$$\boldsymbol{v} \cdot \boldsymbol{w} = xz + xy + yz = \frac{1}{2}(x+y+z)^2 - \frac{1}{2}(x^2+y^2+z^2)$$

This is the same as $\boldsymbol{v} \cdot \boldsymbol{w} = 0 - \frac{1}{2}\,\|\boldsymbol{v}\|\|\boldsymbol{w}\|$. Then $\cos\theta = \frac{1}{2}$.

**32** Wikipedia gives this proof of geometric mean $G = \sqrt[3]{xyz} \leq$ arithmetic mean $A = (x+y+z)/3$. First there is equality in case $x = y = z$. Otherwise $A$ is somewhere between the three positive numbers, say for example $z < A < y$.

Use the known inequality $g \leq a$ for the *two* positive numbers $x$ and $y + z - A$. Their mean $a = \frac{1}{2}(x + y + z - A)$ is $\frac{1}{2}(3A - A)$ = same as $A$! So $a \geq g$ says that $A^3 \geq g^2 A = x(y + z - A)A$. But $(y + z - A)A = (y - A)(A - z) + yz > yz$. Substitute to find $A^3 > xyz = G^3$ as we wanted to prove. Not easy!

There are many proofs of $G = (x_1 x_2 \cdots x_n)^{1/n} \leq A = (x_1 + x_2 + \cdots + x_n)/n$. In calculus you are maximizing $G$ on the plane $x_1 + x_2 + \cdots + x_n = n$. The maximum occurs when all $x$'s are equal.

**33** The columns of the 4 by 4 "Hadamard matrix" (times $\frac{1}{2}$) are perpendicular unit vectors:

$$\frac{1}{2}H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

**34** The commands $V =$ **randn** $(3, 30)$; $D =$ **sqrt** (**diag** $(V' * V)$); $U = V \backslash D$; will give 30 random unit vectors in the columns of $U$. Then $\boldsymbol{u}' * U$ is a row matrix of 30 dot products whose average absolute value should be close to $2/\pi$.

## Problem Set 1.3, page 29

**1** $3\boldsymbol{s}_1 + 4\boldsymbol{s}_2 + 5\boldsymbol{s}_3 = (3, 7, 12)$. The same vector $\boldsymbol{b}$ comes from $S$ times $\boldsymbol{x} = (3, 4, 5)$:

$$\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 3 \\ 4 \\ 5 \end{bmatrix} = \begin{bmatrix} (\text{row } 1) \cdot \boldsymbol{x} \\ (\text{row } 2) \cdot \boldsymbol{x} \\ (\text{row } 2) \cdot \boldsymbol{x} \end{bmatrix} = \begin{bmatrix} 3 \\ 7 \\ 12 \end{bmatrix}.$$

**2** The solutions are $y_1 = 1$, $y_2 = 0$, $y_3 = 0$ (right side $=$ column 1) and $y_1 = 1$, $y_2 = 3$, $y_3 = 5$. That second example illustrates that the first $n$ odd numbers add to $n^2$.

**3** $$\begin{aligned} y_1 \quad\quad &= B_1 \\ y_1 + y_2 \quad &= B_2 \\ y_1 + y_2 + y_3 &= B_3 \end{aligned} \quad \text{gives} \quad \begin{aligned} y_1 &= B_1 \\ y_2 &= -B_1 + B_2 \\ y_3 &= \quad\quad -B_2 + B_3 \end{aligned} \quad = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} B_1 \\ B_2 \\ B_3 \end{bmatrix}$$

The inverse of $S = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$ is $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ : **independent** columns in $A$ and $S$!

**4** The combination $0\boldsymbol{w}_1 + 0\boldsymbol{w}_2 + 0\boldsymbol{w}_3$ always gives the zero vector, but this problem looks for other *zero* combinations (then the vectors are *dependent*, they lie in a plane): $\boldsymbol{w}_2 = (\boldsymbol{w}_1 + \boldsymbol{w}_3)/2$ so one combination that gives zero is $\boldsymbol{w}_1 - 2\boldsymbol{w}_2 + \boldsymbol{w}_3 = \mathbf{0}$.

**5** The rows of the 3 by 3 matrix in Problem 4 must also be *dependent*: $\boldsymbol{r}_2 = \frac{1}{2}(\boldsymbol{r}_1 + \boldsymbol{r}_3)$. The column and row combinations that produce $\mathbf{0}$ are the same: this is unusual. Two solutions to $y_1\boldsymbol{r}_1 + y_2\boldsymbol{r}_2 + y_3\boldsymbol{r}_3 = \mathbf{0}$ are $(Y_1, Y_2, Y_3) = (1, -2, 1)$ and $(2, -4, 2)$.

**6** $c = \mathbf{3}$ $\quad \begin{bmatrix} 1 & 1 & 0 \\ 3 & 2 & 1 \\ 7 & 4 & \mathbf{3} \end{bmatrix}$ has column $3 =$ column $1 -$ column 2

$c = \mathbf{-1}$ $\quad \begin{bmatrix} 1 & 0 & \mathbf{-1} \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ has column $3 = -$ column $1 +$ column 2

$c = \mathbf{0}$ $\quad \begin{bmatrix} \mathbf{0} & \mathbf{0} & \mathbf{0} \\ 2 & 1 & 5 \\ 3 & 3 & 6 \end{bmatrix}$ has column $3 = 3$ (column 1) $-$ column 2

**7** All three rows are perpendicular to the solution $\boldsymbol{x}$ (the three equations $\boldsymbol{r}_1 \cdot \boldsymbol{x} = 0$ and $\boldsymbol{r}_2 \cdot \boldsymbol{x} = 0$ and $\boldsymbol{r}_3 \cdot \boldsymbol{x} = 0$ tell us this). Then the whole plane of the rows is perpendicular to $\boldsymbol{x}$ (the plane is also perpendicular to all multiples $c\boldsymbol{x}$).

**8**
$$\begin{aligned} x_1 - 0 &= b_1 \\ x_2 - x_1 &= b_2 \\ x_3 - x_2 &= b_3 \\ x_4 - x_3 &= b_4 \end{aligned} \qquad \begin{aligned} x_1 &= b_1 \\ x_2 &= b_1 + b_2 \\ x_3 &= b_1 + b_2 + b_3 \\ x_4 &= b_1 + b_2 + b_3 + b_4 \end{aligned} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = A^{-1}\boldsymbol{b}$$

**9** The cyclic difference matrix $C$ has a line of solutions (in $4$ dimensions) to $C\boldsymbol{x} = \mathbf{0}$:

$$\begin{bmatrix} 1 & 0 & 0 & -1 \\ -1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 0 & 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \text{ when } \boldsymbol{x} = \begin{bmatrix} c \\ c \\ c \\ c \end{bmatrix} = \text{any constant vector.}$$

**10**
$$\begin{aligned} z_2 - z_1 &= b_1 \\ z_3 - z_2 &= b_2 \\ 0 - z_3 &= b_3 \end{aligned} \qquad \begin{aligned} z_1 &= -b_1 - b_2 - b_3 \\ z_2 &= \phantom{-b_1} - b_2 - b_3 \\ z_3 &= \phantom{-b_1 - b_2} - b_3 \end{aligned} = \begin{bmatrix} -1 & -1 & -1 \\ 0 & -1 & -1 \\ 0 & 0 & -1 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \Delta^{-1}\boldsymbol{b}$$

**11** The forward differences of the squares are $(t+1)^2 - t^2 = t^2 + 2t + 1 - t^2 = 2t + 1$. Differences of the $n$th power are $(t+1)^n - t^n = t^n - t^n + nt^{n-1} + \cdots$. The leading term is the derivative $nt^{n-1}$. The binomial theorem gives all the terms of $(t+1)^n$.

**12** Centered difference matrices of *even size* seem to be invertible. Look at eqns. $1$ and $4$:

$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} \quad \begin{matrix} \text{First} \\ \text{solve} \\ x_2 = b_1 \\ -x_3 = b_4 \end{matrix} \quad \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} -b_2 - b_4 \\ b_1 \\ -b_4 \\ b_1 + b_3 \end{bmatrix}$$

**13** *Odd size*: The five centered difference equations lead to $b_1 + b_3 + b_5 = 0$.

$$\begin{aligned} x_2 \qquad &= b_1 \\ x_3 - x_1 &= b_2 \\ x_4 - x_2 &= b_3 \\ x_5 - x_3 &= b_4 \\ - x_4 &= b_5 \end{aligned}$$

Add equations $1, 3, 5$

The left side of the sum is zero

The right side is $b_1 + b_3 + b_5$

There cannot be a solution unless $b_1 + b_3 + b_5 = 0$.

**14** An example is $(a, b) = (3, 6)$ and $(c, d) = (1, 2)$. We are given that the ratios $a/c$ and $b/d$ are equal. Then $ad = bc$. Then (when you divide by $bd$) the ratios $a/b$ and $c/d$ must also be equal!