(a) Some linear combination of the 100 rows is **the row of 100 zeros**.

(b) Some linear combination of the 100 **columns** is **the column of zeros**.

(c) A very singular matrix has all ones: $A =$ **ones** (100). A better example has 99 random rows (or the numbers $1^i, \ldots, 100^i$ in those rows). The 100th row could be the sum of the first 99 rows (or any other combination of those rows with no zeros).

(d) The row picture has 100 planes **meeting along a common line through** $\mathbf{0}$. The column picture has 100 vectors all in the same 99-dimensional hyperplane.

Problem Set 2.3, page 66

1 $E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -5 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 7 & 1 \end{bmatrix}$, $P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$.

2 $E_{32}E_{21}\boldsymbol{b} = (1, -5, -35)$ but $E_{21}E_{32}\boldsymbol{b} = (1, -5, 0)$. When E_{32} comes first, row 3 feels no effect from row 1.

3 $\begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 2 & 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix} \quad M = E_{32}E_{31}E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -4 & 1 & 0 \\ 10 & -2 & 1 \end{bmatrix}$.

Those E's are in the right order to give $MA = U$.

4 Elimination on column 4: $\boldsymbol{b} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} \xrightarrow{E_{21}} \begin{bmatrix} 1 \\ -4 \\ 0 \end{bmatrix} \xrightarrow{E_{31}} \begin{bmatrix} 1 \\ -4 \\ 2 \end{bmatrix} \xrightarrow{E_{32}} \begin{bmatrix} 1 \\ -4 \\ 10 \end{bmatrix}$. The original $A\boldsymbol{x} = \boldsymbol{b}$ has become $U\boldsymbol{x} = \boldsymbol{c} = (1, -4, 10)$. Then back substitution gives $\boldsymbol{z} = -5, y = \frac{1}{2}, x = \frac{1}{2}$. This solves $A\boldsymbol{x} = (1, 0, 0)$.

5 Changing a_{33} from 7 to 11 will change the third pivot from 5 to 9. Changing a_{33} from 7 to 2 will change the pivot from 5 to *no pivot*.

6 Example: $\begin{bmatrix} 2 & 3 & 7 \\ 2 & 3 & 7 \\ 2 & 3 & 7 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ -1 \end{bmatrix} = \begin{bmatrix} 4 \\ 4 \\ 4 \end{bmatrix}$. If all columns are multiples of column 1, there is no second pivot.

7 To reverse E_{31}, **add** 7 times row **1** to row **3**. The inverse of the elimination matrix $E = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{-7} & 0 & 1 \end{bmatrix}$ is $E^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ \mathbf{7} & 0 & 1 \end{bmatrix}$. Multiplication confirms $EE^{-1} = I$.

8 $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$ and $M* = \begin{bmatrix} a & b \\ c - \ell a & d - \ell b \end{bmatrix}$. $\det M* = a(d - \ell b) - b(c - \ell a)$ reduces to $ad - bc$! Subtracting row 1 from row 2 doesn't change $\det M$.

9 $M = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ -1 & 1 & 0 \end{bmatrix}$. After the exchange, we need E_{31} (not E_{21}) to act on the new row 3.

10 $E_{13} = \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$; $\begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$; $E_{31}E_{13} = \begin{bmatrix} 2 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$. Test on the identity matrix!

11 An example with two negative pivots is $A = \begin{bmatrix} 1 & 2 & 2 \\ 1 & 1 & 2 \\ 1 & 2 & 1 \end{bmatrix}$. The diagonal entries can change sign during elimination.

12 The first product is $\begin{bmatrix} 9 & 8 & 7 \\ 6 & 5 & 4 \\ 3 & 2 & 1 \end{bmatrix}$ rows and also columns reversed. The second product is $\begin{bmatrix} 1 & 2 & 3 \\ 0 & 1 & -2 \\ 0 & 2 & -3 \end{bmatrix}$.

13 (a) E times the third column of B is the third column of EB. A column that starts at zero will stay at zero. (b) E could add row 2 to row 3 to change a zero row to a nonzero row.

14 E_{21} has $-\ell_{21}=\frac{1}{2}$, E_{32} has $-\ell_{32}=\frac{2}{3}$, E_{43} has $-\ell_{43}=\frac{3}{4}$. Otherwise the E's match I.

15 $a_{ij} = 2i - 3j$: $A = \begin{bmatrix} -1 & -4 & -7 \\ 1 & -2 & -5 \\ 3 & \mathbf{0} & -3 \end{bmatrix} \to \begin{bmatrix} -1 & -4 & -7 \\ 0 & -6 & -12 \\ 0 & -\mathbf{12} & -24 \end{bmatrix}$. The zero became -12, an example of ***fill-in***. To remove that -12, choose $E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -2 & 1 \end{bmatrix}$.

Every 3 by 3 matrix with entries $a_{ij} = ci + dj$ is **singular** !

16 (a) The ages of X and Y are x and y: $x - 2y = 0$ and $x + y = 33$; $x = 22$ and $y = 11$ (b) The line $y = mx + c$ contains $x = 2$, $y = 5$ and $x = 3$, $y = 7$ when $2m + c = 5$ and $3m + c = 7$. Then $m = 2$ is the slope.

17 The parabola $y = a + bx + cx^2$ goes through the 3 given points when $\begin{aligned} a + \ b + \ c &= 4 \\ a + 2b + 4c &= 8 \\ a + 3b + 9c &= 14 \end{aligned}$.

Then $a = 2$, $b = 1$, and $c = 1$. This matrix with columns $(1,1,1)$, $(1,2,3)$, $(1,4,9)$ is a "Vandermonde matrix."

18 $EF = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b & c & 1 \end{bmatrix}$, $FE = \begin{bmatrix} 1 & 0 & 0 \\ a & 1 & 0 \\ b+ac & c & 1 \end{bmatrix}$, $E^2 = \begin{bmatrix} 1 & 0 & 0 \\ 2a & 1 & 0 \\ 2b & 0 & 1 \end{bmatrix}$, $F^3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3c & 1 \end{bmatrix}$.

19 $PQ = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$. In the opposite order, two row exchanges give $QP = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $P^2 = I$. If M exchanges rows 2 and 3 then $M^2 = I$ (also $(-M)^2 = I$). There are many square roots of I: Any matrix $M = \begin{bmatrix} a & b \\ c & -a \end{bmatrix}$ has $M^2 = I$ if $a^2 + bc = 1$.

20 (a) Each column of EB is E times a column of B (b) $\begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}\begin{bmatrix}\mathbf{1} & \mathbf{2} & \mathbf{4}\\\mathbf{1} & \mathbf{2} & \mathbf{4}\end{bmatrix} = \begin{bmatrix}1 & 2 & 4\\\mathbf{2} & \mathbf{4} & \mathbf{8}\end{bmatrix}$. All rows of EB are ***multiples*** of $\begin{bmatrix}1 & 2 & 4\end{bmatrix}$.

21 **No.** $E = \begin{bmatrix}1 & 0\\1 & 1\end{bmatrix}$ and $F = \begin{bmatrix}1 & 1\\0 & 1\end{bmatrix}$ give $EF = \begin{bmatrix}1 & 1\\1 & 2\end{bmatrix}$ but $FE = \begin{bmatrix}2 & 1\\1 & 1\end{bmatrix}$.

22 (a) $\sum a_{3j}x_j$ (b) $a_{21} - a_{11}$ (c) $a_{21} - 2a_{11}$ (d) $(EA\boldsymbol{x})_1 = (A\boldsymbol{x})_1 = \sum a_{1j}x_j$.

23 $E(EA)$ subtracts 4 times row 1 from row 2 (EEA does the row operation twice). AE subtracts 2 times column 2 of A from column 1 (multiplication by E on the right side acts on **columns** instead of rows).

24 $\begin{bmatrix}A & \boldsymbol{b}\end{bmatrix} = \begin{bmatrix}2 & 3 & \mathbf{1}\\4 & 1 & \mathbf{17}\end{bmatrix} \to \begin{bmatrix}2 & 3 & \mathbf{1}\\0 & -5 & \mathbf{15}\end{bmatrix}$. The triangular system is $\begin{aligned}2x_1 + 3x_2 &= 1\\-5x_2 &= 15\end{aligned}$

Back substitution gives $x_1 = 5$ and $x_2 = -3$.

25 The last equation becomes $0 = 3$. If the original 6 is 3, then row $1 +$ row $2 =$ row 3. Then the last equation is $0 = 0$ and the system has infinitely many solutions.

26 (a) Add two columns $\boldsymbol{b}$ and $\boldsymbol{b}^*$ to get $[A\ \ \boldsymbol{b}\ \ \boldsymbol{b}^*]$. The example has

$$\begin{bmatrix}1 & 4 & 1 & 0\\2 & 7 & 0 & 1\end{bmatrix} \to \begin{bmatrix}1 & 4 & 1 & 0\\0 & -1 & -2 & 1\end{bmatrix} \to \boldsymbol{x} = \begin{bmatrix}-7\\2\end{bmatrix} \text{ and } \boldsymbol{x}^* = \begin{bmatrix}4\\-1\end{bmatrix}.$$

27 (a) No solution if $d=0$ and $c\neq 0$ (b) Many solutions if $d=0=c$. No effect from a, b.

28 $A = AI = A(BC) = (AB)C = IC = C$. That middle equation is crucial.

29 $E = \begin{bmatrix}1 & 0 & 0 & 0\\-1 & 1 & 0 & 0\\0 & -1 & 1 & 0\\0 & 0 & -1 & 1\end{bmatrix}$ subtracts each row from the next row. The result $\begin{bmatrix}1 & 0 & 0 & 0\\0 & 1 & 0 & 0\\0 & 1 & 1 & 0\\0 & 1 & 2 & 1\end{bmatrix}$ still has multipliers $= 1$ in a 3 by 3 Pascal matrix. The product M of all elimination matrices is $\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & -2 & 1 & 0 \\ -1 & 3 & -3 & 1 \end{bmatrix}$. This "alternating sign Pascal matrix" is on page 91.

30 (a) $E = A^{-1} = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ will reduce row 2 of EM to $[2 \;\; 3]$.

(b) Then $F = B^{-1} = \begin{bmatrix} 1 & -1 \\ 0 & 1 \end{bmatrix}$ will reduce row 1 of FEM to $[1 \;\; 1]$.

(c) Then $E = A^{-1}$ twice will reduce row 2 of $EEFEM$ to $[0 \;\; 1]$

(d) Now $EEFEM = B$. Move E's and F's to get $\boldsymbol{M} = \boldsymbol{ABAAB}$. This question focuses on positive integer matrices M with $ad - bc = 1$. The same steps make the entries smaller and smaller until M is a product of A's and B's.

31 $E_{21} = \begin{bmatrix} 1 & & & \\ a & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_{32} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & b & 1 & \\ 0 & 0 & 0 & 1 \end{bmatrix}$, $E_{43} = \begin{bmatrix} 1 & & & \\ 0 & 1 & & \\ 0 & 0 & 1 & \\ 0 & 0 & c & 1 \end{bmatrix}$,

$$E_{43}\,E_{32}\,E_{21} = \begin{bmatrix} 1 & & & \\ a & 1 & & \\ ab & b & 1 & \\ abc & bc & c & 1 \end{bmatrix}$$

Problem Set 2.4, page 77

1 If all entries of A, B, C, D are 1, then $BA = 3$ **ones**(5) is 5 by 5; $AB = 5$ **ones**(3) is 3 by 3; $ABD = 15$ **ones**$(3, 1)$ is 3 by 1. DC and $A(B + C)$ are not defined.

2 (a) A (column 2 of B) (b) (Row 1 of A) B (c) (Row 3 of A)(column 5 of B)
(d) (Row 1 of C)D(column 1 of E). (Part (c) assumed 5 columns in B)