Intro to Probability
Chapter 2

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Probability and Statistics

There's an 80% chance of 0.1 to 0.3 inches of rain today between 2 PM and 4 PM.

- Probability: 80%
- Statistics:
  1. 0.1-0.3 inches of rain
  2. 2 PM to 4 PM when rain may occur
- Probability and statistics are not separate topics.
  - Probability measures the uncertainty of our estimates, which also estimates the likelihood of the event itself.
  - Statistics are numerical summaries that estimate a random event such as weather.
  - Estimates have variability because the true outcome is usually unknown.

Basics: What is Probability?

Definition: Probability

The measure of the likelihood or chance of a random event.

- Random events are denoted using capital letters such as A, B, C, D, E, ...
- The probability of event E is denoted as $P(E)$
- In general,

$$P(E) = \frac{\text{number of ways E can occur}}{\text{number of possible outcomes}}$$

Examples

- Given a 6-sided die, what is the probability of rolling a number less than 3?

$$P(< 3) = \frac{2}{6} = \frac{1}{3}$$

- Given that I toss a fair coin once, what is the probability of tails?

$$P(tails) = \frac{1}{2}$$

Examples

- From a survey of 4776 college students who were asked "How often do you wear a seatbelt?"

Response	Frequency
Never	125
Rarely	324
Sometimes	552
Most of the time	1257
Always	2518

What is the probability that a randomly select student always wears a seatbelt?

$$P(always) = \frac{2518}{4776} \approx 0.527$$

Examples

Note, we can convert this table of observed responses from the survey to a probability distribution by dividing each frequency by the total number of students:

Response	Probability
Never	0.026
Rarely	0.068
Sometimes	0.116
Most of the time	0.263
Always	0.527

Sample Space

Definition: Sample Space

The sample space (denoted by Ω) is the collection of all possible outcomes of an experiment.

- $N(\Omega)$ = the number of possible outcomes in Ω.
- Then, the probability of event E can be rewritten as

$$P(E) = \frac{N(E)}{N(\Omega)}$$

- $P(\Omega) = 1$. In this case, the event $E = \Omega$

$$\implies P(E) = \frac{N(E)}{N(\Omega)} = \frac{N(\Omega)}{N(\Omega)} = 1$$

Examples

1. A six-sided die:

$$\Omega = \{1, 2, 3, 4, 5, 6\}$$

2. One coin:

$$\Omega = \{H, T\}$$

3. Two coins:

$$\Omega = \{(H, H), (H, T), (T, H), (T, T)\}$$

4. COVID test results:

$\Omega = \{$True positive, false positive, true negative, false negative$\}$

Empty Set

Definition: Empty Set

The empty or null set (denoted as $\varnothing$) indicates that an event E is impossible.

- If E is impossible, we write $E = \{\varnothing\}$.
- The probability of an impossible event (the empty set) is zero:

$$P(E) = P(\varnothing) = 0$$

- Example: You roll a six-sided die. What is the probability of rolling a 7?

$$P(7) = P(\varnothing) = 0$$

Classifying Probability

Probability can be classified based on its value. Generally,

Probability	Decision
$= 0$	Impossible
< 0.05	Unusual
Between 0.05 and 0.50	Less likely
$= 0.50$	Fair
Between 0.50 and 1	More likely
$= 1$	Certain

Types of Probability: Empirical vs. Theoretical

1. Empirical probability (EP)
   - What is observed
   - Data collection
   - Estimation of some unknown truth
2. Theoretical probability (TP)
   - What is expected
   - Calculated using mathematical reasoning or computation

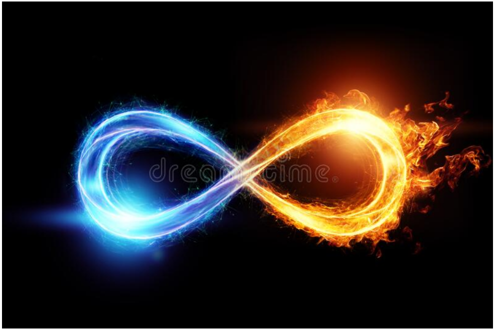
Law of Large Numbers

If I collect a sufficient amount of data, then the observed should be a reasonable estimate of the expected. In other words,

$$EP \to TP \text{ as } n \to \infty$$

where n is the number of data points.

Coin Toss Simulator



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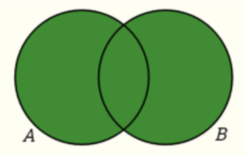
Union of Joint Events

Definition 1: Union of Joint Events

The union of two joint events A and B (denoted as $A \cup B$) is the event that occurs if either A or B or both occur on a single measurement.

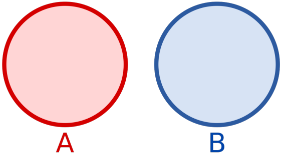


1. $A \cup B$ is everything shaded in green.
2. $A \cup B =$ "A or B".
3. Two events are joint if they intersect or overlap. The intersection represents the occurrence of both A and B on a single measurement.

Union of Disjoint Events

Definition 2: Union of Disjoint Events

The union of two disjoint events A and B is the event that occurs if either A or B occur on a single measurement.



- $A \cup B$ is everything shaded in red and blue.
- Two events are disjoint if they do not intersect or overlap. So, A and B cannot both occur on a single measurement.
- Disjoint events are also called "mutually exclusive" events
- In general for unions, E_1 or E_2 or ... or E_n is equivalent to $E_1 \cup E_2 \cup \ldots \cup E_n$.

Example

Let $A =$ vehicle with two doors, $B =$ red vehicle, $C =$ pickup truck, $D =$ sports car

- $A \cup B$
  1. The event that a vehicle has two doors or is red
  2. Can the vehicle be both A and B? Yes.
  3. A and B are joint.
- $C \cup D$
  1. The event that a vehicle is a pickup truck or a sports car.
  2. Can the vehicle be both C and D? No.
  3. C and D are disjoint.

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Defining a Probability Measure

For any event $E \in \Omega$, the following are the Axioms of Probability:

1. Probability is non-negative.

$$0 \leq P(E) \leq 1$$

2. The probability of the sample space is $P(\Omega) = 1$.
3. For any n disjoint events,
$P(E_1 \cup E_2 \cup \ldots \cup E_n) = P(E_1) + P(E_2) + \ldots + P(E_n)$
which can be condensed as

$$P\left(\cup_{i=1}^{n} E_i\right) = \sum_{i=1}^{n} P(E_i)$$

Example

In a large bag of M&M's, the observed proportion of colors is given in the table below.

Color	Proportion
Brown	0.13
Yellow	0.14
Red	0.13
Blue	0.24
Orange	0.20
Green	0.16

Table: M&M's

1. Axiom #1: All probabilities are contained in $[0, 1]$.
2. Axiom #2: The sum of all probabilities is $0.13+0.14+0.13+0.24+0.20+0.16 = 1$, so $P(\Omega) = 1$.
3. Axiom #3: The M&M colors are disjoint. So then

$$P(Brown \cup Yellow \cup \ldots \cup Green)$$

can be written as

$$P(Brown)+P(Yellow)+\ldots+P(Green)$$

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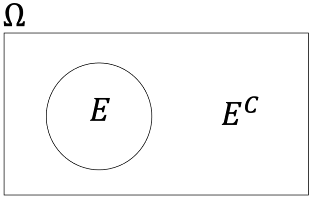
Complement of an Event

Definition: Complement Rule

The complement of event E (denoted as E^C or $\overline{E}$) is the event that occurs when E does not occur.



1. E and E^C are disjoint.
2. E and E^C partition Ω into 2 parts so that $E \cup E^C = \Omega$ which gives

$$P(E \cup E^C) = P(E) + P(E^C) = 1$$

3. Two useful equations result:

$$P(E) = 1 - P(E^C)$$

$$P(E^C) = 1 - P(E)$$

Examples

1. On a six-sided die, suppose E is the event "rolling a 6" or $E = \{6\}$.
   - $E^C = \{1, 2, 3, 4, 5\}$
   - $E \cup E^C = \{1, 2, 3, 4, 5, 6\} = \Omega$
2. The complement of "none" is "at least 1".
   - Suppose a computer code has no errors with probability of 0.45. Let E be the number of errors in the computer code.
     (a) What is Ω? $\Omega = \{0, 1, 2, \ldots, \infty\}$
     (b) What is the probability of at least one error?

$$E(\text{at least 1 error}) = \{1, 2, \ldots, \infty\}$$

$$E^C(\text{none}) = \{0\}$$

$$P(none) = 1 - P(\text{at least 1})$$
$$P(E^C) = 1 - P(E)$$
$$P(E^C) = 1 - 0.45 = \boxed{0.55}$$

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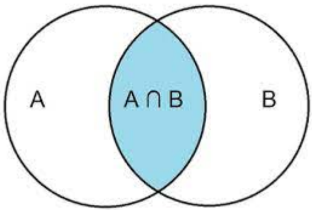
Intersection of Events

Definition: Intersection

The intersection of two events (denoted $A \cap B$) is the set of all events that are in both A and B.

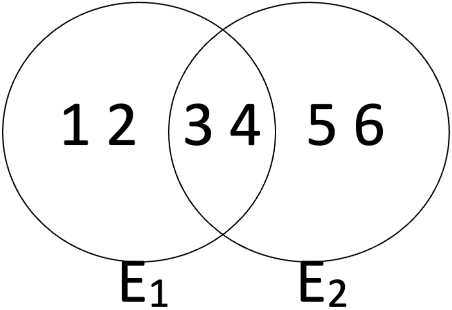


1. The intersection is shaded in blue.
2. $A \cap B =$ "A and B".
3. In general, E_1 and E_2 and ... and E_n is equivalent to $E_1 \cap E_2 \cap \ldots \cap E_n$.
4. If A and B are disjoint, then $A \cap B = \varnothing$.

Example 1

- Let $E_1 = \{1, 2, 3, 4\}$ and $E_2 = \{3, 4, 5, 6\}$.

$$\therefore E_1 \cap E_2 = \{3, 4\}$$



Example 2

- A standard deck of 52 cards. Let $A =$ "draw a card with hearts" and $B =$ "draw an odd number". What is $A \cap B$?

- Solution: any card that has both hearts and odd numbers.

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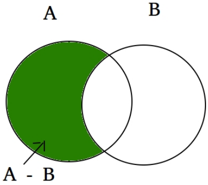
Difference of Two Events

Definition: Difference of Two Events

The difference of events A and B (denoted as $A \setminus B$) is any event in A that is not an event in B where

$$A \setminus B = A - B$$



- The part of A that does not include B :

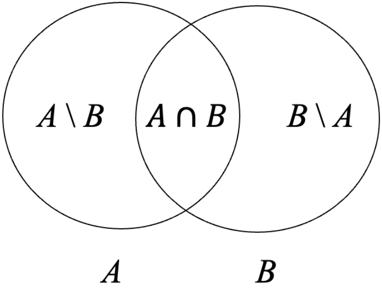
$$A \setminus B = A - A \cap B$$

$$\boxed{P(A \setminus B) = P(A) - P(A \cap B)}$$

$$\boxed{P(B \setminus A) = P(B) - P(A \cap B)}$$

Difference of Two Events

- When A and B are joint events, they intersect each other.
- A is partitioned into two disjoint events:
  1. The part of A that intersects B.
  2. The part of A that does not intersect B.
- Likewise for B.
- $A \setminus B \neq B \setminus A$



Example

Let $E_1 = \{1, 2, 3, 4\}$ and $E_2 = \{3, 4, 5, 6\}$.

1. $E_1 \cap E_2 = \{3, 4\}$
2. $E_1 \setminus E_2$?

$$\begin{aligned} &= E_1 - E_1 \cap E_2 \\ &= \{1, 2, 3, 4\} - \{3, 4\} \\ &= \{1, 2\} \end{aligned}$$

3. $E_2 \setminus E_1$?

$$\begin{aligned} &= E_2 - E_1 \cap E_2 \\ &= \{3, 4, 5, 6\} - \{3, 4\} \\ &= \{5, 6\} \end{aligned}$$