# CHAPTER 8
# Advanced Counting Techniques

## SECTION 8.1 Applications of Recurrence Relations

*This section picks up where Section 2.4 left off; recurrence relations were first introduced there. In addition this section is related to Section 5.3, in that recurrence relations are in some sense really recursive or inductive definitions. Many of the exercises in this set provide practice in setting up such relations from a given applied situation. In each problem of this type, ask yourself how the $n^{\text{th}}$ term of the sequence can be related to one or more previous terms; the answer is the desired recurrence relation.*

*Exercise 21 is interesting and challenging, and shows that the inductive step may be quite nontrivial. Exercise 29 deals with onto functions; another—totally different—approach to counting onto functions is given in Section 8.6. Exercises 30–45 deal with additional interesting applications.*

**1.** We want to show that $H_n = 2^n - 1$ is a solution to the recurrence relation $H_n = 2H_{n-1} + 1$ with initial condition $H_1 = 1$. For $n = 1$ (the base case), this is simply the calculation that $2^1 - 1 = 1$. Assume that $H_n = 2^n - 1$. Then by the recurrence relation we have $H_{n+1} = 2H_n + 1$, whereupon if we substitute on the basis of the inductive hypothesis we obtain $2(2^n - 1) + 1 = 2^{n+1} - 2 + 1 = 2^{n+1} - 1$, exactly the formula for the case of $n + 1$. Thus we have shown that if the formula is correct for $n$, then it is also correct for $n + 1$, and our proof by induction is complete.

**3.** **a)** Let $a_n$ be the number of ways to deposit $n$ dollars in the vending machine. We must express $a_n$ in terms of earlier terms in the sequence. If we want to deposit $n$ dollars, we may start with a dollar coin and then deposit $n - 1$ dollars. This gives us $a_{n-1}$ ways to deposit $n$ dollars. We can also start with a dollar bill and then deposit $n - 1$ dollars. This gives us $a_{n-1}$ more ways to deposit $n$ dollars. Finally, we can deposit a five-dollar bill and follow that with $n - 5$ dollars; there are $a_{n-5}$ ways to do this. Therefore the recurrence relation is $a_n = 2a_{n-1} + a_{n-5}$. Note that this is valid for $n \geq 5$, since otherwise $a_{n-5}$ makes no sense.

**b)** We need initial conditions for all subscripts from 0 to 4. It is clear that $a_0 = 1$ (deposit nothing) and $a_1 = 2$ (deposit either the dollar coin or the dollar bill). It is also not hard to see that $a_2 = 2^2 = 4$, $a_3 = 2^3 = 8$, and $a_4 = 2^4 = 16$, since each sequence of $n$ C's and B's corresponds to a way to deposit $n$ dollars—a C meaning to deposit a coin and a B meaning to deposit a bill.

**c)** We will compute $a_5$ through $a_{10}$ using the recurrence relation:

$$\begin{aligned}
a_5 &= 2a_4 + a_0 = 2 \cdot 16 + 1 = 33 \\
a_6 &= 2a_5 + a_1 = 2 \cdot 33 + 2 = 68 \\
a_7 &= 2a_6 + a_2 = 2 \cdot 68 + 4 = 140 \\
a_8 &= 2a_7 + a_3 = 2 \cdot 140 + 8 = 288 \\
a_9 &= 2a_8 + a_4 = 2 \cdot 288 + 16 = 592 \\
a_{10} &= 2a_9 + a_5 = 2 \cdot 592 + 33 = 1217
\end{aligned}$$

Thus there are 1217 ways to deposit \$10.

**5.** Since this problem concerns a bill of 17 pesos, we can ignore all denominations greater than 17. Therefore we assume that we have coins for 1, 2, 5, and 10 pesos, and bills for 5 and 10 pesos. Then we proceed as in

Exercise 13 to write down a recurrence relation and initial conditions for $a_n$, the number of ways to pay a bill of $n$ pesos (order mattering). If we want to achieve a total of $n$ pesos, we can start with a 1-peso coin and then pay out $n-1$ pesos. This gives us $a_{n-1}$ ways to pay $n$ pesos. Similarly, there are $a_{n-2}$ ways to pay starting with a 2-peso coin, $a_{n-5}$ ways to pay starting with a 5-peso coin, $a_{n-10}$ ways to pay starting with a 10-peso coin, $a_{n-5}$ ways to pay starting with a 5-peso bill, and $a_{n-10}$ ways to pay starting with a 10-peso bill. This gives the recurrence relation $a_n = a_{n-1} + a_{n-2} + 2a_{n-5} + 2a_{n-10}$, valid for all $n \geq 10$. As for initial conditions, we see immediately that $a_0 = 1$ (there is one way to pay nothing, namely by using no coins or bills), $a_1 = 1$ (use a 1-peso coin), $a_2 = 2$ (use a 2-peso coin or two 1-peso coins), $a_3 = 3$ (use only 1-peso coins, or use a 2-peso coin either first or second), and $a_4 = 5$ (the bill can be paid using the schemes 1111, 112, 121, 211, or 22, with the obvious notation). For $n = 5$ through $n = 9$, we can iterate the recurrence relation $a_n = a_{n-1} + a_{n-2} + 2a_{n-5}$, since no 10-peso bills are involved. This yields:

$$a_5 = a_4 + a_3 + 2a_0 = 5 + 3 + 2 \cdot 1 = 10$$
$$a_6 = a_5 + a_4 + 2a_1 = 10 + 5 + 2 \cdot 1 = 17$$
$$a_7 = a_6 + a_5 + 2a_2 = 17 + 10 + 2 \cdot 2 = 31$$
$$a_8 = a_7 + a_6 + 2a_3 = 31 + 17 + 2 \cdot 3 = 54$$
$$a_9 = a_8 + a_7 + 2a_4 = 54 + 31 + 2 \cdot 5 = 95$$

Next we iterate the full recurrence relation to get up to $n = 17$:

$$a_{10} = a_9 + a_8 + 2a_5 + 2a_0 = 95 + 54 + 2 \cdot 10 + 2 \cdot 1 = 171$$
$$a_{11} = a_{10} + a_9 + 2a_6 + 2a_1 = 171 + 95 + 2 \cdot 17 + 2 \cdot 1 = 302$$
$$a_{12} = a_{11} + a_{10} + 2a_7 + 2a_2 = 302 + 171 + 2 \cdot 31 + 2 \cdot 2 = 539$$
$$a_{13} = a_{12} + a_{11} + 2a_8 + 2a_3 = 539 + 302 + 2 \cdot 54 + 2 \cdot 3 = 955$$
$$a_{14} = a_{13} + a_{12} + 2a_9 + 2a_4 = 955 + 539 + 2 \cdot 95 + 2 \cdot 5 = 1694$$
$$a_{15} = a_{14} + a_{13} + 2a_{10} + 2a_5 = 1694 + 955 + 2 \cdot 171 + 2 \cdot 10 = 3011$$
$$a_{16} = a_{15} + a_{14} + 2a_{11} + 2a_6 = 3011 + 1694 + 2 \cdot 302 + 2 \cdot 17 = 5343$$
$$a_{17} = a_{16} + a_{15} + 2a_{12} + 2a_7 = 5343 + 3011 + 2 \cdot 539 + 2 \cdot 31 = 9494$$

Thus the final answer is that there are 9494 ways to pay a 17-peso debt using the coins and bills described here, assuming that order matters.

**7.** **a)** Let $a_n$ be the number of bit strings of length $n$ containing a pair of consecutive 0's. In order to construct a bit string of length $n$ containing a pair of consecutive 0's we could start with 1 and follow with a string of length $n-1$ containing a pair of consecutive 0's, or we could start with a 01 and follow with a string of length $n-2$ containing a pair of consecutive 0's, or we could start with a 00 and follow with any string of length $n-2$. These three cases are mutually exclusive and exhaust the possibilities for how the string might start. From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 2$: $a_n = a_{n-1} + a_{n-2} + 2^{n-2}$. (Recall that there are $2^k$ bit strings of length $k$.)

**b)** There are no bit strings of length 0 or 1 containing a pair of consecutive 0's, so the initial conditions are $a_0 = a_1 = 0$.

**c)** We will compute $a_2$ through $a_7$ using the recurrence relation:

$$a_2 = a_1 + a_0 + 2^0 = 0 + 0 + 1 = 1$$
$$a_3 = a_2 + a_1 + 2^1 = 1 + 0 + 2 = 3$$
$$a_4 = a_3 + a_2 + 2^2 = 3 + 1 + 4 = 8$$
$$a_5 = a_4 + a_3 + 2^3 = 8 + 3 + 8 = 19$$
$$a_6 = a_5 + a_4 + 2^4 = 19 + 8 + 16 = 43$$
$$a_7 = a_6 + a_5 + 2^5 = 43 + 19 + 32 = 94$$

Thus there are 94 bit strings of length 7 containing two consecutive 0's.

**9.** **a)** This problem is very similar to Example 3, with the recurrence required to go one level deeper. Let $a_n$ be the number of bit strings of length $n$ that do not contain three consecutive 0's. In order to construct a bit string of length $n$ of this type we could start with 1 and follow with a string of length $n-1$ not containing three consecutive 0's, or we could start with a 01 and follow with a string of length $n-2$ not containing three consecutive 0's, or we could start with a 001 and follow with a string of length $n-2$ not containing three consecutive 0's. These three cases are mutually exclusive and exhaust the possibilities for how the string might start, since it cannot start 000. From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 3$: $a_n = a_{n-1} + a_{n-2} + a_{n-3}$.

**b)** The initial conditions are $a_0 = 1$, $a_1 = 2$, and $a_2 = 4$, since all strings of length less than 3 satisfy the conditions (recall that the empty string has length 0).

**c)** We will compute $a_3$ through $a_7$ using the recurrence relation:

$$\begin{aligned}
a_3 &= a_2 + a_1 + a_0 = 4 + 2 + 1 = 7 \\
a_4 &= a_3 + a_2 + a_1 = 7 + 4 + 2 = 13 \\
a_5 &= a_4 + a_3 + a_2 = 13 + 7 + 4 = 24 \\
a_6 &= a_5 + a_4 + a_3 = 24 + 13 + 7 = 44 \\
a_7 &= a_6 + a_5 + a_4 = 44 + 24 + 13 = 81
\end{aligned}$$

Thus there are 81 bit strings of length 7 that do not contain three consecutive 0's.

**11.** **a)** Let $a_n$ be the number of ways to climb $n$ stairs. In order to climb $n$ stairs, a person must either start with a step of one stair and then climb $n-1$ stairs (and this can be done in $a_{n-1}$ ways) or else start with a step of two stairs and then climb $n-2$ stairs (and this can be done in $a_{n-2}$ ways). From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 2$: $a_n = a_{n-1} + a_{n-2}$.

**b)** The initial conditions are $a_0 = 1$ and $a_1 = 1$, since there is one way to climb no stairs (do nothing) and clearly only one way to climb one stair. Note that the recurrence relation is the same as that for the Fibonacci sequence, and the initial conditions are that $a_0 = f_1$ and $a_1 = f_2$, so it must be that $a_n = f_{n+1}$ for all $n$.

**c)** Each term in our sequence $\{a_n\}$ is the sum of the previous two terms, so the sequence begins $a_0 = 1$, $a_1 = 1$, $a_2 = 2$, $a_3 = 3$, $a_4 = 5$, $a_5 = 8$, $a_6 = 13$, $a_7 = 21$, $a_8 = 34$. Thus a person can climb a flight of 8 stairs in 34 ways under the restrictions in this problem.

**13.** **a)** Let $a_n$ be the number of ternary strings of length $n$ that do not contain two consecutive 0's. In order to construct a bit string of length $n$ of this type we could start with a 1 or a 2 and follow with a string of length $n-1$ not containing two consecutive 0's, or we could start with 01 or 02 and follow with a string of length $n-2$ not containing two consecutive 0's. There are clearly $2a_{n-1}$ possibilities in the first case and $2a_{n-2}$ possibilities in the second. These two cases are mutually exclusive and exhaust the possibilities for how the string might start, since it cannot start 00. From this analysis we can immediately write down the recurrence relation, valid for all $n \geq 2$: $a_n = 2a_{n-1} + 2a_{n-2}$.

**b)** The initial conditions are $a_0 = 1$ (for the empty string) and $a_1 = 3$ (all three strings of length 1 fail to contain two consecutive 0's).

**c)** We will compute $a_2$ through $a_6$ using the recurrence relation:

$$\begin{aligned}
a_2 &= 2a_1 + 2a_0 = 2 \cdot 3 + 2 \cdot 1 = 8 \\
a_3 &= 2a_2 + 2a_1 = 2 \cdot 8 + 2 \cdot 3 = 22 \\
a_4 &= 2a_3 + 2a_2 = 2 \cdot 22 + 2 \cdot 8 = 60 \\
a_5 &= 2a_4 + 2a_3 = 2 \cdot 60 + 2 \cdot 22 = 164 \\
a_6 &= 2a_5 + 2a_4 = 2 \cdot 164 + 2 \cdot 60 = 448
\end{aligned}$$

Thus there are 448 ternary strings of length 6 that do not contain two consecutive 0's.

**15.** **a)** Let $a_n$ be the number of ternary strings of length $n$ that do not contain two consecutive 0's or two consecutive 1's. In order to construct a bit string of length $n$ of this type we could start with a 2 and follow with a string of length $n-1$ not containing two consecutive 0's or two consecutive 1's, or we could start with 02 or 12 and follow with a string of length $n-2$ not containing two consecutive 0's or two consecutive 1's, or we could start with 012 or 102 and follow with a string of length $n-3$ not containing two consecutive 0's or two consecutive 1's, or we could start with 0102 or 1012 and follow with a string of length $n-4$ not containing two consecutive 0's or two consecutive 1's, and so on. In other words, once we encounter a 2, we can, in effect, start fresh, but the first 2 may not appear for a long time. Before the first 2 there are always two possibilities—the sequence must alternate between 0's and 1's, starting with either a 0 or a 1. Furthermore, there is one more possibility—that the sequence contains no 2's at all, and there are two cases in which this can happen: 0101... and 1010.... Putting this all together we can write down the recurrence relation, valid for all $n \geq 2$:

$$a_n = a_{n-1} + 2a_{n-2} + 2a_{n-3} + 2a_{n-4} + \cdots + 2a_0 + 2$$

(It turns out that the sequence also satisfies the recurrence relation $a_n = 2a_{n-1} + a_{n-2}$, which can be derived algebraically from the recurrence relation we just gave by subtracting the recurrence for $a_{n-1}$ from the recurrence for $a_n$. Can you find a direct argument for it?)

**b)** The initial conditions are that $a_0 = 1$ (the empty string satisfies the conditions) and $a_1 = 3$ (the condition cannot be violated in so short a string).

**c)** We will compute $a_2$ through $a_6$ using the recurrence relation:

$$a_2 = a_1 + 2a_0 + 2 = 3 + 2 \cdot 1 + 2 = 7$$
$$a_3 = a_2 + 2a_1 + 2a_0 + 2 = 7 + 2 \cdot 3 + 2 \cdot 1 + 2 = 17$$
$$a_4 = a_3 + 2a_2 + 2a_1 + 2a_0 + 2 = 17 + 2 \cdot 7 + 2 \cdot 3 + 2 \cdot 1 + 2 = 41$$
$$a_5 = a_4 + 2a_3 + 2a_2 + 2a_1 + 2a_0 + 2 = 41 + 2 \cdot 17 + 2 \cdot 7 + 2 \cdot 3 + 2 \cdot 1 + 2 = 99$$
$$a_6 = a_5 + 2a_4 + 2a_3 + 2a_2 + 2a_1 + 2a_0 + 2 = 99 + 2 \cdot 41 + 2 \cdot 17 + 2 \cdot 7 + 2 \cdot 3 + 2 \cdot 1 + 2 = 239$$

Thus there are 239 ternary strings of length 6 that do not contain two consecutive 0's or two consecutive 1's.

**17.** **a)** Let $a_n$ be the number of ternary strings that do not contain consecutive symbols that are the same. By symmetry we know that $a_n/3$ of these must start with each of the symbols 0, 1, and 2. Now to construct such a string, we can begin with any symbol (3 choices), but we must follow it with a string of length $n-1$ not containing two consecutive symbols that are the same and not beginning with the symbol with which we began ($\frac{2}{3}a_{n-1}$ choices). This tells us that $a_n = 3 \cdot \frac{2}{3}a_{n-1}$, or more simply $a_n = 2a_{n-1}$, valid for every $n \geq 2$.

**b)** The initial condition is clearly that $a_1 = 3$. (We could also mention that $a_0 = 1$, but the recurrence only goes one level deep.)

**c)** Here it is easy to compute the terms in the sequence, since each is just 2 times the previous one. Thus $a_6 = 2a_5 = 2^2a_4 = 2^3a_3 = 2^4a_2 = 2^5a_1 = 2^5 \cdot 3 = 96$.

**19.** **a)** This problem is really the same as ("isomorphic to," as a mathematician would say) Exercise 11, since a sequence of signals exactly corresponds to a sequence of steps in that exercise. Therefore the recurrence relation is $a_n = a_{n-1} + a_{n-2}$ for all $n \geq 2$.

**b)** The initial conditions are again the same as in Exercise 11, namely $a_0 = 1$ (the empty message) and $a_1 = 1$.

**c)** Continuing where we left off our calculation in Exercise 11, we find that $a_9 = a_8 + a_7 = 34 + 21 = 55$ and then $a_{10} = a_9 + a_8 = 55 + 34 = 89$. (If we allow only part of the time period to be used, and if we rule out the empty message, then the answer will be $1 + 2 + 3 + 5 + 8 + 13 + 21 + 34 + 55 + 89 = 231$.)

**21.** **a)** This problem is related to Exercise 62 in Section 5.1. Consider the plane already divided by $n-1$ lines into $R_{n-1}$ regions. The $n^{\text{th}}$ line is now added, intersecting each of the other $n-1$ lines in exactly one point, $n-1$ intersections in all. Think of drawing that line, beginning at one of its ends (out at "infinity"). (You should be drawing a picture as you read these words!) As we move toward the first point of intersection, we are dividing the unbounded region of the plane through which it is passing into two regions; the division is complete when we reach the first point of intersection. Then as we draw from the first point of intersection to the second, we cut off another region (in other words we divide another of the regions that were already there into two regions). This process continues as we encounter each point of intersection. By the time we have reached the last point of intersection, the number of regions has increased by $n-1$ (one for each point of intersection). Finally, as we move off to infinity, we divide the unbounded region through which we pass into two regions, increasing the count by yet 1 more. Thus there are exactly $n$ more regions than there were before the $n^{\text{th}}$ line was added. The analysis we have just completed shows that the recurrence relation we seek is $R_n = R_{n-1} + n$. The initial condition is $R_0 = 1$ (since there is just one region—the whole plane—when there are no lines). Alternately, we could specify $R_1 = 2$ as the initial condition.

**b)** The recurrence relation and initial condition we have are precisely those in Exercise 9c, so the solution is $R_n = (n^2+n+2)/2$.

**23.** This problem is intimately related to Exercise 50 in the supplementary set of exercises in Chapter 5. It also will use the result of Exercise 21 of the present section.

**a)** Imagine $n-1$ planes meeting the stated conditions, dividing space into $S_{n-1}$ solid regions. (This may be hard to visualize once $n$ gets to be more than 2 or 3, but you should try to see it in your mind, even if the picture is blurred.) Now a new plane is drawn, intersecting each of the previous $n-1$ planes in a line. Look at the pattern these lines form on the new plane. There are $n-1$ lines, each two of which intersect and no three of which pass through the same point (because of the requirement on the "general position" of the planes). According to the result of Exercise 21b, they form $((n-1)^2+(n-1)+2)/2 = (n^2-n+2)/2$ regions in the new plane. Now each of these planar regions is actually splitting a former solid region into two. Thus the number of new solid regions this new plane creates is $(n^2-2n+2)/2$. In other words, we have our recurrence relation: $S_n = S_{n-1} + (n^2-n+2)/2$. The initial condition is $S_0 = 1$ (if there are no planes, we get one region). Let us verify this for some small values of $n$. If $n=1$, then the recurrence relation gives $S_1 = S_0 + (1^2-1+2)/2 = 1+1 = 2$, which is correct (one plane divides space into two half-spaces). Next $S_2 = S_1 + (2^2-2+2)/2 = 2+2 = 4$, and again it is easy to see that this is correct. Similarly, $S_3 = S_2 + (3^2-3+2)/2 = 4+4 = 8$, and we know that this is right from our familiarity with 3-dimensional graphing (space has eight octants). The first surprising case is $n=4$, when we have $S_4 = S_3 + (4^2-4+2)/2 = 8+7 = 15$. This takes some concentration to see (consider the plane $x+y+z=1$ passing through space. It splits each octant into two parts except for the octant in which all coordinates are negative, because it does not pass through that octant. Thus 7 regions become 14, and the additional region makes a total of 15.

**b)** The iteration here gets a little messy. We need to invoke two summation formulae from Table 2 in Section 2.4: $1+2+3+\cdots+n = n(n+1)/2$ and $1^2+2^2+3^2+\cdots+n^2 = n(n+1)(2n+1)/6$. We proceed as follows:

$$\begin{aligned}
S_n &= \frac{n^2}{2} - \frac{n}{2} + 1 + S_{n-1} \\
&= \frac{n^2}{2} - \frac{n}{2} + 1 + \left(\frac{(n-1)^2}{2} - \frac{(n-1)}{2} + 1\right) + S_{n-2} \\
&= \frac{n^2}{2} - \frac{n}{2} + 1 + \left(\frac{(n-1)^2}{2} - \frac{(n-1)}{2} + 1\right) + \left(\frac{(n-2)^2}{2} - \frac{(n-2)}{2} + 1\right) + S_{n-3} \\
&\;\;\vdots
\end{aligned}$$

$$
\begin{aligned}
&= \frac{n^2}{2} - \frac{n}{2} + 1 + \left(\frac{(n-1)^2}{2} - \frac{(n-1)}{2} + 1\right) + \left(\frac{(n-2)^2}{2} - \frac{(n-2)}{2} + 1\right) + \cdots \\
&\quad + \left(\frac{1^2}{2} - \frac{1}{2} + 1\right) + S_0 \\
&= \frac{1}{2}\left((n^2 + (n-1)^2 + \cdots + 1^2) - (n + (n-1) + \cdots + 1)\right) + (1 + 1 + \cdots + 1) + 1 \\
&= \frac{1}{2}\left(\frac{n(n+1)(2n+1)}{6} - \frac{n(n+1)}{2}\right) + n + 1 \\
&= \frac{n^3 + 5n + 6}{6} \quad \text{(after a little algebra).}
\end{aligned}
$$

Note that this answer agrees with that given in supplementary Exercise 50 of Chapter 5.

**25.** The easy way to do this problem is to invoke symmetry. A bit string of length 7 has an even number of 0's if and only if it has an odd number of 1's, since the sum of the number of 0's and the number of 1's, namely 7, is odd. Because of the symmetric role of 0 and 1, there must be just as many 7-bit strings with an even number of 0's as there are with an odd number of 0's, each therefore being $2^7/2$ (since there are $2^7$ bit strings altogether). Thus the answer is $2^{7-1} = 64$.

The solution can also be found using recurrence relations. Let $e_n$ be the number of bit strings of length $n$ with an even number of 0's. A bit string of length $n$ with an even number of 0's is either a bit string that starts with a 1 and is then followed by a bit string of length $n-1$ with an even number of 0's (of which there are $e_{n-1}$), or else it starts with a 0 and is then followed by a bit string of length $n-1$ with an odd number of 0's (of which there are $2^{n-1} - e_{n-1}$). Therefore we have the recurrence relation $e_n = e_{n-1} + (2^{n-1} - e_{n-1}) = 2^{n-1}$. In other words, it is a recurrence relation that is its own solution. In our case, $n = 7$, so there are $2^{7-1} = 64$ such strings. (See also Exercise 31 in Section 6.4.)

**27.** We assume that the walkway is one tile in width and $n$ tiles long, from start to finish. Thus we are talking about ternary sequences of length $n$ that do not contain two consecutive 0's, say. This was studied in Exercise 13, so the answers obtained there apply. We let $a_n$ represent the desired quantity.

**a)** As in Exercise 13, we find the recurrence relation to be $a_n = 2a_{n-1} + 2a_{n-2}$.

**b)** As in Exercise 13, the initial conditions are $a_0 = 1$ and $a_1 = 3$.

**c)** Continuing the computation started in the solution to Exercise 13, we find

$$a_7 = 2a_6 + 2a_5 = 2 \cdot 448 + 2 \cdot 164 = 1224.$$

Thus there are 1224 such colored paths.

**29.** If the codomain has only one element, then there is only one function (namely the function that takes each element of the domain to the unique element of the codomain). Therefore when $n = 1$ we have $S(m,n) = S(m,1) = 1$, the initial condition we are asked to verify. Now assume that $m \geq n > 1$, and we want to count $S(m,n)$, the number of functions from a domain with $m$ elements *onto* a codomain with $n$ elements. The form of the recurrence relation we are supposed to verify suggests that what we want to do is to look at the non-onto functions. There are $n^m$ functions from the $m$-set to the $n$-set altogether (by the product rule, since we need to choose an element from the $n$-set, which can be done in $n$ ways, a total of $m$ times). Therefore we must show that there are $\sum_{k=1}^{n-1} C(n,k)S(m,k)$ functions from the domain to the codomain that are *not* onto. First we use the sum rule and break this count down into the disjoint cases determined by the number of elements—let us call it $k$—in the range of the function. Since we want the function not to be onto, $k$ can have any value from 1 to $n-1$, but $k$ cannot equal $n$. Once we have specified $k$, in order to specify a function we need to first specify the actual range, and this can be done in $C(n,k)$ ways, namely choosing the subset of $k$ elements from the codomain that are to constitute the range; and second choose an *onto* function from the domain to this set of $k$ elements. This latter task can be done in $S(m,k)$ ways, since (and here is the key recursive point) we are defining $S(m,k)$ to be precisely this number. Therefore by the product rule there are $C(n,k)S(m,k)$ different functions with our original domain having a range of $k$ elements, and so by the sum rule there are $\sum_{k=1}^{n-1} C(n,k)S(m,k)$ non-onto functions from our original domain to our original codomain. Note that this two-dimensional recurrence relation can be used to compute $S(m,n)$ for any desired positive integers $m$ and $n$. Using it is much easier than trying to list all onto functions.

**31.** We will see that the answer is too large for us to list all the possibilities by hand with a reasonable amount of effort.

**a)** We know from Example 5 that $C_0 = 1$, $C_1 = 1$, and $C_3 = 5$. It is also easy to see that $C_2 = 2$, since there are only two ways to parenthesize the product of three numbers. We know from Exercise 30 that $C_4 = 14$. Therefore the recurrence relation tells us that $C_5 = C_0C_4 + C_1C_3 + C_2C_2 + C_3C_1 + C_4C_0 = 1 \cdot 14 + 1 \cdot 5 + 2 \cdot 2 + 5 \cdot 1 + 14 \cdot 1 = 42$.

**b)** Here $n = 5$, so the formula gives $\frac{1}{6}C(10,5) = \frac{1}{6} \cdot 10 \cdot 9 \cdot 8 \cdot 7 \cdot 6/5! = 42$.

**33.** Obviously $J(1) = 1$. When $n = 2$, the second person is killed, so $J(2) = 1$. When $n = 3$, person 2 is killed off, then person 3 is skipped, so person 1 is killed, making $J(3) = 3$. When $n = 4$, the order of death is 2, 4, 3; so $J(4) = 1$. For $n = 5$, the order of death is 2, 4, 1, 5; so $J(5) = 3$. With pencil and paper (or a computer, if we feel like writing a little program), we find the remaining values:

| $n$ | $J(n)$ | $n$ | $J(n)$ |
|---|---|---|---|
| 1 | 1 | 9 | 3 |
| 2 | 1 | 10 | 5 |
| 3 | 3 | 11 | 7 |
| 4 | 1 | 12 | 9 |
| 5 | 3 | 13 | 11 |
| 6 | 5 | 14 | 13 |
| 7 | 7 | 15 | 15 |
| 8 | 1 | 16 | 1 |

**35.** If the number of players is even (call it $2n$), then after we have gone around the circle once we are back at the beginning, with two changes. First, the number assigned to every player has been changed, since all the even numbers are now missing. The first remaining player is 1, the second remaining player is 3, the third remaining player is 5, and so on. In general, the player in location $i$ at this point is the player whose original number was $2i - 1$. Second, the number of players is half of what it used to be; it's now $n$. Therefore we know that the survivor will be the player currently occupying spot $J(n)$, namely $2J(n) - 1$. Thus we have shown that $J(2n) = 2J(n) - 1$. The argument when there are an odd number of players is similar. If there are $2n+1$ players, then after we have gone around the circle once and then killed off player 1, we will have $n$ players left. The first remaining spot is occupied by player 3, the second remaining player is 5, and so on—the $i^{\text{th}}$ remaining player is $2i + 1$. Therefore we know that the survivor will be the player currently occupying spot $J(n)$, namely $2J(n) + 1$. Thus we have shown that $J(2n+1) = 2J(n) + 1$. The base case is clearly $J(1) = 1$.

**37.** We use the conjecture from Exercise 34: If $n = 2^m + k$, where $k < 2^m$, then $J(n) = 2k + 1$. Thus $J(100) = J(2^6 + 36) = 2 \cdot 36 + 1 = 73$; $J(1000) = J(2^9 + 488) = 2 \cdot 488 + 1 = 977$; and $J(10000) = J(2^{13} + 1808) = 2 \cdot 1808 + 1 = 3617$.

**39.** It is not too hard to find the winning moves (here $a \xrightarrow{b} c$ means to move disk $b$ from peg $a$ to peg $c$, where we label the smallest disk 1 and the largest disk 4): $1 \xrightarrow{1} 2$, $1 \xrightarrow{2} 3$, $2 \xrightarrow{1} 3$, $1 \xrightarrow{3} 2$, $1 \xrightarrow{4} 4$, $2 \xrightarrow{3} 4$, $3 \xrightarrow{1} 2$,

$3 \xrightarrow{2} 4$, $2 \xrightarrow{1} 4$. We can argue that at least seven moves are required no matter how many pegs we have: three to unstack the disks, one to move disk 4, and three more to restack them. We need to show that at least two additional moves are required because of the congestion caused by there being only four pegs. Note that in order to move disk 4 from peg 1 to peg 4, the other three disks must reside on pegs 2 and 3. That requires at least one move to restack them and one move to unstack them. This completes the argument.

**41.** It is helpful to do Exercise 40 first to get a feeling for what is going on. The base cases are obvious. If $n > 1$, then the algorithm consists of three stages. In the first stage, by the inductive hypothesis, it takes $R(n-k)$ moves to transfer the smallest $n-k$ disks to peg 2. Then by the usual Tower of Hanoi algorithm, it takes $2^k - 1$ moves to transfer the largest $k$ disks (i.e., the rest of them) to peg 4, avoiding peg 2. Then again by induction, it takes $R(n-k)$ moves to transfer the smallest $n-k$ disks to peg 4; all the pegs are available for this work, since the largest disks, now residing on peg 4, do not interfere. The recurrence relation is therefore established.

**43.** First write $R(n) = \sum_{j=1}^{n}(R(j) - R(j-1))$, which is immediate from the telescoping nature of the sum (and the fact that $R(0) = 0$. By the result from Exercise 42, this is the sum of $2^{k'-1}$ for this range of values of $j$ (here $j$ is playing the role that $n$ played in Exercise 42, and $k'$ is the value selected by the algorithm for $j$). But $k'$ is constant (call this constant $i$) for $i$ successive values of $j$. Therefore this sum is $\sum_{i=1}^{k} i2^{i-1}$, except that if $n$ is not a triangular number, then the last few values when $i = k$ are missing, and that is what the final term in the given expression accounts for.

**45.** By Exercise 43, $R(n)$ is no greater than $\sum_{i=1}^{k} i2^{i-1}$. By using algebra and calculus, we can show that this equals $(k+1)2^k - 2^{k+1} + 1$, so it is no greater than $(k+1)2^k$. (The proof is to write the formula for a geometric series $\sum_{i=0}^{k} x^i = (1 - x^{k+1})/(1-x)$, differentiate both sides, and simplify.) Since $n > k(k-1)/2$, we see from the quadratic formula that $k < \frac{1}{2} + \sqrt{2n + \frac{1}{4}} < 1 + \sqrt{2n}$ for all $n > 1$. Therefore $R(n)$ is bounded above by $(1 + \sqrt{2n} + 1)2^{1+\sqrt{2n}} \le 8 \cdot \sqrt{n}\,2^{\sqrt{2n}}$ for all $n > 2$. This shows that $R(n)$ is $O(\sqrt{n}\,2^{\sqrt{2n}})$, as desired.

**47.** We have to do Exercise 46 before we can do this exercise.
**a)** We found that the first differences were $\nabla a_n = 0$. Therefore the second differences are given by $\nabla^2 a_n = 0 - 0 = 0$.
**b)** We found that the first differences were $\nabla a_n = 2n - 2(n-1) = 2$. Therefore the second differences are given by $\nabla^2 a_n = 2 - 2 = 0$.
**c)** We found that the first differences were $\nabla a_n = n^2 - (n-1)^2 = 2n - 1$. Therefore the second differences are given by $\nabla^2 a_n = (2n-1) - (2(n-1) - 1) = 2$.
**d)** We found that the first differences were $\nabla a_n = 2^n - 2^{n-1} = 2^{n-1}$. Therefore the second differences are given by $\nabla^2 a_n = 2^{n-1} - 2^{n-2} = 2^{n-2}$.

**49.** This is just an exercise in algebra. The right-hand side of the given expression is by definition $a_n - 2\nabla a_n + \nabla a_n - \nabla a_{n-1} = a_n - \nabla a_n - \nabla a_{n-1} = a_n - (a_n - a_{n-1}) - (a_{n-1} - a_{n-2})$. Everything in this expression cancels except the last term, yielding $a_{n-2}$, as desired.

**51.** In order to express the recurrence relation $a_n = a_{n-1} + a_{n-2}$ in terms of $a_n$, $\nabla a_n$, and $\nabla^2 a_n$, we use the results of Exercise 48 (that $a_{n-1} = a_n - \nabla a_n$) and Exercise 49 (that $a_{n-2} = a_n - 2\nabla a_n + \nabla^2 a_n$). Thus the given recurrence relation is equivalent to $a_n = (a_n - \nabla a_n) + (a_n - 2\nabla a_n + \nabla^2 a_n)$, which simplifies algebraically to $a_n = 3\nabla a_n - \nabla^2 a_n$.

**53.** Modify Algorithm 1 so that in addition to storing the value $T(j)$, the maximum number of attendees possible for an optimal schedule of the first $j$ talks, we also store, for each $j$, a set $S(j)$, which consists of the subscripts of a set of talks among the first $j$ talks that can be scheduled to achieve a total of $T(j)$ attendees. Initially we set $S(0)$ equal to the empty set. At the point at which we compute $T(j)$, we set $S(j)$ equal to $S(j-1)$ if $T(j)$ was set to $T(j-1)$, and we set $S(j)$ equal to $S(p(j)) \cup \{j\}$ otherwise. (If those two values are the same, then we have our choice—from a practical point of view, it would probably be better to see which choice resulted in more talks.)

**55.** We record the information in tabular form, filling in $T(j)$ and $S(j)$ line by line. For each $j$, if $w_j + T(p(j)) \leq T(j-1)$, then in the $j^{\text{th}}$ line we just copy the information from the previous line. If $w_j + T(p(j)) > T(j-1)$, then we put $w_j + T(p(j))$ in the $T(j)$ column and put $S(p(j)) \cup \{j\}$ in the $S(j)$ column.

**a)**

| $j$ | $p(j)$ | $w(j)$ | $T(j)$ | $S(j)$ |
|---|---|---|---|---|
| 0 | — | — | 0 | $\emptyset$ |
| 1 | 0 | 20 | 20 | $\{1\}$ |
| 2 | 0 | 10 | 20 | $\{1\}$ |
| 3 | 1 | 50 | 70 | $\{1,3\}$ |
| 4 | 0 | 30 | 70 | $\{1,3\}$ |
| 5 | 0 | 15 | 70 | $\{1,3\}$ |
| 6 | 2 | 25 | 70 | $\{1,3\}$ |
| 7 | 4 | 40 | 110 | $\{1,3,7\}$ |

**b)** This time the answer is not unique. When we had to compute $T(7)$ and $S(7)$ we could achieve an attendance of 140 either with talks 1 and 6 or with talks 1, 3, and 7.

| $j$ | $p(j)$ | $w(j)$ | $T(j)$ | $S(j)$ |
|---|---|---|---|---|
| 0 | — | — | 0 | $\emptyset$ |
| 1 | 0 | 100 | 100 | $\{1\}$ |
| 2 | 0 | 5 | 100 | $\{1\}$ |
| 3 | 1 | 10 | 110 | $\{1,3\}$ |
| 4 | 0 | 20 | 110 | $\{1,3\}$ |
| 5 | 0 | 25 | 110 | $\{1,3\}$ |
| 6 | 2 | 40 | 140 | $\{1,6\}$ |
| 7 | 4 | 30 | 140 | $\{1,6\}$ |

**c)** Here when we had to compute $T(6)$ and $S(6)$ we could achieve an attendance of 10 either with talks 1 and 3 or with talks 2 and 6. But the final answer is unique.

| $j$ | $p(j)$ | $w(j)$ | $T(j)$ | $S(j)$ |
|---|---|---|---|---|
| 0 | — | — | 0 | $\emptyset$ |
| 1 | 0 | 2 | 2 | $\{1\}$ |
| 2 | 0 | 3 | 3 | $\{2\}$ |
| 3 | 1 | 8 | 10 | $\{1,3\}$ |
| 4 | 0 | 5 | 10 | $\{1,3\}$ |
| 5 | 0 | 4 | 10 | $\{1,3\}$ |
| 6 | 2 | 7 | 10 | $\{1,3\}$ |
| 7 | 4 | 10 | 20 | $\{1,3,7\}$ |

d)

| $j$ | $p(j)$ | $w(j)$ | $T(j)$ | $S(j)$ |
|---|---|---|---|---|
| 0 | — | — | 0 | $\emptyset$ |
| 1 | 0 | 10 | 10 | $\{1\}$ |
| 2 | 0 | 8 | 10 | $\{1\}$ |
| 3 | 1 | 7 | 17 | $\{1,3\}$ |
| 4 | 0 | 25 | 25 | $\{4\}$ |
| 5 | 0 | 20 | 25 | $\{4\}$ |
| 6 | 2 | 30 | 40 | $\{1,6\}$ |
| 7 | 4 | 5 | 40 | $\{1,6\}$ |

**57. a)** By Example 5 the number of ways to parenthesize the product to determine the order in which to perform the multiplications is the Catalan number $C_n$. By Exercise 41 in Section 8.4, $C_n > 2^{n-1}$. Therefore the brute-force method (to test all possible orders of multiplication) has exponential time complexity.

**b)** The last step in computing $\mathbf{A}_{ij}$ is to multiply the $m_i \times m_{k+1}$ matrix $\mathbf{A}_{ik}$ by the $m_{k+1} \times m_{j+1}$ matrix $\mathbf{A}_{k+1,j}$ for some $k$ between $i$ and $j-1$ inclusive, which will require $m_i m_{k+1} m_{j+1}$ integer multiplications, independent of the manner in which $\mathbf{A}_{ik}$ and $\mathbf{A}_{k+1,j}$ are computed. Therefore to minimize the total number of integer multiplications, each of those two factors must be computed in the most efficient manner.

**c)** This follows immediately from part **(b)** and the definition of $M(i,j)$. We just try all $j-i$ possible ways of splitting the product into two factors.

**d)** In the algorithm shown here, $M(i,j)$ is as defined in part **(b)**, and $where(i,j)$ is the value of $k$ such that the calculation of $\mathbf{A}_{ij}$ should be broken into the calculation of $\mathbf{A}_{ik}\mathbf{A}_{k+1,j}$ in order to achieve the fewest possible number of integer multiplications. Once all these values are computed, it is a simple matter to read off, using $where(i,j)$, the proper order in which to carry out the matrix multiplications. (First, $where(1,n)$ tells us which two subproducts to compute first, say $\mathbf{A}_{1r}$ and $\mathbf{A}_{r+1,n}$. Then $where(1,r)$ and $where(r+1,n)$ tell us where to break those subproducts, and so on.) Notice that the main loop is indexed by $d$, which represents $j-i$ in the notation of part **(c)**.

```
procedure matrixorder(m_1, ..., m_{n+1} : positive integers)
for i := 1 to n
        M(i, i) := 0
for d := 1 to n − 1
        for i := 1 to n − d
                min := 0
                for k := i to i + d
                        new := M(i, k) + M(k + 1, i + d) + m_i m_{k+1} m_{i+d+1}
                        if new < min then
                                min := new
                                where(i, i + d) := k
                M(i, i + d) := min
```

**e)** The work in this algorithm is done by three nested loops, each of which is indexed over at most $n$ values.

## SECTION 8.2 Solving Linear Recurrence Relations

*In many ways this section is extremely straightforward. Theorems 1–6 give an algorithm for solving linear homogeneous recurrence relations with constant coefficients. The only difficulty that sometimes occurs is that the algebra involved becomes messy or impossible. (Although the fundamental theorem of algebra says that every $n^{\text{th}}$ degree polynomial equation has exactly $n$ roots (counting multiplicities), there is in general no way to find their exact values. For example, there is nothing analogous to the quadratic formula for equations of degree 5. Also, the roots may be irrational, as we saw in Example 4, or complex, as is discussed in Exercises 38 and 39. Patience is required with the algebra in such cases.) Many other techniques are available in other special cases, in analogy to the situation with differential equations; see Exercises 48–50, for example. If you have access to a computer algebra package, you should investigate how good it is at solving recurrences. See the solution to Exercise 49 for the kind of command to use in* Maple.

**1.** a) This is linear (the terms $a_i$ all appear to the first power), has constant coefficients ($3$, $4$, and $5$), and is homogeneous (no terms are functions of just $n$). It has degree 3, since $a_n$ is expressed in terms of $a_{n-1}$, $a_{n-2}$, and $a_{n-3}$.

b) This does not have constant coefficients, since the coefficient of $a_{n-1}$ is the nonconstant $2n$.

c) This is linear, homogeneous, with constant coefficients. It has degree 4, since $a_n$ is expressed in terms of $a_{n-1}$, $a_{n-2}$, $a_{n-3}$ and $a_{n-4}$ (the fact that the coefficient of $a_{n-2}$, for example, is 0 is irrelevant—the degree is the largest $k$ such that $a_{n-k}$ is present).

d) This is not homogeneous because of the 2.

e) This is not linear, since the term $a_{n-1}^2$ appears.

f) This is linear, homogeneous, with constant coefficients. It has degree 2.

g) This is linear but not homogeneous because of the $n$.

**3.** a) We can solve this problem by iteration (or even by inspection), but let us use the techniques in this section instead. The characteristic equation is $r-2=0$, so the only root is $r=2$. Therefore the general solution to the recurrence relation, by Theorem 3 (with $k=1$), is $a_n=\alpha 2^n$ for some constant $\alpha$. We plug in the initial condition to solve for $\alpha$. Since $a_0=3$ we have $3=\alpha 2^0$, whence $\alpha=3$. Therefore the solution is $a_n=3\cdot 2^n$.

b) Again this is trivial to solve by inspection, but let us use the algorithm. The characteristic equation is $r-1=0$, so the only root is $r=1$. Therefore the general solution to the recurrence relation, by Theorem 3 (with $k=1$), is $a_n=\alpha 1^n=\alpha$ for some constant $\alpha$. In other words, the sequence is constant. We plug in the initial condition to solve for $\alpha$. Since $a_0=2$ we have $\alpha=2$. Therefore the solution is $a_n=2$ for all $n$.

c) The characteristic equation is $r^2-5r+6=0$, which factors as $(r-2)(r-3)=0$, so the roots are $r=2$ and $r=3$. Therefore by Theorem 1 the general solution to the recurrence relation is $a_n=\alpha_1 2^n+\alpha_2 3^n$ for some constants $\alpha_1$ and $\alpha_2$. We plug in the initial condition to solve for the $\alpha$'s. Since $a_0=1$ we have $1=\alpha_1+\alpha_2$, and since $a_1=0$ we have $0=2\alpha_1+3\alpha_2$. These linear equations are easily solved to yield $\alpha_1=3$ and $\alpha_2=-2$. Therefore the solution is $a_n=3\cdot 2^n-2\cdot 3^n$.

d) The characteristic equation is $r^2-4r+4=0$, which factors as $(r-2)^2=0$, so there is only one root, $r=2$, which occurs with multiplicity 2. Therefore by Theorem 2 the general solution to the recurrence relation is $a_n=\alpha_1 2^n+\alpha_2 n2^n$ for some constants $\alpha_1$ and $\alpha_2$. We plug in the initial conditions to solve for the $\alpha$'s. Since $a_0=6$ we have $6=\alpha_1$, and since $a_1=8$ we have $8=2\alpha_1+2\alpha_2$. These linear equations are easily solved to yield $\alpha_1=6$ and $\alpha_2=-2$. Therefore the solution is $a_n=6\cdot 2^n-2\cdot n2^n=(6-2n)2^n$. Incidentally, there is a good way to check a solution to a recurrence relation problem, namely by calculating the next term in two ways. In this exercise, the recurrence relation tells us that $a_2=4a_1-4a_0=4\cdot 8-4\cdot 6=8$, whereas the solution tells us that $a_2=(6-2\cdot 2)2^2=8$. Since these answers agree, we are somewhat confident that our solution is correct. We could calculate $a_3$ in two ways for another confirmation.

e) This time the characteristic equation is $r^2+4r+4=0$, which factors as $(r+2)^2=0$, so again there is only one root, $r=-2$, which occurs with multiplicity 2. Therefore by Theorem 2 the general solution to the recurrence relation is $a_n = \alpha_1(-2)^n + \alpha_2 n(-2)^n$ for some constants $\alpha_1$ and $\alpha_2$. We plug in the initial conditions to solve for the $\alpha$'s. Since $a_0 = 0$ we have $0 = \alpha_1$, and since $a_1 = 1$ we have $1 = -2\alpha_1 - 2\alpha_2$. These linear equations are easily solved to yield $\alpha_1 = 0$ and $\alpha_2 = -1/2$. Therefore the solution is $a_n = (-1/2)n(-2)^n = n(-2)^{n-1}$.

**f)** The characteristic equation is $r^2 - 4 = 0$, so the roots are $r = 2$ and $r = -2$. Therefore the solution is $a_n = \alpha_1 2^n + \alpha_2(-2)^n$ for some constants $\alpha_1$ and $\alpha_2$. We plug in the initial conditions to solve for the $\alpha$'s. We have $0 = \alpha_1 + \alpha_2$, and $4 = 2\alpha_1 - 2\alpha_2$. These linear equations are easily solved to yield $\alpha_1 = 1$ and $\alpha_2 = -1$. Therefore the solution is $a_n = 2^n - (-2)^n$.

**g)** The characteristic equation is $r^2 - 1/4 = 0$, so the roots are $r = 1/2$ and $r = -1/2$. Therefore the solution is $a_n = \alpha_1(1/2)^n + \alpha_2(-1/2)^n$ for some constants $\alpha_1$ and $\alpha_2$. We plug in the initial conditions to solve for the $\alpha$'s. We have $1 = \alpha_1 + \alpha_2$, and $0 = \alpha_1/2 - \alpha_2/2$. These linear equations are easily solved to yield $\alpha_1 = \alpha_2 = 1/2$. Therefore the solution is $a_n = (1/2)(1/2)^n + (1/2)(-1/2)^n = (1/2)^{n+1} - (-1/2)^{n+1}$.

**5.** The recurrence relation found in Exercise 19 of Section 8.1 was $a_n = a_{n-1} + a_{n-2}$, with initial conditions $a_0 = a_1 = 1$. To solve this, we look at the characteristic equation $r^2 - r - 1 = 0$ (exactly as in Example 4) and obtain, by the quadratic formula, the roots $r_1 = (1+\sqrt{5})/2$ and $r_2 = (1-\sqrt{5})/2$. Therefore from Theorem 1 we know that the solution is given by

$$a_n = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right)^n,$$

for some constants $\alpha_1$ and $\alpha_2$. The initial conditions $a_0 = 1$ and $a_1 = 1$ allow us to determine these constants. We plug them into the equation displayed above and obtain

$$\begin{aligned} 1 &= a_0 = \alpha_1 + \alpha_2 \\ 1 &= a_1 = \alpha_1\left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2\left(\frac{1-\sqrt{5}}{2}\right). \end{aligned}$$

By algebra we solve these equations (one way is to solve the first for $\alpha_2$ in terms of $\alpha_1$, and plug that into the second equation to get one equation in $\alpha_1$, which can then be solved—the fact that these coefficients are messy irrational numbers involving $\sqrt{5}$ does not change the rules of algebra, of course). The solutions are $\alpha_1 = (5+\sqrt{5})/10$ and $\alpha_2 = (5-\sqrt{5})/10$. Therefore the specific solution is given by

$$a_n = \frac{5+\sqrt{5}}{10}\left(\frac{1+\sqrt{5}}{2}\right)^n + \frac{5-\sqrt{5}}{10}\left(\frac{1-\sqrt{5}}{2}\right)^n.$$

Alternatively, by not rationalizing the denominators when we solve for $\alpha_1$ and $\alpha_2$, we get $\alpha_1 = (1+\sqrt{5})/(2\sqrt{5})$ and $\alpha_2 = -(1-\sqrt{5})/(2\sqrt{5})$. With these expressions, we can write our solution as

$$a_n = \frac{1}{\sqrt{5}}\left(\frac{1+\sqrt{5}}{2}\right)^{n+1} - \frac{1}{\sqrt{5}}\left(\frac{1-\sqrt{5}}{2}\right)^{n+1}.$$

**7.** First we need to find a recurrence relation and initial conditions for the problem. Let $t_n$ be the number of ways to tile a $2 \times n$ board with $1 \times 2$ and $2 \times 2$ pieces. To obtain the recurrence relation, imagine what tiles are placed at the left-hand end of the board. We can place a $2 \times 2$ tile there, leaving a $2 \times (n-2)$ board to be tiled, which of course can be done in $t_{n-2}$ ways. We can place a $1 \times 2$ tile at the edge, oriented vertically, leaving a $2 \times (n-1)$ board to be tiled, which of course can be done in $t_{n-1}$ ways. Finally, we can place two $1 \times 2$ tiles horizontally, one above the other, leaving a $2 \times (n-2)$ board to be tiled, which of course can be done in $t_{n-2}$ ways. These three possibilities are disjoint. Therefore our recurrence relation is $t_n = t_{n-1} + 2t_{n-2}$. The initial conditions are $t_0 = t_1 = 1$, since there is only one way to tile a $2 \times 0$ board (the way that uses no tiles) and only one way to tile a $2 \times 1$ board. This recurrence relation is the same one that appeared in Example 3; it has characteristic roots 2 and $-1$, so the general solution is

$$t_n = \alpha_1 2^n + \alpha_2(-1)^n .$$

To determine the coefficients we plug in the initial conditions, giving us the equations

$$1 = t_0 = \alpha_1 + \alpha_2$$
$$1 = t_1 = 2\alpha_1 - \alpha_2 .$$

Solving these yields $\alpha_1 = 2/3$ and $\alpha_2 = 1/3$, so our final solution is $t_n = 2^{n+1}/3 + (-1)^n/3$.

**9.** **a)** The amount $P_n$ in the account at the end of the $n^{\text{th}}$ year is equal to the amount at the end of the previous year ($P_{n-1}$), plus the 20% dividend on that amount ($0.2P_{n-1}$) plus the 45% dividend on the amount at the end of the year before that ($0.45P_{n-2}$). Thus we have $P_n = 1.2P_{n-1} + 0.45P_{n-2}$. We need two initial conditions, since the equation has degree 2. Clearly $P_0 = 100000$. The other initial condition is that $P_1 = 120000$, since there is only one dividend at the end of the first year.

**b)** Solving this recurrence relation requires looking at the characteristic equation $r^2 - 1.2r - 0.45 = 0$. By the quadratic formula, the roots are $r_1 = 1.5$ and $r_2 = -0.3$. Therefore the general solution of the recurrence relation is $P_n = \alpha_1(1.5)^n + \alpha_2(-0.3)^n$. Plugging in the initial conditions gives us the equations $100000 = \alpha_1 + \alpha_2$ and $120000 = 1.5\alpha_1 - 0.3\alpha_2$. These are easily solved to give $\alpha_1 = 250000/3$ and $\alpha_2 = 50000/3$. Therefore the solution of our problem is

$$P_n = \frac{250000}{3}(1.5)^n + \frac{50000}{3}(-0.3)^n .$$

**11.** **a)** We prove this by induction on $n$. We need to verify two base cases. For $n = 1$ we have $L_1 = 1 = 0 + 1 = f_0 + f_2$; and for $n = 2$ we have $L_2 = 3 = 1 + 2 = f_1 + f_3$. Assume the inductive hypothesis that $L_k = f_{k-1} + f_{k+1}$ for $k < n$. We must show that $L_n = f_{n-1} + f_{n+1}$. To do this, we let $k = n - 1$ and $k = n - 2$:

$$L_{n-1} = f_{n-2} + f_n$$
$$L_{n-2} = f_{n-3} + f_{n-1} .$$

If we add these two equations, we obtain

$$L_{n-1} + L_{n-2} = (f_{n-2} + f_{n-3}) + (f_n + f_{n-1}) ,$$

which is the same as

$$L_n = f_{n-1} + f_{n+1}$$

as desired, using the recurrence relations for the Lucas and Fibonacci numbers.

**b)** To find an explicit formula for the Lucas numbers, we need to solve the recurrence relation and initial conditions. Since the recurrence relation is the same as that of the Fibonacci numbers, we get the same general solution as in Example 4, namely

$$L_n = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right)^n + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right)^n ,$$

for some constants $\alpha_1$ and $\alpha_2$. The initial conditions are different, though. When we plug them in we get the system

$$2 = L_0 = \alpha_1 + \alpha_2$$
$$1 = L_1 = \alpha_1 \left(\frac{1+\sqrt{5}}{2}\right) + \alpha_2 \left(\frac{1-\sqrt{5}}{2}\right) .$$

By algebra we solve these equations, yielding $\alpha_1 = \alpha_1 = 1$. Therefore the specific solution is given by

$$L_n = \left(\frac{1+\sqrt{5}}{2}\right)^n + \left(\frac{1-\sqrt{5}}{2}\right)^n .$$

**13.** This is a third degree equation. The characteristic equation is $r^3 - 7r - 6 = 0$. Assuming the composer of the problem has arranged that the roots are nice numbers, we use the rational root test, which says that rational roots must be of the form $\pm p/q$, where $p$ is a factor of the constant term (6 in this case) and $q$ is a factor of the coefficient of the leading term (the coefficient of $r^3$ is 1 in this case). Hence the possible rational roots are $\pm1, \pm2, \pm3, \pm6$. We find that $r = -1$ is a root, so one factor of $r^3 - 7r + 6$ is $r + 1$. Dividing $r + 1$ into $r^3 - 7r - 6$ by long (or synthetic) division, we find that $r^3 - 7r - 6 = (r+1)(r^2 - r - 6)$. By inspection we factor the rest, obtaining $r^3 - 7r - 6 = (r+1)(r-3)(r+2)$. Hence the roots are $-1$, $3$, and $-2$, so the general solution is $a_n = \alpha_1(-1)^n + \alpha_2 3^n + \alpha_3(-2)^n$. To find these coefficients, we plug in the initial conditions:

$$\begin{aligned}9 &= a_0 = \alpha_1 + \alpha_2 + \alpha_3\\ 10 &= a_1 = -\alpha_1 + 3\alpha_2 - 2\alpha_3\\ 32 &= a_2 = \alpha_1 + 9\alpha_2 + 4\alpha_3\,.\end{aligned}$$

Solving this system of equations (by elimination, for instance), we get $\alpha_1 = 8$, $\alpha_2 = 4$, and $\alpha_3 = -3$. Therefore the specific solution is $a_n = 8(-1)^n + 4 \cdot 3^n - 3(-2)^n$.

**15.** This is a third degree recurrence relation. The characteristic equation is $r^3 - 2r^2 - 5r + 6 = 0$. By the rational root test, the possible rational roots are $\pm1, \pm2, \pm3, \pm6$. We find that $r = 1$ is a root. Dividing $r - 1$ into $r^3 - 2r^2 - 5r + 6$, we find that $r^3 - 2r^2 - 5r + 6 = (r-1)(r^2 - r - 6)$. By inspection we factor the rest, obtaining $r^3 - 2r^2 - 5r + 6 = (r-1)(r-3)(r+2)$. Hence the roots are $1$, $3$, and $-2$, so the general solution is $a_n = \alpha_1 1^n + \alpha_2 3^n + \alpha_3(-2)^n$, or more simply $a_n = \alpha_1 + \alpha_2 3^n + \alpha_3(-2)^n$. To find these coefficients, we plug in the initial conditions:

$$\begin{aligned}7 &= a_0 = \alpha_1 + \alpha_2 + \alpha_3\\ -4 &= a_1 = \alpha_1 + 3\alpha_2 - 2\alpha_3\\ 8 &= a_2 = \alpha_1 + 9\alpha_2 + 4\alpha_3\,.\end{aligned}$$

Solving this system of equations, we get $\alpha_1 = 5$, $\alpha_2 = -1$, and $\alpha_3 = 3$. Therefore the specific solution is $a_n = 5 - 3^n + 3(-2)^n$.

**17.** We almost follow the hint and let $a_{n+1}$ be the right-hand side of the stated identity. Clearly $a_1 = C(0,0) = 1$ and $a_2 = C(1,0) = 1$. Thus $a_1 = f_1$ and $a_2 = f_2$. Now if we can show that the sequence $\{a_n\}$ satisfies the same recurrence relation that the Fibonacci numbers do, namely $a_{n+1} = a_n + a_{n-1}$, then we will know that $a_n = f_n$ for all $n \geq 1$ (precisely what we want to show), since the solution of a second degree recurrence relation with two initial conditions is unique.

To show that $a_{n+1} = a_n + a_{n-1}$, we start with the right-hand side, which is, by our definition, $C(n-1,0) + C(n-2,1) + \cdots + C(n-1-k,k) + C(n-2,0) + C(n-3,1) + \cdots + C(n-2-l,l)$, where $k = \lfloor (n-1)/2 \rfloor$ and $l = \lfloor (n-2)/2 \rfloor$. Note that $k = l$ if $n$ is even, and $k = l + 1$ if $n$ is odd. Let us first take the case in which $k = l = (n-2)/2$. By Pascal's identity, we regroup the sum above and rewrite it as

$$\begin{aligned}&C(n-1,0) + [C(n-2,0) + C(n-2,1)] + [C(n-3,1) + C(n-3,2)] + \cdots\\ &\qquad + [C(n-2-((n-2)/2-1),(n-2)/2-1) + C(n-1-(n-2)/2,(n-2)/2)]\\ &\qquad + C(n-2-(n-2)/2,(n-2)/2)\\ &\quad = C(n-1,0) + C(n-1,1) + C(n-2,2) + \cdots\\ &\qquad + C(n-(n-2)/2,(n-2)/2) + C(n-2-(n-2)/2,(n-2)/2)\\ &\quad = 1 + C(n-1,1) + C(n-2,2) + \cdots + C(n-(n-2)/2,(n-2)/2) + 1\\ &\quad = C(n,0) + C(n-1,1) + C(n-2,2) + \cdots + C(n-(n-2)/2,(n-2)/2) + C(n-n/2,n/2)\\ &\quad = C(n,0) + C(n-1,1) + C(n-2,2) + \cdots + C(n-j,j)\,,\end{aligned}$$

where $j = n/2 = \lfloor n/2 \rfloor$. This is precisely $a_{n+1}$, as desired. In case $n$ is odd, so that $k = (n-1)/2$ and $l = (n-3)/2$, we have a similar calculation (in this case the sum involving $k$ has one more term than the sum involving $l$):

$$\begin{aligned}
&C(n-1,0)+[C(n-2,0)+C(n-2,1)]+[C(n-3,1)+C(n-3,2)]+\cdots\\
&\qquad +[C(n-2-(n-3)/2,(n-3)/2)+C(n-1-(n-1)/2,(n-1)/2)]\\
&\quad = C(n-1,0)+C(n-1,1)+C(n-2,2)+\cdots+C(n-(n-1)/2,(n-1)/2)\\
&\quad = 1+C(n-1,1)+C(n-2,2)+\cdots+C(n-(n-1)/2,(n-1)/2)\\
&\quad = C(n,0)+C(n-1,1)+C(n-2,2)+\cdots+C(n-j,j)\,,
\end{aligned}$$

where $j=(n-1)/2=\lfloor n/2\rfloor$. Again, this is precisely $a_{n+1}$, as desired.

**19.** This is a third degree recurrence relation. The characteristic equation is $r^3+3r^2+3r+1=0$. We easily recognize this polynomial as $(r+1)^3$. Hence the only root is $-1$, with multiplicity 3, so the general solution is (by Theorem 4) $a_n=\alpha_1(-1)^n+\alpha_2 n(-1)^n+\alpha_3 n^2(-1)^n$. To find these coefficients, we plug in the initial conditions:

$$\begin{aligned}
5 &= a_0 = \alpha_1\\
-9 &= a_1 = -\alpha_1-\alpha_2-\alpha_3\\
15 &= a_2 = \alpha_1+2\alpha_2+4\alpha_3
\end{aligned}$$

Solving this system of equations, we get $\alpha_1=5$, $\alpha_2=3$, and $\alpha_3=1$. Therefore the answer is $a_n=5(-1)^n+3n(-1)^n+n^2(-1)^n$. We could also write this in factored form, of course, as $a_n=(n^2+3n+5)(-1)^n$. As a check of our answer, we can calculate $a_3$ both from the recurrence and from our formula, and we find that it comes out to be $-23$ in both cases.

**21.** This is similar to Example 6. We can immediately write down the general solution using Theorem 4. In this case there are four distinct roots, so $t=4$. The multiplicities are 4, 3, 2, and 1. So the general solution is $a_n=(\alpha_{1,0}+\alpha_{1,1}n+\alpha_{1,2}n^2+\alpha_{1,3}n^3)+(\alpha_{2,0}+\alpha_{2,1}n+\alpha_{2,2}n^2)(-2)^n+(\alpha_{3,0}+\alpha_{3,1}n)3^n+\alpha_{4,0}(-4)^n$.

**23.** Theorem 5 tells us that the general solution to the inhomogeneous linear recurrence relation

$$a_n=c_1a_{n-1}+c_2a_{n-2}+\cdots+c_ka_{n-k}+F(n)$$

can be found by finding one particular solution of this recurrence relation and adding it to the general solution of the corresponding homogeneous recurrence relation

$$a_n=c_1a_{n-1}+c_2a_{n-2}+\cdots+c_ka_{n-k}\,.$$

If we let $f_n$ be the particular solution to the inhomogeneous recurrence relation and $g_n$ be the general solution to the homogeneous recurrence relation (which will have some unspecified parameters $\alpha_1$, $\alpha_2$, ..., $\alpha_k$), then the general solution to the inhomogeneous recurrence relation is $f_n+g_n$ (so it, too, will have some unspecified parameters $\alpha_1$, $\alpha_2$, ..., $\alpha_k$).

**a)** To show that $a_n=-2^{n+1}$ is a solution to $a_n=3a_{n-1}+2^n$, we simply substitute it in and see if we get a true statement. Upon substituting into the right-hand side we get $3a_{n-1}+2^n=3(-2^n)+2^n=2^n(-3+1)=-2^{n+1}$, which is precisely the left-hand side.

**b)** By Theorem 5 and the comments above, we need to find the general solution to the corresponding homogeneous recurrence relation $a_n=3a_{n-1}$. This is easily seen to be $a_n=\alpha 3^n$ (either by the iterative method or by the method of this section with a linear characteristic equation). Putting these together as discussed above, we find the general solution to the given recurrence relation: $a_n=\alpha 3^n-2^{n+1}$.

**c)** To find the solution with $a_0=1$, we need to plug this initial condition (where $n=0$) into our answer to part **(b)**. Doing so gives the equation $1=\alpha-2$, whence $\alpha=3$. Therefore the solution to the given recurrence relation and initial condition is $a_n=3\cdot 3^n-2^{n+1}=3^{n+1}-2^{n+1}$.

**25.** See the introductory remarks to Exercise 23, which apply here as well.

**a)** We solve this problem by wishful thinking. Suppose that $a_n = An + B$, and substitute into the given recurrence relation. This gives us $An+B = 2\big(A(n-1)+B\big)+n+5$, which simplifies to $(A+1)n+(-2A+B+5) = 0$. Now if this is going to be true for all $n$, then both of the quantities in parentheses will have to be $0$. In other words, we need to solve the simultaneous equations $A+1=0$ and $-2A+B+5=0$. The solution is $A=-1$ and $B=-7$. Therefore a solution to the recurrence relation is $a_n = -n-7$.

**b)** By Theorem 5 and the comments at the beginning of Exercise 23, we need to find the general solution to the corresponding homogeneous recurrence relation $a_n = 2a_{n-1}$. This is easily seen to be $a_n = \alpha 2^n$ (either by the iterative method or by the method of this section with a linear characteristic equation). Putting these together as discussed above, we find the general solution to the given recurrence relation: $a_n = \alpha 2^n - n - 7$.

**c)** To find the solution with $a_0 = 4$, we need to plug this initial condition (where $n=0$) into our answer to part **(b)**. Doing so gives the equation $4 = \alpha - 7$, whence $\alpha = 11$. Therefore the solution to the given recurrence relation and initial condition is $a_n = 11\cdot 2^n - n - 7$.

**27.** We need to use Theorem 6, and so we need to find the roots of the characteristic polynomial of the associated homogeneous recurrence relation. The characteristic equation is $r^4-8r^2+16=0$, and as we saw in Exercise 20, $r=\pm 2$ are the only roots, each with multiplicity 2.

**a)** Since 1 is not a root of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $p_3n^3+p_2n^2+p_1n+p_0$. Note that $s=1$ here, in the notation of Theorem 6.

**b)** Since $-2$ is a root with multiplicity 2 of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $n^2p_0(-2)^n$.

**c)** Since 2 is a root with multiplicity 2 of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $n^2(p_1n+p_0)2^n$.

**d)** Since 4 is not a root of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $(p_2n^2+p_1n+p_0)4^n$.

**e)** Since $-2$ is a root with multiplicity 2 of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $n^2(p_2n^2+p_1n+p_0)(-2)^n$. Note that we needed a second degree polynomial inside the parenthetical expression because the polynomial in $F(n)$ was second degree.

**f)** Since 2 is a root with multiplicity 2 of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $n^2(p_4n^4+p_3n^3+p_2n^2+p_1n+p_0)2^n$.

**g)** Since 1 is not a root of the characteristic polynomial of the associated homogeneous recurrence relation, Theorem 6 tells us that the particular solution will be of the form $p_0$. Note that $s=1$ here, in the notation of Theorem 6.

**29. a)** The associated homogeneous recurrence relation is $a_n = 2a_{n-1}$. We easily solve it to obtain $a_n^{(h)} = \alpha 2^n$. Next we need a particular solution to the given recurrence relation. By Theorem 6 we want to look for a function of the form $a_n = c\cdot 3^n$. We plug this into our recurrence relation and obtain $c\cdot 3^n = 2c\cdot 3^{n-1}+3^n$. We divide through by $3^{n-1}$ and simplify, to find easily that $c=3$. Therefore the particular solution we seek is $a_n^{(p)} = 3\cdot 3^n = 3^{n+1}$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = \alpha 2^n + 3^{n+1}$.

**b)** We plug the initial condition into our solution from part **(a)** to obtain $5 = a_1 = 2\alpha + 9$. This tells us that $\alpha = -2$. So the solution is $a_n = -2\cdot 2^n + 3^{n+1} = -2^{n+1}+3^{n+1}$. At this point it would be very useful to check our answer. One method is to let a computer do the work; a computer algebra package such as *Maple* will solve equations of this type (see Exercise 49 for the syntax of the command). Alternatively, we can compute the next term of the sequence in two ways and verify that we obtain the same answer in each case. From the recurrence relation, we expect that $a_2 = 2 \cdot a_1 + 3^2 = 2 \cdot 5 + 9 = 19$. On the other hand, our solution tells us that $a_2 = -2^{2+1} + 3^{2+1} = -8 + 27 = 19$. Since the values agree, we can be fairly confident that our solution is correct.

**31.** The associated homogeneous recurrence relation is $a_n = 5a_{n-1} - 6a_{n-2}$. To solve it we find the characteristic equation $r^2 - 5r + 6 = 0$, find that $r = 2$ and $r = 3$ are its solutions, and therefore obtain the homogeneous solution $a_n^{(h)} = \alpha 2^n + \beta 3^n$. Next we need a particular solution to the given recurrence relation. By using the idea in Theorem 6 twice (or following the hint), we want to look for a function of the form $a_n = cn \cdot 2^n + dn + e$. (The reason for the factor $n$ in front of $2^n$ is that $2^n$ was already a solution of the homogeneous equation. The reason for the term $dn + e$ is the first degree polynomial $3n$.) We plug this into our recurrence relation and obtain $cn \cdot 2^n + dn + e = 5c(n-1) \cdot 2^{n-1} + 5d(n-1) + 5e - 6c(n-2) \cdot 2^{n-2} - 6d(n-2) - 6e + 2^n + 3n$. In order for this equation to be true, the exponential parts must be equal, and the polynomial parts must be equal. Therefore we have $c \cdot 2^n = 5c(n-1) \cdot 2^{n-1} - 6c(n-2) \cdot 2^{n-2} + 2^n$ and $dn + e = 5d(n-1) + 5e - 6d(n-2) - 6e + 3n$. To solve the first of these equations, we divide through by $2^{n-1}$, obtaining $2c = 5c(n-1) - 3c(n-2) + 2$, whence a little algebra yields $c = -2$. To solve the second equation, we note that the coefficients of $n$ as well as the constant terms must be equal on each side, so we know that $d = 5d - 6d + 3$ and $e = -5d + 5e + 12d - 6e$. This tells us that $d = 3/2$ and $e = 21/4$. Therefore the particular solution we seek is $a_n^{(p)} = -2n \cdot 2^n + 3n/2 + 21/4$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = \alpha 2^n + \beta 3^n - 2n \cdot 2^n + 3n/2 + 21/4 = \alpha 2^n + \beta 3^n - n \cdot 2^{n+1} + 3n/2 + 21/4$.

**33.** The associated homogeneous recurrence relation is $a_n = 4a_{n-1} - 4a_{n-2}$. To solve it we find the characteristic equation $r^2 - 4r + 4 = 0$, find that $r = 2$ is a repeated root, and therefore obtain the homogeneous solution $a_n^{(h)} = \alpha 2^n + \beta n \cdot 2^n$. Next we need a particular solution to the given recurrence relation. By Theorem 6 we want to look for a function of the form $a_n = n^2(cn + d)2^n$. (The reason for the factor $cn + d$ is that there is a linear polynomial factor in front of $2^n$ in the nonhomogeneous term; the reason for the factor $n^2$ is that the root $r = 2$ already appears twice in the associated homogeneous relation.) We plug this into our recurrence relation and obtain $n^2(cn+d)2^n = 4(n-1)^2(cn-c+d)2^{n-1} - 4(n-2)^2(cn-2c+d)2^{n-2} + (n+1)2^n$. We divide through by $2^n$, obtaining $n^2(cn + d) = 2(n-1)^2(cn - c + d) - (n-2)^2(cn - 2c + d) + (n + 1)$. Some algebra transforms this into $cn^3 + dn^2 = cn^3 + dn^2 + (-6c + 1)n + (6c - 2d + 1)$. Equating like powers of $n$ tells us that $c = 1/6$ and $d = 1$. Therefore the particular solution we seek is $a_n^{(p)} = n^2(n/6 + 1)2^n$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = (\alpha + \beta n + n^2 + n^3/6)2^n$.

**35.** The associated homogeneous recurrence relation is $a_n = 4a_{n-1} - 3a_{n-2}$. To solve it we find the characteristic equation $r^2 - 4r + 3 = 0$, find that $r = 1$ and $r = 3$ are its solutions, and therefore obtain the homogeneous solution $a_n^{(h)} = \alpha + \beta 3^n$. Next we need a particular solution to the given recurrence relation. By using the idea in Theorem 6 twice, we want to look for a function of the form $a_n = c \cdot 2^n + n(dn + e) = c \cdot 2^n + dn^2 + en$. (The factor $n$ in front of $(dn + e)$ is needed since 1 is already a root of the characteristic polynomial.) We plug this into our recurrence relation and obtain $c \cdot 2^n + dn^2 + en = 4c \cdot 2^{n-1} + 4d(n-1)^2 + 4e(n-1) - 3c \cdot 2^{n-2} - 3d(n-2)^2 - 3e(n-2) + 2^n + n + 3$. A lot of messy algebra transforms this into the following equation, where we group by function of $n$: $2^{n-2}(-c - 4) + n^2 \cdot 0 + n(-4d - 1) + (8d - 2e - 3) = 0$. The coefficients must therefore all be 0, whence $c = -4$, $d = -1/4$, and $e = -5/2$. Therefore the particular solution we seek is $a_n^{(p)} = -4 \cdot 2^n - n^2/4 - 5n/2$. So the general solution is the sum of the homogeneous solution and this particular solution, namely $a_n = -4 \cdot 2^n - n^2/4 - 5n/2 + \alpha + \beta 3^n$. Next we plug in the initial conditions to obtain $1 = a_0 = -4 + \alpha + \beta$ and $4 = a_1 = -8 - 11/4 + \alpha + 3\beta$. We solve this system of equations to obtain $\alpha = 1/8$ and $\beta = 39/8$. So the final solution is $a_n = -4 \cdot 2^n - n^2/4 - 5n/2 + 1/8 + (39/8)3^n$. As a check of our work (it would be too much to hope that we could always get this far without making an algebraic error), we can compute $a_2$ both from the recurrence and from the solution, and we find that $a_2 = 22$ both ways.

**37.** Obviously the $n^{\text{th}}$ term of the sequence comes from the $(n-1)^{\text{st}}$ term by adding the $n^{\text{th}}$ triangular number; in symbols, $a_{n-1} + n(n+1)/2 = \left(\sum_{k=1}^{n-1} k(k+1)/2\right) + n(n+1)/2 = \sum_{k=1}^{n} k(k+1)/2 = a_n$. Also, the sum of the first triangular number is clearly $1$. To solve this recurrence relation, we easily see that the homogeneous solution is $a_n^{(h)} = \alpha$, so since the nonhomogeneous term is a second degree polynomial, we need a particular solution of the form $a_n = cn^3 + dn^2 + en$. Plugging this into the recurrence relation gives $cn^3 + dn^2 + en = c(n-1)^3 + d(n-1)^2 + e(n-1) + n(n+1)/2$. Expanding and collecting terms, we have $(3c - \frac{1}{2})n^2 + (-3c + 2d - \frac{1}{2})n + (c - d + e) = 0$, whence $c = \frac{1}{6}$, $d = \frac{1}{2}$, and $e = \frac{1}{3}$. Thus $a_n^{(p)} = \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$. So the general solution is $a_n = \alpha + \frac{1}{6}n^3 + \frac{1}{2}n^2 + \frac{1}{3}n$. It is now a simple matter to plug in the initial condition $a_1 = 1$ to see that $\alpha = 0$. Note that we can find a common denominator and write our solution in the nice form $a_n = n(n+1)(n+2)/6$, which is the binomial coefficient $C(n+2, 3)$.

**39.** Nothing in the discussion of solving recurrence relations by the methods of this section relies on the roots of the characteristic equation being real numbers. Sometimes the roots are complex numbers (involving $i = \sqrt{-1}$). The situation is analogous to the fact that we sometimes get irrational numbers when solving the characteristic equation (for example, for the Fibonacci numbers), even though the coefficients are all integers and the terms in the sequence are all integers. It is just that we need irrational numbers in order to write down an algebraic solution. Here we need complex numbers in order to write down an algebraic solution, even though all the terms in the sequence are real.

**a)** The characteristic equation is $r^4 - 1 = 0$. This factors as $(r-1)(r+1)(r^2+1) = 0$, so the roots are $r = 1$ and $r = -1$ (from the first two factors) and $r = i$ and $r = -i$ (from the third factor).

**b)** By our work in part **(a)**, the general solution to the recurrence relation is $a_n = \alpha_1 + \alpha_2(-1)^n + \alpha_3 i^n + \alpha_4(-i)^n$. In order to figure out the $\alpha$'s we plug in the initial conditions, yielding the following system of linear equations:

$$\begin{aligned} 1 &= a_0 = \alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \\ 0 &= a_1 = \alpha_1 - \alpha_2 + i\alpha_3 - i\alpha_4 \\ -1 &= a_2 = \alpha_1 + \alpha_2 - \alpha_3 - \alpha_4 \\ 1 &= a_3 = \alpha_1 - \alpha_2 - i\alpha_3 + i\alpha_4 \end{aligned}$$

Remembering that $i$ is just a constant, we solve this system by elimination or other means. For instance, we could begin by subtracting the third equation from the first, to give $2 = 2\alpha_3 + 2\alpha_4$ and subtracting the fourth from the second to give $-1 = 2i\alpha_3 - 2i\alpha_4$. This gives us two equation in two unknowns. Solving them yields $\alpha_3 = (2i-1)/(4i)$ which can be put into nicer form by multiplying by $i/i$, so $\alpha_3 = (2+i)/4$; and then $\alpha_4 = 1 - \alpha_3 = (2-i)/4$. We plug these values back into the first and fourth equations, obtaining $\alpha_1 + \alpha_2 = 0$ and $\alpha_1 - \alpha_2 = 1/2$. These tell us that $\alpha_1 = 1/4$ and $\alpha_2 = -1/4$. Therefore the answer to the problem is

$$a_n = \frac{1}{4} - \frac{1}{4}(-1)^n + \frac{2+i}{4}i^n + \frac{2-i}{4}(-i)^n.$$

**41. a)** To say that $f_n$ is the integer closest to $\dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^n$ is to say that the absolute difference between these two numbers is less than $\dfrac{1}{2}$. But the difference is just $\left|\dfrac{1}{\sqrt{5}}\left(\dfrac{1-\sqrt{5}}{2}\right)^n\right|$. Thus we are asked to show that this latter number is less than $\frac{1}{2}$. The value within the parentheses is about $-0.62$. When raised to the $n^{\text{th}}$ power, for $n \geq 0$, we get a number of absolute value less than or equal to $1$. When we then divide by $\sqrt{5}$ (which is greater than $2$), we get a number less than $\frac{1}{2}$, as desired.

**b)** Clearly the second term in the formula for $f_n$ alternates sign as $n$ increases: a positive number is being subtracted for $n$ even, and a negative number is being subtracted for $n$ odd. Therefore $f_n$ is less than $\dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^n$ for even $n$ and greater than $\dfrac{1}{\sqrt{5}}\left(\dfrac{1+\sqrt{5}}{2}\right)^n$ for odd $n$.

**43.** We follow the hint and let $b_n = a_n + 1$, or, equivalently, $a_n = b_n - 1$. Then the recurrence relation becomes $b_n - 1 = b_{n-1} - 1 + b_{n-2} - 1 + 1$, or $b_n = b_{n-1} + b_{n-2}$; and the initial conditions become $b_0 = a_0 + 1 = 0 + 1 = 1$ and $b_1 = a_1 + 1 = 1 + 1 = 2$. We now apply the result of Exercise 42, with $b$ playing the role of $a$, and $s = 1$ and $t = 2$, to get $b_n = f_{n-1} + 2f_n$. Therefore $a_n = f_{n-1} + 2f_n - 1$. We can check this with a few small values of $n$: for $n = 2$, our solution predicts that $a_2 = f_1 + 2f_2 - 1 = 1 + 2 \cdot 1 - 1 = 2$; similarly, $a_3 = f_2 + 2f_3 - 1 = 1 + 2 \cdot 2 - 1 = 4$ and $a_4 = 7$. These are precisely the values we would get by applying directly the recurrence relation defining $a_n$ in this problem. A reality check like this is a good way to increase the chances that we haven't made a mistake.

An alternative answer is $a_n = f_{n+2} - 1$. We can prove this as follows:

$$f_{n-1} + 2f_n - 1 = f_{n-1} + f_n + f_n - 1 = f_{n+1} + f_n - 1 = f_{n+2} - 1$$

**45.** Let $a_n$ be the desired quantity, the number of pairs of rabbits on the island after $n$ months. So $a_0 = 1$, since one pair is there initially. We need to read the problem carefully and decide how we will interpret what it says. Since a pair produces two new pairs "at the age of one month" and six new pairs "at the age of two months" and every month thereafter, the original pair has already produced two new pairs at the end of one month, so $a_1 = 3$ (the original pair plus two new pairs), and $a_2 = 3 + 6 + 4 = 13$ (the three pairs that were already there, six new pairs produced by the original inhabitants, and two new pairs produced by each of the two pairs born at the end of the first month). If you interpret the wording to imply that births do not occur until after the month has finished, then naturally you will get different answers from those we are about to find.

**a)** We already have stated the initial conditions $a_0 = 1$ and $a_1 = 3$. To obtain a recurrence relation for $a_n$, the number of pairs of rabbits present at the end of the $n^{\text{th}}$ month, we observe (as was the case in analyzing Fibonacci's example) that all the rabbit pairs who were present at the end of the $(n-2)^{\text{nd}}$ month will give rise to six new ones, giving us $6a_{n-2}$ new pairs; and all the rabbit pairs who were present at the end of the $(n-1)^{\text{st}}$ month but not at the end of the $(n-2)^{\text{nd}}$ month will give rise to two new ones, namely $2(a_{n-1} - a_{n-2})$ new pairs. Of course, the $a_{n-1}$ pairs who were there stay around as well. Thus our recurrence relation is $a_n = 6a_{n-2} + 2(a_{n-1} - a_{n-2}) + a_{n-1}$, or, more simply, $a_n = 3a_{n-1} + 4a_{n-2}$. As a check, we compute that $a_2 = 3 \cdot 3 + 4 \cdot 1 = 13$, which is the number we got above.

**b)** We proceed by the method of this section, as we did in, say, Exercise 3. The characteristic equation is $r^2 - 3r - 4 = 0$, which factors as $(r-4)(r+1) = 0$, so we get roots 4 and $-1$. Thus the general solution is $a_n = \alpha_1 4^n + \alpha_2(-1)^n$. Plugging in the initial conditions $a_0 = 1$ and $a_1 = 3$, we find $1 = \alpha_1 + \alpha_2$ and $3 = 4\alpha_1 - \alpha_2$, which are easily solved to yield $\alpha_1 = 4/5$ and $\alpha_2 = 1/5$. Therefore the number of pairs of rabbits on the island after $n$ months is $a_n = 4 \cdot 4^n/5 + (-1)^n/5 = \left(4^{n+1} + (-1)^n\right)/5$. As a check, we see that $a_2 = (4^3 + 1)/5 = 65/5 = 13$, the number we found above.

**47.** Let $a_n$ be the employee's salary for the $n^{\text{th}}$ year of employment, in tens of thousands of dollars (this makes the numbers easier to work with). Thus we are told that $a_1 = 5$, and applying the given rule for raises, we have $a_2 = 2 \cdot 5 + 1 = 11$, $a_3 = 2 \cdot 11 + 2 = 24$, and so on.

**a)** For her $n^{\text{th}}$ year of employment, she has $n - 1$ years of experience, so the raise rule says that $a_n = 2a_{n-1} + (n-1)$. (Remember that we are using \$10,000 as the unit of pay here.)

**b)** The associated homogeneous recurrence relation is $a_n = 2a_{n-1}$, which clearly has the solution $a_n^{(h)} = \alpha 2^n$. For the particular solution of the given relation, we note that the nonhomogeneous term is a linear function of $n$ and try $a_n = cn + d$. Plugging into the relation yields $cn + d = 2c(n-1) + 2d + n - 1$, which, upon grouping like terms, becomes $(-c-1)n + (2c - d + 1) = 0$. Therefore $c = -1$ and $d = -1$, so $a_n^{(p)} = -n - 1$. Therefore the general solution is $a_n = \alpha 2^n - n - 1$. Plugging in the initial condition gives $5 = a_1 = 2\alpha - 2$, whence $\alpha = 7/2$. Our solution is therefore $a_n = 7 \cdot 2^{n-1} - n - 1$. We can check that this gives the correct salary for the first few years, as computed above.

**49.** Using the notation of Exercise 48 we have $f(n) = n+1$, $g(n) = n+3$, $h(n) = n$, and $C = 1$. Therefore

$$Q(n) \cdot n = \frac{(2 \cdot 3 \cdot 4 \cdots n) \cdot n}{4 \cdot 5 \cdot 6 \cdots (n+3)} = \frac{6n}{(n+1)(n+2)(n+3)} = \frac{-3}{n+1} + \frac{12}{n+2} + \frac{-9}{n+3}.$$

The last decomposition was by standard partial fractions techniques from calculus (write the fraction as $A/(n+1) + B/(n+2) + C/(n+3)$ and solve for $A$, $B$, and $C$ by multiplying it out and equating like powers of $n$ with the original fraction). Now we can give a closed form for $\sum_{i=1}^{n} Q(i)i$, since almost all the terms cancel out in a telescoping manner:

$$\begin{aligned}
\sum_{i=1}^{n} Q(i)i &= \sum_{i=1}^{n} \frac{-3}{i+1} + \frac{12}{i+2} + \frac{-9}{i+3} \\
&= -\frac{3}{2} + \frac{12}{3} - \frac{9}{4} - \frac{3}{3} + \frac{12}{4} - \frac{9}{5} - \frac{3}{4} + \frac{12}{5} - \frac{9}{6} - \frac{3}{5} + \frac{12}{6} - \frac{9}{7} + \cdots \\
&\quad - \frac{3}{n-1} + \frac{12}{n} - \frac{9}{n+1} - \frac{3}{n} + \frac{12}{n+1} - \frac{9}{n+2} - \frac{3}{n+1} + \frac{12}{n+2} - \frac{9}{n+3} \\
&= -\frac{3}{2} + \frac{12}{3} - \frac{3}{3} - \frac{9}{n+2} + \frac{12}{n+2} - \frac{9}{n+3} = \frac{3}{2} - \frac{6n+9}{(n+2)(n+3)}
\end{aligned}$$

This plus 1 gives us the numerator for $a_n$, according to the formula given in Exercise 48. For the denominator, we need

$$g(n+1)Q(n+1) = \frac{(n+4) \cdot 2 \cdot 3 \cdots (n+1)}{4 \cdot 5 \cdots (n+4)} = \frac{6}{(n+2)(n+3)}.$$

Putting this all together algebraically, we obtain $a_n = (5n^2 + 13n + 12)/12$. We can (and should!) check that this conforms to the recurrence when we calculate $a_1$, $a_2$, and so on. Indeed, we get $a_1 = 5/2$ and $a_2 = 29/6$ both ways. It is interesting to note that asking *Maple* to do this with the command

```
rsolve({a(n) = ((n + 3) * a(n - 1) + n)/(n + 1), a(0) = 1}, a(n));
```

produces the correct answer.

**51.** A proof of this theorem can be found in textbooks such as *Discrete Mathematics with Applications* by H. E. Mattson, Jr. (Wiley, 1993), Chapter 11.

**53.** We follow the hint, letting $n = 2^k$ and $a_k = \log T(n) = \log T(2^k)$. We take the log (base 2, of course) of both sides of the given recurrence relation and use the properties of logarithms to obtain

$$\log T(n) = \log n + 2 \log T(n/2),$$

so we have

$$\log T(2^k) = k + 2 \log T(2^{k-1})$$

or

$$a_k = k + 2a_{k-1}.$$

The initial condition becomes $a_0 = \log 6$. Using the techniques in this section, we find that the general solution of the recurrence relation is $a_k = c \cdot 2^k - k - 2$. Plugging in the initial condition leads to $c = 2 + \log 6$. Now we have to translate this back into terms involving $T$. Since $T(n) = 2^{a_k}$ and $n = 2^k$ we have

$$T(n) = 2^{(2+\log 6) \cdot 2^k - k - 2} = (2^{\log 6})^{2^k} (2^{2 \cdot 2^k - 2})(2^{-k}) = \frac{6^n \cdot 4^{n-1}}{n}.$$

## SECTION 8.3 Divide-and-Conquer Algorithms and Recurrence Relations

*Many of these exercises are fairly straightforward applications of Theorem 2 (or its special case, Theorem 1). The messiness of the algebra and analysis in this section is indicative of what often happens when trying to get reasonably precise estimates for the efficiency of complicated or clever algorithms.*

**1.** Let $f(n)$ be the number of comparisons needed in a binary search of a list of $n$ elements. From Example 1 we know that $f$ satisfies the divide-and-conquer recurrence relation $f(n) = f(n/2) + 2$. Also, 2 comparisons are needed for a list with one element, i.e., $f(1) = 2$ (see Example 3 in Section 3.3 for further discussion). Thus $f(64) = f(32) + 2 = f(16) + 4 = f(8) + 6 = f(4) + 8 = f(2) + 10 = f(1) + 12 = 2 + 12 = 14$.

**3.** In the notation of Example 4 (all numerals in base 2), we want to multiply $a = 1110$ by $b = 1010$. Note that $n = 2$. Therefore $A_1 = 11$, $A_0 = 10$, $B_1 = 10$ and $B_0 = 10$. We need to form $A_1 - A_0 = 11 - 10 = 01$ and $B_0 - B_1 = 00$. Then we need the following three products: $A_1B_1 = (11)(10)$, $(A_1 - A_0)(B_0 - B_1) = (01)(00)$, and $A_0B_0 = (10)(10)$. In order to from these products, the algorithm would in fact recurse, but let us not worry about that, assuming instead that we have these answers, namely $A_1B_1 = 0110$, $(A_1 - A_0)(B_0 - B_1) = 0000$, and $A_0B_0 = 0100$. Now we need to shift these products various numbers of places to the left. We shift $A_1B_1$ $2n = 4$ places and also $n = 2$ places, obtaining 01100000 and 011000; we shift $(A_1 - A_0)(B_0 - B_1)$ $n = 2$ places, obtaining 000000, and we shift $A_0B_0$ $n = 2$ places and also no places, obtaining 010000 and 0100. Finally we add all five of these binary numbers, obtaining 10001100.

**5.** This problem is asking us to estimate the number of bit operations needed to do the shifts, additions, and subtractions in multiplying two $2n$-bit integers by the algorithm in Example 4. First recall from Example 9 in Section 4.2 that the number of bit operations needed for an addition of two $k$-bit numbers is at most $3k$; the same bound holds for subtraction. Let us assume that to shift a number $k$ bits also requires $k$ bit operations. Thus we need to count the number of additions and shifts of various sizes that occur in the fast multiplication algorithm. First, we need to perform two subtractions of $n$-bit numbers to get $A_1 - A_0$ and $B_0 - B_1$; these will take up to $6n$ bit operations altogether. We need to shift $A_1B_1$ $2n$ places (requiring $2n$ bit operations), and also $n$ places (requiring $n$ bit operations); we need to shift $(A_1 - A_0)(B_0 - B_1)$ $n$ places (requiring $n$ bit operations); and we need to shift $A_0B_0$ $n$ places, also requiring $n$ bit operations. This makes a total of $5n$ bit operations for the shifting. Finally we need to worry about the additions (which actually might include a subtraction if the middle term is negative). If we are clever, we can add the four terms that involve at most $3n$ bits first (that is, everything except the $2^{2n}A_1B_1$). Three additions are required, each taking $9n$ bit operations, for a total of $27n$ bit operations. Finally we need to perform one addition involving a $4n$-bit number, taking $12n$ operations. This makes a total of $39n$ bit operations for the additions.

Putting all these operations together, we need perhaps a total of $6n + 5n + 39n = 50n$ bit operations to perform all the additions, subtractions, and shifts. Obviously this bound is not exact; it depends on the actual implementation of these binary operations.

Using $C = 50$ as estimated above, the recurrence relation for fast multiplication is $f(2n) = 3f(n) + 50n$, with $f(1) = 1$ (one multiplication of bits is all that is needed if we have 1-bit numbers). Thus we can compute $f(64)$ as follows: $f(2) = 3 \cdot 1 + 50 = 53$; $f(4) = 3 \cdot 53 + 100 = 259$; $f(8) = 3 \cdot 259 + 200 = 977$; $f(16) = 3 \cdot 977 + 400 = 3331$; $f(32) = 3 \cdot 3331 + 800 = 10793$; and finally $f(64) = 3 \cdot 10793 + 1600 = 33979$. Thus about 34,000 bit operations are needed.

**7.** We compute these from the bottom up. (In fact, it is easy to see by induction that $f(3^k) = k + 1$, so no computation is really needed at all.)

**a)** $f(3) = f(1) + 1 = 1 + 1 = 2$ **b)** $f(9) = f(3) + 1 = 2 + 1 = 3$; $f(27) = f(9) + 1 = 3 + 1 = 4$

**c)** $f(81) = f(27) + 1 = 4 + 1 = 5$; $f(243) = f(81) + 1 = 5 + 1 = 6$; $f(729) = f(243) + 1 = 6 + 1 = 7$

**9.** We compute these from the bottom up.

**a)** $f(5) = f(1) + 3 \cdot 5^2 = 4 + 75 = 79$

**b)** $f(25) = f(5) + 3 \cdot 25^2 = 79 + 1875 = 1954$; $f(125) = f(25) + 3 \cdot 125^2 = 1954 + 46875 = 48{,}829$

**c)** $f(625) = f(125) + 3 \cdot 625^2 = 48829 + 1171875 = 1220704$; $f(3125) = f(625) + 3 \cdot 3125^2 = 1220704 + 29296875 = 30{,}517{,}579$

**11.** We apply Theorem 2, with $a = 1$, $b = 2$, $c = 1$, and $d = 0$. Since $a = b^d$, we have that $f(n)$ is $O(n^d \log n) = O(\log n)$.

**13.** We apply Theorem 2, with $a = 2$, $b = 3$, $c = 4$, and $d = 0$. Since $a > b^d$, we have that $f(n)$ is $O(n^{\log_b a}) = O(n^{\log_3 2}) \approx O(n^{0.63})$.

**15.** After 1 round, there are 16 teams left; after 2 rounds, 8 teams; after 3 rounds, 4 teams; after 4 rounds, 2 teams; and after 5 rounds, only 1 team remains, so the tournament is over. In general, $k$ rounds are needed if there are $2^k$ teams (easily proved by induction).

**17. a)** Our recursive algorithm will take a sequence of names and determine whether one name occurs as more than half of the elements of the sequence, and if so, which name that is. If the sequence has just one element, then the one person on the list is the winner. For the recursive step, divide the list into two parts—the first half and the second half—as equally as possible. As is pointed out in the hint, no one could have gotten a majority of the votes on this list without having a majority in one half or the other, since if a candidate got less than or equal to half the votes in each half, then he got less than or equal to half the votes in all (this is essentially just the distributive law). Apply the algorithm recursively to each half to come up with at most two names. Then run through the entire list to count the number of occurrences of each of those names to decide which, if either, is the winner. This requires at most $2n$ additional comparisons for a list of length $n$.

**b)** We apply the master theorem with $a = 2$, $b = 2$, $c = 2$, and $d = 1$. Since $a = b^d$, we know that the number of comparisons is $O(n^d \log n) = O(n \log n)$.

**19. a)** We compute $x^n$ when $n$ is even by first computing $y = x^{n/2}$ recursively and then doing one multiplication, namely $y \cdot y$. When $n$ is odd, we first compute $y = x^{(n-1)/2}$ recursively and then do two multiplications, namely $y \cdot y \cdot x$. So if $f(n)$ is the number of multiplications required, assuming the worst, then we have essentially $f(n) = f(n/2) + 2$.

**b)** By the master theorem, with $a = 1$, $b = 2$, $c = 2$, and $d = 0$, we see that $f(n)$ is $O(n^0 \log n) = O(\log n)$.

**21. a)** $f(16) = 2f(4) + 1 = 2(2f(2) + 1) + 1 = 2(2 \cdot 1 + 1) + 1 = 7$

**b)** Let $m = \log n$, so that $n = 2^m$. Also, let $g(m) = f(2^m)$. Then our recurrence becomes $f(2^m) = 2f(2^{m/2}) + 1$, since $\sqrt{2^m} = (2^m)^{1/2} = 2^{m/2}$. Rewriting this in terms of $g$ we have $g(m) = 2g(m/2) + 1$. Theorem 1 now tells us that $g(m)$ is $O(m^{\log_2 2}) = O(m)$. Since $m = \log n$, this says that our function is $O(\log n)$.

**23. a)** The messiest part of this is just the bookkeeping. Note that we start with *best* set to $0$, since the empty subsequence has a sum of $0$, and this is the best we can do if all the terms are negative. Also note that it would be an easy improvement to keep track of where the subsequence is located, as well as what its sum is.

**procedure** *largest sum*($a_1, \ldots, a_n$ : real numbers)
*best* := 0 {empty subsequence has sum 0}
**for** $i := 1$ **to** $n$
    *sum* := 0
    **for** $j := i$ **to** $n$
        *sum* := *sum* + $a_j$
        **if** *sum* > *best* **then** *best* := *sum*
**return** *best* {the maximum possible sum of numbers in the list}

**b)** One sum and one comparison are made inside the inner loop. This loop is executed $C(n+1,2)$ times—once for each choice of a pair $(i,j)$ of endpoints of the sequence of consecutive terms being examined (this is a combination with repetition allowed, since $i = j$ when we are examining one term by itself). Since $C(n+1,2) = n(n+1)/2$ the computational complexity is $O(n^2)$.

**c)** We divide the list into a first half and a second half and apply the algorithm recursively to find the largest sum of consecutive terms for each half. The largest sum of consecutive terms in the entire sequence is either one of these two numbers or the sum of a sequence of consecutive terms that crosses the middle of the list. To find the largest possible sum of a sequence of consecutive terms that crosses the middle of the list, we start at the middle and move forward to find the largest possible sum in the second half of the list, and move backward to find the largest possible sum in the first half of the list; the desired sum is the sum of these two quantities. The final answer is then the largest of this sum and the two answers obtained recursively. The base case is that the largest sum of a sequence of one term is the larger of that number and 0.

**d)** (i) Split the list into the first half, $-2, 4, -1, 3$, and the second half, $5, -6, 1, 2$. Apply the algorithm recursively to each half (we omit the details of this step) to find that the largest sum in the first half is 6 and the largest sum in the second half is 5. Now find the largest sum of a sequence of consecutive terms that crosses the middle of the list. Moving forward, the best we can do is 5; moving backward, the best we can do is 6. Therefore we can get a sum of 11 by adding the second through fifth terms. This is better than either recursive answer, so the desired answer is 11. (ii) Split the list into the first half, $4, 1, -3, 7$, and the second half, $-1, -5, 3, -2$. Apply the algorithm recursively to each half (we omit the details of this step) to find that the largest sum in the first half is 9 and the largest sum in the second half is 3. Now find the largest sum of a sequence of consecutive terms that crosses the middle of the list. Moving forward, the best we can do is $-1$; moving backward, the best we can do is 9. Therefore we can get a sum of 8 by crossing the middle. The best of these three possibilities is 9, which we get from the first through fourth terms. (iii) Split the list into the first half, $-1, 6, 3, -4$, and the second half, $-5, 8, -1, 7$. Apply the algorithm recursively to each half (we omit the details of this step) to find that the largest sum in the first half is 9 and the largest sum in the second half is 14. Now find the largest sum of a sequence of consecutive terms that crosses the middle of the list. Moving forward, the best we can do is 9; moving backward, the best we can do is 5. Therefore we can get a sum of 14 by adding the second through eighth terms. The best of these three is actually a tie, between the second through eighth terms and the sixth through eighth terms, with a sum of 14 in each case.

**e)** Let $S(n)$ be the number of sums and $C(n)$ the number of comparisons used. Since the "conquer" step requires $n$ sums and $n+2$ comparisons (two extra comparisons to determine the winner among the three possible largest sums), we have $S(n) = 2S(n/2) + n$ and $C(n) = 2C(n/2) + n + 2$. The basis step is $C(1) = 1$ and $S(1) = 0$.

**f)** By the master theorem with $a = b = 2$ and $d = 1$, we see that we need only $O(n \log n)$ of each type of operation. This is a significant improvement over the $O(n^2)$ complexity we found in part **(b)** for the algorithm in part **(a)**.

**25.** To carry this down to its base level would require applying the algorithm seven times, so to keep things within reason, we will show only the outermost step. The points are already sorted for us, and so we divide them into two groups, using $x$ coordinate. The left side will have the first eight points listed in it (they all have $x$ coordinates less than 4.5), and the right side will have the rest, all of which have $x$ coordinates greater than 4.5. Thus our vertical line will be taken to be $x = 4.5$. Now assume that we have already applied the algorithm recursively to find the minimum distance between two points on the left, and the minimum distance on the right. It turns out that $d = d_L = d_R = 2$. This is achieved, for example, by the points $(1,6)$ and $(3,6)$. Thus we want to concentrate on the strip from $x = 2.5$ to $x = 6.5$ of width $2d$. The only points in this strip are $(3,1)$, $(3,6)$, $(3,10)$, $(4,3)$, $(5,1)$, $(5,5)$, $(5,9)$, and $(6,7)$, Working from the bottom up, we compute distances from these points to points as much as $d = 2$ vertical units above them. According to the discussion in the text, there can never be more than seven such computations for each point in the strip. The distances we need, then, are $\overline{(3,1)(5,1)}$, $\overline{(3,1)(4,3)}$, $\overline{(5,1)(4,3)}$, $\overline{(4,3)(5,5)}$, $\overline{(5,5)(3,6)}$, $\overline{(5,5)(6,7)}$, $\overline{(3,6)(6,7)}$, $\overline{(6,7)(5,9)}$, and $\overline{(5,9)(3,10)}$. The smallest of these turns out to be $2$, so the minimum distance $d = 2$ in fact is the smallest distance among all the points. (Actually we did not need to compute the distances between points that were already on the same side of the dividing line, since their distance had already been computed in the recursive step, but checking whether they are on opposite sides of the vertical line would entail additional computation anyway.)

**27.** The algorithm is essentially the same as the algorithm given in Example 12. The only difference is in constructing the boxes centered on the vertical line that divides the two halves of the set of points. In this variation, our strip still has width $2d$ (i.e., $d$ units to the left and $d$ units to the right of the vertical line), because it would be possible for two points within this box, one on each side of the line, to lie at a distance less than $d$ from each other, but no point outside this strip has a chance to contribute to a small "across the line" distance. In this variation, however, we do not need to construct eight boxes of size $(d/2) \times (d/2)$, but rather just two boxes of size $d \times d$. The reason for this is that there can be at most one point in each of those boxes using the distance formula given in this exercise—two points within such a box (which is on the same side of the dividing line) can be at most $d$ units apart and so would already have been considered in the recursive step. Thus the recurrence relation is the same as the recurrence relation in Example 12, except that the coefficient 7 is replaced by 1. The analysis via the master theorem remains the same, and again we get a $O(n \log n)$ algorithm.

**29.** Suppose $n = b^k$, so that $k = \log_b n$. We will prove by induction on $k$ that $f(b^k) = f(1)(b^k)^d + c(b^k)^d k$, which is what we are asked to prove, translated into this notation. If $k = 0$, then the equation reduces to $f(1) = f(1)$, which is certainly true. We assume the inductive hypothesis, that $f(b^k) = f(1)(b^k)^d + c(b^k)^d k$, and we try to prove that $f(b^{k+1}) = f(1)(b^{k+1})^d + c(b^{k+1})^d(k+1)$. By the recurrence relation for $f(n)$ in terms of $f(n/b)$, we have $f(b^{k+1}) = b^d f(b^k) + c(b^{k+1})^d$. Then we invoke the inductive hypothesis and work through the algebra:

$$\begin{aligned} b^d f(b^k) + c(b^{k+1})^d &= b^d\big(f(1)(b^k)^d + c(b^k)^d k\big) + c(b^{k+1})^d \\ &= f(1)b^{kd+d} + cb^{kd+d}k + c(b^{k+1})^d \\ &= f(1)(b^{k+1})^d + c(b^{k+1})^d k + c(b^{k+1})^d \\ &= f(1)(b^{k+1})^d + c(b^{k+1})^d(k+1) \end{aligned}$$

**31.** The algebra is quite messy, but this is a straightforward proof by induction on $k = \log_b n$. If $k = 0$, so that $n = 1$, then we have the true statement

$$f(1) = C_1 + C_2 = \frac{b^d c}{b^d - a} + f(1) + \frac{b^d c}{a - b^d}$$

(since the fractions cancel each other out). Assume the inductive hypothesis, that for $n = b^k$ we have

$$f(n) = \frac{b^d c}{b^d - a} n^d + \left(f(1) + \frac{b^d c}{a - b^d}\right) n^{\log_b a}.$$

Then for $n = b^{k+1}$ we apply first the recurrence relation, then the inductive hypothesis, and finally some algebra:

$$\begin{aligned}
f(n) &= af\left(\frac{n}{b}\right) + cn^d \\
&= a\left(\frac{b^d c}{b^d - a}\left(\frac{n}{b}\right)^d + \left(f(1) + \frac{b^d c}{a - b^d}\right)\left(\frac{n}{b}\right)^{\log_b a}\right) + cn^d \\
&= \frac{b^d c}{b^d - a} \cdot n^d \cdot \frac{a}{b^d} + \left(f(1) + \frac{b^d c}{a - b^d}\right) n^{\log_b a} + cn^d \\
&= n^d\left(\frac{ac}{b^d - a} + \frac{c(b^d - a)}{b^d - a}\right) + \left(f(1) + \frac{b^d c}{a - b^d}\right) n^{\log_b a} \\
&= \frac{b^d c}{b^d - a} \cdot n^d + \left(f(1) + \frac{b^d c}{a - b^d}\right) n^{\log_b a}
\end{aligned}$$

Thus we have verified that the equation holds for $k+1$, and our induction proof is complete.

**33.** The equation given in Exercise 31 says that $f(n)$ is the sum of a constant times $n^d$ and a constant times $n^{\log_b a}$. Therefore we need to determine which term dominates, i.e., whether $d$ or $\log_b a$ is larger. But we are given $a > b^d$; hence $\log_b a > \log_b b^d = d$. It therefore follows (we are also using the fact that $f$ is increasing) that $f(n)$ is $O(n^{\log_b a})$.

**35.** We use the result of Exercise 33, since $a = 5 > 4^1 = b^d$. Therefore $f(n)$ is $O(n^{\log_b a}) = O(n^{\log_4 5}) \approx O(n^{1.16})$.

**37.** We use the result of Exercise 33, since $a = 8 > 2^2 = b^d$. Therefore $f(n)$ is $O(n^{\log_b a}) = O(n^{\log_2 8}) = O(n^3)$.

## SECTION 8.4 Generating Functions

*Generating functions are an extremely powerful tool in mathematics (not just in discrete mathematics). This section, as well as some material introduced in these exercises, gives you an introduction to them. The algebra in many of these exercises gets very messy, and you probably want to check your answers, either by computing values when solving recurrence relation problems, or by using a computer algebra package. See the solution to Exercise 11, for example, to learn how to get* Maple *to produce the sequence for a given generating function. For more information on generating functions, consult reference [Wi2] given at the end of this* Guide *(in the List of References for the Writing Projects).*

**1.** By definition we want the function $f(x) = 2 + 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 = 2(1 + x + x^2 + x^3 + x^4 + x^5)$. From Example 2, we see that the expression in parentheses can also be written as $(x^6 - 1)/(x - 1)$. Thus we can write the answer as $f(x) = 2(x^6 - 1)/(x - 1)$.

**3.** We will use Table 1 in much of this solution.

**a)** Apparently all the terms are 0 except for the six 2's shown. Thus $f(x) = 2x + 2x^2 + 2x^3 + 2x^4 + 2x^5 + 2x^6$. This is already in closed form, but we can also write it more compactly as $f(x) = 2x(1 - x^6)/(1 - x)$ by factoring out $2x$, or as $f(x) = 2(1 - x^7)/(1 - x) - 2$ by subtracting away the missing term. In each case we use the identity from Example 2.

**b)** Apparently all the terms beyond the first three are 1's. Since $1/(1-x) = 1 + x + x^2 + x^3 + \cdots$, we can write this generating function as $1/(1-x) - 1 - x - x^2$, or we can write it as $x^3/(1-x)$, depending on whether we want to simplify by adding back the missing terms or by factoring out $x^3$.

**c)** This generating function is the sequence $x + x^4 + x^7 + x^{10} + \cdots$. If we factor out an $x$, we have $x(1 + (x^3) + (x^3)^2 + (x^3)^3 + \cdots) = x/(1 - x^3)$, from Table 1.

**d)** We factor out a 2 and then include the remaining factors of 2 along with the $x$ terms. Thus our generating function is $2(1 + (2x) + (2x)^2 + (2x)^3 + \cdots) = 2/(1 - 2x)$, again using Table 1.

**e)** By the binomial theorem (see also Table 1), the generating function is $(1+x)^7$.

**f)** From Table 1 we know that $1/(1-ax) = 1 + ax + a^2x^2 + a^3x^3 + \cdots$. That is what we have here, with $a = -1$ (and a factor of 2 in front of it all). Therefore the generating function is $2/(1+x)$.

**g)** This sequence is all 1's except for a 0 where the $x^2$ coefficient should be. Therefore the generating function is $(1/(1-x)) - x^2$.

**h)** If we factor out $x^3$, then we can use a formula from Table 1: $x^3 + 2x^4 + 3x^5 + \cdots = x^3(1 + 2x + 3x^2 + \cdots) = x^3/(1-x)^2$.

**5.** As in Exercise 3, we make extensive use of Table 1.

**a)** Since the sequence with $a_n = 1$ for all $n$ has generating function $1/(1-x)$, this sequence has generating function $5/(1-x)$.

**b)** By Table 1 the answer is $1/(1-3x)$.

**c)** We can either subtract the missing terms and write this generating function as $(2/(1-x)) - 2 - 2x - 2x^2$, or we can factor out $x^3$ and write it as $2x^3/(1-x)$. Note that these two algebraic expressions are equivalent.

**d)** We need to split this into two parts. Since we know that the generating function for the sequence $\{n+1\}$ is $1/(1-x)^2$, we write $2n+3 = 2(n+1)+1$. Therefore the generating function is $2/(1-x)^2 + 1/(1-x)$. We can combine terms and write this function as $(3-x)/(1-x)^2$, but there is no particular reason to prefer that form in general.

**e)** By Table 1 the answer is $(1+x)^8$. Note that $C(8,n) = 0$ by definition for all $n > 8$.

**f)** By Table 1 the generating function is $1/(1-x)^5$.

**7. a)** We can rewrite this as $(-4(1-\frac{3}{4}x))^3 = -64(1-\frac{3}{4}x)^3$ and then apply the binomial theorem (the second line of Table 1) to get $a_n = -64C(3,n)(-\frac{3}{4})^n$. Explicitly, this says that $a_0 = -64$, $a_1 = 144$, $a_2 = -108$, $a_3 = 27$, and $a_n = 0$ for all $n \geq 4$. Alternatively, we could (by hand or with *Maple*) just multiply out this finite polynomial and note the coefficients.

**b)** This is like part **(a)**. By the binomial theorem (the third line of Table 1) we get $a_{3n} = C(3,n)$, and the other coefficients are all $0$. Alternatively, we could just multiply out this finite polynomial and note the nonzero coefficients: $a_0 = 1$, $a_3 = 3$, $a_6 = 3$, $a_9 = 1$.

**c)** By Table 1, $a_n = 5^n$.

**d)** Note that $x^3/(1+3x) = x^3\sum_{n=0}^{\infty}(-3)^n x^n = \sum_{n=0}^{\infty}(-3)^n x^{n+3} = \sum_{n=3}^{\infty}(-3)^{n-3}x^n$. So $a_n = (-3)^{n-3}$ for $n \geq 3$, and $a_0 = a_1 = a_2 = 0$.

**e)** We know what the coefficients are for the power series of $1/(1-x^2)$, namely 0 for the odd ones and 1 for the even ones. The first three terms of this function force us to adjust the values of $a_0$, $a_1$ and $a_2$. So we have $a_0 = 7+1 = 8$, $a_1 = 3+0 = 3$, $a_2 = 1+1 = 2$, $a_n = 0$ for odd $n$ greater than 2, and $a_n = 1$ for even $n$ greater than 2.

**f)** Perhaps this is easiest to see if we write it out: $x^4(1 + x^4 + x^8 + x^{12} + \cdots) - x^3 - x^2 - x - 1 = x^4 + x^8 + x^{12} + \cdots - x^3 - x^2 - x - 1$. Therefore we have $a_n = 1$ if $n$ is a positive multiple of 4; $a_n = -1$ if $n < 4$, and $a_n = 0$ otherwise.

**g)** We know that $x^2/(1-x)^2 = x^2\sum_{n=0}^{\infty}(n+1)x^n = \sum_{n=0}^{\infty}(n+1)x^{n+2} = \sum_{n=2}^{\infty}(n-1)x^n$. Therefore $a_n = n-1$ for $n \geq 2$ and $a_0 = a_1 = 0$.

**h)** We know that $2e^{2x} = 2\sum_{n=0}^{\infty}(2x)^n/n! = \sum_{n=0}^{\infty}(2^{n+1}/n!)x^n$. Therefore $a_n = 2^{n+1}/n!$.

**9.** Different approaches are possible for obtaining these answers. One can use brute force algebra and just multiply everything out, either by hand or with computer algebra software such as *Maple*. One can view the problem as asking for the solution to a particular combinatorial problem and solve the problem by other means (e.g., listing all the possibilities). Or one can get a closed form expression for the coefficients, using the generating function theory developed in this section.

**a)** First we view this combinatorially. To obtain a term $x^{10}$ when multiplying out these three factors, we could either take two $x^5$'s and one $x^0$, or we could take two $x^0$'s and one $x^{10}$. In each case there are $C(3,1)=3$ choices for the factor from which to pick the single value. Therefore the answer is $3+3=6$. Second, it is clear that we can view this problem as asking for the coefficient of $x^2$ in $(1+x+x^2+x^3+\cdots)^3$, since each $x^5$ in the original is playing the role of $x$ here. Since $(1+x+x^2+x^3+\cdots)^3 = 1/(1-x)^3 = \sum_{n=0}^{\infty} C(n+2,2)x^n$, the answer is clearly $C(2+2,2)=C(4,2)=6$. A third way to get the answer is to ask *Maple* to compute $(1+x^5+x^{10})^3$ and look at the coefficient of $x^{10}$, which will turn out to be $6$. Note that we don't have to go beyond $x^{10}$ in each factor, because the higher terms can't contribute to an $x^{10}$ term in the answer.

**b)** If we factor out $x^3$ from each factor, we can write this as $x^9(1+x+x^2+\cdots)^3$. Thus we are seeking the coefficient of $x$ in $(1+x+x^2+\cdots)^3 = \sum_{n=0}^{\infty} C(n+2,2)x^n$, so the answer is $C(1+2,2)=3$. The other two methods explained in part **(a)** work here as well.

**c)** If we factor out as high a power of $x$ from each factor as we can, then we can write this as

$$x^7(1+x+x^2)(1+x+x^2+x^3+x^4)(1+x+x^2+x^3+\cdots)\,,$$

and so we seek the coefficient of $x^3$ in $(1+x+x^2)(1+x+x^2+x^3+x^4)(1+x+x^2+x^3+\cdots)$. By brute force we can list the nine ways to obtain $x^3$ in this product (where "$ijk$" means choose an $x^i$ term from the first factor, an $x^j$ term from the second factor, and an $x^k$ term from the third factor): 003, 012, 021, 030, 102, 111, 120, 201, 210. If we want to do this more analytically, let us write our expression in closed form as

$$\frac{1-x^3}{1-x}\cdot\frac{1-x^5}{1-x}\cdot\frac{1}{1-x} = \frac{1-x^3-x^5+x^8}{(1-x)^3} = \frac{1}{(1-x)^3} - x^3\cdot\frac{1}{(1-x)^3} + \text{irrelevant terms}\,.$$

Now the coefficient of $x^n$ in $1/(1-x)^3$ is $C(n+2,2)$. Furthermore, the coefficient of $x^3$ in this power series comes either from the coefficient of $x^3$ in the first term in the final expression displayed above, or from the coefficient of $x^0$ in the second factor of the second term of that expression. Therefore our answer is $C(3+2,2)-C(0+2,2)=10-1=9$.

**d)** Note that only even powers appear in the first and third factor, so to get $x^{10}$ when we multiply this out, we can only choose the $x^6$ term in the second factor. But this would require terms from the first and third factors with a total exponent of 4, and clearly that is not possible. Therefore the desired coefficient is 0.

**e)** The easiest approach here might be brute force. Using the same notation as explained in part **(c)** above, the ways to get $x^{10}$ are 046, 280, 406, 640, and (10)00. Therefore the answer is 5. We can check this with *Maple*. An analytic approach would be rather messy for this problem.

**11. a)** By Table 1 the coefficient of $x^n$ in this power series is $2^n$. Therefore the answer is $2^{10}=1024$.

**b)** By Table 1 the coefficient of $x^n$ in this power series is $(-1)^nC(n+1,1)$. Therefore the answer is $(-1)^{10}C(10+1,1)=11$.

**c)** By Table 1 the coefficient of $x^n$ in this power series is $C(n+2,2)$. Therefore the answer is $C(10+2,2)=66$.

**d)** By Table 1 the coefficient of $x^n$ in this power series is $(-2)^nC(n+3,3)$. Therefore the answer is $(-2)^{10}C(10+3,3)=292{,}864$. Incidentally, *Maple* can do this kind of problem as well. Typing

```
series(1/(1 + 2 * x)^4, x = 0, 11);
```

will cause *Maple* to give the terms of the power series for this function, including all terms less than $x^{11}$. The output looks like

$$1-8\,x+40\,x^2-160\,x^3+560\,x^4-1792\,x^5+5376\,x^6-15360\,x^7+42240\,x^8-112640\,x^9+292864\,x^{10}+O(x^{11})\,.$$

(You might wonder why *Maple* says that the terms involving $x^{11}$, $x^{12}$, and so on are big-$O$ of $x^{11}$. That seems backward! The reason is that one thinks of $x$ as approaching 0 here, rather than infinity. Then, indeed, each term with a higher power of $x$ (greater than 11) is smaller than $x^{11}$, up to a constant multiple.)

**e)** This is really asking for the coefficient of $x^6$ in $1/(1-3x)^3$. Following the same idea as in part **(d)**, we see that the answer is $3^6C(6+2,2)=20{,}412$.

**13.** Each child will correspond to a factor in our generating function. We can give any number of balloons to the child, as long as it is at least 2; therefore the generating function for each child is $x^2 + x^3 + x^4 + \cdots$. We want to find the coefficient of $x^{10}$ in the expansion of $(x^2 + x^3 + x^4 + \cdots)^4$. This function is the same as $x^8(1 + x + x^2 + x^3 + \cdots)^4 = x^8/(1-x)^4$. Therefore we want the coefficient of $x^2$ in the generating function for $1/(1-x)^4$, which we know from Table 1 is $C(2+3,3) = 10$. Alternatively, to find the coefficient of $x^2$ in $(1+x+x^2+x^3+\cdots)^4$, we can multiply out $(1+x+x^2)^4$ (perhaps with a computer algebra package such as *Maple*), and the coefficient of $x^2$ turns out to be $10$. Note that we truncated the series to be multiplied out, since terms higher than $x^2$ can't contribute to the $x^2$ term.

**15.** Each child will correspond to a factor in our generating function. We can give 1, 2, or 3 animals to the child; therefore the generating function for each child is $x + x^2 + x^3$. We want to find the coefficient of $x^{15}$ in the expansion of $(x + x^2 + x^3)^6$. Factoring out an $x$ from each term, we see that this is the same as the coefficient of $x^9$ in $(1 + x + x^2)^6$. We can multiply this out (preferably with a computer algebra package such as *Maple*), and the coefficient of $x^9$ turns out to be $50$. To solve it analytically, we write our generating function $(1 + x + x^2)^6$ as

$$\left(\frac{1-x^3}{1-x}\right)^6 = \frac{1 - 6x^3 + 15x^6 - 20x^9 + \text{higher order terms}}{(1-x)^6}.$$

There are four contributions to the coefficient of $x^9$, one for each listed term in the numerator, from the power series for $1/(1-x)^6$. Since the coefficient of $x^n$ in $1/(1-x)^6$ is $C(n+5,5)$, our answer is $C(9+5,5) - 6C(6+5,5) + 15C(3+5,5) - 20C(0+5,5) = 2002 - 2772 + 840 - 20 = 50$.

**17.** The factor in the generating function for choosing the donuts for each policeman is $x^3 + x^4 + x^5 + x^6 + x^7$. Therefore the generating function for this problem is $(x^3+x^4+x^5+x^6+x^7)^4$. We want to find the coefficient of $x^{25}$, since we want 25 donuts in all. This is equivalent to finding the coefficient of $x^{13}$ in $(1+x+x^2+x^3+x^4)^4$, since we can factor out $(x^3)^4 = x^{12}$. At this point, we could multiply it out (perhaps with *Maple*), and see that the desired coefficient is 20. Alternatively, we can write our generating function as

$$\left(\frac{1-x^5}{1-x}\right)^4 = \frac{1 - 4x^5 + 6x^{10} + \text{higher order terms}}{(1-x)^4}.$$

There are three contributions to the coefficient of $x^{13}$, one for each term in the numerator, from the power series for $1/(1-x)^4$. Since the coefficient of $x^n$ in $1/(1-x)^4$ is $C(n+3,3)$, our answer is $C(13+3,3) - 4C(8+3,3) + 6C(3+3,3) = 560 - 660 + 120 = 20$.

**19.** We want the coefficient of $x^k$ to be the number of ways to make change for $k$ dollars. One-dollar bills contribute 1 each to the exponent of $x$. Thus we can model the choice of the number of one-dollar bills by the choice of a term from $1+x+x^2+x^3+\cdots$. Two-dollar bills contribute 2 each to the exponent of $x$. Thus we can model the choice of the number of two-dollar bills by the choice of a term from $1+x^2+x^4+x^6+\cdots$. Similarly, five-dollar bills contribute 5 each to the exponent of $x$, so we can model the choice of the number of five-dollar bills by the choice of a term from $1+x^5+x^{10}+x^{15}+\cdots$. Similar reasoning applies to ten-dollar bills. Thus the generating function is $f(x) = (1+x+x^2+x^3+\cdots)(1+x^2+x^4+x^6+\cdots)(1+x^5+x^{10}+x^{15}+\cdots)(1+x^{10}+x^{20}+x^{30}+\cdots)$, which can also be written (see Table 1) as

$$f(x) = \frac{1}{(1-x)(1-x^2)(1-x^5)(1-x^{10})}.$$

**21.** Let $e_i$, for $i = 1, 2, 3$, be the exponent of $x$ taken from the $i^{\text{th}}$ factor in forming a term $x^4$ in the expansion. Thus $e_1 + e_2 + e_3 = 4$. The coefficient of $x^4$ is therefore the number of ways to solve this equation with nonnegative integers, which, from Section 6.5, is $C(3+4-1,4) = C(6,4) = 15$.

**23.** **a)** The restriction on $x_1$ gives us the factor $x^2+x^3+x^4+\cdots$. The restriction on $x_2$ gives us the factor $1+x+x^2+x^3$. The restriction on $x_3$ gives us the factor $x^2+x^3+x^4+x^5$. Thus the answer is the product of these: $(x^2+x^3+x^4+\cdots)(1+x+x^2+x^3)(x^2+x^3+x^4+x^5)$. We can use algebra and Table 1 to rewrite this in closed form as $x^4(1+x+x^2+x^3)^2/(1-x)$.
**b)** We want the coefficient of $x^6$ in this series, which is the same as the coefficient of $x^2$ in the series for

$$\frac{(1+x+x^2+x^3)^2}{1-x}=\frac{1+2x+3x^2+\text{higher order terms}}{1-x}.$$

Since the coefficient of $x^n$ in $1/(1-x)$ is $1$, our answer is $1+2+3=6$.

**25.** This problem reinforces the point that "and" corresponds to multiplication and "or" corresponds to addition.
**a)** The only issue is how many stamps of each denomination we choose. The exponent on $x$ will be the number of cents. So the generating function for choosing 3-cent stamps is $1+x^3+x^6+x^9+\cdots$, the generating function for 4-cent stamps is $1+x^4+x^8+x^{12}+\cdots$, and the generating function for 20-cent stamps is $1+x^{20}+x^{40}+x^{60}+\cdots$. In closed form this is $1/((1-x^3)(1-x^4)(1-x^{20}))$. The coefficient of $x^r$ gives the answer—the number of ways to choose stamps totaling $r$ cents of postage.
**b)** Again the exponent on $x$ will be the number of cents, but this time we paste one stamp at a time. For the first pasting, we can choose a 3-cent stamp, a 4-cent stamp, or a 20-cent stamp. Hence the generating function for the number of ways to paste one stamp is $x^3+x^4+x^{20}$. For the second pasting, we can make these same choices, so the generating function for the number of ways to paste two stamps is $(x^3+x^4+x^{20})^2$. In general, if we use $n$ stamps, the generating function is $(x^3+x^4+x^{20})^n$. Since a pasting consists of a pasting of zero or more stamps, the entire generating function will be

$$\sum_{n=0}^{\infty}(x^3+x^4+x^{20})^n=\frac{1}{1-x^3-x^4-x^{20}}.$$

**c)** We seek the coefficient of $x^{46}$ in the power series for our answer to part **(a)**, $1/((1-x^3)(1-x^4)(1-x^{20}))$. Other than working this out by brute force (enumerating the combinations), the best way to get the answer is probably asking *Maple* or another computer algebra package to multiply out these series. If we do so, the answer turns out to be $7$. (The choices are $2\cdot 20+2\cdot 3$, $20+5\cdot 4+2\cdot 3$, $20+2\cdot 4+6\cdot 3$, $10\cdot 4+2\cdot 3$, $7\cdot 4+6\cdot 3$, $4\cdot 4+10\cdot 3$, and $1\cdot 4+14\cdot 3$.)
**d)** We seek the coefficient of $x^{46}$ in the power series for our answer to part **(b)**, $1/(1-x^3-x^4-x^{20})$. The best way to get the answer is probably asking *Maple* or another computer algebra package to find this power series using calculus. If we do so, the answer turns out to be $3224$. Alternatively, for each of the seven combinations in our answer to part **(c)**, we can find the number of ordered arrangements, as in Section 6.5. Thus the answer is

$$\frac{4!}{2!2!}+\frac{8!}{1!5!2!}+\frac{9!}{1!2!6!}+\frac{12!}{10!2!}+\frac{13!}{7!6!}+\frac{14!}{4!10!}+\frac{15!}{1!14!}=6+168+252+66+1716+1001+15=3224.$$

**27.** We will write down the generating function in each case and then use a computer algebra package to find the desired coefficients. As a check, one could carefully enumerate these by hand. In making change, one usually considers order irrelevant.
**a)** The generating function for the dimes is $1+x^{10}+x^{20}+x^{30}+\cdots=1/(1-x^{10})$, and the generating function for the quarters is $1+x^{25}+x^{50}+x^{75}+\cdots=1/(1-x^{25})$, so the generating function for the whole problem is $1/((1-x^{10})(1-x^{25}))$. The coefficient of $x^k$ gives the number of ways to make change for $k$ cents, so we seek the coefficient of $x^{100}$. If we ask a computer algebra system to find this coefficient (it uses calculus to get the power series), we find that the answer is $3$. In fact, this is correct, since we can use four quarters, two quarters, or no quarters (and the number of dimes is uniquely determined by this choice).

**b)** This is identical to part **(a)** except for a factor for the nickels. Thus we seek the coefficient of $x^{100}$ in $1/((1-x^5)(1-x^{10})(1-x^{25}))$, which turns out to be $29$. (If we wanted to list these systematically, we could organize our work by the number of quarters, and within that by the number of dimes.)

**c)** This is identical to part **(a)** except for a factor for the pennies. Thus we seek the coefficient of $x^{100}$ in $1/((1-x)(1-x^{10})(1-x^{25}))$, which turns out to be 29 again. (In retrospect, this is obvious. The only difference between parts **(b)** and **(c)** is that five pennies are substituted for each nickel.)

**d)** This is identical to part **(a)** except for factors for the pennies and nickels. Thus we seek the coefficient of $x^{100}$ in $1/((1-x)(1-x^5)(1-x^{10})(1-x^{25}))$, which turns out to be $242$.

**29.** We will write down the generating function in each case and then use a computer algebra package to find the desired coefficients. In making change, one usually considers order irrelevant.

**a)** The generating function for the \$10 bills is $1+x^{10}+x^{20}+x^{30}+\cdots=1/(1-x^{10})$, the generating function for the \$20 bills is $1+x^{20}+x^{40}+x^{60}+\cdots=1/(1-x^{20})$, and the generating function for the \$50 bills is $1+x^{50}+x^{100}+x^{150}+\cdots=1/(1-x^{50})$, so the generating function for the whole problem is $1/((1-x^{10})(1-x^{20})(1-x^{50}))$. The coefficient of $x^k$ gives the number of ways to make change for $k$ dollars, so we seek the coefficient of $x^{100}$. If we ask a computer algebra system to find this coefficient (it uses calculus to get the power series), we find that the answer is $10$. In fact, this is correct, since there is one way in which we can use two \$50 bills, three ways in which we use one \$50 bill (using either two, one, or no \$20 bills), and six ways to use no \$50 bills (using zero through five \$20's).

**b)** This is identical to part **(a)** except for a factor for the \$5 bills. Thus we seek the coefficient of $x^{100}$ in $1/((1-x^5)(1-x^{10})(1-x^{20})(1-x^{50}))$, which turns out to be $49$.

**c)** In part **(b)** we saw that the generating function for this problem is $1/((1-x^5)(1-x^{10})(1-x^{20})(1-x^{50}))$. If at least one of each bill must be used, let us assume that this $\$50+\$20+\$10+\$5=\$85$ has already been dispersed. Then we seek the coefficient of $x^{15}$. The computer algebra package tells us that the answer is $2$, but it is trivial to see that there are only two ways to make \$15 with these bills.

**d)** This time the generating function is $(x^5+x^{10}+x^{15}+x^{20})(x^{10}+x^{20}+x^{30}+x^{40})(x^{20}+x^{40}+x^{60}+x^{80})$. When the computer multiplies this out, it tells us that the coefficient of $x^{100}$ is $4$, so that is the answer. (In retrospect, we see that the only solutions are $4\cdot\$20+1\cdot\$10+2\cdot\$5$, $3\cdot\$20+3\cdot\$10+2\cdot\$5$, $3\cdot\$20+2\cdot\$10+4\cdot\$5$, and $2\cdot\$20+4\cdot\$10+4\cdot\$5$.)

**31. a)** The terms involving $a_0$, $a_1$, and $a_2$ are missing; $G(x)-a_0-a_1x-a_2x^2=a_3x^3+a_4x^4+\cdots$. That is the generating function for precisely the sequence we are given. Thus the answer is $G(x)-a_0-a_1x-a_2x^2$.

**b)** Every other term is missing, and the old coefficient of $x^n$ is now the coefficient of $x^{2n}$. This suggests that maybe $x^2$ should be used in place of $x$. Indeed, this works; the answer is $G(x^2)=a_0+a_1x^2+a_2x^4+\cdots$.

**c)** If we want $a_0$ to be the coefficient of $x^4$ (and similarly for the other powers), we must throw in an extra factor. Thus the answer is $x^4G(x)$. Note that $x^4(a_0+a_1x+a_2x^2+\cdots)=a_0x^4+a_1x^5+a_2x^6+\cdots$.

**d)** Extra factors of 2 are applied to each term, with the power of 2 matching the subscript (which, of course, gives us the power of $x$). Thus the answer must be $G(2x)=a_0+a_1(2x)+a_2(2x)^2+a_3(2x)^3\cdots=a_0+2a_1x+4a_2x^2+8a_3x^3\cdots$.

**e)** Following the hint, we integrate $G(t)=\sum_{n=0}^{\infty}a_nt^n$ from 0 to $x$, to obtain $\int_0^x G(t)\,dt=\sum_{n=0}^{\infty}a_n\int_0^x t^n\,dt=\sum_{n=0}^{\infty}a_nx^{n+1}/(n+1)$. (If we had tried differentiating first, we'd see that that didn't work. It is a theorem of advanced calculus that it is legal to integrate inside the summation within the open interval of convergence.) This is the series

$$a_0x+\frac{a_1}{2}x^2+\frac{a_2}{3}x^3+\cdots,$$

precisely the sequence we are given (note that the constant term is 0). Thus $\int_0^x G(t)\,dt$ is the generating function for this sequence.

**f)** If we look at Theorem 1, it is not hard to see that the sequence shown here is precisely the coefficients of $G(x)\cdot(1+x+x^2+\cdots) = G(x)/(1-x)$.

**33.** This problem is like Example 16. First let $G(x)=\sum_{k=0}^{\infty} a_k x^k$. Then $xG(x)=\sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1}x^k$ (by changing the name of the variable from $k$ to $k+1$). Thus

$$G(x)-3xG(x)=\sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} 3a_{k-1}x^k = a_0 + \sum_{k=1}^{\infty}(a_k - 3a_{k-1})x^k = a_0 + \sum_{k=1}^{\infty} 2x^k$$
$$= 1 + \frac{2}{1-x} - 2 = \frac{1+x}{1-x},$$

because of the given recurrence relation, the initial condition, and the fact from Table 1 that $\sum_{k=0}^{\infty} 2x^k = 2/(1-x)$. Thus $G(x)(1-3x)=(1+x)/(1-x)$, so $G(x)=(1+x)/((1-3x)(1-x))$. At this point we must use partial fractions to break up the denominator. Setting

$$\frac{1+x}{(1-3x)(1-x)} = \frac{A}{1-3x}+\frac{B}{1-x},$$

multiplying through by the common denominator, and equating coefficients, we find that $A=2$ and $B=-1$. Thus

$$G(x)=\frac{2}{1-3x}+\frac{-1}{1-x}=\sum_{k=0}^{\infty}(2\cdot 3^k - 1)x^k$$

(the last equality came from using Table 1). Therefore $a_k = 2\cdot 3^k - 1$.

**35.** Let $G(x)=\sum_{k=0}^{\infty} a_k x^k$. Then $xG(x)=\sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1}x^k$ (by changing the name of the variable from $k$ to $k+1$), and similarly $x^2G(x)=\sum_{k=0}^{\infty} a_k x^{k+2} = \sum_{k=2}^{\infty} a_{k-2}x^k$. Thus

$$G(x)-5xG(x)+6x^2G(x)=\sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} 5a_{k-1}x^k + \sum_{k=2}^{\infty} 6a_{k-2}x^k = a_0 + a_1x - 5a_0x + \sum_{k=2}^{\infty} 0\cdot x^k = 6,$$

because of the given recurrence relation and the initial conditions. Thus $G(x)(1-5x+6x^2)=6$, so $G(x)=6/((1-3x)(1-2x))$. At this point we must use partial fractions to break up the denominator. Setting

$$\frac{6}{(1-3x)(1-2x)} = \frac{A}{1-3x}+\frac{B}{1-2x},$$

multiplying through by the common denominator, and equating coefficients, we find that $A=18$ and $B=-12$. Thus

$$G(x)=\frac{18}{1-3x}+\frac{-12}{1-2x}=\sum_{k=0}^{\infty}(18\cdot 3^k - 12\cdot 2^k)x^k$$

(the last equality came from using Table 1). Therefore $a_k = 18\cdot 3^k - 12\cdot 2^k$. Incidentally, it would be wise to check our answers, either with a computer algebra package (see the solution to Exercise 37 for the syntax in *Maple*) or by computing the next term of the sequence from both the recurrence and the formula (here $a_2 = 114$ both ways).

**37.** Let $G(x)=\sum_{k=0}^{\infty} a_k x^k$. Then $xG(x)=\sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1}x^k$ (by changing the name of the variable from $k$ to $k+1$), and similarly $x^2G(x)=\sum_{k=0}^{\infty} a_k x^{k+2} = \sum_{k=2}^{\infty} a_{k-2}x^k$. Thus

$$G(x)-4xG(x)+4x^2G(x)=\sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} 4a_{k-1}x^k + \sum_{k=2}^{\infty} 4a_{k-2}x^k = a_0 + a_1x - 4a_0x + \sum_{k=2}^{\infty} k^2\cdot x^k$$
$$= 2 - 3x + \frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{1}{1-x} - x$$
$$= 2 - 4x + \frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{1}{1-x},$$

because of the given recurrence relation, the initial conditions, Table 1, and a calculation of the generating function for $\{k^2\}$ (the last "$-x$" comes from the fact that the $k^2$ sum starts at 2). (To find the generating function for $\{k^2\}$, start with the fact that $1/(1-x)^3$ is the generating function for $\{C(k,2) = (k+2)(k+1)/2\}$, that $1/(1-x)^2$ is the generating function for $\{k+1\}$, and that $1/(1-x)$ is the generating function for $\{1\}$, and take an appropriate linear combination of these to get the generating function for $\{k^2\}$.) Thus

$$G(x)(1-4x+4x^2) = 2 - 4x + \frac{2}{(1-x)^3} - \frac{3}{(1-x)^2} + \frac{1}{1-x},$$

so

$$G(x) = \frac{2-4x}{(1-2x)^2} + \frac{2}{(1-2x)^2(1-x)^3} - \frac{3}{(1-2x)^2(1-x)^2} + \frac{1}{(1-2x)^2(1-x)}.$$

At this point we must use partial fractions to break up the denominators. Setting the previous expression equal to

$$\frac{A}{1-x} + \frac{B}{(1-x)^2} + \frac{C}{(1-x)^3} + \frac{D}{1-2x} + \frac{E}{(1-2x)^2},$$

multiplying through by the common denominator, and equating coefficients, we find (after a lot of algebra) that $A = 13$, $B = 5$, $C = 2$, $D = -24$, and $E = 6$. (Alternatively, one can ask *Maple* to produce the partial fraction decomposition, with the command

convert(*expression*, parfrac, x);

where the expression is $G(x)$.) Thus

$$\begin{aligned} G(x) &= \frac{13}{1-x} + \frac{5}{(1-x)^2} + \frac{2}{(1-x)^3} + \frac{-24}{1-2x} + \frac{6}{(1-2x)^2} \\ &= \sum_{k=0}^{\infty} \left(13 + 5(k+1) + 2(k+2)(k+1)/2 - 24 \cdot 2^k + 6(k+1)2^k\right) x^k \end{aligned}$$

(from Table 1). Therefore $a_k = k^2 + 8k + 20 + (6k-18)2^k$. Incidentally, it would be most wise to check our answers, either with a computer algebra package, or by computing the next term of the sequence from both the recurrence and the formula (here $a_2 = 16$ both ways). The command in *Maple* for solving this recurrence is this:

```
rsolve({a(k) = 4 * a(k - 1) - 4 * a(k - 2) + k^2, a(0) = 2, a(1) = 5}, a(k));
```

**39.** In principle this exercise is similar to the examples and previous exercises. In fact, the algebra is quite a bit messier. We want to solve the recurrence relation $f_k = f_{k-1} + f_{k-2}$, with initial conditions $f_0 = 0$ and $f_1 = 1$. Let $G$ be the generating function for $f_k$, so that $G(x) = \sum_{k=0}^{\infty} f_k x^k$. We look at $G(x) - xG(x) - x^2G(x)$ in order to take advantage of the recurrence relation:

$$\begin{aligned} G(x) - xG(x) - x^2G(x) &= \sum_{k=0}^{\infty} f_k x^k - \sum_{k=1}^{\infty} f_{k-1} x^k - \sum_{k=2}^{\infty} f_{k-2} x^k \\ &= f_0 + f_1 x - f_0 x + \sum_{k=2}^{\infty} (f_k - f_{k-1} - f_{k-2}) x^k \\ &= 0 + x - 0 + 0 = x \end{aligned}$$

Thus $G$ satisfies the equation

$$G(x) = \frac{x}{1-x-x^2}.$$

To write this more usefully, we need to use partial fractions. The roots of the denominator are $r_1 = (-1+\sqrt{5})/2$ and $r_2 = (-1-\sqrt{5})/2$. We want to find constants $A$ and $B$ such that

$$\frac{x}{1-x-x^2} = \frac{-x}{x^2+x-1} = \frac{A}{x-r_1} + \frac{B}{x-r_2}.$$

This means that $A$ and $B$ have to satisfy the simultaneous equations $A+B=-1$ and $r_2A+r_1B=0$ (multiply the last displayed equation through by the denominator and equate like powers of $x$). Solving, we obtain $A=(1-\sqrt{5})/(2\sqrt{5})$ and $B=(-1-\sqrt{5})/(2\sqrt{5})$. Now we have

$$\begin{aligned} G(x) &= \frac{A}{x-r_1}+\frac{B}{x-r_2} \\ &= \frac{-A}{r_1}\frac{1}{1-(x/r_1)}+\frac{-B}{r_2}\frac{1}{1-(x/r_2)} \\ &= \frac{-A}{r_1}\sum_{k=0}^{\infty}\left(\frac{1}{r_1}\right)^k x^k+\frac{-B}{r_2}\sum_{k=0}^{\infty}\left(\frac{1}{r_2}\right)^k x^k . \end{aligned}$$

Therefore

$$\begin{aligned} f_k &= -A\left(\frac{1}{r_1}\right)^{k+1} - B\left(\frac{1}{r_2}\right)^{k+1} \\ &= \frac{1}{\sqrt{5}}\left(\frac{2}{-1+\sqrt{5}}\right)^k-\frac{1}{\sqrt{5}}\left(\frac{2}{-1-\sqrt{5}}\right)^k . \end{aligned}$$

We can check our answer by computing the first few terms with a calculator, and indeed we find that $f_2=1$, $f_3=2$, $f_4=3$, $f_5=5$, and so on.

**41.** **a)** Let $G(x)=\sum_{n=0}^{\infty}C_nx^n$ be the generating function for the sequence of Catalan numbers. Then by Theorem 1 a change of variable in the middle, and the recurrence relation for the Catalan numbers,

$$G(x)^2=\sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}C_kC_{n-k}\right)x^n=\sum_{n=1}^{\infty}\left(\sum_{k=0}^{n-1}C_kC_{n-1-k}\right)x^{n-1}=\sum_{n=1}^{\infty}C_nx^{n-1} .$$

So $xG(x)^2=\sum_{n=1}^{\infty}C_nx^n$. Therefore,

$$xG(x)^2-G(x)+1=\left(\sum_{n=1}^{\infty}C_nx^n\right)-\left(\sum_{n=0}^{\infty}C_nx^n\right)+1=-C_0+1=0 .$$

We now apply the quadratic formula to solve for $G(x)$:

$$G(x)=\frac{1\pm\sqrt{1-4x}}{2x}$$

We must decide whether to use the plus sign or the minus sign. If we use the plus sign, then trying to calculate $G(0)$, which, after all, is supposed to be $C_0$, gives us the undefined value $2/0$. Therefore we must use the minus sign, and indeed one can find using calculus that the indeterminate form $0/0$ equals 1 here, since $\lim_{x\to 0}G(x)=1$.

**b)** By Exercise 40 we know that

$$(1-4x)^{-1/2}=\sum_{n=0}^{\infty}\binom{2n}{n}x^n ,$$

so by integrating term by term (which is valid) we have

$$\int_0^x(1-4t)^{-1/2}\,dt=\frac{1-\sqrt{1-4x}}{2}=x\cdot\frac{1-\sqrt{1-4x}}{2x}=\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^{n+1}=x\sum_{n=0}^{\infty}\frac{1}{n+1}\binom{2n}{n}x^n .$$

Since $G(x)=(1-\sqrt{1-4x})/(2x)$, equating coefficients of the power series tells us that $C_n=\frac{1}{n+1}\binom{2n}{n}$.

**c)** It is natural to try a proof by strong induction, because the sequence is defined recursively. We need to check the base cases: $C_1=1\ge 2^{1-1}$, $C_2=2\ge 2^{2-1}$, $C_3=5\ge 2^{3-1}$, $C_4=14\ge 2^{4-1}$, $C_5=42\ge 2^{5-1}$. For the inductive step assume that $C_j\ge 2^{j-1}$ for $1\le j<n$, where $n\ge 6$. Then

$$C_n=\sum_{k=0}^{n-1}C_kC_{n-k-1}\ge\sum_{k=1}^{n-2}C_kC_{n-k-1}\ge(n-2)2^{k-1}2^{n-k-2}=\frac{n-2}{4}\cdot 2^{n-1}\ge 2^{n-1} .$$

**43.** Following the hint, we note that $(1+x)^{m+n} = (1+x)^m(1+x)^n$. Then applying the binomial theorem, we have

$$\sum_{k=0}^{m+n} C(m+n,r)x^r = \sum_{r=0}^{m} C(m,r)x^r \cdot \sum_{r=0}^{m} C(n,r)x^r = \sum_{r=0}^{m+n}\Big(\sum_{k=0}^{r} C(m,r-k)C(n,k)\Big)x^r$$

by Theorem 1. Comparing coefficients gives us the desired identity.

**45.** We will make heavy use of the identity $e^x = \sum_{n=0}^{\infty} \frac{1}{n!}x^n$.

**a)** $\sum_{n=0}^{\infty} \frac{2}{n!}x^n = 2\sum_{n=0}^{\infty}\frac{1}{n!}x^n = 2e^x$

**b)** $\sum_{n=0}^{\infty} \frac{(-1)^n}{n!}x^n = \sum_{n=0}^{\infty}\frac{1}{n!}(-x)^n = e^{-x}$

**c)** $\sum_{n=0}^{\infty} \frac{3^n}{n!}x^n = \sum_{n=0}^{\infty}\frac{1}{n!}(3x)^n = e^{3x}$

**d)** This generating function can be obtained either with calculus or without. To do it without calculus, write

$$\sum_{n=0}^{\infty}\frac{n+1}{n!}x^n = \sum_{n=0}^{\infty}\frac{n}{n!}x^n + \sum_{n=0}^{\infty}\frac{1}{n!}x^n = \sum_{n=1}^{\infty}\frac{1}{(n-1)!}x^n + e^x = x\sum_{n=1}^{\infty}\frac{1}{(n-1)!}x^{n-1} + e^x$$
$$= x\sum_{n=0}^{\infty}\frac{1}{n!}x^n + e^x = xe^x + e^x\,.$$

To do it with calculus, differentiate both sides of $xe^x = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n!}$ to obtain $xe^x + e^x = \sum_{n=0}^{\infty}(n+1)\frac{x^n}{n!}$.

**e)** This generating function can be obtained either with calculus or without. To do it without calculus, write

$$\sum_{n=0}^{\infty}\frac{1}{n+1}\cdot\frac{x^n}{n!} = \sum_{n=0}^{\infty}\frac{x^n}{(n+1)!} = \frac{1}{x}\sum_{n=0}^{\infty}\frac{x^{n+1}}{(n+1)!} = \frac{1}{x}\sum_{n=1}^{\infty}\frac{x^n}{n!} = \frac{1}{x}(e^x-1)\,.$$

To do it with calculus, integrate $e^t = \sum_{n=0}^{\infty}\frac{t^n}{n!}$ from 0 to $x$ to obtain

$$e^x - 1 = \sum_{n=0}^{\infty}\frac{x^{n+1}}{n+1}\cdot\frac{1}{n!} = x\sum_{n=0}^{\infty}\frac{1}{(n+1)}\frac{x^n}{n!}\,.$$

Therefore $\sum_{n=0}^{\infty}\frac{1}{(n+1)}\frac{x^n}{n!} = (e^x-1)/x$.

**47.** In many of these cases, it's a matter of plugging the exponent of $e$ into the generating function for $e^x$. We let $a_n$ denote the $n^{\text{th}}$ term of the sequence whose generating function is given.

**a)** The generating function is $e^{-x} = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!} = \sum_{n=0}^{\infty}(-1)^n\frac{x^n}{n!}$, so the sequence is $a_n = (-1)^n$.

**b)** The generating function is $3e^{2x} = 3\sum_{n=0}^{\infty}\frac{(2x)^n}{n!} = \sum_{n=0}^{\infty}(3\cdot 2^n)\frac{x^n}{n!}$, so the sequence is $a_n = 3\cdot 2^n$.

**c)** The generating function is $e^{3x} - 3e^{2x} = \sum_{n=0}^{\infty}\frac{(3x)^n}{n!} - 3\sum_{n=0}^{\infty}\frac{(2x)^n}{n!} = \sum_{n=0}^{\infty}(3^n - 3\cdot 2^n)\frac{x^n}{n!}$, so the sequence is $a_n = 3^n - 3\cdot 2^n$.

**d)** The sequence whose exponential generating function is $e^{-2x}$ is clearly $\{(-2)^n\}$, as in the previous parts of this exercise. Since

$$1 - x = \frac{1}{0!}x^0 + \frac{-1}{1!}x^1 + \sum_{n=2}^{\infty}\frac{0}{n!}x^n\,,$$

we know that $a_n = (-2)^n$ for $n \geq 2$, with $a_1 = (-2)^1 - 1 = -3$ and $a_0 = (-2)^0 + 1 = 2$.

**e)** We know that

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = \sum_{n=0}^{\infty} \frac{n!}{n!} x^n,$$

so the sequence for which $1/(1-x)$ is the exponential generating function is $\{n!\}$. Combining this with the rest of the function (similar to previous parts of this exercise), we have $a_n = (-2)^n + n!$.

**f)** This is similar to part **(e)**. The three functions being added here are the exponential generating functions for $\{(-3)^n\}$, $(-1, -1, 0, 0, 0, \ldots)$, and $\{n! \cdot 2^n\}$. Therefore $a_n = (-3)^n + n! \cdot 2^n$ for $n \geq 2$, with $a_0 = (-3)^0 - 1 + 0! \cdot 2^0 = 1$ and $a_1 = (-3)^1 - 1 + 1! \cdot 2^1 = -2$.

**g)** First we note that

$$\begin{aligned} e^{x^2} &= \sum_{n=0}^{\infty} \frac{(x^2)^n}{n!} = 1 + \frac{x^2}{1!} + \frac{x^4}{2!} + \frac{x^6}{3!} + \cdots \\ &= \frac{x^0}{0!} \cdot \frac{0!}{0!} + \frac{x^2}{2!} \cdot \frac{2!}{1!} + \frac{x^4}{4!} \cdot \frac{4!}{2!} + \frac{x^6}{6!} \cdot \frac{6!}{3!} + \cdots . \end{aligned}$$

Therefore we see that $a_n = 0$ if $n$ is odd, and $a_n = n!/(n/2)!$ if $n$ is even.

**49. a)** Let $a_n$ be the number of codewords of length $n$. There are $8^n$ strings of length $n$ in all, and only those that contain an even number of 7's are code words. The initial condition is clearly $a_0 = 1$ (the empty string has an even number of 7's); if that seems too obscure, one can take $a_1 = 7$, since one of the eight strings of length 1 (namely the string 7) is disallowed. To write down a recurrence, we observe that a valid string of length $n$ consists either of a valid string of length $n-1$ followed by a digit other than 7 (so that there will still be an even number of 7's), and there are $7a_{n-1}$ of these; or of an invalid string of length $n-1$ followed by a 7 (so that there will still be an even number of 7's), and there are $8^{n-1} - a_{n-1}$ of these. Putting these together, we have the recurrence relation $a_n = 7a_{n-1} + 8^{n-1} - a_{n-1} = 6a_{n-1} + 8^{n-1}$. For example, $a_2 = 6 \cdot 7 + 8 = 50$.

**b)** Using the techniques of Section 8.2, we note that the general solution to the associated homogeneous recurrence relation is $a_n^{(h)} = \alpha 6^n$, and then we seek a particular solution of the form $a_n = c \cdot 8^n$. Plugging this into the recurrence relation, we have $c \cdot 8^n = 6c \cdot 8^{n-1} + 8^{n-1}$, which is easily solved to yield $c = \frac{1}{2}$ (first divide through by $8^{n-1}$). This gives $a_n^{(p)} = \frac{1}{2} \cdot 8^n$, so the general solution is $a_n = \alpha 6^n + \frac{1}{2} \cdot 8^n$. Next we plug in the initial condition and easily find that $\alpha = \frac{1}{2}$. Therefore the solution is $a_n = (6^n + 8^n)/2$. We can check that this gives the correct answer when $n = 2$.

**c)** We proceed as in Example 17. Let $G(x) = \sum_{k=0}^{\infty} a_k x^k$. Then $xG(x) = \sum_{k=0}^{\infty} a_k x^{k+1} = \sum_{k=1}^{\infty} a_{k-1} x^k$ (by a change of variable). Thus

$$\begin{aligned} G(x) - 6xG(x) &= \sum_{k=0}^{\infty} a_k x^k - \sum_{k=1}^{\infty} 6a_{k-1} x^k = a_0 + \sum_{k=1}^{\infty} (a_k - 6a_{k-1}) x^k = 1 + \sum_{k=1}^{\infty} 8^{k-1} x^k \\ &= 1 + x \sum_{k=1}^{\infty} 8^{k-1} x^{k-1} = 1 + x \sum_{k=0}^{\infty} 8^k x^k = 1 + x \cdot \frac{1}{1-8x} = \frac{1-7x}{1-8x} . \end{aligned}$$

Thus $G(x)(1-6x) = (1-7x)/(1-8x)$, so $G(x) = (1-7x)/((1-6x)(1-8x))$. At this point we need to use partial fractions to break this up (see, for example, Exercise 35):

$$G(x) = \frac{1-7x}{(1-6x)(1-8x)} = \frac{1/2}{(1-6x)} + \frac{1/2}{(1-8x)}$$

Therefore, with the help of Table 1, $a_n = (6^n + 8^n)/2$, as we found in part **(b)**.

**51.** To form a partition of $n$, we must choose some 1's, some 2's, some 3's, and so on. The generating function for choosing 1's is

$$1 + x + x^2 + x^3 + \cdots = \frac{1}{1-x}$$

(the exponent gives the number so obtained). Similarly, the generating function for choosing 2's is

$$1 + x^2 + x^4 + x^6 + \cdots = \frac{1}{1-x^2}$$

(again the exponent gives the number so obtained). The other choices have analogous generating functions. Therefore the generating function for the entire problem, so that the coefficient of $x^n$ will give $p(n)$, the number of partitions of $n$, is the infinite product

$$\frac{1}{1-x} \cdot \frac{1}{1-x^2} \cdot \frac{1}{1-x^3} \cdots .$$

**53.** This is similar to Exercise 51. Since all the parts have to be of different sizes, we can choose only no 1's or one 1; thus the generating function for choosing 1's is $1 + x$ (the exponent gives the number so obtained). Similarly the generating function for choosing 2's is $1 + x^2$, and analogously for higher choices. Therefore the generating function for the entire problem, so that the coefficient of $x^n$ will give $p_d(n)$, the number of partitions of $n$ into distinct-sized parts, is the infinite product

$$(1+x)(1+x^2)(1+x^3)\cdots .$$

**55.** It suffices to show that the generating functions obtained in Exercises 53 and 52 are equal, that is, that

$$(1+x)(1+x^2)(1+x^3)\cdots = \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots .$$

Assuming that the symbol-pushing we are about to do with infinite products is valid, we simply rewrite the left-hand side using the trivial algebraic identity $(1-x^{2r})/(1-x^r) = 1 + x^r$ and cancel common factors:

$$\begin{aligned}(1+x)(1+x^2)(1+x^3)\cdots &= \frac{1-x^2}{1-x} \cdot \frac{1-x^4}{1-x^2} \cdot \frac{1-x^6}{1-x^3} \cdot \frac{1-x^8}{1-x^4} \cdots \\ &= \frac{1}{1-x} \cdot \frac{1}{1-x^3} \cdot \frac{1}{1-x^5} \cdots\end{aligned}$$

**57.** These follow fairly easily from the definitions.

**a)** $G_X(1) = \sum_{k=0}^{\infty} p(X=k) \cdot 1^k = \sum_{k=0}^{\infty} p(X=k) = 1$, since $X$ has to take on some nonnegative integer value. (That the sum of the probabilities is 1 is one of the axioms of a sample space; see Section 7.2.)

**b)** $G'_X(1) = \frac{d}{dx} \sum_{k=0}^{\infty} p(X=k) \cdot x^k \big|_{x=1} = \sum_{k=0}^{\infty} p(X=k) \cdot k \cdot x^{k-1} \big|_{x=1} = \sum_{k=0}^{\infty} p(X=k) \cdot k = E(X)$, by the definition of expected value from Section 7.4.

**c)** $G''_X(1) = \frac{d^2}{dx^2} \sum_{k=0}^{\infty} p(X=k) \cdot x^k \big|_{x=1} = \sum_{k=0}^{\infty} p(X=k) \cdot k(k-1) \cdot x^{k-2} \big|_{x=1} = \sum_{k=0}^{\infty} p(X=k) \cdot (k^2 - k) = V(X) + E(X)^2 - E(X)$, since, by Theorem 6 in Section 7.4, $V(X) = E(X^2) - E(X)^2$. Combining this with the result of part **(b)** gives the desired equality.

**59.** **a)** In order to have the $m^{\text{th}}$ success on the $(m+n)^{\text{th}}$ trial, where $n \geq 0$, we must have $m-1$ successes and $n$ failures in any order among the first $m+n-1$ trials, followed by a success. The probability of each such ordered arrangement is clearly $q^n p^m$, where $p$ is the probability of success and $q = 1-p$ is the probability of failure; and there are $C(n+m-1, n)$ such orders. Therefore $p(X=n) = C(n+m-1, n)q^n p^m$. (This was Exercise 32 in the Supplementary Exercises for Chapter 7.) Therefore the probability generating function is

$$G(x) = \sum_{n=0}^{\infty} C(n+m-1, n) q^n p^m x^n = p^m \sum_{n=0}^{\infty} C(n+m-1, n)(qx)^n = p^m \frac{1}{(1-qx)^m}$$

by Table 1.

**b)** By Exercise 57, $E(X)$ is the derivative of $G(x)$ at $x = 1$. Here we have

$$G'(x) = \frac{p^m mq}{(1-qx)^{m+1}}, \quad \text{so} \quad G'(1) = \frac{p^m mq}{(1-q)^{m+1}} = \frac{p^m mq}{p^{m+1}} = \frac{mq}{p}.$$

From the same exercise, we know that the variance is $G''(1) + G'(1) - G'(1)^2$; so we compute:

$$G''(x) = \frac{p^m m(m+1)q^2}{(1-qx)^{m+2}}, \quad \text{so} \quad G''(1) = \frac{p^m m(m+1)q^2}{(1-q)^{m+2}} = \frac{m(m+1)q^2}{p^2},$$

and therefore

$$V(X) = G''(1) + G'(1) - G'(1)^2 = \frac{m(m+1)q^2}{p^2} + \frac{mq}{p} - \left(\frac{mq}{p}\right)^2 = \frac{mq}{p^2}.$$

## SECTION 8.5 Inclusion–Exclusion

*Inclusion–exclusion is not a nice compact formula in practice, but it is often the best that is available. In Exercise 19, for example, the answer contains over 30 terms. The applications in this section are somewhat contrived, but much more interesting applications are presented in Section 8.6. The inclusion–exclusion principle in some sense gives a methodical way to apply common sense. Presumably anyone could solve a problem such as Exercise 9 by trial and error or other ad hoc techniques, given enough time; the inclusion–exclusion principle makes the solution straightforward. Be careful when using the inclusion–exclusion principle to get the signs right—some terms need to be subtracted and others need to be added. In general the sign changes when the size of the expression changes.*

**1.** In all cases we use the fact that $|A_1 \cup A_2| = |A_1| + |A_2| - |A_1 \cap A_2| = 12 + 18 - |A_1 \cap A_2| = 30 - |A_1 \cap A_2|$.

**a)** Here $|A_1 \cap A_2| = 0$, so the answer is $30 - 0 = 30$.

**b)** This time we are told that $|A_1 \cap A_2| = 1$, so the answer is $30 - 1 = 29$.

**c)** This time we are told that $|A_1 \cap A_2| = 6$, so the answer is $30 - 6 = 24$.

**d)** If $A_1 \subseteq A_2$, then $A_1 \cap A_2 = A_1$, so $|A_1 \cap A_2| = |A_1| = 12$. Therefore the answer is $30 - 12 = 18$.

**3.** We may as well treat percentages as if they were cardinalities—as if the population were exactly 100. Let $V$ be the set of households with television sets, and let $P$ be the set of households with phones. Then we are given $|V| = 96$, $|P| = 98$, and $|V \cap P| = 95$. Therefore $|V \cup P| = 96 + 98 - 95 = 99$, so only 1% of the households have neither telephones nor televisions.

**5.** For all parts we need to use the formula $|A_1 \cup A_2 \cup A_3| = |A_1| + |A_2| + |A_3| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_2 \cap A_3| + |A_1 \cap A_2 \cap A_3|$.

**a)** If the sets are pairwise disjoint, then the cardinality of the union is the sum of the cardinalities, namely 300, since all but the first three terms on the right-hand side of the formula are equal to 0.

**b)** Using the formula, we have $100 + 100 + 100 - 50 - 50 - 50 + 0 = 150$.

**c)** Using the formula, we have $100 + 100 + 100 - 50 - 50 - 50 + 25 = 175$.

**d)** In this case the answer is obviously 100. By the formula, the cardinality of each set on the right-hand side is 100, so we can arrive at this answer through the computation $100+100+100-100-100-100+100 = 100$.

**7.** We need to use the formula $|J \cup L \cup C| = |J| + |L| + |C| - |J \cap L| - |J \cap C| - |L \cap C| + |J \cap L \cap C|$, where, for example, $J$ is the set of students who have taken a course in Java. Thus we have $|J \cup L \cup C| = 1876 + 999 + 345 - 876 - 290 - 231 + 189 = 2012$. Therefore, since there are 2504 students altogether, we know that $2504 - 2012 = 492$ have taken none of these courses.

**9.** We need to use the inclusion–exclusion formula for four sets, $C$ (the students taking calculus), $D$ (the students taking discrete mathematics), $S$ (those taking data structures), and $L$ (those taking programming languages). The formula says $|C \cup D \cup S \cup L| = |C| + |D| + |S| + |L| - |C \cap D| - |C \cap S| - |C \cap L| - |D \cap S| - |D \cap L| - |S \cap L| + |C \cap D \cap S| + |C \cap D \cap L| + |C \cap S \cap L| + |D \cap S \cap L| - |C \cap D \cap S \cap L|$. Plugging the given information into this formula gives us a total of $507 + 292 + 312 + 344 - 0 - 14 - 213 - 211 - 43 - 0 + 0 + 0 + 0 + 0 - 0 = 974$.

**11.** There are clearly 50 odd positive integers not exceeding 100 (half of these 100 numbers are odd), and there are 10 squares (from $1^2$ to $10^2$). Furthermore, half of these squares are odd. Thus we compute the cardinality of the set in question to be $50 + 10 - 5 = 55$.

**13.** Let us count the strings that have 6 or more consecutive 0's. There are 4 strings that have 0's in the first six places, since there are $2 \cdot 2 = 4$ ways to specify the last two bits. Similarly, there are 4 strings that have 0's in bits 2 through 7, and there are 4 strings that have 0's in bits 3 through 8. We have overcounted, though. There are 2 strings that have 0's in bits 1 through 7 (the intersection of the first two sets mentioned above); 2 strings that have 0's in bits 2 through 8 (the intersection of the last two sets mentioned above); and 1 string that has 0's in all bits (the intersection of the first and last sets mentioned above). Moreover, there is 1 string with 0's in bits 1 through 8, the intersection of all three sets mentioned above. Putting this all together, we know that the number of strings with 6 consecutive 0's is $4 + 4 + 4 - 2 - 2 - 1 + 1 = 8$. Since there are $2^8 = 256$ strings in all, there must be $256 - 8 = 248$ that do not contain 6 consecutive 0's.

**15.** We need to use inclusion–exclusion with three sets. There are $7!$ permutations that begin 987, since there are 7 digits free to be permuted among the last 7 spaces (we are assuming that it is meant that the permutations are to start with 987 *in that order*, not with 897, for instance). Similarly, there are $8!$ permutations that have 45 in the fifth and sixth positions, and there are $7!$ that end with 123. (We assume that the intent is that these digits are to appear in the order given.) There are $5!$ permutations that begin with 987 and have 45 in the fifth and sixth positions; $4!$ that begin with 987 and end with 123; and $5!$ that have 45 in the fifth and sixth positions and end with 123. Finally, there are $2!$ permutations that begin with 987, have 45 in the fifth and sixth positions, and end with 123 (since only the 0 and the 6 are left to place). Therefore the total number of permutations meeting any of these conditions is $7! + 8! + 7! - 5! - 4! - 5! + 2! = 50{,}138$.

**17.** By inclusion–exclusion, the answer is $50+60+70+80-6\cdot 5+4\cdot 1-0 = 234$. Note that there were $C(4,2) = 6$ pairs to worry about (each with 5 elements in common) and $C(4,1) = 4$ triples to worry about (each with 1 element in common).

**19.** $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5| = |A_1| + |A_2| + |A_3| + |A_4| + |A_5| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_1 \cap A_5| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_2 \cap A_5| - |A_3 \cap A_4| - |A_3 \cap A_5| - |A_4 \cap A_5| + |A_1 \cap A_2 \cap A_3| + |A_1 \cap A_2 \cap A_4| + |A_1 \cap A_2 \cap A_5| + |A_1 \cap A_3 \cap A_4| + |A_1 \cap A_3 \cap A_5| + |A_1 \cap A_4 \cap A_5| + |A_2 \cap A_3 \cap A_4| + |A_2 \cap A_3 \cap A_5| + |A_2 \cap A_4 \cap A_5| + |A_3 \cap A_4 \cap A_5| - |A_1 \cap A_2 \cap A_3 \cap A_4| - |A_1 \cap A_2 \cap A_3 \cap A_5| - |A_1 \cap A_2 \cap A_4 \cap A_5| - |A_1 \cap A_3 \cap A_4 \cap A_5| - |A_2 \cap A_3 \cap A_4 \cap A_5| + |A_1 \cap A_2 \cap A_3 \cap A_4 \cap A_5|$

**21.** Since no three of the sets have a common intersection, we need only carry our expression out as far as pairs. Thus we have $|A_1 \cup A_2 \cup A_3 \cup A_4 \cup A_5 \cup A_6| = |A_1| + |A_2| + |A_3| + |A_4| + |A_5| + |A_6| - |A_1 \cap A_2| - |A_1 \cap A_3| - |A_1 \cap A_4| - |A_1 \cap A_5| - |A_1 \cap A_6| - |A_2 \cap A_3| - |A_2 \cap A_4| - |A_2 \cap A_5| - |A_2 \cap A_6| - |A_3 \cap A_4| - |A_3 \cap A_5| - |A_3 \cap A_6| - |A_4 \cap A_5| - |A_4 \cap A_6| - |A_5 \cap A_6|$.

**23.** Since the probability of an event (i.e., a set) $E$ is proportional to the number of elements in the set $E$, this problem is just asking about cardinalities, and so inclusion–exclusion gives us the answer. Thus $p(E_1 \cup E_2 \cup E_3) = p(E_1) + p(E_2) + p(E_3) - p(E_1 \cap E_2) - p(E_1 \cap E_3) - p(E_2 \cap E_3) + p(E_1 \cap E_2 \cap E_3)$.

**25.** We can do this problem either by working directly with probabilities or by counting ways to satisfy the condition. We choose to do the former. First we need to determine the probability that all the numbers are odd. There are $C(100,4)$ ways to choose the numbers, and there are $C(50,4)$ ways to choose them all to be odd (since there are 50 odd numbers in the given interval). Therefore the probability that they are all odd is $C(50,4)/C(100,4)$. Similarly, since there are 33 multiples of 3 in the given interval, the probability of having all four numbers divisible by 3 is $C(33,4)/C(100,4)$. Finally, the probability that all four are divisible by 5 is $C(20,4)/C(100,4)$.

Next we need to know the probabilities that two of these events occur simultaneously. A number is both odd and divisible by 3 if and only if it is divisible by 3 but not by 6; therefore, since there are $\lfloor 100/6 \rfloor = 16$ multiples of 6 in the given interval, there are $33 - 16 = 17$ numbers that are both odd and divisible by 3. Thus the probability is $C(17,4)/C(100,4)$. Similarly there are 10 odd numbers divisible by 5, so the probability that all four numbers meet those conditions is $C(10,4)/C(100,4)$. Finally, the probability that all four numbers are divisible by both 3 and 5 is $C(6,4)/C(100,4)$, since there are only $\lfloor 100/15 \rfloor = 6$ such numbers.

Finally, the only numbers satisfying all three conditions are the odd multiplies of 15, namely 15, 45, and 75. Since there are only 3 such numbers, it is impossible that all chosen four numbers are divisible by 2, 3, and 5; in other words, the probability of that event is 0. We are now ready to apply the result of Exercise 23 (i.e., inclusion–exclusion viewed in terms of probabilities). We get

$$\begin{aligned}
&\frac{C(50,4)}{C(100,4)}+\frac{C(33,4)}{C(100,4)}+\frac{C(20,4)}{C(100,4)}-\frac{C(17,4)}{C(100,4)}-\frac{C(10,4)}{C(100,4)}-\frac{C(6,4)}{C(100,4)}+0\\
&\quad=\frac{230300+40920+4845-2380-210-15}{3921225}\\
&\quad=\frac{273460}{3921225}=\frac{4972}{71295}\approx 0.0697\,.
\end{aligned}$$

**27.** We are asked to write down inclusion–exclusion for five sets, just as in Exercise 19, except that intersections of more than three sets can be omitted. Furthermore, we are to use event notation, rather than set notation. Thus we have $p(E_1 \cup E_2 \cup E_3 \cup E_4 \cup E_5) = p(E_1) + p(E_2) + p(E_3) + p(E_4) + p(E_5) - p(E_1 \cap E_2) - p(E_1 \cap E_3) - p(E_1 \cap E_4) - p(E_1 \cap E_5) - p(E_2 \cap E_3) - p(E_2 \cap E_4) - p(E_2 \cap E_5) - p(E_3 \cap E_4) - p(E_3 \cap E_5) - p(E_4 \cap E_5) + p(E_1 \cap E_2 \cap E_3) + p(E_1 \cap E_2 \cap E_4) + p(E_1 \cap E_2 \cap E_5) + p(E_1 \cap E_3 \cap E_4) + p(E_1 \cap E_3 \cap E_5) + p(E_1 \cap E_4 \cap E_5) + p(E_2 \cap E_3 \cap E_4) + p(E_2 \cap E_3 \cap E_5) + p(E_2 \cap E_4 \cap E_5) + p(E_3 \cap E_4 \cap E_5)$.

**29.** We are simply asked to rephrase Theorem 1 in terms of probabilities of events. Thus we have

$$p(E_1 \cup E_2 \cup \cdots \cup E_n) = \sum_{1\le i\le n} p(E_i) - \sum_{1\le i<j\le n} p(E_i \cap E_j) + \sum_{1\le i<j<k\le n} p(E_i \cap E_j \cap E_k)$$
$$\cdots + (-1)^{n+1} p(E_1 \cap E_2 \cap \cdots \cap E_n)\,.$$

## SECTION 8.6 Applications of Inclusion–Exclusion

*Some of these applications are quite subtle and not easy to understand on first encounter. They do point out the power of the inclusion–exclusion principle. Many of the exercises are closely tied to the examples, so additional study of the examples should be helpful in doing the exercises. It is often helpful, in organizing your work, to write down (in complete English sentences) exactly what the properties of interest are, calling them the $P_i$'s. To find the number of elements lacking all the properties (as you need to do in Exercise 2, for example), use the formula above Example 1.*

**1.** We want to find the number of apples that have neither of the properties of having worms or of having bruises. By inclusion–exclusion, we know that this is equal to the number of apples, minus the numbers with each of the properties, plus the number with both properties. In this case, this is $100 - 20 - 15 + 10 = 75$.

**3.** We need first to find the number of solutions with no restrictions. By the results of Section 6.5, there are $C(3+13-1, 13) = C(15,13) = C(15,2) = 105$. Next we need to find the number of solutions in which each restriction is violated. There are three variables that can fail to be less than 6, and the situation is symmetric, so the total number of solutions in which each restriction is violated is 3 times the number of solutions in which $x_1 \geq 6$. By the trick we used in Section 6.5, this is the same as the number of nonnegative integer solutions to $x_1' + x_2 + x_3 = 7$, where $x_1 = x_1' + 6$. This of course is $C(3+7-1, 7) = C(9,7) = C(9,2) = 36$. Therefore there are $3 \cdot 36 = 108$ solutions in which at least one of the restrictions is violated (with some of these counted more than once).

Next we need to find the number of solutions with at least two of the restrictions violated. There are $C(3,2) = 3$ ways to choose the pair to be violated, so the number we are seeking is 3 times the number of solutions in which $x_1 \geq 6$ and $x_2 \geq 6$. Again by the trick we used in Section 6.5, this is the same as the number of nonnegative integer solutions to $x_1' + x_2' + x_3 = 1$, where $x_1 = x_1' + 6$ and $x_2 = x_2' + 6$. This of course is $C(3+1-1, 1) = C(3,1) = 3$. Therefore there are $3 \cdot 3 = 9$ solutions in which two of the restrictions are violated. Finally, we note that there are no solutions in which all three of the solutions are violated, since if each of the variables is at least 6, then their sum is at least 18, and hence cannot equal 13.

Thus by inclusion–exclusion, we see that there are $105 - 108 + 9 = 6$ solutions to the original problem. (We can check this on an ad hoc basis. The only way the sum of three numbers, not as big as 6, can be 13, is to have either two 5's and one 3, or else one 5 and two 4's. There are three variables that can be the "odd man out" in each case, for a total of 6 solutions.)

**5.** We follow the procedure described in the text. There are 198 positive integers less than 200 and greater than 1. The ones that are not prime are divisible by at least one of the primes in the set $\{2, 3, 5, 7, 11, 13\}$. The number of integers in the given range divisible by the prime $p$ is given by $\lfloor 199/p \rfloor$. Therefore we apply inclusion–exclusion and obtain the following number of integers from 2 to 199 that are not divisible by at least one of the primes in our set. (We have only listed those terms that contribute to the result, deleting all those that equal 0.)

$$\begin{aligned}
198 &- \left\lfloor\frac{199}{2}\right\rfloor - \left\lfloor\frac{199}{3}\right\rfloor - \left\lfloor\frac{199}{5}\right\rfloor - \left\lfloor\frac{199}{7}\right\rfloor - \left\lfloor\frac{199}{11}\right\rfloor - \left\lfloor\frac{199}{13}\right\rfloor + \left\lfloor\frac{199}{2\cdot 3}\right\rfloor \\
&+ \left\lfloor\frac{199}{2\cdot 5}\right\rfloor + \left\lfloor\frac{199}{2\cdot 7}\right\rfloor + \left\lfloor\frac{199}{2\cdot 11}\right\rfloor + \left\lfloor\frac{199}{2\cdot 13}\right\rfloor + \left\lfloor\frac{199}{3\cdot 5}\right\rfloor + \left\lfloor\frac{199}{3\cdot 7}\right\rfloor + \left\lfloor\frac{199}{3\cdot 11}\right\rfloor \\
&+ \left\lfloor\frac{199}{3\cdot 13}\right\rfloor + \left\lfloor\frac{199}{5\cdot 7}\right\rfloor + \left\lfloor\frac{199}{5\cdot 11}\right\rfloor + \left\lfloor\frac{199}{5\cdot 13}\right\rfloor + \left\lfloor\frac{199}{7\cdot 11}\right\rfloor + \left\lfloor\frac{199}{7\cdot 13}\right\rfloor + \left\lfloor\frac{199}{11\cdot 13}\right\rfloor \\
&- \left\lfloor\frac{199}{2\cdot 3\cdot 5}\right\rfloor - \left\lfloor\frac{199}{2\cdot 3\cdot 7}\right\rfloor - \left\lfloor\frac{199}{2\cdot 3\cdot 11}\right\rfloor - \left\lfloor\frac{199}{2\cdot 3\cdot 13}\right\rfloor - \left\lfloor\frac{199}{2\cdot 5\cdot 7}\right\rfloor - \left\lfloor\frac{199}{2\cdot 5\cdot 11}\right\rfloor \\
&- \left\lfloor\frac{199}{2\cdot 5\cdot 13}\right\rfloor - \left\lfloor\frac{199}{2\cdot 7\cdot 11}\right\rfloor - \left\lfloor\frac{199}{2\cdot 7\cdot 13}\right\rfloor - \left\lfloor\frac{199}{3\cdot 5\cdot 7}\right\rfloor - \left\lfloor\frac{199}{3\cdot 5\cdot 11}\right\rfloor - \left\lfloor\frac{199}{3\cdot 5\cdot 13}\right\rfloor \\
= 198 &- 99 - 66 - 39 - 28 - 18 - 15 + 33 + 19 + 14 + 9 + 7 + 13 + 9 + 6 + 5 \\
&+ 5 + 3 + 3 + 2 + 2 + 1 - 6 - 4 - 3 - 2 - 2 - 1 - 1 - 1 - 1 - 1 - 1 - 1 = 40
\end{aligned}$$

These 40 numbers are therefore all prime, as are the 6 numbers in our set. Therefore there are exactly 46 prime numbers less than 200.

**7.** We can apply inclusion–exclusion if we reason as follows. First, we restrict ourselves to numbers greater than 1. If the number $N$ is the power of an integer, then it is certainly the prime power of an integer, since if

$N = x^k$, where $k = mp$, with $p$ prime, then $N = (x^m)^p$. Thus we need to count the number of perfect second powers, the number of perfect third powers, the number of perfect fifth powers, etc., less than 10,000. Let us first determine how many positive integers greater than 1 and less than 10,000 are the square of an integer. Since $\lfloor\sqrt{9999}\rfloor = 99$, there must be $99 - 1 = 98$ such numbers (namely $2^2$ through $99^2$). Similarly, since $\lfloor\sqrt[3]{9999}\rfloor - 1 = 20$, there are 20 cubes of integers less than 10,000. Similarly, there are $\lfloor\sqrt[5]{9999}\rfloor - 1 = 5$ fifth powers, $\lfloor\sqrt[7]{9999}\rfloor - 1 = 2$ seventh powers, $\lfloor\sqrt[11]{9999}\rfloor - 1 = 1$ eleventh power, and $\lfloor\sqrt[13]{9999}\rfloor - 1 = 1$ thirteenth power. There are no higher prime powers, since $\lfloor\sqrt[17]{9999}\rfloor - 1 = 0$ (and indeed, $2^{17} = 131072 > 9999$).

Now we need to account for the double counting. There are $\lfloor\sqrt[6]{9999}\rfloor - 1 = 3$ sixth powers, and these were counted as both second powers and third powers. Similarly, there is $\lfloor\sqrt[10]{9999}\rfloor - 1 = 1$ tenth power ($10 = 2\cdot 5$). These are the only two cases of double counting, since all other combinations give a count of 0. Therefore among the 9998 numbers from 2 to 9999, inclusive, we found that there were $98+20+5+2+1+1-3-1 = 123$ powers. Therefore there are $9998 - 123 = 9875$ numbers that are not powers.

**9.** This exercise is just asking for the number of onto functions from a set with 6 elements (the toys) to a set with 3 elements (the children), since each toy is assigned a unique child. By Theorem 1 there are $3^6 - C(3,1)2^6 + C(3,2)1^6 = 540$ such functions.

**11.** Here is one approach. Let us ignore temporarily the stipulation about the most difficult job being assigned to the best employee (we assume that this language uniquely specifies a job and an employee). Then we are looking for the number of onto functions from the set of 7 jobs to the set of 4 employees. By Theorem 1 there are $4^7 - C(4,1)3^7 + C(4,2)2^7 - C(4,1)1^7 = 8400$ such functions. Now by symmetry, in exactly one fourth of those assignments should the most difficult job be given to the best employee, as opposed to one of the other three employees. Therefore the answer is $8400/4 = 2100$.

**13.** We simply apply Theorem 2:

$$\begin{aligned} D_7 &= 7!\left(1 - \frac{1}{1!} + \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} + \frac{1}{6!} - \frac{1}{7!}\right) \\ &= 5040 - 5040 + 2520 - 840 + 210 - 42 + 7 - 1 = 1854 \end{aligned}$$

**15.** **a)** An arrangement in which no letter is put into the correct envelope is a derangement. There are by definition $D_{100}$ derangements. Since there are $P(100,100) = 100!$ equally likely permutations altogether, the probability of a derangement is $D_{100}/100!$. Numerically, this is almost exactly equal to $1/e$, which is about 0.368.

**b)** We need to count the number of ways to put exactly one letter into the correct envelope. First, there are $C(100,1) = 100$ ways to choose the letter that is to be correctly stuffed. Then there are $D_{99}$ ways to insert the remaining 99 letters so that none of them go into their correct envelopes. By the product rule, there are $100D_{99}$ such arrangements. As in part **(a)** the denominator is $P(100,100) = 100!$. Therefore the answer is $100D_{99}/100! = D_{99}/99!$. Again this is almost exactly $1/e \approx 0.368$.

**c)** This time, to count the number of ways that exactly 98 letters can be put into their correct envelopes, we need simply to choose the two letters that are to be misplaced, since there is only one way to misplace them. There are of course $C(100,2) = 4950$ ways to do this. As in part **(a)** the denominator is $P(100,100) = 100!$. Therefore the answer is $4950/100!$. This is substantially less than $10^{-100}$, so for all practical purposes, the answer is 0.

**d)** There is no way that exactly 99 letters can be inserted into their correct envelopes, since as soon as 99 letters have been correctly inserted, there is only one envelope left for the remaining letter, and it is the correct one. Therefore the answer is exactly 0. (The probability of an event that cannot happen is 0.)

**e)** Only one of the 100! permutations is the correct stuffing, so the answer is $1/100!$. As in part **(c)** this is 0 for all practical purposes.

**17.** We can derive this answer by mimicking the derivation of the formula for the number of derangements, but worrying only about the even digits. There are 10! permutations altogether. Let $e$ be one of the 5 even digits. The number of permutations in which $e$ is in its original position is 9! (the other 9 digits need to be permuted). Therefore we need to subtract from 10! the $5 \cdot 9!$ ways in which the even digits can end up in their original positions. However, we have overcounted, since there are $C(5,2)8!$ ways in which 2 of the even digits can end up in their original positions, $C(5,3)7!$ ways in which 3 of them can, $C(5,4)6!$ ways in which 4 of them can, and $C(5,5)5!$ ways in which they can all retain their original positions. Applying inclusion–exclusion, we therefore have the answer

$$10! - 5 \cdot 9! + 10 \cdot 8! - 10 \cdot 7! + 5 \cdot 6! - 5! = 2{,}170{,}680\,.$$

**19.** We want to show that $D_n - nD_{n-1} = (-1)^n$. We will use an iterative approach, taking advantage of the result of Exercise 18, which can be rewritten algebraically as $D_k - kD_{k-1} = -\big(D_{k-1} - (k-1)D_{k-2}\big)$ for all $k \geq 2$. We have

$$\begin{aligned} D_n - nD_{n-1} &= -\big(D_{n-1} - (n-1)D_{n-2}\big) \\ &= -\big(-(D_{n-2} - (n-2)D_{n-3})\big) \\ &= (-1)^2\big(D_{n-2} - (n-2)D_{n-3}\big) \\ &\ \ \vdots \\ &= (-1)^{n-2}(D_2 - 2D_1) \\ &= (-1)^n \end{aligned}$$

since $D_2 = 1$ and $D_1 = 0$, and since $(-1)^{n-2} = (-1)^n$.

**21.** We can solve this problem by looking at the explicit formula we have for $D_n$ from Theorem 2 (multiplying through by $n!$):

$$D_n = n! - n! + \frac{n!}{2} - \frac{n!}{3!} + \cdots + (-1)^{n-1}\frac{n!}{(n-1)!} + (-1)^n\frac{n!}{n!}$$

Now all of these terms are even except possibly for the last two, since (after being reduced to natural numbers) they all contain the factors $n$ and $n-1$, at least one of which must be even. Therefore to determine whether $D_n$ is even or odd, we need only look at these last two terms, which are $\pm n \mp 1$. If $n$ is even, then this difference is odd; but if $n$ is odd, then this difference is even. Therefore $D_n$ is even precisely when $n$ is odd.

**23.** Recall that $\phi(n)$, for a positive integer $n > 1$, denotes the number of positive integers less than (or, vacuously, equal to) $n$ and relatively prime to $n$ (in other words, that have no common prime factors with $n$). We will derive a formula for $\phi(n)$ using inclusion–exclusion. We are given that the prime factorization of $n$ is $n = p_1^{a_1}p_2^{a_2}\cdots p_m^{a_m}$. Let $P_i$ be the property that a positive integer less than or equal to $n$ has $p_i$ as a factor. Then $\phi(n)$ is precisely the number of positive integers less than or equal to $n$ that have none of the properties $P_i$. By the alternative form of the principle of inclusion–exclusion, we have the following formula for this quantity:

$$\begin{aligned} N(P_1'P_2'\cdots P_m') = n - \sum_{1\le i\le m} N(P_i) + \sum_{1\le i<j\le m} N(P_iP_j) - \sum_{1\le i<j<k\le m} N(P_iP_jP_k) \\ + \cdots + (-1)^m N(P_1P_2\cdots P_m) \end{aligned}$$

Our only remaining task is to find a formula for each of these sums. This is not hard. First $N(P_i)$, the number of positive integers less than or equal to $n$ divisible by $p_i$, is equal to $n/p_i$, just as in the discussion of the sieve of Eratosthenes (we need no floor function symbols since $n/p_i$ is necessarily an integer). Similarly, $N(P_iP_j)$, the number of positive integers less than or equal to $n$ divisible by both $p_i$ and $p_j$, i.e., by the product $p_ip_j$, is equal to $n/(p_ip_j)$, and so on. Making these substitutions, we can rewrite the formula displayed above as

$$N(P_1'P_2'\cdots P_m') = n - \sum_{1\le i\le m}\frac{n}{p_i} + \sum_{1\le i<j\le m}\frac{n}{p_ip_j} - \sum_{1\le i<j<k\le m}\frac{n}{p_ip_jp_k} + \cdots + (-1)^m\frac{n}{p_1p_2\cdots p_m}.$$

This formula can be written in a more useful form. If we factor out the $n$ from every term, then it is not hard to see that what remains is the product $(1-1/p_1)(1-1/p_2)\cdots(1-1/p_m)$. Therefore our answer is

$$n\prod_{i=1}^{m}\left(1-\frac{1}{p_i}\right).$$

**25.** A permutation meeting these conditions must be a derangement of 123 followed by a derangement of 456 in positions 4, 5, and 6. Since there are $D_3 = 2$ derangements of the first 3 elements to choose from for the first half of our permutation and $D_3 = 2$ derangements of the last 3 elements to choose from for the second half, there are, by the product rule, $2\cdot 2 = 4$ derangements satisfying the given conditions. Indeed, these 4 derangements are 231564, 231645, 312564, and 312645.

**27.** Let $P_i$ be the property that a function from a set with $m$ elements to a set with $n$ elements does not have the $i^{\text{th}}$ element of the codomain included in its range. We want to compute $N(P_1'P_2'\cdots P_n')$. In order to use the principle of inclusion–exclusion we need to determine $\sum N(P_i)$, $\sum N(P_iP_j)$, etc. By the product rule, there are $n^m$ functions from the set with $m$ elements to the set with $n$ elements. If we want the function not to have the $i^{\text{th}}$ element of the codomain in its range, then there are only $n-1$ choices at each stage, rather than $n$, to assign to each element of the domain; therefore $N(P_i) = (n-1)^m$, for each $i$. Furthermore, there are $C(n,1)$ different $i$'s. Therefore $\sum N(P_i) = C(n,1)(n-1)^m$. Similarly, to compute $\sum N(P_iP_j)$, we note that there are $C(n,2)$ ways to specify $i$ and $j$, and that once we have determined which 2 elements are to be omitted from the codomain, there are $(n-2)^m$ different functions with this smaller codomain. Therefore $\sum N(P_iP_j) = C(n,2)(n-2)^m$. We continue in this way, until finally we need to find $N(P_1P_2\cdots P_n)$, which is clearly equal to 0, since the function must have at least one element in its range. The formula given in the statement of Theorem 1 therefore follows from the inclusion–exclusion principle.

## GUIDE TO REVIEW QUESTIONS FOR CHAPTER 8

**1.** **a)** See pp. 158 and 501. **b)** $1,000,000 · $1.09^n$

**2.** See Example 1 in Section 8.1.

**3.** See Example 2 in Section 8.1.

**4.** **a)** See Example 3 in Section 8.1 (interchange the roles of 0 and 1). **b)** See Exercise 11 in Section 8.1.

**5.** **a)** See pp. 407–408. **b)** This application is discussed in detail at the end of Section 8.1.

**6.** an equation of the form $a_n = c_1a_{n-1} + c_2a_{n-2} + \cdots + c_ka_{n-k}$

**7.** **a)** See Theorem 1 and Example 3 in Section 8.2 if the roots of the characteristic equation are distinct; otherwise see Theorem 2 and Example 5.
**b)** The characteristic equation is $r^2 - 13r + 22 = 0$, leading to roots 2 and 11. This gives the general solution $a_n = \alpha_1 2^n + \alpha_2 11^n$. Substituting in the initial conditions gives $\alpha_1 = 2$ and $\alpha_2 = 1$. Therefore the solution is $a_n = 2^{n+1} + 11^n$.
**c)** The characteristic equation is $r^2 - 14r + 49 = 0$, leading to the repeated root 7. This gives the general solution $a_n = \alpha_1 7^n + \alpha_2 n7^n$. Substituting in the initial conditions gives $\alpha_1 = 3$ and $\alpha_2 = 2$. Therefore the solution is $a_n = (3+2n)7^n$.

**8.** **a)** See pp. 527–530. The exact solution is $f(b^k) = a^k f(1) + \sum_{j=0}^{k-1} a^j g(b^{k-j})$. **b)** 1442

**9.** **a)** See Example 1 in Section 8.3. **b)** $O(\log n)$

**10.** **a)** See p. 555. **b)** See pp. 554–555.
**c)** Note that the number of integers not exceeding 1000 that are divisible by $a$ and $b$ is $\lfloor 1000/\text{lcm}(a,b) \rfloor$. Thus the answer is

$$\left\lfloor \frac{1000}{6} \right\rfloor + \left\lfloor \frac{1000}{10} \right\rfloor + \left\lfloor \frac{1000}{15} \right\rfloor - \left\lfloor \frac{1000}{\text{lcm}(6,10)} \right\rfloor - \left\lfloor \frac{1000}{\text{lcm}(6,15)} \right\rfloor - \left\lfloor \frac{1000}{\text{lcm}(10,15)} \right\rfloor + \left\lfloor \frac{1000}{\text{lcm}(6,10,15)} \right\rfloor$$
$$= 166 + 100 + 66 - 33 - 33 - 33 + 33 = 266\,.$$

This is similar to (but slightly harder than) Example 2 in Section 8.5.
**d)** For a similar problem, see Example 1 in Section 8.6. The solution is $C(4+22-1,22) - C(4+14-1,14) - C(4+16-1,16) - C(4+17-1,17) + C(4+8-1,8) + C(4+9-1,9) + C(4+11-1,11) - C(4+3-1,3)$.

**11.** **a)** See Example 5 in Section 8.5. **b)** $4\cdot 25 - 6\cdot 5 + 4\cdot 2 - 1 = 77$

**12.** See Theorem 1 in Section 8.5.

**13.** See pp. 560–561.

**14.** **a)** Count the number of onto functions from an $m$-set to an $n$-set, using Theorem 1 in Section 8.6; see Example 3 in Section 8.6.
**b)** $3^7 - 3\cdot 2^7 + 3\cdot 1^7 = 1806$

**15.** See the discussion of the Sieve of Eratosthenes on p. 560.

**16.** **a)** See p. 562.
**b)** Think of the hats permuted among the heads (which are the positions for the objects being permuted).
**c)** See Theorem 2 in Section 8.6.

## SUPPLEMENTARY EXERCISES FOR CHAPTER 8

**1.** Let $L_n$ be the number of chain letters sent at the $n^{\text{th}}$ stage.
**a)** Since each person receiving a letter sends it to 4 new people, there will be 4 times as many letters sent at the $n^{\text{th}}$ stage as were sent at the $(n-1)^{\text{st}}$ stage. Therefore the recurrence relation is $L_n = 4L_{n-1}$.
**b)** The initial condition is that at the first stage 40 letters are sent (each of the original 10 people sent it to 4 others), i.e., $L_1 = 40$.
**c)** We need to solve this recurrence relation. We do so easily by iteration, since $L_n = 4L_{n-1} = 4^2L_{n-2} = \cdots = 4^{n-1}L_1 = 4^{n-1}\cdot 40$, or more simply $L_n = 10\cdot 4^n$.

**3.** Let $M_n$ be the amount of money (in dollars) that the government prints in the $n^{\text{th}}$ hour.
**a)** According to the given information, the amount of money printed in the $n^{\text{th}}$ hour is \$10,000 in \$1 bills, \$20,000 in \$5 bills, \$30,000 in \$10 bills, \$50,000 in \$20 bills, and \$50,000 in \$50 bills, for a total of \$160,000. Therefore our recurrence relation is $M_n = M_{n-1} + 160000$.
**b)** Since 1000 of each bill was produced in the first hour, we know that $M_1 = 1000(1+5+10+20+50+100) = 186000$.

**c)** We solve the recurrence relation by iteration:

$$\begin{aligned}M_n &= 160000 + M_{n-1}\\ &= 160000 + 160000 + M_{n-2} = 2 \cdot 160000 + M_{n-2}\\ &\ \ \vdots\\ &= (n-1)\cdot 160000 + M_1\\ &= 160000(n-1) + 186000 = 160000n + 26000\end{aligned}$$

**d)** Let $T_n$ be the total amount of money produced in the first $n$ hours. Then $T_n = T_{n-1} + M_n$, since the total amount of money produced in the first $n$ hours is the same as the total amount of money produced in the first $n-1$ hours, plus the amount of money produced in the $n^{\text{th}}$ hour. Thus, from our result in part **(c)**, the recurrence relation is $T_n = T_{n-1} + 160000n + 26000$, with initial condition $T_0 = 0$ (no money is produced in $0$ hours).

**e)** We solve the recurrence relation from part **(d)** by iteration:

$$\begin{aligned}T_n &= 26000 + 160000n + T_{n-1}\\ &= 26000 + 160000n + 26000 + 160000(n-1) + T_{n-2}\\ &= 2\cdot 26000 + \big(n + (n-1)\big)\cdot 160000 + T_{n-2}\\ &\ \ \vdots\\ &= n\cdot 26000 + 160000\cdot\big(n + (n-1) + \cdots + 1\big) + T_0\\ &= 26000n + 160000\cdot\frac{n(n+1)}{2} = 80000n^2 + 106000n\end{aligned}$$

**5.** This problem is similar to Exercise 19 in Section 8.1. Let $m_n$ be the number of messages that can be sent in $n$ microseconds.

**a)** A message must begin with either the two-microsecond signal or the three-microsecond signal. If it begins with the two-microsecond signal, then the rest of the message is of length $n-2$; if it begins with the three-microsecond signal, then the message continues as a message of length $n-3$. Therefore the recurrence relation is $m_n = m_{n-2} + m_{n-3}$.

**b)** We need initial conditions for $n = 0$, $1$, and $2$, since the recurrence relation has degree $3$. Clearly $m_0 = 1$, since the empty message is the one and only message of length $0$. Also $m_1 = 0$, since every nonempty message contains at least one signal, and the shortest signal has length $2$. Finally $m_2 = 1$, since there is only one message of length $2$, namely the one that uses one of the shorter signals and none of the longer signals.

**c)** There are two approaches here. One is to solve the recurrence relation, using the methods of Section 8.2. Unfortunately, the characteristic equation is $r^3 - r - 1 = 0$, and it has no rational roots. It is possible to find real roots, but the formula for solving third degree equations is messy, and the algebra in completing the solution this way would not be pleasant. (Alternatively, one could get approximations to the roots, then get approximations to the coefficients in the solution, plug in $n = 12$, and round to the nearest integer; again the calculation involved would be unpleasant.)

The other approach is simply to use the recurrence relation to compute $m_3, m_4, \ldots, m_{12}$. First $m_3 = m_1 + m_0 = 0 + 1 = 1$; then $m_4 = m_2 + m_1 = 1 + 0 = 1$, then $m_5 = m_3 + m_2 = 1 + 1 = 2$, and so on. Starting with $m_6$, the sequence continues $2, 3, 4, 5, 7, 9, 12$. Therefore there are $12$ different messages that can be sent in exactly $12$ microseconds. (If we wanted to find the number of nonempty messages that could be sent in at most $12$ microseconds—which is certainly one interpretation of the question—then we would add $m_1$ through $m_{12}$, obtaining $47$ as our answer.)

**7.** The recurrence relation found in Exercise 6 is of degree 10, namely $a_n = a_{n-4} + a_{n-6} + a_{n-10}$. It needs 10 initial conditions, namely $a_0 = 1$, $a_1 = a_2 = a_3 = a_5 = a_7 = a_9 = 0$, and $a_4 = a_6 = a_8 = 1$.

**a)** $a_{12} = a_8 + a_6 + a_2 = 1 + 1 + 0 = 2$ (Indeed, the 2 ways to affix 12 cents postage is either to use 3 4-cent stamps or to use 2 6-cent stamps.)

**b)** First we need to compute $a_{10} = a_6 + a_4 + a_0 = 1 + 1 + 1 = 3$. Then $a_{14} = a_{10} + a_8 + a_4 = 3 + 1 + 1 = 5$.

**c)** We use the results of previous parts here: $a_{18} = a_{14} + a_{12} + a_8 = 5 + 2 + 1 = 8$.

**d)** First we need to compute $a_{16} = a_{12} + a_{10} + a_6 = 2 + 3 + 1 = 6$. Using this (and previous parts), we have $a_{22} = a_{18} + a_{16} + a_{12} = 8 + 6 + 2 = 16$.

**9.** Following the hint, let $b_n = \log a_n$ (remember that we mean log base 2). Then using the property that the log of a quotient is the difference of the logs and the log of a power is the multiple of the log, we take the logarithm of both sides of the recurrence relation for $a_n$ to obtain $b_n = 2b_{n-1} - b_{n-2}$. The initial conditions translate into $b_0 = \log a_0 = \log 1 = 0$ and $b_1 = \log a_1 = \log 2 = 1$. Thus we have transformed our problem into a linear, homogeneous, second degree recurrence relation with constant coefficients.

To solve $b_n = 2b_{n-1} - b_{n-2}$, we form the characteristic equation $r^2 - 2r + 1 = 0$, which has the repeated root $r = 1$. By Theorem 2 in Section 8.2, the general solution is $b_n = \alpha_1 1^n + \alpha_2 n 1^n = \alpha_1 + \alpha_2 n$. Plugging in the initial conditions gives the equations $\alpha_1 = 0$ and $\alpha_1 + \alpha_2 = 1$, whence $\alpha_2 = 1$. Therefore the solution is $b_n = n$. Finally, $b_n = \log a_n$ implies that $a_n = 2^{b_n}$. Therefore our solution to the original problem is $a_n = 2^n$.

**11.** The characteristic equation of the associated homogeneous equation is $r^3 - 3r^2 + 3r - 1 = 0$. This factors as $(r-1)^3 = 0$, so there is only one root, 1, and its multiplicity is 3. Therefore the general solution is $a_n^{(h)} = \alpha + \beta n + \gamma n^2$. Since the nonhomogeneous term is 1, Theorem 6 in Section 8.2 tells us to look for a particular solution of the form $a_n = c \cdot n^3$. Plugging this into the recurrence gives $c \cdot n^3 = 3c(n-1)^3 - 3c(n-2)^3 + (n-3)^3 + 1$. Simplifying this by multiplying it out and collecting like powers of $n$ gives us $6c = 1$ (all the other terms cancel out), so $c = 1/6$. Thus $a_n^{(p)} = n^3/6$. Plugging in the initial conditions to the general solution $a_n = \alpha + \beta n + \gamma n^2 + n^3/6$ gives us $2 = \alpha$, $4 = \alpha + \beta + \gamma + 1/6$, and $8 = \alpha + 2\beta + 4\gamma + 4/3$. Solving yields $\alpha = 2$, $\beta = 4/3$, and $\gamma_3 = 1/2$. Therefore the solution is $a_n = 2 + 4n/3 + n^2/2 + n^3/6$. As a check we can compute $a_3$ both from the recurrence and from the formula, and we get 15 in both cases.

**13.** One way to approach this problem is by temporarily using three variables. We assume that rabbits are born at the beginning of the month. Let $a_n$ be the number of pairs of $\frac{1}{2}$-month-old rabbits present in the middle of the $n^{\text{th}}$ month, let $b_n$ be the number of pairs of $1\frac{1}{2}$-month-old rabbits present in the middle of the $n^{\text{th}}$ month, and let $c_n$ be the number of pairs of $2\frac{1}{2}$-month-old rabbits present in the middle of the $n^{\text{th}}$ month. All the older rabbits have left the island, by the conditions of the exercise. Let us see how each of these depends on previous values. First note that $b_n = a_{n-1}$, since these rabbits are one month older. Similarly $c_n = b_{n-1}$. Combining these two equations gives $c_n = a_{n-2}$. Finally, $a_n = b_{n-1} + c_{n-1}$, since newborns come from these two groups of rabbits. Writing this last equation totally in terms of $a_n$ (using the previous equations) gives $a_n = a_{n-2} + a_{n-3}$.

Now we are interested in $T_n = a_n + b_n + c_n$, the total number of pairs of rabbits in the middle of the $n^{\text{th}}$ month. Since we have seen that the sequences $\{b_n\}$ and $\{c_n\}$ are the same as the sequence $\{a_n\}$, just shifted by one or two months, they must satisfy the same recurrence relation, so we have $b_n = b_{n-2} + b_{n-3}$ and $c_n = c_{n-2} + c_{n-3}$. If we add these three recurrence relations, we obtain $T_n = T_{n-2} + T_{n-3}$. We can take as the initial conditions $T_1 = T_2 = 1$ and $T_3 = 2$.

(We are not asked to solve this recurrence relation, and fortunately so. The characteristic equation, $r^3 - r - 1 = 0$ has no nice roots—one is irrational and two are complex. The roots are distinct, however, so let us call them $r_1$, $r_2$, and $r_3$. Then the general solution to the recurrence relation is $T_n = \alpha_1 r_1^n + \alpha_2 r_2^n + \alpha_3 r_3^n$.

We could in principle determine the values of the $\alpha$'s by plugging in the initial conditions, thereby obtaining an explicit solution. We will not do this.)

**15.** **a)** Under the given conditions, one longest common subsequence clearly ends at the last term in each sequence, so $a_m = b_n = c_p$. Furthermore, a longest common subsequence of what is left of the $a$-sequence and the $b$-sequence after those last terms are deleted has to form the beginning of a longest common subsequence of the original sequences.

**b)** If $c_p \neq a_m$, then the longest common subsequence's appearance in the $a$-sequence must terminate before the end; therefore the $c$-sequence must be a longest common subsequence of $a_1, a_2, \ldots, a_{m-1}$ and $b_1, b_2, \ldots, b_n$. The other half is similar.

**17.** The following algorithm uses the initial conditions and recurrences given in Exercise 16.

```
procedure howlong(a_1, ..., a_m, b_1, ..., b_n : sequences)
for i := 1 to m
        L(i, 0) := 0
for j := 1 to n
        L(0, j) := 0
for i := 1 to m
        for j := 1 to n
                if a_i = b_j then L(i, j) := L(i − 1, j − 1) + 1
                else L(i, j) := max(L(i, j − 1), L(i − 1, j))
return L(m, n)
```

**19.** Define a new function $g(k)$ by $g(k) = f(2^k)$. Then the initial condition for $f$ becomes the initial condition $g(0) = 1$, and the recurrence relation for $f$ becomes $g(k) = g(k-1) + 4^k$. Therefore $g(k) = 4^k + 4^{k-1} + \cdots + 4^1 + 1$, a geometric series. Its sum (see Table 2 in Section 2.4) is $(4^{k+1} - 1)/(4-1) = (4(2^k)^2 - 1)/3$, so $f(n) = (4n^2 - 1)/3$.

**21.** We use Theorem 2 in Section 8.3, with $a = 3$, $b = 5$, $c = 2$ and $d = 4$. Since $a < b^d$, we have that $f(n)$ is $O(n^d) = O(n^4)$.

**23.** In the algorithm in Exercise 22, we need two comparisons to determine the largest and second largest elements of the sequence, knowing the largest and second largest elements of the first half and the second half. Thus letting $f(n)$ be the number of comparisons needed for a list with $n$ elements, and assuming that $n$ is even, we have $f(n) = 2f(n/2) + 2$. Now by Theorem 2 in Section 8.3, with $a = 2$, $b = 2$, $c = 2$ and $d = 0$, we know that $f(n)$ is $O(n^{\log_b a}) = O(n^1) = O(n)$.

**25.** The idea of the algorithm in Exercise 24 is that it looks at the middle three terms of the sequence and based on their relative sizes (needing at most two comparisons) concludes that the index $m$ has to be either in the second half of the sequence or in the first half of the sequence. A contiguous subsequence of a unimodal sequence is again unimodal, so the same algorithm can then be applied to a sequence only about half as big as the original. Since the length of the sequence is cut in half each time, only about $2\log_2 n$ comparisons are needed.

**27.** First we have to find $\Delta a_n$. By definition we have $\Delta a_n = a_{n+1} - a_n = 3(n+1)^3 + (n+1) + 2 - (3n^3 + n + 2) = 9n^2 + 9n + 4$.

**a)** By definition $\Delta^2 a_n = \Delta a_{n+1} - \Delta a_n = 9(n+1)^2 + 9(n+1) + 4 - (9n^2 + 9n + 4) = 18n + 18$.

**b)** By definition $\Delta^3 a_n = \Delta^2 a_{n+1} - \Delta^2 a_n = 18(n+1) + 18 - (18n + 18) = 18$.

**c)** By definition $\Delta^4 a_n = \Delta^3 a_{n+1} - \Delta^3 a_n = 18 - 18 = 0$.

**29.** We apply the definition, starting with the right-hand side:

$$\begin{aligned} a_{n+1}(\Delta b_n) + b_n(\Delta a_n) &= a_{n+1}(b_{n+1} - b_n) + b_n(a_{n+1} - a_n) \\ &= a_{n+1}b_{n+1} - a_n b_n \quad \text{(by algebra)} \\ &= \Delta(a_n b_n) \quad \text{(by definition)} \end{aligned}$$

**31. a)** Let $G(x) = \sum_{n=0}^{\infty} a_n x^n$. Then $G'(x) = \sum_{n=1}^{\infty} n a_n x^{n-1} = \sum_{n=0}^{\infty}(n+1)a_{n+1}x^n$. Therefore

$$G'(x) - G(x) = \sum_{n=0}^{\infty}((n+1)a_{n+1} - a_n)x^n = \sum_{n=0}^{\infty} \frac{x^n}{n!} = e^x,$$

as desired. That $G(0) = a_0 = 1$ is given.

**b)** We compute the indicated derivative:

$$(e^{-x}G(x))' = e^{-x}G'(x) - e^{-x}G(x) = e^{-x}(G'(x) - G(x)) = e^{-x} \cdot e^x = 1$$

This means that $e^{-x}G(x)$ is $x$ plus a constant, say $x + c$. So $G(x) = xe^x + ce^x$. Plugging in the initial condition shows that $c = 1$, and we are done.

**c)** We work with the generating function for the exponential function:

$$G(x) = \sum_{n=0}^{\infty} \frac{x^{n+1}}{n!} + \sum_{n=0}^{\infty} \frac{x^n}{n!} = \sum_{n=1}^{\infty} \frac{x^n}{(n-1)!} + \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

Therefore $a_n = 1/(n-1)! + 1/n!$ for all $n \geq 1$ (and $a_0 = 1$). As a check we can compute the first few terms of the sequence both from this solution and from the recurrence, and in each case we find the sequence $a_0 = 1$, $a_1 = 2$, $a_2 = 3/2$, $a_3 = 2/3$, $a_4 = 5/24$, $\ldots$.

**33.** Let $H$, $C$, and $S$ stand for the sets of farms that have horses, cows, and sheep, respectively. We are told that $|H \cup C \cup S| = 323$, $|H| = 224$, $|C| = 85$, $|S| = 57$, and $|H \cap C \cap S| = 18$. We are asked to find $|H \cap C| + |H \cap S| + |C \cap S| - 3|H \cap C \cap S|$ (the reason for the subtraction is that the indicated sum counts the farms with all three animals 3 times, and we wish to count it no times). By the principle of inclusion–exclusion we know that $|H \cup C \cup S| = |H| + |C| + |S| - |H \cap C| - |H \cap S| - |C \cap S| + |H \cap C \cap S|$. Solving for the expression we are interested in, we get $|H \cap C| + |H \cap S| + |C \cap S| - 3|H \cap C \cap S| = |H| + |C| + |S| - |H \cup C \cup S| - 2|H \cap C \cap S| = 224 + 85 + 57 - 323 - 2 \cdot 18 = 7$. Thus 7 farms have exactly two of the three types of animals.

**35.** We apply the principle of inclusion–exclusion: $|AM \cup PM \cup OR \cup CS| = 23 + 17 + 44 + 63 - 5 - 8 - 4 - 6 - 5 - 14 + 2 + 2 + 1 + 1 - 1 = 110$.

**37.** Since the largest possible value for $x_1 + x_2 + x_3$ under these constraints is $5 + 9 + 4 = 18$, there are no solutions to the given equation.

**39. a)** We solve this problem in the same manner as we solved Exercise 7 in Section 8.6. As explained in our solution there, we need only look at prime powers. Let us restrict ourselves to integers greater than 1, and add 1 at the end. There are $\lfloor \sqrt{199} \rfloor - 1 = 13$ perfect second powers in the given range, namely $2^2$ through $14^2$. There are $\lfloor \sqrt[3]{199} \rfloor - 1 = 4$ perfect third powers, $\lfloor \sqrt[5]{199} \rfloor - 1 = 1$ perfect fifth power, and $\lfloor \sqrt[7]{199} \rfloor - 1 = 1$ perfect seventh power. Furthermore, there is $\lfloor \sqrt[6]{199} \rfloor - 1 = 1$ perfect sixth power, which is both a perfect square and a perfect cube. Therefore by inclusion–exclusion, the number of numbers between 2 and 199 inclusive that are powers greater than the first power of an integer is $13 + 4 + 1 + 1 - 1 = 18$; adding on the number 1 itself (since $1 = 1^2$), we get the answer 19.

**b)** We saw in Exercise 5 in Section 8.6 that there are 46 primes less than 200, and we just saw above that there are 19 powers. Since these two sets are disjoint, we just add the cardinalities, obtaining $19 + 46 = 65$.

**c)** Solving this problem is like counting prime numbers, except that the squares of primes play the role of the primes themselves. The squares of primes relevant to the problem are 4, 9, 25, 49, 121, and 169. The number of positive integers less than 200 divisible by $p^2$ is $\lfloor 199/p^2 \rfloor$. There is overcounting, however, since a number divisible by a number like $36 = 2^2 \cdot 3^2$ is counted in both $\lfloor 199/2^2 \rfloor$ and $\lfloor 199/3^2 \rfloor$; hence we need to subtract $\lfloor 199/6^2 \rfloor$. The number of numbers divisible by squares of primes is therefore

$$\left\lfloor \frac{199}{2^2} \right\rfloor + \left\lfloor \frac{199}{3^2} \right\rfloor + \left\lfloor \frac{199}{5^2} \right\rfloor + \left\lfloor \frac{199}{7^2} \right\rfloor + \left\lfloor \frac{199}{11^2} \right\rfloor + \left\lfloor \frac{199}{13^2} \right\rfloor - \left\lfloor \frac{199}{6^2} \right\rfloor - \left\lfloor \frac{199}{10^2} \right\rfloor - \left\lfloor \frac{199}{14^2} \right\rfloor ,$$

which is just $49 + 22 + 7 + 4 + 1 + 1 - 5 - 1 - 1 = 77$. Therefore there are $199 - 77 = 122$ positive integers less than 200 that are not divisible by the square of an integer greater than 1.

**d)** This is similar to part (c), with cubes in place of squares. Reasoning the same way, we get

$$199 - \left( \left\lfloor \frac{199}{2^3} \right\rfloor + \left\lfloor \frac{199}{3^3} \right\rfloor + \left\lfloor \frac{199}{5^3} \right\rfloor \right) = 199 - (24 + 7 + 1) = 167 .$$

**e)** For each set of three prime numbers $\{p, q, r\}$, the number of positive integers less than 200 divisible by $p$, $q$, and $r$ is given by $\lfloor 200/(pqr) \rfloor$. There is no overcounting to worry about in this problem, since no number less than 200 is divisible by four primes (the smallest such number is $2 \cdot 3 \cdot 5 \cdot 7 = 210$). Therefore the number of positive integers less than 200 divisible by three primes is the sum of $\lfloor 200/(pqr) \rfloor$ over all triples of distinct primes whose product is at most 200. A tedious listing shows that there are 19 such triples, and when we form the sum we get 31. Therefore there are $199 - 31 = 168$ positive integers less than 200 that are not divisible by three or more primes.

**41.** There are $n$ ways to choose which person is to receive the correct hat, and there are $D_{n-1}$ ways to have the remaining hats returned totally incorrectly (where $D_{n-1}$ is the number of derangements of $n-1$ objects). On the other hand there are $n!$ possible ways to return the hats. Therefore the probability is $nD_{n-1}/n! = D_{n-1}/(n-1)!$. Note that this happens to be the same as the probability that none of $n-1$ people is given the correct hat; therefore it is approximately $1/e \approx 0.368$ for large $n$.

**43.** There are $2^6 = 64$ bit strings of length 6. We need to find the number that contain at least 4 1's. The number that contain exactly $i$ 1's is $C(6, i)$, since such a string is determined by choosing $i$ of the 6 positions to contain the 1's. Therefore there are $C(6,4)+C(6,5)+C(6,6) = C(6,2)+C(6,1)+C(6,0) = 15+6+1 = 22$ strings with at least 4 1's. Hence the probability in question is $22/64 = 11/32$.

## WRITING PROJECTS FOR CHAPTER 8

*Books and articles indicated by bracketed symbols below are listed near the end of this manual. You should also read the general comments and advice you will find there about researching and writing these essays.*

**1.** Obviously you will need to find a translated version if you want to read what Fibonacci actually said. The search technique of gradually working your way backwards usually works: If you can't find what you want in the place you start (here, for example, maybe with a standard mathematics history textbook), then search the references provided by that work, then check the references in the references, and so on backwards.

**2.** Articles and books at all levels have dealt with this subject. You might find something in, say, *Scientific American* (which is indexed in hard-copy and electronic versions of *Readers' Guide*); you might find some articles in materials for high-school students (see, for example, *Mathematics Teacher*, a magazine for high school teachers); and just browsing through the mathematics section of a public library or popular bookstore might yield something on this topic. Talk to someone who teaches a "math for poets" course at your school (i.e., a course with almost no mathematical prerequisite that deals with appreciating the beauty or applications of mathematics); some of the textbooks for that kind of course have material on this topic, as well as references to other sources of information.

3. Paul Stockmeyer, a professor at the College of William and Mary, describes the Tower of Hanoi problem and its variations as his "main professional hobby." Consult his website (`http://www.cs.wm.edu/~pkstoc`) for some of his papers on the subject, as well as some great links.

4. There are many articles about the Catalan numbers, as well as treatments in textbooks. One article to start with might be [HiPe2].

5. Bellman wrote a book on the subject [Be1].

6. Here is a set of online lectures that has relevant information:
`lectures.molgen.mpg.de/online_lectures.html`

7. An early article by Bellman for RAND discusses economics applications:
`www.rand.org/pubs/research_memoranda/2006/RM3539.pdf`
The Wikipedia article on dynamic programming mentions optimal consumption and saving:
`en.wikipedia.org/wiki/Dynamic_programming`

8. The Wikipedia article on dynamic programming mentions the egg-dropping puzzle:
`en.wikipedia.org/wiki/Dynamic_programming`

9. Obviously, consult the reference mentioned in that exercise.

10. Andrzej Pelc has written several papers on this topic; search for his Web page.

11. Numerous websites discuss this, and several of them have working demos, as well. Search for the key words.

12. One book on sieve methods is [HaRi].

13. See the article [Da2]. There is also relevant material in the chapter on arrangements with forbidden positions in [MiRo].

14. A wonderful book on generating functions is [Wi2], and [GrKn] also has a lot of relevant material. There are sections on generating functions in [Gr2] as well as any of the advanced combinatorics books mentioned in the general suggestions at the back of this *Guide*.

15. See advanced combinatorics texts, such as [Ro1] or [Tu1]. For another writing project, find out about George Polyá, a fascinating figure in 20th century mathematics and mathematics education.

16. See [BoDo], which should also have pointers to historical sources.

17. Advanced combinatorics texts, such as [Br2] or [Ro1], discuss this topic.