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1 (a) Row and column space dimensions $= 5$, nullspace dimension $= 4$, $\dim(\boldsymbol{N}(A^{\mathrm{T}}))$ $= 2$ sum $5+5+4+2 = 16 = m+n$

(b) Column space is $\mathbf{R}^3$; left nullspace contains only $\mathbf{0}$.

2 A: Row space basis $=$ row $1 = (1,2,4)$; nullspace $(-2,1,0)$ and $(-4,0,1)$; column space basis $=$ column $1 = (1,2)$; left nullspace $(-2,1)$. B: Row space basis $=$ both rows $= (1,2,4)$ and $(2,5,8)$; column space basis $=$ two columns $= (1,2)$ and $(2,5)$; nullspace $(-4,0,1)$; left nullspace basis is empty because the space contains only $\boldsymbol{y} = \mathbf{0}$: the rows of B are independent.

3 Row space basis $=$ first two rows of U; column space basis $=$ pivot columns (of A not U) $= (1,1,0)$ and $(3,4,1)$; nullspace basis $(1,0,0,0,0)$, $(0,2,-1,0,0)$, $(0,2,0,-2,1)$; left nullspace $(1,-1,1) =$ last row of $E^{-1} = L$.

4 (a) $\begin{bmatrix} 1 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$ (b) Impossible: $r+(n-r)$ must be 3 (c) $\begin{bmatrix} 1 & 1 \end{bmatrix}$ (d) $\begin{bmatrix} 9 & -3 \\ 3 & -1 \end{bmatrix}$

(e) *Impossible* Row space $=$ column space requires $m = n$. Then $m - r = n - r$; nullspaces have the same dimension. Section 4.1 will prove $\boldsymbol{N}(A)$ and $\boldsymbol{N}(A^{\mathrm{T}})$ orthogonal to the row and column spaces respectively—here those are the same space.

5 $A = \begin{bmatrix} 1 & 1 & 1 \\ 2 & 1 & 0 \end{bmatrix}$ has those rows spanning its row space. $B = \begin{bmatrix} 1 & -2 & 1 \end{bmatrix}$ has the same rows spanning its nullspace and $AB^{\mathrm{T}} = 0$.

6 A: dim $\mathbf{2}, \mathbf{2}, \mathbf{2}, \mathbf{1}$: Rows $(0,3,3,3)$ and $(0,1,0,1)$; columns $(3,0,1)$ and $(3,0,0)$; nullspace $(1,0,0,0)$ and $(0,-1,0,1)$; $\boldsymbol{N}(A^{\mathrm{T}})$ $(0,1,0)$. B: dim $\mathbf{1}, \mathbf{1}, \mathbf{0}, \mathbf{2}$ Row space (1), column space $(1,4,5)$, nullspace: empty basis, $\boldsymbol{N}(A^{\mathrm{T}})$ $(-4,1,0)$ and $(-5,0,1)$.

7 Invertible 3 by 3 matrix A: row space basis $=$ column space basis $= (1,0,0)$, $(0,1,0)$, $(0,0,1)$; nullspace basis and left nullspace basis are *empty*. Matrix $B = \begin{bmatrix} A & A \end{bmatrix}$: row space basis $(1,0,0,1,0,0)$, $(0,1,0,0,1,0)$ and $(0,0,1,0,0,1)$; column space basis

$(1,0,0)$, $(0,1,0)$, $(0,0,1)$; nullspace basis $(-1,0,0,1,0,0)$ and $(0,-1,0,0,1,0)$ and $(0,0,-1,0,0,1)$; left nullspace basis is empty.

8 $\begin{bmatrix} I & 0 \end{bmatrix}$ and $\begin{bmatrix} I & I; & 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 \end{bmatrix} = 3$ by 2 have *row space dimensions* $= 3, 3, 0 =$ column space dimensions; ***nullspace dimensions*** $2, 3, 2$; ***left nullspace dimensions*** $0, 2, 3$.

9 (a) Same row space and nullspace. So rank (dimension of row space) is the same

(b) Same column space and left nullspace. Same rank (dimension of column space).

10 For **rand** (3), almost surely rank$=3$, nullspace and left nullspace contain only $(0,0,0)$. For **rand** $(3,5)$ the rank is almost surely 3 and the dimension of the nullspace is 2.

11 (a) No solution means that $r < m$. Always $r \le n$. Can't compare m and n here.

(b) Since $m - r > 0$, the left nullspace must contain a nonzero vector.

12 A neat choice is $\begin{bmatrix} 1 & 1 \\ 0 & 2 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ 1 & 2 & 0 \end{bmatrix} = \begin{bmatrix} 2 & 2 & 1 \\ 2 & 4 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $r + (n - r) = n = 3$ does not match $2 + 2 = 4$. Only $\boldsymbol{v} = \mathbf{0}$ is in both $\boldsymbol{N}(A)$ and $\boldsymbol{C}(A^{\mathrm{T}})$.

13 (a) *False*: Usually row space $\neq$ column space (they do not have the same dimension!)

(b) *True*: A and $-A$ have the same four subspaces

(c) *False* (choose A and B same size and invertible: then they have the same four subspaces)

14 Row space basis can be the nonzero rows of U: $(1,2,3,4)$, $(0,1,2,3)$, $(0,0,1,2)$; nullspace basis $(0,1,-2,1)$ as for U; column space basis $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ (happen to have $\boldsymbol{C}(A) = \boldsymbol{C}(U) = \mathbf{R}^3$); left nullspace has empty basis.

15 After a row exchange, the row space and nullspace stay the same; $(2,1,3,4)$ is in the new left nullspace after the row exchange.

16 If $A\boldsymbol{v} = \mathbf{0}$ and $\boldsymbol{v}$ is a row of A then $\boldsymbol{v} \cdot \boldsymbol{v} = 0$. So $\boldsymbol{v} = \mathbf{0}$.

17 Row space $= yz$ plane; column space $= xy$ plane; nullspace $= x$ axis; left nullspace $= z$ axis. For $I + A$: Row space $=$ column space $= \mathbf{R}^3$, both nullspaces contain only the zero vector.

18 Row $3 - 2$ row $2 +$ row $1 =$ zero row so the vectors $c(1, -2, 1)$ are in the left nullspace. The same vectors happen to be in the nullspace (an accident for this matrix).

19 (a) Elimination on $A\boldsymbol{x} = \mathbf{0}$ leads to $0 = b_3 - b_2 - b_1$ so $(-1, -1, 1)$ is in the left nullspace. (b) 4 by 3: Elimination leads to $b_3 - 2b_1 = 0$ and $b_4 + b_2 - 4b_1 = 0$, so $(-2, 0, 1, 0)$ and $(-4, 1, 0, 1)$ are in the left nullspace. ***Why***? Those vectors multiply the matrix to give ***zero rows*** in $\boldsymbol{v}A$. Section 4.1 will show another approach: $A\boldsymbol{x} = \boldsymbol{b}$ is solvable ($\boldsymbol{b}$ is in $\boldsymbol{C}(A)$) exactly when $\boldsymbol{b}$ is orthogonal to the left nullspace.

20 (a) Special solutions $(-1, 2, 0, 0)$ and $(-\frac{1}{4}, 0, -3, 1)$ are perpendicular to the rows of R (and rows of ER). (b) $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ has 1 independent solution $=$ last row of E^{-1}. ($E^{-1}A = R$ has a zero row, which is just the transpose of $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$).

21 (a) $\boldsymbol{u}$ and $\boldsymbol{w}$ (b) $\boldsymbol{v}$ and $\boldsymbol{z}$ (c) rank < 2 if $\boldsymbol{u}$ and $\boldsymbol{w}$ are dependent or if $\boldsymbol{v}$ and $\boldsymbol{z}$ are dependent (d) The rank of $\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} + \boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}$ is 2.

22 $$A = \begin{bmatrix} & \\ \boldsymbol{u} & \boldsymbol{w} \\ & \end{bmatrix} \begin{bmatrix} \boldsymbol{v}^{\mathrm{T}} \\ \boldsymbol{z}^{\mathrm{T}} \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 2 \\ 4 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 4 & 2 \\ 5 & 1 \end{bmatrix}$$ $\boldsymbol{u}, \boldsymbol{w}$ span column space; $\boldsymbol{v}, \boldsymbol{z}$ span row space

23 As in Problem 22: Row space basis $(3, 0, 3), (1, 1, 2)$; column space basis $(1, 4, 2)$, $(2, 5, 7)$; the rank of (3 by 2) times (2 by 3) cannot be larger than the rank of either factor, so rank ≤ 2 and the 3 by 3 product is not invertible.

24 $A^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{d}$ puts $\boldsymbol{d}$ in the ***row space*** of A; unique solution if the ***left nullspace*** (nullspace of A^{T}) contains only $\boldsymbol{y} = \mathbf{0}$.

25 (a) ***True*** (A and A^{T} have the same rank) (b) ***False*** $A = [\,1 \quad 0\,]$ and A^{T} have very different left nullspaces (c) ***False*** (A can be invertible and unsymmetric even if $\boldsymbol{C}(A) = \boldsymbol{C}(A^{\mathrm{T}})$) (d) ***True*** (The subspaces for A and $-A$ are always the same. If $A^{\mathrm{T}} = A$ or $A^{\mathrm{T}} = -A$ they are also the same for A^{T})

26 Choose $d = bc/a$ to make $\left[\begin{smallmatrix} \mathbf{a} & \mathbf{b} \\ \mathbf{c} & \mathbf{d} \end{smallmatrix}\right]$ a rank-1 matrix. Then the row space has basis (a, b) and the nullspace has basis $(-b, a)$. Those two vectors are perpendicular !

27 B and C (checkers and chess) both have rank 2 if $p \neq 0$. Row 1 and 2 are a basis for the row space of C, $B^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ has 6 special solutions with -1 and 1 separated by a zero;

$\boldsymbol{N}(C^{\mathrm{T}})$ has $(-1,0,0,0,0,0,0,1)$ and $(0,-1,0,0,0,0,1,0)$ and columns $3,4,5,6$ of I; $\boldsymbol{N}(C)$ is a challenge: one vector in $\boldsymbol{N}(C)$ is $(1,0,\ldots,0,-1)$.

28 $a_{11}=1, a_{12}=0, a_{13}=1, a_{22}=0, a_{32}=1, a_{31}=0, a_{23}=1, a_{33}=0, a_{21}=1$. (Need to specify the five moves).

29 The subspaces for $A=\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ are pairs of orthogonal lines ($\boldsymbol{v}$ and $\boldsymbol{v}^{\perp}$, $\boldsymbol{u}$ and $\boldsymbol{u}^{\perp}$). If B has those same four subspaces then $B=cA$ with $c\neq 0$.

30 (a) $AX=0$ if each column of X is a multiple of $(1,1,1)$; $\dim(\text{nullspace})=3$. (b) If $AX=B$ then all columns of B add to zero; dimension of the B's $=6$. (c) $3+6=\dim(M^{3\times 3})=9$ entries in a 3 by 3 matrix.

31 The key is equal row spaces. First row of $A=$ combination of the rows of B: only possible combination (notice I) is 1 (row 1 of B). Same for each row so $F=G$.