Problem Set 1.2, page 18

1 $\boldsymbol{u}\cdot\boldsymbol{v} = -2.4 + 2.4 = 0$, $\boldsymbol{u}\cdot\boldsymbol{w} = -.6 + 1.6 = 1$, $\boldsymbol{u}\cdot(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{u}\cdot\boldsymbol{v} + \boldsymbol{u}\cdot\boldsymbol{w} = 0 + 1$, $\boldsymbol{w}\cdot\boldsymbol{v} = 4 + 6 = 10 = \boldsymbol{v}\cdot\boldsymbol{w}$.

2 $\|\boldsymbol{u}\| = 1$ and $\|\boldsymbol{v}\| = 5$ and $\|\boldsymbol{w}\| = \sqrt{5}$. Then $|\boldsymbol{u}\cdot\boldsymbol{v}| = 0 < (1)(5)$ and $|\boldsymbol{v}\cdot\boldsymbol{w}| = 10 < 5\sqrt{5}$, confirming the Schwarz inequality.

3 Unit vectors $\boldsymbol{v}/\|\boldsymbol{v}\| = (\frac{4}{5}, \frac{3}{5}) = (0.8, 0.6)$. The vectors $\boldsymbol{w}, (2, -1)$, and $-\boldsymbol{w}$ make $0\,°, 90\,°, 180\,°$ angles with $\boldsymbol{w}$ and $\boldsymbol{w}/\|\boldsymbol{w}\| = (1/\sqrt{5}, 2/\sqrt{5})$. The cosine of θ is $\frac{\boldsymbol{v}}{\|\boldsymbol{v}\|}\cdot\frac{\boldsymbol{w}}{\|\boldsymbol{w}\|} = 10/5\sqrt{5}$.

4 (a) $\boldsymbol{v}\cdot(-\boldsymbol{v}) = \mathbf{-1}$ (b) $(\boldsymbol{v}+\boldsymbol{w})\cdot(\boldsymbol{v}-\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v} + \boldsymbol{w}\cdot\boldsymbol{v} - \boldsymbol{v}\cdot\boldsymbol{w} - \boldsymbol{w}\cdot\boldsymbol{w} = 1 + (\quad) - (\quad) - 1 = \mathbf{0}$ so $\theta = 90°$ (notice $\boldsymbol{v}\cdot\boldsymbol{w} = \boldsymbol{w}\cdot\boldsymbol{v}$) (c) $(\boldsymbol{v}-2\boldsymbol{w})\cdot(\boldsymbol{v}+2\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v} - 4\boldsymbol{w}\cdot\boldsymbol{w} = 1 - 4 = \mathbf{-3}$.

5 $\boldsymbol{u}_1 = \boldsymbol{v}/\|\boldsymbol{v}\| = (1, 3)/\sqrt{10}$ and $\boldsymbol{u}_2 = \boldsymbol{w}/\|\boldsymbol{w}\| = (2, 1, 2)/3$. $\boldsymbol{U}_1 = (3, -1)/\sqrt{10}$ is perpendicular to $\boldsymbol{u}_1$ (and so is $(-3, 1)/\sqrt{10}$). $\boldsymbol{U}_2$ could be $(1, -2, 0)/\sqrt{5}$: There is a whole plane of vectors perpendicular to $\boldsymbol{u}_2$, and a whole circle of unit vectors in that plane.

6 All vectors $\boldsymbol{w} = (c, 2c)$ are perpendicular to $\boldsymbol{v}$. They lie on a line. All vectors (x, y, z) with $x + y + z = 0$ lie on a ***plane***. All vectors perpendicular to $(1, 1, 1)$ and $(1, 2, 3)$ lie on a ***line*** in 3-dimensional space.

7 (a) $\cos\theta = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = 1/(2)(1)$ so $\theta = 60°$ or $\pi/3$ radians (b) $\cos\theta = 0$ so $\theta = 90°$ or $\pi/2$ radians (c) $\cos\theta = 2/(2)(2) = 1/2$ so $\theta = 60°$ or $\pi/3$ (d) $\cos\theta = -1/\sqrt{2}$ so $\theta = 135°$ or $3\pi/4$.

8 (a) False: $\boldsymbol{v}$ and $\boldsymbol{w}$ are any vectors in the plane perpendicular to $\boldsymbol{u}$ (b) True: $\boldsymbol{u}\cdot(\boldsymbol{v}+2\boldsymbol{w}) = \boldsymbol{u}\cdot\boldsymbol{v} + 2\boldsymbol{u}\cdot\boldsymbol{w} = 0$ (c) True, $\|\boldsymbol{u}-\boldsymbol{v}\|^2 = (\boldsymbol{u}-\boldsymbol{v})\cdot(\boldsymbol{u}-\boldsymbol{v})$ splits into $\boldsymbol{u}\cdot\boldsymbol{u} + \boldsymbol{v}\cdot\boldsymbol{v} = \mathbf{2}$ when $\boldsymbol{u}\cdot\boldsymbol{v} = \boldsymbol{v}\cdot\boldsymbol{u} = 0$.

9 If $v_2w_2/v_1w_1 = -1$ then $v_2w_2 = -v_1w_1$ or $v_1w_1 + v_2w_2 = \boldsymbol{v}\cdot\boldsymbol{w} = 0$: perpendicular! The vectors $(1, 4)$ and $(1, -\frac{1}{4})$ are perpendicular.

10 Slopes $2/1$ and $-1/2$ multiply to give -1: then $\boldsymbol{v} \cdot \boldsymbol{w} = 0$ and the vectors (the directions) are perpendicular.

11 $\boldsymbol{v} \cdot \boldsymbol{w} < 0$ means angle $> 90°$; these $\boldsymbol{w}$'s fill half of 3-dimensional space.

12 $(1,1)$ perpendicular to $(1,5) - c(1,1)$ if $(1,1) \cdot (1,5) - c(1,1) \cdot (1,1) = 6 - 2c = 0$ or $\boldsymbol{c = 3}$; $\boldsymbol{v} \cdot (\boldsymbol{w} - c\boldsymbol{v}) = 0$ if $c = \boldsymbol{v} \cdot \boldsymbol{w} / \boldsymbol{v} \cdot \boldsymbol{v}$. Subtracting $c\boldsymbol{v}$ is the key to constructing a perpendicular vector.

13 The plane perpendicular to $(1,0,1)$ contains all vectors $(c, d, -c)$. In that plane, $\boldsymbol{v} = (1, 0, -1)$ and $\boldsymbol{w} = (0, 1, 0)$ are perpendicular.

14 One possibility among many: $\boldsymbol{u} = (1, -1, 0, 0)$, $\boldsymbol{v} = (0, 0, 1, -1)$, $\boldsymbol{w} = (1, 1, -1, -1)$ and $(1, 1, 1, 1)$ are perpendicular to each other. "We can rotate those $\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}$ in their 3D hyperplane and they will stay perpendicular."

15 $\frac{1}{2}(x + y) = (2 + 8)/2 = 5$ and $5 > 4$; $\cos\theta = 2\sqrt{16}/\sqrt{10}\sqrt{10} = \boldsymbol{8/10}$.

16 $\|\boldsymbol{v}\|^2 = 1 + 1 + \cdots + 1 = 9$ so $\|\boldsymbol{v}\| = \boldsymbol{3}$; $\boldsymbol{u} = \boldsymbol{v}/3 = (\frac{1}{3}, \ldots, \frac{1}{3})$ is a unit vector in 9D; $\boldsymbol{w} = (1, -1, 0, \ldots, 0)/\sqrt{2}$ is a unit vector in the 8D hyperplane perpendicular to $\boldsymbol{v}$.

17 $\cos\alpha = 1/\sqrt{2}$, $\cos\beta = 0$, $\cos\gamma = -1/\sqrt{2}$. For any vector $\boldsymbol{v} = (\boldsymbol{v}_1, \boldsymbol{v}_2, \boldsymbol{v}_3)$ the cosines with $(1,0,0)$ and $(0,0,1)$ are $\cos^2\alpha + \cos^2\beta + \cos^2\gamma = (v_1^2 + v_2^2 + v_3^2)/\|\boldsymbol{v}\|^2 = 1$.

18 $\|\boldsymbol{v}\|^2 = 4^2 + 2^2 = 20$ and $\|\boldsymbol{w}\|^2 = (-1)^2 + 2^2 = 5$. Pythagoras is $\|(3,4)\|^2 = 25 = 20 + 5$ for the length of the hypotenuse $\boldsymbol{v} + \boldsymbol{w} = (3, 4)$.

19 Start from the rules $(1), (2), (3)$ for $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$ and $\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w})$ and $(c\boldsymbol{v}) \cdot \boldsymbol{w}$. Use rule (2) for $(\boldsymbol{v} + \boldsymbol{w}) \cdot (\boldsymbol{v} + \boldsymbol{w}) = (\boldsymbol{v} + \boldsymbol{w}) \cdot \boldsymbol{v} + (\boldsymbol{v} + \boldsymbol{w}) \cdot \boldsymbol{w}$. By rule (1) this is $\boldsymbol{v} \cdot (\boldsymbol{v} + \boldsymbol{w}) + \boldsymbol{w} \cdot (\boldsymbol{v} + \boldsymbol{w})$. Rule (2) again gives $\boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w} = \boldsymbol{v} \cdot \boldsymbol{v} + 2\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{w}$. Notice $\boldsymbol{v} \cdot \boldsymbol{w} = \boldsymbol{w} \cdot \boldsymbol{v}$! The main point is to feel free to open up parentheses.

20 We know that $(\boldsymbol{v} - \boldsymbol{w}) \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{v} \cdot \boldsymbol{v} - 2\boldsymbol{v} \cdot \boldsymbol{w} + \boldsymbol{w} \cdot \boldsymbol{w}$. The Law of Cosines writes $\|\boldsymbol{v}\|\|\boldsymbol{w}\| \cos\theta$ for $\boldsymbol{v} \cdot \boldsymbol{w}$. Here θ is the angle between $\boldsymbol{v}$ and $\boldsymbol{w}$. When $\theta < 90°$ this $\boldsymbol{v} \cdot \boldsymbol{w}$ is positive, so in this case $\boldsymbol{v} \cdot \boldsymbol{v} + \boldsymbol{w} \cdot \boldsymbol{w}$ is larger than $\|\boldsymbol{v} - \boldsymbol{w}\|^2$.

Pythagoras changes from equality $a^2 + b^2 = c^2$ to ***inequality*** when $\theta < 90°$ or $\theta > 90°$.

21 $2\boldsymbol{v}\cdot\boldsymbol{w} \le 2\|\boldsymbol{v}\|\|\boldsymbol{w}\|$ leads to $\|\boldsymbol{v}+\boldsymbol{w}\|^2 = \boldsymbol{v}\cdot\boldsymbol{v}+2\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{w}\cdot\boldsymbol{w} \le \|\boldsymbol{v}\|^2+2\|\boldsymbol{v}\|\|\boldsymbol{w}\|+\|\boldsymbol{w}\|^2$. This is $(\|\boldsymbol{v}\|+\|\boldsymbol{w}\|)^2$. Taking square roots gives $\|\boldsymbol{v}+\boldsymbol{w}\| \le \|\boldsymbol{v}\|+\|\boldsymbol{w}\|$.

22 $v_1^2w_1^2 + 2v_1w_1v_2w_2 + v_2^2w_2^2 \le v_1^2w_1^2 + v_1^2w_2^2 + v_2^2w_1^2 + v_2^2w_2^2$ is true (cancel 4 terms) because the difference is $v_1^2w_2^2 + v_2^2w_1^2 - 2v_1w_1v_2w_2$ which is $(v_1w_2 - v_2w_1)^2 \ge 0$.

23 $\cos\beta = w_1/\|\boldsymbol{w}\|$ and $\sin\beta = w_2/\|\boldsymbol{w}\|$. Then $\cos(\beta-a) = \cos\beta\cos\alpha+\sin\beta\sin\alpha = v_1w_1/\|\boldsymbol{v}\|\|\boldsymbol{w}\| + v_2w_2/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\|$. This is $\cos\theta$ because $\beta-\alpha=\theta$.

24 Example 6 gives $|u_1||U_1| \le \frac{1}{2}(u_1^2+U_1^2)$ and $|u_2||U_2| \le \frac{1}{2}(u_2^2+U_2^2)$. The whole line becomes $.96 \le (.6)(.8)+(.8)(.6) \le \frac{1}{2}(.6^2+.8^2)+\frac{1}{2}(.8^2+.6^2) = 1$. True: $.96 < 1$.

25 The cosine of θ is $x/\sqrt{x^2+y^2}$, near side over hypotenuse. Then $|\cos\theta|^2$ is not greater than 1: $x^2/(x^2+y^2) \le 1$.

26–27 (with apologies for that typo !) These two lines add to $2||\boldsymbol{v}||^2 + 2||\boldsymbol{w}||^2$:

$$||\boldsymbol{v}+\boldsymbol{w}||^2 = (\boldsymbol{v}+\boldsymbol{w})\cdot(\boldsymbol{v}+\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v}+\boldsymbol{v}\cdot\boldsymbol{w}+\boldsymbol{w}\cdot\boldsymbol{v}+\boldsymbol{w}\cdot\boldsymbol{w}$$
$$||\boldsymbol{v}-\boldsymbol{w}||^2 = (\boldsymbol{v}-\boldsymbol{w})\cdot(\boldsymbol{v}-\boldsymbol{w}) = \boldsymbol{v}\cdot\boldsymbol{v}-\boldsymbol{v}\cdot\boldsymbol{w}-\boldsymbol{w}\cdot\boldsymbol{v}+\boldsymbol{w}\cdot\boldsymbol{w}$$

28 The vectors $\boldsymbol{w} = (x,y)$ with $(1,2)\cdot\boldsymbol{w} = x+2y = 5$ lie on a line in the xy plane. The shortest $\boldsymbol{w}$ on that line is $(1,2)$. (The Schwarz inequality $\|\boldsymbol{w}\| \ge \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\| = \sqrt{5}$ is an equality when $\cos\theta = 0$ and $\boldsymbol{w} = (1,2)$ and $\|\boldsymbol{w}\| = \sqrt{5}$.)

29 The length $\|\boldsymbol{v}-\boldsymbol{w}\|$ is between 2 and 8 (triangle inequality when $\|\boldsymbol{v}\| = 5$ and $\|\boldsymbol{w}\| = 3$). The dot product $\boldsymbol{v}\cdot\boldsymbol{w}$ is between -15 and 15 by the Schwarz inequality.

30 Three vectors in the plane could make angles greater than $90°$ with each other: for example $(1,0),(-1,4),(-1,-4)$. Four vectors could *not* do this ($360°$ total angle). How many can do this in $\mathbf{R}^3$ or $\mathbf{R}^n$? Ben Harris and Greg Marks showed me that the answer is $n+1$. The vectors from the center of a regular simplex in $\mathbf{R}^n$ to its $n+1$ vertices all have negative dot products. If $n+2$ vectors in $\mathbf{R}^n$ had negative dot products, project them onto the plane orthogonal to the last one. Now you have $n+1$ vectors in $\mathbf{R}^{n-1}$ with negative dot products. Keep going to 4 vectors in $\mathbf{R}^2$: no way!

31 For a specific example, pick $\boldsymbol{v} = (1,2,-3)$ and then $\boldsymbol{w} = (-3,1,2)$. In this example $\cos\theta = \boldsymbol{v}\cdot\boldsymbol{w}/\|\boldsymbol{v}\|\|\boldsymbol{w}\| = -7/\sqrt{14}\sqrt{14} = -1/2$ and $\theta = 120°$. This always happens when $x+y+z=0$:

$$\boldsymbol{v}\cdot\boldsymbol{w} = xz + xy + yz = \frac{1}{2}(x+y+z)^2 - \frac{1}{2}(x^2+y^2+z^2)$$

This is the same as $\boldsymbol{v}\cdot\boldsymbol{w} = 0 - \frac{1}{2}\,\|\boldsymbol{v}\|\|\boldsymbol{w}\|$. Then $\cos\theta = \frac{1}{2}$.

32 Wikipedia gives this proof of geometric mean $G = \sqrt[3]{xyz} \le$ arithmetic mean $A = (x+y+z)/3$. First there is equality in case $x = y = z$. Otherwise A is somewhere between the three positive numbers, say for example $z < A < y$.

Use the known inequality $g \le a$ for the *two* positive numbers x and $y + z - A$. Their mean $a = \frac{1}{2}(x + y + z - A)$ is $\frac{1}{2}(3A - A)$ = same as A! So $a \ge g$ says that $A^3 \ge g^2 A = x(y+z-A)A$. But $(y+z-A)A = (y-A)(A-z) + yz > yz$. Substitute to find $A^3 > xyz = G^3$ as we wanted to prove. Not easy!

There are many proofs of $G = (x_1 x_2 \cdots x_n)^{1/n} \le A = (x_1 + x_2 + \cdots + x_n)/n$. In calculus you are maximizing G on the plane $x_1 + x_2 + \cdots + x_n = n$. The maximum occurs when all x's are equal.

33 The columns of the 4 by 4 "Hadamard matrix" (times $\frac{1}{2}$) are perpendicular unit vectors:

$$\frac{1}{2}H = \frac{1}{2}\begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & 1 & -1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -1 & 1 \end{bmatrix}.$$

34 The commands $V = $ **randn** $(3,30)$; $D = $ **sqrt** (**diag** $(V' * V)$); $U = V\backslash D$; will give 30 random unit vectors in the columns of U. Then $\boldsymbol{u}' * U$ is a row matrix of 30 dot products whose average absolute value should be close to $2/\pi$.

Problem Set 1.3, page 29

1 $3\boldsymbol{s}_1 + 4\boldsymbol{s}_2 + 5\boldsymbol{s}_3 = (3, 7, 12)$. The same vector $\boldsymbol{b}$ comes from S times $\boldsymbol{x} = (3, 4, 5)$: