Chapter 4: Continuous Random Variables
Probability Density Functions

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FTC Part 1

Fundamental Theorem of Calculus, Part 1

If f is a continuous function on $[a, b]$, then the integral function F given by

$$F(x) = \int_a^x f(t)\ dt$$

is continuous on $[a, b]$ and differentiable on (a, b). Further,

$$\begin{aligned} F'(x) &= \frac{d}{dx}\int_a^x f(t)\ dt \\ F'(x) &= f(x) \end{aligned}$$

In other words, F is the antiderivative of f.

FTC Part 2

Fundamental Theorem of Calculus, Part 2

If f is a continuous function on $[a, b]$, then

$$\int_a^b f(x)\ dx = F(b) - F(a)$$

where F is the antiderivative of f.

- This computes the area under $f(x)$

Example 1: Compute the integrals.

(a) $\int_{1}^{3} 6x^2 + 1 \; dx$

(b) $\int_{-1}^{0} e^{-3x} \; dx$

Derivative Notation

- y as a function of x
  1. Function notation: $y = f(x)$
  2. Derivative notation: $y' = f'(x)$ or $\frac{dy}{dx} = f'(x)$
- x as a function of y
  1. Function notation: $x = f(y)$
  2. Derivative notation: $x' = f'(y)$ or $\frac{dx}{dy} = f'(y)$

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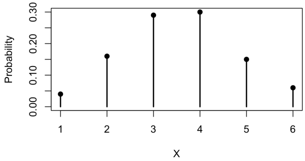
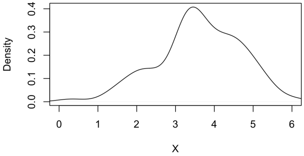
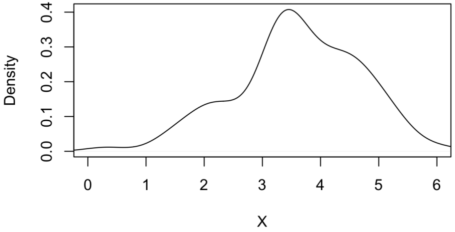
Comparison

Discrete	**Continuous**
Countable/finite	Measurable/infinite
Defined on the set $\mathbb{N}$	Defined on an interval in $\mathbb{R}$
Plot: points (mass) 	Plot: continuous curve (density) 
$P(2 \leq X \leq 5) =$ $P(2) + P(3) + P(4) + P(5)$	How do I sum the probabilities on a continuous interval? $\Longrightarrow$ Use an integral!

Probability and Density for a Continuous RV



- $f(x)$ is a probability density function (p.d.f.).
- The calculated area under $f(x)$ measures probability.
- Density $\neq$ probability

- For this example,

$$P(2 \leq X \leq 5) = \int_{2}^{5} f(x)\ dx$$

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Property #1: Continuous RV

Definition: Density

The probability density function or p.d.f. is the derivative of the cumulative distribution function (c.d.f.), $f(x) = F'(x)$ and $f(x) \geq 0 \;\; \forall x \in \mathbb{R}$.

- The y or $f(x)$ values are densities, not probabilities.
- Example:
  (a) Let $f_X(x) = 3x^2$ defined on $0 \leq X \leq 1$ be the pdf of X. Calculate $f_X(1)$.

Example (continued)

(b) We can write the pdf as a piecewise function defined on $(-\infty, \infty)$:

(c) Plot $f_X(x)$

Property #2: Continuous RV

Total Probability

The sum of the probabilities of a continuous r.v. X is one and is equivalent to

$$\int_{-\infty}^{\infty} f_X(x)\ dx = 1$$

where $f_X(x)$ is the p.d.f. of X.

(d) Show that $f_X(x) = 3x^2$ on $[0, 1]$ is a valid p.d.f.

Property #3: Continuous RV

Calculating Probability

The integral of the density function $f(x)$ from a to b calculates the probability that X is between a and b, which is written as

$$P(a \leq X \leq b) = \int_a^b f(x)\ dx = F(b) - F(a)$$

where F is the antiderivative *and* cdf of f.

(e) Given $f_X(x) = 3x^2$ on $[0, 1]$, calculate $P(1/2 \leq X \leq 1)$.

Property #4: Continuous RV

Probability at a Single Point

The probability that $X = a$ is given by $P(X = a)$ where

$$P(a \leq X \leq a) = \int_a^a f(x)\,dx = F(a) - F(a) = 0$$

and so $P(X = x) = 0 \ \forall x \in \mathbb{R}$.

(f) Given $f_X(x) = 3x^2$ on $[0, 1]$, calculate $P(X = 1/2)$.

Property #5: Continuous RV

Cumulative Distribution Function (cdf)

The cumulative distribution function, $F_x(x)$, of the continuous r.v. X calculates $P(X \leq x) = P(X < x)$ (by Property 4) and is computed as

$$F_X(x) = \int_{-\infty}^{x} f(t)\ dt$$

(g) Find the cdf of $f(x) = 3x^2$ defined on [0,1].

Expected Value and Variance

Definition: Mean of a Continuous RV

The expected value ($E(X)$ or μ) of a continuous random variable is given by

$$\mu = \int x \cdot f(x)\ dx$$

Definition: Variance of a Continuous RV

The variance of a continuous r.v. is given by

$$\sigma^2 = E(X^2) - [E(X)]^2$$

where $E(X^2) = \int x^2 \cdot f(x)\ dx$.

Example

(h) Calculate the mean and variance of $f(x) = 3x^2$ on $[0, 1]$.

Summary/Comparisons

Property	**Discrete**	**Continuous**
1. Definition	$f(x) = P(X = x)$ pmf	$f(x) = F'(x)$ pdf
2. Total Probability	$\sum_{i=1}^{n} P(X_i) = 1$	$\int_{-\infty}^{\infty} f(x)\ dx = 1$
3. Cumulative Probability	$P(X \leq x) = \sum_{a \leq x} P(a)$	$P(X < x) = \int_{-\infty}^{x} f(x)\ dx$
4. Computing Probability	$P(x \in A) = \sum_{x \in A} P(x)$	$P(x \in A) = \int_A f(x)\ dx$
5. Expected Value	$\mu = \sum_x x \cdot P(x)$	$\mu = \int x \cdot f(x)\ dx$
6. Variance	$\sigma^2 = E(X^2) - [E(X)]^2 = \sum_x x^2 \cdot P(x) - \mu^2$	$\sigma^2 = E(X^2) - [E(X)]^2 = \int_x x^2 \cdot f(x)\ dx - \mu^2$

Example 2

Suppose the pdf of a continuous r.v. X is defined as:

$$f(x) = \begin{cases} 0 & x \leq -1 \\ x+1 & -1 < x \leq 0 \\ -x+1 & 0 < x \leq 1 \\ 0 & x \geq 1 \end{cases}$$

(a) Plot $f(x)$

Example 2

$$f(x) = \begin{cases} 0 & x \leq -1 \\ x+1 & -1 < x \leq 0 \\ -x+1 & 0 < x \leq 1 \\ 0 & x \geq 1 \end{cases}$$

(b) Show that $f(x)$ is a valid pdf.

Example 2

$$f(x) = \begin{cases} 0 & x \leq -1 \\ x+1 & -1 < x \leq 0 \\ -x+1 & 0 < x \leq 1 \\ 0 & x \geq 1 \end{cases}$$

(c) Calculate the mean and variance of X.

Example 2

$$f(x) = \begin{cases} 0 & x \leq -1 \\ x+1 & -1 < x \leq 0 \\ -x+1 & 0 < x \leq 1 \\ 0 & x \geq 1 \end{cases}$$

(d) Compute $P(-1/2 < X \leq 1/2)$.

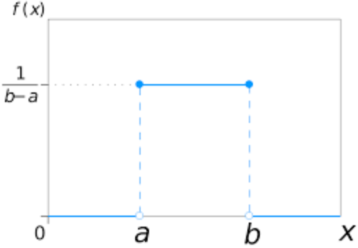
Example 2

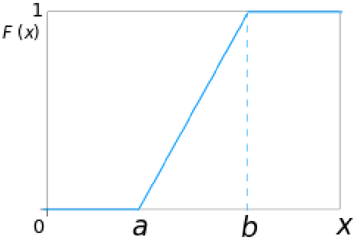
$$f(x) = \begin{cases} 0 & x \leq -1 \\ x+1 & -1 < x \leq 0 \\ -x+1 & 0 < x \leq 1 \\ 0 & x \geq 1 \end{cases}$$

(e) Determine the cdf of X.

The Uniform Distribution

- Uniform: all outcomes are equally likely
- Defined on an interval $[a, b] \implies X \sim \text{Uniform}(a, b)$





$$f_x(x) = \begin{cases} \frac{1}{b-a} & a \leq X \leq b \\ 0 & elsewhere \end{cases} \qquad F_X(x) = \begin{cases} \frac{x-a}{b-a} & a \leq X \leq b \\ 0 & elsewhere \end{cases}$$

Summary: Uniform Distribution

If $X \sim \text{Uniform}(a, b)$, then:

Term	**Notation**	**Formula**
Density pdf	$f_X(x)$	$\frac{1}{b-a}$
Cumulative Density cdf	$F_X(x) = P(X < x)$	$\frac{x-a}{b-a}$
Mean	μ	$\frac{a+b}{2}$
Variance	σ^2	$\frac{(b-a)^2}{12}$

Example 3

Given that $X \sim \text{Uniform}(0, 1)$, find the following:

(a) Determine and sketch the pdf and cdf.

Example 3

Given that $X \sim$ Uniform$(0, 1)$, find the following:

(b) Calculate the mean and variance of X.

Example 3

Given that $X \sim \text{Uniform}(0, 1)$, find the following:

(c) Calculate $P(X < 3/4)$.

Example 3

Given that $X \sim$ Uniform$(0, 1)$, find the following:

(d) Calculate $P(1/4 < X < 3/4)$.

Transformations of Random Variables I

Suppose X has pdf $f_X(x)$ and cdf $F_X(x)$. Find the pdf of Y in terms of f_X if $Y = aX + b$ for $a > 0$.

$$\begin{aligned}
\Longrightarrow \quad F_Y(y) &= P(Y \leq y) \\
&= P(aX + b \leq y) \\
&= P\left(X \leq \frac{y-b}{a}\right) \\
&= F_X\left(\frac{y-b}{a}\right) \\
\Longrightarrow \quad \frac{d}{dy} F_Y(y) &= \frac{d}{dy} F_X\left(\frac{y-b}{a}\right)
\end{aligned}$$

Transformations of Random Variables II

Recall that $F' = f$:

$$\begin{aligned} f_Y(y) &= F_X'\left(\frac{y-b}{a}\right) \cdot \frac{d}{dy}\left(\frac{y-b}{a}\right) \\ f_Y(y) &= \frac{1}{a} \cdot f_X\left(\frac{y-b}{a}\right) \end{aligned}$$

In general,

$$\boxed{f_Y(y) = f_X\left(g(y)\right) \cdot \left|\frac{dx}{dy}\right|}$$

Example 4

Suppose $X \sim \text{Uniform}(0, 1)$ and $y = e^x$. Determine the pdf of Y.