37 The proof of $(AB)\boldsymbol{c} = A(B\boldsymbol{c})$ used the column rule for matrix multiplication. "The same is true for all other *columns* of C."

Even for nonlinear transformations, $A(B(\boldsymbol{c}))$ would be the ***"composition"*** of A with B (applying B then A). This composition $A \circ B$ is just written as AB for matrices.

One of many uses for the associative law: The left-inverse $B =$ the right-inverse C because $B = B(AC) = (BA)C = C$.

38 (a) Multiply the columns $\boldsymbol{a}_1, \ldots, \boldsymbol{a}_m$ by the rows $\boldsymbol{a}_1^{\mathrm{T}}, \ldots, \boldsymbol{a}_m^{\mathrm{T}}$ and add the resulting matrices.

(b) $A^{\mathrm{T}}CA = c_1\boldsymbol{a}_1\boldsymbol{a}_1^{\mathrm{T}} + \cdots + c_m\boldsymbol{a}_m\boldsymbol{a}_m^{\mathrm{T}}$. Diagonal C makes it neat.

Problem Set 2.5, page 92

1 $A^{-1} = \begin{bmatrix} 0 & \frac{1}{4} \\ \frac{1}{3} & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} \frac{1}{2} & 0 \\ -1 & \frac{1}{2} \end{bmatrix}$ and $C^{-1} = \begin{bmatrix} 7 & -4 \\ -5 & 3 \end{bmatrix}$.

2 For the first, a simple row exchange has $P^2 = I$ so $P^{-1} = P$. For the second, $P^{-1} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$. Always $P^{-1} =$ "transpose" of P, coming in Section 2.7.

3 $\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} .5 \\ -.2 \end{bmatrix}$ and $\begin{bmatrix} t \\ z \end{bmatrix} = \begin{bmatrix} -.2 \\ .1 \end{bmatrix}$ so $A^{-1} = \frac{1}{10}\begin{bmatrix} 5 & -2 \\ -2 & 1 \end{bmatrix}$. This question solved $AA^{-1} = I$ column by column, the main idea of Gauss-Jordan elimination. For a different matrix $A = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, you could find a first column for A^{-1} but not a second column—so A would be singular (*no inverse*).

4 The equations are $x + 2y = 1$ and $3x + 6y = 0$. No solution because 3 times equation 1 gives $3x + 6y = 3$.

5 An upper triangular U with $U^2 = I$ is $U = \begin{bmatrix} 1 & a \\ 0 & -1 \end{bmatrix}$ for any a. And also $-U$.

6 (a) Multiply $AB = AC$ by A^{-1} to find $B = C$ (since A is invertible) (b) As long as $B - C$ has the form $\begin{bmatrix} x & y \\ -x & -y \end{bmatrix}$, we have $AB = AC$ for $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$.

7 (a) In $A\boldsymbol{x} = (1, 0, 0)$, equation 1 + equation 2 − equation 3 is $0 = 1$ (b) Right sides must satisfy $b_1 + b_2 = b_3$ (c) Row 3 becomes a row of zeros—no third pivot.

8 (a) The vector $\boldsymbol{x} = (1, 1, -1)$ solves $A\boldsymbol{x} = \mathbf{0}$ (b) After elimination, columns 1 and 2 end in zeros. Then so does column $3 =$ column $1 + 2$: no third pivot.

9 Yes, B is invertible (A was just multiplied by a permutation matrix P). If you exchange rows 1 and 2 of A to reach B, you exchange **columns** 1 and 2 of A^{-1} to reach B^{-1}. In matrix notation, $B = PA$ has $B^{-1} = A^{-1}P^{-1} = A^{-1}P$ for this P.

10 $A^{-1} = \begin{bmatrix} 0 & 0 & 0 & 1/5 \\ 0 & 0 & 1/4 & 0 \\ 0 & 1/3 & 0 & 0 \\ 1/2 & 0 & 0 & 0 \end{bmatrix}$ and $B^{-1} = \begin{bmatrix} 3 & -2 & 0 & 0 \\ -4 & 3 & 0 & 0 \\ 0 & 0 & 6 & -5 \\ 0 & 0 & -7 & 6 \end{bmatrix}$ (invert each block of B)

11 (a) If $B = -A$ then certainly $A + B$ = zero matrix is not invertible.

(b) $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$ are both singular but $A + B = I$ is invertible.

12 Multiply $C = AB$ on the left by A^{-1} and on the right by C^{-1}. Then $A^{-1} = BC^{-1}$.

13 $M^{-1} = C^{-1}B^{-1}A^{-1}$ so multiply on the left by C and the right by A : $B^{-1} = CM^{-1}A$.

14 $B^{-1} = A^{-1}\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}^{-1} = A^{-1}\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$: subtract *column* 2 of A^{-1} from *column* 1.

15 If A has a column of zeros, so does BA. Then $BA = I$ is impossible. There is no A^{-1}.

16 $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$. The inverse of each matrix is the other divided by $ad - bc$

17 $E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ & 1 & \\ -1 & & 1 \end{bmatrix} \begin{bmatrix} 1 & & \\ -1 & 1 & \\ & & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -1 & 1 & \\ 0 & -1 & 1 \end{bmatrix} = E.$

Reverse the order and change -1 to $+1$ to get inverses $E_{21}^{-1}E_{31}^{-1}E_{32}^{-1} = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 1 & 1 & 1 \end{bmatrix} =$

$L = E^{-1}$. Notice that the 1's are unchanged by multiplying inverses in this order.

18 $A^2B = I$ can also be written as $A(AB) = I$. Therefore A^{-1} is AB.

19 The $(1, 1)$ entry requires $4a - 3b = 1$; the $(1, 2)$ entry requires $2b - a = 0$. Then $b = \frac{1}{5}$ and $a = \frac{2}{5}$. For the 5 by 5 case $5a - 4b = 1$ and $2b = a$ give $b = \frac{1}{6}$ and $a = \frac{2}{6}$.

20 $A * \mathsf{ones}(4, 1) = A$ (column of 1's) is the zero vector so A cannot be invertible.

21 Six of the sixteen $0 - 1$ matrices are invertible: I and P and all four with three 1's.

22 $\begin{bmatrix} 1 & 3 & 1 & 0 \\ 2 & 7 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 3 & 1 & 0 \\ 0 & 1 & -2 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 7 & -3 \\ 0 & 1 & -2 & 1 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix};$

$\begin{bmatrix} 1 & 4 & 1 & 0 \\ 3 & 9 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 4 & 1 & 0 \\ 0 & -3 & -3 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -3 & 4/3 \\ 0 & 1 & 1 & -1/3 \end{bmatrix} = \begin{bmatrix} I & A^{-1} \end{bmatrix}.$

23 $[\boldsymbol{A} \ \ \boldsymbol{I}] = \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 1 & 2 & 1 & 0 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 1 & 2 & 0 & 0 & 1 \end{array}\right] \to$

$\left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 1 & -1/2 & 1 & 0 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 2 & 1 & 0 & 1 & 0 & 0 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & -3/4 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array}\right] \to$

$$\left[\begin{array}{ccc|ccc} 2 & 0 & 0 & 3/2 & -1 & 1/2 \\ 0 & 3/2 & 0 & -3/4 & 3/2 & -3/4 \\ 0 & 0 & 4/3 & 1/3 & -2/3 & 1 \end{array}\right] \to \left[\begin{array}{ccc|ccc} 1 & 0 & 0 & 3/4 & -1/2 & 1/4 \\ 0 & 1 & 0 & -1/2 & 1 & -1/2 \\ 0 & 0 & 1 & 1/4 & -1/2 & 3/4 \end{array}\right] =$$

$[\boldsymbol{I}\ \boldsymbol{A}^{-1}]$.

24 $$\begin{bmatrix} 1 & a & b & 1 & 0 & 0 \\ 0 & 1 & c & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & a & 0 & 1 & 0 & -b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 & 1 & -a & ac-b \\ 0 & 1 & 0 & 0 & 1 & -c \\ 0 & 0 & 1 & 0 & 0 & 1 \end{bmatrix}.$$

25 $$\begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \\ 1 & 1 & 2 \end{bmatrix}^{-\mathbf{1}} = \frac{1}{4}\begin{bmatrix} 3 & -1 & -1 \\ -1 & 3 & -1 \\ -1 & -1 & 3 \end{bmatrix};\ B\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 & -1 & -1 \\ -1 & 2 & -1 \\ -1 & -1 & 2 \end{bmatrix}\begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

so B^{-1} does not exist.

26 $E_{21}A = \begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 \\ 2 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix}$. $E_{12}E_{21}A = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 \\ -2 & 1 \end{bmatrix} A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}$.

Multiply by $D = \begin{bmatrix} 1 & 0 \\ 0 & 1/2 \end{bmatrix}$ to reach $DE_{12}E_{21}A = I$. Then $A^{-1} = DE_{12}E_{21} = \frac{1}{2}\begin{bmatrix} 6 & -2 \\ -2 & 1 \end{bmatrix}$.

27 $A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & -3 \\ 0 & 0 & 1 \end{bmatrix}$ (notice the sign changes); $A^{-1} = \begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 1 \end{bmatrix}$.

28 $$\begin{bmatrix} 0 & 2 & 1 & 0 \\ 2 & 2 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 2 & 2 & 0 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 2 & 0 & -1 & 1 \\ 0 & 2 & 1 & 0 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & -1/2 & 1/2 \\ 0 & 1 & 1/2 & 0 \end{bmatrix}.$$

This is $\begin{bmatrix} I & A^{-1} \end{bmatrix}$: row exchanges are certainly allowed in Gauss-Jordan.

29 (a) True (If A has a row of zeros, then every AB has too, and $AB = I$ is impossible).

(b) False (the matrix of all ones is singular even with diagonal 1's.

(c) True (the inverse of A^{-1} is A and the inverse of A^2 is $(A^{-1})^2$).

30 Elimination produces the pivots a and $a-b$ and $a-b$. $A^{-1} = \dfrac{1}{a(a-b)}\begin{bmatrix} a & 0 & -b \\ -a & a & 0 \\ 0 & -a & a \end{bmatrix}$.

The matrix C is not invertible if $c = 0$ or $c = 7$ or $c = 2$.

31 $A^{-1} = \begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{bmatrix}$ and $\boldsymbol{x} = A^{-1}\begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 2 \\ 1 \end{bmatrix}$. When the triangular A alternates 1 and -1 on its diagonals, A^{-1} has 1's on the diagonal and first superdiagonal.

32 $\boldsymbol{x} = (1, 1, \ldots, 1)$ has $\boldsymbol{x} = P\boldsymbol{x} = Q\boldsymbol{x}$ so $(P - Q)\boldsymbol{x} = \mathbf{0}$. Permutations do not change this all-ones vector.

33 $\begin{bmatrix} I & 0 \\ -C & I \end{bmatrix}$ and $\begin{bmatrix} A^{-1} & 0 \\ -D^{-1}CA^{-1} & D^{-1} \end{bmatrix}$ and $\begin{bmatrix} -D & I \\ I & 0 \end{bmatrix}$.

34 A can be invertible with diagonal zeros (example to find). B is singular because each row adds to zero. The all-ones vector $\boldsymbol{x}$ has $B\boldsymbol{x} = \mathbf{0}$.

35 The equation $LDLD = I$ says that $LD =$ pascal $(4, 1)$ is its own inverse.

36 hilb(6) is not the exact Hilbert matrix because fractions are rounded off. So inv(hilb(6)) is not the exact inverse either.

37 The three Pascal matrices have $P = LU = LL^{\mathrm{T}}$ and then inv$(P) =$ inv$(L^{\mathrm{T}})*$inv(L).

38 $A\boldsymbol{x} = \boldsymbol{b}$ has many solutions when $A =$ ones $(4, 4) =$ ***singular*** and $\boldsymbol{b} =$ ones $(4, 1)$. $A\backslash\boldsymbol{b}$ in MATLAB will pick the shortest solution $\boldsymbol{x} = (1, 1, 1, 1)/4$. This is the only solution that is a combination of the rows of A (later it comes from the "pseudoinverse" $A^{+} =$ pinv(A) which replaces A^{-1} when A is singular). Any vector that solves $A\boldsymbol{x} = \mathbf{0}$ could be added to this particular solution $\boldsymbol{x}$.

39 The inverse of $A = \begin{bmatrix} 1 & -a & 0 & 0 \\ 0 & 1 & -b & 0 \\ 0 & 0 & 1 & -c \\ 0 & 0 & 0 & 1 \end{bmatrix}$ is $A^{-1} = \begin{bmatrix} 1 & a & ab & abc \\ 0 & 1 & b & bc \\ 0 & 0 & 1 & c \\ 0 & 0 & 0 & 1 \end{bmatrix}$. (This would be a good example for the cofactor formula $A^{-1} = C^{\mathrm{T}} / \det A$ in Section 5.3)

40 $$\begin{bmatrix} 1 & & & \\ a & 1 & & \\ b & 0 & 1 & \\ c & 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & d & 1 & \\ 0 & e & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & 1 & \\ & & f & 1 \end{bmatrix} = \begin{bmatrix} 1 & & & \\ a & 1 & & \\ b & d & 1 & \\ c & e & f & 1 \end{bmatrix}$$

In this order the multipliers a, b, c, d, e, f are unchanged in the product (**important for** $A = LU$ **in Section 2.6**).

41 4 by 4 still with $T_{11} = 1$ has pivots $1, 1, 1, 1$; reversing to $T^* = UL$ makes $T^*_{44} = 1$.

$$T = \begin{bmatrix} 1 & -1 & 0 & 0 \\ -1 & 2 & -1 & 0 \\ 0 & -1 & 2 & -1 \\ 0 & 0 & -1 & 2 \end{bmatrix} \quad \text{and} \quad T^{-1} = \begin{bmatrix} 4 & 3 & 2 & 1 \\ 3 & 3 & 2 & 1 \\ 2 & 2 & 2 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

42 Add the equations $C\boldsymbol{x} = \boldsymbol{b}$ to find $0 = b_1 + b_2 + b_3 + b_4$. So C is singular. Same for $F\boldsymbol{x} = \boldsymbol{b}$.

43 The block pivots are A and $S = D - CA^{-1}B$ (and $d - cb/a$ is the correct second pivot of an ordinary 2 by 2 matrix). The example problem has Schur complement $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} - \begin{bmatrix} 4 \\ 4 \end{bmatrix} \frac{1}{2} \begin{bmatrix} 3 & 3 \end{bmatrix} = \begin{bmatrix} -5 & -6 \\ -6 & -5 \end{bmatrix}$.

44 Inverting the identity $A(I + BA) = (I + AB)A$ gives $(I + BA)^{-1}A^{-1} = A^{-1}(I + AB)^{-1}$. So $I + BA$ and $I + AB$ are both invertible or both singular when A is invertible. (This remains true also when A is singular : Chapter 6 will show that AB and BA have the same nonzero eigenvalues, and we are looking here at the eigenvalue -1.)