & 3 & 2 & b_0 \\ 0 & 1 & 1 & b_2 \\ 0 & 0 & 0 & -b_0 + b_1 + 3b_2 \end{array}\right]$$

The row of zeros at the end indicates that f_1, f_2, f_3 do not span P_2, as we are not able to solve for c_1, c_2, c_3 for all polynomials b in P_2. We can say that $b(x) = b_0 + b_1x + b_2x^2$ is in the span of f_1, f_2, f_3 if and only if $-b_0 + b_1 + 3b_2 = 0$. ◇

Summary of Procedure

To check if $\mathbf{v}_1, \ldots, \mathbf{v}_k$ span V:

1. Set $c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k$ equal to an arbitrary vector $\mathbf{b}$ in V:

$$c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k = \mathbf{b}.$$

2. Convert into a system of equations and try to solve for $c_1, \ldots, c_k$.
3. If you get a row of zeros after row reducing, the vectors do not span V. Otherwise, they do.

5.3.2 Spans are Subspaces

It turns out that the span of any collection of vectors is always a subspace of the space they reside in. To understand why, let V be a vector space, let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors in V, and let W denote the span of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. As we hopefully know by now, the span of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is the set of all linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. To show that W is a subspace of V, we need to check closure under addition and closure under scalar multiplication.

W is closed under addition because the sum of any two linear combinations of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is still a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$:

$$(c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k) + (d_1\mathbf{v}_1 + \ldots + d_k\mathbf{v}_k) = (c_1 + d_1)\mathbf{v}_1 + \ldots + (c_k + d_k)\mathbf{v}_k.$$

Likewise, W is closed under scalar multiplication because a scalar multiple of a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is still a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$:

$$c(c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k) = (cc_1)\mathbf{v}_1 + \ldots + (cc_k)\mathbf{v}_k.$$

Since $W = \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k)$ is closed under addition and scalar multiplication, it follows that the span of any collection of vectors is a subspace of the space they reside in.

5.3.3 The Columnspace of a Matrix

Let A be an $m \times n$ matrix. The ***columnspace*** of A, denoted $C(A)$, is defined to be the span of its column vectors. In other words, $\mathbf{b}$ is in $C(A)$ if and only if $\mathbf{b}$ is a linear combination of the columns of A. Note that the columns of A would be vectors in $\mathbb{R}^m$, hence $C(A)$ would be a subspace of $\mathbb{R}^m$.

Recall (Section 2.3) that matrix-vector multiplication was defined as a linear combination of the columns of the matrix. Specifically,

$$\text{if} \quad A = \overbrace{[\mathbf{v}_1\ \mathbf{v}_2\ \ldots\ \mathbf{v}_n]}^{\text{columns of } A} \quad \text{and} \quad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

$$\text{then} \quad A\mathbf{x} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \ldots + x_n\mathbf{v}_n.$$

From this, we see that a vector $\mathbf{b}$ would be a linear combination of the columns of A, and hence in $C(A)$, if and only if there exists a solution to the equation $A\mathbf{x} = \mathbf{b}$. This is important enough to highlight below.

Proposition. A vector $\mathbf{b}$ will be in the columnspace of A if and only if any of the following equivalent conditions hold:

- $\mathbf{b}$ is a linear combination of the columns of A
- the equation $A\mathbf{x} = \mathbf{b}$ has a solution.

Example. Let's go all the way back to the first example of this section. We were asked if the vector $\mathbf{b} = (2, -1, 1)$ is a linear combination of $\mathbf{v}_1 = (1, 0, 1)$, $\mathbf{v}_2 = (2, 1, 3)$, and $\mathbf{v}_3 = (1, 2, 3)$. To answer this question, ultimately we set up the following augmented matrix,

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 2 \\ 0 & 1 & 2 & -1 \\ 1 & 3 & 3 & 1 \end{array}\right],$$

and row reduced to find the solutions. Notice that the columns of the coefficient matrix, which we'll call A, are $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$, while the right hand side is $\mathbf{b}$.

Now that we have introduced the columnspace, we can look at everything we did with another perspective. By putting $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ into a matrix A as columns, the question of whether $\mathbf{b}$ is a linear combination of them becomes equivalent to checking whether $\mathbf{b}$ is in the columnspace of A, and to check whether $\mathbf{b}$ is in $C(A)$, according to the above proposition, we need to check if the equation $A\mathbf{x} = \mathbf{b}$ has a solution. This is precisely what we were doing in row reducing the augmented matrix.

We also saw in the first example that for the equation $A\mathbf{x} = \mathbf{b}$ to have a solution, the vector $\mathbf{b}$ needed to satisfy $-b_1 - b_2 + b_3 = 0$. Hence, we can say $\mathbf{b}$ would be in $C(A)$ if and only if

$$-b_1 - b_2 + b_3 = 0.$$

Let's go ahead and write out the solutions to this equation. We have $b_3 = b_1 + b_2$. If we parametrize this by setting $b_1 = s$, $b_2 = t$, then the solutions to this equation can be expressed as

$$\begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = s \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \qquad s, t \in \mathbb{R}.$$

This shows that the columnspace of A is the span of the vectors $(1, 0, 1)$ and $(0, 1, 1)$. Since these vectors are not multiples of each other, they would span a plane in $\mathbb{R}^3$. Hence, for this matrix A, $C(A)$ turns out to be a plane in $\mathbb{R}^3$. $\diamond$

Anyways, we will say more about columnspaces when we talk about the four fundamental subspaces of a matrix.

5.4 Linear Independence

In Section 1.5, we talked about linear independence, but again, back then we were doing everything strictly in $\mathbb{R}^n$. Now, we revisit the topic again in the more general context of vector spaces.

Definition. Let V be a vector space, and let $\mathbf{v}_1, \ldots, \mathbf{v}_k$ be vectors in V. The vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are said to be ***linearly independent*** if the only linear combination of them that equals $\mathbf{0}$ is the trivial combination, where all the coefficients are zero. In other words, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent if and only if the only solution to the equation

$$c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k = \mathbf{0}$$

is the trivial solution where $c_1 = \ldots = c_k = 0$.

5.4.1 Examples

Example. Are the vectors $\mathbf{v}_1 = (1, 1, 0)$, $\mathbf{v}_2 = (2, 3, 2)$, $\mathbf{v}_3 = (1, 3, 4)$ linearly independent?

Solution. We need to check if the only solution to the equation

$$c_1 \overbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}}^{\mathbf{v}_1} + c_2 \overbrace{\begin{bmatrix} 2 \\ 3 \\ 2 \end{bmatrix}}^{\mathbf{v}_2} + c_3 \overbrace{\begin{bmatrix} 1 \\ 3 \\ 4 \end{bmatrix}}^{\mathbf{v}_3} = \overbrace{\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}}^{\mathbf{0}}$$

is the trivial solution $c_1 = c_2 = c_3 = 0$. We can rewrite the above vector equation as the following system of equations:

$$\begin{aligned} c_1 + 2c_2 + \ c_2 \ &= 0 \\ c_1 + 3c_2 + 3c_2 &= 0 \\ 2c_2 + 4c_3 &= 0 \end{aligned}$$

If the only solution to this system of equations is $c_1 = c_2 = c_3 = 0$, then the vectors are linearly independent. Otherwise, they are linearly dependent. Let's set up the augmented matrix and row reduce:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 1 & 3 & 3 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right] \xrightarrow{R_2 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 2 & 4 & 0 \end{array}\right] \xrightarrow{R_3 - 2R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Notice that the third column is missing a pivot, indicating that we'll have a free variable and thus infinitely many solutions. Since we have infinitely many solutions, this means that $c_1 = c_2 = c_3 = 0$ is NOT the only solution, so the $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly dependent.

As a side note, the row of zeros tells us that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ do not span $\mathbb{R}^3$ (as we talked about in the last section). It doesn't say anything about being linearly independent or dependent. ◇

Let's reflect a bit on some things before continuing with more examples. First, notice from this example how we converted the vector equation

$$c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k = \mathbf{0}$$

into a linear system of equations. As you'll recall, a linear system of equations will have either no solution, a unique solution, or infinitely many solutions. We can rule out having no solution to the above equation, since we have the trivial solution $c_1 = \ldots = c_k = 0$. Either this is the only solution, in which case the vectors are independent, or we have infinitely many solutions, in which case they are dependent. Whether we have infinitely many solutions or not depends on if we have a column missing a pivot or not. A column missing a pivot means infinitely many solutions. If every column has a pivot, then we have a unique solution.

Example. Are the matrices

$$A_1 = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}, \qquad A_2 = \begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}, \qquad A_3 = \begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix},$$

viewed as vectors in the vector space M_2, linearly independent?

Solution. We need to check if the only solution to the equation

$$c_1 \overbrace{\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}}^{A_1} + c_2 \overbrace{\begin{bmatrix} 2 & 1 \\ 3 & 0 \end{bmatrix}}^{A_2} + c_3 \overbrace{\begin{bmatrix} 1 & 1 \\ 0 & 2 \end{bmatrix}}^{A_3} = \overbrace{\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}}^{\mathbf{0}}$$

is the trivial solution $c_1 = c_2 = c_3 = 0$. Combining everything on the left gives us

$$\begin{bmatrix} c_1 + 2c_2 + c_3 & c_2 + c_3 \\ c_1 + 3c_2 & c_1 + 2c_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

Hence, we end up with the following system of equations:

$$\begin{aligned} c_1 + 2c_2 + \ c_3 &= 0 \\ c_2 + \ c_3 &= 0 \\ c_1 + 3c_2 \qquad &= 0 \\ c_1 \qquad + 2c_3 &= 0 \end{aligned}$$

If the only solution to this is $c_1 = c_2 = c_3 = 0$, then the matrices are linearly independent. Otherwise, they are linearly dependent. As usual, we'll set up the augmented matrix and row reduce:

$$\left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 3 & 0 & 0 \\ 1 & 0 & 2 & 0 \end{array}\right] \xrightarrow[R_4 - R_1]{R_3 - R_1} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & -1 & 0 \\ 0 & -2 & 1 & 0 \end{array}\right] \xrightarrow[R_4 + 2R_2]{R_3 - R_2} \left[\begin{array}{ccc|c} 1 & 2 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & -2 & 0 \\ 0 & 0 & 3 & 0 \end{array}\right].$$

Notice that this time, every column has a pivot, so the trivial solution $c_1 = c_2 = c_3 = 0$ must be the only solution. Hence, A_1, A_2, A_3 are linearly independent.

As a side note, notice that if we do one more row operation, $R_4 + \frac{3}{2}R_3$, we would get a row of zeros. Again, this says nothing in regards to linear independence vs. dependence, but it does tell us that A_1, A_2, A_3 fail to span all of M_2. $\diamond$

Example. Are the polynomials

$$f_1(x) = 1 - x + x^2, \qquad f_2(x) = x + 2x^2, \qquad f_3(x) = 2 + 3x - x^2,$$

viewed as vectors in the vector space P_2, linearly independent?

Solution. We need to check if the only solution to the equation

$$c_1 \overbrace{(1 - x + x^2)}^{f_1(x)} + c_2 \overbrace{(x + 2x^2)}^{f_2(x)} + c_3 \overbrace{(2 + 3x - x^2)}^{f_3(x)} = \overbrace{0 + 0x + 0x^2}^{\mathbf{0}}.$$

is the trivial solution $c_1 = c_2 = c_3 = 0$. Note how on the right, the zero vector in P_2 is the polynomial whose coefficients are all zero. Combining like terms on the left gives us

$$(c_1 + 2c_3) + (-c_1 + c_2 + 3c_3)x + (c_1 + 2c_2 - c_3)x^2 = 0 + 0x + 0x^2.$$

For the two polynomials on the left and right to be equal, the coefficients of like powers of x must coincide. This yields the following system of equations:

$$\begin{aligned} c_1 \qquad + 2c_3 &= 0 \\ -c_1 + \ c_2 + 3c_3 &= 0 \\ c_1 + 2c_2 - \ c_3 &= 0 \end{aligned}$$

Again, if the only solution to this system of equation is $c_1 = c_2 = c_3 = 0$, then the polynomials are linearly independent. Otherwise, they are linearly dependent. Below, we set up and perform the usual elimination on the augmented matrix.:

$$\left[\begin{array}{rrr|r} 1 & 0 & 2 & 0 \\ -1 & 1 & 3 & 0 \\ 1 & 2 & -1 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{R_2+R_1} \\ \xrightarrow{R_3-R_1} \end{array} \left[\begin{array}{rrr|r} 1 & 0 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 2 & -3 & 0 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{rrr|r} 1 & 0 & 2 & 0 \\ 0 & 1 & 5 & 0 \\ 0 & 0 & -13 & 0 \end{array}\right].$$

Since every pivot has a column, we see that system has a unique solution, namely the trivial one. Hence, we conclude that f_1, f_2, f_3 are linearly independent.

As a side note, since there are no rows of zeros, f_1, f_2, f_3 also span all of P_2. ◇

Example. Suppose $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent vectors in some vector space V. Are the vectors

$$\mathbf{w}_1 = \mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3, \qquad \mathbf{w}_2 = \mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3, \qquad \mathbf{w}_3 = \mathbf{v}_2 + 2\mathbf{v}_3$$

linearly independent?

Solution. We need to check if the only solution to the equation

$$c_1 \overbrace{(\mathbf{v}_1 + \mathbf{v}_2 + \mathbf{v}_3)}^{\mathbf{w}_1} + c_2 \overbrace{(\mathbf{v}_1 + 2\mathbf{v}_2 + 3\mathbf{v}_3)}^{\mathbf{w}_2} + c_3 \overbrace{(\mathbf{v}_2 + 2\mathbf{v}_3)}^{\mathbf{w}_3} = \mathbf{0}$$

is the trivial solution where $c_1 = c_2 = c_3 = 0$. Rearranging everything on the left by grouping all the terms with a $\mathbf{v}_1$ in them together, and then likewise for $\mathbf{v}_2$ and $\mathbf{v}_3$, gives us

$$(c_1 + c_2)\mathbf{v}_1 + (c_1 + 2c_2 + c_3)\mathbf{v}_2 + (c_1 + 3c_2 + 2c_3)\mathbf{v}_3 = \mathbf{0}.$$

Since $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are linearly independent, the only linear combination of them that equals $\mathbf{0}$ is the trivial combination where all the coefficients are zero. Setting the coefficients of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ equal to zero gives us the following system of equations:

$$\begin{aligned} c_1 + c_2 \phantom{{}+2c_3} &= 0 \\ c_1 + 2c_2 + c_3 &= 0 \\ c_1 + 3c_2 + 2c_3 &= 0 \end{aligned}$$

If the only solution to this system is $c_1 = c_2 = c_3 = 0$, then $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are linearly independent. Otherwise, they are linearly dependent. Performing elimination on the augmented matrix yields

$$\left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 \\ 1 & 3 & 2 & 0 \end{array}\right] \begin{array}{c} \xrightarrow{R_2-R_1} \\ \xrightarrow{R_3-R_1} \end{array} \left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 2 & 2 & 0 \end{array}\right] \xrightarrow{R_3-2R_2} \left[\begin{array}{rrr|r} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

Since we have a column missing a pivot, there are infinitely many solutions to this system of equations. In particular, the trivial solution is NOT the only solution, so the vectors $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$ are linearly dependent. ◇

5.5 Bases for Vector Spaces

Be sure that you've gone through the previous sections on span and linear independence, and/or that you have some understanding of these concepts, because you'll need to be familiar with them to understand the next definition.

Definition. A ***basis*** for a vector space V is a set of vectors in V which span V and are linearly independent.

5.5.1 A Specific Case Study: $\mathbb{R}^n$

In order to get a better sense for why bases are special, let's focus our attention on the vector space $\mathbb{R}^n$, which for us is by far the most important case. Much of what we are going to discuss here has already been mentioned in previous sections.

Suppose $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a basis for $\mathbb{R}^n$, and let A be the $n \times k$ matrix whose columns are $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Recall how matrix-vector multiplication forms a linear combination of the columns of A:

$$A\mathbf{x} = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_k] \begin{bmatrix} x_1 \\ \vdots \\ x_k \end{bmatrix} = x_1\mathbf{v}_1 + \ldots + x_k\mathbf{v}_k. \tag{$*$}$$

Since they form a basis for $\mathbb{R}^n$, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ must span $\mathbb{R}^n$. This means that every vector $\mathbf{b}$ in $\mathbb{R}^n$ is a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_k$. Thus, for every $\mathbf{b}$ in $\mathbb{R}^n$, there exists scalars $x_1, \ldots, x_k$ such that

$$x_1\mathbf{v}_1 + \ldots + x_k\mathbf{v}_k = \mathbf{b}.$$

From $(*)$, we see that this is equivalent to saying

$$\text{the equation } A\mathbf{x} = \mathbf{b} \text{ has a solution for every } \mathbf{b} \text{ in } \mathbb{R}^n.$$

For this to be the case, there cannot be any rows of zeros when we row reduce A to row echelon form. In other words then, every row of A must have a pivot. Since A is size $n \times k$, in order to have a pivot in every row, k must be greater than or equal to n. To see why, consider an example of when this is not the case - for instance, if we consider a 3×2 matrix:

$$\begin{bmatrix} a & b \\ c & d \\ e & f \end{bmatrix},$$

this matrix could have at most two pivots since it has only two column, meaning one row for sure would be missing a pivot and would thus zero out when row reducing. Therefore, since $\mathbf{v}_1, \ldots, \mathbf{v}_k$ span $\mathbb{R}^n$, every row of A must have a pivot, and it must be that $k \geq n$.

It's worth noting the fact that $k \geq n$ can also be reasoned geometrically and intuitively: k vectors can span at most k dimensions, and $\mathbb{R}^n$ is n-dimensional, so k would have to $\geq n$ for $\mathbf{v}_1, \ldots, \mathbf{v}_k$ to span $\mathbb{R}^n$.

On the other hand, since they form a basis for $\mathbb{R}^n$, the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are also linearly independent. This means that the only solution to the equation

$$x_1\mathbf{v}_1 + \ldots + x_k\mathbf{v}_k = \mathbf{0}$$

is the trivial solution $x_1 = \ldots = x_k = 0$. From $(*)$, we see that this is equivalent to saying

$$\text{the only solution to } A\mathbf{x} = \mathbf{0} \text{ is the trivial solution } \mathbf{x} = \mathbf{0}.$$

In particular, since $A\mathbf{x} = \mathbf{0}$ has a unique solution, there cannot have been any free variables, so every column of A must have a pivot. For this to be the case with A being $n \times k$, it must be that

k is less than or equal to n. To see why, again we can consider an example of when this is not the case - for instance, if we consider a 2×3 matrix:

$$\begin{bmatrix} a & b & c \\ d & e & f \end{bmatrix},$$

this matrix could have at most two pivots since it has only two rows, meaning one column for sure would be missing a pivot, indicating at least one free variable and thus infinitely many solutions. Therefore, since $\mathbf{v}_1, \ldots, \mathbf{v}_k$ are linearly independent, every column of A must have a pivot, and it must be that $k \leq n$.

Again, the fact that $k \leq n$ can be reasoned geometrically/intuitively: k linearly independent vectors will span k-dimensions, and since $\mathbb{R}^n$ is n-dimensional, $\mathbf{v}_1, \ldots, \mathbf{v}_k$ could not span more than n dimensions. Hence, $k \leq n$.

Thus, we see that if $\mathbf{v}_1, \ldots, \mathbf{v}_k$ is a basis for $\mathbb{R}^n$, then it must be that both $k \geq n$ and $k \leq n$. Hence, we see that k must be equal to n. Furthermore, we see that the matrix $A = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_n]$, which we now see is $n \times n$ since $k = n$, must have a pivot in every row and in every column. Hence, A has a full set of pivots and is thus invertible.

Conversely, if A is an $n \times n$ invertible matrix with columns $\mathbf{v}_1, \ldots, \mathbf{v}_n$, then we know that the equation $A\mathbf{x} = \mathbf{b}$ has a unique solution for all $\mathbf{b}$, namely $\mathbf{x} = A^{-1}\mathbf{b}$. The fact that a solution exists for all $\mathbf{b}$ tells us that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ span $\mathbb{R}^n$, and the fact that the solution is unique tells us that $A\mathbf{x} = \mathbf{0}$ has only the trivial solution, and hence that $\mathbf{v}_1, \ldots, \mathbf{v}_n$ are linearly independent. Hence, $\mathbf{v}_1, \ldots, \mathbf{v}_n$ is a basis for $\mathbb{R}^n$.

We summarize some of the main results of our discussion in the following theorem.

Theorem. Every basis for $\mathbb{R}^n$ consists of n vectors, and vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ will form a basis for $\mathbb{R}^n$ if and only if the matrix $A = [\mathbf{v}_1 \ \ldots \ \mathbf{v}_n]$ is invertible.

Here are some other highlights/consequences of our discussion:

- k vectors in $\mathbb{R}^n$ will fail to span $\mathbb{R}^n$ if $k < n$. This is because we saw if they span $\mathbb{R}^n$, then $k \geq n$.
- k vectors in $\mathbb{R}^n$ will fail to be linearly independent if $k > n$. This is because we saw if they are linearly independent, then $k \leq n$.
- For n vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ in $\mathbb{R}^n$, the following are equivalent (meaning if one is true, then all are true; and if one is false, then all are false):
  1. They span $\mathbb{R}^n$.
  2. They are linearly independent.
  3. They form a basis for $\mathbb{R}^n$.
  4. Every vector in $\mathbb{R}^n$ can be uniquely represented as a linear combination of $\mathbf{v}_1, \ldots, \mathbf{v}_n$.

  This is true because they are all equivalent to saying $A = [\mathbf{v}_1 \ \cdots \ \mathbf{v}_n]$ is invertible. Specifically, span means every row A has a pivot, while being linearly independent means every column of A has a pivot. Both of these are equivalent to A being invertible since A is $n \times n$. We already noted in the above theorem that being a basis is equivalent to A being invertible. And by $(*)$, the last statement is equivalent to saying that $A\mathbf{x} = \mathbf{b}$ always has a unique solution, which is equivalent to saying A is invertible.

5.5.2 Generalizing to Other Vector Spaces

Everything that we just proved for $\mathbb{R}^n$ generalizes to the other vector spaces that we occasionally work with in this class. I'll give a very quick explanation of how to generalize the above arguments, but I will not provide any details.

We know that for any vector space V, vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$ span V if and only if for all $\mathbf{b}$ in V, the equation

$$c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k = \mathbf{b}$$

has a solution, and they are linearly independent if and only if the only solution to the equation

$$c_1\mathbf{v}_1 + \ldots + c_k\mathbf{v}_k = \mathbf{0}$$

is the trivial solution $c_1 = \ldots = c_k = 0$. As we have seen in the previous two sections, the above two equations can be converted into matrix equations of the form $A\mathbf{x} = \mathbf{b}$ and $A\mathbf{x} = \mathbf{0}$, respectively, and one can reason in a similar fashion that we did for $\mathbb{R}^n$ that the vectors will be a basis for V if and only A is invertible. One can then prove results about V analogous to what we just proved for $\mathbb{R}^n$.

5.6 Dimension of a Vector Space

We have talked about dimension intuitively at times, and have mentioned that $\mathbb{R}^n$ is n-dimensional. However, we have never actually defined what dimension is. Now, we will finally get around to doing that.

The key to defining the dimension of a vector space lies in one of the facts that we just talked about regarding bases. We saw how in $\mathbb{R}^n$, every basis will consist of n vectors. This fact generalizes to *all* vector spaces, in the sense that every basis for a vector space V will have the same number of vectors in it.

Definition. The ***dimension*** of a vector space V is defined to be the number of vectors in any basis for V. Note: This definition makes sense, in light of the fact that we just mentioned that every basis for a vector space will have the same number of vectors in it.

Example. Since we proved that every basis for $\mathbb{R}^n$ has n vectors in it, this definition now makes it official that the dimension of $\mathbb{R}^n$ is, in fact, equal to n. $\diamond$

Example. Consider the vector space M_2 of all 2×2 matrices. Notice that

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} = a\overbrace{\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}}^{A_1} + b\overbrace{\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}}^{A_2} + c\overbrace{\begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}}^{A_3} + d\overbrace{\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}}^{A_4}.$$

This equation immediately shows that every matrix in M_2 is a linear combination of A_1, A_2, A_3, A_4, and hence these four matrices span M_2. Also, it is clear by looking at the left side of the equation that the only way for everything to equal to zero matrix is if $a = b = c = d = 0$. Hence, the four matrices are also linearly independent.

It follows that A_1, A_2, A_3, A_4 form a basis for M_2. We can also now conclude that every basis for M_2 will consist of four matrices, and that the dimension of M_2 is 4. $\diamond$

Example. One can easily generalize the previous example to conclude that the dimension of $M_{m\times n}$, the vector space of all $m \times n$ matrices, is mn. $\diamond$

Example. Consider vector space S_3 of all 3×3 symmetric matrices. A typical 3×3 symmetric matrix has the form

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix}.$$

We can express this as a linear combination of six symmetric matrices, as follows:

$$\begin{bmatrix} a & b & c \\ b & d & e \\ c & e & f \end{bmatrix} = a\overbrace{\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{A_1} + b\overbrace{\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{A_2} + c\overbrace{\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}}^{A_3} + d\overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}}^{A_4} + e\overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}}^{A_5} + f\overbrace{\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}}^{A_6}$$

Notice that the matrices $A_1, \ldots, A_6$ are indeed symmetric. Furthermore, the above equation shows that every 3×3 symmetric matrix is a linear combination of $A_1, \ldots, A_6$. Hence, these six matrices span S_3. In addition, it is clear by looking at the left side of the equation that the only way for everything to equal zero is for $a = b = c = d = e = f = 0$. Hence, these six matrices are also linearly independent.

It follows that $A_1, \ldots, A_6$ is a basis for S_3. We can also conclude that every basis for S_3 will consist of six (symmetric) matrices, and that the dimension of S_3 is 6. $\diamond$

Example. Once can generalize the previous example to find a basis and the dimension of S_n, the vector space of all $n \times n$ matrices. Let's come up with a formula for the dimension of S_n.

Notice in the previous example that the dimension of S_3 turned out to be 6 because we needed six parameters to represent a typical 3×3 symmetric matrix: a, b, c, d, e, and f. To find the dimension of S_n, we need to count how many parameters would we need to represent a typical $n \times n$ symmetric matrix.

An $n \times n$ matrix has a total of n^2 entries. However, to represent a symmetric matrix, we only need a parameter for each diagonal entry, plus a parameter for each entry above the diagonal. There are $n^2 - n$ entries off the diagonal. Half of them would be above the diagonal, and half would be below. Thus, the total number of parameters we would need to represent an $n \times n$ symmetric matrix is

$$\underset{\substack{\#\text{ of entries}\\\text{above diagonal}}}{\tfrac{1}{2}(n^2-n)} \quad + \quad \underset{\substack{\text{number of}\\\text{diagonal entries}}}{n} \quad = \tfrac{1}{2}(n^2+n)$$

Hence, the dimension of S_n equals $\frac{1}{2}(n^2+n)$. ◇

We'll do just a few more examples, but first, let's try to summarize the general strategy for these types of problems. Basically, you want to write out what a typical element of the given vector space looks like. Intuitively, the number of parameters that you use to do this will give you the dimension of the vector space. To obtain a basis for the vector space, take your expression for a typical element, and write it as a linear combination using the parameters as coefficients. The vectors of the vector space that each parameter is a coefficient of should form a basis, and the number of basis vectors gives the dimension.

Example. Consider the vector space P_n of all polynomials of degree $\leq n$. A typical element of P_n has the form

$$a_0 + a_1 x + \ldots + a_n x^n.$$

We need $n+1$ parameters (we are referring to $a_0, a_1, \ldots, a_n$) to represent a typical element of P_n. So intuitively, the dimension of P_n is $n+1$.

Every element of P_n is already naturally expressed as a linear combination of the polynomials $1, x, \ldots, x^n$ in P_n, since

$$a_0 + a_1 x + \ldots + a_n x^n \quad = \quad a_0 \cdot 1 + a_1 \cdot x + \ldots + a_n \cdot x^n.$$

Hence, these polynomials span P_n. It is also clear that the only way for everything to equal the zero polynomial, $0 + 0x + \ldots + 0x^n$, is for all the coefficients to equal zero, so these polynomials are also linearly independent.

It follows that $1, x, \ldots, x^n$ is a basis for P_n. Since there are $n+1$ polynomials in this basis, we can state definitively that the dimension of P_n is $n+1$. ◇

Example. Consider the subset W of P_4 consisting of those polynomials $f(x)$ for which $f'(2) = 0$. Let's verify that W is a subspace of P_4, and then let's find the dimension of W.

To show that W is a subspace, we need to check closure under addition and scalar multiplication. For closure under addition, let $f(x)$ and $g(x)$ be in W, so that

$$f'(2) = 0 \qquad \text{and} \qquad g'(2) = 0.$$

We know from calculus that $(f+g)'(x) = f'(x) + g'(x)$. It follows that

$$(f+g)'(2) = f'(2) + g'(2) = 0 + 0 = 0.$$

This shows that the sum two polynomials in W is also in W, so W is closed under addition. For closure under scalar multiplication, let f be as above, and let c be a scalar. We know from calculus that $(cf)'(x) = c \cdot f'(x)$, so that

$$(cf)'(2) = c \cdot f'(2) = c \cdot 0 = 0.$$

This shows that a scalar multiple of a polynomial in W is also in W, so W is closed under scalar multiplication. Since W is closed under addition and scalar multiplication, it is a subspace of P_4.

As for the dimension of W, we need to know what a typical element of W looks like. First, as a subspace of P_4, we know every polynomial $f(x)$ in W has the form

$$f(x) = a + bx + cx^2 + dx^3 + ex^4.$$

However, we also know that $f'(2)$ should equal 0. Let's compute $f'(x)$, $f'(2)$, and set the latter equal to 0:

$$f'(x) = b + 2cx + 3dx^2 + 4ex^3 \qquad \Longrightarrow \qquad f'(2) = b + 4c + 12d + 32e = 0.$$

Let's solve for b in the last equation to get

$$b = -4c - 12d - 32e,$$

and then let's substitute this into the above expression for $f(x)$:

$$f(x) = a + (-4c - 12d - 32e)x + cx^2 + dx^3 + ex^4.$$

Now we'll rearrange this last expression into a linear combination using the parameters a, c, d, e as coefficients:

$$f(x) = a \cdot 1 + c(x^2 - 4x) + d(x^3 - 12x) + e(x^4 - 32x). \tag{$*$}$$

From this, we see that any polynomial $f(x)$ in W is a linear combination of the polynomials

$$f_1(x) = 1, \quad f_2(x) = x^2 - 4x, \quad f_3(x) = x^3 - 12x, \quad f_4(x) = x^4 - 32x.$$

These four polynomials are also in W, since their derivatives evaluated at 2 are all zero. Hence, these four polynomials span W. Regarding linear independence, recall that for a polynomial to equal the zero polynomial, the coefficients of each power of x must equal zero. And since a, c, d, e are the coefficients of x^0, x^2, x^3, x^4 in $(*)$, we see that the only way for $(*)$ to equal zero is if $a = c = d = e = 0$. Hence, f_1, f_2, f_3, f_4 are also linearly independent.

Thus, f_1, f_2, f_3, f_4 form a basis for W, and since there are four vectors in this basis, the dimension of W is 4. $\diamond$

5.7 Four Fundamental Subspaces

Let A be an $m \times n$ matrix. There are two subspaces of $\mathbb{R}^m$ and two subspaces of $\mathbb{R}^n$ that we associate with the matrix. These four subspaces together are commonly referred to as the four fundamental subspaces of A. They are:

1. $C(A)$ - the column space of A. This is defined to be the span of the columns of A. In other words, $C(A)$ consists of all linear combinations of the columns of A.
2. $N(A)$ - the nullspace of A. This is defined to be the set of all solutions to the homogeneous equation $Ax = \mathbf{0}$.
3. $C(A^T)$ - the column space of A^T. Since the columns of A^T are the rows of A, $C(A^T)$ is also called the row space of A; it is equal to the span of the rows of A.
4. $N(A^T)$ - the nullspace of A^T, also called the left nullspace of A.

Example. Suppose A is a 4×5 matrix.

$$A = \begin{bmatrix} * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \\ * & * & * & * & * \end{bmatrix}.$$

Notice that the columns of A are vectors in $\mathbb{R}^4$. Hence, the column space of A is a subspace of $\mathbb{R}^4$. The rows of A can be thought of as being vectors in $\mathbb{R}^5$, and so the rowspace of A is a subspace of $\mathbb{R}^5$.

The nullspace of A consists of all solutions $\mathbf{x}$ to the equation $A\mathbf{x} = \mathbf{0}$. Since A is 4×5, $\mathbf{x}$ must be a 5×1 vector. Thus, the nullspace of A is a subspace of $\mathbb{R}^5$.

The left nullspace consists of all solutions $\mathbf{y}$ to the equation $A^T\mathbf{y} = \mathbf{0}$. Since A is 4×5, A^T is 5×4, so $\mathbf{y}$ must be a 4×1 vector. Thus, the left nullspace of A is a subspace of $\mathbb{R}^4$. $\diamond$

In general, if A is $m \times n$, then $C(A)$ and $N(A^T)$ are subspaces of $\mathbb{R}^m$, while $C(A^T)$ and $N(A)$ are subspaces of $\mathbb{R}^n$.

5.7.1 Quick Review of Column Space

We've mentioned this several times by now, but it is important so we will keep repeating ourselves. Recall that if $A = [\mathbf{a}_1\ \mathbf{a}_2\ \ldots\ \mathbf{a}_n]$ and

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix},$$

then

$$A\mathbf{x} = x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + \ldots + x_n\mathbf{a}_n.$$

This is just a typical linear combination of the columns of A. Thus, to say that $\mathbf{b}$ is a linear combination of the columns of A and hence in $C(A)$ is to say that $\mathbf{b} = A\mathbf{x}$ for some $\mathbf{x}$. The takeaway from this is that $\mathbf{b}$ is in the columnspace of A if and only if the equation $A\mathbf{x} = \mathbf{b}$ has a solution. Thus, we can view $C(A)$ as the span of the columns of A, and we can also view it as the set of all vectors $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has a solution.

5.7.2 Why "Left" Nullspace?

Recall how transposing a product works: $(BC)^T = C^TB^T$. Now, suppose $\mathbf{y}$ is in $N(A^T)$. Since $\mathbf{y}$ is in the nullspace of A^T, $A^T\mathbf{y} = \mathbf{0}$. If we transpose both sides of the equation $A^T\mathbf{y} = \mathbf{0}$, we would get

$$\mathbf{y}^TA = \mathbf{0}^T.$$

So it kind of looks like multiplying A *on the left* by $\mathbf{y}$ gives us $\mathbf{0}$. This is why $N(A^T)$ is referred to as the *left* nullspace of A.

5.7.3 RREF & Finding Bases for the Fundamental Subspaces

Information about the fundamental subspaces of a matrix is most transparent after row reducing A to ***reduced row echelon form*** (RREF). Recall (Section 2.2) that a matrix is in row echelon form (REF) if

- Any and all rows of zero appear on the bottom.
- All entries beneath each pivot are 0.
- Each pivot is located to the right of the pivots in the previous rows.

A matrix is in RREF if, in addition to being in REF:

- All pivots are equal to 1. These pivots are often referred to as "leading 1's", since they are the first nonzero entry in each nonzero row.
- All entries above each pivot are 0.

To reduce a matrix to RREF, you basically do what you usually do to reduce it to REF, except at each step, you'll use the pivots to eliminate everything above them in addition to eliminating everything below them. And for each pivot not equal to 1, you'll divide that row by the pivot at some point to make the pivot equal to 1.

Once we reduce a matrix to RREF, we can obtain a basis for each of the four fundamental subspaces as follows:

- For the nullspace, just finish finding the solutions to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ by parametrizing the free variables, etc, as we usually do. Then take the general expression for the solutions $\mathbf{x}$ and write is as a linear combination of vectors using the parameters as coefficients. These vectors will give you a basis for $N(A)$. (This is basically what we did in the previous section to find bases and dimensions of various vector spaces).
- For the columnspace, identify which columns in the RREF have pivots. The corresponding columns in the original matrix A will form a basis for $C(A)$.
- For the rowspace, the nonzero rows of the RREF will give a basis for $C(A^T)$.
- For the left nullspace, take the basis for $C(A)$ that was found and put those vectors into a new matrix as rows. The nullspace of this new matrix will equal the left nullspace of A, so finding a basis for the nullspace of this new matrix will give you a basis for $N(A^T)$.

We will illustrate all of this in the next example, and we will also use the example to illustrate why everything works. Also, it is admittedly not necessary to reduce A to RREF. Just any REF will suffice. However, in my opinion, reducing A to RREF tends to make finding $N(A)$ easier. Also, the arguments for why everything works are simplest when A has been reduced to RREF, and since I want to explain how everything works, I will be reducing to RREF in the next example.

5.7.4 Explanatory Example

Example. Let

$$A = \begin{bmatrix} 1 & 0 & 1 & 1 & -2 \\ 2 & 1 & 0 & 2 & -3 \\ -1 & 2 & -5 & 0 & 3 \\ 0 & 1 & -2 & -1 & 2 \end{bmatrix}.$$

We are going to find a basis for each of the four fundamental subspaces of A. First, let's row reduce A to reduced row echelon form.

$$\begin{bmatrix} 1 & 0 & 1 & 1 & -2 \\ 2 & 1 & 0 & 2 & -3 \\ -1 & 2 & -5 & 0 & 3 \\ 0 & 1 & -2 & -1 & 2 \end{bmatrix} \begin{matrix} \xrightarrow{R_2-2R_1} \\ \xrightarrow{R_3+R_1} \end{matrix} \begin{bmatrix} 1 & 0 & 1 & 1 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 2 & -4 & 1 & 1 \\ 0 & 1 & -2 & -1 & 2 \end{bmatrix} \begin{matrix} \xrightarrow{R_3-2R_2} \\ \xrightarrow{R_4-R_2} \end{matrix}$$

$$\begin{bmatrix} 1 & 0 & 1 & 1 & -2 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -1 & 1 \end{bmatrix} \begin{array}{c} \xrightarrow{R_1 - R_3} \\ \\ \\ \xrightarrow{R_4 + R_3} \end{array} \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Finding a Basis for $N(A)$

We'll start with finding a basis for the nullspace of A. Since $N(A)$ is the set of solutions to the homogeneous equation, we should start by finding the solutions to $A\mathbf{x} = \mathbf{0}$. From the RREF of A, we have the following homogeneous system of equations:

$$\begin{aligned} x_1 \quad + \ x_3 \quad - x_5 &= 0 \\ x_2 - 2x_3 \quad + x_5 &= 0 \\ x_4 - x_5 &= 0. \end{aligned}$$

Since columns 3 and 5 do not have pivots, we'll choose x_3 and x_5 to be our free variables. If we assign parameters s and t to x_3 and x_5, respectively, and then solve for the remaining variables in terms of these parameters, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s+t \\ 2s-t \\ s \\ t \\ t \end{bmatrix}.$$

Let's take this expression for the general solution to $A\mathbf{x} = \mathbf{0}$ and write it as a linear combination of some vectors, using the parameters as coefficients:

$$\begin{bmatrix} -s+t \\ 2s-t \\ s \\ t \\ t \end{bmatrix} = s\begin{bmatrix} -1 \\ 2 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t\begin{bmatrix} 1 \\ -1 \\ 0 \\ 1 \\ 1 \end{bmatrix}.$$

We see that every solution to the homogeneous equation is a linear combination of the vectors $(-1, 2, 1, 0, 0)$ and $(1, -1, 0, 1, 1)$. Thus, these two vectors span the nullspace of A. It is also clear by looking at the expression on the left that the only way for everything to equal $\mathbf{0}$ is if $s = t = 0$. This shows that the vectors are also linearly independent.

Since the vectors $(-1, 2, 1, 0, 0)$ and $(1, -1, 0, 1, 1)$ span $N(A)$ and are linearly independent, it follows that they form a basis for $N(A)$.

Finding a Basis for $C(A)$

As mentioned earlier, to get a basis for the columspace, locate the columns in RREF(A) with a pivot. The corresponding columns in the original matrix will form a basis for $C(A)$. In this example, the pivots are located in columns 1, 2, and 4. Thus, the first, second, and fourth columns in the original matrix, which are

$$\mathbf{a}_1 = \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix} \qquad \mathbf{a}_2 = \begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{a}_4 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix},$$

form a basis for $C(A)$.

Let's take a closer look now to see why this is the case. In order to prove this claim, we will argue that the third and fifth columns, $\mathbf{a}_3$ and $\mathbf{a}_5$, are linear combinations of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$. Hence, any linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3, \mathbf{a}_4, \mathbf{a}_5$ is going to be a linear combination of just $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$, meaning that $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$ span $C(A)$. Then, we will argue that $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$ are linearly independent.

First, to check that $\mathbf{a}_3$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$, we would try to solve the equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_4\mathbf{a}_4 = \mathbf{a}_3$$

for x_1, x_2, x_4. To solve this, we would set up the augmented matrix $[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_4|\mathbf{a}_3]$ and row reduce. But we have already done all the row operations when we reduced A to RREF(A)! For reference, here is what we did earlier:

$$\begin{array}{ccccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 & \mathbf{a}_4 & \mathbf{a}_5 \end{array}$$
$$\begin{bmatrix} 1 & 0 & 1 & 1 & -2 \\ 2 & 1 & 0 & 2 & -3 \\ -1 & 2 & -5 & 0 & 3 \\ 0 & 1 & -2 & -1 & 2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 & 0 & -1 \\ 0 & 1 & -2 & 0 & 1 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now, if we take the augmented matrix $[\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_4\,|\,\mathbf{a}_3]$ and perform the exact same row operations as we did earlier, what happens is we'll get what we got in RREF(A), except the third and fourth columns are swapped (and there wouldn't be a fifth column):

$$\begin{array}{cccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_4 & \mathbf{a}_3 \end{array}$$
$$\left[\begin{array}{ccc|c} 1 & 0 & 1 & 1 \\ 2 & 1 & 2 & 0 \\ -1 & 2 & 0 & -5 \\ 0 & 1 & -1 & -2 \end{array}\right] \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & -2 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

The rows in the reduced matrix give us the equations $x_1 = 1$, $x_2 = -2$, and $x_4 = 0$. In other words,

$$1\mathbf{a}_1 - 2\mathbf{a}_2 + 0\mathbf{a}_4 = \mathbf{a}_3.$$

You can check that this is indeed the case:

$$1\underset{}{\overset{\mathbf{a}_1}{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}}} - 2\overset{\mathbf{a}_2}{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}} + 0\overset{\mathbf{a}_4}{\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}} = \overset{\mathbf{a}_3}{\begin{bmatrix} 1 \\ 0 \\ -5 \\ -2 \end{bmatrix}}$$

Thus, $\mathbf{a}_3$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$, so it doesn't contribute anything to the column space of A that $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$ don't already contribute.

Essentially then, the entries in the third column of RREF(A) tell us how $\mathbf{a}_3$ is a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$. The same reasoning applies to the fifth column. Namely, the fifth column in RREF(A) is

$$\begin{bmatrix} -1 \\ 1 \\ -1 \\ 0 \end{bmatrix}.$$

This tells us that $\mathbf{a}_5 = -1\mathbf{a}_1 + 1\mathbf{a}_2 - 1\mathbf{a}_4$. Again, you can check that this is indeed the case:

$$-1\overset{\mathbf{a}_1}{\begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}} + 1\overset{\mathbf{a}_2}{\begin{bmatrix} 0 \\ 1 \\ 2 \\ 1 \end{bmatrix}} - 1\overset{\mathbf{a}_4}{\begin{bmatrix} 1 \\ 2 \\ 0 \\ -1 \end{bmatrix}} = \overset{\mathbf{a}_5}{\begin{bmatrix} -2 \\ -3 \\ 3 \\ 2 \end{bmatrix}}$$

Thus, $\mathbf{a}_5$ is also a linear combination of $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$ and therefore doesn't contribute anything to the column space that $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$ don't already contribute. In particular, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$ span the column space of A.

Let's see why $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$ are linearly independent now. Consider the homogeneous equation

$$x_1\mathbf{a}_1 + x_2\mathbf{a}_2 + x_4\mathbf{a}_4 = \mathbf{0}.$$

We want to show that the only solution to this is the trivial solution where $x_1 = x_2 = x_4 = 0$. To solve for x_1, x_2, and x_4, we would set up the following augmented matrix $[\,\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_4 \,|\, \mathbf{0}\,]$ and row reduce. But again, we have already done the row operations! So we know that we would get

$$\begin{array}{c} \begin{array}{ccc} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_4 \end{array} \\ \left[\begin{array}{rrr|r} 1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 0 \\ -1 & 2 & 0 & 0 \\ 0 & 1 & -1 & 0 \end{array}\right] \end{array} \longrightarrow \left[\begin{array}{ccc|c} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{array}\right].$$

The RREF tells us that $x_1 = x_2 = x_4 = 0$. Thus, $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$ are linearly independent.

Since $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_4$ span the columnspace and are linearly independent, it follows that they form a basis for $C(A)$.

To summarize: find which columns in RREF(A) have pivots in them, and the corresponding columns in the original A will form a basis for $C(A)$. The arguments that we gave in this particular example for why this procedure works can easily be generalized for any $m \times n$ matrix.

Finding a Basis for $C(A^T)$

As we mentioned just prior to this example, the nonzero rows in RREF(A) give a basis for $C(A^T)$. In this example, the nonzero rows of RREF(A), written as vectors, are

$$(1, 0, 1, 0, -1) \qquad (0, 1, -2, 0, 1) \qquad \text{and} \qquad (0, 0, 0, 1, -1).$$

Thus, these three vectors form a basis for $C(A^T)$.

Now for a little explanation on why this works. First, we note that A and RREF(A) have the same row space. To see why this is so, note that all you are doing with row operations is replacing rows in the original matrix with linear combinations of rows in the original matrix. So all of the rows of RREF(A) are linear combinations of the rows of A. The reverse is true as well; that is, the rows of A are linear combinations of the rows of RREF(A). This is because we can invert all of the row operations performed to go from RREF(A) back to A. Hence, A and RREF(A) have the same row space. In general, elementary row operations preserve the row space of a matrix. This explains why we are able to extract a basis for the row space of A from the rows of RREF(A). On the other hand, row operations do not preserve the column space of a matrix. This is why we could not use the columns of RREF(A) to get a basis for $C(A)$, and instead, we had to take the columns from the original matrix.

Next, we observe that the nonzero rows of RREF(A) are linearly independent. Let's verify that this is the case in our example. Again, the nonzero rows of RREF(A), written as vectors, are

$$(1, 0, 1, 0, -1) \qquad (0, 1, -2, 0, 1) \qquad \text{and} \qquad (0, 0, 0, 1, -1).$$

Consider the equation

$$c_1 \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix} + c_2 \begin{bmatrix} 0 \\ 1 \\ -2 \\ 0 \\ 1 \end{bmatrix} + c_3 \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Notice that the first, second, and fourth equations immediately give us $c_1 = c_2 = c_3 = 0$. These equations are actually coming from the first, second, and fourth columns of the RREF, which contain the pivots (= leading 1's) and zeros everywhere else. It is because of this structure in the RREF that we immediately got the trivial solution.

Thus, the nonzero rows of RREF(A) are linearly independent, and of course they span the row space of RREF(A), which is the same as the row space of A. Hence, the nonzero rows of RREF(A) give a basis for $C(A^T)$.

Basis for $N(A^T)$

Finally, we'll find a basis for the left nullspace of A. Unfortunately, we cannot get a basis for $N(A^T)$ directly from RREF(A). So we'll do what we did with finding a basis for the nullspace of A, but with A^T instead of A. There is a small simplification that we can make, however.

The first step would be to reduce A^T to RREF(A^T). This is where the small simplification comes in. The rows of A^T are the columns of A, and we saw earlier that columns 3 and 5 in A are linear combinations of columns 1, 2, and 4. Thus, rows 3 and 5 in A^T are linear combinations of rows 1, 2, and 4, which means that they will reduce to rows of zeros when we perform the row operations. Because of this, we can ignore the third and fifth rows of A^T, and just work with rows 1, 2, and 4 instead.

We'll take columns 1, 2, and 4 from A (the basis for $C(A)$) and put them into a matrix as rows, and then row reduce:

$$\begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 2 & 1 \\ 1 & 2 & 0 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 & -7 \\ 0 & 1 & 0 & 3 \\ 0 & 0 & 1 & -1 \end{bmatrix}.$$

Since the fourth column is missing a pivot, we choose x_4 to be a free variable. If we assign the parameter t to x_4 and solve the homogeneous equation, we get

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = t \begin{bmatrix} 7 \\ -3 \\ 1 \\ 1 \end{bmatrix}.$$

Then the vector $(7, -3, 1, 1)$ forms a basis for the left nullspace of A. $\diamond$

5.7.5 Summary of Procedure

We summarize the procedure for finding bases for each of the fundamental subspaces:

1. First, reduce A to REF or RREF.

2. For $N(A)$, use the REF or RREF to solve the homogeneous equation. Parametrize the free variables, and express the general solution to the homogeneous equation as a linear combination of vectors with the parameters as the coefficients. The vectors that you end up with form a basis for $N(A)$.

3. For $C(A)$, identify the columns in the REF or RREF that have pivots in them. The corresponding columns in the original matrix form a basis for $C(A)$. We'll call these the basic columns.

   If necessary, you can express the non-basic columns as a linear combination of the basic columns by looking at the entries in RREF(A). For each non-basic column, the entries in the corresponding column in RREF(A) give the coefficients needed to express the non-basic column as a linear combination of the basic columns. Note: This part only works when using the RREF, it does not work when using the REF.

4. The nonzero rows in the REF or RREF form a basis for $C(A^T)$.

5. To get a basis for $N(A^T)$, take the basic columns of A and put them into a matrix as rows. We'll call this new matrix B. Then the nullspace of B will be equal to the nullspace of A^T. Find a basis for the nullspace of B, and that will be a basis for $N(A^T)$.

5.7.6 Rank and Nullity

Definition. For any matrix A,

- the ***rank*** of A is defined to be the dimension of its column space, $C(A)$;
- the ***nullity*** of A is defined to be the dimension of its nullspace, $N(A)$.

Recall that the dimension of a vector space is equal to the number of vectors in any basis for the space. We know that when finding a basis for $C(A)$, we get one basis vector for each column with a pivot. Thus, the rank of A is also equal to the number of pivots. Similarly, we get one basis vector for $C(A^T)$, the row space of A, for each row with a pivot. So the rank of A is also equal to the dimension of $C(A^T)$, which in turn is equal to the rank of A^T by definition. We summarize our findings in the following theorem.

Theorem. For any matrix A, the rank of A is equal to all of the following:

- The dimension of $C(A)$.
- The number of pivots.
- The dimension of $C(A^T)$.
- The rank of A^T.

On the other hand, we know that when finding a basis for $N(A)$, we get one basis vector for each parameter in the general solution to the homogeneous equation. So the dimension of $N(A)$ is equal to the number of parameters in the general solution to the homogeneous equation. We know that the number of parameters is equal to the number of free variables, and we get one free variable for each column without a pivot. In summary then, we have:

Theorem. For any matrix A, the nullity of A is equal to all of the following:

- The dimension of $N(A)$.
- The number of parameters in the general solution to the homogeneous equation.
- The number of free variables.
- The number of columns without a pivot.

In particular, since $\text{rank}(A)$ is equal to the number of columns with pivots, and $\text{nullity}(A)$ is equal to the number of columns without pivots, it follows that

$$\text{rank}(A) + \text{nullity}(A) = \text{number of columns in } A$$

This result is typically called the rank-nullity theorem. If we apply this result to A^T, we get

$$\text{rank}(A) + \text{nullity}(A^T) = \text{number of rows in } A.$$

If A is $m \times n$, then we can rewrite these two as

$$\text{rank}(A) + \text{nullity}(A) = n$$

and

$$\text{rank}(A) + \text{nullity}(A^T) = m.$$

5.7.7 Orthogonality of the Fundamental Subspaces

Let W be a subspace of $\mathbb{R}^n$. Recall that the orthogonal complement of W, denoted $W^\perp$, is defined to be the set of all vectors in $\mathbb{R}^n$ that are orthogonal to every vector in W.

Properties of Orthogonal Complements

- $W^\perp$ is always a subspace of $\mathbb{R}^n$.
- $(W^\perp)^\perp = W$.
- The dimension of W plus the dimension of $W^\perp$ is always equal to n.

Now let A be an $m \times n$ matrix. Then we have the following relationships between the fundamental subspaces of A:

- The row space of A and the nullspace of A are orthogonal complements, i.e. $C(A^T)^\perp = N(A)$ and $N(A)^\perp = C(A^T)$.
- The column space of A and the left nullspace of A are orthogonal complements, i.e. $C(A)^\perp = N(A^T)$ and $N(A^T)^\perp = C(A)$.

Let's try and understand why the row space and nullspace are orthogonal complements. For clarity, let's suppose that A has m rows. Recall that $A\mathbf{x}$ can be computed by taking the dot product between the rows of A and $\mathbf{x}$:

$$A\mathbf{x} = \begin{bmatrix} (\text{Row 1 of } A) \cdot \mathbf{x} \\ (\text{Row 2 of } A) \cdot \mathbf{x} \\ \vdots \\ (\text{Row } m \text{ of } A) \cdot \mathbf{x} \end{bmatrix}$$

To say that $\mathbf{x}$ is in $N(A)$ is equivalent to saying that $A\mathbf{x} = \mathbf{0}$, which in turn is equivalent to saying that

$$\begin{bmatrix} (\text{Row 1 of } A) \cdot \mathbf{x} \\ (\text{Row 2 of } A) \cdot \mathbf{x} \\ \vdots \\ (\text{Row } m \text{ of } A) \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 0 \end{bmatrix}.$$

Since two vectors are orthogonal if and only if their dot product is zero, this in turn is equivalent to saying that $\mathbf{x}$ is orthogonal to all of the rows of A. Thus, we see that $\mathbf{x}$ is in $N(A)$ if and only if $\mathbf{x}$ is orthogonal to all of the rows of A. This in turn is equivalent to saying that $\mathbf{x}$ is orthogonal to the row space of A. Hence, $N(A)$ and $C(A^T)$ are orthogonal complements.

The same argument applied to A^T shows that $C(A)$ and $N(A^T)$ are orthogonal complements.

Example. Let A be a 3×2 matrix, and suppose $\text{rank}(A) = 2$. Describe geometrically the four fundamental subspaces of A. What can you say about the equation $A\mathbf{x} = \mathbf{b}$?

$$A = \begin{bmatrix} * & * \\ * & * \\ * & * \end{bmatrix}$$

Solution. First, notice that the columns of A are vectors in $\mathbb{R}^3$. So the column space and its orthogonal complement (the left nullspace) are subspaces of $\mathbb{R}^3$. On the other hand, the rows of A are vectors in $\mathbb{R}^2$. So the row space and its orthogonal complement (the nullspace) are subspaces of $\mathbb{R}^2$.

Since $\text{rank}(A) = 2$, it follows that $C(A)$ and $C(A^T)$ are both two-dimensional. So $C(A)$ would be a plane passing through the origin in $\mathbb{R}^3$, while $C(A^T)$ would be all of $\mathbb{R}^2$.

From the rank-nullity theorem, we see that $\text{nullity}(A) = 0$, so $N(A)$ contains only the origin in $\mathbb{R}^2$; while $\text{nullity}(A^T) = 1$, so $N(A^T)$ is a line passing through the origin in $\mathbb{R}^3$.

Since the column space and the left nullspace are orthogonal complements, we can be more specific and say that $C(A)$ is a plane passing through the origin in $\mathbb{R}^3$, while $N(A^T)$ is a line passing through the origin in $\mathbb{R}^3$ that is orthogonal to the plane $C(A)$.

The equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in $C(A)$. We saw earlier that $C(A)$ is only a plane in $\mathbb{R}^3$. So for any vector $\mathbf{b}$ in $\mathbb{R}^3$ that does not lie in this plane, the equation $A\mathbf{x} = \mathbf{b}$ would have no solution. In particular, the equation $A\mathbf{x} = \mathbf{b}$ is not always solvable.

On the other hand, if $\mathbf{b}$ is in $C(A)$, then not only will $A\mathbf{x} = \mathbf{b}$ have a solution, but the solution will also be unique. One way to see this is to notice that since $\text{nullity}(A) = 0$, there are no free variables. Hence, there would not be infinitely many solutions, so instead the solution would be unique. $\diamond$

Example. Let A be a 2×3 matrix, and suppose $\text{rank}(A) = 2$. Describe geometrically the four fundamental subspaces of A. What can you say regarding the equation $A\mathbf{x} = \mathbf{b}$?

$$A = \begin{bmatrix} * & * & * \\ * & * & * \end{bmatrix}$$

Solution. First, notice that the columns of A are vectors in $\mathbb{R}^2$. So the column space and its orthogonal complement (the left nullspace) are subspaces of $\mathbb{R}^2$. On the other hand, the rows of A are vectors in $\mathbb{R}^3$. So the row space and its orthogonal complement (the nullspace) are subspaces of $\mathbb{R}^3$.

Since $\text{rank}(A) = 2$, it follows that $C(A)$ and $C(A^T)$ are both two-dimensional. So $C(A)$ is all of $\mathbb{R}^2$, while $C(A^T)$ is a plane passing through the origin in $\mathbb{R}^3$.

From the rank-nullity theorem, we see that $\text{nullity}(A) = 1$, so $N(A)$ is a line passing through the origin in $\mathbb{R}^3$; we also see that $\text{nullity}(A^T) = 0$, so $N(A^T)$ contains only the origin in $\mathbb{R}^2$.

Since the row space and the nullspace are orthogonal complements, we can be more specific and say that $C(A^T)$ is a plane passing through the origin in $\mathbb{R}^3$, while $N(A)$ is a line passing through the origin in $\mathbb{R}^3$ that is orthogonal to the plane $C(A^T)$.

Since the equation $A\mathbf{x} = \mathbf{b}$ has a solution if and only if $\mathbf{b}$ is in $C(A)$, and since $C(A)$ is equal to all of $\mathbb{R}^2$, it follows that the equation $A\mathbf{x} = \mathbf{b}$ always has a solution. Furthermore, it will always have infinitely many solutions. One way to see why this is the case is to observe that since $\text{nullity}(A) = 1$, there is a free variable. Since we have at least one free variable, there will be infinitely many solutions. $\diamond$

5.7.8 The General Solution to $A\mathbf{x} = \mathbf{b}$

Suppose $\mathbf{x}$ satisfies $A\mathbf{x} = \mathbf{b}$ and that $\mathbf{w}$ is in $N(A)$, so $A\mathbf{w} = \mathbf{0}$. Then

$$A(\mathbf{x} + \mathbf{w}) = A\mathbf{x} + A\mathbf{w} = \mathbf{b} + \mathbf{0} = \mathbf{b}.$$

Thus, if $\mathbf{x}$ is a solution to $A\mathbf{x} = \mathbf{b}$, then so too is $\mathbf{x} + \mathbf{w}$ for any $\mathbf{w}$ in $N(A)$.

On the other hand, suppose $\mathbf{x}_1$ and $\mathbf{x}_2$ are two solutions to $A\mathbf{x} = \mathbf{b}$, so that

$$A\mathbf{x}_1 = \mathbf{b} \qquad \text{and} \qquad A\mathbf{x}_2 = \mathbf{b}.$$

Then

$$A(\mathbf{x}_2 - \mathbf{x}_1) = A\mathbf{x}_2 - A\mathbf{x}_1 = \mathbf{b} - \mathbf{b} = \mathbf{0}.$$

If we set $\mathbf{w} = \mathbf{x}_2 - \mathbf{x}_1$, then we see that $\mathbf{w}$ is in $N(A)$. Furthermore, we have $\mathbf{x}_2 = \mathbf{x}_1 + \mathbf{w}$. This shows us that if $\mathbf{x}_1$ is a solution to $A\mathbf{x} = \mathbf{b}$, then any other solution $\mathbf{x}_2$ has the form $\mathbf{x}_1 + \mathbf{w}$, where $\mathbf{w}$ is in $N(A)$.

Thus, we see that once we find one particular solution $\mathbf{x}_p$ (p for particular) to $A\mathbf{x} = \mathbf{b}$, that the general solution $\mathbf{x}$ can be expressed as

$$\mathbf{x} = \mathbf{x}_p \ + \ \mathbf{x}_h,$$

where $\mathbf{x}_h$ represents the general solution to the homogeneous equation $A\mathbf{x} = \mathbf{0}$ and a typical element of $N(A)$.

Example. Let A be a 4×6 matrix, and denote the j^{th} column of A by $\mathbf{a}_j$. Suppose that columns $\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5$ form a basis for $C(A)$, and that

$$\begin{aligned}
\mathbf{a}_2 &= -2\mathbf{a}_1 \\
\mathbf{a}_4 &= \mathbf{a}_1 - 3\mathbf{a}_3 \\
\mathbf{a}_6 &= -\mathbf{a}_1 + 2\mathbf{a}_3 - 4\mathbf{a}_5.
\end{aligned}$$

Answer the following questions.

a. What is the rank of A?

b. What is the nullity of A?

c. Find a basis for the orthogonal complement of the nullspace of A.

d. Find a basis for the orthogonal complement of the row space of A.

e. Suppose $\mathbf{b} = 2\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_4 - 3\mathbf{a}_5$. Does the equation $A\mathbf{x} = \mathbf{b}$ have a solution? If so, write out the general solution to this equation.

Solution.

a. Since there are three vectors in the basis for $C(A)$, it follows that $C(A)$ is three-dimensional. Hence, $\text{rank}(A) = 3$.

b. By the rank-nullity theorem,

$$\text{rank}(A) + \text{nullity}(A) = \text{number of columns in } A.$$

Since $\text{rank}(A) = 3$ and A has six columns, it follows that $\text{nullity}(A) = 3$.

c. The orthogonal complement of the nullspace of A is $C(A^T)$, the row space of A. A basis for $C(A^T)$ is given by the nonzero rows in $\text{RREF}(A)$. Thus, we need to find $\text{RREF}(A)$ first.

Since $\mathbf{a}_1, \mathbf{a}_3, \mathbf{a}_5$ form a basis for $C(A)$, we know that the first, third, and fifth columns of $\text{RREF}(A)$ will be

$$\begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

respectively, since these will be the columns with pivots.

Next, since $\mathbf{a}_2 = -2\mathbf{a}_1$, the second column of $\text{RREF}(A)$ must be

$$\begin{bmatrix} -2 \\ 0 \\ 0 \\ 0 \end{bmatrix}.$$

Since $\mathbf{a}_4 = \mathbf{a}_1 - 3\mathbf{a}_3$, the fourth column of $\text{RREF}(A)$ must be

$$\begin{bmatrix} 1 \\ -3 \\ 0 \\ 0 \end{bmatrix}.$$

And since $\mathbf{a}_6 = -\mathbf{a}_1 + 2\mathbf{a}_3 - 4\mathbf{a}_5$, the sixth column of $\text{RREF}(A)$ must be

$$\begin{bmatrix} -1 \\ 2 \\ -4 \\ 0 \end{bmatrix}.$$

Putting everything together then, we see that

$$\mathrm{RREF}(A) = \begin{bmatrix} 1 & -2 & 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & -3 & 0 & 2 \\ 0 & 0 & 0 & 0 & 1 & -4 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}.$$

Now we just take the nonzero rows in RREF(A) to get the basis for the rowspace of A:

$$\text{Basis for } C(A^T): \begin{bmatrix} 1 \\ -2 \\ 0 \\ 1 \\ 0 \\ -1 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 1 \\ -3 \\ 0 \\ 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ -4 \end{bmatrix}.$$

d. The orthogonal complement of the row space of A is the nullspace of A. So this question is asking us to find a basis for $N(A)$. This requires us to solve the homogeneous equation. From the nonzero rows of RREF(A), we get the following system of equations:

$$\begin{aligned} x_1 - 2x_2 \quad + x_4 \quad - x_6 &= 0 \\ x_3 - 3x_4 \quad + 2x_6 &= 0 \\ x_5 - 4x_6 &= 0. \end{aligned}$$

Since the second, fourth, and sixth columns do not have pivots, we choose x_2, x_4, x_6 to be free variables. If we parametrize them as $x_2 = r$, $x_4 = s$, and $x_6 = t$, and then solve for x_1, x_3, x_5 in terms of these parameters, we get

$$\begin{aligned} x_1 &= 2r - s + t \\ x_2 &= r \\ x_3 &= 3s - 2t \\ x_4 &= s \\ x_5 &= 4t \\ x_6 &= t. \end{aligned}$$

Equivalently,

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \\ x_6 \end{bmatrix} = r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 4 \\ 1 \end{bmatrix}.$$

Thus, we have

$$\text{Basis for } N(A): \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 4 \\ 1 \end{bmatrix}.$$

e. Since the given vector $\mathbf{b}$ is a linear combination of the columns of A, the equation $A\mathbf{x} = \mathbf{b}$

does have a solution. In fact, since $\mathbf{b} = 2\mathbf{a}_1 - \mathbf{a}_2 + \mathbf{a}_4 - 3\mathbf{a}_5$, we know that

$$\mathbf{x}_p = \begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \\ -3 \\ 0 \end{bmatrix}$$

is a particular solution to the equation $A\mathbf{x} = \mathbf{b}$.

Recall that the general solution to $A\mathbf{x} = \mathbf{b}$ can be expressed as $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_h$, where $\mathbf{x}_p$ is a particular solution and $\mathbf{x}_h$ is the general solution to the homogeneous equation. We already found the general solution of the homogeneous equation in Part d. Thus, the general solution to the equation $A\mathbf{x} = \mathbf{b}$ is given by

$$\mathbf{x} = \overbrace{\begin{bmatrix} 2 \\ -1 \\ 0 \\ 1 \\ -3 \\ 0 \end{bmatrix}}^{\mathbf{x}_p} \overbrace{+ r \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} + s \begin{bmatrix} -1 \\ 0 \\ 3 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ -2 \\ 0 \\ 4 \\ 1 \end{bmatrix}}^{\mathbf{x}_h}.$$

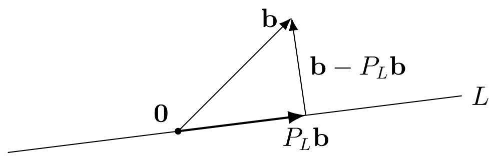
That concludes the example. ◇

Chapter 6

Orthogonal Projections

6.1 Projection onto a Line

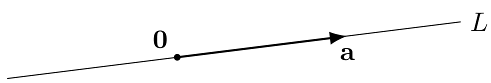
Consider the following picture of a line L that passes through the origin $\mathbf{0}$ and a vector $\mathbf{b}$:



The vector labeled as $P_L\mathbf{b}$ is called the projection of $\mathbf{b}$ onto the line L. It satisfies the following two properties:

1. The vector $\mathbf{b} - P_L\mathbf{b}$ is orthogonal to the line L.
2. Among all vectors in L, the vector $P_L\mathbf{b}$ is the one that is closest to $\mathbf{b}$.

Let's derive a formula for $P_L\mathbf{b}$. Start by choosing any nonzero vector $\mathbf{a}$ that lies on the line L:



Since $P_L\mathbf{b}$ and $\mathbf{a}$ lie along the same line, they must be scalar multiples of each other. Let k be the scalar such that

$$P_L\mathbf{b} = k\mathbf{a}.$$

Now, since $\mathbf{b} - P_L\mathbf{b}$ is orthogonal to L, it also follows that $\mathbf{b} - P_L\mathbf{b}$ is orthogonal to $\mathbf{a}$. Thus,

$$\big(\mathbf{b} - P_L\mathbf{b}\big) \cdot \mathbf{a} = 0.$$

Substitute $P_L\mathbf{b} = k\mathbf{a}$ into the above equation to get

$$(\mathbf{b} - k\mathbf{a}) \cdot \mathbf{a} = 0.$$

Distribute the dot product:

$$\mathbf{b} \cdot \mathbf{a} - k(\mathbf{a} \cdot \mathbf{a}) = 0.$$

And then solve for k to get

$$k = \frac{\mathbf{b} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}} = \frac{\mathbf{b} \cdot \mathbf{a}}{\|\mathbf{a}\|^2}.$$

Plugging this into $P_L\mathbf{b} = k\mathbf{a}$ gives us the following formula:

$$P_L\mathbf{b} = \left(\frac{\mathbf{b} \cdot \mathbf{a}}{\mathbf{a} \cdot \mathbf{a}}\right)\mathbf{a}.$$

Some of you might recognize this as the formula for the projection of $\mathbf{b}$ onto $\mathbf{a}$ from calculus.

6.2 Projection onto a Subspace

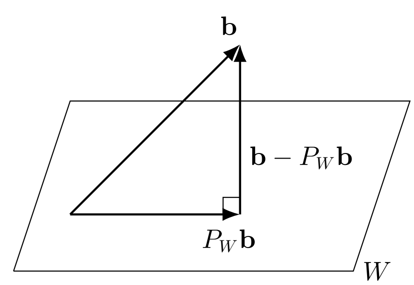
Remember that lines through the origin are one-dimensional subspaces. We want to generalize the notion of projection onto lines by considering the notion of projection onto a subspace. Suppose W is a subspace of $\mathbb{R}^n$, and let $\mathbf{b}$ be a vector in $\mathbb{R}^n$. We define the projection of $\mathbf{b}$ onto W, denoted $P_W\mathbf{b}$, to be the vector in W satisfying the following two properties:

1. The vector $\mathbf{b} - P_W\mathbf{b}$ is orthogonal to W.
2. Among all vectors in W, the vector $P_W\mathbf{b}$ is the one that is closest to $\mathbf{b}$.

It actually turns out that these two properties are equivalent to each other, in the sense that any vector in W that satisfies the first property will also satisfy the second property, and vice versa.

If you are wondering how do we know that a vector satisfying the above properties exists and is unique, worry not, because we will derive a formula for it later on.

Here is a good picture to keep in mind:



6.3 Least Squares

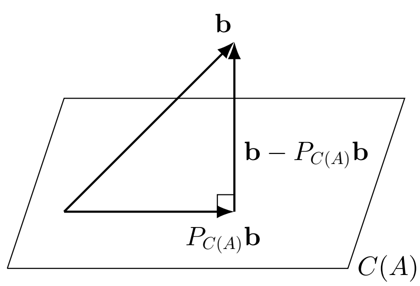
Probably most of you are familiar with Newton's second law, which says that the net force acting on an object is equal its mass times its acceleration, or, in short, $\mathbf{F} = m\mathbf{a}$. Notice that this is a linear relationship between force and acceleration.

For those of you who have taken or are currently taking Physics I, you might remember one of the labs had you measure the force and acceleration on an object. Using the measurements for acceleration and force, you had to compute what the mass of the object was. To do this, you were asked to create a scatter plot of Force vs. Acceleration. In theory, from the equation $\mathbf{F} = m\mathbf{a}$, we would expect the data points to all lie on the same line, and the slope of this line would be the mass of the object. However, due to measurement inaccuracies, human error, certain simplifying assumptions (no friction forces), etc, the data points did not lie on the same line. So then what you did was you found the equation of the line that best fit through the data, and used the slope of the best fit line as an estimate for the object's mass.

More generally, one can imagine an experiment where certain quantities are being measured (e.g. force and acceleration) in order to estimate another quantity (e.g. mass). For one reason or another, we choose a linear model to describe the relationship between these quantities. So, the measurements from the experiment might be stored into some matrix A and some vector $\mathbf{b}$, and the quantity $\mathbf{x}$ that we are trying to estimate is expected to satisfy the equation $A\mathbf{x} = \mathbf{b}$. However, due to measurement errors, the equation we end up with has no solution. We still want to solve the equation $A\mathbf{x} = \mathbf{b}$ as best as we can though, in order to estimate $\mathbf{x}$.

This leads us to the following question: if $A\mathbf{x} = \mathbf{b}$ has no solution, then can we find a vector $\hat{\mathbf{x}}$ that is somehow closest to being a solution to the original equation?

Well, if the equation $A\mathbf{x} = \mathbf{b}$ has no solution, then the vector $\mathbf{b}$ must be outside $C(A)$, the column space of A.



We know that the vector in $C(A)$ that is closest to $\mathbf{b}$ is its projection, $P_{C(A)}\mathbf{b}$. So in some sense, the closest we can get to solving $A\mathbf{x} = \mathbf{b}$ is to solve $A\mathbf{x} = P_{C(A)}\mathbf{b}$.

Definition. Any solution $\hat{\mathbf{x}}$ to the equation $A\hat{\mathbf{x}} = P_{C(A)}\mathbf{b}$ is called a least squares solution to the equation $A\mathbf{x} = \mathbf{b}$.

Again, you should think of a least squares solution to $A\mathbf{x} = \mathbf{b}$ as the closest thing to an actual solution of $A\mathbf{x} = \mathbf{b}$.

In order to find a least squares solution $\hat{\mathbf{x}}$, it looks like we need to solve the equation $A\hat{\mathbf{x}} = P_{C(A)}\mathbf{b}$. However, we currently don't have a formula for computing $P_{C(A)}\mathbf{b}$, so how can we solve for $\hat{\mathbf{x}}$? Well, it turns out that with a bit of reasoning, we can solve for $\hat{\mathbf{x}}$ without computing $P_{C(A)}\mathbf{b}$. First, to simplify the notation, let

$$\mathbf{e} = \mathbf{b} - P_{C(A)}\mathbf{b}.$$

The key principle is that $\mathbf{e}$ should be orthogonal to $C(A)$. This means that $\mathbf{e}$ is in $N(A^T)$, since $N(A^T)$ is the orthogonal complement of $C(A)$. As a vector in $N(A^T)$, $\mathbf{e}$ must be a solution to the homogeneous equation for A^T, i.e., $A^T\mathbf{e} = \mathbf{0}$. Rewriting $\mathbf{e}$ as $\mathbf{b} - P_{C(A)}\mathbf{b}$, we have

$$A^T(\mathbf{b} - P_{C(A)}\mathbf{b}) = \mathbf{0}.$$

If $\hat{\mathbf{x}}$ is a least squares solution to $A\mathbf{x} = \mathbf{b}$, then $A\hat{\mathbf{x}} = P_{C(A)}\mathbf{b}$. Substituting this into the above equation gives us

$$A^T(\mathbf{b} - A\hat{\mathbf{x}}) = \mathbf{0}.$$

Rearranging this gives us the so-called normal equations:

$$A^T A\hat{\mathbf{x}} = A^T\mathbf{b}.$$

What we see here is that, in order to find $\hat{\mathbf{x}}$, we can solve the normal equations instead of solving the equation $A\hat{\mathbf{x}} = P_{C(A)}\mathbf{b}$. This way, we manage to avoid the computation of the projection vector.

Example. Suppose the following data represents measurements obtained from some experiment:

x	1	1	2	3
y	2	3	3	4

We want to model the relationship between x and y as a "linear" relationship of the form

$$y = cx + d$$

for some constants c and d. Assuming such a relationship holds, the data points (x, y) in the above table should satisfy this equation. Plugging the four data points in yields the following system of equations:

$$\begin{aligned} 2 &= c + d \\ 3 &= c + d \\ 3 &= 2c + d \\ 4 &= 3c + d \end{aligned}$$

It is immediately obvious that this system of equations has no solution (just look at the first and second equations). We can rewrite this as the matrix equation $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} \qquad \mathbf{x} = \begin{bmatrix} c \\ d \end{bmatrix} \qquad \mathbf{b} = \begin{bmatrix} 2 \\ 3 \\ 3 \\ 4 \end{bmatrix}.$$

Since $A\mathbf{x} = \mathbf{b}$ has no solution, we will instead look for a least squares solution

$$\hat{\mathbf{x}} = \begin{bmatrix} \hat{c} \\ \hat{d} \end{bmatrix}.$$

The entries of $\hat{\mathbf{x}}$ will represent the slope and intercept of the line that best fits our data, in the least square sense. We solve for $\hat{\mathbf{x}}$ by solving the normal equations

$$A^T A\hat{\mathbf{x}} = A^T\mathbf{b}.$$

We compute:

$$A^T A = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \\ 2 & 1 \\ 3 & 1 \end{bmatrix} = \begin{bmatrix} 15 & 7 \\ 7 & 4 \end{bmatrix}.$$

$$A^T\mathbf{b} = \begin{bmatrix} 1 & 1 & 2 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \\ 3 \\ 4 \end{bmatrix} = \begin{bmatrix} 23 \\ 12 \end{bmatrix}.$$

Then the normal equations are

$$\begin{bmatrix} 15 & 7 \\ 7 & 4 \end{bmatrix} \begin{bmatrix} \hat{c} \\ \hat{d} \end{bmatrix} = \begin{bmatrix} 23 \\ 12 \end{bmatrix}.$$

Using the formula

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix},$$

we find that

$$\begin{bmatrix} \hat{c} \\ \hat{d} \end{bmatrix} = \frac{1}{11}\begin{bmatrix} 4 & -7 \\ -7 & 15 \end{bmatrix}\begin{bmatrix} 23 \\ 12 \end{bmatrix} = \begin{bmatrix} \frac{8}{11} \\ \frac{19}{11} \end{bmatrix}.$$

Thus, the equation of the line that best fits our data is given by

$$y = \tfrac{8}{11}x + \tfrac{19}{11}.$$

◇

6.3.1 Some Remarks

In the previous example, we obtained the least squares solution $\hat{\mathbf{x}}$ by using the following formula:

$$\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}.$$

In particular, the least squares solution was unique because A^TA was invertible. It is worth noting that if A^TA is not invertible, then there will be infinitely many least squares solutions.

It is a fact that the matrix A^TA will be invertible if and only if the columns of A are linearly independent (meaning every column of A would have a pivot). One argument for why this is so is based off the following:

Proposition. For any matrix A, the nullspace of A^TA and the nullspace of A coincide. In short, $N(A^TA) = N(A)$.

Proof. Suppose $\mathbf{x}$ is in $N(A)$, so that $A\mathbf{x} = \mathbf{0}$. Then $\mathbf{x}$ is also in $N(A^TA)$, since

$$(A^TA)\mathbf{x} = A^T(A\mathbf{x}) = A^T(\mathbf{0}) = \mathbf{0}.$$

This proves that every vector in $N(A)$ is also in $N(A^TA)$; hence, we see that $N(A)$ is contained in $N(A^TA)$.

On the other hand, suppose now that $\mathbf{x}$ is in $N(A^TA)$, so that $A^TA\mathbf{x} = \mathbf{0}$. Notice that this equation says $A\mathbf{x}$ is in $N(A^T)$. On the other hand, $A\mathbf{x}$ is also in $C(A)$, as matrix-vector multiplication results in a linear combination of the columns of the matrix. Thus, we have a vector, $A\mathbf{x}$, that lies in both $N(A^T)$ and $C(A)$. Since $N(A^T)$ and $C(A)$ are orthogonal complements, the only vector common to both of them would be the zero vector. Thus, $A\mathbf{x} = \mathbf{0}$, meaning $\mathbf{x}$ lies in $N(A)$. This proves that every vector in $N(A^TA)$ is also in $N(A)$, i.e. that $N(A^TA)$ is contained in $N(A)$.

Since $N(A)$ and $N(A^TA)$ are contained in each other, they must be equal to each other. This concludes the proof. □

We can now prove that A^TA is invertible if and only if the columns of A are linearly independent. The argument is as follows:

1. A^TA is invertible if and only if $N(A^TA)$ consists only of the zero vector.
2. Since $N(A^TA) = N(A)$, it follows that A^TA is invertible if and only if $N(A)$ consists only of the zero vector.
3. $N(A)$ consists only of the zero vector if and only if the columns of A are linearly independent.
4. QED.

6.4 Projection Formula

Suppose A^TA is invertible. Then the least squares solution to $A\mathbf{x} = \mathbf{b}$ is given by the formula

$$\hat{\mathbf{x}} = (A^TA)^{-1}A^T\mathbf{b}.$$

Recall however that $\hat{\mathbf{x}}$ is a solution to the equation

$$A\hat{\mathbf{x}} = P_{C(A)}\mathbf{b}.$$

Since know $\hat{\mathbf{x}}$, we can substitute it into the above equation to solve for the projection! Doing so yields the following formula for computing the projection of $\mathbf{b}$ onto $C(A)$:

$$A(A^TA)^{-1}A^T\mathbf{b} = P_{C(A)}\mathbf{b}.$$

The matrix

$$A(A^TA)^{-1}A^T,$$

which we will denote by $P_{C(A)}$, is called the ***projection matrix*** onto $C(A)$. Multiplying a vector $\mathbf{b}$ by this projection matrix has the effect of projecting $\mathbf{b}$ onto $C(A)$. It is important to note that this formula specifically gives the projection onto the column space of A.

Now suppose that we have an arbitrary subspace W of $\mathbb{R}^n$. We can obtain the projection matrix onto W with the following procedure:

1. Find a basis for W.
2. Put the basis vectors found in Step 1 into the columns of a matrix. Call this matrix A. Since the columns of A form a basis for W, it follows that the column space of A is equal to W, and that A^TA is invertible.
3. The matrix $A(A^TA)^{-1}A^T$ is equal to the projection matrix onto $C(A)$. Since $C(A) = W$, it follows that $P_W = A(A^TA)^{-1}A^T$ is the projection matrix onto W. Multiplying a vector $\mathbf{b}$ by P_W has the effect of projecting $\mathbf{b}$ onto the subspace W.

Example. Let

$$A = \begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \end{bmatrix}.$$

Find the projection matrix onto the nullspace of A. Use it to find the vector in $N(A)$ that is closest to the vector $\mathbf{b} = (2, 1, -1)$.

Solution. First, we need to find a basis for $N(A)$.

$$\begin{bmatrix} 1 & 1 & -1 \\ 2 & 2 & -2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}$$

We choose x_2 and x_3 to be free variables since the second and third columns have no pivot. Letting $x_2 = s$ and $x_3 = t$, and solving the homogeneous equation for x_1 yields

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

The vectors $(-1, 1, 0)$ and $(1, 0, 1)$ form a basis for $N(A)$. Put them into a matrix B as columns:

$$B = \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then the projection matrix onto $N(A)$ is given by $B(B^TB)^{-1}B^T$. So now we compute:

$$B^TB = \begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}.$$

Using the formula for a 2×2 inverse gives us

$$(B^T B)^{-1} = \frac{1}{3}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

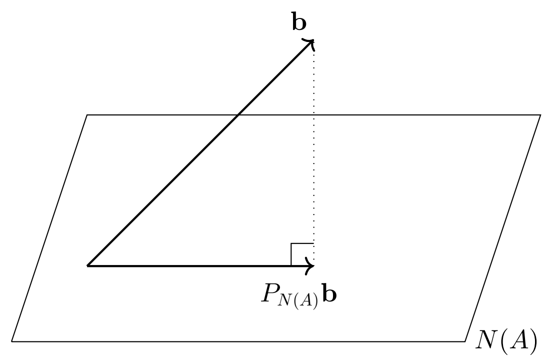
Putting it all together,

$$B(B^T B)^{-1}B^T = \frac{1}{3}\begin{bmatrix} -1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}\begin{bmatrix} -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Thus, the projection matrix onto $N(A)$ is given by

$$P_{N(A)} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}.$$

Next, we need to find the vector in $N(A)$ that is closest to the vector $\mathbf{b} = (2, 1, -1)$. This is just going to be the projection of $\mathbf{b}$ onto $N(A)$.



We can calculate the projection by multiplying $P_{N(A)}$ with $\mathbf{b}$:

$$P_{N(A)}\mathbf{b} = \frac{1}{3}\begin{bmatrix} 2 & -1 & 1 \\ -1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \\ -1 \end{bmatrix} = \frac{1}{3}\begin{bmatrix} 2 \\ -1 \\ 1 \end{bmatrix}.$$

Thus, the projection of $\mathbf{b}$ onto $N(A)$ is equal to $\frac{1}{3}(2, -1, 1)$. This is the vector in $N(A)$ that is closest to $\mathbf{b}$. $\diamond$

Example. Find the projection of $\mathbf{b} = (1,2,1,2)$ onto the space spanned by the vectors $\mathbf{a}_1 = (1,1,0,0)$, $\mathbf{a}_2 = (1,0,1,0)$, and $\mathbf{a}_3 = (1,0,0,1)$.

Solution. Let W denote the space spanned by $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$. If we take the vectors $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$, and put them into a matrix A as columns, then the projection of $\mathbf{b}$ onto W is given by

$$A(A^T A)^{-1} A^T \mathbf{b}.$$

This is very straightforward and not difficult. However, $A^T A$ will be a 3×3 matrix, so its inverse is slightly annoying to compute. The question doesn't ask us to find the projection matrix, so we don't really need to compute its inverse. Instead, we can compute the projection of $\mathbf{b}$ onto W by using the idea of least squares. Namely, let $\hat{\mathbf{x}}$ be the least squares solution to $A\mathbf{x} = \mathbf{b}$. Then

$$A\hat{\mathbf{x}} = P_W \mathbf{b}.$$

Remember that $\hat{\mathbf{x}}$ is also a solution to the normal equations:

$$A^T A\hat{\mathbf{x}} = A^T \mathbf{b}.$$

Thus, we could solve for $\hat{\mathbf{x}}$ by performing elimination on the augmented matrix

$$[A^T A | A^T \mathbf{b}]$$

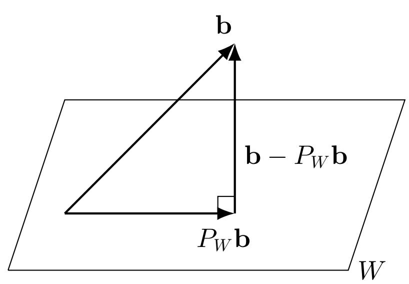
and then back-substitution. Once we find $\hat{\mathbf{x}}$, then we can compute $P_W \mathbf{b}$ as

$$P_W \mathbf{b} = A\hat{\mathbf{x}}.$$

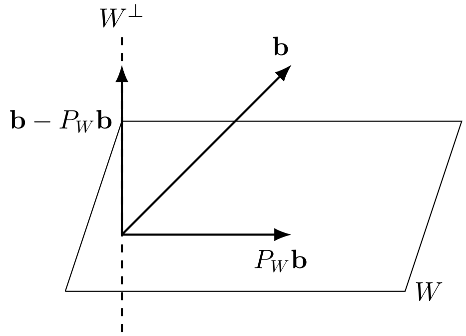
The details are left to the reader. The point of this example is that you don't need to compute the projection matrix if the question only asks for the projection of a vector. $\square$

6.4.1 Projection onto Orthogonal Complements

Let W be a subspace of $\mathbb{R}^n$, and let P_W be the projection matrix onto W. We recall that for any vector $\mathbf{b}$, $\mathbf{b} - P_W \mathbf{b}$ is orthogonal to W.



Thus, $\mathbf{b} - P_W \mathbf{b}$ is in $W^\perp$, the orthogonal complement of W. What is really happening here is that $\mathbf{b} - P_W \mathbf{b}$ is just the projection of $\mathbf{b}$ onto $W^\perp$.



In particular then, the projection matrix $P_{W^\perp}$ is given by

$$P_{W^\perp} = I - P_W,$$

since

$$(I - P_W)\mathbf{b} = \mathbf{b} - P_W\mathbf{b} = \begin{array}{c}\text{projection of } \mathbf{b}\\ \text{onto } W^\perp\end{array}.$$

We state this result formally below.

Proposition. For any subspace W of $\mathbb{R}^n$,

$$P_{W^\perp} = I - P_W.$$

Example. Let A be a matrix such that the vector $(1, 1, 0)$ forms a basis for the nullspace of A. Find the projection matrix onto $C(A^T)$, the row space of A.

Solution. Let

$$B = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

Then the projection matrix onto $N(A)$ is given by $P_{N(A)} = B(B^TB)^{-1}B^T$. Since $C(A^T)$ is the orthogonal complement of $N(A)$, it follows that the projection matrix onto $C(A^T)$ is given by

$$I - P_{N(A)}.$$

So we compute:

$$B^TB = \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = 2.$$

Then

$$(B^TB)^{-1} = \frac{1}{2}.$$

Putting it all together, we have

$$B(B^TB)^{-1}B^T = \frac{1}{2}\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus,

$$P_{N(A)} = \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Then the projection matrix onto $C(A^T)$ is given by $P_{C(A^T)} = I - P_{N(A)}$:

$$P_{C(A^T)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} - \frac{1}{2}\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}.$$

◇

6.5 Orthogonal Bases and Gram-Schmidt

6.5.1 Orthogonal Bases

Let W be a subspace of $\mathbb{R}^n$, and suppose $\mathbf{w}_1, \ldots, \mathbf{w}_k$ is a basis for W. Then any vector $\mathbf{b}$ in W can be expressed uniquely as a linear combination of the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$. Consider the problem of actually finding the scalars $c_1, \ldots, c_k$ such that

$$\mathbf{b} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \ldots + c_k\mathbf{w}_k.$$

In general, we would need to solve the equation $A\mathbf{x} = \mathbf{b}$, where $A = [\mathbf{a}_1|\ldots|\mathbf{a}_k]$ and $\mathbf{x} = (c_1, \ldots, c_k)$.

However, suppose the vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are orthogonal to each other. Then we can actually derive a simple formula for the coefficients $c_1, \ldots, c_k$. Starting with

$$\mathbf{b} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \ldots + c_k\mathbf{w}_k,$$

if we dot product both sides of the equation by the vector $\mathbf{w}_1$, we would get

$$\mathbf{b}\cdot\mathbf{w}_1 = (c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \ldots + c_k\mathbf{w}_k)\cdot\mathbf{w}_1.$$

If we distribute the dot product on the right hand side, we end up with

$$\mathbf{b}\cdot\mathbf{w}_1 = c_1(\mathbf{w}_1\cdot\mathbf{w}_1).$$

Do you see why every term after the first term vanished? The vectors $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$ are assumed to be orthogonal to each other, and so $\mathbf{w}_1\cdot\mathbf{w}_j = 0$ for $j \neq 1$. So we are left with

$$\mathbf{b}\cdot\mathbf{w}_1 = c_1(\mathbf{w}_1\cdot\mathbf{w}_1).$$

Divide both sides of the equation by $\mathbf{w}_1\cdot\mathbf{w}_1$ to find that

$$c_1 = \frac{\mathbf{b}\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1}.$$

If we take the dot product of both sides of the equation

$$\mathbf{b} = c_1\mathbf{w}_1 + c_2\mathbf{w}_2 + \ldots + c_k\mathbf{w}_k$$

with $\mathbf{w}_j$ instead of $\mathbf{w}_1$, we would end up with

$$c_j = \frac{\mathbf{b}\cdot\mathbf{w}_j}{\mathbf{w}_j\cdot\mathbf{w}_j}.$$

We summarize these results in the following

Proposition. Let $\mathbf{w}_1, \ldots, \mathbf{w}_k$ be a set of nonzero orthogonal vectors in $\mathbb{R}^n$. Suppose $\mathbf{b}$ is in the span of $\mathbf{w}_1, \ldots, \mathbf{w}_k$. Then

$$\mathbf{b} = \left(\frac{\mathbf{b}\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1}\right)\mathbf{w}_1 + \left(\frac{\mathbf{b}\cdot\mathbf{w}_2}{\mathbf{w}_2\cdot\mathbf{w}_2}\right)\mathbf{w}_2 + \ldots + \left(\frac{\mathbf{b}\cdot\mathbf{w}_k}{\mathbf{w}_k\cdot\mathbf{w}_k}\right)\mathbf{w}_k.$$

Example. Notice that the vectors

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \qquad \mathbf{w}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \qquad \mathbf{w}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$$

are orthogonal to each other; hence, they form an orthogonal basis for $\mathbb{R}^3$. Express the vector $\mathbf{b} = (2, 1, 3)$ as a linear combination of $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$.

Solution. From the above proposition,

$$\mathbf{b} = \left(\frac{\mathbf{b}\cdot\mathbf{w}_1}{\mathbf{w}_1\cdot\mathbf{w}_1}\right)\mathbf{w}_1 + \left(\frac{\mathbf{b}\cdot\mathbf{w}_2}{\mathbf{w}_2\cdot\mathbf{w}_2}\right)\mathbf{w}_2 + \left(\frac{\mathbf{b}\cdot\mathbf{w}_3}{\mathbf{w}_3\cdot\mathbf{w}_3}\right)\mathbf{w}_3.$$

We compute all of the dot products:

$$\begin{aligned} \mathbf{b}\cdot\mathbf{w}_1 &= 6 & \mathbf{b}\cdot\mathbf{w}_2 &= 1 & \mathbf{b}\cdot\mathbf{w}_3 &= 3 \\ \mathbf{w}_1\cdot\mathbf{w}_1 &= 3 & \mathbf{w}_2\cdot\mathbf{w}_2 &= 2 & \mathbf{w}_3\cdot\mathbf{w}_3 &= 6. \end{aligned}$$

Plugging everything in, we find that

$$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = \frac{6}{3}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \frac{3}{6}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix},$$

which simplifies as

$$\begin{bmatrix} 2 \\ 1 \\ 3 \end{bmatrix} = 2\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}.$$

You should check that the right hand side does add up to $(2, 1, 3)$. ◇

Recall that the formula for projecting $\mathbf{b}$ onto $\mathbf{w}$ is given by

$$\operatorname{proj}_{\mathbf{w}}(\mathbf{b}) = \left(\frac{\mathbf{b}\cdot\mathbf{w}}{\mathbf{w}\cdot\mathbf{w}}\right)\mathbf{w}.$$

So the above proposition actually just says that $\mathbf{b}$ is equal to the sum of its projections onto $\mathbf{w}_1, \mathbf{w}_2, \ldots, \mathbf{w}_k$.

A simplification to the formula occurs if $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are orthonormal. Since $\mathbf{w}\cdot\mathbf{w} = \|\mathbf{w}\|^2$, we can rewrite the formula in the above proposition as

$$\mathbf{b} = \left(\frac{\mathbf{b}\cdot\mathbf{w}_1}{\|\mathbf{w}_1\|^2}\right)\mathbf{w}_1 + \left(\frac{\mathbf{b}\cdot\mathbf{w}_2}{\|\mathbf{w}_2\|^2}\right)\mathbf{w}_2 + \ldots + \left(\frac{\mathbf{b}\cdot\mathbf{w}_k}{\|\mathbf{w}_k\|^2}\right)\mathbf{w}_k.$$

If $\mathbf{w}_1, \ldots, \mathbf{w}_k$ are orthonormal, then $\|\mathbf{w}_j\| = 1$ for every j, and this formula simplifies to

$$\mathbf{b} = (\mathbf{b}\cdot\mathbf{w}_1)\,\mathbf{w}_1 + (\mathbf{b}\cdot\mathbf{w}_2)\,\mathbf{w}_2 + \ldots + (\mathbf{b}\cdot\mathbf{w}_k)\,\mathbf{w}_k.$$

6.5.2 Gram-Schmidt

We have seen one advantage of working with orthogonal bases: namely, we have an explicit formula for expressing a vector as a linear combination of an orthogonal basis. There are other advantages as well; for instance, some numerical algorithms are more stable when using an orthogonal basis.

A natural question to ask now is: how can we find an orthogonal basis for a given space? The answer is provided by the Gram-Schmidt algorithm.

Given a set of vectors $\mathbf{w}_1, \ldots, \mathbf{w}_k$, the Gram-Schmidt algorithm produces a new set of vectors $\mathbf{u}_1, \ldots, \mathbf{u}_k$ satisfying the following properties:

1. $\mathbf{u}_1, \ldots, \mathbf{u}_k$ are orthogonal to each other.

2. For $j = 1, 2, \ldots, k$, the span of $\mathbf{w}_1, \ldots, \mathbf{w}_j$ and the span of $\mathbf{u}_1, \ldots, \mathbf{u}_j$ coincide. In other words, the Gram-Schmidt algorithm preserves the span of the original vectors at each step.

Here are the formulas:

$$\begin{aligned}
\mathbf{u}_1 &= \mathbf{w}_1 \\
\mathbf{u}_2 &= \mathbf{w}_2 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 \\
\mathbf{u}_3 &= \mathbf{w}_3 - \left(\frac{\mathbf{w}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 - \left(\frac{\mathbf{w}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\right)\mathbf{u}_2
\end{aligned}$$

Hopefully you won't have more than three vectors to deal with, but just in case, the general formula is

$$\mathbf{u}_j = \mathbf{w}_j - \left(\frac{\mathbf{w}_j \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 - \ldots - \left(\frac{\mathbf{w}_j \cdot \mathbf{u}_{j-1}}{\mathbf{u}_{j-1} \cdot \mathbf{u}_{j-1}}\right)\mathbf{u}_{j-1}.$$

Intuitively, the way Gram-Schmidt works is to take $\mathbf{w}_j$ and subtract its projections onto all of the previously constructed vectors to get a vector that is orthogonal to all of the previously constructed vectors.

Once we obtain an orthogonal basis via Gram-Schmidt, if we want an orthonormal basis, then all we need to do is normalize all of the vectors to make them orthonormal.

Example. Find an orthonormal basis for the span of the vectors

$$\mathbf{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{w}_2 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \qquad \mathbf{w}_3 = \begin{bmatrix} -1 \\ 2 \\ 3 \\ 2 \end{bmatrix}.$$

Solution. Let W denote the span of $\mathbf{w}_1, \mathbf{w}_2, \mathbf{w}_3$. We start by applying the Gram-Schmidt algorithm to find an orthogonal basis for W. The first step is $\mathbf{u}_1 = \mathbf{w}_1$:

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.$$

Next,

$$\begin{aligned}
\mathbf{u}_2 &= \mathbf{w}_2 - \left(\frac{\mathbf{w}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 \\
&= \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} - \frac{-2}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}.
\end{aligned}$$

Adding everything up, we find that

$$\mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.$$

You should check that $\mathbf{u}_2$ is orthogonal to $\mathbf{u}_1$. Finally,

$$\begin{aligned}
\mathbf{u}_3 &= \mathbf{w}_3 - \left(\frac{\mathbf{w}_3 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1}\right)\mathbf{u}_1 - \left(\frac{\mathbf{w}_3 \cdot \mathbf{u}_2}{\mathbf{u}_2 \cdot \mathbf{u}_2}\right)\mathbf{u}_2 \\
&= \begin{bmatrix} -1 \\ 2 \\ 3 \\ 2 \end{bmatrix} - \frac{2}{2}\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} - \frac{6}{3}\begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}.
\end{aligned}$$

Finishing the computations, we find that

$$\mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}.$$

Again, you should check that $\mathbf{u}_3$ is orthogonal to $\mathbf{u}_1$ and $\mathbf{u}_2$.

The vectors

$$\mathbf{u}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{u}_2 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix} \qquad \mathbf{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 2 \end{bmatrix}$$

form an orthogonal basis for W. To get an orthonormal basis for W, we just need to normalize each of these vectors. Thus,

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

is an orthonormal basis for W. $\diamond$

Chapter 7

Determinants

7.1 Generalizing Vectors to Higher Dimensions

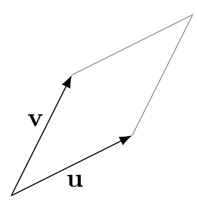
Recall way back in Chapter 1 how we talked about vectors as being arrows or directed line segments. From this geometric perspective, a vector is characterized by its length and its direction, in the sense that two vectors are considered equal if they have the same length and point in the same direction.

Vectors, as a geometric object, are inherently 1-dimensional, since they have length (let's ignore the zero vector throughout our discussion). Even when considering a vector $\mathbf{v}$ in $\mathbb{R}^n$, although $\mathbb{R}^n$ is an n-dimensional space, the vector $\mathbf{v}$ itself is still 1-dimensional; it would just be a 1-dimensional object sitting inside of an n-dimensional universe.

It seems like it would be pretty limiting if we stopped at one dimension. Why not consider higher dimensional "vectors" as well? Thinking of a vector as an oriented length, with orientation being the vector's direction, the next step up would be *oriented areas.*

7.1.1 Oriented Areas & 2-vectors

Suppose we have two linearly independent vectors $\mathbf{u}, \mathbf{v}$, and consider the parallelogram they determine:



We can assign a notion of orientation to this parallelogram as follows. Imagine yourself positioned at the point where $\mathbf{u}, \mathbf{v}$ are connected. You can move along the edges of the parallelogram in two ways: either you start moving along $\mathbf{u}$ or you start moving along $\mathbf{v}$. If you starting moving along $\mathbf{u}$, you would be moving counterclockwise. On the other hand, if you start moving along $\mathbf{v}$, you would be moving clockwise instead. Thus, we can kind of think of orientation of areas in terms of being either counterclockwise or clockwise.

To distinguish between the two parallelograms above, we will use the notation $\mathbf{u} \wedge \mathbf{v}$ for the parallelogram on the left, whose counterclockwise orientation comes from moving around the parallelogram by starting along $\mathbf{u}$. Likewise, we can use the notation $\mathbf{v} \wedge \mathbf{u}$ for the parallelogram on the right, whose clockwise orientation comes from moving around the parallelogram by starting along $\mathbf{v}$. The $\wedge$ symbol is read as "wedge", so $\mathbf{u}\wedge\mathbf{v}$ would be read as "u wedge v". We will refer to objects like $\mathbf{u}\wedge\mathbf{v}$ as ***2-vectors***, since they are like 2-dimensional analogs of the traditional 1-dimensional vector.

To represent the fact that $\mathbf{u}\wedge\mathbf{v}$ and $\mathbf{v}\wedge\mathbf{u}$ are the same, except with opposite orientation, we'd want

$$\mathbf{u}\wedge\mathbf{v} = -(\mathbf{v}\wedge\mathbf{u}),$$

analogous to how two vectors which are the same except they point in opposite directions are negatives of each other. If we take this property to hold for any two vectors, then we would also conclude that

$$\mathbf{v}\wedge\mathbf{v} = -(\mathbf{v}\wedge\mathbf{v})$$

for any vector $\mathbf{v}$. This seems to suggest that

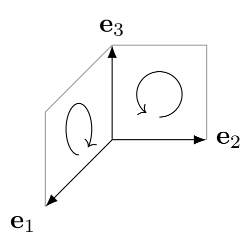
$$\mathbf{v}\wedge\mathbf{v} = 0,$$

since the primary thing that equals the negative of itself is zero. Since the "parallelogram" determined by $\mathbf{v}$ and itself would have zero area, this conforms nicely with the geometry, so let's take it to be true.

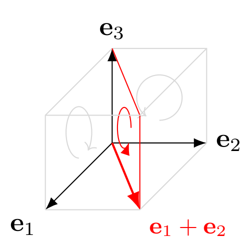
Similar to how two vectors are equal if they have the same length and point in the same direction, we will consider two 2-vectors to be equal if *they lie in the same plane* and have the same area and the same orientation. The reason we want them to lie in the same plane is so we can compare their orientations. For instance, the two parallelograms below lie in the same plane and have the same area. They both have a counterclockwise orientation when viewed from the "front", but if we could look at them from "behind", they would both be clockwise instead. Regardless of how one looks at them, we can say they have the orientation. Thus, we consider them equal.

$$\mathbf{u}_1\wedge\mathbf{v}_1 \;=\; \mathbf{u}_2\wedge\mathbf{v}_2$$

However, in the picture below, the two squares have the same area, but we can't really compare their orientations because they lie in different planes. For instance $\mathbf{e}_2\wedge\mathbf{e}_3$ has a counterclockwise orientation when viewed from the "front", and $\mathbf{e}_1\wedge\mathbf{e}_3$ has a counterclockwise orientation when viewed from the "left". So it seems like they are both counterclockwise. However, if we view $\mathbf{e}_1\wedge\mathbf{e}_3$ from the "right", it would be clockwise. None of these different perspectives is more special than the other, so it would be wrong to pick one in particular. Therefore, we say that they simply can't be compared, and therefore $\mathbf{e}_1\wedge\mathbf{e}_3 \neq \mathbf{e}_2\wedge\mathbf{e}_3$.



Since I have already created the above picture, it takes little work for me to add the vector $\mathbf{e}_1+\mathbf{e}_2$ in the picture, and also the oriented parallelogram $(\mathbf{e}_1+\mathbf{e}_2)\wedge\mathbf{e}_3$:



Notice how $(\mathbf{e}_1 + \mathbf{e}_2) \wedge \mathbf{e}_3$ is like the diagonal of the parallelepiped formed by $\mathbf{e}_1 \wedge \mathbf{e}_3$ and $\mathbf{e}_2 \wedge \mathbf{e}_3$. Recalling how the sum of two vectors is the diagonal of the parallelogram that they form, this suggests that the sum of the two 2-vectors $\mathbf{e}_1 \wedge \mathbf{e}_3$ and $\mathbf{e}_2 \wedge \mathbf{e}_3$ should be $(\mathbf{e}_1 + \mathbf{e}_2) \wedge \mathbf{e}_3$, i.e.

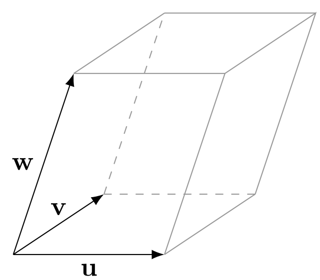
$$(\mathbf{e}_1 + \mathbf{e}_2) \wedge \mathbf{e}_3 = \mathbf{e}_1 \wedge \mathbf{e}_3 + \mathbf{e}_2 \wedge \mathbf{e}_3.$$

Similar pictures would suggest that for any three linearly independent vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$,

$$(\mathbf{u} + \mathbf{v}) \wedge \mathbf{w} = \mathbf{u} \wedge \mathbf{w} + \mathbf{v} \wedge \mathbf{w}$$
$$\mathbf{u} \wedge (\mathbf{v} + \mathbf{w}) = \mathbf{u} \wedge \mathbf{v} + \mathbf{u} \wedge \mathbf{w}.$$

7.1.2 Oriented Volumes & 3-vectors

Moving up another dimension, suppose we have three linearly vectors $\mathbf{u}, \mathbf{v}, \mathbf{w}$. These three vectors would form a parallelepiped:



We can assign to this parallelepiped either a "left-handed" or "right-handed" orientation. Below we list a few different ways of assigning to it one of these orientations.

- Let's start with the oriented area $\mathbf{u} \wedge \mathbf{v}$, which would have a counterclockwise orientation when viewed from top-down. Take both hands and curl your fingers in what would be a counterclockwise direction when viewed from top-down. Your right thumb points upwards, towards the direction of $\mathbf{w}$. Thus, the parallelepiped would be right-handed. We could represent this oriented volume as $(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w}$, since we started with $\mathbf{u} \wedge \mathbf{v}$.

- Alternatively, we could start with the oriented area $\mathbf{v} \wedge \mathbf{u}$, which would have a clockwise orientation when viewed from top-down. If you take both hands and curl your fingers in what would be a clockwise direction when viewed from top-down, this time your left thumb points upwards, towards the direction of $\mathbf{w}$. Hence, this oriented volume, which we could represent as $(\mathbf{v} \wedge \mathbf{u}) \wedge \mathbf{w}$, would be left-handed.

- Yet another way of looking at things is to start by taking your right thumb and pointing it in the direction of $\mathbf{u}$. Your left fingers should curl as if they were going around the parallelogram determined by $\mathbf{v}, \mathbf{w}$, starting along the direction of $\mathbf{v}$. Hence, our right hand gives this parallelogram the orientation of $\mathbf{v} \wedge \mathbf{w}$. With this perspective, we could represent the oriented parallelepiped as $\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w})$, since we started with $\mathbf{u}$, and it would be right-handed since we used our right hand.

Notice that $(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w}$ and $\mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w})$ represent the same parallelepiped with the same (right-handed) orientation. Thus,

$$(\mathbf{u} \wedge \mathbf{v}) \wedge \mathbf{w} = \mathbf{u} \wedge (\mathbf{v} \wedge \mathbf{w}).$$

Hence, this $\wedge$ operation appears to be associative, meaning we can drop the parentheses and just right $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$. We will refer to objects such as $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ as ***3-vectors***, since they are like 3-dimensional analogs of vectors.

If you want to practice a bit, you can verify on your own that

$$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} = \mathbf{w} \wedge \mathbf{u} \wedge \mathbf{v} = \mathbf{v} \wedge \mathbf{w} \wedge \mathbf{u},$$

since they represent the same right-handed parallelepiped, while

$$\mathbf{v} \wedge \mathbf{u} \wedge \mathbf{w} = \mathbf{w} \wedge \mathbf{v} \wedge \mathbf{u} = \mathbf{u} \wedge \mathbf{w} \wedge \mathbf{v},$$

since they represent the same left-handed parallelepiped. You can also verify things like $(\mathbf{w}\wedge\mathbf{v})\wedge\mathbf{u} = \mathbf{w} \wedge (\mathbf{v} \wedge \mathbf{u})$, etc.

Oriented n-Volumes & n-Vectors

We can generalize all of this to higher dimensions. For instance, let's keep $\mathbf{u}, \mathbf{v}, \mathbf{w}$ as before, and suppose we add a fourth vector $\mathbf{x}$ to the mix, which is independent of the first three. These four vectors would form the 4-dimensional analog of parallelograms and parallelepipeds, which we will call a 4-parallelepiped. We can view this 4-parallelepiped as having two possible orientations: one orientation coming from

$$\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w} \wedge \mathbf{x} = (\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}) \wedge \mathbf{x},$$

and an opposite orientation coming from

$$\mathbf{v} \wedge \mathbf{u} \wedge \mathbf{w} \wedge \mathbf{x} = (\mathbf{v} \wedge \mathbf{u} \wedge \mathbf{w}) \wedge \mathbf{x}.$$

These two would give the 4-parallelepiped opposite orientations because $\mathbf{u} \wedge \mathbf{v} \wedge \mathbf{w}$ and $\mathbf{v} \wedge \mathbf{u} \wedge \mathbf{w}$ have opposite orientations, as we noted earlier. Any other ordering of the four vectors $\wedge$'d together would be equivalent to one of these two.

In general, n vectors $\mathbf{v}_1, \ldots, \mathbf{v}_n$ would form an n-parallelepiped (with n-volume equal to zero if they are linearly dependent) which could have two possible orientations. Objects of the form $\mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_n$ are known as ***n-vectors***, as well as ***n-blades***, and are used as algebraic representations of oriented n-dimensional volumes, or n-volumes. If the vectors involved are linearly independent, then two n-vectors

$$\mathbf{u}_1 \wedge \ldots \wedge \mathbf{u}_n \qquad \text{and} \qquad \mathbf{v}_1 \wedge \ldots \wedge \mathbf{v}_n$$

are considered equal if

- $\operatorname{span}(\mathbf{u}_1, \ldots, \mathbf{u}_n) = \operatorname{span}(\mathbf{v}_1, \ldots, \mathbf{v}_n)$, and
- the two n-parallelepipeds that these two n-vectors represent have the same n-volume and the same orientation.

Algebraic operations on n-vectors then correspond to geometric operations on the corresponding n-volumes or subspaces.

If you want to learn more about these ideas, the $\wedge$ symbol denotes something commonly called the ***exterior product***, and there is an area of mathematics/physics often referred to as ***Geometric Algebra*** that delves more into these ideas. You can look them up for more info, but be warned that a rigorous treatment of these topics will likely require more "mathematical maturity" than you currently have. We will not be discussing this stuff much further, as we don't need it for this course. What you should take away from this section is that n linearly independent vectors will form an n-parallelepiped with two possible orientations which we view as opposites of each other (and if they are linearly dependent, then the "n-parallelepiped" that they form would have n-volume equal to 0).

7.2 The "Big Formula" For Determinants

7.2.1 The Determinant in 2 Dimensions

Let $A = [\mathbf{u}\ \mathbf{v}]$ be a 2×2 matrix with columns $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^2$. These vectors will form a parallelogram in $\mathbb{R}^2$, which can then be oriented counterclockwise or clockwise. In the last section, we learned to represent the two oriented parallelograms as the 2-vectors $\mathbf{u} \wedge \mathbf{v}$ and $\mathbf{v} \wedge \mathbf{u}$. We define the determinant of A, denoted $\det(A)$, to be the signed area of $\mathbf{u} \wedge \mathbf{v}$, with the sign being positive if $\mathbf{u} \wedge \mathbf{v}$ has a counterclockwise orientation, and negative otherwise.

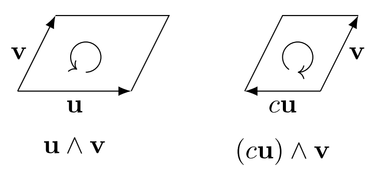
We would like to come up with a formula for $\det(A)$. To do this, first we will explore some of the properties that the determinant should satisfy, based off its definition as signed area. Most of these properties are going to reference the columns of A, so we will use the notation $\det(\mathbf{u}, \mathbf{v})$ interchangeably with $\det(A)$, as convenient.

Properties of Determinant

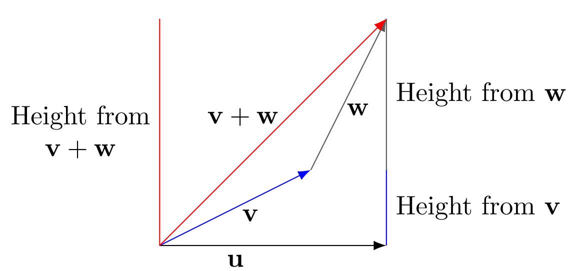
1. For any $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^2$,
$$\det(\mathbf{u}, \mathbf{v}) = -\det(\mathbf{v}, \mathbf{u}).$$
This should be true because $\det(\mathbf{u}, \mathbf{v})$ is the signed area of $\mathbf{u} \wedge \mathbf{v}$, while $\det(\mathbf{v}, \mathbf{u})$ is the signed area of $\mathbf{v} \wedge \mathbf{u}$. Since $\mathbf{u} \wedge \mathbf{v}$ and $\mathbf{v} \wedge \mathbf{u}$ have opposite orientations, their signed areas are opposite in sign.

2. For any $\mathbf{v}$ in $\mathbb{R}^2$,
$$\det(\mathbf{v}, \mathbf{v}) = 0.$$
This is because the area of the "parallelogram" formed by $\mathbf{v}$ and itself is zero. Alternatively, this follows from the first property, since
$$\det(\mathbf{v}, \mathbf{v}) = -\det(\mathbf{v}, \mathbf{v}) \quad \Longrightarrow \quad \det(\mathbf{v}, \mathbf{v}) = 0.$$

3. For any scalar c,
$$\det(c\mathbf{u}, \mathbf{v}) = c\det(\mathbf{u}, \mathbf{v}) \qquad \text{and} \qquad \det(\mathbf{u}, c\mathbf{v}) = c\det(\mathbf{u}, \mathbf{v}).$$
Essentially, the area of a parallelogram can be computed as base$\times$height. When $c > 0$, scaling $\mathbf{u}$ or $\mathbf{v}$ by c will scale either the base or height by c, thus scaling the area by c. When c is negative, it reverses the orientation as well:



4. For any $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^2$,
$$\det(\mathbf{u}, \mathbf{v} + \mathbf{w}) = \det(\mathbf{u}, \mathbf{v}) + \det(\mathbf{u}, \mathbf{w})$$
$$\det(\mathbf{u} + \mathbf{v}, \mathbf{w}) = \det(\mathbf{u}, \mathbf{w}) + \det(\mathbf{v}, \mathbf{w}).$$
Focusing on the first of these equations, here is the idea: if we think of $\mathbf{u}$ as representing the base of the three parallelograms, and if h_1 and h_2 are the heights determined by $\mathbf{v}$ and $\mathbf{w}$, then the height determined by $\mathbf{v} + \mathbf{w}$ will be $h_1 + h_2$:

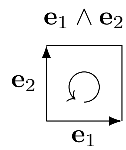


Computing the areas as base$\times$height gives us
$$\det(\mathbf{u}, \mathbf{v} + \mathbf{w}) = b(h_1 + h_2) = bh_1 + bh_2 = \det(\mathbf{u}, \mathbf{v}) + \det(\mathbf{u}, \mathbf{w}).$$

5. Let $\mathbf{e}_1 = (1, 0)$, $\mathbf{e}_2 = (0, 1)$ be the standard basis vectors of $\mathbb{R}^2$. Then

$$\det(\mathbf{e}_1, \mathbf{e}_2) = \det \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = 1.$$

The oriented parallelogram $\mathbf{e}_1 \wedge \mathbf{e}_2$ has area equal to 1 and a counterclockwise orientation:



These five properties allow us to derive a formula for the determinant of a 2×2, as follows:

$$\begin{aligned}
\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} &= \det\left(\begin{bmatrix} a \\ c \end{bmatrix}, \begin{bmatrix} b \\ d \end{bmatrix} \right) \\
&= \det(a\mathbf{e}_1 + c\mathbf{e}_2\ ,\ b\mathbf{e}_1 + d\mathbf{e}_2) \\
&= ab\det(\mathbf{e}_1, \mathbf{e}_1) + ad\det(\mathbf{e}_1, \mathbf{e}_2) + bc\det(\mathbf{e}_2, \mathbf{e}_1) + cd\det(\mathbf{e}_2, \mathbf{e}_2) \qquad (*) \\
&= \quad 0 \quad + \quad ad \cdot 1 \quad + \quad bc \cdot (-1) \quad + \quad 0 \\
&= ad - bc.
\end{aligned}$$

In $(*)$, we "foiled" everything out using Property 4 and pulled out all the constants using Property 3. We used Properties 1, 2, and 5 in the next line to evaluate the determinants.

In summary, we have the following formula for the determinant of a 2×2 matrix:

$$\det \begin{bmatrix} a & b \\ c & d \end{bmatrix} = ad - bc.$$

The determinant is equal to the signed area of the oriented parallelogram formed by the columns of the matrix.

7.2.2 The Determinant in 3 Dimensions

Let $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]$ be a 3×3 matrix with columns $\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3$ in $\mathbb{R}^3$. The determinant of A, which we will denote interchangeably as $\det(A)$ and $\det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3)$, is defined to be the signed volume of the oriented parallelepiped $\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3$, with the sign being positive if $\mathbf{a}_1 \wedge \mathbf{a}_2 \wedge \mathbf{a}_3$ has a right-handed orientation, and negative otherwise.

The determinant for 3×3 matrices satisfies similar properties as in the 2×2 case:

1. Interchanging any two columns changes the sign of the determinant, e.g.

$$\det(\mathbf{u}, \mathbf{v}, \mathbf{w}) = -\det(\mathbf{w}, \mathbf{v}, \mathbf{u}) \qquad \text{etc.}$$

This is because swapping two columns changes the orientation of the parallelepiped.

2. The determinant is zero whenever two columns coincide, e.g.

$$\det(\mathbf{u}, \mathbf{v}, \mathbf{v}) = 0 \qquad \text{etc.}$$

This is because the "parallelepiped" determined by two vectors has zero volume, and it also follows as a consequence of the first property.

3. For any $\mathbf{u}, \mathbf{v}, \mathbf{w}$ in $\mathbb{R}^3$ and for any scalar c,

$$\det(c\mathbf{u}, \mathbf{v}, \mathbf{w}) = \det(\mathbf{u}, c\mathbf{v}, \mathbf{w}) = \det(\mathbf{u}, \mathbf{v}, c\mathbf{w}) = c\det(\mathbf{u}, \mathbf{v}, \mathbf{w}).$$

Essentially, volume is length$\times$width$\times$height. Scaling either of these by c scales the signed volume by c.

4. For any $\mathbf{u}, \mathbf{v}, \mathbf{w}, \mathbf{x}$ in $\mathbb{R}^3$,

$$\det(\mathbf{u}+\mathbf{x}, \mathbf{v}, \mathbf{w}) = \det(\mathbf{u}, \mathbf{v}, \mathbf{w}) + \det(\mathbf{x}, \mathbf{v}, \mathbf{w})$$
$$\det(\mathbf{u}, \mathbf{v}+\mathbf{x}, \mathbf{w}) = \det(\mathbf{u}, \mathbf{v}, \mathbf{w}) + \det(\mathbf{u}, \mathbf{x}, \mathbf{w})$$
$$\det(\mathbf{u}, \mathbf{v}, \mathbf{w}+\mathbf{x}) = \det(\mathbf{u}, \mathbf{v}, \mathbf{w}) + \det(\mathbf{u}, \mathbf{v}, \mathbf{x})$$

   Focusing on the first one, the idea is if l_1 and l_2 denote the lengths determined by $\mathbf{u}$ and $\mathbf{x}$, then $l_1 + l_2$ is the length determined by $\mathbf{u}+\mathbf{x}$, so

$$\det(\mathbf{u}+\mathbf{x}, \mathbf{v}, \mathbf{w}) = (l_1 + l_2) \cdot wh = l_1 wh + l_2 wh = \det(\mathbf{u}, \mathbf{v}, \mathbf{w}) + \det(\mathbf{x}, \mathbf{v}, \mathbf{w}).$$

5. If $\mathbf{e}_1 = (1, 0, 0)$, $\mathbf{e}_2 = (0, 1, 0)$, $\mathbf{e}_3 = (0, 0, 1)$ is the standard basis of $\mathbb{R}^3$, then

$$\det \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \det(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) = 1.$$

   The oriented parallelepiped $\mathbf{e}_1 \wedge \mathbf{e}_2 \wedge \mathbf{e}_3$ has volume equal to 1 and is right-handed, so the signed volume should be positive.

Just as in the 2×2 case, we can use these five properties to derive a formula for 3×3 determinants:

$$\begin{aligned}
\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} &= \det \left(\begin{bmatrix} a_{11} \\ a_{21} \\ a_{31} \end{bmatrix}, \begin{bmatrix} a_{12} \\ a_{22} \\ a_{32} \end{bmatrix}, \begin{bmatrix} a_{13} \\ a_{23} \\ a_{33} \end{bmatrix} \right) \\
&= \det(a_{11}\mathbf{e}_1 + a_{21}\mathbf{e}_2 + a_{31}\mathbf{e}_3\, ,\, a_{12}\mathbf{e}_1 + a_{22}\mathbf{e}_2 + a_{32}\mathbf{e}_3\, , \\
&\qquad a_{13}\mathbf{e}_1 + a_{23}\mathbf{e}_2 + a_{33}\mathbf{e}_3) \\
&= a_{11}a_{22}a_{33} \det(\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3) + a_{11}a_{32}a_{23} \det(\mathbf{e}_1, \mathbf{e}_3, \mathbf{e}_2) \\
&\quad + a_{21}a_{12}a_{33} \det(\mathbf{e}_2, \mathbf{e}_1, \mathbf{e}_3) + a_{21}a_{32}a_{13} \det(\mathbf{e}_2, \mathbf{e}_3, \mathbf{e}_1) \\
&\quad + a_{31}a_{12}a_{23} \det(\mathbf{e}_3, \mathbf{e}_1, \mathbf{e}_2) + a_{31}a_{22}a_{13} \det(\mathbf{e}_3, \mathbf{e}_2, \mathbf{e}_1) \\
&= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} \\
&\quad + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}.
\end{aligned}$$

The expression after the third "equals" sign comes from using Property 4 to distribute out everything in the expression before it; we pulled out the scalars using Property 3, and all the determinants that had a repeated argument zeroed out by Property 2, so we did not write them. In the last expression, we evaluated the determinants using Properties 1 and 5.

In summary, we have the following formula for a 3×3 determinant:

$$\begin{aligned}
\det \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} &= a_{11}a_{22}a_{33} - a_{11}a_{32}a_{23} - a_{21}a_{12}a_{33} \\
&\quad + a_{21}a_{32}a_{13} + a_{31}a_{12}a_{23} - a_{31}a_{22}a_{13}.
\end{aligned}$$

The determinant is equal to the signed volume of the oriented parallelepiped formed by the columns of the matrix.

As you can see, the formula for a 3×3 determinant is quite a bit more complicated than the formula for a 2×2. We would like to write out a general formula for the determinant of an $n \times n$ matrix. To do that, let's examine more closely the formula for a 3×3.

Notice that each of the six terms that are being added or substracted have the form

$$a_{i1}a_{j2}a_{k3},$$

where (i, j, k) is a permutation, or reordering, of $(1, 2, 3)$. There are six permutations of $(1, 2, 3)$ overall:

$$(1, 2, 3),\ (1, 3, 2),\ (2, 1, 3),\ (2, 3, 1),\ (3, 1, 2),\ (3, 2, 1),$$

and each term in the formula for a 3×3 determinant corresponds to one of these permutations. As for which terms are being added and which are being subtracted, looking back at our derivation of the formula, each term $a_{i1}a_{j2}a_{k3}$ in the determinant formula came from the expression

$$a_{i1}a_{j2}a_{k3} \det(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)$$

in the line before. Then by Properties 1 and 5, $\det(\mathbf{e}_i, \mathbf{e}_j, \mathbf{e}_k)$ is then equal to 1 or -1, depending on the number of column swaps needed to reorder (i, j, k) back to $(1, 2, 3)$. Specifically, the terms being subtracted come from the permutations for which an odd number of swaps are needed to bring them back to $(1, 2, 3)$, and the terms being added come from the permutations for which an even number of swaps are needed to bring them back to $(1, 2, 3)$.

7.2.3 Permutations of $(1, \ldots, n)$

Since the 3×3 determinant formula involves permutations of $(1, 2, 3)$, it seems like the general $n \times n$ determinant formula will involve permutations of $(1, \ldots, n)$. It will be useful to talk a bit about permutations then, and to establish some notation.

Focusing on when $n = 3$, we can represent each permutation of $(1, 2, 3)$ as a function in the following manner: for example, the permutation $(2, 3, 1)$ is represented by the function σ defined as

$$\sigma(1) = 2 \qquad \sigma(2) = 3 \qquad \sigma(3) = 1.$$

(The σ symbol is a lowercase "sigma"). As another example, the function

$$\sigma(1) = 3 \qquad \sigma(2) = 2 \qquad \sigma(3) = 1$$

represents the permutation $(3, 2, 1)$. With this idea, each term in the 3×3 determinant formula is of the form

$$\pm a_{\sigma(1)1}a_{\sigma(2)2}a_{\sigma(3)3}$$

for some permutation σ of $(1, 2, 3)$. Regarding the $\pm$ sign, we define the sign of a permutation, denoted $\operatorname{sgn}(\sigma)$, to be 1 if an even number of swaps are needed to order it back to $(1, 2, 3)$, and -1 if an odd number of swaps are needed. If we define $\operatorname{Perm}(3)$ to be the set of all permutations of $(1, 2, 3)$, then we can rewrite the 3×3 determinant formula as

$$\det(A) = \sum_{\substack{\sigma \text{ in} \\ \operatorname{Perm}(3)}} \operatorname{sgn}(\sigma)\, a_{\sigma(1)1}a_{\sigma(2)2}a_{\sigma(3)3}$$

In general, we define $\operatorname{Perm}(n)$ to be the set of all permutations of $(1, \ldots, n)$, and for each permutation σ in $\operatorname{Perm}(n)$, we define its sign, $\operatorname{sgn}(\sigma)$, to be 1 if an even number of swaps are needed to reorder $(\sigma(1), \ldots, \sigma(n))$ back to $(1, \ldots, n)$, and -1 if an odd number of swaps are needed.

7.2.4 The Big Formula for Determinants

We are now going to derive a general formula for $n \times n$ determinants. I think Strang calls this the big formula, and since we are using his textbook, we have to adopt his terminology. To be fair, the formula is indeed big.

Mimicking our treatment of the 2×2 and 3×3 cases, let $A = [\mathbf{a}_1 \ \ldots \ \mathbf{a}_n]$ be an $n \times n$ matrix with columns $\mathbf{a}_1, \ldots, \mathbf{a}_n$ in $\mathbb{R}^n$. These vectors form an oriented n-parallelepiped, which we represent algebraically as the n-vector $\mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_n$. The ***determinant*** of A is defined to be the signed n-volume of $\mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_n$, with the sign being positive if $\mathbf{a}_1 \wedge \ldots \wedge \mathbf{a}_n$ has the same orientation as $\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n$. Note that by definition, we are choosing $\mathbf{e}_1 \wedge \ldots, \wedge \mathbf{e}_n$ to have a positive orientation.

We'll denote the determinant of A as $\det(A)$ and $\det(\mathbf{a}_1, \ldots, \mathbf{a}_n)$ interchangeably. We have the following properties that the determinant must satisfy, as a function which measure signed n-volumes:

1. For any permutation σ in $\mathrm{Perm}(n)$, and for any $\mathbf{a}_1, \ldots, \mathbf{a}_n$ in $\mathbb{R}^n$,

$$\det(\mathbf{a}_{\sigma(1)}, \ldots, \mathbf{a}_{\sigma(n)}) = \mathrm{sgn}(\sigma) \det(\mathbf{a}_1, \ldots, \mathbf{a}_n).$$

   This is a consequence of the fact that swapping any two columns changes the sign of the determinant.

2. If any two columns are the same, then the determinant is equal to zero. This is because the n-volume of such an n-parallelepiped would be zero, and this is also a consequence of Property 1.

3. Scaling a column by a scalar c scales the determinant by c as well. More explicitly, for any scalar c, and for any j satisfying $1 \leq j \leq n$, and for any $\mathbf{a}_1, \ldots, \mathbf{a}_n$ in $\mathbb{R}^n$,

$$\det(\mathbf{a}_1, \ldots, c\mathbf{a}_j, \ldots, \mathbf{a}_n) = c \det(\mathbf{a}_1, \ldots, \mathbf{a}_j, \ldots, \mathbf{a}_n).$$

4. The determinant is additive/distributive along each column. More explicitly, for any j satisfying $1 \leq j \leq n$, and for any vectors $\mathbf{a}_1, \ldots, \mathbf{a}_n$ and $\mathbf{b}_j$ in $\mathbb{R}^n$,

$$\det(\mathbf{a}_1, \ldots, \mathbf{a}_j + \mathbf{b}_j, \ldots, \mathbf{a}_n) = \det(\mathbf{a}_1, \ldots, \mathbf{a}_j, \ldots, \mathbf{a}_n) + \det(\mathbf{a}_1, \ldots, \mathbf{b}_j, \ldots, \mathbf{a}_n).$$

   Properties 3 and 4 can be summarized by saying that the determinant is a ***multilinear*** function.

5. The determinant of the identity matrix equals 1.

$$\det(I) = \det(\mathbf{e}_1, \ldots, \mathbf{e}_n) = 1.$$

   The n-volume of $\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n$ is 1, and by our definition of determinant, $\mathbf{e}_1 \wedge \ldots \wedge \mathbf{e}_n$ has a positive orientation, so its signed n-volume is positive.

Using these five properties, let's derive the general formula for an $n \times n$ determinant. To compute $\det(\mathbf{a}_1, \ldots, \mathbf{a}_n)$, we first rewrite $\mathbf{a}_1, \ldots, \mathbf{a}_n$ in terms of the standard basis vectors:

$$\mathbf{a}_1 = \sum_{i_1=1}^{n} a_{i_1 1} \mathbf{e}_{i_1} \qquad \cdots \qquad \mathbf{a}_n = \sum_{i_n=1}^{n} a_{i_n n} \mathbf{e}_{i_n}.$$

Then, using multilinearity of the determinant (Properties 3 and 4), we have

$$\det(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \sum_{1 \leq i_1, \ldots, i_n \leq n} a_{i_1 1} \ldots a_{i_n n} \cdot \det(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_n}).$$

Right now, the sum runs over all possible combinations of $i_1, \ldots, i_n$. However, by Property 2, $\det(\mathbf{e}_{i_1}, \ldots, \mathbf{e}_{i_n}) = 0$ whenever any two indices coincide. Ignoring these cases then, the sum runs over all combinations where $i_1, \ldots, i_n$ are distinct. Equivalently, the sum runs over all permutations of $1, 2, \ldots, n$, i.e.

$$\det(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \sum_{\substack{\sigma \text{ in} \\ \mathrm{Perm}(n)}} a_{\sigma(1)1} \ldots a_{\sigma(n)n} \cdot \det(\mathbf{e}_{\sigma(1)}, \ldots, \mathbf{e}_{\sigma(n)}).$$

By Property 1, this becomes

$$\det(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \sum_{\substack{\sigma \text{ in} \\ \mathrm{Perm}(n)}} \mathrm{sgn}(\sigma)\, a_{\sigma(1)1} \ldots a_{\sigma(n)n} \cdot \det(\mathbf{e}_1, \ldots, \mathbf{e}_n).$$

By Property 5, $\det(\mathbf{e}_1, \ldots, \mathbf{e}_n) = 1$, so

$$\det(\mathbf{a}_1, \ldots, \mathbf{a}_n) = \sum_{\substack{\sigma \text{ in} \\ \mathrm{Perm}(n)}} \mathrm{sgn}(\sigma)\, a_{\sigma(1)1} \ldots a_{\sigma(n)n}.$$

This is the big formula for determinants!

In summary, for an $n \times n$ matrix A, its determinant can be computed with the formula

$$\det(A) = \sum_{\substack{\sigma \text{ in} \\ \mathrm{Perm}(n)}} \mathrm{sgn}(\sigma)\, a_{\sigma(1)1} \dots a_{\sigma(n)n}.$$

The determinant of A is equal to the signed n-volume of the oriented n-parallelepiped formed by its columns.

The reason why this formula is "big" is because the number of permutations of $(1, \dots, n)$ is $n!$. When $n = 5$, $n! = 120$, so there would already be 120 terms in the summation to compute if you wanted to find the determinant of a 5×5 matrix! In the next section, we will learn some simpler ways of computing determinants, so that we won't need to use this big formula much.

Example. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 2 & 3 & 0 \\ 1 & 4 & 0 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}.$$

Let's compute $\det(A)$ using the big formula. Since $4! = 24$, the big formula for 4×4 matrices has 24 terms in it. However, since A has a lot of zeros in it, this means that a lot of the terms in the summation will be zero, so we need only focus on the nonzero terms.

Let's start by figuring out for which permutations is

$$a_{\sigma(1)1} a_{\sigma(2)2} a_{\sigma(3)3} a_{\sigma(4)4}$$

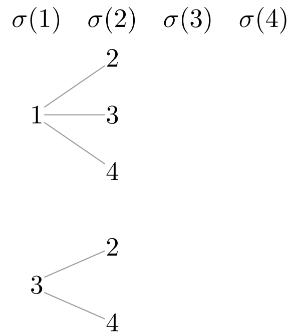
nonzero. Looking at $a_{\sigma(1)1}$ first, this is referencing the entry in the $\sigma(1)^{\text{th}}$ row and the first column of A. Since the only nonzero entries in the first column are in the first and third row, this means $a_{\sigma(1)1}$ is nonzero only when $\sigma(1)$ equals 1 or 3. To help organize our analysis, below I have started a table with columns for the relevant values of $\sigma(1)$, $\sigma(2)$, etc. I have entered the numbers 1 and 3 already into the first column for $\sigma(1)$.

$\sigma(1)$	$\sigma(2)$	$\sigma(3)$	$\sigma(4)$
1			
3			

Let's consider the case where $\sigma(1) = 1$ first. Then $\sigma(2)$ could only be 2, 3, or 4 now. Let's look at the second column of A to see if any of these values would make $a_{\sigma(2)2}$ equal 0. Unfortunately, it seems that the second, third, and fourth entries are all nonzero. Therefore, we need to consider all three possibilities for $\sigma(2)$.

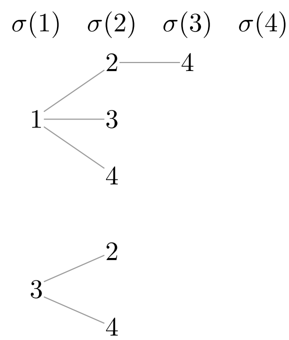
$$\begin{tikzcd} \sigma(1) & \sigma(2) & \sigma(3) & \sigma(4) \\ & 2 & & \\ 1 \arrow[ur, no head] \arrow[r, no head] \arrow[dr, no head] & 3 & & \\ & 4 & & \\ 3 & & & \end{tikzcd}$$

On the other hand, if $\sigma(1) = 3$, then $\sigma(2)$ could only be 1, 2, or 4 now. But the first entry in the second column of A is 0, so $\mathbf{a}_{\sigma(2)2}$ would equal zero if $\sigma(2)$ equals 1. Hence, we need only consider if $\sigma(2)$ equals 2 or 4.

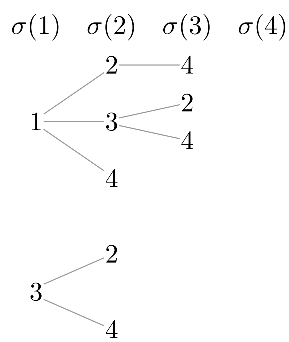


We now have 5 "branches" in our "tree". Let's go through the remaining possibilities, branch by branch.

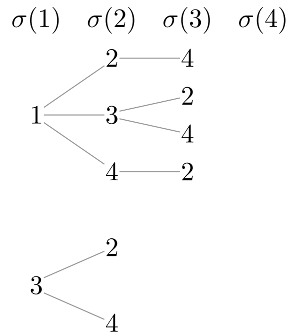
- If $\sigma(1) = 1$, $\sigma(2) = 2$, then $\sigma(3)$ could only be 3 or 4 now. Looking at the third column of A, the third entry is zero, meaning $a_{\sigma(3)3}$ would be zero if $\sigma(3) = 3$. So we need only consider if $\sigma(3) = 4$.



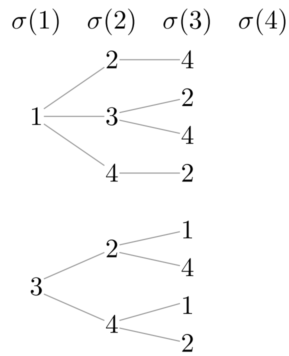
- If $\sigma(1) = 1$, $\sigma(2) = 3$, then $\sigma(3)$ could be 2 or 4. Both the second and fourth entries of the third column are nonzero, so we need to consider both possibilities.



- If $\sigma(1) = 1$, $\sigma(2) = 4$, then $\sigma(3)$ could be 2 or 3. However, since the third entry of the third column is 0, we need only consider if $\sigma(3) = 2$.

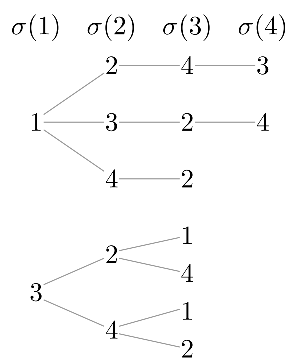


- If $\sigma(1) = 3$, $\sigma(2) = 2$, then $\sigma(3)$ could be 1 or 4. Both the first and fourth entries of the third column are nonzero, so we need to consider both possibilities.
- If $\sigma(1) = 3$, $\sigma(2) = 4$, then $\sigma(3)$ could be 1 or 2. Both the first and second entries in the third column are nonzero, so we need to consider both possibilities.



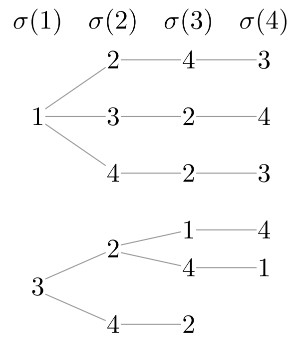
Next, we'll go through each of the branches again to determine the values of $\sigma(4)$ that need to be considered.

- If $\sigma(1) = 1$, $\sigma(2) = 2$, $\sigma(3) = 4$, then $\sigma(4)$ has to be 3. The third entry of the fourth column is nonzero, so we'll end the first branch with a 3.
- If $\sigma(1) = 1$, $\sigma(2) = 3$, $\sigma(3) = 2$, then $\sigma(4)$ has to be 4. The fourth entry of the fourth column is nonzero, so we'll end the second branch with a 4.
- If $\sigma(1) = 1$, $\sigma(2) = 3$, $\sigma(3) = 4$, then $\sigma(4)$ has to be 2. However, the second entry of the fourth column is zero, so we can actually remove this entire branch from the tree.

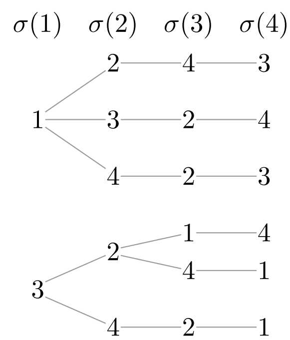


- If $\sigma(1) = 1$, $\sigma(2) = 4$, $\sigma(3) = 2$, then $\sigma(4)$ has to be 3. The third entry of the fourth column is nonzero, so we'll end this branch with a 3.

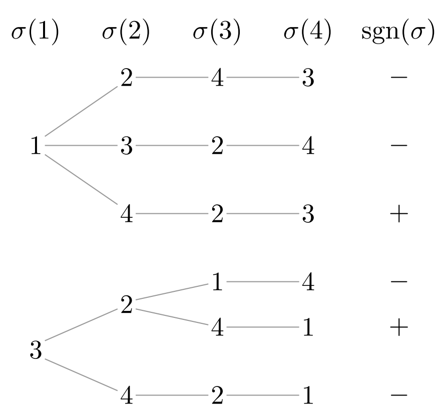
- If $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 1$, then $\sigma(4)$ has to be 4. The fourth entry of the fourth column is nonzero, so we'll end this branch with a 4.
- If $\sigma(1) = 3$, $\sigma(2) = 2$, $\sigma(3) = 4$, then $\sigma(4)$ has to be 1. The first entry of the fourth column is nonzero, so we'll end this branch with a 1.
- If $\sigma(1) = 3$, $\sigma(2) = 4$, $\sigma(3) = 1$, then $\sigma(4)$ has to be 2. But the second entry of the fourth column is zero, so we can actually remove this entire branch from the tree.



- If $\sigma(1) = 3$, $\sigma(2) = 4$, $\sigma(3) = 2$, then $\sigma(4)$ has to be 1. The first entry of the fourth column is nonzero, so we'll end this last branch with a 1.



Thus, we see there are only six nonzero terms in the big formula for our particular example. Next, I will add another column to our picture for the sign of each permutation.



To understand how I filled in this newest column:

- The first permutation in the table is $(1, 2, 4, 3)$. By swapping the 3 and the 4, we could bring it back to the natural order of $(1, 2, 3, 4)$. It took an odd number of swaps to do this, so the sign of the this first permutation is -1.

- The second permutation is $(1, 3, 2, 4)$. Swapping the 3 and the 2 brings it back to the natural order. It took an odd number of swaps for this, so the sign of this permutation is -1.
- The third permutation is $(1, 4, 2, 3)$. Swapping the 4 and the 2 gives us $(1, 2, 4, 3)$. From here, swapping the 4 and the 3 brings it back to the natural order. It took an even number of swaps for this, so the sign of this permutation is $+1$.
- The fourth permutation is $(3, 2, 1, 4)$. Swapping the 3 and the 1 brings it back to the natural order. It took an odd number of swaps to do this, so the sign of the this first permutation is -1.
- The fifth permutation is $(3, 2, 4, 1)$. Swapping the 3 and the 4 gives us $(4, 2, 3, 1)$. From here, swapping the 4 and the 1 brings it back to the natural order. It took an even number of swaps for this, so the sign of this permutation is $+1$.
- The last permutation is $(3, 4, 2, 1)$. Swapping the 3 and the 1 gives us $(1, 4, 2, 3)$. From here, swapping the 4 and the 2 gives us $(1, 2, 4, 3)$. From here, swapping the 4 and the 3 brings it back to the natural order. It took an odd number of swaps to do this, so the sign of the this first permutation is -1.

Finally, for each of the six permutations, we need to compute

$$\operatorname{sgn}(\sigma) a_{\sigma(1)1} a_{\sigma(2)2} a_{\sigma(3)3} a_{\sigma(4)4} \tag{$*$}$$

and then add everything up. Below, I've added one final column with the computations of $(*)$.

$\sigma(1)$	$\sigma(2)$	$\sigma(3)$	$\sigma(4)$	$\mathrm{sgn}(\sigma)$	$\mathrm{sgn}(\sigma)a_{\sigma(1)1}a_{\sigma(2)2}a_{\sigma(3)3}a_{\sigma(4)4}$
	2	4	3	$-$	$-(1)(2)(2)(1) = -4$
1	3	2	4	$-$	$-(1)(4)(3)(1) = -12$
	4	2	3	$+$	$(1)(1)(3)(1) = 3$
		1	4	$-$	$-(1)(2)(2)(1) = -4$
	2	4	1	$+$	$(1)(2)(2)(3) = 12$
3					
	4	2	1	$-$	$-(1)(1)(3)(3) = -9$

Then

$$\det(A) = -4 - 12 + 3 - 4 + 12 - 9 = -14.$$

This means that the columns of A,

$$\begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \quad \begin{bmatrix} 0 \\ 2 \\ 4 \\ 1 \end{bmatrix}, \quad \begin{bmatrix} 2 \\ 3 \\ 0 \\ 2 \end{bmatrix}, \quad \begin{bmatrix} 3 \\ 0 \\ 1 \\ 1 \end{bmatrix},$$

form a 4-parallelepiped in $\mathbb{R}^4$ whose 4-volume is 14, and since the determinant is negative, its orientation is opposite that of the standard basis in $\mathbb{R}^4$. $\diamond$

The idea of this last example can be used to prove the following

Proposition. The determinant of a triangular matrix (upper triangular, lower triangular, or diagonal) is equal to the product of its diagonal entries.

Proof. Since a triangular matrix has a lot of zeros in it, a lot of the terms in the big formula zero out. Let's figure out for which permutations is

$$\mathrm{sgn}(\sigma)a_{\sigma(1)1}a_{\sigma(2)2}\dots a_{\sigma(n)n}$$

nonzero. We'll do this for upper triangular matrices only, as the argument for lower triangular is similar, and diagonal matrices are both upper and lower triangular. If A is upper triangular, then A looks like this:

$$\begin{bmatrix} * & * & * & \cdots & * \\ 0 & * & * & \cdots & * \\ 0 & 0 & * & \cdots & * \\ \vdots & \vdots & \ddots & \vdots & \\ 0 & 0 & 0 & \cdots & * \end{bmatrix}$$

Since all entries in the first column are zero aside from the first entry, the only value of $\sigma(1)$ for which $a_{\sigma(1)1}$ might not be zero is $\sigma(1) = 1$. Next, since $\sigma(1) = 1$, $\sigma(2)$ can only be $2, 3, \dots, n$ now. However, all entries after the second entry in the second column are zero, so the only value of $\sigma(2)$ for which $a_{\sigma(2)2}$ might not be zero is $\sigma(2) = 2$. Continuing this analysis, one finds that the only possibility for

$$\mathrm{sgn}(\sigma)a_{\sigma(1)1}a_{\sigma(2)2}\dots a_{\sigma(n)n}$$

to be nonzero is if

$$\sigma(1) = 1, \qquad \sigma(2) = 2, \quad \dots \quad \sigma(n) = n.$$

Thus, the big formula reduces to

$$\det(A) \;=\; a_{11}a_{22}\dots a_{nn} \;=\; \substack{\text{product of diagonal}\\ \text{entries of } A.} \qquad \square$$

$\square$

Finally, we mention the most important fact of the whole chapter. We've kind of stated it a few times already, but we state it explicitly below.

Theorem. A square matrix is invertible if and only if its determinant is nonzero. Equivalently, a square matrix is not invertible, or singular, if and only if its determinant is zero.

Proof. Suppose $A = [\mathbf{a}_1 \ \dots \ \mathbf{a}_n]$ is $n \times n$ with columns $\mathbf{a}_1, \dots, \mathbf{a}_n$ in $\mathbb{R}^n$. Recall from Chapter 5 that A is invertible if and only if its columns are linearly independent, in which case the n-parallelepiped formed by $\mathbf{a}_1, \dots, \mathbf{a}_n$ will have nonzero n-volume. Hence, if A is invertible, then its determinant is nonzero.

If A is not invertible, then the columns are linearly dependent, so $\mathbf{a}_1, \dots, \mathbf{a}_n$ will span less than n-dimensions, meaning they will form an "n-parallelepiped" that has n-volume equal to 0. $\square$

A good visual for this is if we have two linearly dependent vectors, then they would lie on a line which has no area, i.e. the "parallelogram" that they form would have zero area. Likewise, three linearly dependent vectors lie on a line or in a plane, so they would form a "parallelepiped" with no volume.

7.3 Cofactors & Other Ways To Compute Determinants

7.3.1 Quick Review, Notation

Recall that for a 2×2 matrix

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

the formula for the determinant of A is

$$\det(A) = ad - bc.$$

The above notation for determinants, $\det(A)$, requires that we name our matrix A first. Having some notation to denote the determinant of a matrix without the need of naming it will be convenient. We will adopt the convention that using vertical lines for the borders of a matrix rather than brackets denotes the determinant of that matrix. Thus, for instance,

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix}$$

denotes the determinant of the matrix

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

7.3.2 Cofactors

The cofactor expansion method of computing determinants is probably the method that you will be using the most for computing determinants in this class. Let A be an $n \times n$ matrix, and let (i, j) be a pair of integers where $1 \leq i, j \leq n$. Let A_{ij} denote the matrix obtained by deleting the i^{th} row and the j^{th} column of A. We define the (i, j)***-cofactor*** of A, denoted C_{ij}, as follows:

$$C_{ij} = (-1)^{i+j} \det(A_{ij}).$$

Example. Consider

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix}.$$

Let's compute C_{11} and C_{23}. For C_{11}, first cross out the first row and first column of A to get

$$A_{11} = \begin{bmatrix} e & f \\ h & i \end{bmatrix}.$$

Then

$$C_{11} = (-1)^{1_1} \det(A_{11}) = ei - fh.$$

For C_{23}, first cross out the second row and third column of A to get

$$A_{23} = \begin{bmatrix} a & b \\ g & h \end{bmatrix}.$$

Then

$$C_{23} = (-1)^{2+3} \det(A_{23}) = -(ah - bg). \qquad \diamond$$

The following visuals show for which pairs (i, j) will $(-1)^{i+j}$ be 1 or -1. These are for 3×3 and 4×4 matrices specifically, but one can easily generalize.

$$\begin{bmatrix} + & - & + \\ - & + & - \\ + & - & + \end{bmatrix} \qquad \begin{bmatrix} + & - & + & - \\ - & + & - & + \\ + & - & + & - \\ - & + & - & + \end{bmatrix}$$

Example. Let

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 5 \\ 4 & 0 & 1 \end{bmatrix}.$$

Let's compute all of the cofactors of A.

$$\begin{array}{lll}
C_{11} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 & C_{12} = -\begin{vmatrix} 1 & 5 \\ 4 & 1 \end{vmatrix} = 19 & C_{13} = \begin{vmatrix} 1 & 3 \\ 4 & 0 \end{vmatrix} = -12 \\[2ex]
C_{21} = -\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = -1 & C_{22} = \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = -10 & C_{23} = -\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} = 4 \\[2ex]
C_{31} = \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} = -4 & C_{32} = -\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = -7 & C_{33} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5
\end{array}$$

$\diamond$

Cofactor Expansion Method for Computing Determinants

Let A be an $n \times n$ matrix. For any i between 1 and n, we can compute the determinant of A as follows:

$$\det(A) = a_{i1}C_{i1} + a_{i2}C_{i2} + \ldots + a_{in}C_{in}.$$

In words, we multiply each entry in the i^{th} row of A with the corresponding cofactor, and then add everything up. We refer to this method of computing determinants as cofactor expansion along the i^{th} row. Similarly, for any j between 1 and n, we can compute the determinant of A as

$$\det(A) = a_{1j}C_{1j} + a_{2j}C_{2j} + \ldots + a_{nj}C_{nj}.$$

In words, we multiply each entry in the j^{th} column of A with the corresponding cofactor, and then add everything up. We refer to this method of computing determinants as cofactor expansion along the j^{th} column.

Example. Using the matrix in the previous example, cofactor expansion along the first row gives us

$$\det(A) \ = \ a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13} \ = \ 2(3) + 1(19) + 3(-12) = -11.$$

Cofactor expansion along the third row gives us

$$\det(A) \ = \ a_{31}C_{31} + a_{32}C_{32} + a_{33}C_{33} \ = \ 4(-4) + 0 + 1(5) = -11.$$

Cofactor expansion along the second column gives us

$$\det(A) \ = \ a_{12}C_{12} + a_{22}C_{22} + a_{32}C_{32} \ = \ 1(19) + 3(-10) + 0 = -11.$$

$\diamond$

As the last example helps to illustrate, you can do cofactor expansion along any row or column, and as long as you do the computations correctly, you will always end up with the same answer in the end for $\det(A)$. We also see that choosing a row or column with zeros in it can simplify computations, since one of the terms in the sum zeros out.

Example. Consider

$$A = \begin{bmatrix} 1 & 3 & 2 \\ 2 & 0 & 5 \\ 1 & 2 & 1 \end{bmatrix}.$$

Cofactor expansion along the first row gives us

$$\begin{aligned}
\det(A) &= 1\begin{vmatrix} 0 & 5 \\ 2 & 1 \end{vmatrix} - 3\begin{vmatrix} 2 & 5 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 2 & 0 \\ 1 & 2 \end{vmatrix} \\
&= 1(-10) \ - \ 3(-3) \ + \ 2(4) \\
&= 7.
\end{aligned}$$

Cofactor expansion along the second row gives us

$$\begin{aligned}\det(A) &= -2\begin{vmatrix}3 & 2\\ 2 & 1\end{vmatrix} + 0 - 5\begin{vmatrix}1 & 3\\ 1 & 2\end{vmatrix}\\ &= -2(-1) - 5(-1)\\ &= \quad 7.\end{aligned}$$

◇

We can also use the cofactor expansion method to compute determinants of larger matrices.

Example. Recall in the last section that we used the big formula to compute the determinant of

$$A = \begin{bmatrix}1 & 0 & 2 & 3\\ 0 & 2 & 3 & 0\\ 1 & 4 & 0 & 1\\ 0 & 1 & 2 & 1\end{bmatrix},$$

and we found that $\det(A) = -14$. Let's verify that we get the same result using cofactor expansion. We'll do cofactor expansion along the first column, since there are two zeros in it.

$$\det(A) = 1\begin{vmatrix}2 & 3 & 0\\ 4 & 0 & 1\\ 1 & 2 & 1\end{vmatrix} - \ 0 \ + 1\begin{vmatrix}0 & 2 & 3\\ 2 & 3 & 0\\ 1 & 2 & 1\end{vmatrix} - \ 0.$$

Now we need to evaluate these 3×3 determinants. We'll do this using cofactor expansion along the first row:

$$\begin{aligned}\det(A) &= \left(2\begin{vmatrix}0 & 1\\ 2 & 1\end{vmatrix} - 3\begin{vmatrix}4 & 1\\ 1 & 1\end{vmatrix} + 0\right) + \left(0 - 2\begin{vmatrix}2 & 0\\ 1 & 1\end{vmatrix} + 3\begin{vmatrix}2 & 3\\ 1 & 2\end{vmatrix}\right)\\ &= \qquad \big(2(-2) - 3(3)\big) \qquad + \qquad \big(-2(2) + 3(1)\big)\\ &= \quad -14.\end{aligned}$$

Thus, using cofactor expansion gives us the same result. ◇

To be clear, cofactor expansion and the big formula are equivalent to each other, meaning one should always get the same answer for the determinant no matter which method they use. I will omit the proof that they are equivalent, but you can look it up yourself if you are interested.

Cofactor Formula for A^{-1}

While we are still discussing cofactors, let us mention one application of them relating to finding the inverse of a matrix. For illustration purposes, let A be a 3×3 matrix and let C be the matrix of cofactors of A:

$$A = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix} \qquad C = \begin{bmatrix}C_{11} & C_{12} & C_{13}\\ C_{21} & C_{22} & C_{23}\\ C_{31} & C_{32} & C_{33}\end{bmatrix}$$

Let's consider what happens when we multiply A with C^T:

$$AC^T = \begin{bmatrix}a_{11} & a_{12} & a_{13}\\ a_{21} & a_{22} & a_{23}\\ a_{31} & a_{32} & a_{33}\end{bmatrix}\begin{bmatrix}C_{11} & C_{21} & C_{31}\\ C_{12} & C_{22} & C_{32}\\ C_{13} & C_{23} & C_{33}\end{bmatrix}$$

The $(1, 1)$-entry of the product would be

$$a_{11}C_{11} + a_{12}C_{12} + a_{13}C_{13},$$

which happens to equal $\det(A)$, since it is just cofactor expansion along the first row. So right now, we have

$$AC^T = \begin{bmatrix}\det(A) & ? & ?\\ ? & ? & ?\\ ? & ? & ?\end{bmatrix}.$$

The $(1,2)$-entry of the AC^T would be

$$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23}.$$

This does not equal $\det(A)$. It would be if the cofactors were multiplied by the entries of A in the second row, since that would be cofactor expansion along the second row. Instead, the cofactors are multiplied by the entries in the first row of A. So actually, this is computing the determinant, using cofactor expansion along the second row, not of A but of the matrix obtained by replacing the second row of A with the first row of A:

$$a_{11}C_{21} + a_{12}C_{22} + a_{13}C_{23} = \begin{vmatrix} a_{11} & a_{12} & a_{13} \\ a_{11} & a_{12} & a_{13} \\ a_{31} & a_{32} & a_{33} \end{vmatrix}$$

However, this matrix is not invertible since it has two identical rows. Thus, its determinant equals 0. So now we have

$$AC^T = \begin{bmatrix} \det(A) & 0 & ? \\ ? & ? & ? \\ ? & ? & ? \end{bmatrix}.$$

After a similar analysis for the remaining entries, one finds that

$$AC^T = \begin{bmatrix} \det(A) & 0 & 0 \\ 0 & \det(A) & 0 \\ 0 & 0 & \det(A) \end{bmatrix},$$

and essentially the same analysis for any size square matrix gives

$$AC^T \;=\; C^TA \;=\; \det(A) \cdot I.$$

Now, if $\det(A)$ is nonzero, then we could divide the above equation through by $\det(A)$, giving us

$$A \cdot \left(\tfrac{1}{\det(A)} C^T\right) = \left(\tfrac{1}{\det(A)} C^T\right) \cdot A = I.$$

Since $\frac{1}{\det(A)} C^T$ multiplies with A to equal I, it must be that

$$A^{-1} = \tfrac{1}{\det(A)} C^T.$$

Example. Earlier, we computed all of the cofactors as well as the determinant of

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 5 \\ 4 & 0 & 1 \end{bmatrix}.$$

The determinant of A was -11, and here are the cofactors of A

$$C_{11} = \begin{vmatrix} 3 & 5 \\ 0 & 1 \end{vmatrix} = 3 \qquad C_{12} = -\begin{vmatrix} 1 & 5 \\ 4 & 1 \end{vmatrix} = 19 \qquad C_{13} = \begin{vmatrix} 1 & 3 \\ 4 & 0 \end{vmatrix} = -12$$

$$C_{21} = -\begin{vmatrix} 1 & 3 \\ 0 & 1 \end{vmatrix} = -1 \qquad C_{22} = \begin{vmatrix} 2 & 3 \\ 4 & 1 \end{vmatrix} = -10 \qquad C_{23} = -\begin{vmatrix} 2 & 1 \\ 4 & 0 \end{vmatrix} = 4$$

$$C_{31} = \begin{vmatrix} 1 & 3 \\ 3 & 5 \end{vmatrix} = -4 \qquad C_{32} = -\begin{vmatrix} 2 & 3 \\ 1 & 5 \end{vmatrix} = -7 \qquad C_{33} = \begin{vmatrix} 2 & 1 \\ 1 & 3 \end{vmatrix} = 5$$

If we put these cofactors into a matrix,

$$C = \begin{bmatrix} 3 & 19 & -12 \\ -1 & -10 & 4 \\ -4 & -7 & 5 \end{bmatrix},$$

then

$$A^{-1} = \tfrac{1}{\det(A)} C^T \;=\; -\frac{1}{11}\begin{bmatrix} 3 & -1 & -4 \\ 19 & -10 & -7 \\ -12 & 4 & 5 \end{bmatrix}$$

We can confirm this by checking that multiplying with A gives us the identity matrix:

$$-\frac{1}{11}\begin{bmatrix} 3 & -1 & -4 \\ 19 & -10 & -7 \\ -12 & 4 & 5 \end{bmatrix}\begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 5 \\ 4 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}. \qquad \diamond$$

7.3.3 Computing Determinants Using Row/Column Operations

Another method of computing determinants is with row and column operations. Recall the following properties of determinants:

- Swapping two columns of a matrix changes the sign of the determinant.
- If a matrix has two identical columns, then its determinant is 0.
- Scaling a column by a scalar c scales the determinant by c.
- The determinant is additive/distributive, in the sense that

$$\det(\mathbf{a}_1, \ldots, \mathbf{a}_j + \mathbf{b}_j, \ldots, \mathbf{a}_n) = \det(\mathbf{a}_1, \ldots, \mathbf{a}_j, \ldots, \mathbf{a}_n) + \det(\mathbf{a}_1, \ldots, \mathbf{b}_j, \ldots, \mathbf{a}_n).$$

Now, consider what happens when we add a multiple of a column to another column. For clarity, suppose $A = [\mathbf{a}_1\ \mathbf{a}_2\ \mathbf{a}_3]$, and suppose we add c of column 1 to column 2 to get

$$[\mathbf{a}_1\ \ c\mathbf{a}_1 + \mathbf{a}_2\ \ \mathbf{a}_3].$$

Using the properties of determinant listed above, we have

$$\begin{aligned} \det(\mathbf{a}_1, c\mathbf{a}_1 + \mathbf{a}_2, \mathbf{a}_3) &= c\det(\mathbf{a}_1, \mathbf{a}_1, \mathbf{a}_3) + \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \\ &= \qquad 0 \qquad + \det(\mathbf{a}_1, \mathbf{a}_2, \mathbf{a}_3) \\ &= \quad \det(A). \end{aligned}$$

Thus, we see that adding c of column 1 to column 2 did not affect the determinant. In general, for any size square matrix, adding a multiple of one column to another column does not change the determinant. This fact, plus the first and third properties listed above, tell us how performing column operations affects the determinant of a matrix. We summarize our findings below:

Proposition.

- Swapping two columns of a matrix changes the sign of the determinant.
- Scaling a column by c scales the determinant by c.
- Adding a multiple of a column to another column does not affect the determinant.

Using the fact that $\det(A^T) = \det(A)$ (which we will talk more about in the next section), all of these results apply to row operations as well. Thus, we have

Proposition.

- Swapping two rows of a matrix changes the sign of the determinant.
- Scaling a row by c scales the determinant by c.
- Adding a multiple of a row to another row does not affect the determinant.

One strategy for computing determinants then would be to perform row or column operations to get more zeros in the matrix, keeping track of how these operations affect the determinant. Once you have enough zeros in the matrix, you can compute the determinant using cofactor expansion, etc, and then see the determinant of the reduced matrix relates to the determinant of the original.

Example. Consider the matrix

$$A = \begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 5 \\ 4 & 0 & 1 \end{bmatrix}$$

whose inverse we computed in the previous example. Let's do some row operations to get more zeros in the matrix. Specifically, we see that there is already a zero in the second column, so let's do a row operation to get one more zero in the second column.

$$\begin{bmatrix} 2 & 1 & 3 \\ 1 & 3 & 5 \\ 4 & 0 & 1 \end{bmatrix} \xrightarrow{R_2 - 3R_1} \begin{bmatrix} 2 & 1 & 3 \\ -5 & 0 & -4 \\ 4 & 0 & 1 \end{bmatrix}$$

Since adding a multiple of a row to another row does not affect the determinant, $\det(A)$ is equal to the determinant of the reduced matrix, which we can easily compute doing cofactor expansion along the second column:

$$\det(A) = \begin{vmatrix} 2 & 1 & 3 \\ -5 & 0 & -4 \\ 4 & 0 & 1 \end{vmatrix} = -1 \begin{vmatrix} -5 & -4 \\ 4 & 1 \end{vmatrix} = -(-5+16) = -11. \qquad \diamond$$

Performing row or column operations first is particularly efficient for finding determinants of larger matrices without a lot of zeros already.

Example. Let's do some row/column operations to find the determinant of

$$A = \begin{bmatrix} 1 & 0 & 3 & 2 \\ 2 & 1 & 4 & 2 \\ 1 & 3 & 5 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

I will do some row operations to get more zeros in the fourth column.

$$\begin{bmatrix} 1 & 0 & 3 & 2 \\ 2 & 1 & 4 & 2 \\ 1 & 3 & 5 & 2 \\ 2 & 1 & 2 & 0 \end{bmatrix} \begin{matrix} \xrightarrow{R_2 - R_1} \\ \xrightarrow{R_3 - R_1} \end{matrix} \begin{bmatrix} 1 & 0 & 3 & 2 \\ 1 & 1 & 1 & 0 \\ 0 & 3 & 2 & 0 \\ 2 & 1 & 2 & 0 \end{bmatrix}$$

Since adding multiples of a row to other rows does not affect the determinant, $\det(A)$ is equal to the determinant of the reduced matrix, which we can start computing with cofactor expansion along the fourth column:

$$\det(A) = -2 \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 2 & 1 & 2 \end{vmatrix}$$

We could compute this 3×3 determinant fairly easily using cofactor expansion, but let's do one row operation instead to get an extra zero in the first column:

$$\begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 2 & 1 & 2 \end{bmatrix} \xrightarrow{R_3 - 2R_1} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & -1 & 0 \end{bmatrix}.$$

The determinants of the above two matrices are equal, since adding a multiple of a row to another row does not affect determinants. Combining this with what we had earlier, now we have

$$\det(A) = -2 \begin{vmatrix} 1 & 1 & 1 \\ 0 & 3 & 2 \\ 0 & -1 & 0 \end{vmatrix} = -2(1) \begin{vmatrix} 3 & 2 \\ -1 & 0 \end{vmatrix} = -4. \qquad \diamond$$

7.4 Additional Properties of Determinants

Most of the properties of determinants that we have discussed so far have been in terms of the columns of the matrix as opposed to the actual matrix itself (e.g. swapping two columns changes the sign of the determinant. Below, we mention some properties of determinants that are centered around the matrix itself rather than its columns.

Additional Properties of Determinants

1. If A is $n \times n$, then for any scalar c,
$$\det(cA) = c^n \det(A).$$
2. For any square matrix A,
$$\det(A^T) = \det(A).$$
3. For any square matrices A, B of the same size,
$$\det(AB) = \det(A)\det(B).$$
This result generalizes to any number of matrices, i.e.
$$\det(A_1 \cdots A_k) = \det(A_1) \cdots \det(A_k).$$
4. For any nonnegative integer k, and for any square matrix A,
$$\det(A^k) = \det(A)^k.$$
If A is invertible, then we also have
$$\det(A^{-k}) = \det(A)^{-k}.$$

Let's try to understand why each of these are true.

Starting with Property 1, we know that scaling a column by c scales the determinant by c. Scaling A by c scales all n columns by c, so the determinant gets scaled by c^n.

Let's look at Property 4 next.

- First, we consider A^0 to be equal to I, so
$$\det(A^0) = \det(I) = 1 = \det(A)^0.$$
- Next, when $k > 0$, we can use Property 3:
$$\det(A^k) = \det(\overbrace{A \cdots A}^{k \text{ times}}) = \overbrace{\det(A) \cdots \det(A)}^{k \text{ times}} = \det(A)^k.$$
- If A is invertible, then we have
$$AA^{-1} = I \quad \Longrightarrow \quad \det(AA^{-1}) = \det(I).$$
Applying Property 3, and the fact that $\det(I) = 1$,
$$\det(A)\det(A^{-1}) = 1 \quad \Longrightarrow \quad \det(A^{-1}) = \frac{1}{\det(A)},$$
or equivalently,
$$\det(A^{-1}) = \det(A)^{-1}.$$
Then, since $A^{-k} = (A^{-1})^k$, applying the $k > 0$ case to A^{-1} gives
$$\det(A^{-k}) = \det(A)^{-k}.$$

Let's now consider Property 3. First, notice that if $\det(AB) = \det(A)\det(B)$ holds for any two square matrices A, B of the same size, then if A, B, C are square of the same size, we would have

$$\det(ABC) = \det((AB)C) = \det(AB)\det(C) = \det(A)\det(B)\det(C).$$

So once we prove the result for two matrices, it follows that the result holds for three matrices, and we can generalize to any number of matrices with similar reasoning.

Part of our argument for why $\det(AB) = \det(A)\det(B)$ is true is based off elementary matrices (from Chapter 4). Recall that an elementary matrix is a matrix obtained by performing exactly one row operation on an identity matrix, and that every elementary is also one column operation from an identity matrix. Multiplying on the left by an elementary matrix has the effect of performing the corresponding row operation, while multiplying on the right has the effect of performing the corresponding column operation. Since determinants were defined in terms of columns, we'll focus on the connection between elementary matrices and column operations rather than row operations. We'll break the argument into two cases.

- First, suppose either A or B is singular, so that $\det(A) = 0$ or $\det(B) = 0$. We will prove that AB is singular, so that $\det(AB) = 0$ as well. It would then follow that $\det(AB) = \det(A)\det(B)$.

  If B is singular, then we know that the nullspace of B is nontrivial, meaning there are nonzero solutions to the equation $B\mathbf{x} = \mathbf{0}$. (For instance, there would be at least one column missing a pivot, meaning at least one free variable, meaning infinitely many solutions to $B\mathbf{x} = \mathbf{0}$.) If $\mathbf{x}$ is a nonzero vector such that $B\mathbf{x} = \mathbf{0}$, then $AB\mathbf{x} = \mathbf{0}$ as well. In other words, the nullspace of AB is also nontrivial, so AB must be singular.

  If A is singular, then we know that there is some vector $\mathbf{b}$ for which $A\mathbf{x} = \mathbf{b}$ has no solution. (For instance, there would be at least one row of zeros in any row echelon form of A, leading to equations of the form 0 equals something nonzero.) For this same vector $\mathbf{b}$, there would also be no solution to the equation $AB\mathbf{x} = \mathbf{b}$; otherwise, if $\mathbf{v}$ satisfies $AB\mathbf{v} = \mathbf{b}$, then $\mathbf{x} = B\mathbf{v}$ would satisfy $A\mathbf{x} = \mathbf{b}$, contradicting $A\mathbf{x} = \mathbf{b}$ having no solution. Thus, there is some vector $\mathbf{b}$ for which $AB\mathbf{x} = \mathbf{b}$ has no solution, meaning AB is singular.

  In conclusion, if A or B is singular, then AB is singular, and $\det(AB) = \det(A)\det(B)$.

- It remains to consider the case where A and B are invertible. Recall from Chapter 4 that every invertible matrix is a product of elementary matrices. Therefore, we only need to prove the result for elementary matrices, since then if

  $$A = A_1 \cdots A_m \qquad\qquad B = B_1 \cdots B_n$$

  are factorizations of A, B into a product of elementary matrices, we would have

  $$\det(AB) = \overbrace{\det(A_1)\cdots\det(A_m)}^{\det(A)}\,\overbrace{\det(B_1)\cdots\det(B_n)}^{\det(B)}.$$

  There are three types of elementary matrices, corresponding to the three types of column operations:

  - If E is an elementary permutation matrix, i.e. E is obtained from I by swapping two columns, then since $\det(I) = 1$ and swapping two columns changes the sign of the determinant, $\det(E) = -1$. If A is any square matrix, then AE is A with two columns swapped, so $\det(AE) = -\det(A)$, or in particular,

    $$\det(AE) = \det(A)\det(E).$$

  - If E is an elementary scaling matrix, i.e. E is obtained from I by scaling a column by some scalar c, then since $\det(I) = 1$ an scaling a column by c scales the determinant by c, $\det(E) = c$. If A is any square matrix, then AE is A but with a column scaled by c, so $\det(AE) = c\det(A)$, or in particular,

    $$\det(AE) = \det(A)\det(E).$$

- If E is an elementary elimination matrix, i.e. E is obtained from I by adding a multiple of column to another column, then since $\det(I) = 1$ and adding a multiple of a column to another column doesn't affect the determinant, $\det(E) = 1$. If A is any square matrix, then AE is A but with a multiple of a column added to another column, so $\det(AE) = \det(A)$, or in particular,

$$\det(AE) = \det(A)\det(E).$$

  Thus, we see for any square matrix A and for any elementary matrix E,

$$\det(AE) = \det(A)\det(E).$$

  In particular, for any two elementary matrices E_1, E_2,

$$\det(E_1E_2) = \det(E_1)\det(E_2).$$

Thus, for any two square matrices AB of the same size,

$$\det(AB) = \det(A)\det(B).$$

□

- As for Property 2, $\det(A^T) = \det(A)$, we can again break it into two cases. If A is singular, then so too is A^T so $\det(A^T) = \det(A) = 0$. If A is invertible, then we can factor A into a product elementary matrices, which then gives us a factorization of A^T into a product of elementary matrices as well (since the transpose of an elementary matrix is also an elementary matrix):

$$A = A_1 \cdots A_k \qquad \Longrightarrow \qquad A^T = A_k^T \cdots A_1^T.$$

  You can verify that for any elementary matrix, its determinant is equal to the determinant of its transpose. Then

$$\det(A^T) \;=\; \det(A_k^T)\cdots\det(A_1^T) \;=\; \det(A_k)\cdots\det(A_1) \;=\; \det(A).$$

  Thus, for any square matrix A,

$$\det(A^T) = \det(A).$$

Example. Suppose A, B, C are 4×4 with $\det(A) = 3, \det(B) = -2, \det(C) = 4$. Compute $\det(2A^TB^3C^{-2})$.

Solution. Using the properties of determinants discussed in this section,

$$\det(2A^TB^3C^{-2}) = 2^4 \cdot \frac{\det(A)\det(B)^3}{\det(C)^2} = 16 \cdot \frac{3(-2)^3}{4^2} = -24.$$

◇

7.5 Geometric Interpretation of $\det(AB) = \det(A)\det(B)$

This is an optional section, in which we discuss a geometric interpretation of the property $\det(AB) = \det(A)\det(B)$. Recall that the determinant of an $n \times n$ matrix is the signed n-volume of the n-parallelepiped formed by the columns of A. We will denote the n-parallelepipeds formed by the columns of A, B, and AB by P_A, P_B, and P_{AB}. Recall also that the columns of AB are equal to A times the columns of B:

$$B = [\mathbf{b}_1\ \mathbf{b}_2\ \cdots\ \mathbf{b}_n] \qquad \Longrightarrow \qquad AB = [A\mathbf{b}_1\ \ A\mathbf{b}_2\ \ \cdots\ \ A\mathbf{b}_n].$$

Let $T : \mathbb{R}^n \to \mathbb{R}^n$ be the linear transformation associated with A, i.e. $T(\mathbf{x}) = A\mathbf{x}$. Then the columns of AB are the result of applying the transformation T to the columns of B.

$$AB = [T(\mathbf{b}_1)\ \ T(\mathbf{b}_2)\ \ \cdots\ \ T(\mathbf{b}_n)].$$

Hence, the P_{AB} is the image of P_B under the transformation T, i.e. $P_{AB} = T(P_B)$. The fact that $\det(AB) = \det(A)\det(B)$ then tells us that when we use T to transform the n-parallelepiped P_B, the result is another n-parallelepiped, P_{AB}, whose n-volume is $|\det(A)|$ times the n-volume of the original P_B. Since this argument holds for any B, it follows that *for any n-parallelepiped P, the n-volume of $T(P)$ is equal to $|det(A)|$ times the n-volume of the original P.*

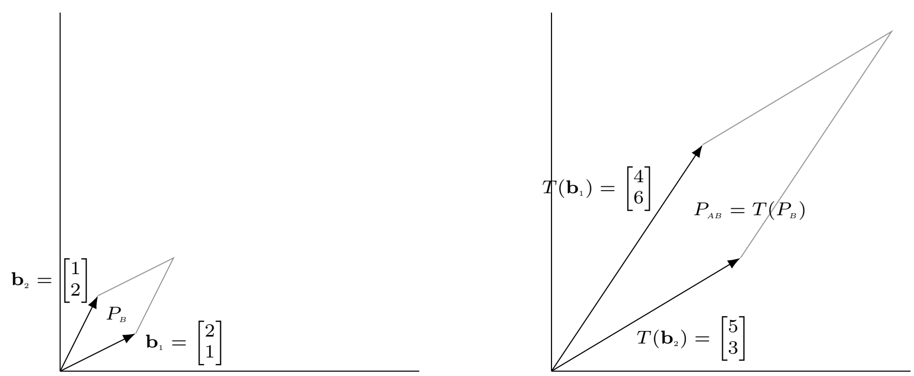
To understand things better, let's look at an example. Consider the matrices

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix} \qquad \text{and} \qquad B = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}$$

The columns of AB are

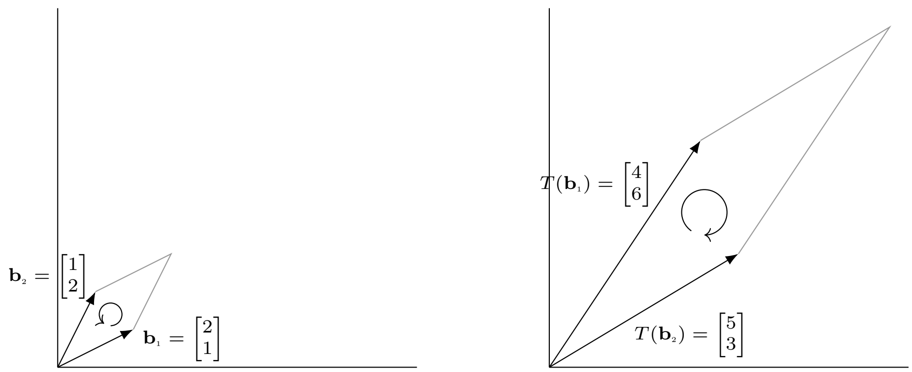
$$T(\mathbf{b}_1) = A\mathbf{b}_1 = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 2 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 6 \end{bmatrix} \qquad \text{and} \qquad T(\mathbf{b}_2) = A\mathbf{b}_2 = \begin{bmatrix} 1 & 2 \\ 3 & 0 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 3 \end{bmatrix}$$

Below is a picture of the parallelograms P_B and P_{AB}:



The area of P_B is $|\det(B)| = 3$, and after being transformed by T, the area of $T(P_B) = P_{AB}$ is $|\det(AB)| = |\det(A)\det(B)| = 18$. Since this analysis works for any other 2×2 matrix B, it follows that for any parallelogram P, the area of $T(P)$, its image under T, is equal to $|\det(A)|$ times the area of the original P.

Continuing with the example, if we ignore the absolute value signs, then the signed area of P_B is $\det(B) = 3$, while the signed area of $T(P_B) = P_{AB}$ is $\det(AB) = \det(A)\det(B) = -18$. The difference in sign tells us that the oriented parallelograms represented by $\mathbf{b}_1 \wedge \mathbf{b}_2$ and $T(\mathbf{b}_1) \wedge T(\mathbf{b}_2)$ have opposite orientations (see the picture below). The reason why the orientation changed is because $\det(A)$ is negative.



In conclusion, if A is $n \times n$ and $T : \mathbb{R}^n \to \mathbb{R}^n, T(\mathbf{x}) = A\mathbf{x}$ is the associated linear transformation, then the fact that $\det(AB) = \det(A)\det(B)$ for all $n \times n$ matrices B tells us that

- for any n-parallelepiped P, the n-volume of $T(P)$ is equal to $|\det(A)|$ times the n-volume of P, and
- if $\det(A) > 0$, then T is orientation-preserving, i.e. for any oriented n-parallelepiped P, the oriented n-parallelepiped $T(P)$ will have the same orientation as P; likewise, if $\det(A) < 0$, then T is orientation-reversing, i.e. for any oriented n-parrallelepiped P, the oriented n-parallelepiped $T(P)$ will have the opposite orientation of P.

These facts generalize beyond just n-parallelepipeds to any orientable object $\mathbb{R}^n$ with n-volume. For instance, for any subset D of $\mathbb{R}^n$ with n-volume, the n-volume of $T(D)$ will be $|\det(A)|$ times the n-volume of D. This can be proven using the Change of Variables Theorem from multivariable calculus (covered somewhat in MATH 2415 and MATH 3351):

$$\substack{n\text{-volume}\\ \text{of } T(D)} = \int_{T(D)} 1\,\mathrm{d}\mathbf{x} = \int_D |\det(A)|\,\mathrm{d}\mathbf{u} = |\det(A)| \times \left(\substack{n\text{-volume}\\ \text{of } D}\right).$$

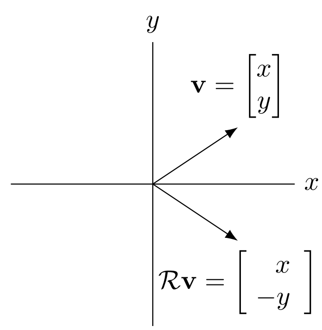
This fact can be succinctly summarized by saying the linear transformation T scales n-volumes by a factor of $|\det(A)|$. Because of these facts, we also see that the determinant of A is actually an intrinsic property of the associated linear transformation T, and therefore we should consider $\det(A)$ to be the determinant of T as well rather than just the determinant of A, i.e. $\det(T) = \det(A)$.

Chapter 8

Eigenvalues & Eigenvectors, Diagonalization, Spectral Theorem

8.1 Eigenvalues and Eigenvectors - Motivation

Let $\mathcal{R}$ denote the linear operator on $\mathbb{R}^2$ that reflects a vector $\mathbf{v}$ about the x-axis.



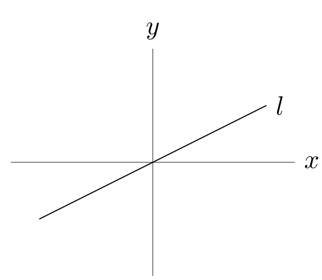
The matrix of $\mathcal{R}$ is given by

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$

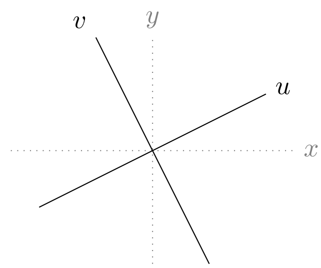
since

$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x \\ -y \end{bmatrix}.$$

Now let $\mathcal{R}$ denote the linear operator on $\mathbb{R}^2$ that reflects vectors about some arbitrary line L passing through the origin.



We worked out a general formula for the matrix of $\mathcal{R}$ in Section 3.1.4. Presently, I would like to consider something else. What if we were to consider a different coordinate system on $\mathbb{R}^2$? In the picture below, I have "faded out" the x- and y- axes, and have added two new, perpendicular axes, which I label as the u- and v- axes.

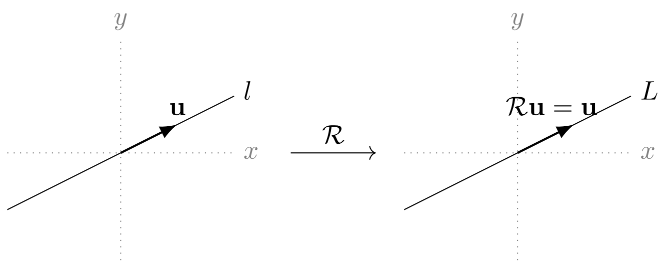


We can imagine that the u axis plays the role of the x-axis in this new coordinate system, and similarly, the v-axis now plays the role of the y-axis. Now, I have actually chosen the u-axis so that it coincides with the line L. Essentially then, in this new coordinate system, the line L is our "x" axis! So, although it maybe isn't obvious what the matrix of $\mathcal{R}$ is in the (x, y) coordinate system, in the (u, v) coordinate system, we know exactly what the matrix of $\mathcal{R}$ is! Namely, it should be the same as the matrix for the reflection about the usual x-axis, which we earlier saw is given by

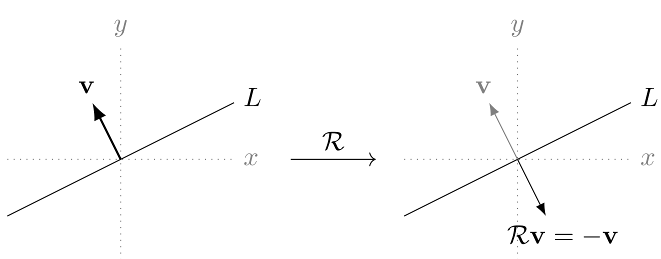
$$\begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

The takeaway from the above discussion is that a linear operator might not be easy to study in the standard coordinate system, but there may be another coordinate system in which the linear operator takes on a very simple form. So now the question is, how can we find a coordinate system in which a given linear operator is easy to visualize? To answer this question, let's revisit our example from earlier.

We were considering the reflection $\mathcal{R}$ on $\mathbb{R}^2$ about some arbitrary line L. Let $\mathbf{u}$ be any vector that lies on L; in the (u, v) coordinate system, $\mathbf{u}$ would be lying on the u-axis. If we apply the transformation $\mathcal{R}$ to $\mathbf{u}$, that is, if we reflect $\mathbf{u}$ about the line L, then nothing would happen. In other words, $\mathcal{R}\mathbf{u} = \mathbf{u}$.



Now let $\mathbf{v}$ be any vector that is orthogonal to L; in the (u, v) coordinate system, $\mathbf{v}$ would be lying on the v-axis. If we apply the transformation $\mathcal{R}$ to $\mathbf{v}$, that is, if we reflect $\mathbf{v}$ about the line L, then the result will be $-\mathbf{v}$. In other words, $\mathcal{R}\mathbf{v} = -\mathbf{v}$.

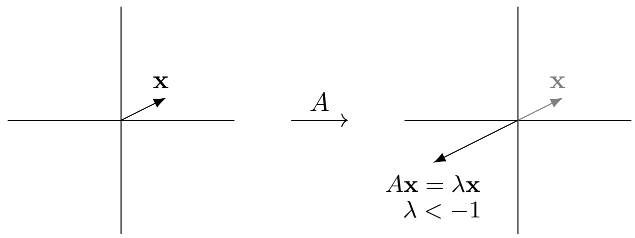


One thing that the two equations $\mathcal{R}\mathbf{u} = \mathbf{u}$ and $\mathcal{R}\mathbf{v} = -\mathbf{v}$ have in common is that, after applying the transformation $\mathcal{R}$, the result turns out to be a scalar multiple of the input.

More generally, suppose A is some linear transformation, and $\mathbf{x}$ is a vector such that

$$A\mathbf{x} = \lambda\mathbf{x}$$

for some scalar λ. Note that this equation always holds when $\mathbf{x}$ is the zero vector, so it's only interesting when $\mathbf{x}$ is nonzero. If $\lambda > 1$, then this equation says that the transformation A just stretches the vector $\mathbf{x}$. If λ is between zero and one, then this equation says that A just shrinks the vector $\mathbf{x}$ instead. And if λ is negative, then the transformation A also reverses the direction of $\mathbf{x}$. Thus, the action of A on $\mathbf{x}$ is very easy to visualize.



Given an operator A, a scalar λ is said to be an ***eigenvalue*** of A if there exists a nonzero vector $\mathbf{x}$ such that

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Any nonzero vector $\mathbf{x}$ satisfying the above equation is said to be an ***eigenvector*** of A, corresponding to the eigenvalue λ.

So, as an example, the numbers 1 and -1 are eigenvalues of our reflection operator $\mathcal{R}$, and the vectors $\mathbf{u}$ and $\mathbf{v}$ happen to be corresponding eigenvectors.

Now remember, we were trying to answer the question: how can we find a coordinate system in which a linear transformation A is easy to visualize? Well, we just saw that the action of a linear transformation on any eigenvector is easy to visualize. Thus, one answer to our question would be to choose, if possible, a coordinate system in which the coordinate axes are spanned by eigenvectors of A. In fact, this is exactly what we did for our reflection operator! And this gives us a good reason to learn more about eigenvalues and eigenvectors.

In the next few sections, we will learn, among other things: how to compute, in theory, eigenvalues and eigenvectors for a given transformation (i.e. matrix) A; some additional applications of all this eigenstuff; and we will also study the properties of eigenvalues and eigenvectors of some classes of linear transformations (i.e. matrices).

8.2 Finding Eigenvalues and Eigenvectors

First, let's recall the relevant definitions.

Definition. Let A be an $n \times n$ matrix. A scalar λ is said to be an ***eigenvalue*** of A if there exists a nonzero vector $\mathbf{x}$ satisfying

$$A\mathbf{x} = \lambda\mathbf{x}.$$

Any nonzero vector $\mathbf{x}$ satisfying the above equation is said to be an ***eigenvector*** of A, with corresponding eigenvalue λ.

Alright. Now the question is: given a matrix A, how would we go about finding its eigenvalues and eigenvectors? Let's take a closer look at the defining equation:

$$A\mathbf{x} = \lambda\mathbf{x}$$

By moving things around, we can rewrite the above equation as

$$\lambda\mathbf{x} - A\mathbf{x} = \mathbf{0}. \tag{8.1}$$

Now, on the left side of the equation, it looks like we can factor an $\mathbf{x}$ out from every term, and we'd be left with $(\lambda - A)\mathbf{x} = \mathbf{0}$. However, this is actually not quite right, because in the parentheses we have a scalar minus a matrix, which doesn't make sense. We can resolve this issue though by inserting an identity matrix into the equation; in particular, since $I\mathbf{x} = \mathbf{x}$, we can rewrite equation (8.1) as

$$\lambda I\mathbf{x} - A\mathbf{x} = \mathbf{0}.$$

Now we can factor an $\mathbf{x}$ out from everything, leaving us with

$$(\lambda I - A)\mathbf{x} = \mathbf{0}.$$

Now we have in the parentheses a matrix minus a matrix, which does make sense.

Thus, we see that if λ is an eigenvalue of A, then the eigenvectors satisfy the equation

$$(\lambda I - A)\mathbf{x} = \mathbf{0}.$$

In other words, the eigenvectors can be found by finding the nullspace of $\lambda I - A$. Because of this, we refer to the nullspace of $\lambda I - A$ as the ***eigenspace*** corresponding to the eigenvalue λ. Since we know how to find the nullspace of a matrix, this means we now know how to find eigenvectors, once we have eigenvalues. So now we need a way of finding the eigenvalues.

Recall from the definition that eigenvectors are nonzero. So for λ to be an eigenvalue, we are saying that the nullspace of $\lambda I - A$ must be nontrivial. We know that this happens when the matrix $\lambda I - A$ is missing a pivot, as this would indicate the presence of free variables. In particular, since it is missing a pivot, the matrix $\lambda I - A$ must be non-invertible. This is equivalent to saying that $\det(\lambda I - A)$ must equal zero. The takeaway from all this is that the eigenvalues of A are the solutions to the equation

$$\det(\lambda I - A) = 0.$$

This equation is called the ***characteristic equation***.

In summary, here is the procedure for finding eigenvalues and eigenvectors:

- Find the eigenvalues by solving the characteristic equation. Spelling this out more, find the values of λ for which $\det(\lambda I - A)$ equals zero. This gives the eigenvalues of A.
- For each eigenvalue of A that was found, find the nullspace of $\lambda I - A$ to get the corresponding eigenvectors.

Now for some examples.

Example. Let's find the eigenvalues and eigenvectors of the matrix

$$A = \begin{bmatrix} -1 & -2 \\ 4 & 5 \end{bmatrix}.$$

We start with the characteristic equation $\det(\lambda I - A) = 0$ to find the eigenvalues. First,

$$\lambda I - A = \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix} - \begin{bmatrix} -1 & -2 \\ 4 & 5 \end{bmatrix} = \begin{bmatrix} \lambda + 1 & 2 \\ -4 & \lambda - 5 \end{bmatrix}.$$

Thus,

$$\begin{aligned} \det(\lambda I - A) &= (\lambda + 1)(\lambda - 5) + 8 \\ &= \lambda^2 - 4\lambda + 3 \\ &= (\lambda - 3)(\lambda - 1). \end{aligned}$$

Setting this equal to zero and solving for λ, we find the eigenvalues to be

$$\lambda = 3 \quad \text{and} \quad \lambda = 1.$$

Next, for each eigenvalue λ that was found, we need to find the nullspace of $\lambda I - A$ in order to find the eigenvectors corresponding to the eigenvalue λ. Earlier, we saw that

$$\lambda I - A = \begin{bmatrix} \lambda + 1 & 2 \\ -4 & \lambda - 5 \end{bmatrix}.$$

Let's start with $\lambda = 3$. Plugging $\lambda = 3$ into the above matrix, we get

$$3I - A = \begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix}.$$

The nullspace of this matrix will give us the eigenspace corresponding to $\lambda = 3$. To find the nullspace, first we row reduce:

$$\begin{bmatrix} 4 & 2 \\ -4 & -2 \end{bmatrix} \longrightarrow \begin{bmatrix} 2 & 1 \\ 0 & 0 \end{bmatrix}.$$

From this, we see that the nullspace consists of all solutions to the equation

$$2x + y = 0.$$

Usually, we would choose y to be the variable since the second column is missing a pivot. We get nicer numbers though if we choose x to be free instead. Setting $x = t$ and solving for y, we get $y = -2t$. Thus, the nullspace is all vectors of the form

$$\begin{bmatrix} t \\ -2t \end{bmatrix} = t \begin{bmatrix} 1 \\ -2 \end{bmatrix}.$$

From this, we see that the *nonzero* multiples of the vector $(1, -2)$ are the eigenvectors of A corresponding to the eigenvalue $\lambda = 3$. We emphasize nonzero since by definition eigenvectors are nonzero.

Now rinse and repeat with $\lambda = 1$. First,

$$1I - A = \begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix}.$$

Row reducing, we get

$$\begin{bmatrix} 2 & -2 \\ 4 & -4 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

The corresponding eigenspace is all solutions to the equation

$$x - y = 0.$$

Letting $y = t$, we get $x = t$. Thus, the eigenspace corresponding to $\lambda = 1$ is all vectors of the form

$$t \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The eigenvectors of A corresponding to $\lambda = 1$ are thus all nonzero multiples of the vector $(1, 1)$. ◇

Example. Let's do a 3×3 example next. Consider

$$A = \begin{bmatrix} 2 & 3 & -3 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}.$$

To find the eigenvalues, we start by computing

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 2 & 3 & -3 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix} = \begin{bmatrix} \lambda - 2 & -3 & 3 \\ 0 & \lambda - 1 & 0 \\ 0 & -2 & \lambda + 1 \end{bmatrix}.$$

Next, we need the determinant of this. Using cofactor expansion along the first column, we find that

$$\det(\lambda I - A) = (\lambda - 2)(\lambda - 1)(\lambda + 1).$$

Setting this equal to zero and solving for λ, we find that the eigenvalues of A are

$$\lambda = 2, \quad \lambda = 1, \quad \lambda = -1.$$

For each of these eigenvalues, we need to find the nullspace of $\lambda I - A$ for the corresponding eigenspaces.

Starting with $\lambda = 2$, we have

$$2I - A = \begin{bmatrix} 0 & -3 & 3 \\ 0 & 1 & 0 \\ 0 & -2 & 3 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 0 & 1 & -1 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenspace is thus all solutions to the system of equations

$$\begin{aligned} y - z &= 0 \\ z &= 0. \end{aligned}$$

This immediately gives us $y = z = 0$. The first column is missing a pivot, so x is free. The eigenspace corresponding to $\lambda = 2$ is thus all vectors of the form

$$t \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}.$$

For $\lambda = 1$, we have

$$1I - A = \begin{bmatrix} -1 & -3 & 3 \\ 0 & 0 & 0 \\ 0 & -2 & 2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The corresponding eigenspace is thus all solutions the system of equations

$$\begin{aligned} x &= 0 \\ y - z &= 0. \end{aligned}$$

Since the third column is missing a pivot, we choose z to be free. Setting $z = t$, we get $x = 0$ and $y = t$. Thus, the eigenspace is all vectors of the form

$$t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

Finally, for $\lambda = -1$, we have

$$-1I - A = \begin{bmatrix} -3 & -3 & 3 \\ 0 & -2 & 0 \\ 0 & -2 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The corresponding eigenspace is all solutions to the system of equations

$$\begin{aligned} x - z &= 0 \\ y &= 0. \end{aligned}$$

Choosing z to be free and setting $z = t$, we get $x = t$ and $y = 0$. Thus, the eigenspace is all vectors of the form

$$t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

◇

Let's pause for a moment to go over some stuff. Notice that in our first example with a 2×2 matrix, $\det(\lambda I - A)$ was a polynomial of degree 2; in our second example with a 3×3 matrix, $\det(\lambda I - A)$ was a polynomial of degree 3. In general, for any $n \times n$ matrix A, $\det(\lambda I - A)$ will be a polynomial of degree n. We call this polynomial the ***characteristic polynomial*** of A. We can then say that the eigenvalues of A are the roots/zeroes of its characteristic polynomial.

Next, we'll note the following fact without proof:

Theorem. *Eigenvectors corresponding to distinct eigenvalues of a matrix are always linearly independent.*

For instance, in our last example, we had three distinct eigenvalues: 2, 1, and -1; with $(1, 0, 0)$, $(0, 1, 1)$, and $(1, 0, 1)$ being corresponding eigenvectors, respectively. You can check that these three vectors are linearly independent. The theorem says that this must be the case, because they were eigenvectors corresponding to distinct eigenvalues of a matrix.

Finally, notice that every time we row-reduced $\lambda I - A$ to find the eigenspaces, we always ended up with a row of zeroes. This should always happen, since $\det(\lambda I - A) = 0$ indicates that the matrix $\lambda I - A$ should be non-invertible. Thus, if you do not get any rows of zeroes when looking for eigenspaces, then you've made a mistake somewhere.

Example. Consider

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}.$$

To find the eigenvalues, first we compute

$$\lambda I - A = \begin{bmatrix} \lambda & 0 & 0 \\ 0 & \lambda & 0 \\ 0 & 0 & \lambda \end{bmatrix} - \begin{bmatrix} 3 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix} = \begin{bmatrix} \lambda - 3 & 0 & -2 \\ -1 & \lambda - 1 & -1 \\ 1 & 0 & \lambda \end{bmatrix}.$$

Next, we compute $\det(\lambda I - A)$ using cofactor expansion along the second column:

$$\begin{aligned} \det(\lambda I - A) &= (\lambda - 1) \begin{vmatrix} \lambda - 3 & -2 \\ 1 & \lambda \end{vmatrix} \\ &= (\lambda - 1)(\lambda^2 - 3\lambda + 2) \\ &= (\lambda - 1)(\lambda - 2)(\lambda - 1). \end{aligned}$$

Thus, we see that the eigenvalues of A are

$$\lambda = 2 \quad \text{and} \quad \lambda = 1.$$

Notice that $\lambda = 1$ is a root of the characteristic polynomial of multiplicity 2. We say that the eigenvalue $\lambda = 1$ has ***algebraic multiplicity*** equal to 2. In general, the algebraic multiplicity of an eigenvalue is its multiplicity as a root of the characteristic polynomial.

Next, let's find the eigenspace corresponding to $\lambda = 2$. To do this, row reduce the matrix $2I - A$:

$$2I - A = \begin{bmatrix} -1 & 0 & -2 \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenspace is thus all solutions to the system of equations

$$\begin{aligned} x + 2z &= 0 \\ y + z &= 0 \end{aligned}$$

Since the third column was missing a pivot, we'll choose z to be the free variable. Setting $z = t$, we get $x = -2t$ and $y = -t$. Thus, the eigenspace of A corresponding to $\lambda = 2$ is all vectors of the form

$$t \begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}$$

In particular, the actual eigenvectors of A corresponding to its eigenvalue 2 are the nonzero multiplies of the vector $(-2, -1, 1)$.

Finally, let's find the eigenspace corresponding to $\lambda = 1$. To do this, row reduce the matrix $1I - A$:

$$1I - A = \begin{bmatrix} -2 & 0 & -2 \\ -1 & 0 & -1 \\ 1 & 0 & 1 \end{bmatrix} \longrightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenspace is thus all solutions to the equation

$$x + 0y + z = 0.$$

Since the second and third columns don't have pivots, we'll choose y and z to be free. Setting $y = s$ and $z = t$, we get $x = -t$. The eigenspace is then all vectors of the form

$$\begin{bmatrix} -t \\ s \\ t \end{bmatrix} = s \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Notice in this example that $\lambda = 1$ had an algebraic multiplicity of 2, and its corresponding eigenspace is 2-dimensional.

Looking at the two eigenspaces that we found, we see that even though our 3×3 matrix had only two distinct eigenvalues, we still managed to find three linearly independent eigenvectors, namely:

$$\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}. \qquad \diamond$$

Example. Consider

$$A = \begin{bmatrix} -1 & 2 & -1 \\ -2 & 3 & -1 \\ -1 & 1 & 0 \end{bmatrix}.$$

We have

$$\lambda I - A = \begin{bmatrix} \lambda + 1 & -2 & 1 \\ 2 & \lambda - 3 & 1 \\ 1 & -1 & \lambda \end{bmatrix}.$$

Doing cofactor expansion along the first row,

$$\begin{aligned} \det(\lambda I - A) &= (\lambda + 1) \begin{vmatrix} \lambda - 3 & 1 \\ -1 & \lambda \end{vmatrix} + 2 \begin{vmatrix} 2 & 1 \\ 1 & \lambda \end{vmatrix} + 1 \begin{vmatrix} 2 & \lambda - 3 \\ 1 & -1 \end{vmatrix} \\ &= (\lambda + 1)(\lambda^2 - 3\lambda + 1) + 2(2\lambda - 1) + [-2 - (\lambda - 3)] \\ &= \lambda^3 - 2\lambda^2 + \lambda \\ &= \lambda(\lambda^2 - 2\lambda + 1) \\ &= \lambda(\lambda - 1)^2. \end{aligned}$$

Thus, we see that the eigenvalues are

$$\lambda = 0 \quad \text{and} \quad \lambda = 1,$$

and that $\lambda = 1$ has an algebraic multiplicity of 2.

For $\lambda = 0$, we have

$$0I - A = \begin{bmatrix} 1 & -2 & 1 \\ 2 & -3 & 1 \\ 1 & -1 & 0 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This gives us $x = 0$ and $y = z$. If we set $z = t$, then we find the corresponding eigenspace to be all vectors of the form

$$t \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}.$$

For $\lambda = 1$, the eigenvalue with an algebraic multiplicity of 2, we have

$$1I - A = \begin{bmatrix} 2 & -2 & 1 \\ 2 & -2 & 1 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & -1 \\ 0 & 0 & 0 \end{bmatrix}.$$

This gives us $z = 0$ and $x = y$. Since the second column is missing a pivot, we'll set $y = t$. Then the corresponding eigenspace is all vectors of the form

$$t \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}.$$

◇

Let's focus on the last two examples. Both involved a 3×3 matrix that had an eigenvalue of algebraic multiplicity 2. In the first one, the corresponding eigenspace was 2-dimensional, and overall, we were able to find 3 linearly independent eigenvectors. However, in the second of the two examples, the eigenvalue of algebraic multiplicity 2 yielded an eigenspace which was only 1-dimensional, and so overall, we were only able to find 2 linearly independent eigenvectors.

The dimension of the eigenspace corresponding to an eigenvalue is referred to as the ***geometric multiplicity*** of the eigenvalue. Thus, from the last two examples, we see that an eigenvalue with algebraic multiplicity 2 could have a geometric multiplicity of 1 or 2.

In general, the geometric multiplicity of an eigenvalue is always less than or equal to the algebraic multiplicity. For an $n \times n$ matrix, if all of the geometric multiplicities are equal to the corresponding algebraic multiplicities, then the matrix will yield n linearly independent eigenvectors. Otherwise, if some of the geometric multiplicities are smaller than the corresponding algebraic multiplicities, then the matrix will yield fewer than n linearly independent eigenvectors. This distinction is important for the next topic that we discuss.

One final comment. As you probably know, it is possible for the roots of a polynomial to be imaginary or complex. Thus, it is possible that a matrix has complex eigenvalues. At the time of my writing this, I do not plan to go into this, since currently, the linear algebra course here at UTD doesn't really do much with complex eigenvalues.

8.3 Diagonalization

8.3.1 Quick Review

Before we talk about what the heck diagonalization is, I want to note/review some facts about basic matrix multiplication. Let P be a matrix whose columns we will denote as $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_n$. We can express P as

$$P = [\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \cdots \;\; \mathbf{v}_n].$$

First, recall that when multiplying two matrices A and P, the j^{th} column of the product AP is equal to A times the j^{th} column of P. Symbolically, we can express this by saying

$$AP = A[\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \cdots \;\; \mathbf{v}_n] = [A\mathbf{v}_1 \;\; A\mathbf{v}_2 \;\; \cdots \;\; A\mathbf{v}_n].$$

As a small illustration, let

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \qquad\qquad P = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix}.$$

Multiplying A with each column of P, we have

$$\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} 4 \\ 1 \end{bmatrix} \qquad\qquad \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 3 \\ 2 \end{bmatrix},$$

while

$$AP = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} = \begin{bmatrix} 4 & 3 \\ 1 & 2 \end{bmatrix}.$$

You can see that the columns of AP are obtained by multiplying A with each column of P. If you actually do the computation to verify, you can probably see how this generalizes beyond this specific example.

There is one more fact that I want to note. If D is a diagonal matrix with diagonal entries $\lambda_1, \lambda_2, \ldots, \lambda_n$, then

$$PD = [\lambda_1\mathbf{v}_1 \;\; \lambda_2\mathbf{v}_2 \;\; \cdots \;\; \lambda_n\mathbf{v}_n].$$

In words, the first column of PD is the first diagonal entry of D times the first column of P, and similar for every other column. Here is a simple example to illustrate. Let

$$P = \begin{bmatrix} 3 & 1 \\ 1 & 2 \end{bmatrix} \qquad\qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 4 \end{bmatrix}.$$

Then

$$PD = \begin{bmatrix} 6 & 4 \\ 2 & 8 \end{bmatrix}.$$

You can see that the first column of PD is the first diagonal entry of D, namely 2, times the first column of P. Similarly, the second column of PD is the second diagonaly entry of D, namely 4, times the second column of P. Again, if you actually do the computations to verify this, you can probably see how this generalizes beyond this specific example as well.

8.3.2 Diagonalization

Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ (some of these eigenvalues may be repeated). Let $\mathbf{v}_1, \ldots, \mathbf{v}_n$ be corresponding eigenvectors, and let's put these eigenvectors into a matrix P as columns:

$$P = [\mathbf{v}_1 \;\; \mathbf{v}_2 \;\; \cdots \;\; \mathbf{v}_n].$$

Let's also put the eigenvalues into a diagonal matrix D along the diagonal:

$$D = \begin{bmatrix} \lambda_1 & 0 & \cdots & 0 \\ 0 & \lambda_2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \lambda_n \end{bmatrix}$$

Note that by definition of eigenvalues and eigenvectors, we have

$$A\mathbf{v}_1 = \lambda_1 \mathbf{v}_1 \qquad A\mathbf{v}_2 = \lambda_2 \mathbf{v}_2 \qquad \cdots \qquad A\mathbf{v}_n = \lambda_n \mathbf{v}_n.$$

Combining this with the two facts from the previous page, we have

$$\begin{aligned} AP &= A[\mathbf{v}_1 \ \ \mathbf{v}_2 \ \ \cdots \ \ \mathbf{v}_n] \\ &= [A\mathbf{v}_1 \ \ A\mathbf{v}_2 \ \ \cdots \ \ A\mathbf{v}_n] \\ &= [\lambda_1\mathbf{v}_1 \ \ \lambda_2\mathbf{v}_2 \ \ \cdots \lambda_n\mathbf{v}_n] \\ &= PD. \end{aligned}$$

Focusing on the beginning and end of this chain of equalities, we have that

$$AP = PD,$$

where the columns of P are eigenvectors of A, and D is diagonal with the corresponding eigenvalues along its diagonal. The chain of equalities shows that the converse is also true: that is, if $AP = PD$ where D is diagonal, then the columns of P are necessarily eigenvectors of A, and the diagonal entries of D are necessarily corresponding eigenvalues.

Now, if the eigenvectors are linearly independent, that would make the matrix P invertible. We could then multiply both sides of the above equation by P^{-1} on the left, or on the right. This would yield the following equivalent equations:

$$P^{-1}AP = D \qquad \Longleftrightarrow \qquad A = PDP^{-1}.$$

The last equation, $A = PDP^{-1}$, gives us a factorization/decomposition of A in terms of its eigenvalues and eigenvectors, and motivates the following:

Definition. We say that an $n \times n$ matrix A is ***diagonalizable*** if there exists an invertible matrix P and a diagonal matrix D such that

$$A = PDP^{-1}.$$

We summarize our discussion succinctly in the following theorem.

Theorem. An $n \times n$ matrix A is diagonalizable precisely when it has n linearly independent eigenvectors. The columns of the matrix P will be a set of n such linearly independent eigenvectors, and the diagonal entries of the diagonal matrix D are the corresponding eigenvalues.

Regarding some terminology, to ***diagonalize*** a matrix refers to the act of finding its eigenvalues and eigenvectors in order to find a matrix P and diagonal matrix D such that $A = PDP^{-1}$.

We remarked in the previous section, Section 7.2, that an $n \times n$ matrix will have n linearly independent eigenvectors precisely when the geometric multiplicity of any of its eigenvalues equals the corresponding algebraic multiplicity. Thus, we have

Theorem. An $n \times n$ matrix is diagonalizable if and only if the geometric multiplicity of each of its eigenvalues equals the corresponding algebraic multiplicity.

We also remarked in the previous section that the eigenvectors corresponding to distinct eigenvalues are always linearly independent. This gives us another theorem:

Theorem. If an $n \times n$ matrix A has n distinct eigenvalues, then A is diagonalizable.

Note that if A does not have n distinct eigenvalues, it does NOT mean that A is not diagonalizable. What matters is that A has n linearly independent eigenvectors, and we know this can happen even when A has fewer than n distinct eigenvalues.

Example. In the previous section, we saw that the matrix

$$A = \begin{bmatrix} -1 & -2 \\ 4 & 5 \end{bmatrix}$$

has eigenvalues $\lambda = 3$ and $\lambda = 1$. As a 2×2 matrix with 2 distinct eigenvalues, A is diagonalizable. The eigenspace corresponding to $\lambda = 3$ was found to be the span of the vector

$$\begin{bmatrix} 1 \\ -2 \end{bmatrix},$$

while the eigenspace corresponding to $\lambda = 1$ was found to be the span of the vector

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

If we let

$$P = \begin{bmatrix} 1 & 1 \\ -2 & 1 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 3 & 0 \\ 0 & 1 \end{bmatrix},$$

then we have $A = PDP^{-1}$. You can compute P^{-1} and verify that this is the case. $\diamond$

Example. In the previous section, we saw that the matrix

$$A = \begin{bmatrix} 2 & 3 & -3 \\ 0 & 1 & 0 \\ 0 & 2 & -1 \end{bmatrix}$$

has eigenvalues $\lambda = 2$, $\lambda = 1$, and $\lambda = -1$. As a 3×3 matrix with 3 distinct eigenvalues, A is diagonalizable. The eigenspace corresponding to $\lambda = 2$ was found to be the span of the vector

$$\begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix};$$

the eigenspace corresponding to $\lambda = 1$ was found to be the span of the vector

$$\begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix};$$

and the eigenspace corresponding to $\lambda = -1$ was found to be the span of the vector

$$\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

Let

$$P = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

You can check that the inverse of P is

$$\begin{bmatrix} 1 & 1 & -1 \\ 0 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix},$$

and then verify that the equation $A = PDP^{-1}$ is indeed satisfied.

Note that the order in which we place the eigenvectors into P doesn't matter, so long as the order of the eigenvalues in D corresponds to the order of the eigenvectors in P. For instance, if we swap the first and third columns of P, that's okay as long as swap the first and third diagonal entries of D.

Also, we could have used the $(2,0,0)$ or $(-2020,0,0)$ as an eigenvector corresponding to $\lambda = 2$ instead of (1,0,0). Any nonzero multiple of $(1,0,0)$ would have worked. Similar for the other eigenvalues. Thus, for instance,

$$P = \begin{bmatrix} 2 & 0 & 4 \\ 0 & 3 & 0 \\ 0 & 3 & 4 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

would also diagonalize A, in the sense that we would have $A = PDP^{-1}$. ◇

Example. In the previous section, we saw that the matrix

$$A = \begin{bmatrix} 3 & 0 & 2 \\ 1 & 1 & 1 \\ -1 & 0 & 0 \end{bmatrix}$$

has only two distinct eigenvalues: $\lambda = 2$ and $\lambda = 1$. The eigenvalue $\lambda = 1$ had an algebraic multiplicity of 2. The eigenspace corresponding to $\lambda = 2$ was found to be the span of the vector

$$\begin{bmatrix} -2 \\ -1 \\ 1 \end{bmatrix},$$

while the eigenspace corresponding to $\lambda = 1$ was found to be the span of the vectors

$$\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} \qquad \text{and} \qquad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

Although A is a 3×3 matrix with only two distinct eigenvalues, it still yields 3 linearly independent eigenvectors and is thus diagonalizable. If we let

$$P = \begin{bmatrix} -2 & 0 & -1 \\ -1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \qquad \text{and} \qquad D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$

then again the equation $A = PDP^{-1}$ will be satisfied. You can verify that

$$P^{-1} = \begin{bmatrix} -1 & 0 & -1 \\ -1 & 1 & -1 \\ 1 & 0 & 2 \end{bmatrix},$$

and then multiply out PDP^{-1} and confirm that it does equal A. ◇

Example. In the previous section, we saw that the matrix

$$A = \begin{bmatrix} -1 & 2 & -1 \\ -2 & 3 & -1 \\ -1 & 1 & 0 \end{bmatrix}$$

has only two distinct eigenvalues: $\lambda = 0$ and $\lambda = 1$. We also saw that the corresponding eigenspaces were each 1-dimensional, so we were only able to find 2 linearly independent eigenvectors. Since A is 3×3 but does not yield 3 linearly independent eigenvectors, it follows that A is not diagonalizable. ◇

Diagonal matrices are extremely easy to work with. One sense in which this is true is when multiplying diagonal matrices together - you just multiply the diagonal entries together. For instance, you can verify that

$$\begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix} = \begin{bmatrix} ac & 0 \\ 0 & bd \end{bmatrix}$$

is true by doing the matrix multiplication. As a consequence of this, raising a diagonal matrix to a power is also easy - just raise each diagonal entry to that power, i.e. if

$$D = \begin{bmatrix} \lambda_1 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n \end{bmatrix},$$

then

$$D^k = \begin{bmatrix} \lambda_1^k & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & \lambda_n^k \end{bmatrix},$$

Think about how complicated it is to raise an arbitrary matrix to a high power: to compute A^2, you have to compute A times A; to compute A^{50}, you'll have to do quite a few matrix multiplications.

If a matrix is diagonalizable, then it turns out we can also easily raise it to a high power. To see how this is so, suppose A is diagonalizable with $A = PDP^{-1}$. Notice that

$$A^2 = A \cdot A = PDP^{-1} \cdot PDP^{-1} = PD^2P^{-1}.$$

Similarly, one finds that for any k,

$$A^k = PD^kP^{-1}.$$

Since computing D^k is relatively easy, this makes computing A^k relatively easy well. There are many applications, for instance some algorithms for solving certain equations, that involve needing to raise a matrix to a high power. In such applications, working with diagonalizable matrices makes things much easier.

8.4 Miscallaneous Eigen-Related Facts

8.4.1 The Trace of a Square Matrix

For an $n \times n$ matrix A, we define the ***trace*** of A, denoted $\mathrm{Tr}(A)$ or simply $\mathrm{Tr}A$, to be the sum of the diagonal entries of A.

Example. If

$$A = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

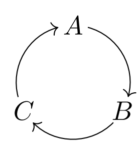
then

$$\mathrm{Tr}(A) = a + e + i. \qquad \diamond$$

Properties of Trace

- If A and B are $n \times n$, then $\mathrm{Tr}(A + B) = \mathrm{Tr}(A) + \mathrm{Tr}(B)$.
- If A is $n \times n$, then $\mathrm{Tr}(cA) = c \cdot \mathrm{Tr}(A)$.
- If A and B are matrices, not necessarily square, for which AB and BA are square, then $\mathrm{Tr}(AB) = \mathrm{Tr}(BA)$.
- If A, B, and C are matrices, not necessarily square, for which ABC, CAB, and BCA are all square, then $\mathrm{Tr}(ABC) = \mathrm{Tr}(CAB) = \mathrm{Tr}(BCA)$.

This last property is referred to as the cyclic invariance of trace, since one can remember the ordering of the letters A, B, C in the three products by arranging them into a cycle, with B following A, C following B, and A following C.



8.4.2 Relation between Eigenvalues, Determinant, and Trace

Theorem. We have the following facts relating the eigenvalues of a matrix to its determinant and trace.

- The determinant of a matrix is equal to the product of its eigenvalues.
- The trace of a matrix is equal to the sum of its eigenvalues.

Both statements can be proved by looking at the characteristic polynomial of a matrix, which we will be denoting as $p(\lambda)$.

Proof. Let's start by proving the first statement. Let A be an $n \times n$ matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$. Considering the character polynomial $p(\lambda) = \det(\lambda I - A)$, notice that if we evaluate $p(\lambda)$ at $\lambda = 0$, we get

$$p(0) = \det(0I - A) = \det(-A) = (-1)^n \det(A).$$

On the other hand, since the eigenvalues of A are the roots of its characteristic polynomial,

$$p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \cdot \ldots \cdot (\lambda - \lambda_n).$$

Using the above expression to evaluate $p(\lambda)$ at $\lambda = 0$ again, we get

$$p(\lambda) = (-\lambda_1)(-\lambda_2) \cdot \ldots \cdot (-\lambda_n) = (-1)^n \lambda_1 \lambda_2 \cdot \ldots \cdot \lambda_n.$$

Thus, we have

$$p(0) = (-1)^n \det(A) \qquad \text{and} \qquad p(0) = (-1)^n \lambda_1 \cdot \ldots \cdot \lambda_n.$$

From this, it follows that $\det(A) = \lambda_1 \cdot \ldots \cdot \lambda_n$, i.e. the determinant of A is equal to the product of its eigenvalues.

Proving that the trace of a matrix is equal to the sum of its eigenvalues is a little bit harder. Let us illustrate the idea by examining the 2×2 case first.

First, consider

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

By computing $\det(\lambda I - A)$, one finds that

$$p(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc),$$

or equivalently,

$$p(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A).$$

On the other hand, if λ_1, λ_2 are the eigenvalues of A, then

$$\begin{aligned} & p(\lambda) = (\lambda - \lambda_1)(\lambda - \lambda_2) \\ \implies \quad & p(\lambda) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2. \end{aligned}$$

Thus, we have the following two expressions for $p(\lambda)$:

$$p(\lambda) = \lambda^2 - \mathrm{Tr}(A)\lambda + \det(A) \qquad \text{and} \qquad p(\lambda) = \lambda^2 - (\lambda_1 + \lambda_2)\lambda + \lambda_1\lambda_2.$$

Since these expressions are equal to each other, the coefficients of like powers of λ must coincide. In particular, the coefficients of λ, which are $-\mathrm{Tr}(A)$ and $-(\lambda_1 + \lambda_2)$, must be equal to each other. This proves that the trace of A is equal to the sum of its eigenvalues.

In the general $n \times n$ case, when you compute $p(\lambda)$, you'll find that

$$p(\lambda) = \lambda^n - \mathrm{Tr}(A)\lambda^{n-1} + \text{ other stuff...}$$

On the other hand, if $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A, then we know that

$$p(\lambda) = (\lambda - \lambda_1) \cdot \ldots \cdot (\lambda - \lambda_n).$$

If you begin to foil this out, you'll find that

$$p(\lambda) = \lambda^n - (\lambda_1 + \ldots + \lambda_n)\lambda^{n-1} + \text{ other stuff...}$$

Again, we end up with two polynomial expressions for $p(\lambda)$, and since the expressions are equal to each other, the coefficients of like powers of λ must coincide. The coefficients of λ^{n-1} are $-\mathrm{Tr}(A)$ and $-(\lambda_1 + \ldots + \lambda_n)$, and so equating these to each other once again gives us that the trace of A is equal to the sum of its eigenvalues. $\square$

Note that both statements hold even if the eigenvalues are complex.

One consequence of the fact that $\det(A)$ is the product of its eigenvalues is that $\det(A) = 0$ if and only if 0 is an eigenvalue of A. Thus, A is invertible if and only if all of its eigenvalues are nonzero.

8.4.3 Polynomials of Matrices

Let A be a square matrix. For any polynomial

$$f(t) = b_m t^m + b_{m-1}t^{m-1} + \ldots + b_1 t + b_0,$$

we define

$$f(A) = b_m A^m + b_{m-1}A^{m-1} + \ldots + b_1 A + b_0 I.$$

Note that if A is $n \times n$, then so too is any polynomial of A.

Example. Let

$$A = \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} \qquad \text{and} \qquad f(t) = t^2 - 3t + 2.$$

Then

$$\begin{aligned} f(A) &= \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}^2 - 3\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} + 2\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 0 & 3 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 8 \\ 0 & 9 \end{bmatrix} - \begin{bmatrix} 3 & 6 \\ 0 & 9 \end{bmatrix} + \begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 0 & 2 \\ 0 & 2 \end{bmatrix}. \end{aligned}$$

◇

Theorem. Let A be a square matrix, and $f(t)$ some polynomial. If $\mathbf{x}$ is an eigenvector of A with corresponding eigenvalue λ, then $\mathbf{x}$ is also an eigenvector of $f(A)$ with corresponding eigenvalue $f(\lambda)$.

Proof. Since $\mathbf{x}$ is an eigenvector with corresponding eigenvalue λ, we have $A\mathbf{x} = \lambda\mathbf{x}$. Multiplying both sides on the left by A, we get

$$A^2 x = \lambda \cdot A\mathbf{x} = \lambda \cdot \lambda\mathbf{x} = \lambda^2\mathbf{x}.$$

Similar reasoning shows that in general, $A^k\mathbf{x} = \lambda^k\mathbf{x}$.

Now suppose

$$f(t) = b_m t^m + b_{m-1}t^{m-1} + \ldots + b_1 t + b_0.$$

Then

$$\begin{aligned} f(A)\mathbf{x} &= (b_m A^m + b_{m-1}A^{m-1} + \ldots + b_1 A + b_0 I)\mathbf{x} \\ &= b_m A^m\mathbf{x} + b_{m-1}A^{m-1}\mathbf{x} + \ldots + b_1 A\mathbf{x} + b_0 I\mathbf{x} && \text{(Distributed } \mathbf{x}) \\ &= b_m\lambda^m\mathbf{x} + b_{m-1}\lambda^{m-1}\mathbf{x} + \ldots + b_1\lambda\mathbf{x} + b_0\mathbf{x} && (A^k\mathbf{x} = \lambda^k\mathbf{x}) \\ &= (b_m\lambda^m + b_{m-1}\lambda^{m-1} + \ldots + b_1\lambda + b_0)\mathbf{x} && \text{(Factored } \mathbf{x} \text{ back out)} \\ &= f(\lambda)\mathbf{x}. \end{aligned}$$

□

Corollary. Let A be a square matrix and $f(t)$ some polynomial. If λ is an eigenvalue of A, then $f(\lambda)$ is an eigenvalue of $f(A)$.

Example. Suppose A is 3×3 with eigenvalues 2, 1, and 0. What are the eigenvalues of $3A + 4I$? According to the corollary, the eigenvalues would be

$$\begin{aligned} 3(2) + 4 &= 10 \\ 3(1) + 4 &= 7 \\ 3(0) + 4 &= 4. \end{aligned}$$

◇

Example. Suppose A is 3×3 with eigenvalues 3,0, and -2. What are the eigenvalues of $A^2 - 5A + I$? According to the corollary, the eigenvalues would be

$$\begin{aligned} (3)^2 - 5(3) + 1 &= -5 \\ (0)^2 - 5(0) + 1 &= 1 \\ (-2)^2 - 5(-2) + 1 &= 15. \end{aligned}$$

◇

Theorem (Cayley-Hamilton). Let A be an $n \times n$ matrix and let $p(\lambda)$ be its characteristic polynomial. Then $p(A) = 0$. (Note: The "0" is denoting the $n \times n$ zero matrix.)

I think the proof of this result is a bit too advanced for the level of MATH 2418 at UTD, plus this theorem doesn't seem to be important in the course, so I will omit the proof.

Example. In Section 8.2, we found the characteristic polynomial of

$$A = \begin{bmatrix} -1 & -2 \\ 4 & 5 \end{bmatrix}$$

to be $\lambda^2 - 4\lambda + 3$. According to the Cayley-Hamilton Theorem, $A^2 - 4A + 3I$ should be equal to the 2×2 zero matrix. Let's confirm that this is the case.

$$A^2 - 4A + 3I = \begin{bmatrix} -7 & -8 \\ 16 & 17 \end{bmatrix} - 4 \begin{bmatrix} -1 & -2 \\ 4 & 5 \end{bmatrix} + 3 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix} \qquad \diamond$$

8.5 Matrix Exponential

8.5.1 Definition

Recall from Calculus II the Taylor series for e^x centered at 0:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + x + \frac{x^2}{2} + \frac{x^3}{3!} + \dots \tag{8.2}$$

This series converges to e^x for all x.

Similarly, we define e^A for any square matrix A by the following:

$$e^A = \sum_{n=0}^{\infty} \frac{A^n}{n!} = I + A + \frac{1}{2}A^2 + \frac{1}{3!}A^3 + \dots$$

This series also converges for any square matrix A; note that e^A itself will be a square matrix, the same size as A.

So how can we compute e^A? It requires being able to raise the matrix A to any power. As we touched on in the last section, this can be difficult in general, but for diagonalizable matrices, e^A is relatively easy to compute. Let us first consider the case where $A = D$ is already diagonal. Let's also suppose D is 2×2 for convenience:

$$D = \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix}.$$

Then

$$\begin{aligned} e^D &= I + D + \tfrac{1}{2}D^2 + \dots \\ &= \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + \begin{bmatrix} d_1 & 0 \\ 0 & d_2 \end{bmatrix} + \frac{1}{2}\begin{bmatrix} d_1^2 & 0 \\ 0 & d_2^2 \end{bmatrix} + \dots \end{aligned}$$

Adding the matrices on the right gives us

$$e^D = \begin{bmatrix} 1 + d_1 + \frac{1}{2}d_1^2 + \dots & 0 \\ 0 & 1 + d_2 + \frac{1}{2}d_2^2 + \dots \end{bmatrix} = \begin{bmatrix} e^{d_1} & 0 \\ 0 & e^{d_2} \end{bmatrix},$$

where the last equality comes from (8.2). In general, for diagonal matrices we have

$$D = \begin{bmatrix} d_1 & & & \\ & d_2 & & \\ & & \ddots & \\ & & & d_n \end{bmatrix} \quad \Longrightarrow \quad e^D = \begin{bmatrix} e^{d_1} & & & \\ & e^{d_2} & & \\ & & \ddots & \\ & & & e^{d_n} \end{bmatrix}.$$

Now, if A is diagonalizable with $A = PDP^{-1}$, then $A^n = PD^nP^{-1}$, and so

$$\begin{aligned} e^A &= I + A + \tfrac{1}{2}A^2 + \dots \\ &= PP^{-1} + PDP^{-1} + \tfrac{1}{2}PD^2P^{-1} + \dots \end{aligned}$$

We can factor P out towards the left, and P^{-1} out towards the right. Then

$$e^A = P[I + D + \tfrac{1}{2}D^2 + \dots]P^{-1} = Pe^DP^{-1}.$$

In summary, if A is diagonalizable with $A = PDP^{-1}$ then

$$e^A = Pe^DP^{-1}.$$

8.5.2 Review of some Differential Equations

Consider the differential equation

$$\frac{dx}{dt} = ax. \tag{8.3}$$

Here, a is a constant. To solve this, first divide both sides by x:

$$\frac{1}{x}\frac{dx}{dt} = a.$$

Then integrate both sides with respect to t to get:

$$\ln|x| = at + C.$$

Here, C is an arbitrary constant of integration. Solving for x, we get

$$|x| = e^{at+C} = e^{at}e^{C} = ce^{at}.$$

Notice that e^C is just a constant, and so we renamed it c. We can drop the absolute value by allowing c to be positive or negative. Thus, the general solution to the differential equation (8.3) is

$$x(t) = ce^{at}.$$

Notice that $x(0) = c$, so if we are given an initial condition $x(0) = x_0$, then the particular solution becomes

$$x(t) = x_0 e^{at}.$$

8.5.3 Solving a System of Diff. Eqs. via Diagonalization

Let's start using the notation $\frac{dx}{dt} = \dot{x}$ for convenience. Consider the system of differential equations

$$\begin{aligned}\dot{x}_1 &= ax_1 + bx_2\\ \dot{x}_2 &= cx_1 + dx_2.\end{aligned}$$

Here, x_1 and x_2 are unknown functions which we are solving for, and a, b, c, d are constants. If we let

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \qquad A = \begin{bmatrix} a & b \\ c & d \end{bmatrix},$$

then we can rewrite the above system of differential equations as

$$\dot{\mathbf{x}} = A\mathbf{x}. \tag{8.4}$$

Notice the similarity to (8.3). Although we are not able to solve (8.44) the same way we did (8.3), it turns out that the solution to (8.4) will be very similar to the solution to (8.3).

We will assume that A is diagonalizable with $A = PDP^{-1}$. Thus, equation (4) can be rewritten as

$$\dot{\mathbf{x}} = PDP^{-1}\mathbf{x}.$$

It would look nice if we multiplied both sides of this equation on the left by P^{-1}, so we'll do that next:

$$P^{-1}\dot{\mathbf{x}} = DP^{-1}\mathbf{x}.$$

If we now set

$$\mathbf{y} = P^{-1}\mathbf{x},$$

then $\dot{\mathbf{y}} = P^{-1}\dot{\mathbf{x}}$, and our equation becomes

$$\dot{\mathbf{y}} = D\mathbf{y}.$$

We now write this out more explicitly as

$$\begin{bmatrix} \dot{y}_1 \\ \dot{y}_2 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix},$$

which gives us the following system of differential equations:

$$\dot{y}_1 = \lambda_1 y_1 \qquad \dot{y}_2 = \lambda_2 y_2.$$

Each of these equations has the form $\dot{x} = ax$ as in (3), so we know the solution has the form $x = ce^{at}$. Thus,

$$y_1(t) = c_1 e^{\lambda_1 t} \qquad y_2(t) = c_2 e^{\lambda_2 t}.$$

Putting everything back into a vector gives us

$$\mathbf{y}(t) = \begin{bmatrix} y_1(t) \\ y_2(t) \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix}.$$

We can rewrite this as

$$\mathbf{y}(t) = \begin{bmatrix} e^{\lambda_1 t} & 0 \\ 0 & e^{\lambda_2 t} \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Notice that the diagonal matrix is just e^{Dt}, so we can rewrite this as

$$\mathbf{y}(t) = e^{Dt}\,\mathbf{c},$$

where $\mathbf{c} = (c_1, c_2)$. Since $\mathbf{y} = P^{-1}\mathbf{x}$, we can now solve for $\mathbf{x}$ as $\mathbf{x} = P\mathbf{y}$:

$$\mathbf{x}(t) = Pe^{Dt}\,\mathbf{c}.$$

Let's rewrite this as

$$\mathbf{x}(t) = Pe^{Dt}P^{-1}\,P\mathbf{c}.$$

Now, we saw earlier that $Pe^{Dt}P^{-1}$ is just e^{At}, and since $\mathbf{c}$ is just an arbitrary constant, so too is $P\mathbf{c}$. Renaming $P\mathbf{c}$ as $\mathbf{c}$, we find the general solution to (4) has the following form:

$$\mathbf{x}(t) = e^{At}\,\mathbf{c}.$$

This is extremely similar to what we found for the general solution to (3)! And again, notice that $\mathbf{x}(0) = \mathbf{c}$, so if we are given an initial condition $\mathbf{x}(0) = \mathbf{x}_0$, then the particular solution is

$$\mathbf{x}(t) = e^{At}\,\mathbf{x}_0.$$

In summary, the general solution to the differential equation

$$\dot{\mathbf{x}} = A\mathbf{x}$$

has the form

$$\mathbf{x}(t) = e^{At}\,\mathbf{c}$$

for an arbitrary constant vector $\mathbf{c}$. The particular solution to the initial value problem

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \mathbf{x}_0$$

is given by

$$\mathbf{x}(t) = e^{At}\,\mathbf{x}_0.$$

8.5.4 Example

Example. Consider the initial value problem (ivp)

$$\dot{\mathbf{x}} = A\mathbf{x}, \qquad \mathbf{x}(0) = \begin{bmatrix} 1 \\ 2 \end{bmatrix},$$

where

$$A = \begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

As discussed on the previous page, the solution to this ivp is given by

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0.$$

We are given that $\mathbf{x}_0 = (1, 2)$. Now we just need

$$e^{At} = Pe^{Dt}P^{-1},$$

and thus we need to find the eigenvalues and eigenvectors of A.

Starting with the eigenvalues, we have

$$\det(\lambda I - A) = \begin{vmatrix} \lambda - 2 & -1 \\ -1 & \lambda - 2 \end{vmatrix} = (\lambda - 2)^2 - 1.$$

Set this equal to zero, and solve for λ by moving the one over and then square rooting, to get

$$\lambda = 2 \pm 1 \qquad \Longrightarrow \qquad \lambda = 1, 3.$$

Now we find the eigenspace corresponding to $\lambda = 1$:

$$1I - A = \begin{bmatrix} -1 & -1 \\ -1 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

From the RREF, we find the eigenspace corresponding to $\lambda = 1$ consists of all solutions to the equation

$$x_1 + x_2 = 0.$$

Choosing x_2 to be free, we find the eigenspace is given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

We will use the eigenvector $(-1, 1)$ for the first column of P.

Similarly, when $\lambda = 3$, we have

$$3I - A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

The corresponding eigenspace is given by

$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = t \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

We will use the eigenvector $(1, 1)$ for the second column of P.

Thus, we have

$$P = \begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix} \qquad \Longrightarrow \qquad P^{-1} = \frac{1}{2}\begin{bmatrix} -1 & 1 \\ 1 & 1 \end{bmatrix}$$

and

$$D = \begin{bmatrix} 1 & 0 \\ 0 & 3 \end{bmatrix} \qquad \Longrightarrow \qquad e^{Dt} = \begin{bmatrix} e^t & 0 \\ 0 & e^{3t} \end{bmatrix}.$$

Then

$$e^{At} = Pe^{Dt}P^{-1} = \begin{bmatrix} -e^t + e^{3t} & -e^t + e^{3t} \\ e^t + e^{3t} & e^t + e^{3t} \end{bmatrix},$$

and the solution to our ivp is

$$\mathbf{x}(t) = e^{At}\mathbf{x}_0 = \begin{bmatrix} -e^t + e^{3t} & -e^t + e^{3t} \\ e^t + e^{3t} & e^t + e^{3t} \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix}.$$

Multiplying everything out gives us

$$\mathbf{x}(t) = \begin{bmatrix} -3e^t + 3e^{3t} \\ 3e^t + 3e^{3t} \end{bmatrix},$$

or, if one prefers,

$$x_1(t) = -3e^t + 3e^{et} \qquad \text{and} \qquad x_2(t) = 3e^t + 3e^{3t}.$$ $\diamond$

8.6 Orthogonal Matrices

Let Q be an $n \times n$ matrix. We say that Q is an orthogonal matrix if it satisfies $Q^T = Q^{-1}$. It turns out that the following are equivalent:

1. Q is orthogonal.
2. $Q^T Q = I$.
3. $Q^T = Q^{-1}$.
4. The columns of Q form an orthonormal basis for $\mathbb{R}^n$.
5. The rows of Q form an orthonormal basis for $\mathbb{R}^n$.

There is a lot more that can be said, but I think that should be good.

Here are some properties of orthogonal matrices. Recall that if $\mathbf{u}, \mathbf{v}$ are vectors in $\mathbb{R}^n$, then the dot product $\mathbf{u} \cdot \mathbf{v}$ can be expressed as matrix multiplication by $\mathbf{u} \cdot \mathbf{v} = \mathbf{u}^T \mathbf{v}$.

a. Orthogonal matrices preserve the dot product, i.e. if Q is orthogonal, then $Q\mathbf{u} \cdot Q\mathbf{v} = \mathbf{u} \cdot \mathbf{v}$ for any $\mathbf{u}, \mathbf{v}$.

   Proof.

$$\begin{aligned} Q\mathbf{u} \cdot Q\mathbf{v} &= (Q\mathbf{u})^T (Q\mathbf{v}) \\ &= \mathbf{u}^T Q^T Q \mathbf{v} \\ &= \mathbf{u}^T I \mathbf{v} \\ &= \mathbf{u}^T \mathbf{v} \\ &= \mathbf{u} \cdot \mathbf{v}. \end{aligned}$$

   □

b. Orthogonal matrices preserve length, i.e. $\|Q\mathbf{v}\| = \|\mathbf{v}\|$ for any $\mathbf{v}$.

   Proof. Using the results of Part a,

$$\begin{aligned} \|Q\mathbf{v}\|^2 &= Q\mathbf{v} \cdot Q\mathbf{v} \\ &= \mathbf{v} \cdot \mathbf{v} \\ &= \|\mathbf{v}\|^2. \end{aligned}$$

   □

c. Orthogonal matrices preserve angles, i.e. the angle between $\mathbf{u}$ and $\mathbf{v}$ is the same as the angle between $Q\mathbf{u}$ and $Q\mathbf{v}$.

   Proof. Let α denote the angle between $\mathbf{u}$ and $\mathbf{v}$, and let β denote the angle between $Q\mathbf{u}$ and $Q\mathbf{v}$. We know that

$$\begin{aligned} \mathbf{u} \cdot \mathbf{v} &= \|\mathbf{u}\| \, \|\mathbf{v}\| \, \cos\alpha \\ Q\mathbf{u} \cdot Q\mathbf{v} &= \|Q\mathbf{u}\| \, \|Q\mathbf{v}\| \, \cos\beta. \end{aligned}$$

   From Part a, $Q\mathbf{u} \cdot Q\mathbf{v} = \mathbf{u} \cdot \mathbf{v}$, and so the two expressions are equal to each other:

$$\|\mathbf{u}\| \, \|\mathbf{v}\| \, \cos\alpha = \|Q\mathbf{u}\| \, \|Q\mathbf{v}\| \, \cos\beta.$$

   From Part b, $\|\mathbf{u}\| = \|Q\mathbf{u}\|$ and $\|\mathbf{v}\| = \|Q\mathbf{v}\|$. Cancelling these from both sides of the equation leaves us with

$$\cos\alpha = \cos\beta.$$

   Since α and β are between 0 and π, it follows that $\alpha = \beta$. □

   It turns out that orthogonal matrices are nothing more than rotation matrices and reflection matrices! With this in mind, it makes sense that they would preserve length and angles.

d. The determinant of an orthogonal matrix is 1 or -1.

Proof. If Q is orthogonal, then $Q^TQ = I$. Taking the determinant of both sides gives us

$$\det(Q^TQ) = 1.$$

By properties of determinants,

$$\det(Q^TQ) = \det(Q^T)\det(Q) = \det(Q)^2.$$

Hence,

$$\det(Q)^2 = 1 \implies \det(Q) = \pm 1. \qquad \square$$

It turns out that rotations have determinant equal to 1, and reflections have determinant equal to -1.

e. The eigenvalues of an orthogonal matrix satisfy $|\lambda| = 1$.

Proof. Suppose λ is an eigenvalue of Q, and $\mathbf{x}$ is a corresponding eigenvector, so $Q\mathbf{x} = \lambda\mathbf{x}$. Then

$$\|Q\mathbf{x}\| = \|\lambda\mathbf{x}\| = |\lambda|\|\mathbf{x}\|.$$

From Part b, $\|Q\mathbf{x}\| = \|\mathbf{x}\|$, and so

$$\|\mathbf{x}\| = |\lambda|\|\mathbf{x}\|.$$

It follows that $|\lambda| = 1$. $\square$

The fact that $|\lambda| = 1$ does not mean that $\lambda = \pm 1$, as the eigenvalues might be complex. For instance, rotation matrices in $\mathbb{R}^2$,

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix},$$

have eigenvalues of $e^{\pm i\theta}$.

f. If Q_1, Q_2 are orthogonal matrices, then Q_1Q_2 is also orthogonal.

Proof. We just need to show that $(Q_1Q_2)^T(Q_1Q_2) = I$.

$$\begin{aligned} (Q_1Q_2)^T(Q_1Q_2) &= Q_2^TQ_1^TQ_1Q_2 \\ &= Q_2^TQ_2 \\ &= I. \end{aligned} \qquad \square$$

g. If Q is orthogonal, then so is its inverse.

Proof. By definition, since Q is orthogonal, its inverse is its transpose, so

$$QQ^T = Q^TQ = I.$$

These equations also tell us that the inverse of Q^T is Q, i.e. the inverse of Q^T is its own transpose. Hence, $Q^T = Q^{-1}$ is also orthogonal. $\square$

Orthogonal matrices are needed for the spectral decomposition, which we talk about in the next section.

8.7 The Spectral Decomposition for Symmetric Matrices

In Section 8.5, we computed the eigenvalues and eigenspaces for the matrix

$$\begin{bmatrix} 2 & 1 \\ 1 & 2 \end{bmatrix}.$$

The eigenvalues were 1 and 3, with corresponding eigenspaces

$$t\begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \text{and} \qquad t\begin{bmatrix} 1 \\ 1 \end{bmatrix},$$

respectively (t can be any real number). Notice that our matrix is symmetric, and that the eigenspaces are orthogonal to each other. If you look at the other matrices that we computed the eigenvalues and eigenspaces of, they were not symmetric, and the eigenspaces were not orthogonal to each other. It turns out that this is no coincidence! This is part of the following theorem:

Theorem. Let A be a symmetric matrix. Then the following are all true:

- All eigenvalues of A are real.
- The eigenspaces corresponding to distinct eigenvalues of A are orthogonal to each other.
- A is diagonalizable, and can be diagonalized using n orthonormal eigenvectors.

In fact, the following theorem, known as the Spectral Theorem, says that symmetric matrices are the only (real) matrices satisfying the last property.

Spectral Theorem. An $n \times n$ matrix has n orthonormal eigenvectors if and only if it is symmetric.

Proofs of some of these results are available at the end of this section. If we put the orthonormal eigenvectors into a matrix Q as columns, then we would have $A = QDQ^{-1}$ as usual. But because the columns of Q are orthonormal, Q is in fact an orthogonal matrix, meaning $Q^{-1} = Q^T$. Thus, we have $A = QDQ^T$. This factorization of A is known as a ***spectral decomposition*** of A. Note that a spectral decomposition is only possible for symmetric matrices.

Example. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 1 \\ 0 & -2 & 0 \\ 1 & 0 & 1 \end{bmatrix}.$$

Since A is symmetric, we expect its eigenvalues to be real, and that the eigenspaces corresponding to distinct eigenvalues will be orthogonal to each other. Let's verify that this is the case, and let's also find a spectral decomposition of A while we are at it.

First, the eigenvalues. We start with computing the characteristic polynomial:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda-1 & 0 & -1 \\ 0 & \lambda+2 & 0 \\ -1 & 0 & \lambda-1 \end{vmatrix}.$$

This determinant is easily computed via cofactor expansion along the second column:

$$\det(\lambda I - A) = (\lambda+2)[(\lambda-1)^2 - 1] = \lambda(\lambda+2)(\lambda-2).$$

The eigenvalues are the roots of the characteristic polynomial, so we get $\lambda = -2, 0, 2$.

The eigenspace corresponding to $\lambda = 2$ is the nullspace of $2I - A$, so we compute:

$$2I - A = \begin{bmatrix} 1 & 0 & -1 \\ 0 & 4 & 0 \\ -1 & 0 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenspace is thus the solutions to the following system of equations:

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 &= 0. \end{aligned}$$

Choosing x_3 to be free and letting $x_3 = t$, we find the eigenspace corresponding to $\lambda = 2$ to be

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

The eigenspace corresponding to $\lambda = 0$ is the nullspace of $0I - A$, so we compute:

$$0I - A = \begin{bmatrix} -1 & 0 & -1 \\ 0 & 2 & 0 \\ -1 & 0 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenspace is thus the solutions to the following system of equations:

$$\begin{aligned} x_1 + x_3 &= 0 \\ x_2 &= 0. \end{aligned}$$

Choosing x_3 to be free and letting $x_3 = t$, we find the eigenspace corresponding to $\lambda = 2$ to be

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}, \quad t \in \mathbb{R}.$$

The eigenspace corresponding to $\lambda = -2$ is the nullspace of $-2I - A$, so we compute:

$$-2I - A = \begin{bmatrix} -3 & 0 & -1 \\ 0 & 0 & 0 \\ -1 & 0 & -3 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

The eigenspace is thus the solutions to the following system of equations:

$$\begin{aligned} x_1 &= 0 \\ x_3 &= 0. \end{aligned}$$

In this case, x_2 is free, and we find the eigenspace corresponding to $\lambda = -2$ to be

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad t \in \mathbb{R}.$$

Summarizing our results so far, the eigenvalues of A are $2, 0, -2$, and the corresponding eigenspaces are, respectively: the line spanned by $(1, 0, 1)$; the line spanned by $(-1, 0, 1)$; and the line spanned by $(0, 1, 0)$. Notice that these three vectors are all orthogonal to each other, as expected, since eigenspaces corresponding to distinct eigenvalues should be orthogonal to each other when the matrix is symmetric. Let us normalize these vectors to obtain an orthonormal set of eigenvectors of A:

$$\mathbf{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \mathbf{q}_2 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \mathbf{q}_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

If we put these into a matrix Q as columns, then we obtain the spectral decomposition of A:

$$A = QDQ^T,$$

where D is diagonal with the eigenvalues $2, 0, -2$ for its diagonal entries. ◇

Note that if a symmetric matrix has repeated eigenvalues, then in order to obtain an orthonormal set of eigenvectors, one might need to perform Gram-Schmidt, as the next example illustrates.

Example. Consider the symmetric matrix

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}.$$

We want to find a spectral decomposition for A, which entails finding the eigenvalues and an orthonormal set of eigenvectors of A. We begin with the eigenvalues:

$$\det(\lambda I - A) = \begin{vmatrix} \lambda & -1 & 1 \\ -1 & \lambda & -1 \\ 1 & -1 & \lambda \end{vmatrix}.$$

Using cofactor expansion along the first row, we get

$$\begin{aligned} \det(\lambda I - A) &= \lambda(\lambda^2 - 1) + (-\lambda + 1) + (1 - \lambda) \\ &= \lambda(\lambda - 1)(\lambda + 1) - (\lambda - 1) - (\lambda - 1) \\ &= (\lambda - 1)[\lambda(\lambda + 1) - 2] \\ &= (\lambda - 1)[(\lambda^2 + \lambda - 2] \\ &= (\lambda - 1)(\lambda - 1)(\lambda + 2). \end{aligned}$$

Thus, we see that the eigenvalues are 1, 1, and -2.

Next, we proceed to find the nullspace of $1I - A$ to get some eigenvectors corresponding to $\lambda = 1$.

$$1I - A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

From the RREF, we see that the eigenspace corresponding to $\lambda = 1$ is the set of solutions to the equation

$$x_1 - x_2 + x_3 = 0.$$

Choosing x_2 and x_3 to be free, and setting $x_2 = s$ and $x_3 = t$, we find that the eigenspace is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = s \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \quad ; \quad s, t \in \mathbb{R}.$$

In particular, the vectors $(1, 1, 0)$ and $(-1, 0, 1)$ provide us with two linearly independent eigenvectors corresponding to $\lambda = 1$.

Now we find the eigenspace corresponding to $\lambda = -2$:

$$-2I - A = \begin{bmatrix} -2 & -1 & 1 \\ -1 & -2 & -1 \\ 1 & -1 & -2 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 0 & -1 \\ 0 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix}.$$

From the RREF, we see that the eigenspace corresponding to $\lambda = -2$ is the set of solutions to the system of equations

$$\begin{aligned} x_1 - x_3 &= 0 \\ x_2 + x_3 &= 0. \end{aligned}$$

Choosing x_3 to be free, and setting $x_3 = t$, we find that the eigenspace is given by

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = t \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix} \quad ; \quad t \in \mathbb{R}.$$

In particular, the vector $(1, -1, 1)$ provides us with an eigenvector corresponding to $\lambda = -2$.

Summarizing the results so far, the eigenvalues and some corresponding eigenvectors are given by:

$$\lambda = 1: \quad \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

$$\lambda = -2: \quad \begin{bmatrix} 1 \\ -1 \\ 1 \end{bmatrix}.$$

Note that $(1, -1, 1)$ is orthogonal to $(1, 1, 0)$ and $(-1, 0, 1)$, as expected, since eigenvectors of a symmetric matrix corresponding to distinct eigenvalues should be orthogonal. However, the eigenvectors corresponding to $\lambda = 1$ that we are currently working with, namely $(1, 1, 0)$ and $(-1, 0, 1)$, are not orthogonal to each other. If we want to obtain an orthonormal set of eigenvectors of A, then we first need to obtain two eigenvectors corresponding to $\lambda = 1$ that are orthogonal to each other. We can do this by performing Gram-Schmidt to the two eigenvectors corresponding to $\lambda = 1$ that we already have. Let

$$\mathbf{v}_1 = (1, 1, 0) \qquad \mathbf{v}_2 = (-1, 0, 1) \qquad \mathbf{v}_3 = (1, -1, 1).$$

Gram-Schmidt on $\mathbf{v}_1, \mathbf{v}_2$ will yield two orthogonal eigenvectors $\mathbf{u}_1, \mathbf{u}_2$ corresponding to $\lambda = 1$. Recall the formulas for Gram-Schmidt:

$$\mathbf{u}_1 = \mathbf{v}_1 \qquad \mathbf{u}_2 = \mathbf{v}_2 - \left(\frac{\mathbf{v}_2 \cdot \mathbf{u}_1}{\mathbf{u}_1 \cdot \mathbf{u}_1} \right) \mathbf{u}_1.$$

Plugging everything in, we get

$$\mathbf{u}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} - \frac{-1}{2} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}.$$

Then $\mathbf{u}_1$ and $\mathbf{u}_2$ are orthogonal eigenvectors corresponding to $\lambda = 1$. Even better, the vectors $\mathbf{u}_1$ and $2\mathbf{u}_2$ are orthogonal eigenvectors corresponding to $\lambda = 1$. Note that $\mathbf{u}_1$ and $2\mathbf{u}_2$ are still orthogonal to $\mathbf{v}_3$, as expected. If we now normalize $\mathbf{u}_1, 2\mathbf{u}_2, \mathbf{v}_3$, that will give us an orthonormal set of eigenvectors of A.

$$\mathbf{q}_1 = \frac{\mathbf{u}_1}{|\mathbf{u}_1|} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \qquad \mathbf{q}_2 = \frac{2\mathbf{u}_2}{|2\mathbf{u}_2|} = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \qquad \mathbf{q}_3 = \frac{\mathbf{v}_3}{|\mathbf{v}_3|} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}.$$

Then, if we set $Q = [\mathbf{q}_1 | \mathbf{q}_2 | \mathbf{q}_3]$ and D to be a diagonal matrix whose diagonal entries are the eigenvalues $1, 1, -2$, then a spectral decomposition of A is given by $A = QDQ^T$. ◇

8.7.1 Alternate Form of Spectral Decomposition

Let A be a symmetric matrix with eigenvalues $\lambda_1, \ldots, \lambda_n$ and corresponding orthonormal eigenvectors $\mathbf{q}_1, \ldots, \mathbf{q}_n$. It turns out that

$$A = \lambda_1 \mathbf{q}_1 \mathbf{q}_1^T + \lambda_2 \mathbf{q}_2 \mathbf{q}_2^T + \ldots + \lambda_n \mathbf{q}_n \mathbf{q}_n^T.$$

This is an alternate way of writing out the spectral decomposition $A = QDQ^T$. Each of the matrices $\lambda_i \mathbf{q}_i \mathbf{q}_i^T$ are rank 1 matrices, so in particular this alternate form of the spectral decomposition expresses A as a sum of rank 1 matrices. While this is far from the only way of expressing A as a sum of rank 1 matrices, this particular way of doing so has some advantages over the others.

Example. In the last example, we found a spectral decomposition for the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$$

In particular, we found the following eigenvalues and corresponding orthonormal eigenvectors:

$$\lambda_1 = 1,\ \mathbf{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \qquad \lambda_2 = 1,\ \mathbf{q}_2 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \qquad \lambda_3 = -2,\ \mathbf{q}_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

The alternate form of the spectral decomposition would be:

$$A = \lambda_1 \mathbf{q}_1 \mathbf{q}_1^T + \lambda_2 \mathbf{q}_2 \mathbf{q}_2^T + \lambda_3 \mathbf{q}_3 \mathbf{q}_3^T,$$

which, written explicitly, gives us

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \end{bmatrix} + \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \end{bmatrix}$$

$$+(-2) \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix}.$$

◇

8.7.2 Some Proofs

As promised, here are some lovely proofs, written by one of our tutors at the time who was helping me out with these notes.

Theorem. The eigenvalues of a (real) symmetric matrix A are real numbers.

Proof. Let λ be an eigenvalue of A. Then there is an eigenvector $\mathbf{x}$ such that $A\mathbf{x} = \lambda\mathbf{x}$. Multiplying both sides by the row vector $(A\mathbf{x})^T$ on the left, we get

$$\begin{aligned}(A\mathbf{x})^T(A\mathbf{x}) &= (A\mathbf{x})^T(\lambda\mathbf{x}) \\ &= \lambda(\mathbf{x}^T A^T \mathbf{x}) \\ &= \lambda(\mathbf{x}^T A\mathbf{x}) \quad (\text{since } A = A^T) \\ &= \lambda\mathbf{x}^T(\lambda x) \quad (\text{since } A\mathbf{x} = \lambda\mathbf{x}) \\ &= \lambda^2\mathbf{x}^T\mathbf{x} \\ &= \lambda^2\|\mathbf{x}\|^2\end{aligned}$$

so if we solve for λ^2 then we get

$$\begin{aligned}\lambda^2 &= \frac{(A\mathbf{x})^T A\mathbf{x}}{\|\mathbf{x}\|^2} \\ &= \frac{\|A\mathbf{x}\|^2}{\|\mathbf{x}\|^2}\end{aligned}$$

Notice that the numerator and denominator of this fraction are both non-negative real numbers. Thus, after taking the square root, λ will be a real number. $\square$

Theorem. If A is a symmetric matrix, then eigenvectors corresponding to distinct eigenvalues are orthogonal.

Proof. Let λ_1 and λ_2 be two distinct eigenvalues of A, and let $\mathbf{v}_1$ and $\mathbf{v}_2$ be corresponding eigenvectors. Then we have

$$A\mathbf{v}_1 = \lambda_1\mathbf{v}_1 \tag{8.5}$$
$$A\mathbf{v}_2 = \lambda_2\mathbf{v}_2 \tag{8.6}$$

Multiply (1) on the left by $\mathbf{v}_2^T$ and (2) on the left by $\mathbf{v}_1^T$. We get

$$\begin{aligned}\mathbf{v}_2^T A\mathbf{v}_1 &= \mathbf{v}_2^T(\lambda_1\mathbf{v}_1) && (1') \\ &= \lambda_1\mathbf{v}_2^T\mathbf{v}_1 \\ \mathbf{v}_1^T A\mathbf{v}_2 &= \mathbf{v}_1^T(\lambda_2\mathbf{v}_2) && (2') \\ &= \lambda_2\mathbf{v}_1^T\mathbf{v}_2\end{aligned}$$

Notice that the left-hand side of $(1')$ is the transpose of the left-hand side of $(2')$. That is, $(\mathbf{v}_2^T A\mathbf{v}_1)^T = \mathbf{v}_1^T A\mathbf{v}_2$ because A is symmetric. Therefore, the right-hand sides have the same property: $(\lambda_1\mathbf{v}_2^T\mathbf{v}_1)^T = \lambda_2\mathbf{v}_1^T\mathbf{v}_2$. As a result,

$$\lambda_1\mathbf{v}_1^T\mathbf{v}_2 = (\lambda_1\mathbf{v}_2^T\mathbf{v}_1)^T = \lambda_2\mathbf{v}_1^T\mathbf{v}_2,$$

so $(\lambda_1 - \lambda_2)\mathbf{v}_1^T\mathbf{v}_2 = 0$. But we assumed that $\lambda_1 \neq \lambda_2$ so $\lambda_1 - \lambda_2 \neq 0$. Therefore $\mathbf{v}_1^T\mathbf{v}_2 = 0$. Since this is exactly the dot product of $\mathbf{v}_1$ and $\mathbf{v}_2$, they must be orthogonal. $\square$

I feel that a proof of the Spectral Theorem itself is a little too advanced for MATH 2418 at UTD, so I will omit it.

8.8 Theory of Quadratic Forms

8.8.1 Definition & Relation to Symmetric Matrices

Let

$$A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{12} & a_{22} & a_{23} \\ a_{12} & a_{23} & a_{33} \end{bmatrix} \qquad \text{and} \qquad \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}.$$

Notice that A is symmetric. On your own, compute $\mathbf{x}^T A\mathbf{x}$. You should find that

$$\mathbf{x}^T A\mathbf{x} \;=\; a_{11}x_1^2 \;+\; a_{22}x_2^2 \;+\; a_{33}x_3^2 \;+\; 2a_{12}x_1x_2 \;+\; 2a_{13}x_1x_3 \;+\; 2a_{23}x_2x_3.$$

Let's call the right-hand side $f(x_1, x_2, x_3)$. Notice that f is a multivariate polynomial, and every term of f is of degree 2 (of course, x_1^2, x_2^2, and x_3^2 are degree 2; we also consider polynomials like x_1x_2, x_1x_3, and x_2x_3 degree 2 as well).

In general, a polynomial function $f : \mathbb{R}^n \to \mathbb{R}$ is called a ***quadratic form*** if all of its terms are of degree 2. As the above computation suggests, it is a fact that every quadratic form $f : \mathbb{R}^n \to \mathbb{R}$ can be expressed as

$$f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$$

for some symmetric matrix A, and furthermore this symmetric matrix is unique.

Example. Consider the quadratic form f on $\mathbb{R}^4$ given by

$$f(\mathbf{x}) = 2x_1^2 - x_3^2 + 3x_4^2 + 6x_1x_3 - 4x_2x_4 + 2x_3x_4.$$

We want to find the symmetric matrix A associated with this quadratic form. Here is how it works. The coefficients of x_1^2, x_2^2, x_3^2, x_4^2 are 2, 0, -1, 3. These go into the diagonal of A:

$$A = \begin{bmatrix} 2 & & & \\ & 0 & & \\ & & -1 & \\ & & & 3 \end{bmatrix}$$

The coefficient of x_1x_3 is 6; we will halve this and put 3 in the $(1,3)$ and $(3,1)$ positions of A:

$$A = \begin{bmatrix} 2 & & 3 & \\ & 0 & & \\ 3 & & -1 & \\ & & & 3 \end{bmatrix}$$

The coefficient of x_2x_4 is -4; we will halve this and put -2 in the $(2,4)$ and $(4,2)$ entries:

$$A = \begin{bmatrix} 2 & & 3 & \\ & 0 & & -2 \\ 3 & & -1 & \\ & -2 & & 3 \end{bmatrix}$$

The coefficient of x_3x_4 is 2; we will halve this and put 1 in the $(3,4)$ and $(4,3)$ entries

$$A = \begin{bmatrix} 2 & & 3 & \\ & 0 & & -2 \\ 3 & & -1 & 1 \\ & -2 & 1 & 3 \end{bmatrix}$$

Since there are no other terms in f, the remaining entries of A are zero:

$$A = \begin{bmatrix} 2 & 0 & 3 & 0 \\ 0 & 0 & 0 & -2 \\ 3 & 0 & -1 & 1 \\ 0 & -2 & 1 & 3 \end{bmatrix}$$

You can multiply out $\mathbf{x}^T A\mathbf{x}$ to confirm that it equals $f(\mathbf{x})$. $\diamond$

Given a quadratic form on $\mathbb{R}^n$, one can mimick the procedure in this last example to obtain the associated symmetric matrix. We summarize the discussion so far:

Theorem. Given a quadratic form $f : \mathbb{R}^n \to \mathbb{R}$, there exists a unique $n \times n$ symmetric matrix A such that $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$.

We can gain some insight into quadratic forms by using the spectral decomposition, as we will see in the next subsection.

8.8.2 "Diagonalizing" a Quadratic Form Using Spectral Decomposition

Suppose D is a diagonal matrix whose diagonal entries are $\lambda_1, \ldots, \lambda_n$. The associated quadratic form is

$$f(\mathbf{x}) \;=\; \mathbf{x}^T D\mathbf{x} \;=\; \lambda_1 x_1^2 \,+\, \lambda_2 x_2^2 \,+\ldots+\, \lambda_n x_n^2.$$

Notice that the formula for f consists only of the squared terms, and none of the "mixed" terms like x_1x_2, etc. This makes the study of f easier in many aspects. For instance, if all of the coefficients $\lambda_1, \ldots, \lambda_n$ are positive, then we can see that $f(\mathbf{x}) > 0$ for all nonzero $\mathbf{x}$. Contrast this with a quadratic form like

$$f(x_1, x_2) = x_1^2 + x_2^2 - x_1x_2.$$

For this particular example, can you tell if $f(\mathbf{x}) > 0$ whenever $\mathbf{x} \neq 0$?

Using the spectral decomposition, we can "diagonalize" any quadratic form as follows. Let f be a quadratic form on $\mathbb{R}^n$ with associated symmetric matrix A, so that $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$. Let $A = QDQ^T$ be a spectral decomposition of A. Then we can write

$$f(\mathbf{x}) \;=\; \mathbf{x}^T QDQ^T\mathbf{x} \;=\; (Q^T\mathbf{x})^T D(Q^T\mathbf{x}).$$

If we set $\mathbf{y} = Q^T\mathbf{x}$, then in terms of the new variable $\mathbf{y}$, we see that f has the form

$$f(\mathbf{y}) \;=\; \mathbf{y}^T D\mathbf{y} \;=\; \lambda_1 y_1^2 + \ldots + \lambda_n y_n^2.$$

Thus, we can eliminate the "mixed" terms in any quadratic form using the spectral decomposition.

Example. Previously, for the matrix

$$A = \begin{bmatrix} 0 & 1 & -1 \\ 1 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix},$$

we found the following eigenvalues and corresponding orthonormal eigenvectors:

$$\lambda_1 = 1,\; \mathbf{q}_1 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix} \qquad \lambda_2 = 1,\; \mathbf{q}_2 = \begin{bmatrix} -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{6}} \\ \frac{2}{\sqrt{6}} \end{bmatrix} \qquad \lambda_3 = -2,\; \mathbf{q}_3 = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \end{bmatrix}$$

The quadratic form associated with A is

$$f(\mathbf{x}) \;=\; \mathbf{x}^T A\mathbf{x} \;=\; 2x_1x_2 - 2x_1x_3 + 2x_2x_3.$$

Let's make the change of variables $\mathbf{y} = Q^T\mathbf{x}$:

$$\mathbf{y} \;=\; \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \;=\; Q^T\mathbf{x} \;=\; \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & \frac{2}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;=\; \begin{bmatrix} \frac{1}{\sqrt{2}}x_1 + \frac{1}{\sqrt{2}}x_2 \\ -\frac{1}{\sqrt{6}}x_1 + \frac{1}{\sqrt{6}}x_2 + \frac{2}{\sqrt{6}}x_3 \\ \frac{1}{\sqrt{3}}x_1 - \frac{1}{\sqrt{3}}x_2 + \frac{1}{\sqrt{3}}x_3 \end{bmatrix}$$

Per our discussion earlier, in terms of $\mathbf{y}$, f has the form

$$f(\mathbf{y}) = \lambda_1 y_1^2 + \lambda_2 y_2^2 + \lambda_3 y_3^2 \qquad \Longleftrightarrow \qquad f(y_1, y_2, y_3) \;=\; y_1^2 + y_2^2 - 2y_3^2.$$

You can confirm for yourself that $f(\mathbf{x})$ and $f(\mathbf{y})$ are equivalent by checking that

$$\overbrace{2x_1x_2 - 2x_1x_3 + 2x_2x_3}^{f(\mathbf{x})} = \overbrace{\Big(\overbrace{\tfrac{1}{\sqrt{2}}x_1 + \tfrac{1}{\sqrt{2}}x_2}^{y_1}\Big)^2 + \Big(\overbrace{-\tfrac{1}{\sqrt{6}}x_1 + \tfrac{1}{\sqrt{6}}x_2 + \tfrac{2}{\sqrt{6}}x_3}^{y_2}\Big)^2 - 2\Big(\overbrace{\tfrac{1}{\sqrt{3}}x_1 - \tfrac{1}{\sqrt{3}}x_2 + \tfrac{1}{\sqrt{3}}x_3}^{y_3}\Big)^2}^{f(\mathbf{y})}$$

◇

8.8.3 Positive Definite Quadratic Forms & Related Notions

Let's see what insight on quadratic forms we can gain from all of this. Once again, suppose $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ for some symmetric matrix A. Let $A = QDQ^T$ be a spectral decomposition of A. After the change of variables $\mathbf{y} = Q^T\mathbf{x}$, f has the form

$$f(\mathbf{y}) \;=\; \mathbf{y}^T D\mathbf{y} \;=\; \lambda_1 y_1^2 + \ldots + \lambda_n y_n^2,$$

where $\lambda_1, \ldots, \lambda_n$ are the eigenvalues of A. We list a few observations below.

- Notice that $f(\mathbf{y}) > 0$ for all nonzero $\mathbf{y}$ if and only if all of the eigenvalues are positive. Since $\mathbf{y} = Q^T\mathbf{x}$ and Q is invertible, we can also conclude that $f(\mathbf{x}) > 0$ for all nonzero $\mathbf{x}$ in this case.

  A quadratic form f is said to be ***positive definite*** if $f(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$. We see that a quadratic form is positive definite if and only if all eigenvalues of the associated symmetric matrix are positive.

- Notice that $f(\mathbf{y}) \geq 0$ for all $\mathbf{y}$, and hence $f(\mathbf{x}) \geq 0$ for all $\mathbf{x}$, if and only if all of the eigenvalues are nonnegative. Note however that if one of the eigenvalues is zero, then unlike the first case, f would equal zero for some nonzero inputs. For instance, if

  $$f(\mathbf{y}) = 0y_1^2 + y_2^2,$$

  then $f(1, 0) = 0$.

  A quadratic form f is said to be ***positive semidefinite*** if $f(\mathbf{x}) \geq 0$ for all $\mathbf{x}$. We see that a quadratic form is positive semidefinite if and only if all eigenvalues of the associated symmetric matrix are nonnegative.

- Notice that $f(\mathbf{y}) < 0$ for all nonzero $\mathbf{y}$, and hence $f(\mathbf{x}) < 0$ for all nonzero $\mathbf{x}$, if and only if all the eigenvalues are negative.

  A quadratic form f is said to be ***negative definite*** if $f(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$. We see that a quadratic form is negative definite if and only if all eigenvalues of the associated symmetric matrix are negative.

- Notice that $f(\mathbf{y}) \leq 0$ for all $\mathbf{y}$, and hence $f(\mathbf{x}) \leq 0$ for all $\mathbf{x}$, if and only if all of the eigenvalues are nonpositive.

  A quadratic form f is said to be ***negative semidefinite*** if $f(\mathbf{x}) \leq 0$ for all $\mathbf{x}$. We see that a quadratic form is negative semidefinite if and only if all eigenvalues of the associated symmetric matrix are nonpositive.

Definition. A symmetric matrix A is said to be ***positive definite*** if the associated quadratic form is positive definite. Similar for ***positive semidefinite***, etc.

We will have more to say on positive definite matrices in a later section.

8.8.4 "Diagonalizing" a Quadratic Form Using LDL^T Decomposition

Using the spectral decomposition is not the only way of diagonalizing a quadratic form. Suppose we have quadratic form $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ with A symmetric. Let $A = LDL^T$ be an LDL^T decomposition of A (see Section 4.6 for a refresher). We can rewrite $f(\mathbf{x})$ as

$$f(\mathbf{x}) \;=\; \mathbf{x}^T LDL^T\mathbf{x} \;=\; (L^T\mathbf{x})^T D(L^T\mathbf{x}).$$

Thus, if we make the change of variables $\mathbf{y} = L^T\mathbf{x}$, then in term of $\mathbf{y}$, we have

$$f(\mathbf{y}) \;=\; \mathbf{y}^T D\mathbf{y} \;=\; d_1 y_1^2 + \ldots + d_n y_n^2,$$

where $d_1, \ldots, d_n$ are the diagonal entries of D. Note that these are not eigenvalues of A in general; rather, if you'll recall how the LDL^T decomposition works, the diagonal entries of D are the pivots from performing elimination on A.

We can make similar observations from this as we did using the spectral decomposition regarding positive definiteness, etc.

- The quadratic form f and the associated symmetric matrix A are positive definite if and only if all pivots from performing elimination on A are positivediagona.
- The quadratic form f and the associated symmetric matrix A are positive semidefinite if and only if all pivots from performing elimination on A are nonnegative.
- The quadratic form f and the associated symmetric matrix A are negative definite if and only if all pivots from performing elimination on A are negative.
- The quadratic form f and the associated symmetric matrix A are negative semidefinite if and only if all pivots from performing elimination on A are nonpositive.

Example. Back in Section 4.6, we found the LDL^T decomposition of

$$A = \begin{bmatrix} 1 & 2 & -2 \\ 2 & 3 & -1 \\ -2 & -1 & -1 \end{bmatrix}.$$

The matrices L and D are

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ -2 & -3 & 1 \end{bmatrix} \qquad\qquad D = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 4 \end{bmatrix}.$$

The quadratic form associated with A is

$$f(\mathbf{x}) \;=\; \mathbf{x}^T A\mathbf{x} \;=\; x_1^2 + 3x_2^2 - x_3^2 + 4x_1x_2 - 4x_1x_3 - 2x_2x_3.$$

Let's make the change of variables $\mathbf{y} = L^T\mathbf{x}$:

$$\mathbf{y} \;=\; \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} \;=\; L^T\mathbf{x} \;=\; \begin{bmatrix} 1 & 2 & -2 \\ 0 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} \;=\; \begin{bmatrix} x_1 + 2x_2 - 2x_3 \\ x_2 - 3x_3 \\ x_3 \end{bmatrix}.$$

In terms of $\mathbf{y}$, f is given by

$$f(\mathbf{y}) \;=\; \mathbf{y}^T D\mathbf{y} \;=\; y_1^2 - y_2^2 + 4y_3^2.$$

You can confirm that $f(\mathbf{x})$ and $f(\mathbf{y})$ are equivalent by checking that

$$\overbrace{x_1^2 + 3x_2^2 - x_3^2 + 4x_1x_2 - 4x_1x_3 - 2x_2x_3}^{f(\mathbf{x})} = \overbrace{\Big(\overbrace{x_1 + 2x_2 - 2x_3}^{y_1}\Big)^2 - \Big(\overbrace{x_2 - 3x_3}^{y_2}\Big)^2 + 4\Big(\overbrace{x_3}^{y_3}\Big)^2}^{f(\mathbf{y})}. \qquad \diamond$$

Since an LDL^T decomposition is easier to find than a spectral decomposition, and since the numbers are often nicer as well, you might prefer to use the LDL^T decomposition over the spectral decomposition to diagonalize a quadratic form.

8.8.5 Max and Min of a Quadratic Form Among Unit Vectors

In this section, we derive a result about quadratic forms using the spectral decomposition that we could not derive using the LDL^T decomposition. First, let S be unit "sphere" in $\mathbb{R}^n$:

$$S = \{\mathbf{x} \in \mathbb{R}^n \ : \ \|\mathbf{x}\| = 1\}.$$

Consider the quadratic form $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ with $A = QDQ^T$ being a spectral decomposition of A. With $\mathbf{y} = Q^T\mathbf{x}$, we have

$$f(\mathbf{y}) \ = \ \mathbf{y}^T D\mathbf{y} \ = \ \lambda_1 y_1^2 + \lambda_2 y_2^2 + \ldots + \lambda_n y_n^2. \tag{$*$}$$

Suppose we have ordered the eigenvalues so that λ_1 is the largest eigenvalue and λ_n is the smallest. Replacing each of the the eigenvalues in $(*)$ with λ_1 would give us a larger expression then:

$$\begin{aligned} & f(\mathbf{y}) \leq \lambda_1 y_1^2 + \lambda_1 y_2^2 + \ldots + \lambda_1 y_n^2. \\ \iff \ & f(\mathbf{y}) \leq \lambda_1 (y_1^2 + y_2^2 + \ldots + y_n^2) \\ \iff \ & f(\mathbf{y}) \leq \lambda_1 \|\mathbf{y}\|^2. \end{aligned}$$

Likewise, replacing each of the eigenvalues in $(*)$ with λ_n would give us a smaller value, leading to the inequality

$$\lambda_n \|\mathbf{y}\|^2 \leq f(\mathbf{y}).$$

Putting these last two inequalities together, we find that

$$\lambda_n \|\mathbf{y}\|^2 \ \leq \ f(\mathbf{y}) \ \leq \ \lambda_1 \|\mathbf{y}\|^2.$$

Now, let's restrict the quadratic form $f(\mathbf{y})$ to the unit sphere S, so that $\|\mathbf{y}\| = 1$. Then we would get

$$\lambda_n \ \leq \ f(\mathbf{y}) \ \leq \ \lambda_1, \qquad \|\mathbf{y}\| = 1.$$

Thus, when $\mathbf{y}$ is restricted to S, we see that the values of $f(\mathbf{y})$ are bounded below by the smallest eigenvalue, λ_n, and bounded above by the largest eigenvalue, λ_1. These in fact turn out to be the minimum and maximum values of $f(\mathbf{y})$ when restricted to S. Indeed, when

$$\mathbf{y} = \mathbf{e}_1 \iff (y_1, y_2, \ldots, y_n) = (1, 0, \ldots, 0),$$

we see from $(*)$ that

$$f(\mathbf{y}) = \lambda_1.$$

Similarly, when

$$\mathbf{y} = \mathbf{e}_n \iff (y_1, y_2, \ldots, y_n) = (0, 0, \ldots, 1),$$

we see from $(*)$ that

$$f(\mathbf{y}) = \lambda_n.$$

Thus, when restricted to the unit sphere, $f(\mathbf{y})$ achieves a maximual value of λ_1 when $\mathbf{y} = \mathbf{e}_1$, and a minimal value of λ_n when $\mathbf{y} = \mathbf{e}_n$.

What can we say in terms of the original variable $\mathbf{x}$? Recall that $\mathbf{y} = Q^T\mathbf{x}$. Since Q is an orthogonal matrix, $Q^T = Q^{-1}$ is also orthgonal. Since orthogonal matrices are length-preserving (see Section 8.6), it follows that restricting $\mathbf{y}$ to S is equivalent to restricting $\mathbf{x}$ to S as well. Furthermore, from $\mathbf{y} = Q^T\mathbf{x}$, we get $\mathbf{x} = Q\mathbf{y}$ since Q and Q^T are inverses. When $\mathbf{y} = \mathbf{e}_1$, we get

$$\mathbf{x} = Q\mathbf{e}_1 = \text{ first column of } Q \ = \mathbf{q}_1,$$

where $\mathbf{q}_1$ is a unit eigenvector corresponding to λ_1. Likewise, when $\mathbf{y} = \mathbf{e}_n$, we get $\mathbf{x} = \mathbf{q}_n$, where $\mathbf{q}_n$ is a unit eigenvector corresponding to λ_n. Hence, we get the following results:

Theorem. Let $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ be a quadratic form with associated symmetric matrix A. When restricted to the unit sphere, $f(\mathbf{x})$ achieves a maximal value of λ_1 when $\mathbf{x} = \mathbf{q}_1$, and a minmal value of λ_n when $\mathbf{x} = \mathbf{q}_n$, where λ_1 and λ_n are the largest and smallest eigenvalues of A and $\mathbf{q}_1$ and $\mathbf{q}_n$ are corresponding unit length eigenvectors, respectively.

8.9 Positive Definite/Semidefinite Matrices

Recall the following definitions.

Definition. Let A be a symmetric matrix, and let $f(\mathbf{x}) = \mathbf{x}^T A\mathbf{x}$ be the corresponding quadratic form.

- f is positive definite if $f(\mathbf{x}) > 0$ for all $\mathbf{x} \neq \mathbf{0}$, and A is positive definite if f is positive definite.
- f is positive semidefinite if $f(\mathbf{x}) \geq 0$ for all $\mathbf{x}$, and A is positive semidefinite if f is positive semidefinite.
- f is negative definite if $f(\mathbf{x}) < 0$ for all $\mathbf{x} \neq \mathbf{0}$, and A is negative definite if f is negative definite.
- f is negative semidefinite if $f(\mathbf{x}) \leq 0$ for all $\mathbf{x}$, and A is negative semidefinite if f is negative semidefinite.

We have also proven the following theorem already.

Theorem. Let A be a symmetric matrix.

- The following are equivalent.
  - A is positive definite.
  - $\mathbf{x}^T A\mathbf{x} > 0$ for all $\mathbf{x} \neq \mathbf{0}$.
  - All eigenvalues of A are > 0.
  - All pivots from performing elimination on A are > 0.
- The following are equivalent.
  - A is positive semidefinite.
  - $\mathbf{x}^T A\mathbf{x} \geq 0$ for all $\mathbf{x}$.
  - All eigenvalues of A are ≥ 0.
  - All pivots from performing elimination on A are ≥ 0.
- The following are equivalent.
  - A is negative definite.
  - $\mathbf{x}^T A\mathbf{x} < 0$ for all $\mathbf{x} \neq \mathbf{0}$.
  - All eigenvalues of A are < 0.
  - All pivots from performing elimination on A are < 0.
- The following are equivalent.
  - A is negative semidefinite.
  - $\mathbf{x}^T A\mathbf{x} \leq 0$ for all $\mathbf{x}$.
  - All eigenvalues of A are ≤ 0.
  - All pivots from performing elimination on A are ≤ 0.

To be clear, performing elimination on A is the process of reducing A to upper triangular form, using the pivots to produce zeros underneath them by adding multiples of a row to another row. Furthermore, these are the only types of row operations that can be performed, so no swapping rows or multilying a row by a nonzero number, because you cannot obtain an LDL^T decomposition (as we have defined it) if you do any of these other types of row operations.

8.9.1 Positive Semidefiniteness of A^TA and AA^T

Let A be an an arbtrary matrix (not necessarily symmetric or even square). Then the matrices A^TA and AA^T are always positive semidefinite. Let's consider A^TA specifically. First, note that A^TA is symmetric, since it is its own transpose:

$$(A^TA)^T = A^T(A^T)^T = A^TA.$$

To see why it is positive semidefinite, we will show that $\mathbf{x}^T(A^TA)\mathbf{x} \geq 0$ for all $\mathbf{x}$. Recall that for any $\mathbf{u}, \mathbf{v}$ in $\mathbb{R}^n$, that $\mathbf{u}^T\mathbf{v}$ is equivalent to the dot product $\mathbf{u} \cdot \mathbf{v}$. With this in mind, we have

$$\mathbf{x}^TA^TA\mathbf{x} \;=\; (A\mathbf{x})^T(A\mathbf{x}) \;=\; (A\mathbf{x}) \cdot (A\mathbf{x}) \;=\; \|A\mathbf{x}\|^2 \;\geq\; 0.$$

Hence, A^TA is positive semidefinite. One can show in a very similar manner that AA^T is also positive semidefinite.

The matrices A^TA and AA^T play a primary role in the singular value decomposition of a matrix A, which is the subject of the next chapter. The facts that both of these matrices are symmetric and positive semidefinite come into play there.

8.9.2 Two Factorizations for Positive Semidefinite Matrices

We saw in the last section that for any matrix A, the matrices A^TA and AA^T are positive semidefinite. One might wonder if the reverse of this is true: namely, if A is positive semidefinite, is A equal to B^TB for some matrix B? The answer is yes, and we can use both the spectral decomposition as well as the LDL^T decomposition to find such a matrix. Furthermore, this matrix will have certain special properties depending on which of these decompositions we use.

Finding a Positive Semidefinite Square Root

To start with, let's consider the spectral decomposition: $A = QDQ^T$. Since A is positive semidefinite, all of its eigenvalues are nonnegative. This allows us to compute the square root of D:

$$D = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \qquad \Longrightarrow \qquad D^{1/2} = \begin{bmatrix} \sqrt{\lambda_1} & & \\ & \ddots & \\ & & \sqrt{\lambda_n} \end{bmatrix}$$

We then have

$$A \;=\; QDQ^T \;=\; QD^{1/2}\,D^{1/2}Q^T.$$

Now, since Q is orthogonal, $Q^TQ = I$, so we can insert a Q^TQ in the middle to get

$$A \;=\; (QD^{1/2}Q^T)(QD^{1/2}Q^T).$$

Let $B = QD^{1/2}Q^T$. Then right now we see that

$$A = B^2.$$

Notice that B is symmetric, since

$$B^T \;=\; (QD^{1/2}Q^T)^T \;=\; (Q^T)^T(D^{1/2})^TQ^T \;=\; QD^{1/2}Q^T \;=\; B.$$

In fact, $B = QD^{1/2}Q^T$ happens to be a spectral decomposition of B, and since its eigenvalues (= the diagonal entries of $D^{1/2}$) are nonnegative, we see that B is even positive semidefinite. Finally, since $B = B^T$, then $A = B^2$ is equivalent to saying

$$A = B^TB.$$

Thus, using the spectral decomposition, we can factor a positive semidefinite matrix A as B^TB, where B is also positive semidefinite. Since $A = B^2$, we can view B as a positive semidefinite square root of A. We will denote it as $A^{1/2}$. If A is positive definite, then $A^{1/2}$ is the unique positive definite square root of A. If A is only positive semidefinite, then it will have more than one positive semidefinite square root.

The Cholesky Factorization

Turning our attention to the $A = LDL^T$ decomposition, since A is positive semidefinite, the diagonal entries of D (= the pivots from performing elimination on A) are nonnegative. Thus, we can again take the square root of D by square rooting each of the diagaonal entries. We then have

$$A = LDL^T = LD^{1/2}D^{1/2}L^T.$$

Let $C = D^{1/2}L^T$. Then we have

$$A = C^TC.$$

This is known as the ***Cholesky factorization*** of a positive semidefinite matrix.

What special property does the matrix C possess? Well, since $D^{1/2}$ is a diagonal matrix, and since L^T is upper triangular, $C = D^{1/2}L^T$ is also upper triangular. Thus, using the LDL^T decomposition to obtain the Cholesky factorization, we can factor a positive semidefinite matrix as C^TC, where C is an upper triangular matrix.

In the event that A does not admit an LDL^T decomposition, it turns out we can still find a Cholesky factorization for it by doing something else. We will not get into that though.

Example. Consider the symmetric matrix

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix}.$$

Let's find a spectral decomposition for A first.

$$\begin{aligned} \det(\lambda I - A) &= \begin{vmatrix} \lambda - 2 & -2 \\ -2 & \lambda - 5 \end{vmatrix} \\ &= \lambda^2 - 7\lambda + 6 \\ &= (\lambda - 1)(\lambda - 6). \end{aligned}$$

Thus, the eigenvalues of A are 6 and 1. Since all eigenvalues are positive, A is positive definite.

$$6I - A = \begin{bmatrix} 4 & -2 \\ -2 & 1 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}.$$

This gives the equation $2x_1 - x_2 = 0$. Hence, $(1, 2)$ is an eigenvector corresponding to $\lambda = 6$.

$$1I - A = \begin{bmatrix} -1 & -2 \\ -2 & -4 \end{bmatrix} \xrightarrow{\text{row ops}} \begin{bmatrix} 1 & 2 \\ 0 & 0 \end{bmatrix}.$$

This gives the equation $x_1 + 2x_2 = 0$. Hence, $(-2, 1)$ is an eigenvector corresponding to $\lambda = 1$.

The two eigenvectors we have found are already orthogonal to each other, since they correspdond to different eigenvalues. Normalizing them gives us the orthonormal eigenvectors needed for the spectral decomposition:

$$\mathbf{q}_1 = \tfrac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix} \qquad \mathbf{q}_2 = \tfrac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}.$$

Thus, we obtain the spectral decomposition $A = QDQ^T$ with

$$Q = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \qquad D = \begin{bmatrix} 6 & 0 \\ 0 & 1 \end{bmatrix}.$$

Then we obtain the positive definite square root of A as

$$A^{1/2} = QD^{1/2}Q^T = \frac{1}{5} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \frac{1}{5} \begin{bmatrix} \sqrt{6} + 4 & 2\sqrt{6} - 2 \\ 2\sqrt{6} - 2 & 4\sqrt{6} + 1 \end{bmatrix}.$$

You can confirm that $A = A^{1/2}A^{1/2}$.

Next, let's find an LDL^T decomposition of A.

$$A = \begin{bmatrix} 2 & 2 \\ 2 & 5 \end{bmatrix} \xrightarrow{R_2 - R_1} \begin{bmatrix} 2 & 2 \\ 0 & 3 \end{bmatrix}$$

From this, we see that

$$L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \qquad\qquad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

A Cholesky factorization of A is then $A = C^T C$, where

$$C \;=\; D^{1/2} L^T \;=\; D = \begin{bmatrix} \sqrt{2} & 0 \\ 0 & \sqrt{3} \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \;=\; \begin{bmatrix} \sqrt{2} & \sqrt{2} \\ 0 & \sqrt{3} \end{bmatrix}.$$

You can confirm that $A = C^T C$. ◇

8.9.3 A Test for Positive Definiteness, etc.

We state the following theorem without proof.

Theorem. Let A be a symmetric matrix. Let A_k be the $k \times k$ submatrix in the upper left corner of A, obtained by deleting every row and column after the k^{th} row/column. Then:

- A is positive definite if and only if $\det(A_k) > 0$ for all k.
- A is negative definite if and only if $\det(A_k) < 0$ when k is odd, and $\det(A_k) > 0$ when k is even.

Note: These statements do not generalize for the semidefinite cases. For instance, it is not true that if $\det(A_k) \geq 0$ for all k, that A is positive semidefinite.

Example. Consider

$$A = \begin{bmatrix} 3 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 4 \end{bmatrix}.$$

Then:

$$\det(A_1) = \det(3) = 3.$$

$$\det(A_2) = \begin{vmatrix} 3 & 1 \\ 1 & 2 \end{vmatrix} = 5.$$

$$\det(A_3) = \det(A) = 3 \begin{vmatrix} 2 & 1 \\ 1 & 4 \end{vmatrix} - \begin{vmatrix} 1 & 1 \\ 2 & 4 \end{vmatrix} + 2 \begin{vmatrix} 1 & 2 \\ 2 & 1 \end{vmatrix} = 13.$$

Since $\det(A_k) > 0$ for all k, it follows that A is positive definite. ◇

Chapter 9

Singular Value Decomposition (SVD)

9.1 Some Theory Behind the SVD

In this section, we discuss a little bit of the theory behind the SVD. You'll probably be okay if you omit this section, so consider it optional. We go over how to find the SVD in later sections, and the last section of this chapter gives a quick summary of the important facts that you should know.

In the last chapter, we discussed how to diagonalize a square matrix, and the Spectral Theorem, which says that every (real) symmetric matrix can be orthogonally diagonalized. The SVD is kind of like a generalization of the Spectral Theorem, except it applies to all matrices (even rectangular ones)!

Let A be an $m \times n$ matrix. Associated with A are the two matrices A^TA and AA^T. As noted earlier, both of these matrices are symmetric and, in fact, positive semidefinite. Roughly speaking, we can "diagonalize" A using orthonormal eigenvectors of A^TA and AA^T, and this is the singular value decomposition.

Suppose λ is a *positive* eigenvalue of A^TA with corresponding eigenvector $\mathbf{v}$, so that

$$A^TA\mathbf{v} = \lambda\mathbf{v}.$$

Since $\lambda > 0$ and $\mathbf{v} \neq 0$, $A^TA\mathbf{v} \neq 0$, so in particular, $\mathbf{v}$ is not in the nullspace of A^TA. Recall in Section 6.3.1 that $N(A^TA) = N(A)$. Thus, we can also conclude that $\mathbf{v}$ is not in $N(A)$, so that $A\mathbf{v} \neq 0$. If we multiply both sides of the above equation by A, we get

$$(AA^T)A\mathbf{v} = \lambda(A\mathbf{v}).$$

Since $A\mathbf{v} \neq 0$, this last equation shows us that λ is also an eigenvalue of AA^T, and that $A\mathbf{v}$ is a corresponding eigenvector.

In summary, we see that if λ is a positive eigenvalue of A^TA with corresponding eigenvector $\mathbf{v}$, then λ is also an eigenvalue of AA^T and $A\mathbf{v}$ is a corresponding eigenvector. A similar analysis also shows that if λ is a positive eigenvalue of AA^T with corresponding eigenvector $\mathbf{u}$, then λ is an eigenvalue of A^TA and $A^T\mathbf{u}$ is a corresponding eigenvector. Since every positive eigenvaue of A^TA is an eigenvalue of AA^T, and every positive eigenvalue of AA^T is an eigenvalue of A^TA, it follows that A^TA and AA^T have the same positive eigenvalues.

Let $\lambda_1, \ldots, \lambda_r$ be the positive eigenvalues of A^TA. Since A^TA is symmetric, we can find an orthonormal set of corresponding eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$. Per our discussion so far, we know that $\lambda_1, \ldots, \lambda_r$ are also the positive eigenvalues of AA^T, and that $A\mathbf{v}_1, \ldots, A\mathbf{v}_r$ are corresponding eigenvectors. Are these eigenvectors also orthonormal? Let's check by computing the dot product of $A\mathbf{v}_i$ and $A\mathbf{v}_j$:

$$
\begin{aligned}
(A\mathbf{v}_i)\cdot(A\mathbf{v}_j) &= (A\mathbf{v}_i)^T(A\mathbf{v}_j) \\
&= \mathbf{v}_i^T(A^TA\mathbf{v}_j) && \text{since } (A\mathbf{v}_i)^T = \mathbf{v}_i^TA^T \\
&= \mathbf{v}_i^T(\lambda_j\mathbf{v}_j) && \text{since } \mathbf{v}_i \text{ is e.vector of } A^TA \\
&= \lambda_j(\mathbf{v}_i\cdot\mathbf{v}_j).
\end{aligned}
$$

At this point, there are two cases to consider:

- If $i \neq j$, then $\mathbf{v}_i \cdot \mathbf{v}_j = 0$ since $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are orthonormal. Since this holds for all pairs (i, j) with $i \neq j$, it follows that $A\mathbf{v}_1, \ldots, A\mathbf{v}_r$ are orthogonal.
- If $i = j$, then we have

  $$(A\mathbf{v}_i)\cdot(A\mathbf{v}_i) = \lambda_i(\mathbf{v}_i\cdot\mathbf{v}_i) \qquad \Longleftrightarrow \qquad \|A\mathbf{v}_i\|^2 = \lambda_i\|\mathbf{v}_i\|^2.$$

  Since $\mathbf{v}_1, \ldots, \mathbf{v}_r$ are orthonormal, they are unit length, and we get

  $$\|A\mathbf{v}_i\|^2 = \lambda_i \qquad \Longleftrightarrow \qquad \|A\mathbf{v}_i\| = \sqrt{\lambda_i}.$$

Thus, we see that $A\mathbf{v}_1, \ldots, A\mathbf{v}_r$ are orthogonal eigenvectors of AA^T corresponding to the positive eigenvalues $\lambda_1, \ldots, \lambda_r$, but they are not necessarily orthonormal because their lengths are equal to $\sqrt{\lambda_i}$ for each i. However, we can just normalize them to obtain a set of orthonormal eigenvectors. Thus,

$$\mathbf{u}_i = \tfrac{1}{\sqrt{\lambda_i}}A\mathbf{v}_i, \quad i = 1, \ldots, r$$

is an orthonormal set of eigenvectors of AA^T corresponding to its positive eigenvalues $\lambda_1, \ldots, \lambda_r$. To simplify the notation a little bit, let $\sigma_i = \sqrt{\lambda_i}$. The numbers $\sigma_1, \ldots, \sigma_r$ are known as the ***singular*** values of A. We can then rewrite the above equations as

$$\mathbf{u}_i = \tfrac{1}{\sigma_i}A\mathbf{v}_i,$$

or equivalently,

$$A\mathbf{v}_i = \sigma_i\mathbf{u}_i, \quad i = 1, \ldots, r. \qquad (*)$$

If we now put the two sets of orthonormal eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ and $\mathbf{u}_1, \ldots, \mathbf{u}_r$ into matrices V_+ and U_+ as columns, and if we put the singular values $\sigma_1, \ldots, \sigma_r$ along the diagonal of a diagonal matrix Σ:

$$U_+ = [\mathbf{u}_1 \ \ldots \ \mathbf{u}_r] \qquad \Sigma = \begin{bmatrix} \sigma_1 & & \\ & \ddots & \\ & & \sigma_r \end{bmatrix} \qquad V_+ = [\mathbf{v}_1 \ \ldots \ \mathbf{v}_r],$$

then $(*)$ turns out to be equivalent to the following factorization of A:

$$A = U_+\Sigma V_+^T.$$

This factorization of A is the ***singular value decomposition*** of A.

9.2 Procedure for Finding SVD

Let A be an $m \times n$ matrix. Here is a rough outline of how to find the SVD of A.

1. Compute $A^T A$. Notice that $A^T A$ is $n \times n$. It turns out that $A^T A$ will always be symmetric, and in fact, it will always be positive semidefinite, meaning that its eigenvalues will always be nonnegative.

2. Find the eigenvalues of $A^T A$ and order them from largest to smallest:

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0 = \lambda_{r+1} = \ldots = \lambda_n.$$

   If $r = n$, then all of the eigenvalues of $A^T A$ were positive. Otherwise, if $r < n$, then zero was an eigenvalue of $A^T A$ of multiplicity $n - r$. For the SVD, we do not care about the zero eigenvalues, but keep them in mind because we need them for the full SVD (to be discussed later).

   It's not actually necessary to order the eigenvalues from largest to smallest, but this is useful to do in applications, so we will go ahead and do it.

3. Find orthonormal eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_r$ corresponding to the positive eigenvalues $\lambda_1, \ldots, \lambda_r$ (since $A^T A$ is symmetric, the eigenvectors can be made orthonormal). Put the eigenvectors into a matrix as columns:

$$V_+ = [\mathbf{v}_1 | \mathbf{v}_2 | \ldots | \mathbf{v}_r].$$

4. For $i = 1, \ldots, r$, define $\sigma_i = \sqrt{\lambda_i}$. These are called the singular values of A. Put the singular values into the diagonal entries of a diagonal matrix:

$$\Sigma_+ = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix}.$$

5. For $i = 1, \ldots, r$, define

$$\mathbf{u}_i = \tfrac{1}{\sigma_i} A \mathbf{v}_i.$$

   Put these vectors into a matrix as columns:

$$U_+ = [\mathbf{u}_1 | \mathbf{u}_2 | \ldots | \mathbf{u}_r].$$

   The eigenvalues $\lambda_1, \ldots, \lambda_r$ of $A^T A$ that we found turn out to also be eigenvalues of AA^T, and the vectors $\mathbf{u}_1, \ldots, \mathbf{u}_r$ turn out to be corresponding orthonormal eigenvectors of AA^T.

6. The SVD of A is $A = U_+ \Sigma_+ V_+^T$.

Note: The + subscripts are my own notation. It is to emphasize that we are only looking at the positive eigenvalues of $A^T A$ and AA^T. Also, it is perhaps worth noting the sizes of all the matrices involved: A is $m \times n$, U_+ is $m \times r$, Σ is $r \times r$, and V_+ is $n \times r$.

The procedure outlined on the previous page begins with calculating the eigenvalues and eigenvectors of A^TA. It is possible to instead start by calculating the eigenvalues and eigenvectors of AA^T. The main difference in starting with AA^T instead of A^TA is that one finds the matrix U_+ first, and uses it to calculate V_+. We outline the procedure below, with slightly less details:

1. Compute the eigenvalues of AA^T, and order them from largest to smallest:

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0 = \lambda_{r+1} = \ldots = \lambda_m.$$

If $r = m$, then all of the eigenvalues of A^TA were positive. Otherwise, if $r < m$, then zero was an eigenvalue of AA^T of multiplicity $m - r$. For the SVD, we do not care about the zero eigenvalues, but keep them in mind because we need them for the Full SVD.

3. Find orthonormal eigenvectors $\mathbf{u}_1, \ldots, \mathbf{u}_r$ corresponding to the positive eigenvalues $\lambda_1, \ldots, \lambda_r$. Put the eigenvectors into a matrix as columns:

$$U_+ = [\mathbf{u}_1|\mathbf{u}_2|\ldots|\mathbf{u}_r].$$

4. Again, for $i = 1, 2, \ldots, r$, the singular values of A are $\sigma_i = \sqrt{\lambda_i}$. Put the singular values into the diagonal entries of a diagonal matrix:

$$\Sigma_+ = \begin{bmatrix} \sigma_1 & & & \\ & \sigma_2 & & \\ & & \ddots & \\ & & & \sigma_r \end{bmatrix}.$$

5. For $i = 1, 2, \ldots, r$, define

$$\mathbf{v}_i = \tfrac{1}{\sigma_i} A^T \mathbf{u}_i.$$

Put these vectors into a matrix as columns:

$$V_+ = [\mathbf{v}_1|\mathbf{v}_2|\ldots|\mathbf{v}_r].$$

6. The SVD of A is $A = U_+\Sigma_+V_+^T$.

It probably doesn't matter which of these two procedures you use to compute the SVD, since the matrices you will encounter in problems should be fairly small. However, note that if A is $m \times n$, then A^TA is $n \times n$ while AA^T is $m \times m$. Thus, if $n < m$, then A^TA will be smaller than AA^T. Hence, it may be easier to start with A^TA in this case. Similarly, if $m > n$, it may be easier to start with AA^T instead.

Example. Find an SVD for the following matrix:

$$A = \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 2 & 0 \end{bmatrix}$$

Solution: We will go through each of the six steps, one at a time.

1. Compute A^TA:

$$A^TA = \begin{bmatrix} 0 & 1 & 2 \\ 2 & -1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} 5 & -1 \\ -1 & 5 \end{bmatrix}.$$

2. Find the eigenvalues of A^TA and order them from largest to smallest.

$$\det(\lambda I - A^TA) = \begin{vmatrix} \lambda - 5 & 1 \\ 1 & \lambda - 5 \end{vmatrix} = (\lambda - 5)^2 - 1 = 0.$$

This gives us two equations:

$$\lambda - 5 = \pm 1.$$

Thus,

$$\lambda_1 = 6 \qquad \text{and} \qquad \lambda_2 = 4.$$

Notice how they are ordered from largest to smallest.

3. Find orthonormal eigenvectors corresponding to the positive eigenvalues. In this case, all eigenvalues found were positive, so we need to find all of the eigenspaces.

   Start with $\lambda_1 = 6$:

$$6I - A^T A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

   Letting $x_2 = t$ and solving the homogeneous equation, we find the corresponding eigenspace to be all vectors of the form

$$t \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

   Repeat with $\lambda_2 = 4$:

$$4I - A^T A = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

   Letting $x_2 = t$ and solving the homogeneous equation, we find the corresponding eigenspace to be all vectors of the form

$$t \begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

   The eigenvectors $(-1, 1)$ and $(1, 1)$ that we found are already orthogonal to each other, as expected since they correspond to distinct eigenvalues of the symmetric matrix $A^T A$. So we just need to normalize them to get the orthonormal eigenvectors.

$$\mathbf{v}_1 = \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} \qquad \mathbf{v}_2 = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

   Put them into the matrix V_+ as columns:

$$V_+ = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

4. The singular values are the square roots of the eigenvalues.

$$\sigma_1 = \sqrt{6} \qquad\qquad \sigma_2 = \sqrt{4} = 2.$$

   These are the diagonal entries of the diagonal matrix Σ_+:

$$\Sigma_+ = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 2 \end{bmatrix}$$

5. Use the formula $\mathbf{u}_i = \frac{1}{\sigma_i} A\mathbf{v}_i$ to find U_+.

$$\mathbf{u}_1 = \tfrac{1}{\sigma_1} A\mathbf{v}_1 = \frac{1}{\sqrt{6}} \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}$$

$$\mathbf{u}_2 = \tfrac{1}{\sigma_2} A\mathbf{v}_2 = \frac{1}{2} \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 2 & 0 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}.$$

Thus, we have

$$U_+ = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

6. The SVD of A is $A = U_+\Sigma_+V_+^T$:

$$A = \begin{bmatrix} 0 & 2 \\ 1 & -1 \\ 2 & 0 \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T.$$ ◇

9.3 Alternate Form of SVD

Using the same notation as in the previous section, the SVD of A can also be written as

$$A = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T.$$

Example. Suppose we want to express our answer to the previous example in this alternate form. Since we found only two positive singular values, our answer would be in the form

$$A = \sum_{i=1}^{r} \sigma_i \mathbf{u}_i \mathbf{v}_i^T = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T + \sigma_2 \mathbf{u}_2 \mathbf{v}_2^T.$$

Plugging in what we found for $\sigma_1, \sigma_2, \mathbf{u}_1, \mathbf{u}_2, \mathbf{v}_1, \mathbf{v}_2$ gives us

$$A = \sqrt{6} \begin{bmatrix} \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix} + 2 \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$ ◇

What we have talked about so far is also sometimes referred to as the reduced or compact SVD (I don't think there is any standard terminology). In the next section, we take a look at the "full" SVD.

9.4 The "Full" SVD

The main difference between the reduced and full SVDs are that for the full SVD, you do not ignore the zero eigenvalues. If A is $m \times n$, then the full SVD of A can be written as $A = U\Sigma V^T$, where U is $m \times m$, Σ is $m \times n$, and V is $n \times n$. We outline a procedure below for finding U, Σ, and V.

1. Compute A^TA and find its eigenvalues. Order them from largest to smallest. This time, do not ignore the zero eigenvalues, if there are any.

$$\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_r > 0 = \lambda_{r+1} = \ldots = \lambda_n.$$

2. Find orthonormal eigenvectors $\mathbf{v}_1, \ldots, \mathbf{v}_r, \mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$ of A^TA corresponding to the above eigenvectors. Put them into a matrix as columns to get V:

$$V = [\mathbf{v}_1|\ldots|\mathbf{v}_r|\mathbf{v}_{r+1}|\ldots|\mathbf{v}_n].$$

3. Take the positive singular values $\sigma_1, \ldots, \sigma_r$ and put them into an $r \times r$ diagonal matrix Σ_+ just like in the SVD. Afterwards, add rows and columns of zeros to Σ_+ as needed until you have a matrix the same size as A. Call this new matrix Σ.

4. For $i = 1, 2, \ldots, r$, compute

$$\mathbf{u}_i = \tfrac{1}{\sigma_i} A\mathbf{v}_i$$

as before. If $r = m$, then put these vectors into a matrix U as columns and proceed to Step 5.

Probably, however, r will be less than m. If this happens, then go find an orthonormal basis for the nullspace of A^T. To do this, you'll need to solve the homogeneous equation $A^T\mathbf{x} = \mathbf{0}$ and get a basis for the nullspace of A^T. If the basis vectors are not already orthogonal to each other, then you need to perform Gram-Schmidt on them to obtain an orthogonal basis. Then normalize all of the vectors to get an orthonormal basis. Call these vectors $\mathbf{u}_{r+1}, \mathbf{u}_{r+2}, \ldots, \mathbf{u}_m$.

The vectors $\mathbf{u}_1, \ldots, \mathbf{u}_r, \mathbf{u}_{r+1}, \ldots, \mathbf{u}_m$ turn out to be orthonormal eigenvectors of AA^T, corresponding to the same eigenvalues found in Step 1. Put them into the columns of a matrix to get U.

$$U = [\mathbf{u}_1|\ldots|\mathbf{u}_r|\mathbf{u}_{r+1}|\ldots|\mathbf{u}_m].$$

Example. Find a full SVD for the following matrix:

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Solution: Note that A is 3×2, so U should be 3×3, Σ should be 3×2, and V should be 2×2.

1. First, compute A^TA and find its eigenvalues.

$$A^TA = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & -3 \\ -3 & 3 \end{bmatrix}.$$

Next, we find the eigenvalues:

$$\det(\lambda I - A^TA) = (\lambda - 3)^2 - 9 = 0.$$

Thus, $(\lambda - 3)^2 = 9$, which gives us

$$\lambda_1 = 6 \qquad \text{and} \qquad \lambda_2 = 0.$$

2. Next, we find the corresponding orthonormal eigenvectors. Start with $\lambda_1 = 6$:

$$6I - A^TA = \begin{bmatrix} 3 & 3 \\ 3 & 3 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}.$$

Letting $x_2 = t$ and solving the homogeneous equation, we find the corresponding eigenspace to be

$$t\begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Next, do the same with $\lambda_2 = 0$:

$$0I - A^TA = \begin{bmatrix} -3 & 3 \\ 3 & -3 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -1 \\ 0 & 0 \end{bmatrix}.$$

Letting $x_2 = t$ and solving the homogeneous equation, we find the corresponding eigenspace to be

$$t\begin{bmatrix} 1 \\ 1 \end{bmatrix}.$$

The eigenvectors

$$\begin{bmatrix} -1 \\ 1 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

are already orthogonal, as expected since they correspond to distinct eigenvalues of a symmetric matrix. We just need to normalize them to make them orthonormal.

$$\mathbf{v}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} -1 \\ 1 \end{bmatrix} \qquad \mathbf{v}_2 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

Put these orthonormal eigenvectors into V as columns:

$$V = \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}.$$

3. The only positive eigenvalue we found was $\lambda_1 = 6$, so the only positive singular value is $\sigma_1 = \sqrt{6}$. Put this into the diagonal of a matrix, and then add zeros to the matrix to make it the same size as A:

$$\Sigma = \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}.$$

4. Since we have one positive singular value, we can use the formula $\mathbf{u}_i = \frac{1}{\sigma_i}A\mathbf{v}_i$ to find one vector:

$$\mathbf{u}_1 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix}\begin{bmatrix} -\frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \end{bmatrix} = \frac{1}{\sqrt{12}}\begin{bmatrix} -2 \\ 2 \\ -2 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{3}} \\ \frac{1}{\sqrt{3}} \\ -\frac{1}{\sqrt{3}} \end{bmatrix}.$$

As noted earlier, U should be 3×3. We can get $\mathbf{u}_2$ and $\mathbf{u}_3$ by finding an orthonormal basis of the nullspace of A^T.

$$A^T = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \end{bmatrix} \xrightarrow{RREF} \begin{bmatrix} 1 & -1 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

We see that we can choose x_2 and x_3 to be free variables. Letting $x_2 = s$, $x_3 = t$, and solving the homogeneous equation, we find the solutions to be of the form

$$s\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + t\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}.$$

The vectors

$$\begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$$

are a basis for the nullspace of A^T, but they are not orthogonal to each other, so we perform Gram-Schmidt on these two vectors to get an orthogonal basis first.

Let's call these two vectors $\mathbf{x}$ and $\mathbf{y}$, respectively. Gram-Schmidt will produce the following two vectors:

$$\mathbf{X} = \mathbf{x}$$

$$\mathbf{Y} = \mathbf{y} - \left(\frac{\mathbf{y} \cdot \mathbf{X}}{\mathbf{X} \cdot \mathbf{X}}\right) \mathbf{X}$$

Computing, we get

$$\mathbf{X} = \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{Y} = \begin{bmatrix} -\frac{1}{2} \\ \frac{1}{2} \\ 1 \end{bmatrix}.$$

Now that we have an orthogonal basis for the nullspace of A^T, we normalize them to get an orthonormal basis. The result will be the vectors $\mathbf{u}_2$ and $\mathbf{u}_3$ that we need for the Full SVD:

$$\mathbf{u}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$$

$$\mathbf{u}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}.$$

Finally, put $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ in a matrix U as columns:

$$U = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix}.$$

5. The full SVD of A is $A = U\Sigma V^T$, which written out gives us

$$A = \begin{bmatrix} 1 & -1 \\ -1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} -\frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{6}} \\ \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{6}} \\ -\frac{1}{\sqrt{3}} & 0 & \frac{2}{\sqrt{6}} \end{bmatrix} \begin{bmatrix} \sqrt{6} & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{bmatrix}^T$$ ◇

9.5 Special Case: Rank 1 Matrices

The matrix A in the last example was a rank 1 matrix since all of its rows were multiples of each other (or equivalently, all of its columns are multiples of each other). In general, if A is an $m \times n$ rank 1 matrix, then A can be written as

$$A = \mathbf{u}\mathbf{v}^T$$

for some $m \times 1$ vector $\mathbf{u}$ and some $n \times 1$ vector $\mathbf{v}$. For example, if

$$A = \begin{bmatrix} 2 & 3 & 4 \\ -2 & -3 & -4 \\ 4 & 6 & 8 \\ 2 & 3 & 4 \end{bmatrix},$$

then we can write it as

$$A = \overbrace{\begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix}}^{\mathbf{u}} \overbrace{\begin{bmatrix} 2 & 3 & 4 \end{bmatrix}}^{\mathbf{v}^T}.$$

The choice of $\mathbf{u}$ and $\mathbf{v}$ is not unique, meaning there is more than one way to rewrite A as $\mathbf{u}\mathbf{v}^T$ for some $\mathbf{u}$ and $\mathbf{v}$. Let's normalize the $\mathbf{u}$ and $\mathbf{v}$ that we are using and call the normalized vectors $\mathbf{u}_1$ and $\mathbf{v}_1$:

$$\mathbf{u}_1 = \frac{1}{\sqrt{7}} \begin{bmatrix} 1 \\ -1 \\ 2 \\ 1 \end{bmatrix} \qquad \mathbf{v}_1 = \frac{1}{\sqrt{29}} \begin{bmatrix} 2 & 3 & 4 \end{bmatrix}.$$

Note that $7 \cdot 29 = 203$. We can rewrite A as

$$A = \sigma_1 \mathbf{u}_1 \mathbf{v}_1^T,$$

where $\sigma_1 = \sqrt{203}$. It turns out that this is a reduced SVD of A. If you want to find a full SVD of A, then you'll need to find an orthonormal basis for the nullspace of A^T for $\mathbf{u}_2, \mathbf{u}_3, \mathbf{u}_4$, and an orthonormal basis for the nullspace of A for $\mathbf{v}_2, \mathbf{v}_3$. Thus, we can find the SVD and even the full SVD of a rank 1 matrix A without computing $A^T A$ or AA^T and finding its eigenvalues and eigenvectors.

This generalizes for any rank 1 matrix (including $m \times 1$ and $1 \times n$ matrices). So it is quite a bit easier to find an SVD for rank 1 matrices.

9.6 Quick Facts about SVD (Some Old, Some New)

The matrices A^TA and AA^T are positive semidefinite and have the same positive eigenvalues. The singular values of A are the square roots of these positive eigenvalues.

Let $A = U\Sigma V^T$ be a full SVD of A. Let r be the number of positive singular values of A. Then the rank of A is r.

Let $\mathbf{u}_1, \ldots, \mathbf{u}_r$ be the first r columns of U. These are orthonormal eigenvectors of AA^T corresponding to its positive eigenvalues. They also form an orthonormal basis for the column space of A.

Let $\mathbf{u}_{r+1}, \ldots, \mathbf{u}_m$ be the remaining columns of U. These are orthonormal eigenvectors of AA^T corresponding to any zero eigenvalues. It turns out that they also form an orthonormal basis for the left nullspace of A.

Let $\mathbf{v}_1, \ldots, \mathbf{v}_r$ be the first r columns of V. These are orthonormal eigenvectors of A^TA corresponding to its positive eigenvalues. They also form an orthonormal basis for the row space of A.

Let $\mathbf{v}_{r+1}, \ldots, \mathbf{v}_n$ be the remaining columns of V. These are orthonormal eigenvectors of A^TA corresponding to any zero eigenvalues. They also form an orthonormal basis for the nullspace of A.

We have the formula

$$\mathbf{u}_i = \tfrac{1}{\sigma_i} A\mathbf{v}_i \qquad i = 1, \ldots, r,$$

which can be used to solve for $\mathbf{u}_i$, given $\mathbf{v}_i$. There is an analogous formula,

$$\mathbf{v}_i = \tfrac{1}{\sigma_i} A^T\mathbf{u}_i \qquad i = 1, \ldots, r,$$

which can be used to solve for $\mathbf{v}_i$, given $\mathbf{u}_i$.

Recall from Subsection 8.8.5 that for a symmetric matrix B, the maximal value of $\mathbf{x}^TB\mathbf{x}$ among all unit length vectors $\mathbf{x}$ is equal to the largest eigenvalue of B. Applying this result to A^TA, we find that

$$\max_{\|\mathbf{x}\|=1} \mathbf{x}^TA^TA\mathbf{x} \;=\; \lambda_1.$$

Since

$$\mathbf{x}^TA^TA\mathbf{x} = (A\mathbf{x})^T(A\mathbf{x}) = (A\mathbf{x})\cdot(A\mathbf{x}) = \|A\mathbf{x}\|^2,$$

we get

$$\max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\|^2 \;=\; \lambda_1,$$

or equivalently,

$$\max_{\|\mathbf{x}\|=1} \|A\mathbf{x}\| \;=\; \sigma_1,$$

Thus, the maximum value of $\|A\mathbf{x}\|$ among all unit vectors $\mathbf{x}$ is equal to σ_1, the largest singular value of A.