MATH 2418 Linear Algebra. Week 1

Dr. James W. Miller

August 23 and 25, 2022

Summary of this Week's Goals

Welcome to MATH 2418, Linear Algebra! This week we will discuss the class syllabus and Sections 1.1 (Vectors and Linear Combinations) and 1.2 (Lengths and Dot Products) of our text. By the end of the week, you should have a basic understanding of the class expectations, structure, calendar and grading policies. You should understand what scalars and vectors are, how to multiply a vector by a scalar and how to add vectors to form linear combinations of vectors. You should be able to visualize vectors as arrows in one-, two- and three-dimensional space and describe the set of all possible linear combinations of vectors. You should be able to calculate the length of a vector and the dot product of two vectors. You should know how the dot product of two vectors relates to the angle formed by the vectors.

Syllabus Review

- Final syllabus will be posted on CourseBook and eLearning by the end of the week.
- Contact Information: James W. Miller (James.Miller3@utdallas.edu or jwm170630@utdallas.edu). Contact is by e-mail only, no phone calls.
- Office Hours: Monday 5pm - 6pm in FA 2.106, or by appointment. There will be no regularly scheduled office hours on exam days.
- You must be enrolled in a Problem section (3xx) and the Exam section (701). Many important announcements will be posted on eLearning in the Exam section.
- Your first homework assignment has been posted on eLearning and is due during the problem session next week (on Wed Aug 31 or Fri Sep 2), when you will also take your first quiz. Do all recommended problems. Math is a skill and skills require practice. You cannot learn Linear Algebra by reading the text, listening to lectures and taking notes. Those are helpful, but you must do practice problems in order to master the concepts.
- Midterm exams will occur on Oct 6 and Nov 10 (both Thursdays) from 7:30pm to 8:45pm. Locations TBA.

1.1 Vectors and Linear Combinations

What is a vector?

- An n-tuple of numbers. Examples: (1,0), ($\sqrt{2}$, -1, 0), (0, 0, 1, -1). We can give vectors names and write them as "column vectors."

$$\boldsymbol{u} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \boldsymbol{v} = \begin{bmatrix} \sqrt{2} \\ -1 \\ 0 \end{bmatrix}, \boldsymbol{w} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}$$

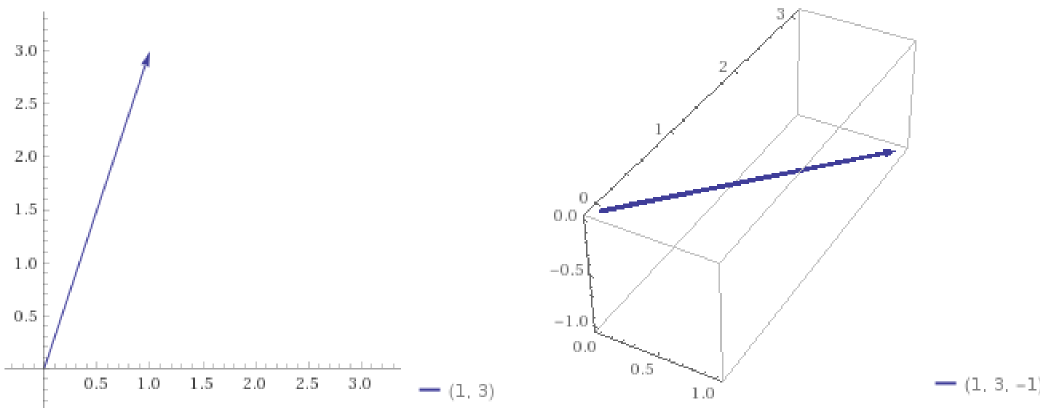

Figure 1: Plots of the vectors $(1, 3)$ and $(1, 3, -1)$

- A point in n-dimensional space ($\mathbb{R}^n$)
- A ray emanating from the origin of an n-dimensional space to a specific point in that space. This ray has both magnitude and direction, unless it is a zero vector. See Figure 1.
- A zero vector has zeros in every component. Its magnitude is zero, and it has no direction.

Vector Addition

- Vectors with the same number of components my be added together by simply adding each value of one vector to the corresponding value of the other vector.
- Vector addition may be visualized as the start of one vector being placed at the endpoint of the other, as shown in Figure 2.

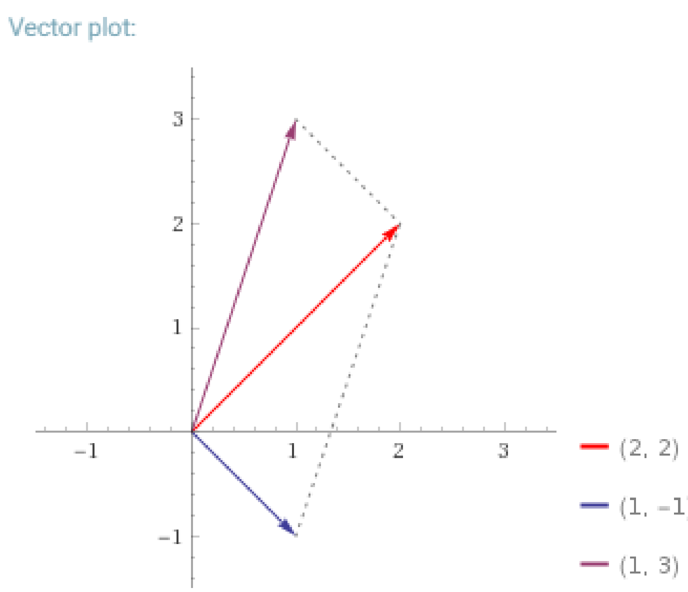

Figure 2: Plot of the vector sum $(1, 3) + (1, -1)$

- Example:
$$\begin{bmatrix}\sqrt{2}\\-1\\0\end{bmatrix}+\begin{bmatrix}-1\\1\\4\end{bmatrix}=\begin{bmatrix}\sqrt{2}-1\\0\\4\end{bmatrix}$$

- Vector addition is commutative ($\boldsymbol{u}+\boldsymbol{v}=\boldsymbol{v}+\boldsymbol{u}$) and associative ($(\boldsymbol{u}+\boldsymbol{v})+\boldsymbol{w}=\boldsymbol{u}+(\boldsymbol{v}+\boldsymbol{w})$)

What is a scalar?

- A scalar is simply a number (in $\mathbb{R}$, for our purposes). We can give them names just as we gives names to vectors.
- You may think of a scalar as a one-dimensional vector, but more often we think of a scalar as something different from a vector.
- Examples: $a=1$, $b=0$, $c=-1/2$, $d=\pi$, $e=-5$, $f=\sqrt{3}$

Scalar Multiplication

- Vectors of any length may be multiplied by a scalar by simply multiplying each value of the vector by the scalar.
- Example:
$$\left(\frac{1}{2}\right)\begin{bmatrix}\sqrt{3}\\-1\end{bmatrix}=\begin{bmatrix}\frac{\sqrt{3}}{2}\\-\frac{1}{2}\end{bmatrix}$$

- Scalar multiplication ($\boldsymbol{v}=c\boldsymbol{u}$) results in increasing (if $|c|>1$) or decreasing (if $|c|<1$) the length of the vector and/or reversing its direction (if $c<0$).
- The set of all possible scalar multiples of a vector fills a one-dimensional space (a line) containing the vector. This set always includes the zero vector.
- Shown in Figure 3 is a vector plot of $2(1,2)$. Multiplying by 2 is equivalent to adding the vector to itself.

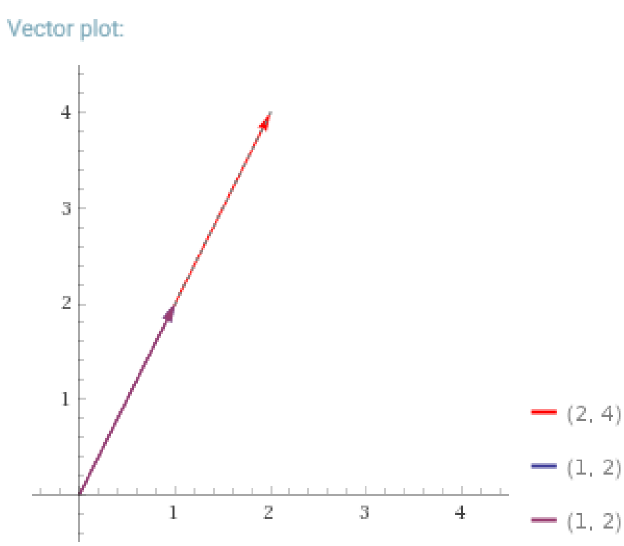

Figure 3: Plot of the vector $2(1,2)$

Linear Combinations

- We can combine the concepts of vector addition and scalar multiplication to form what is called a "linear combination."
- A linear combination of two vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ is any vector having the form $c\boldsymbol{u} + d\boldsymbol{v}$, where c and d are scalars.
- Vector substraction can be expressed as a linear combination: $\boldsymbol{u} - \boldsymbol{v} = (1)\boldsymbol{u} + (-1)\boldsymbol{v}$. Subtracting a vector from itself results in a zero vector.
- As long as the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ do not lie on the same line, the set of all possible linear combinations of the vectors fills a two-dimensional plane containing the vectors. This plane must include the zero vector. If the dimension of $\boldsymbol{u}$ and $\boldsymbol{v}$ is two, then plane is the entire plane of $\mathbb{R}^2$. If the dimension is $n > 2$, the two-dimensional plane is a subset of $\mathbb{R}^n$.
- If the vectors $\boldsymbol{u}$ and $\boldsymbol{v}$ lie on the same line, one can be expressed as a scalar multiple of the other and the set of all possible linear combinations of the vectors consists only of all points on that line. This is the same one-dimensional set which is the set of all scalar multiples of $\boldsymbol{u}$ and $\boldsymbol{v}$.
- A linear combination of three vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ is any vector having the form $c\boldsymbol{u} + d\boldsymbol{v} + e\boldsymbol{w}$, where c, d and e are scalars.
- As long as the vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ do not lie on the same plane, the set of all possible linear combinations of the vectors fills a three-dimensional space containing the vectors. If the dimension of $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ is three, then space is the entire space of $\mathbb{R}^3$. If the dimension is $n > 3$, the three-dimensional space is a subset of $\mathbb{R}^n$.
- If the vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ lie on the same line or plane, at least one of the vectors may be expressed as a linear combination of the other two and the set of all possible linear combinations of the vectors consists only of all points on that line or plane.
- You may combine as many vectors as you like in a linear combination by multiplying each vector by some scalar and adding them together. Exploring and describing the set of all possible linear combinations of vectors will be a key theme in this course.

Standard Basis Vectors

- In $\mathbb{R}^2$, standard basis vectors are $\boldsymbol{i} = (1, 0)$ and $\boldsymbol{j} = (0, 1)$. These vectors point in the directions of the positive x and y axes, respectively. Each vector (x, y) in $\mathbb{R}^2$ can be written as a linear combination of the standard basis vectors via the expression $(x, y) = x\boldsymbol{i} + y\boldsymbol{j}$
- In $\mathbb{R}^3$, standard basis vectors are $\boldsymbol{i} = (1, 0, 0)$, $\boldsymbol{j} = (0, 1, 0)$ and $\boldsymbol{k} = (0, 0, 1)$. These vectors point in the directions of the positive x, y and z axes, respectively. Each vector (x, y, z) in $\mathbb{R}^3$ can be written as a linear combination of the standard basis vectors via the expression $(x, y, z) = x\boldsymbol{i} + y\boldsymbol{j} + z\boldsymbol{k}$

Dot Product

- The dot product (also called "inner product") of two vectors produces a scalar which is the sum of the products of the corresponding terms of each vector. Shown below are formulas for the dot product in $\mathbb{R}^2$ and $\mathbb{R}^3$.

$$\begin{aligned}\boldsymbol{u} \cdot \boldsymbol{v} &= (u_1, u_2) \cdot (v_1, v_2) = u_1 v_1 + u_2 v_2 \\ \boldsymbol{u} \cdot \boldsymbol{v} &= (u_1, u_2, u_3) \cdot (v_1, v_2, v_3) = u_1 v_1 + u_2 v_2 + u_3 v_3\end{aligned}$$

- Examples:

$$\begin{gathered}(1, 0) \cdot (-1, 2) = (1)(-1) + (0)(2) = -1 + 0 = -1 \\ (1, -1) \cdot (1, 1) = (1)(1) + (-1)(1) = 1 - 1 = 0 \\ (0, 2, -1) \cdot (-1, 3, 5) = (0)(-1) + (2)(3) + (-1)(5) = 0 + 6 - 5 = 1\end{gathered}$$

- The dot product operation is commutative ($\boldsymbol{u} \cdot \boldsymbol{v} = \boldsymbol{u} \cdot \boldsymbol{v}$) and distributive over vector addition ($\boldsymbol{u} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{u} \cdot \boldsymbol{w}$ and $(\boldsymbol{u} + \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} + \boldsymbol{v} \cdot \boldsymbol{w}$) and subtraction ($\boldsymbol{u} \cdot (\boldsymbol{v} - \boldsymbol{w}) = \boldsymbol{u} \cdot \boldsymbol{v} - \boldsymbol{u} \cdot \boldsymbol{w}$ and $(\boldsymbol{u} - \boldsymbol{v}) \cdot \boldsymbol{w} = \boldsymbol{u} \cdot \boldsymbol{w} - \boldsymbol{v} \cdot \boldsymbol{w}$).

Vector Length

- The length of a vector $\boldsymbol{u}$ is the distance from the coordinates of the vector to the origin in $\mathbb{R}^n$. The length of $\boldsymbol{u}$ is denoted $||\boldsymbol{u}||$.
- In $\mathbb{R}^2$, the Pythagorean Theorem gives the formula $||\boldsymbol{u}|| = \sqrt{u_1^2 + u_2^2}$.
- Examples: $||(1, 2)|| = \sqrt{5}$ and $||(1, 2, 3)|| = \sqrt{14}$.
- In $\mathbb{R}^n$, repeated applications of the Pythagorean Theorem gives the formula $||\boldsymbol{u}|| = \sqrt{u_1^2 + u_2^2 + ... + u_n^2}$.
- Note that $u_1^2 + u_2^2 + ... + u_n^2 = \boldsymbol{u} \cdot \boldsymbol{u}$, so the dot product of a vector with itself is $||\boldsymbol{u}||^2$.
- A vector whose length is 1 is called a “unit vector.”
- In $\mathbb{R}^2$, unit vectors have the form $(\cos(\theta), \sin(\theta))$, where θ is the angle the vector forms with the positive x-axis.
- For any non-zero vector $\boldsymbol{u}$, the vector $\frac{\boldsymbol{u}}{||\boldsymbol{u}||}$ is a unit vector in the same direction as $\boldsymbol{u}$. Note this this vector is a scalar multiple of $\boldsymbol{u}$ and the scalar being multiplied by $\boldsymbol{u}$ is the reciprocal of $||\boldsymbol{u}||$.

The Cosine Formula and Related Properties

- For any vectors $\boldsymbol{u}$ and $\boldsymbol{v}$, if θ is the angle formed by the vectors at the origin in $\mathbb{R}^2$, then
$$\cos\theta = \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{||\boldsymbol{u}||||\boldsymbol{v}||}$$
- In $\mathbb{R}^2$, let α denote the angle $\boldsymbol{u}$ makes with the positive x-axis and let β denote the angle $\boldsymbol{v}$ makes with the positive x-axis. Then $\cos\alpha = \frac{u_1}{||\boldsymbol{u}||}$, $\sin\alpha = \frac{u_2}{||\boldsymbol{u}||}$, $\cos\beta = \frac{v_1}{||\boldsymbol{v}||}$, and $\sin\beta = \frac{v_2}{||\boldsymbol{v}||}$ and
$$\begin{aligned}\cos\theta &= \cos(\alpha - \beta) = \cos(\beta - \alpha)\\ &= \cos\alpha\cos\beta + \sin\alpha\sin\beta\\ &= \frac{u_1}{||\boldsymbol{u}||}\frac{v_1}{||\boldsymbol{v}||} + \frac{u_2}{||\boldsymbol{u}||}\frac{v_2}{||\boldsymbol{v}||}\\ &= \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{||\boldsymbol{u}||||\boldsymbol{v}||}\end{aligned}$$
- In $\mathbb{R}^n$, imagine a triangle having $\boldsymbol{u}$ and $\boldsymbol{v}$ as two of its sides and a third side which joins the endpoints of the vectors. The third side is congruent with the vector $\boldsymbol{u} - \boldsymbol{v}$, so the lengths of the sides are $||\boldsymbol{u}||$, $||\boldsymbol{v}||$ and $||\boldsymbol{u} - \boldsymbol{v}||$. The Law of Cosines gives
$$||\boldsymbol{u} - \boldsymbol{v}||^2 = ||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 - 2||\boldsymbol{u}||||\boldsymbol{v}||\cos\theta$$
Solving for $\cos\theta$ gives
$$\begin{aligned}\cos\theta &= \frac{||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 - ||\boldsymbol{u} - \boldsymbol{v}||^2}{2||\boldsymbol{u}||||\boldsymbol{v}||}\\ &= \frac{||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 - (\boldsymbol{u} - \boldsymbol{v}) \cdot (\boldsymbol{u} - \boldsymbol{v})}{2||\boldsymbol{u}||||\boldsymbol{v}||}\\ &= \frac{||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 - (\boldsymbol{u} \cdot \boldsymbol{u} - 2\boldsymbol{u} \cdot \boldsymbol{v} + \boldsymbol{v} \cdot \boldsymbol{v})}{2||\boldsymbol{u}||||\boldsymbol{v}||}\\ &= \frac{||\boldsymbol{u}||^2 + ||\boldsymbol{v}||^2 - (||\boldsymbol{u}||^2 - 2\boldsymbol{u} \cdot \boldsymbol{v} + ||\boldsymbol{v}||^2)}{2||\boldsymbol{u}||||\boldsymbol{v}||}\\ &= \frac{\boldsymbol{u} \cdot \boldsymbol{v}}{||\boldsymbol{u}||||\boldsymbol{v}||}\end{aligned}$$

- Vectors whose dot product is zero are perpendicular (Figure 4).

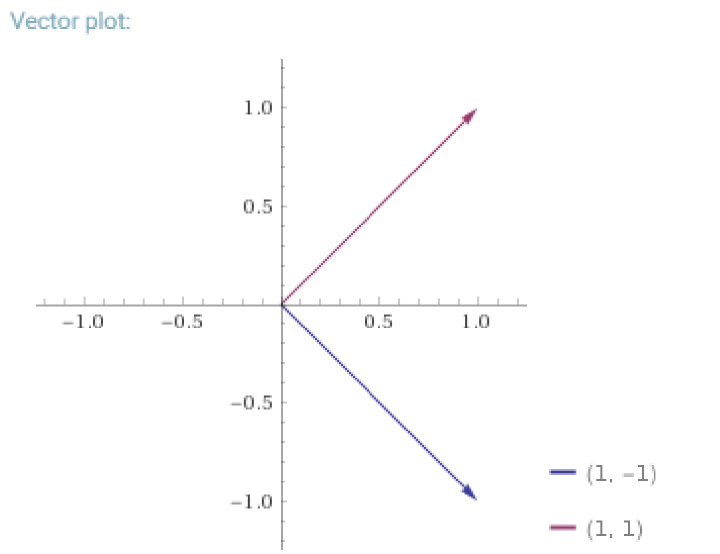

Figure 4: Plots of the vectors $(1,1)$ and $(1,-1)$. Observe that the dot product is zero and the vectors are perpendicular.

- Schwartz Inequality: $|\boldsymbol{u} \cdot \boldsymbol{v}| \leq ||\boldsymbol{u}||||\boldsymbol{v}||$.
- Triangle Inequality: $||\boldsymbol{u}+\boldsymbol{v}|| \leq ||\boldsymbol{u}|| + ||\boldsymbol{v}||$.