MATH 2418 Linear Algebra. Week 7

Dr. James W. Miller

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Summary of this Week's Goals

This week we will cover Sections 3.2 (The Nullspace of A: Solving $A\mathbf{x} = \mathbf{0}$ and $R\mathbf{x} = \mathbf{0}$) and 3.3 (The Complete Solution to $A\mathbf{x} = \mathbf{b}$). You will learn how to find the nullspace of an $m \times n$ matrix A by performing elimination steps to transform the system $A\mathbf{x} = \mathbf{0}$ into the form $R\mathbf{x} = \mathbf{0}$, where the matrix R is in reduced row echelon form. In reduced row echelon form, each column is identified as a pivot column or a free column.

Announcements

- Your first midterm exam is this week on Thursday.

MATH 2418 – MIDTERM 1 – INFORMATION

1. Thursday, OCT/06, 7:30 – 8:45 pm
2. Room assignment:

	Room	Sections
1.	**ECSS 2.412**	301, 302, 303
2.	**GR 4.428**	304, 305, 306
3.	**SCI 1.210**	307, 308, 309
4.	**JSOM 1.212**	311, 314, 315
5.	**SCI 1.220**	316, 317, 318, 320

3. Sections covered: 1.1, 1.2, 1.3, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6
4. Only basic calculators can be used (no calculators with matrix functions or access to the internet are allowed)
5. Scratch paper will be provided. Work on scratch paper will NOT be graded
6. How to practice for the exam: do problems in HW1-5 and recommended problems

Figure 1: Midterm 1 Room Assignments

3.1 Spaces of Vectors (Column Space)

The Column Space of a Matrix

- The column space of an $m \times n$ matrix A, denoted $C(A)$, is the set of all possible linear combinations of the columns of A. This is the set of all vectors $\mathbf{y}$ which can be written as the product $\mathbf{y} = A\mathbf{x}$ for some choice of $\mathbf{x}$.
- The column space of A contains vectors in $\mathbb{R}^m$. It is a set of "all possible results" from the operation $A\mathbf{x}$.
- The column space of A is a vector space (a subspace of $\mathbb{R}^m$). It is closed with respect to vector addition and scalar multiplication.

3.2 The Nullspace of A: Solving $A\mathbf{x} = \mathbf{0}$ and $R\mathbf{x} = \mathbf{0}$

The Nullspace of a Matrix

- The nullspace of an $m \times n$ matrix A, denoted $N(A)$, is the set of all vectors $\mathbf{x}$ which are solutions to the equation $A\mathbf{x} = \mathbf{0}$.
- The nullspace of A contains vectors in $\mathbb{R}^n$. It is a set of "all possible inputs" into the operation $A\mathbf{x}$ which result in the $\mathbb{R}^m$ zero vector output.
- The nullspace of A is a vector space (a subspace of $\mathbb{R}^n$). It is closed with respect to vector addition and scalar multiplication.
  - If $\mathbf{x_1}$ and $\mathbf{x_2}$ are in the nullspace of A, then $A\mathbf{x_1} = \mathbf{0}$ and $A\mathbf{x_2} = \mathbf{0}$. Also $A(\mathbf{x_1} + \mathbf{x_2}) = A\mathbf{x_1} + A\mathbf{x_2} = 0 + 0 = 0$, so $\mathbf{x_1} + \mathbf{x_2}$ is in $N(A)$.
  - If $\mathbf{x_1}$ is in the nullspace of A, then $A(c\mathbf{x_1}) = cA\mathbf{x_1} = c\mathbf{0} = \mathbf{0}$, so $c\mathbf{x_1}$ is in $N(A)$.

Reduced Row Echelon Form

- To find the nullspace of an $m \times n$ matrix A, we first perform row operations on the matrix to transform it to reduced row echelon form.
- In a manner similar to Gauss-Jordan elimination, we force zeros above and below each pivot location and normalize each non-zero row so that a "1" appears in the pivot location.
- When a zero appears in a pivot location, we fix it with a row swap, if possible. If no row swap is possible to move a non-zero value into the pivot position, we pivot on the next column to the right for which a non-zero value appears or can be swapped into the pivot row.
- Elimination ends when no more pivots are possible because only zeros remain on the rows below that last pivot.
- The rank r of A is the number of pivots.
- Example 1

  $$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = R$$

  Reduced row echelon form results in two non-zero pivots. Both columns of the reduced row echelon form contain pivots. The rank of A is $r = 2$.
- Example 2

  $$B = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 \\ 0 & 2 \\ 0 & 0 \\ 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 2 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} = R$$

  Reduced row echelon form results in two non-zero pivots. Both columns of the reduced row echelon form contain pivots. The rank of B is $r = 2$.
- Example 3

  $$C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 2 & 2 & 4 \\ 0 & 2 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 2 & 0 & 4 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix} = R$$

  Reduced row echelon form results in two non-zero pivots. Two columns of the reduced row echelon form contain pivots. The other two columns do not, and are called "free columns." The rank of C is $r = 2$.

- Example 4

$$D = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 1 & 2 & 4 \\ 0 & 1 & 0 & 1 \\ 0 & 2 & 0 & 2 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} = R$$

  Reduced row echelon form results in two non-zero pivots. Two columns of the reduced row echelon form contain pivots. The other two columns do not, and are called "free columns." The rank of D is $r = 2$.

- Example 5

$$E = \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix} \rightarrow \begin{bmatrix} 1 & 3 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = R$$

  Reduced row echelon form results in one non-zero pivot. One column of the reduced row echelon form contains the pivot. The other two columns do not, and are called "free columns." The rank of E is $r = 1$.

Solutions to $A\mathbf{x} = \mathbf{0}$

- When the reduced row echelon form of the matrix A contains no free columns (all columns contain non-zero pivots—that is, $r = n$), the only solution to $A\mathbf{x} = \mathbf{0}$ is the zero vector $\mathbf{x} = \mathbf{0}$.
- When the reduced row echelon form of the matrix A contains at least one free column, a non-zero "special solution" to $A\mathbf{x} = \mathbf{0}$ can be found by setting one free variable equal to 1 and all other free variables equal to zero.
- A non-zero special solution can be created for each free column in the reduced row echelon form matrix derived from A. The equation $A\mathbf{x} = \mathbf{0}$ will have $n - r$ non-zero special solutions.
- If $n > m$, there will always be at least one free column resulting in the existence of a non-zero special solution.
- The complete set of solutions to $A\mathbf{x} = \mathbf{0}$ will be the set of all possible linear combinations of special solutions derived from the free variables in the reduced row echelon form.
- Example 1

$$A = \begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

  The only solution to $A\mathbf{x} = \mathbf{0}$ is the trivial solution, $\mathbf{x} = \mathbf{0} = (0, 0)$.

- Example 2

$$B = \begin{bmatrix} 1 & 2 \\ 3 & 8 \\ 2 & 4 \\ 6 & 16 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix}$$

  The only solution to $B\mathbf{x} = \mathbf{0}$ is the trivial solution, $\mathbf{x} = \mathbf{0} = (0, 0)$.

- Example 3

$$C = \begin{bmatrix} 1 & 2 & 2 & 4 \\ 3 & 8 & 6 & 16 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 2 \end{bmatrix}$$

The reduced row echelon form represents the following set of equations:

$$x_1 + 2x_3 = 0 \rightarrow x_1 = -2x_3$$
$$x_2 + 2x_4 = 0 \rightarrow x_2 = -2x_4$$

Letting $x_3 = 1$ and $x_4 = 0$ gives the special solution $\mathbf{s}_1 = (-2, 0, 1, 0)$. Letting $x_4 = 1$ and $x_3 = 0$ gives the special solution $\mathbf{s}_2 = (0, -2, 0, 1)$. The complete solution is $C\mathbf{x} = \mathbf{0}$ is any linear combination of $\mathbf{s}_1$ and $\mathbf{s}_2$. That is for any scalars c and d, the following is a solution of $C\mathbf{x} = \mathbf{0}$:

$$\mathbf{x} = c \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} 0 \\ -2 \\ 0 \\ 1 \end{bmatrix}$$

- Example 4

$$D = \begin{bmatrix} 1 & 1 & 2 & 4 \\ 1 & 2 & 2 & 5 \\ 1 & 3 & 2 & 6 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 & 2 & 3 \\ 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

Special solutions are $\mathbf{s}_1 = (-2, 0, 1, 0)$ and $\mathbf{s}_2 = (-3, -1, 0, 1)$. The complete solution to $D\mathbf{x} = \mathbf{0}$ has the form:

$$\mathbf{x} = c \begin{bmatrix} -2 \\ 0 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -3 \\ -1 \\ 0 \\ 1 \end{bmatrix}$$

- Example 5

$$E = \begin{bmatrix} 1 & 3 & 10 \\ 2 & 6 & 20 \\ 3 & 9 & 30 \end{bmatrix}, R = \begin{bmatrix} 1 & 3 & 10 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

Special solutions are $\mathbf{s}_1 = (-3, 1, 0)$ and $\mathbf{s}_2 = (-10, 0, 1)$. The complete solution to $E\mathbf{x} = \mathbf{0}$ has the form:

$$\mathbf{x} = c \begin{bmatrix} -3 \\ 1 \\ 0 \end{bmatrix} + d \begin{bmatrix} -10 \\ 0 \\ 1 \end{bmatrix}$$

3.3 The Complete Solution to $A\mathbf{x} = \mathbf{b}$

Overview of the Complete Solution

- In this section, we describe the solution to the matrix equation $A\mathbf{x} = \mathbf{b}$ when A is an $m \times n$ matrix.
- The equation may have no solutions, a unique solution, or an infinite number of solutions.
- We used the reduced row echelon form of A to find the nullspace of A, and we will use the reduced row echelon form of the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ to solve $A\mathbf{x} = \mathbf{b}$.
- To begin, perform row operations on $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ to put it in the reduced row echelon form $\begin{bmatrix} R & \mathbf{d} \end{bmatrix}$.
  - If there are rows in R containing all zeros and the vector $\mathbf{d}$ does not also have zeros on the corresponding rows, there will be **no solution** to the system. The vector $\mathbf{b}$ is not in the column space of A.
  - If a solution exists and R contains only pivot columns, **a unique solution** exists.
  - If a solution exists and R contains both pivot columns and free columns, **an infinite number of solutions** exist.
- If a solution exists, the general form of the solution is $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$, where $\mathbf{x}_p$ is a particular solution and $\mathbf{x}_n$ is any vector in the nullspace of A.
  - The particular solution $\mathbf{x}_p$ may be picked out from $\begin{bmatrix} R & \mathbf{d} \end{bmatrix}$ by setting free variables, if there are any, to zero.
  - Special solutions in the nullspace of A may be determined by setting each of the free variables in R equal to one while holding the others fixed at zero, as we did in the previous section.
  - The vectors $\mathbf{x}_n$ in the nullspace of A are linear combinations of the special solutions in $N(A)$.

Example 1

- Find the complete solution to $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 2 & -2 & 0 \\ 2 & 4 & 1 & -2 \\ 2 & 4 & 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} 3 \\ -3 \\ 4 \end{bmatrix}$$

- First, put the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ in reduced row echelon form:

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 2 & -2 & 0 & 3 \\ 2 & 4 & 1 & -2 & -3 \\ 2 & 4 & 0 & 1 & 4 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & -2 & 0 & 3 \\ 0 & 0 & 5 & -2 & -9 \\ 0 & 0 & 4 & 1 & -2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & -4/5 & -3/5 \\ 0 & 0 & 5 & -2 & -9 \\ 0 & 0 & 0 & 13/5 & 26/5 \end{bmatrix}$$

$$\to \begin{bmatrix} 1 & 2 & 0 & -4/5 & -3/5 \\ 0 & 0 & 1 & -2/5 & -9/5 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & -1 \\ 0 & 0 & 0 & 1 & 2 \end{bmatrix} = \begin{bmatrix} R & \mathbf{d} \end{bmatrix}$$

- Observe that columns 1, 3 and 4 of R are pivot columns. Column 2 is a free column.
- The particular solution $\mathbf{x}_p$ is obtained by setting the free variable x_2 equal to zero. The variables x_1, x_3 and x_4 take their values from the right-most column ($\mathbf{d}$) in reduced row echelon form.

$$\mathbf{x}_p = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix}$$

- The special solution $\mathbf{s}_1$ in the nullspace of A is obtained by setting the free variable x_2 equal to one. The variables x_1, x_3 and x_4 take their values from the free column (column 2) in R.

$$\mathbf{s}_1 = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

- The complete solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \mathbf{x}_p + \mathbf{x}_n$, where $\mathbf{x}_n$ is any vector in $N(A)$. Since $\mathbf{x}_n$ is a scalar multiple of $\mathbf{s}_1$, we may write the complete solution in the following way:

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} + c \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix}$$

  Instead of using c is the scalar in the nullspace term of the solution, one might also choose x_2 as the name of this scalar, since it will be the value of the second component in the vector $\mathbf{x}$.

$$\mathbf{x} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 2 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 - 2x_2 \\ x_2 \\ -1 \\ 2 \end{bmatrix}$$

Example 2

- Find the complete solution to $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$$

- First, put the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ in reduced row echelon form:

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 1 \\ 4 & 1 & 4 \\ 0 & 1 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & -7 & 0 \\ 0 & 1 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 1 \\ 0 & 1 & 0 \\ 0 & 1 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 9 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} R & \mathbf{d} \end{bmatrix}$$

- The bottom row of R contains all zeros but the last row of $\mathbf{d}$ is not zero. This row represents the equation $0 = 1$, which is impossible. The equation $A\mathbf{x} = \mathbf{b}$ has no solutions. The vector $\mathbf{b}$ is not in the column space of A.

Example 3

- Find the complete solution to $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 1 & 2 \\ 4 & 1 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 7 \\ -1 \end{bmatrix}$$

- First, put the augmented matrix $\begin{bmatrix} A & \mathbf{b} \end{bmatrix}$ in reduced row echelon form:

$$\begin{bmatrix} A & \mathbf{b} \end{bmatrix} = \begin{bmatrix} 1 & 2 & 0 \\ 4 & 1 & 7 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & -7 & 7 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 2 & 0 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} R & \mathbf{d} \end{bmatrix}$$

- The bottom row of R contains all zeros and the last row of $\mathbf{d}$ is also zero. This row represents the equation $0 = 0$, which is no contradiction. The vector $\mathbf{b}$ is contained in the column space of A.
- Both columns of R are pivot columns. There are no free columns. The system will have a particular solution, but only the zero vector in the nullspace term.
- The particular solution $\mathbf{x}_p$ is obtained from the right-most column ($\mathbf{d}$) in reduced row echelon form.

$$\mathbf{x}_p = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

- The complete solution to $A\mathbf{x} = \mathbf{b}$ is

$$\mathbf{x} = \mathbf{x}_p + \mathbf{0} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$$

Conclusions

- Properties of matrices A having full column rank ($r = n$).
  - All columns of A are pivot columns.
  - There are no free variables or special solutions.
  - The nullspace $N(A)$ contains only the zero vector $\mathbf{x} = \mathbf{0}$.
  - If $A\mathbf{x} = \mathbf{b}$ has a solution, it has only one solution.
- Properties of matrices A having full row rank ($r = m$).
  - All rows have pivots and R has no zero rows.
  - The equation $A\mathbf{x} = \mathbf{b}$ will have a solution for any choice of $\mathbf{b}$.
  - The column space $C(A)$ is all of $\mathbb{R}^m$.
  - There are $n - r = n - m$ special solutions in the nullspace of A.
- Four possibilities.
  - Full rank, square and invertible ($r = m = n$). $A\mathbf{x} = \mathbf{b}$ will have one solution ($A^{-1}\mathbf{b}$).

$$\begin{bmatrix} 1 & 2 \\ 3 & 8 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix}$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

  - Full row rank, short and wide ($r = m < n$). $A\mathbf{x} = \mathbf{b}$ will have infinite solutions.

$$\begin{bmatrix} 1 & 2 & -1 \\ 3 & 8 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \end{bmatrix}$$
$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} + x_3 \begin{bmatrix} 5 \\ -2 \\ 1 \end{bmatrix}$$

– Full column rank, tall and thin ($r = n < m$). $A\mathbf{x} = \mathbf{b}$ will have no solutions or one solution, depending on whether $\mathbf{b}$ is in the column space of A.

$$\begin{bmatrix} 1 & 2 \\ 3 & 8 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 14 \\ -1 \end{bmatrix}$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

Change the last component of $\mathbf{b}$ ($= -1$) for an example with no solution.

– Not full rank ($r < m$ and $r < n$). $A\mathbf{x} = \mathbf{b}$ will have no solutions or infinite solutions, depending on whether $\mathbf{b}$ is in the column space of A.

$$\begin{bmatrix} 1 & 2 \\ 2 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 8 \end{bmatrix}$$
$$\begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} -2 \\ 1 \end{bmatrix}$$

Change the last component of $\mathbf{b}$ ($= 8$) for an example with no solution.