MATH 2418 Linear Algebra. Week 3

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Summary of this Week's Goals

This week we will discuss Sections 2.2 (The Idea of Elimination) and 2.3 (Elimination Using Matrices) of our text. We will extend the definition of matrix multiplication to define the way we multiply one matrix by another matrix and consider some properties of matrix multiplication. We will define an algorithm for using elimination to triangularize a linear system and backsolve for the solution to $A\boldsymbol{x} = \boldsymbol{b}$. We will learn how to identify multipliers and pivots that are used in the algorithm. We will observe conditions which might cause the algorithm to fail, and we will learn if the algorithm can be adjusted to overcome the failure and produce a unique solution, or if failure is permanent producing no solution or an infinite number of solutions. Finally, we will learn to describe the elimination process as a sequence of matrix multiplications. We will learn to write the entire system $A\boldsymbol{x} = \boldsymbol{b}$ as a single "augmented matrix" and solve the system by performing operations on the augmented matrix.

Announcements

- The final exam is scheduled for Saturday, December 10, 2:00-4:45PM.

2.2 The Idea of Elimination

Using Elimination to Solve $A\mathbf{x} = \mathbf{b}$

- Last week, we began using a process called "elimination" to solve the system $A\mathbf{x} = \mathbf{b}$. Today, we will define the process more carefully and investigate cases where it might fail temporarily or permanently.
- In short, we will systematically transform the system to a triangular form which can quickly be solved by back substitution. We do this by subtracting a multiple of one equation from other equations in such a way that a variable is eliminated from an equation at each step.
- For a 3×3 system, the steps are:
  - Write the system of equations associated with the row picture of the system.
  - Provided that the coefficient of x_1 in equation (1) is nonzero, eliminate x_1 from equation (2) by subtracting a multiple of equation (1) from equation (2). The coefficient a_{11} is called the pivot and the multiplier is a_{21}/a_{11}.
  - Provided that the coefficient of x_1 in equation (1) is nonzero, eliminate x_1 from equation (3) by subtracting a multiple of equation (1) from equation (3). We are still pivoting on the coefficient a_{11} at this step, but now the multiplier is a_{31}/a_{11}. After this step, there should be a 2×2 system remaining in equations (2) and (3) which involves only x_2 and x_3.
  - Provided that the coefficient of x_2 in equation (2) is nonzero (called ℓ_{22}), eliminate x_2 from equation (3) by subtracting a multiple of equation (2) from equation (3). The coefficient ℓ_{22} is called the pivot and the multiplier is ℓ_{32}/ℓ_{22}. (The coefficients ℓ_{22} and ℓ_{32} are coefficients in the system after x_1 has been eliminated in the previous step. They are no longer the same as the elements of the matrix A in the original system.)

- The system should now be triangular. The third equation should contain only the variable x_3. The coefficient of x_3 is called ℓ_{33}, and this is also what we would call the third pivot. This pivot is used to begin the process of back substitution to solve for x_3 first.
- Apply the process of back substitution to solve for x_3 using equation (3), then x_2 using equation (2), then x_1 using equation (1).

• Example. Solve the following system of the form $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

- The row picture considers the equations

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 4 \\ x_1 - x_2 + x_3 &= 2 \\ -x_1 + x_2 + 2x_3 &= 1 \end{aligned}$$

- The first pivot is $a_{11} = 2$ and the first multiplier is $a_{21}/a_{11} = 1/2$. Subtracting 1/2 times equation (1) from equation (2) eliminates x_1 from equation (2) resulting in the following new system:

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 4 \\ -\frac{3}{2}x_2 + \frac{3}{2}x_3 &= 0 \\ -x_1 + x_2 + 2x_3 &= 1 \end{aligned}$$

- We are still pivoting on $a_{11} = 2$ to eliminate x_1 and the second multiplier is $a_{31}/a_{11} = -1/2$. Subtracting -1/2 times equation (1) from equation (3) eliminates x_1 from equation (3) resulting in the following new system:

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 4 \\ -\frac{3}{2}x_2 + \frac{3}{2}x_3 &= 0 \\ \frac{3}{2}x_2 + \frac{3}{2}x_3 &= 3 \end{aligned}$$

- The second pivot is $\ell_{22} = -3/2$ and the third multiplier is $\ell_{32}/\ell_{22} = -1$. Subtracting -1 times equation (2) from equation (3) eliminates x_2 from equation (3) resulting in the following new system:

$$\begin{aligned} 2x_1 + x_2 - x_3 &= 4 \\ -\frac{3}{2}x_2 + \frac{3}{2}x_3 &= 0 \\ 3x_3 &= 3 \end{aligned}$$

- The system is now triangular. We could write the system in the form $U\mathbf{x} = \mathbf{c}$, where U is an upper-triangular matrix. The third pivot is $\ell_{33} = 3$. Use this pivot to begin the process of back substitution, first solving the third equation for x_3.

$$x_3 = 1$$

- Next, solve the second equation for x_2.

$$x_2 = -\frac{2}{3}\left[-\left(\frac{3}{2}\right)(1)\right] = 1$$

– Finally, solve the first equation for x_1.

$$x_1 = \frac{1}{2}[4 - 1 + 1] = 2$$

– Our solution is the vector $(2, 1, 1)$. The pivots used in this process are the coefficients along the diagonal of the triangular system $U\mathbf{x} = \mathbf{c}$: 2, -3/2 and 3.

- For a $n \times n$ system, continue pivoting on each diagonal element until the system is fully diagonalized and ready for back substitution.
  - Use the first equation to create zeros below the first pivot.
  - Use the new equation (2) to create zeros below the second pivot.
  - Keep going to find all n pivots and the upper triangular U.

When Elimination Fails

- The elimination process will fail if a zero occurs along the diagonal of the system where you would like to pivot. The failure may be temporary of permanent.
- If a zero occurs in a pivot position, swap the equation containing the zero in the pivot location with an equation below which does not have a zero in the same position, if such an equation exists. This will solve the problem and you can continue with elimination after swapping rows. This situation is called "temporary failure."
- If a zero occurs in a pivot position and zeros also occur in the same position on all rows below this equation, you have no equations to swap that will resolve the failure. This situation is called "permanent failure." Permanent failure may lead to no solution at all or an infinite number of solutions. This happens when the columns of A are linearly dependent.
- When a permanent failure occurs in the elimination process, continue triangularizing as much of the remaining system as you can. Eventually, you will encounter a row with zero coefficients in front of every variable. If the right hand side of the system also has a zero on this row, the system has an infinite number of solutions. If the right hand side has something other than a zero on that row, the system has no solution.
- Example: Temporary failure.

$$\begin{bmatrix} 0 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 2 \end{bmatrix}$$

  – The row picture considers the following equations. There is a zero in the coefficient a_{11}, so you cannot pivot there. Simply swap equation (1) with equation (2), which does not have a zero in the corresponding position.

$$\begin{aligned} x_2 - x_3 &= -1 \\ x_1 - x_2 + x_3 &= 1 \\ -x_1 + x_2 + &= 2 \end{aligned}$$

  – After swapping rows, continue with the elimination process. The variable x_1 is already eliminated from equation (2), so continue by eliminating x_1 from equation (3).

$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ x_2 - x_3 &= -1 \\ -x_1 + x_2 &= 2 \end{aligned}$$

- After adding equation (1) to equation (3), the system is triangular and ready for back substitution.

$$\begin{aligned} x_1 - x_2 + x_3 &= 1 \\ x_2 - x_3 &= -1 \\ x_3 &= 3 \end{aligned}$$

- Example: Permanent failure with an infinite number of solutions.

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ -2 \end{bmatrix}$$

  - The row picture considers the following equations.

$$\begin{aligned} x_1 - x_2 + x_3 &= -1 \\ 2x_1 + x_2 - x_3 &= 1 \\ -x_1 - 2x_2 + 2x_3 &= -2 \end{aligned}$$

  - To eliminate x_1, subtract 2 times equation (1) from equation (2) and add equation (1) to equation (3).

$$\begin{aligned} x_1 - x_2 + x_3 &= -1 \\ 3x_2 - 3x_3 &= 3 \\ -3x_2 + 3x_3 &= -3 \end{aligned}$$

  - To eliminate x_2, add equation (2) to equation (3).

$$\begin{aligned} x_1 - x_2 + x_3 &= -1 \\ 3x_2 - 3x_3 &= 3 \\ 0 &= 0 \end{aligned}$$

  - The result of this step leaves a zero where we would like to perform our third pivot. So elimination fails. Since there are no equations below the equation $0 = 0$ to swap with, failure is permanent. Since the equation $0 = 0$ is not a contradiction, the system has an infinite number of solutions. We could allow x_3 to be a free variable and solve for x_1 and x_2 using the equations (1) and (2) to describe the solution space.

- Example: Permanent failure with no solutions.

$$\begin{bmatrix} 1 & -1 & 1 \\ 2 & 1 & -1 \\ -1 & -2 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix}$$

  - The row picture considers the following equations.

$$\begin{aligned} x_1 - x_2 + x_3 &= -1 \\ 2x_1 + x_2 - x_3 &= 1 \\ -x_1 - 2x_2 + 2x_3 &= 0 \end{aligned}$$

  - To eliminate x_1, subtract 2 times equation (1) from equation (2) and add equation (1) to equation (3).

$$\begin{aligned} x_1 - x_2 + x_3 &= -1 \\ 3x_2 - 3x_3 &= 3 \\ -3x_2 + 3x_3 &= -1 \end{aligned}$$

– To eliminate x_2, add equation (2) to equation (3).

$$\begin{aligned} x_1 - x_2 + x_3 &= -1 \\ 3x_2 - 3x_3 &= 3 \\ 0 &= -2 \end{aligned}$$

– The result of this step leaves a zero where we would like to perform our third pivot. So elimination fails. Since there are no equations below the equation $0 = -2$ to swap with, failure is permanent. Since the equation $0 = -2$ is a contradiction, the system has no solutions.

2.3 Elimination Using Matrices

Multiplying a Matrix by Another Matrix

- An $m \times n$ matrix A can be multiplied by an $n \times k$ matrix B. The result is an $m \times k$ matrix which we might call C. The number of columns in A must equal the number of rows in B in order for the matrices to be compatible for matrix multiplication.

- To perform matrix multiplication, calculate the dot product of each row of the matrix A with each column of the matrix B. That is, the ij-th element of C (C_{ij}) is the dot product of the i-th row of A with the j-th column of B.

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} (1,-1)\cdot(b_{11},b_{21}) & (1,-1)\cdot(b_{12},b_{22}) \\ (0,2)\cdot(b_{11},b_{21}) & (0,2)\cdot(b_{12},b_{22}) \\ (1,1)\cdot(b_{11},b_{21}) & (1,1)\cdot(b_{12},b_{22}) \end{bmatrix} = \begin{bmatrix} b_{11}-b_{21} & b_{12}-b_{22} \\ 2b_{21} & 2b_{22} \\ b_{11}+b_{21} & b_{12}+b_{22} \end{bmatrix}$$

- We could also imagine multiplying the matrix A by each column of B using the definition of multiplication of a matrix and a vector which we already know, and putting those results into the columns of C. If the columns of B are $\mathbf{b}_1$ and $\mathbf{b}_2$, then

$$AB = A\begin{bmatrix} \mathbf{b}_1 & \mathbf{b}_2 \end{bmatrix} = \begin{bmatrix} A\mathbf{b}_1 & A\mathbf{b}_2 \end{bmatrix}$$

- Examples:

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 3 & -3 \end{bmatrix} = \begin{bmatrix} -1 & 3 \\ 6 & -6 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 & -1 \\ 2 & 0 & 1 & 0 \\ 3 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 7 & -1 & 1 & 3 \\ 7 & 1 & 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 & y_1 \\ x_2 & y_2 \\ x_3 & y_3 \end{bmatrix} = \begin{bmatrix} x_1 & y_1 \\ x_2 - x_1 & y_2 - y_1 \\ x_3 - x_2 & y_3 - y_2 \end{bmatrix}$$

- Some properties of matrix multiplication:
  – Matrix multiplication is associative—that is, $A(BC) = (AB)C$ for matrices A, B and C with dimensions compatible for multiplication.
  – Matrix multiplication is *not* commutative—that is, $AB \neq BA$ for many matrices A and B.
  – When the inverse of a matrix A exists, the product of A with its inverse is an identity matrix—that is, $AA^{-1} = A^{-1}A = I$. Multiplication of A with its inverse is commutative even though matrix multiplication in general is not.

For example,

$$\begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Elimination as a Form of Matrix Multiplication

- Each step in the elimination process of solving $A\mathbf{x} = \mathbf{b}$ may be expressed as a matrix multiplication.
- When we solved the following system earlier, our first step was to subtract $1/2$ times equation (1) from equation (2) to eliminate x_1 from the second equation.

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

- We may express that operation as the matrix multiplication shown below.

$$\begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -1/2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 2 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & -3/2 & 3/2 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

- In effect, we multiplied both sides of the equation $A\mathbf{x} = \mathbf{b}$ by a matrix we might call E_{21}, creating a new system $E_{21}A\mathbf{x} = E_{21}\mathbf{b}$. The matrix E_{21} is an identity matrix with the entry in the second row, first column (the 2-1 position) replaced with the multiplier $-1/2$. (The matrix E_{21} is called an "elimination matrix" or an "elementary matrix.")
- In the next step, we eliminated x_1 from the third equation by adding $1/2$ times equation (1) to equation (3). Similarly, this may be expressed as multiplying our new system on both sides by a matrix we might call E_{31}, whose construction is similar to that of the matrix E_{21}.

$$\begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & -1 \\ 0 & -3/2 & 3/2 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 1/2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 4 \\ 0 \\ 1 \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & -1 \\ 0 & -3/2 & 3/2 \\ 0 & 3/2 & 3/2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 4 \\ 0 \\ 3 \end{bmatrix}$$

  The resulting system may now be expressed as $E_{31}E_{21}A\mathbf{x} = E_{31}E_{21}\mathbf{b}$.

- Finally, to eliminate x_2 from equation (3), we added equation (2) to equation (3). This may be expressed as multiplication by the matrix E_{32}.

$$\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}2&1&-1\\0&-3/2&3/2\\0&3/2&3/2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1&0&0\\0&1&0\\0&1&1\end{bmatrix}\begin{bmatrix}4\\0\\3\end{bmatrix}$$

$$\begin{bmatrix}2&1&-1\\0&-3/2&3/2\\0&0&3\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}4\\0\\3\end{bmatrix}$$

This triangular system, which we described in the form $U\mathbf{x} = \mathbf{c}$ earlier, can also be expressed as $E_{32}E_{31}E_{21}A\mathbf{x} = E_{32}E_{31}E_{21}\mathbf{b}$, so $U = E_{32}E_{31}E_{21}A$ and $\mathbf{c} = E_{32}E_{31}E_{21}\mathbf{b}$.

- When elimination fails temporarily, we resolve the failure by swapping rows to bring a non-zero coefficient into the pivot position. This operation of swapping rows can also be written as a matrix multiplication.

- In an earlier example, we swapped rows (1) and (2) of the following system to bring a non-zero coefficient into the first pivot position.

$$\begin{bmatrix}0&1&-1\\1&-1&1\\-1&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}-1\\1\\2\end{bmatrix}$$

- The operation of swapping rows may be expressed as multiplying both sides of the system by a matrix we may call P_{12} (a "permutation matrix"). The matrix P_{12} is created by swapping the first and second rows of the identity matrix.

$$\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}0&1&-1\\1&-1&1\\-1&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}0&1&0\\1&0&0\\0&0&1\end{bmatrix}\begin{bmatrix}-1\\1\\2\end{bmatrix}$$

$$\begin{bmatrix}1&-1&1\\0&1&-1\\-1&1&0\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\-1\\2\end{bmatrix}$$

- When we triangularize the system $A\mathbf{x} = \mathbf{b}$, the triangular system may be written in the form $EA\mathbf{x} = E\mathbf{b}$, where the matrix E is the product of E_{ij} and P_{ij} matrices involved in the elimination steps. The order of the product is important. Recall that matrix multiplication is not commutative.

Using an Augmented Matrix

- When elimination steps are performed, they operate on both sides of the equation. On the left, they operate on the matrix A of coefficients in the linear system. On the right, they operate on the resulting vector $\mathbf{b}$.

- Sometimes, we will put all known parts of the system (A and $\mathbf{b}$) into a single $n \times (n+1)$ matrix which we call an "augmented matrix" and apply the elimination steps to the augmented matrix.

- For example, we could solve the system

$$\begin{bmatrix}2&1&-1\\1&-1&1\\-1&1&2\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}4\\2\\1\end{bmatrix}$$

by encapsulating the entire system into the augmented matrix shown below:

$$\begin{bmatrix} 2 & 1 & -1 & 4 \\ 1 & -1 & 1 & 2 \\ -1 & 1 & 2 & 1 \end{bmatrix}$$

- By performing the elimination operations on the augmented matrix, we effectively apply the operations to both sides of the original equation, but we do so with a more concise notation. You will find this handy someday when you do your homework problems.

Summary: What do Matrices Do?

- "Matrices *act*. They don't just sit there." - Strang, p 63.
- Matrices perform operations which we can often describe with English verbs (stretch, rotate, flip, project, swap, combine, etc.) Sometimes matrices perform combinations of steps which can be described as products of matrix multiplication.
- In the elimination process, E_{ij} matrices add a multiple of the pivot row j to row i to eliminate a variable in row i. These matrices are made by replacing the ij-th value of an identity matrix with the multiplier needed to eliminate the variable.
- In the elimination process, P_{ij} matrices exchange rows i and j to move a non-zero coefficient into the next pivot position. These matrices are made by swapping the i-th and j-th rows (or columns) of an identity matrix.

LINEAR ALGEBRA IN A NUTSHELL

((***The matrix*** A ***is*** n ***by*** n))

Nonsingular	**Singular**
A is invertible	A is not invertible
The columns are independent	The columns are dependent
The rows are independent	The rows are dependent
The determinant is not zero	The determinant is zero
$A\boldsymbol{x}=\mathbf{0}$ has one solution $\boldsymbol{x}=\mathbf{0}$	$A\boldsymbol{x}=\mathbf{0}$ has infinitely many solutions
$A\boldsymbol{x}=\boldsymbol{b}$ has one solution $\boldsymbol{x}=A^{-1}\boldsymbol{b}$	$A\boldsymbol{x}=\boldsymbol{b}$ has no solution or infinitely many
A has n (nonzero) pivots	A has $r < n$ pivots
A has full rank $r=n$	A has rank $r < n$
The reduced row echelon form is $R=I$	R has at least one zero row
The column space is all of $\mathbf{R}^n$	The column space has dimension $r < n$
The row space is all of $\mathbf{R}^n$	The row space has dimension $r < n$
All eigenvalues are nonzero	Zero is an eigenvalue of A
$A^{\mathrm{T}}A$ is symmetric positive definite	$A^{\mathrm{T}}A$ is only semidefinite
A has n (positive) singular values	A has $r < n$ singular values

Figure 1: Strang, p. 574