Chapter 4: Continuous Random Variables
Exponential and Gamma RV's

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Introduction

- Let $X \sim$ Binomial(n, p) and $X_1, X_2, \ldots, X_n \sim$ Bernoulli(p). The sum of n Bernoulli r.v.'s is a Binomial r.v.

$$X = X_1 + X_2 + \ldots + X_n$$

- Similarly, a Gamma r.v. (Y) is the sum of a bunch of Exponential random variables $(Y_1, Y_2, \ldots, Y_n)$.

$$\text{Gamma}\ Y = \underbrace{Y_1 + Y_2 + \ldots + Y_n}_{\text{Exp}}$$

- Learn/apply properties of Exponential and Gamma rv's.

Introduction

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- Learn/apply properties of Exponential and Gamma rv's.
  - Goal: Probability of the time until the first n successes.
    - $n = 1$: Exponential r.v.
    - $n \geq 1$: Gamma r.v.
    - n successes is fixed; time is the r.v.

Introduction

- The pdf for a Poisson r.v. is

$$f_X(x) = \frac{e^{-\lambda t} \cdot (\lambda t)^x}{x!}$$

    - Time is fixed.
    - X is a discrete r.v. for the number of successes within the given time.
- Waiting/Arrival times are rare events
    - The fixed number of arrivals are a Poisson process.

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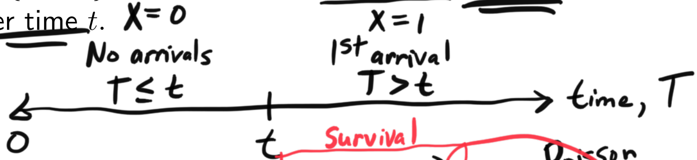
The Exponential Distribution: Intro

- Interested in probability of time until next arrival (i.e., the waiting or arrival time)
    - Time is the r.v.
    - Next arrival (success) is fixed.
- Let $\{T > t\}$ denote that the first arrival does not occur until after time t.



Survival

$$\underline{\text{cdf}}: \; P(T \le t) = 1 - \boxed{P(T > t)} \sim \text{Poisson}$$

$$= 1 - P(X=0 \text{ in } [0,t])$$

$$= 1 - \frac{e^{-\lambda t}\cdot(\lambda t)^0}{0!} = 1 - \boxed{e^{-\lambda t}}$$

$\underline{\text{cdf}}$: $P(T \leq t) = F_T(t) = 1 - e^{-\lambda t}$ $\quad X = T$

$P(X \leq x) = F_X(x) = 1 - e^{-\lambda x}$

$\underline{\text{pdf}}$: $f = F' = \frac{d}{dx}(1 - e^{-\lambda x}) = \lambda e^{-\lambda x}$

$$-e^{-\lambda x} \cdot \underbrace{\frac{d}{dx}(-\lambda x)}_{-\lambda}$$

The Exponential Distribution

The time, X, until the first or next arrival (or success) is an Exponential Random Variable where λ is the number of successes per unit time (λ is the exponential rate). succ/time

$$X \sim \text{Exponential}(\lambda) \text{ where } \boxed{X \geq 0}, \lambda > 0$$

Term	Notation	Formula
Density pdf	$f_X(x)$	$\lambda e^{-\lambda x}$
Cumulative Density cdf	$F_X(x) = P(X < x)$	$1 - e^{-\lambda x}$
Survival Function	$P(X > x)$	$e^{-\lambda x}$
Mean (time)	μ	$1/\lambda$
Variance (time^2)	σ^2	$1/\lambda^2$

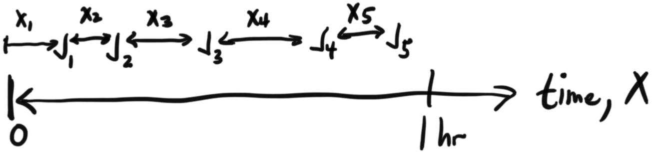
Example: Understanding the Parameters

Jobs are sent to a printer at a rate of 5 jobs per hour.

successes / time

(i) Exponential Rate is $\lambda = 5$ jobs/hr

(ii) Mean $\mu = 1/\lambda = 1/5 = 0.2$ hrs/job $= 12$ min/job



Mean of $X_1, \ldots, X_5 \approx 12$ min/job

Example: Using the mean.

succ/time

Students arrive at a local bar at a rate of 30 students/hour.

(a) What is the expected time until the next student arrives?

$\lambda = 30$ stud/hr

$E(X) = \mu = 1/\lambda = 1/30$ hrs/student

OR

2 min/student

$\lambda = 0.5$ stud/min

Example: Using the pdf.

pdf $\int \lambda e^{-\lambda x} = -e^{-\lambda x}$

Students arrive at a local bar at a rate of 30 students/hour.

(b) What is the probability that the next student will arrive in the next 1 to 3 minutes?

minutes

$P(1 < X < 3)$

$= \int_1^3 0.5 e^{-0.5X} dx$

$= -e^{-0.5x} \Big|_1^3$

$= -e^{-3/2} - (-e^{-1/2})$

$= \boxed{0.3834}$

OR

hours

$P\left(\frac{1}{60} < X < \frac{3}{60}\right)$

$= \int_{1/60}^{3/60} 30 e^{-30x} dx$

$= -e^{-30x} \Big|_{1/60}^{3/60}$

$= -e^{-3/2} - (-e^{-1/2})$

Example: Using the cdf. $1-e^{-\lambda x}$

Students arrive at a local bar at a rate of 30 students/hour.

(c) What is the probability that the next student will arrive in under 2 minutes?

cdf: $P(X<2) = F_X(2) = 1-e^{-0.5(2)} = 0.6321$

or

pdf: $\int_0^2 0.5e^{-0.5x}\,dx = 0.6321$

Example: Using the survival function.

$P(X > x) = e^{-\lambda x}$

Students arrive at a local bar at a rate of 30 students/hour.

(d) What is the probability that the next student will arrive after 4 minutes time?

$$P(X > 4) = e^{-0.5(4)} = e^{-2} = \boxed{0.1353}$$

OR

pdf

$$\int_4^\infty 0.5e^{-0.5x}\,dx = \boxed{0.1353}$$

$[e^{-\infty} \to 0]$

OR

$1 - P(X<4)$

$1 - F(4)$

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What is the memoryless property?

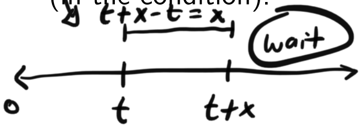
Memoryless Property of an Exponential RV

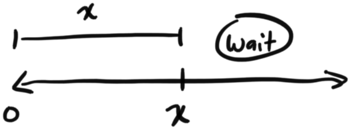
Given that the waiting time exceeds time t, the probability of the time exceeding more than $t + x$ is

$$P(X > t + x | X > t) = P(X > x)$$

where $P(X > x)$ is the probability of exceeding time x.

- The distribution "forgets" that our wait has exceeded time t (in the condition).





Proof of the Memoryless Property

Useful: (i) $P(A|B) = \frac{P(A \cap B)}{P(B)}$ and (ii) $P(X > x) = e^{-\lambda x}$ (survival)

Proof of the Memoryless Property

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$$\text{Pf: } P(X > t + x | X > t) = \frac{P(X > t + x \ \cap \ X > t)}{P(X > t)}$$
$$= \frac{P(X > t + x)}{P(X > t)}$$
$$= \frac{e^{-\lambda(t+x)}}{e^{-\lambda t}}$$
$$= \frac{e^{-\lambda t} \cdot e^{-\lambda x}}{e^{-\lambda t}}$$
$$= e^{-\lambda x}$$
$$= P(X > x) \checkmark$$

Example

Students arrive at a local bar at a rate of 30 students per hour. What is the probability that the bouncer will wait more than 10 minutes given that they have already been waiting 3 minutes for the next student?

$\lambda = 0.5$ stud/min $\qquad \lambda = 30$ stud/hr

minutes $\quad t$

$$P(X > \underset{\substack{3+7 \\ t+x}}{10} \mid X > 3) = P(X > 7)$$

$$= e^{-0.5(7)}$$

$$= \boxed{0.0302}$$

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Introduction

- The gamma function (denoted by Γ) is defined as

$$\boxed{\Gamma(n) = (n-1)!}$$

Ex: $\Gamma(6) = 5!$
$\Gamma(5) = 4!$

  - This is not the pdf or cdf.
  - This is a special factorial number used for counting.
- The gamma distribution is the sum of n independent exponential random variables.
  - Usefulness: applications where the number of arrivals is more than one.
  - Waiting time until the $\boxed{\alpha^{th}}$ event occurs.

The Gamma Distribution

The time, X, until α arrivals (or successes) is a Gamma Random Variable where λ is the number of successes per unit time:

$$\boxed{X \sim \text{Gamma}(\alpha, \lambda)} \text{ where } X > 0, \ \lambda > 0$$

Term	Notation	Formula
Density pdf	$f_X(x)$	$\frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$
Cumulative Density cdf	$F_X(t) = P(X < t)$	$\int_0^t f_X(x)\ dx$
Mean (time)	μ	α/λ
Variance (time^2)	σ^2	α/λ^2

Example

What function is represented by $X \sim$ Gamma(4,2)?

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$$

$$\alpha = 4, \lambda = 2 \implies f_X(x) = \frac{2^4}{\Gamma(4)} x^{4-1} e^{-2x}$$
$$= \frac{16}{3!} x^3 e^{-2x}$$
$$= \boxed{\frac{8}{3} x^3 e^{-2x}}$$

Gamma vs. Exponential

When the number of arrivals is $\alpha = 1$, the gamma distribution reduces to the exponential distribution.

- Exponential is a special case of Gamma!
- Let $X \sim \text{Gamma}(1, \lambda)$

$$f_X(x) = \frac{\lambda^\alpha}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$$

$$\begin{aligned} \alpha = 1 \implies f_X(x) &= \frac{\lambda^1}{\Gamma(1)} x^{1-1} e^{-\lambda x} \\ &= \lambda e^{-\lambda x} \\ &\sim \text{Exponential}(\lambda) \end{aligned}$$

Example: Using the mean and variance

Compilation of a computer program consists of 3 blocks that are processed sequentially, one after other. Each block takes Exponential time with the mean of 5 minutes per block, independent of other blocks.

$\alpha = 3$

(a) Compute the expected value and variance of the total compilation time.

5 min / block

time / success

$$\mu = \frac{\alpha}{\lambda}$$

$$\lambda = \frac{1}{5} \text{ block/min} = 0.2$$

$$\mu = \frac{3 \text{ blocks}}{0.2 \frac{\text{block}}{\text{min}}} = \boxed{15 \text{ minutes}}$$

$$\sigma^2 = \frac{\alpha}{\lambda^2} = \frac{3 \text{ blocks}}{\left(0.2 \frac{\text{blocks}}{\text{min}}\right)^2} = \boxed{75 \text{ min}^2/\text{block}}$$

Example: Finding probability

$\frac{\lambda^{\alpha}}{\Gamma(\alpha)} x^{\alpha-1} e^{-\lambda x}$

Compilation of a computer program consists of 3 blocks that are processed sequentially, one after other. Each block takes Exponential time with the mean of 5 minutes per block, independent of other blocks.

(b) Compute the probability that the entire program will be compiled in less than 12 minutes.

$$P(X<12) = \int_0^{12} \frac{(0.2)^4}{\Gamma(4)} x^{4-1} e^{-0.2x}\, dx$$

$$= \frac{1}{250} \int_0^{12} x^3 e^{-0.2x}\, dx \quad \text{Int. by Parts}$$

$$= \frac{1}{250}\left(-5x^2 e^{-0.2x} - 50 x e^{-0.2x} - 250 e^{-0.2x}\right)\Big|_0^{12} = 0.4303$$

Gamma-Poisson Formula

- Integrating a gamma density is often tedious.
- The Gamma-Poisson formula is a shortcut!
  1. The event $\{T > t\}$ means that the α^{th} event occurs after time t; therefore, fewer than α events occur prior to time t.

     Continuous Discrete

     Gamma(α, λ) Poisson(λt)

     $$P(T > t) = P(X < \alpha)$$

     $\geq$

  2. Likewise but using complements

     Gamma(α, λ) Poisson(λt)

     $$P(T \leq t) = P(X \geq \alpha)$$

     $<$

Gamma-Poisson Formula

- Integrating a gamma density is often tedious.
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$$\begin{array}{cc} \text{Gamma}(\alpha, \lambda) & \text{Poisson}(\lambda t) \end{array}$$
$$P(T > t) = P(X < \alpha)$$
$$P(X \leq \alpha - 1)$$

  2. Likewise but using complements

$$\begin{array}{cc} \text{Gamma}(\alpha, \lambda) & \text{Poisson}(\lambda t) \end{array}$$
$$P(T \leq t) = \boxed{P(X \geq \alpha)} = 1 - P(X < \alpha) = 1 - P(X \leq \alpha - 1)$$

- Use a Poisson cdf *via* calculator: $\boxed{\texttt{poissoncdf}(\lambda t, \alpha - 1)}$.

Example: Using the Gamma-Poisson Formula

Compilation of a computer program consists of 3 blocks that are processed sequentially, one after other. Each block takes Exponential time with the mean of 5 minutes per block, independent of other blocks.

= (b)

(c) Compute the probability that the entire program will be compiled in less than 12 minutes.

$$\underset{\text{Gamma}}{P(\overset{T}{X} < 12)} = P(X \geq \overset{\alpha}{3}) \quad \text{Poisson}$$

$\alpha = 3$

$\lambda = 0.2$

$t = 12$ min

$$1 - P(X \leq 3-1)$$

$$1 - \text{poissoncdf}(\underset{\lambda t}{2.4}, \underset{\alpha - 1}{2})$$

$$= 0.4303$$

Example: Using the Gamma-Poisson Formula

Compilation of a computer program consists of 3 blocks that are processed sequentially, one after other. Each block takes Exponential time with the mean of 5 minutes per block, independent of other blocks.

(d) Compute the probability that it will take at least $\underbrace{\text{15 minutes}}_{t}$ for the entire program to compile.

Gamma ... Poisson

$\lambda = 0.2$
$t = 15$
$\alpha = 3$

$$P(T \geq 15) = P(X < 3)$$
$$= P(X \leq 2)$$
$$= \text{poissoncdf}(\underset{\lambda t}{3}, 2)$$
$$= 0.4232$$