MATH 2418 Linear Algebra. Week 4

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Summary of this Week's Goals

This week we will cover the matrix algebra presented in Sections 2.4 (Rules for Matrix Operations) and 2.5 (Inverse Matrices). By the end of the week, you should be familiar with the rules of matrix addition, scalar multiplication and matrix multiplication. You will know how to describe matrix multiplication four ways. You will also be introduced to block multiplication and elimination by blocks (the Schur complement). You will learn several properties of inverse matrices and three tests for the invertibility of a matrix. You will learn the Gauss-Jordan method for calculating the inverse of a matrix.

Announcements

- Your first midterm exam will be on Thursday, October 6, 7:30-8:45pm.

2.4 Rules for Matrix Operations

Addition and Scalar Multiplication of Matrices

- An $m \times n$ matrix A can be added to an $m \times n$ matrix B. The result is an $m \times n$ matrix which we might call C. The number of rows and columns of A must equal the number of rows and columns of B in order for the matrices to be compatible for matrix addition.

- To perform matrix addition, add each element of the matrix A with each element of the matrix B. That is, the ij-th element of C (c_{ij}) is the sum $a_{ij} + b_{ij}$.

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 1+0 & -1+2 \\ 0+1 & 2+(-2) \\ 1+(-1) & 1+1 \end{bmatrix} = \begin{bmatrix} 1 & 1 \\ 1 & 0 \\ 0 & 2 \end{bmatrix}$$

- An $m \times n$ matrix A can be multiplied by a scalar c. The result is an $m \times n$ matrix which we might call B. There are no constraints on the dimensions of A to be compatible with multiplication by a scalar.

- To perform multiplication of a matrix by a scalar, multiply each element of the matrix A by the scalar c. That is, the ij-th element of B (b_{ij}) is the product ca_{ij}. The scalar may multiply the matrix on either the left or the right and the result is the same.

$$c \begin{bmatrix} 0 & 2 \\ 1 & -2 \\ -1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 2 \\ 1 & -2 \\ -1 & 1 \end{bmatrix} c = \begin{bmatrix} 0 & 2c \\ c & -2c \\ -c & c \end{bmatrix}$$

- Some properties of addition and scalar multiplication for matrices:
  - Matrix addition is commutative: $A + B = B + A$.
  - Scalar multiplication distributes over matrix addition: $c(A+B) = (A+B)c = cA+cB = Ac+Bc$.
  - Matrix addition is associative: $A + B + C = (A + B) + C = A + (B + C)$.

Matrix Multiplication

- An $m \times n$ matrix A can be multiplied by an $n \times k$ matrix B. The result is an $m \times k$ matrix which we might call C. The number of columns in A must equal the number of rows in B in order for the matrices to be compatible for matrix multiplication.
- Let us consider four ways of visualizing what matrix multiplication does. In the examples that follow, let A be a 4×3 matrix and let B be a 3×2 matrix so the the product C is a 4×2 matrix.

  1. Inner Product (Dot Product) Method. The ij-th element of C (c_{ij}) is the dot product of the i-th row of A with the j-th column of B. The "inner product" may be thought of as result of multiplying a row vector on the left by a column vector on the right. The inner product operation produces a scalar which becomes an element of the product matrix.

  $$AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \mathbf{a}_4 \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{a}_1 \cdot \mathbf{b}_1 & \mathbf{a}_1 \cdot \mathbf{b}_2 \\ \mathbf{a}_2 \cdot \mathbf{b}_1 & \mathbf{a}_2 \cdot \mathbf{b}_2 \\ \mathbf{a}_3 \cdot \mathbf{b}_1 & \mathbf{a}_3 \cdot \mathbf{b}_2 \\ \mathbf{a}_4 \cdot \mathbf{b}_1 & \mathbf{a}_4 \cdot \mathbf{b}_2 \end{bmatrix}$$

  2. Matrix A Multiplies Each Column of B on the Left. The j-th column of C is the product of A with the j-th column of B.

  $$AB = A \begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix}$$

  3. Matrix B Multiplies Each Row of A on the Right. The i-th row of C is the product of the i-th row of A with the matrix B.

  $$AB = \begin{bmatrix} \mathbf{a}_1 \\ \mathbf{a}_2 \\ \mathbf{a}_3 \\ \mathbf{a}_4 \end{bmatrix} B = \begin{bmatrix} \mathbf{a}_1 B \\ \mathbf{a}_2 B \\ \mathbf{a}_3 B \\ \mathbf{a}_4 B \end{bmatrix}$$

  4. Outer Product Method. The product C is the sum of outer products of columns of A with the rows of B. An "outer product" is the result of multiplying a column vector on the left by a row vector on the right. The outer product operation produces a matrix which becomes a term of the sum which produces the product matrix C.

  $$AB = \begin{bmatrix} \mathbf{a}_1 & \mathbf{a}_2 & \mathbf{a}_3 \end{bmatrix} \begin{bmatrix} \mathbf{b}_1 \\ \mathbf{b}_2 \\ \mathbf{b}_3 \end{bmatrix} = \mathbf{a}_1\mathbf{b}_1 + \mathbf{a}_2\mathbf{b}_2 + \mathbf{a}_3\mathbf{b}_3$$

- Some properties of matrix multiplication:
  - Matrix multiplication is *not* commutative—that is, $AB \neq BA$ for many matrices A and B.
  - Matrix multiplication distributes over matrix addition from the left: $A(B + C) = AB + AC$.
  - Matrix multiplication distributes over matrix addition from the right: $(A + B)C = AC + BC$.
  - Matrix multiplication is associative—that is, $ABC = A(BC) = (AB)C$.
  - The product of a matrix with an identity matrix multiplied on the left or the right is the matrix unchanged—that is, $IA = A$ and $AI = A$. Suppose the rows of A are $\mathbf{r}_1, \mathbf{r}_2, ..., \mathbf{r}_m$ and the columns of A are $\mathbf{c}_1, \mathbf{c}_2, ..., \mathbf{c}_n$. Then

  $$IA = I \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix} = \begin{bmatrix} I\mathbf{c}_1 & I\mathbf{c}_2 & \cdots & I\mathbf{c}_n \end{bmatrix} = \begin{bmatrix} \mathbf{c}_1 & \mathbf{c}_2 & \cdots & \mathbf{c}_n \end{bmatrix} = A$$

$$AI = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_m \end{bmatrix} I = \begin{bmatrix} \mathbf{r}_1 I \\ \mathbf{r}_2 I \\ \vdots \\ \mathbf{r}_m I \end{bmatrix} = \begin{bmatrix} \mathbf{r}_1 \\ \mathbf{r}_2 \\ \vdots \\ \mathbf{r}_m \end{bmatrix} = A$$

Note that in the above equations, the identity matrix multiplied on the left is $m \times m$ and the identity matrix multiplied on the right is $n \times n$.

- Block Multiplication. If A and B are partitioned into blocks (matrices within a larger matrix) in such a way that the blocks are compatible for multiplication, matrix multiplication works with the blocks just as if they were scalar entries in the matrix. Take note, however, that you must preserve the left-right orientation in the multiplication of the blocks.

  In the examples below, the components of each matrix shown are blocks, not scalars:

$$\begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix} \begin{bmatrix} B_{11} \\ B_{21} \end{bmatrix} = \begin{bmatrix} A_{11}B_{11} + A_{12}B_{21} \\ A_{21}B_{11} + A_{22}B_{21} \end{bmatrix}$$

$$\begin{bmatrix} I & A \end{bmatrix} A^{-1} = \begin{bmatrix} IA^{-1} & AA^{-1} \end{bmatrix} = \begin{bmatrix} A^{-1} & I \end{bmatrix}$$

$$\begin{bmatrix} I & 0 \\ -CA^{-1} & I \end{bmatrix} \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \begin{bmatrix} A & B \\ 0 & D - CA^{-1}B \end{bmatrix}$$

  The last example illustrates the process of elimination by blocks. That is, we pivot on the block A and zero out everything below A in a single step. The matrix $D - CA^{-1}B$ is called a “Schur complement.”

Powers of a Square Matrix

- An $m \times m$ square matrix A can be multiplied by itself p times to produce a matrix which we call A^p. The matrix A^p is also an $m \times m$ square matrix.
- For $p > 0$, the associative law quickly demonstrates the A commutes with A^p.

$$AA^p = A(AAA...) = (AAA...)A = A^pA = A^{p+1}$$

- We may define A^0 as the $m \times m$ identity matrix I. Note that A also commutes with I.
- In the next section, we will show that when A has an inverse, A also commutes with its inverse and all of its negative integer powers $A^{-p} = \left(A^{-1}\right)^p = (A^p)^{-1}$.

2.5 Inverse Matrices

Left Inverses and Right Inverses

- A square matrix A has a “left inverse” if there exists a matrix A^{LI} such that $A^{LI}A = I$.
- A square matrix A has a “right inverse” if there exists a matrix A^{RI} such that $AA^{RI} = I$.
- If a square matrix has both a left inverse and a right inverse, then $A^{LI} = A^{RI}$. That is, the left and right inverses are the same matrix, and we may use the notation A^{-1} to describe them both.

$$A^{LI} = A^{LI}I = A^{LI}(AA^{RI}) = (A^{LI}A)A^{RI} = IA^{RI} = A^{RI}$$
$$A^{-1} = A^{LI} = A^{RI}$$

- We haven’t shown yet the conditions for which left and right inverses exist. We have only shown that if they both exist, they must be identical.

Gauss-Jordan Elimination

- When the system $A\mathbf{x} = \mathbf{b}$ can be triangularized with n non-zero pivots, we can uniquely solve the system for any vector $\mathbf{b}$. In particular, we can solve the system with each column of the identity matrix chosen to take the place of the vector $\mathbf{b}$. Let's call the columns of the identity matrix $\mathbf{e}_1$, $\mathbf{e}_2$, ..., $\mathbf{e}_n$. That is, we can solve the following system for the column vectors $\mathbf{x}_1$, $\mathbf{x}_2$, ..., $\mathbf{x}_n$ which, when multiplied on the left by A, produce each of the columns of the identity matrix.

$$A\begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \cdots & \mathbf{x}_n\end{bmatrix} = \begin{bmatrix}\mathbf{e}_1 & \mathbf{e}_2 & \cdots & \mathbf{e}_n\end{bmatrix} = I$$

- We could solve each system $A\mathbf{x}_i = \mathbf{e}_i$ one at a time by triangularizing the augmented matrix $\begin{bmatrix}A & \mathbf{e}_i\end{bmatrix}$. Alternatively, since the steps are identical, we could solve all n systems at once by triangularizing the augmented matrix $\begin{bmatrix}A & I\end{bmatrix}$.
- The Gauss-Jordan Elimination Method goes beyond simply triangularizing the system so that it can be solved by back substitution. Gauss-Jordan continues with elimination steps to produce zeros in every row above and below each pivot.
- Let's apply the Gauss-Jordan Elimination Method to find the inverse of the following matrix:

$$A = \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 2 \end{bmatrix}$$

  - First, we represent the system $A\begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3\end{bmatrix} = I$ using the augmented matrix $\begin{bmatrix}A & I\end{bmatrix}$ as shown below:

$$\begin{bmatrix} 2 & 1 & -1 & 1 & 0 & 0 \\ 1 & -1 & 1 & 0 & 1 & 0 \\ -1 & 1 & 2 & 0 & 0 & 1 \end{bmatrix}$$

  - Next, pivot on the upper-left coefficent and produce zeros in the first column below the 2:

$$\begin{bmatrix} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & -3/2 & 3/2 & -1/2 & 1 & 0 \\ 0 & 3/2 & 3/2 & 1/2 & 0 & 1 \end{bmatrix}$$

  - Next, pivot on the second diagonal element and produce zeros in the second column below the $-3/2$:

$$\begin{bmatrix} 2 & 1 & -1 & 1 & 0 & 0 \\ 0 & -3/2 & 3/2 & -1/2 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 & 1 \end{bmatrix}$$

  - This is where we could stop the elimination process and begin back substitution. However, let's continue with elimination and put zeros in the columns above the pivot locations as well. We'll get a zero in the first row above the second pivot by adding $2/3$ times the second row to the first row.

$$\begin{bmatrix} 2 & 0 & 0 & 2/3 & 2/3 & 0 \\ 0 & -3/2 & 3/2 & -1/2 & 1 & 0 \\ 0 & 0 & 3 & 0 & 1 & 1 \end{bmatrix}$$

  - We complete elimination by adding $-1/2$ of equation (3) to equation (2) to produce all zeros above the third pivot.

$$\begin{bmatrix} 2 & 0 & 0 & 2/3 & 2/3 & 0 \\ 0 & -3/2 & 0 & -1/2 & 1/2 & -1/2 \\ 0 & 0 & 3 & 0 & 1 & 1 \end{bmatrix}$$

– Our last step will be to multiply each row by the reciprocal of the pivot on each row to produce an identity matrix on the left side of the augmented system.

$$\begin{bmatrix} 1 & 0 & 0 & 1/3 & 1/3 & 0 \\ 0 & 1 & 0 & 1/3 & -1/3 & 1/3 \\ 0 & 0 & 1 & 0 & 1/3 & 1/3 \end{bmatrix}$$

Note: Dividing each row by the reciprocal of the pivot is equivalent to multiplying the augmented matrix on the left by a matrix having the reciprocals of the pivots in the main diagonal and zeros everywhere else. This is a new type of elementary matrix we didn't see last week.

– The column vectors on the right side of this augmented matrix are solutions of the system $A\begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3\end{bmatrix} = I$. Therefore, the matrix $\begin{bmatrix}\mathbf{x}_1 & \mathbf{x}_2 & \mathbf{x}_3\end{bmatrix}$ is a right inverse of the matrix A.

$$A^{RI} = \begin{bmatrix} 1/3 & 1/3 & 0 \\ 1/3 & -1/3 & 1/3 \\ 0 & 1/3 & 1/3 \end{bmatrix}$$

We need only to prove the existence of a left inverse to be sure that the same matrix is both a left inverse and a right inverse and therefore qualifies to be called simply A^{-1}.

– Each step in the elimination process could be expressed as a matrix multiplication. That is, we multiplied the augmented matrix $\begin{bmatrix}A & I\end{bmatrix}$ on the left by several elementary matrices to perform each of the pivot actions. If we call the product of these elementary matrices E, then we can write

$$E\begin{bmatrix}A & I\end{bmatrix} = \begin{bmatrix}I & A^{RI}\end{bmatrix}$$

From this, we can tell $EA = I$, so E is a left inverse of A ($E = A^{LI}$). We can also see that $EI = E = A^{RI}$, further verifying that our left inverse and right inverse are the same. Finally, we have all the information we need to say that

$$A^{-1} = A^{LI} = A^{RI} = \begin{bmatrix} 1/3 & 1/3 & 0 \\ 1/3 & -1/3 & 1/3 \\ 0 & 1/3 & 1/3 \end{bmatrix}$$

- To summarize, we saw that the Gauss-Jordan elimination method allowed us to multiply the augmented matrix $\begin{bmatrix}A & I\end{bmatrix}$ on the left by several matrices, each correpsonding to a step in the elimination process. We can call the product of those elimination matrices E and write

$$E\begin{bmatrix}A & I\end{bmatrix} = \begin{bmatrix}I & A^{RI}\end{bmatrix}$$

From this we can glean that $E = A^{RI} = A^{LI} = A^{-1}$. That is, we have effectively multiplied $\begin{bmatrix}A & I\end{bmatrix}$ by A^{-1} to obtain $\begin{bmatrix}I & A^{-1}\end{bmatrix}$.

Conditions for the Existence of A^{-1}

- The columns of A are linearly independent.
- The linear system $A\mathbf{x} = \mathbf{0}$ has only the trival solution $\mathbf{x} = \mathbf{0}$.
- The matrix A can be triangularized with n non-zero pivots.
- We'll learn more later...

Properties of Matrix Inverses

- $AA^{-1} = A^{-1}A = I$, provided A^{-1} exists.
- $(AB)^{-1} = B^{-1}A^{-1}$, provided A and B are the same size and A^{-1} and B^{-1} exist. Left-right ordering is important here!
- $(AB)^{-1} \neq A^{-1}B^{-1}$ for many matrices A and B.
- The inverse of a 2×2 matrix is shown below, provided $ad - bc \neq 0$.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad - bc} \begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

  The expression $ad - bc$ is called the determinant of the matrix. We'll talk about that more later.
- For $p > 0$, $A^{-p} = \left(A^{-1}\right)^p = (A^p)^{-1}$.
- For all integer values of p (positive, zero and negative), the matrix A commutes with A^p.