Problem Set 3.4, page 175

1 $\begin{bmatrix}1&1&1\\0&1&1\\0&0&1\end{bmatrix}\begin{bmatrix}c_1\\c_2\\c_3\end{bmatrix} = 0$ gives $c_3 = c_2 = c_1 = 0$. So those 3 column vectors are independent. But $\begin{bmatrix}1&1&1&2\\0&1&1&3\\0&0&1&4\end{bmatrix}\begin{bmatrix}\\ \boldsymbol{c}\\ \\ \end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ is solved by $\boldsymbol{c} = (1, 1, -4, 1)$. Then $\boldsymbol{v}_1 + \boldsymbol{v}_2 - 4\boldsymbol{v}_3 + \boldsymbol{v}_4 = \boldsymbol{0}$ (dependent).

2 $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ are independent (the -1's are in different positions). All six vectors in $\mathbf{R}^4$ are on the plane $(1, 1, 1, 1) \cdot \boldsymbol{v} = 0$ so no four of these six vectors can be independent.

3 If $a = 0$ then column $1 = \mathbf{0}$; if $d = 0$ then $b(\text{column } 1) - a(\text{column } 2) = \mathbf{0}$; if $f = 0$ then all columns end in zero (they are all in the xy plane, they must be dependent).

4 $U\boldsymbol{x} = \begin{bmatrix}a&b&c\\0&d&e\\0&0&f\end{bmatrix}\begin{bmatrix}x\\y\\z\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ gives $z = 0$ then $y = 0$ then $x = 0$ (by back substitution). A square triangular matrix has independent columns (invertible matrix) *when its diagonal has no zeros*.

5 (a) $\begin{bmatrix}1&2&3\\3&1&2\\2&3&1\end{bmatrix} \to \begin{bmatrix}1&2&3\\0&-5&-7\\0&-1&-5\end{bmatrix} \to \begin{bmatrix}1&2&3\\0&-5&-7\\0&0&-18/5\end{bmatrix}$: invertible $\Rightarrow$ independent columns.

(b) $\begin{bmatrix}1&2&-3\\-3&1&2\\2&-3&1\end{bmatrix} \to \begin{bmatrix}1&2&-3\\0&7&-7\\0&-7&7\end{bmatrix} \to \begin{bmatrix}1&2&-3\\0&7&-7\\0&0&0\end{bmatrix}$; $A\begin{bmatrix}1\\1\\1\end{bmatrix} = \begin{bmatrix}0\\0\\0\end{bmatrix}$ columns add to $\mathbf{0}$.

6 Columns 1, 2, 4 are independent. Also 1, 3, 4 and 2, 3, 4 and others (but not 1, 2, 3). Same column numbers (not same columns!) for A. This is because $EA = U$ for the matrix E that subtracts 2 times row 1 from row 4. So A and U have the same nullspace (same dependencies of columns).

7 The sum $\boldsymbol{v}_1 - \boldsymbol{v}_2 + \boldsymbol{v}_3 = \mathbf{0}$ because $(\boldsymbol{w}_2 - \boldsymbol{w}_3) - (\boldsymbol{w}_1 - \boldsymbol{w}_3) + (\boldsymbol{w}_1 - \boldsymbol{w}_2) = \mathbf{0}$. So the differences are ***dependent*** and the difference matrix is singular: $A = \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & -1 \\ -1 & -1 & 0 \end{bmatrix}$.

8 If $c_1(\boldsymbol{w}_2 + \boldsymbol{w}_3) + c_2(\boldsymbol{w}_1 + \boldsymbol{w}_3) + c_3(\boldsymbol{w}_1 + \boldsymbol{w}_2) = \mathbf{0}$ then $(c_2 + c_3)\boldsymbol{w}_1 + (c_1 + c_3)\boldsymbol{w}_2 + (c_1 + c_2)\boldsymbol{w}_3 = \mathbf{0}$. Since the $\boldsymbol{w}$'s are independent, $c_2 + c_3 = c_1 + c_3 = c_1 + c_2 = 0$. The only solution is $c_1 = c_2 = c_3 = 0$. Only this combination of $\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3$ gives $\mathbf{0}$.

(changing -1's to 1's for the matrix A in solution **7** above makes A invertible.)

9 (a) The four vectors in $\mathbf{R}^3$ are the columns of a 3 by 4 matrix A. There is a nonzero solution to $A\boldsymbol{x} = \mathbf{0}$ because there is at least one free variable (b) Two vectors are dependent if $[\,\boldsymbol{v}_1 \ \ \boldsymbol{v}_2\,]$ has rank 0 or 1. (OK to say "they are on the same line" or "one is a multiple of the other" but ***not*** "$\boldsymbol{v}_2$ is a multiple of $\boldsymbol{v}_1$" —since $\boldsymbol{v}_1$ might be $\mathbf{0}$.) (c) A nontrivial combination of $\boldsymbol{v}_1$ and $\mathbf{0}$ gives $\mathbf{0}$: $0\boldsymbol{v}_1 + 3(0,0,0) = \mathbf{0}$.

10 The plane is the nullspace of $A = [\,1 \ \ 2 \ \ -3 \ \ -1\,]$. Three free variables give three independent solutions $(x, y, z, t) = (2, -1, 0, 0)$ and $(3, 0, 1, 0)$ and $(1, 0, 0, 1)$. Combinations of those special solutions give more solutions (all solutions).

11 (a) Line in $\mathbf{R}^3$ (b) Plane in $\mathbf{R}^3$ (c) All of $\mathbf{R}^3$ (d) All of $\mathbf{R}^3$.

12 $\boldsymbol{b}$ is in the column space when $A\boldsymbol{x} = \boldsymbol{b}$ has a solution; $\boldsymbol{c}$ is in the row space when $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{c}$ has a solution. ***False***. The zero vector is always in the row space.

13 The column space and row space of A and U all have the same dimension = 2. ***The row spaces of A and U are the same***, because the rows of U are combinations of the rows of A (and vice versa!).

14 $\boldsymbol{v} = \frac{1}{2}(\boldsymbol{v} + \boldsymbol{w}) + \frac{1}{2}(\boldsymbol{v} - \boldsymbol{w})$ and $\boldsymbol{w} = \frac{1}{2}(\boldsymbol{v} + \boldsymbol{w}) - \frac{1}{2}(\boldsymbol{v} - \boldsymbol{w})$. The two pairs ***span*** the same space. They are a basis when $\boldsymbol{v}$ and $\boldsymbol{w}$ are ***independent***.

15 The n independent vectors span a space of dimension n. They are a ***basis*** for that space. If they are the columns of A then m is ***not less*** than n $(m \geq n)$. ***Invertible*** if $m = n$.

16 These bases are not unique! (a) $(1,1,1,1)$ for the space of all constant vectors (c,c,c,c) (b) $(1,-1,0,0),(1,0,-1,0),(1,0,0,-1)$ for the space of vectors with sum of components = 0 (c) $(1,-1,-1,0),(1,-1,0,-1)$ for the space perpendicular to $(1,1,0,0)$ and $(1,0,1,1)$ (d) The columns of I are a basis for its column space, the empty set is a basis (by convention) for $\boldsymbol{N}(I)=\boldsymbol{Z}=\{\text{zero vector}\}$.

17 The column space of $U=\begin{bmatrix}1&0&1&0&1\\0&1&0&1&0\end{bmatrix}$ is $\mathbf{R}^2$ so take any bases for $\mathbf{R}^2$; (row 1 and row 2) or (row 1 and row $1+$ row 2) or (row 1 and $-$ row 2) are bases for the row space of U.

18 (a) The 6 vectors ***might not*** span $\mathbf{R}^4$ (b) The 6 vectors ***are not*** independent (c) Any four ***might be*** a basis.

19 n independent columns $\Rightarrow$ rank n. Columns span $\mathbf{R}^m \Rightarrow$ rank m. Columns are basis for $\mathbf{R}^m \Rightarrow$ ***rank*** $=m=n$. The rank counts the number of ***independent*** columns.

20 One basis is $(2,1,0)$, $(-3,0,1)$. A basis for the intersection with the xy plane is $(2,1,0)$. The normal vector $(1,-2,3)$ is a basis for the line perpendicular to the plane.

21 (a) The only solution to $A\boldsymbol{x}=\mathbf{0}$ is $\boldsymbol{x}=\mathbf{0}$ because ***the columns are independent*** (b) $A\boldsymbol{x}=\boldsymbol{b}$ is solvable because ***the columns span*** $\mathbf{R}^5$. Key point: A basis gives exactly one solution for every $\boldsymbol{b}$.

22 (a) True (b) False because the basis vectors for $\mathbf{R}^6$ might not be in $\mathbf{S}$.

23 Columns 1 and 2 are bases for the (**different**) column spaces of A and U; rows 1 and 2 are bases for the (**equal**) row spaces of A and U; $(1,-1,1)$ is a basis for the (**equal**) nullspaces.

24 (a) ***False*** $A=[\,1\ \ 1\,]$ has dependent columns, independent row (b) ***False*** Column space $\neq$ row space for $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ (c) ***True***: Both dimensions $=2$ if A is invertible, dimensions $=0$ if $A=0$, otherwise dimensions $=1$ (d) ***False***, columns may be dependent, in that case not a basis for $\boldsymbol{C}(A)$.

25 A has rank 2 if $c = 0$ and $d = 2$; $B = \begin{bmatrix} c & d \\ d & c \end{bmatrix}$ has rank 2 except when $c = d$ or $c = -d$.

26 (a) Basis for all diagonal matrices: $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$

(b) Add $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ = basis for symmetric matrices.

(c) $\begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix}$.

These are simple bases (among many others) for (a) diagonal matrices (b) symmetric matrices (c) skew-symmetric matrices. The dimensions are $3, 6, 3$.

27 $I, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$;

echelon matrices do ***not*** form a subspace; they ***span*** the upper triangular matrices (not every U is an echelon matrix).

28 $\begin{bmatrix} 1 & 0 & 0 \\ -1 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 1 & 0 \\ 0 & -1 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & -1 \end{bmatrix}; \begin{bmatrix} 1 & -1 & 0 \\ -1 & 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 & -1 \\ -1 & 0 & 1 \end{bmatrix}$.

29 (a) The invertible matrices span the space of all 3 by 3 matrices (b) The rank one matrices also span the space of all 3 by 3 matrices (c) I by itself spans the space of all multiples cI.

30 $\begin{bmatrix} -1 & 2 & 0 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} -1 & 0 & 2 \\ 0 & 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -1 & 2 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 & 0 \\ -1 & 0 & 2 \end{bmatrix}$. **Dimension** $= 4$.

31 (a) $y(x) =$ constant C (b) $y(x) = 3x$. (c) $y(x) = 3x + C = y_p + y_n$ solves $y' = 3$.

32 $y(0) = 0$ requires $A + B + C = 0$. One basis is $\cos x - \cos 2x$ and $\cos x - \cos 3x$.

33 (a) $y(x) = e^{2x}$ is a basis for all solutions to $y' = 2y$ (b) $y = x$ is a basis for all solutions to $dy/dx = y/x$ (First-order linear equation $\Rightarrow$ 1 basis function in solution space).

34 $y_1(x), y_2(x), y_3(x)$ can be $x, 2x, 3x$ (dim 1) or $x, 2x, x^2$ (dim 2) or x, x^2, x^3 (dim 3).

35 Basis 1, x, x^2, x^3, for cubic polynomials; basis $x-1$, x^2-1, x^3-1 for the subspace with $p(1) = 0$.

36 Basis for $\mathbf{S}$: $(1,0,-1,0)$, $(0,1,0,0)$, $(1,0,0,-1)$; basis for $\mathbf{T}$: $(1,-1,0,0)$ and $(0,0,2,1)$; $\mathbf{S} \cap \mathbf{T} =$ multiples of $(3,-3,2,1) =$ nullspace for 3 equations in $\mathbf{R}^4$ has dimension 1.

37 The subspace of matrices that have $AS = SA$ has dimension ***three***. The 3 numbers a, b, c can be chosen independently in A.

38 (a) No, 2 vectors don't span $\mathbf{R}^3$ (b) No, 4 vectors in $\mathbf{R}^3$ are dependent (c) Yes, a basis (d) No, these three vectors are dependent

39 If the 5 by 5 matrix $[\,A \;\; \boldsymbol{b}\,]$ is invertible, $\boldsymbol{b}$ is not a combination of the columns of A: no solution to $A\boldsymbol{x} = \boldsymbol{b}$. If $[\,A \;\; \boldsymbol{b}\,]$ is singular, and the 4 columns of A are independent (rank 4), $\boldsymbol{b}$ ***is*** a combination of those columns. In this case $A\boldsymbol{x} = \boldsymbol{b}$ has a solution.

40 (a) The functions $y = \sin x$, $y = \cos x$, $y = e^x$, $y = e^{-x}$ are a basis for solutions to $d^4y/dx^4 = y(x)$.

(b) A particular solution to $d^4y/dx^4 = y(x) + 1$ is $y(x) = -\mathbf{1}$. The complete solution is $y(x) = -1 + c_1 \sin x + c_2 \cos x + c_3 e^x + c_4 e^{-x}$ (or use another basis for the nullspace of the 4th derivative).

41 $$\boldsymbol{I} = \begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} - \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} + \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} + \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} - \begin{bmatrix} & & 1 \\ 1 & & \\ & 1 & \end{bmatrix}. \quad \text{The six } P\text{'s are dependent}.$$

Those five are independent: The 4th has $P_{11} = 1$ and cannot be a combination of the others. Then the 2nd cannot be (from $P_{32} = 1$) and also 5th ($P_{32} = 1$). Continuing, a nonzero combination of all five could not be zero. Further challenge: How many independent 4 by 4 permutation matrices?

42 The dimension of $\boldsymbol{S}$ spanned by all rearrangements of $\boldsymbol{x}$ is (a) zero when $\boldsymbol{x} = \mathbf{0}$ (b) one when $\boldsymbol{x} = (1, 1, 1, 1)$ (c) three when $\boldsymbol{x} = (1, 1, -1, -1)$ because all rearrangements of this $\boldsymbol{x}$ are perpendicular to $(1, 1, 1, 1)$ (d) four when the $\boldsymbol{x}$'s are not equal and don't add to zero. **No $\boldsymbol{x}$ gives** dim $\boldsymbol{S} = 2$. I owe this nice problem to Mike Artin—the answers are the same in higher dimensions: $0, 1, n-1, n$.

43 The problem is to show that the $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s together are independent. We know the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s together are a basis for $\boldsymbol{V}$, and the $\boldsymbol{u}$'s and $\boldsymbol{w}$'s together are a basis for $\boldsymbol{W}$. Suppose a combination of $\boldsymbol{u}$'s, $\boldsymbol{v}$'s, $\boldsymbol{w}$'s gives $\mathbf{0}$. **To be proved**: All coefficients $=$ zero.

Key idea: In that combination giving $\mathbf{0}$, the part $\boldsymbol{x}$ from the $\boldsymbol{u}$'s and $\boldsymbol{v}$'s is in $\boldsymbol{V}$. So the part from the $\boldsymbol{w}$'s is $-\boldsymbol{x}$. This part is now in $\boldsymbol{V}$ and also in $\boldsymbol{W}$. But if $-\boldsymbol{x}$ is in $\boldsymbol{V} \cap \boldsymbol{W}$ it is a combination of $\boldsymbol{u}$'s only. Now the combination giving $\mathbf{0}$ uses only $\boldsymbol{u}$'s and $\boldsymbol{v}$'s (independent in $\boldsymbol{V}$!) so all coefficients of $\boldsymbol{u}$'s and $\boldsymbol{v}$'s must be zero. Then $\boldsymbol{x} = \mathbf{0}$ and the coefficients of the $\boldsymbol{w}$'s are also zero.

44 The inputs to multiplication by an m by n matrix fill $\mathbf{R}^n$: dimension n. The outputs (column space!) have dimension r. The nullspace has $n - r$ special solutions. The formula becomes $r + (n - r) = n$.

45 If the left side of $\dim(\mathbf{V}) + \dim(\mathbf{W}) = \dim(\mathbf{V} \cap \mathbf{W}) + \dim(\mathbf{V} + \mathbf{W})$ is greater than n, then $\dim(\mathbf{V} \cap \mathbf{W})$ must be greater than zero. So $\mathbf{V} \cap \mathbf{W}$ contains nonzero vectors.

Oh here is a more basic approach: Put a basis for $\mathbf{V}$ and then a basis for $\mathbf{W}$ in the columns of a matrix A. Then A has more columns than rows and there is a nonzero solution to $A\boldsymbol{x} = \mathbf{0}$. That $\boldsymbol{x}$ gives a combination of the $\mathbf{V}$ columns $=$ a combination of the $\mathbf{W}$ columns.

46 If $A^2 =$ zero matrix, this says that each column of A is in the nullspace of A. If the column space has dimension r, the nullspace has dimension $10 - r$. So we must have $r \leq 10 - r$ and this leads to $r \leq 5$.