matrices is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$. This "alternating sign Pascal matrix" is on page 91.

30 (a) $E = A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ will reduce row 2 of EM to $[2 \;\; 3]$.

(b) Then $F = B^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ will reduce row 1 of FEM to $[1 \;\; 1]$.

(c) Then $E = A^{-1}$ twice will reduce row 2 of $EEFEM$ to $[0 \;\; 1]$

(d) Now $EEFEM = B$. Move E's and F's to get $\boldsymbol{M} = \boldsymbol{ABAAB}$. This question focuses on positive integer matrices M with $ad - bc = 1$. The same steps make the entries smaller and smaller until M is a product of A's and B's.

31 $E_{21} = \begin{bmatrix} 1 & & & \\ a & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & b & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_{43} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & c & 1 \end{bmatrix}$,

$$E_{43}\,E_{32}\,E_{21} = \begin{bmatrix} 1 & & & \\ a & 1 & & \\ ab & b & 1 & \\ abc & bc & c & 1 \end{bmatrix}$$

Problem Set 2.4, page 77

1 If all entries of A, B, C, D are 1, then $BA = 3$ **ones**(5) is 5 by 5; $AB = 5$ **ones**(3) is 3 by 3; $ABD = 15$ **ones**$(3,1)$ is 3 by 1. DC and $A(B+C)$ are not defined.

2 (a) A (column 2 of B) (b) (Row 1 of A) B (c) (Row 3 of A)(column 5 of B)
(d) (Row 1 of C)D(column 1 of E). (Part (c) assumed 5 columns in B)

3 $AB + AC$ is the same as $A(B+C) = \begin{bmatrix} 3 & 8 \\ 6 & 9 \end{bmatrix}$. (*Distributive law*).

4 $A(BC) = (AB)C$ by the *associative law*. In this example both answers are $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$. Column 1 of AB and row 2 of C are zero (then multiply columns times rows).

5 (a) $A^2 = \begin{bmatrix} 1 & 2b \\ 0 & 1 \end{bmatrix}$ and $A^n = \begin{bmatrix} 1 & nb \\ 0 & 1 \end{bmatrix}$. (b) $A^2 = \begin{bmatrix} 4 & 4 \\ 0 & 0 \end{bmatrix}$ and $A^n = \begin{bmatrix} 2^n & 2^n \\ 0 & 0 \end{bmatrix}$.

6 $(A+B)^2 = \begin{bmatrix} 10 & 4 \\ 6 & 6 \end{bmatrix} = A^2 + AB + BA + B^2$. But $A^2 + 2AB + B^2 = \begin{bmatrix} 16 & 2 \\ 3 & 0 \end{bmatrix}$.

7 (a) True (b) False (c) True (d) False: usually $(AB)^2 = ABAB \neq A^2B^2$.

8 The rows of DA are 3 (row 1 of A) and 5 (row 2 of A). Both rows of EA are row 2 of A. The columns of AD are 3 (column 1 of A) and 5 (column 2 of A). The first column of AE is zero, the second is column 1 of A + column 2 of A.

9 $AF = \begin{bmatrix} a & a+b \\ c & c+d \end{bmatrix}$ and $E(AF)$ equals $(EA)F$ because matrix multiplication is *associative*.

10 $FA = \begin{bmatrix} a+c & b+d \\ c & d \end{bmatrix}$ and then $E(FA) = \begin{bmatrix} a+c & b+d \\ a+2c & b+2d \end{bmatrix}$. $E(FA)$ is not the same as $F(EA)$ because multiplication is not commutative: $EF \neq FE$.

11 Suppose EA does the row operation and then $(EA)F$ does the column operation (because F is multiplying from the right). The associative law says that $(EA)F = E(AF)$ so the column operation can be done first !

12 (a) $B = 4I$ (b) $B = 0$ (c) $B = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ (d) Every row of B is $1, 0, 0$.

13 $AB = \begin{bmatrix} a & 0 \\ c & 0 \end{bmatrix} = BA = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix}$ gives $\boldsymbol{b} = \boldsymbol{c} = \mathbf{0}$. Then $AC = CA$ gives $\boldsymbol{a} = \boldsymbol{d}$. The only matrices that commute with B and C (and all other matrices) are multiples of I: $A = aI$.

14 $(A-B)^2 = (B-A)^2 = A(A-B) - B(A-B) = A^2 - AB - BA + B^2$. In a typical case (when $AB \neq BA$) the matrix $A^2 - 2AB + B^2$ is different from $(A-B)^2$.

15 (a) True (A^2 is only defined when A is square).

(b) False (if A is m by n and B is n by m, then AB is m by m and BA is n by n).

(c) True by part (b).

(d) False (take $B = 0$).

16 (a) mn (use every entry of A) (b) $mnp = p\times$part (a) (c) n^3 (n^2 dot products).

17 (a) Use only column 2 of B (b) Use only row 2 of A (c)–(d) Use row 2 of first A.

Column 2 of $AB = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ Row 2 of $AB = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$ Row 2 of $A^2 = \begin{bmatrix} 0 & 1 \end{bmatrix}$

Row 2 of $A^3 = \begin{bmatrix} 3 & -2 \end{bmatrix}$

18 $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 2 \\ 1 & 2 & 3 \end{bmatrix}$ has $a_{ij} = \min(i,j)$. $A = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 1 & -1 \\ 1 & -1 & 1 \end{bmatrix}$ has $a_{ij} = (-1)^{i+j} =$ "alternating sign matrix". $A = \begin{bmatrix} 1/1 & 1/2 & 1/3 \\ 2/1 & 2/2 & 2/3 \\ 3/1 & 3/2 & 3/3 \end{bmatrix}$ has $a_{ij} = i/j$. This will be an example of a ***rank one matrix***: 1 column $\begin{bmatrix} 1 & 2 & 3 \end{bmatrix}^{\mathrm{T}}$ multiplies 1 row $\begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{3} \end{bmatrix}$.

19 Diagonal matrix, lower triangular, symmetric, all rows equal. Zero matrix fits all four.

20 (a) a_{11} (b) $\ell_{31} = a_{31}/a_{11}$ (c) $a_{32} - \left(\dfrac{a_{31}}{a_{11}}\right) a_{12}$ (d) $a_{22} - \left(\dfrac{a_{21}}{a_{11}}\right) a_{12}$.

21 $A^2 = \begin{bmatrix} 0&0&4&0 \\ 0&0&0&4 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$, $A^3 = \begin{bmatrix} 0&0&0&8 \\ 0&0&0&0 \\ 0&0&0&0 \\ 0&0&0&0 \end{bmatrix}$, $A^4 =$ zero matrix for *strictly triangular* A.

Then $A\boldsymbol{v} = A \begin{bmatrix} x \\ y \\ z \\ t \end{bmatrix} = \begin{bmatrix} 2y \\ 2z \\ 2t \\ 0 \end{bmatrix}$, $A^2\boldsymbol{v} = \begin{bmatrix} 4z \\ 4t \\ 0 \\ 0 \end{bmatrix}$, $A^3\boldsymbol{v} = \begin{bmatrix} 8t \\ 0 \\ 0 \\ 0 \end{bmatrix}$, $A^4\boldsymbol{v} = \mathbf{0}$.

22 $A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ has $A^2 = -I$; $BC = \begin{bmatrix} 1 & -1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$;

$DE = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = -ED$. You can find more examples.

23 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $A^2 = 0$. Note: Any matrix $A =$ column times row $= \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}$ will have $A^2 = \boldsymbol{u}\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}} = 0$ if $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{u} = 0$. $A = \begin{bmatrix} 0&1&0 \\ 0&0&1 \\ 0&0&0 \end{bmatrix}$ has $A^2 = \begin{bmatrix} 0&0&1 \\ 0&0&0 \\ 0&0&0 \end{bmatrix}$ but $A^3 = 0$; strictly triangular as in Problem 21.

24 $(A_1)^n = \begin{bmatrix} 2^n & 2^n - 1 \\ 0 & 1 \end{bmatrix}$, $(A_2)^n = 2^{n-1} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $(A_3)^n = \begin{bmatrix} a^n & a^{n-1}b \\ 0 & 0 \end{bmatrix}$.

25 $$\begin{bmatrix} a&b&c \\ d&e&f \\ g&h&i \end{bmatrix} \begin{bmatrix} 1&0&0 \\ 0&1&0 \\ 0&0&1 \end{bmatrix} = \begin{bmatrix} a \\ d \\ g \end{bmatrix} \begin{bmatrix} 1&0&0 \end{bmatrix} + \begin{bmatrix} d \\ e \\ h \end{bmatrix} \begin{bmatrix} 0&1&0 \end{bmatrix} + \begin{bmatrix} c \\ f \\ i \end{bmatrix} \begin{bmatrix} 0&0&1 \end{bmatrix}.$$

26 **Columns of A times rows of B** $\begin{bmatrix}1\\2\\2\end{bmatrix}\begin{bmatrix}3 & 3 & 0\end{bmatrix}+\begin{bmatrix}0\\4\\1\end{bmatrix}\begin{bmatrix}1 & 2 & 1\end{bmatrix}=\begin{bmatrix}3 & 3 & 0\\6 & 6 & 0\\6 & 6 & 0\end{bmatrix}+\begin{bmatrix}0 & 0 & 0\\4 & 8 & 4\\1 & 2 & 1\end{bmatrix}=$
$\begin{bmatrix}3 & 3 & 0\\10 & 14 & 4\\7 & 8 & 1\end{bmatrix}=AB.$

27 (a) (row 3 of A) $\cdot$ (column 1 or 2 of B) and (row 3 of A) $\cdot$ (column 2 of B) are all zero.

(b) $\begin{bmatrix}x\\x\\0\end{bmatrix}\begin{bmatrix}0 & x & x\end{bmatrix}=\begin{bmatrix}0 & x & x\\0 & x & x\\0 & 0 & 0\end{bmatrix}$ and $\begin{bmatrix}x\\x\\x\end{bmatrix}\begin{bmatrix}0 & 0 & x\end{bmatrix}=\begin{bmatrix}0 & 0 & x\\0 & 0 & x\\0 & 0 & x\end{bmatrix}$: **both upper**.

28 A times B with cuts $A\left[\begin{array}{c|c|c|c} & & & \end{array}\right]$, $\left[\begin{array}{c}\\ \hline \\ \end{array}\right]B$, $\left[\begin{array}{c}\\ \hline \\ \end{array}\right]\left[\begin{array}{c|c|c|c} & & & \end{array}\right]$, $\left[\begin{array}{c|c|c} & & \end{array}\right]\left[\begin{array}{c}\\ \hline \\ \hline \\ \end{array}\right]$

4 cols 2 rows 2 rows – 4 cols 3 cols – 3 rows

29 $E_{21}=\begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\0 & 0 & 1\end{bmatrix}$ and $E_{31}=\begin{bmatrix}1 & 0 & 0\\0 & 1 & 0\\-4 & 0 & 1\end{bmatrix}$ produce zeros in the $2,1$ and $3,1$ entries.

Multiply E's to get $E=E_{31}E_{21}=\begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\-4 & 0 & 1\end{bmatrix}$. Then $EA=\begin{bmatrix}2 & 1 & 0\\0 & 1 & 1\\0 & 1 & 3\end{bmatrix}$ is the result of both E's since $(E_{31}E_{21})A=E_{31}(E_{21}A)$.

30 In **29**, $\boldsymbol{c}=\begin{bmatrix}-2\\8\end{bmatrix}$, $D=\begin{bmatrix}0 & 1\\5 & 3\end{bmatrix}$, $D-\boldsymbol{c}\boldsymbol{b}/a=\begin{bmatrix}1 & 1\\1 & 3\end{bmatrix}$ in the lower corner of EA.

31 $\begin{bmatrix}A & -B\\B & A\end{bmatrix}\begin{bmatrix}\boldsymbol{x}\\\boldsymbol{y}\end{bmatrix}=\begin{bmatrix}A\boldsymbol{x}-B\boldsymbol{y}\\B\boldsymbol{x}+A\boldsymbol{y}\end{bmatrix}$ real part / imaginary part. Complex matrix times complex vector needs **4** real times real multiplications.

32 A times $X=[\,\boldsymbol{x}_1\ \ \boldsymbol{x}_2\ \ \boldsymbol{x}_3\,]$ will be the identity matrix $I=[\,A\boldsymbol{x}_1\ \ A\boldsymbol{x}_2\ \ A\boldsymbol{x}_3\,]$.

33 $\boldsymbol{b} = \begin{bmatrix} 3 \\ 5 \\ 8 \end{bmatrix}$ gives $\boldsymbol{x} = 3\boldsymbol{x}_1 + 5\boldsymbol{x}_2 + 8\boldsymbol{x}_3 = \begin{bmatrix} 3 \\ 8 \\ 16 \end{bmatrix}$; $A = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}$ will have those $\boldsymbol{x}_1 = (1,1,1), \boldsymbol{x}_2 = (0,1,1), \boldsymbol{x}_3 = (0,0,1)$ as columns of its "inverse" A^{-1}.

34 $A*$ **ones** $= \begin{bmatrix} a+b & a+b \\ c+d & c+d \end{bmatrix}$ agrees with **ones** $*A = \begin{bmatrix} a+c & b+b \\ a+c & b+d \end{bmatrix}$ when $b = c$ and $a = d$

Then $A = \begin{bmatrix} a & b \\ b & a \end{bmatrix}$. These are the matrices that commute with $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

35 $S = \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$, $S^2 = \begin{bmatrix} 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \\ 2 & 0 & 2 & 0 \\ 0 & 2 & 0 & 2 \end{bmatrix}$,

aba, ada	**cba, cda**	These show
bab, bcb	**dab, dcb**	16 2-step
abc, adc	**cbc, cdc**	paths in
bad, bcd	**dad, dcd**	the graph

36 Multiplying AB =(m by n)(n by p) needs mnp multiplications. Then $(AB)C$ needs mpq more. Multiply $BC =$ (n by p)(p by q) needs npq and then $A(BC)$ needs mnq.

(a) If m, n, p, q are $2, 4, 7, 10$ we compare $(2)(4)(7) + (2)(7)(10) = \mathbf{196}$ with the larger number $(2)(4)(10) + (4)(7)(10) = \mathbf{360}$. So AB first is better, we want to multiply that 7 by 10 matrix by as few rows as possible.

(b) If $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ are N by 1, then $(\boldsymbol{u}^{\mathrm{T}}\boldsymbol{v})\boldsymbol{w}^{\mathrm{T}}$ needs $2N$ multiplications but $\boldsymbol{u}^{\mathrm{T}}(\boldsymbol{v}\boldsymbol{w}^{\mathrm{T}})$ needs N^2 to find $\boldsymbol{v}\boldsymbol{w}^{\mathrm{T}}$ and N^2 more to multiply by the row vector $\boldsymbol{u}^{\mathrm{T}}$. Apologies to use the transpose symbol so early.

(c) We are comparing $mnp + mpq$ with $mnq + npq$. Divide all terms by $mnpq$: Now we are comparing $q^{-1} + n^{-1}$ with $p^{-1} + m^{-1}$. This yields a simple important rule. If matrices A and B are multiplying $\boldsymbol{v}$ for $AB\boldsymbol{v}$, **don't multiply the matrices first**. Better to multiply $B\boldsymbol{v}$ and then $A(B\boldsymbol{v})$.

37 The proof of $(AB)\boldsymbol{c} = A(B\boldsymbol{c})$ used the column rule for matrix multiplication. "The same is true for all other *columns* of C."

Even for nonlinear transformations, $A(B(\boldsymbol{c}))$ would be the ***"composition"*** of A with B (applying B then A). This composition $A \circ B$ is just written as AB for matrices.

One of many uses for the associative law: The left-inverse $B =$ the right-inverse C because $B = B(AC) = (BA)C = C$.

38 (a) Multiply the columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ by the rows $\boldsymbol{a}_1^{\mathrm{T}}, \ldots, \boldsymbol{a}_m^{\mathrm{T}}$ and add the resulting matrices.

(b) $A^{\mathrm{T}}CA = c_1\boldsymbol{a}_1\boldsymbol{a}_1^{\mathrm{T}} + \cdots + c_m\boldsymbol{a}_m\boldsymbol{a}_m^{\mathrm{T}}$. Diagonal C makes it neat.

Problem Set 2.5, page 92

1 $A^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{3} & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$ and $C^{-1} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$.

2 For the first, a simple row exchange has $P^2 = I$ so $P^{-1} = P$. For the second, $P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Always $P^{-1} =$ "transpose" of P, coming in Section 2.7.

3 $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} .5 \\ -.2 \end{bmatrix}$ and $\begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} -.2 \\ .1 \end{bmatrix}$ so $A^{-1} = \frac{1}{10}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$. This question solved $AA^{-1} = I$ column by column, the main idea of Gauss-Jordan elimination. For a different matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, you could find a first column for A^{-1} but not a second column—so A would be singular (*no inverse*).

4 The equations are $x + 2y = 1$ and $3x + 6y = 0$. No solution because 3 times equation 1 gives $3x + 6y = 3$.