MATH 2418: Linear Algebra

Assignment# 2

Due :09/06, Tuesday, 11:59pm Term :Fall 2022

[Last Name] [First Name] [Net ID] [Lab Section]

Recommended Problems:(Do not turn in) **Sec 1.2**: 1, 2,5, 6, 7, 8, 12, 13, 19, 29, 31; **Sec 1.3:** 1, 2, 3,4, 6, 8.

1. Let $\mathbf{u} = (-1, 2, 4)$ and $\mathbf{v} = (-1, 0, -11)$ be two vectors in $\mathbb{R}^3$.

   (a) Calculate the dot product $\mathbf{u} \cdot \mathbf{v}$. What does it say about the angle between $\mathbf{u}$ and $\mathbf{v}$?
   Solution:

   $$\mathbf{u} \cdot \mathbf{v} \;= -1 * (-1) + 2 * 0 + 4 * (-11) = -43$$

   If the dot product is negative, then the cosine of the angle between $\mathbf{u}$ and $\mathbf{v}$ would be negative as well, which means the angle is greater than 90 degrees.

   (b) Compute the lengths $\|\mathbf{u}\|$ and $\|\mathbf{v}\|$ of the vectors.
   Solution:

   $$\|\mathbf{u}\| = \sqrt{(-1)^2 + (2)^2 + (4)^2} = \sqrt{21}$$
   $$\|\mathbf{v}\| = \sqrt{(-1)^2 + 0^2 + (-11)^2} = \sqrt{122}$$

   (c) Let θ be the angle between $\mathbf{u}$ and $\mathbf{v}$. Find $\cos\theta$, where $0 \leq \theta \leq \pi$.
   Solution:

   $$\cos\theta = \frac{\mathbf{u} \cdot \mathbf{v}}{\|\mathbf{u}\|\|\mathbf{v}\|} = \frac{-43}{\sqrt{21} * \sqrt{121}} = \frac{-43}{\sqrt{2541}}$$

   (d) Find the unit vector $\hat{\mathbf{u}}$ in the opposite direction of $\mathbf{u}$.
   Solution:

   $$\widehat{\mathbf{u}} = \frac{-\mathbf{u}}{\|\mathbf{u}\|} = \frac{(1, -2, -4)}{\sqrt{21}}$$

   (e) Find the vector $\hat{\mathbf{v}}$ in the opposite direction of $\mathbf{v}$ and of length 5.
   Solution:

   $$\widehat{\mathbf{v}} = -5 * \frac{\mathbf{v}}{\|\mathbf{v}\|} = -5 * \frac{(-1, 0, -11)}{\sqrt{122}} = \frac{(5, 0, 55)}{\sqrt{122}}$$

   (f) Find a vector $\mathbf{w}$ parallel to $\mathbf{u}$ that has length 2.
   Solution:

   $$\widehat{\mathbf{w}} = 2 * \frac{\mathbf{u}}{\|\mathbf{u}\|} = 2 * \frac{(-1, 2, 4)}{\sqrt{21}} = \frac{(-2, 4, 8)}{\sqrt{21}}$$

   (g) Find a vector $\mathbf{z}$ in the direction of $\mathbf{v}$ and of length 3.
   Solution:

   $$\widehat{\mathbf{z}} = 3 * \frac{\mathbf{v}}{\|\mathbf{v}\|} = 3 * \frac{(-1, 0, -11)}{\sqrt{122}} = \frac{(-3, 0, -33)}{\sqrt{122}}$$

2. For any geometric vectors $\mathbf{u}$ and $\mathbf{v}$, prove the following using triangle inequality.

   (a) $\|\mathbf{u}-\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

   (b) $\Big|\|\mathbf{u}\| - \|\mathbf{v}\|\Big| \leq \|\mathbf{u}-\mathbf{v}\|$

   Solution:

   (a)

$$\begin{aligned}\|\mathbf{u}-\mathbf{v}\| &= \|\mathbf{u}+(-\mathbf{v})\| \\ &\leq \|\mathbf{u}\| + \|-\mathbf{v}\| \quad \text{(by triangle inequality)} \\ &\leq \|\mathbf{u}\| + \|\mathbf{v}\|\end{aligned}$$

   Then $\|\mathbf{u}-\mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

   (b) Observe that

$$\|\mathbf{u}\| = \|(\mathbf{u}-\mathbf{v})+\mathbf{v}\| \leq \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{v}\|$$

   which gives

$$\|\mathbf{u}\| - \|\mathbf{v}\| \leq \|\mathbf{u}-\mathbf{v}\| \tag{1}$$

   Similarly,

$$\|\mathbf{v}\| = \|(\mathbf{v}-\mathbf{u})+\mathbf{u}\| \leq \|\mathbf{v}-\mathbf{u}\| + \|\mathbf{u}\| = \|\mathbf{u}-\mathbf{v}\| + \|\mathbf{u}\|$$

   which gives

$$\|\mathbf{v}\| - \|\mathbf{u}\| \leq \|\mathbf{u}-\mathbf{v}\| \tag{2}$$

   From (1) and (2), we have proved that $\Big|\|\mathbf{u}\| - \|\mathbf{v}\|\Big| \leq \|\mathbf{u}-\mathbf{v}\|$.

3. (a) Let $\mathbf{u}$ and $\mathbf{v}$ be two vectors in $\mathbb{R}^3$ such that $\|\mathbf{u}\| = 3$ and $\|\mathbf{v}\| = 9$.

(i) Find the maximum and minimum possible values of $\mathbf{u} \cdot \mathbf{v}$.

Answer:

The dot product of two fixed length vectors $\mathbf{u}$ and $\mathbf{v}$ will be minimum if the angle between them is π and will be maximum when angle between them is 0.

$\mathbf{u} \cdot \mathbf{v} = \|\mathbf{u}\|\|\mathbf{v}\| \cos(\theta)$

Therefore the maximum value of $\mathbf{u}\cdot\mathbf{v}$ is 27 (when $\theta = 0$) while the minimum is -27 (when $\theta = \pi$).

(ii) Find the maximum and minimum possible values of $\|\mathbf{u} - \mathbf{v}\|$.

Answer:

$\|\mathbf{u}\| - \|\mathbf{v}\| \leq \|\mathbf{u} - \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$, switching $\mathbf{u}$ and $\mathbf{v}$: $\|\mathbf{v}\| - \|\mathbf{u}\| \leq \|\mathbf{v} - \mathbf{u}\| \leq \|\mathbf{v}\| + \|\mathbf{u}\|$ But $\|\mathbf{u} - \mathbf{v}\| = \|\mathbf{v} - \mathbf{u}\|$, therefore,

$$\max(\|\mathbf{u}\| - \|\mathbf{v}\|,\ \|\mathbf{v}\| - \|\mathbf{u}\|) \leq \|\mathbf{u} - \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|.$$

So the maximum is 12 while the minimum is 6.

(b) Let $\mathbf{u} = (1, -3), \mathbf{v} = (2, 4)$ and $\mathbf{w} = (c, d),\ c, d \in \mathbb{R}$, be three vectors in $\mathbb{R}^2$. Find all real values c, d, such that $\mathbf{u}$ and $\mathbf{w}$ are orthogonal, and $\mathbf{v} \cdot \mathbf{w} = 3$.

Answer:

The vector $\mathbf{u}$ is orthogonal to $\mathbf{w}$ if

$$(1, -3) \cdot (c, d) = 0,$$

i.e. if

$$c - 3d = 0.$$

The dot product of vector $\mathbf{v}$ and $\mathbf{w}$ is equal to 3 if

$$(2, 4) \cdot (c, d) = 0,$$

i.e. if

$$2c + 4d = 3.$$

On solving these two equations, we get,

$$c = \frac{9}{10}.$$

$$d = \frac{3}{10}.$$

4. Given a matrix $A = \begin{bmatrix} 3 & 4 & 9 \\ 2 & 1 & 2 \\ 0 & 3 & -1 \\ 5 & -9 & -7 \end{bmatrix}$ and a vector $\mathbf{x} = \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} \in \mathbb{R}^3$, calculate $A\mathbf{x}$

(a) as a linear combination of columns of A.

(b) with entries as dot products of rows of A and $\mathbf{x}$.

Solution:

(a) We need to calculate,

$$A\mathbf{x} = \begin{bmatrix} 3 & 4 & 9 \\ 2 & 1 & 2 \\ 0 & 3 & -1 \\ 5 & -9 & -7 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix}.$$

When this matrix multiplication is carried out, every entry in the first column gets multiplied by -1, every entry in the second column gets multiplied by 5, every entry in the third column gets multiplied by 3. It follows that:

$$A\mathbf{x} = \begin{bmatrix} 3 & 4 & 9 \\ 2 & 1 & 2 \\ 0 & 3 & -1 \\ 5 & -9 & -7 \end{bmatrix} \begin{bmatrix} -2 \\ 5 \\ 3 \end{bmatrix} = -2 \begin{bmatrix} 3 \\ 2 \\ 0 \\ 5 \end{bmatrix} + 5 \begin{bmatrix} 4 \\ 1 \\ 3 \\ -9 \end{bmatrix} + 3 \begin{bmatrix} 9 \\ 2 \\ -1 \\ -7 \end{bmatrix} = \begin{bmatrix} -6+20+27 \\ -4+5+6 \\ 0+15-3 \\ -10-45-21 \end{bmatrix} = \begin{bmatrix} 41 \\ 7 \\ 12 \\ -76 \end{bmatrix}.$$

(b) Notice above, the first entry in the first column is being multiplied by -2, the first entry in the second column is being multiplied by 5, the first entry in the third column is being multiplied by 3. Equivalently this can be calculated by getting the dot product between the first row of A and $\mathbf{x}$. Therefore, we may recognize $A\mathbf{x}$ as:

$$A\mathbf{x} = \begin{bmatrix} (3,4,9) \cdot \mathbf{x} \\ (2,1,2) \cdot \mathbf{x} \\ (0,3,-1) \cdot \mathbf{x} \\ (5,-9,-7) \cdot \mathbf{x} \end{bmatrix} = \begin{bmatrix} -6+20+27 \\ -4+5+6 \\ 0+15-3 \\ -10-45-21 \end{bmatrix} = \begin{bmatrix} 41 \\ 7 \\ 12 \\ -76 \end{bmatrix}.$$

5. Let $A = \begin{bmatrix} 2 & -3 & 1 \\ 2 & 7 & -4 \\ 3 & 9 & 9 \end{bmatrix}$, $\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$ and $\mathbf{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$,

(a) write the linear system corresponding to the matrix equation $A\mathbf{x} = \mathbf{b}$.

Solution: We have that:

$$A\mathbf{x} = \begin{bmatrix} (2,-3,1)\cdot(x_1,x_2,x_3) \\ (2,7,-4)\cdot(x_1,x_2,x_3) \\ (3,9,9)\cdot(x_1,x_2,x_3) \end{bmatrix} = \begin{bmatrix} 2x_1 - 3x_2 + x_3 \\ 2x_1 + 7x_2 - 4x_3 \\ 3x_1 + 9x_2 + 9x_2 \end{bmatrix},$$

hence the linear system $A\mathbf{x} = \mathbf{b}$ is written as:

$$2x_1 - 3x_2 + x_3 = b_1 \tag{3}$$
$$2x_1 + 7x_2 - 4x_3 = b_2 \tag{4}$$
$$3x_1 + 9x_2 + 9x_2 = b_3 \tag{5}$$

(b) solve the linear system.

Solution: Subtracting equation (3) from (4), and next, subtracting equation (3) multiplied by 3/2 from equation (5), yields the equivalent system:

$$2x_1 - 3x_2 + x_3 = b_1 \tag{6}$$
$$\big((4)-(3)\big) \qquad 10x_2 - 5x_3 = b_2 - b_1 \tag{7}$$
$$\big((5)-3/2*(3)\big) \qquad \frac{27}{2}x_2 + \frac{15}{2}x_2 = b_3 - \frac{3}{2}b_1 \tag{8}$$

Now subtracting equation (7) multiplied by 27/20 from (8) gives:

$$2x_1 - 3x_2 + x_3 = b_1 \tag{9}$$
$$10x_2 - 5x_3 = b_2 - b_1 \tag{10}$$
$$\big((8)-27/20*(7)\big) \qquad \frac{57}{4}x_3 = b_3 - \frac{3}{2}b_1 - \frac{27}{20}(b_2 - b_1) \tag{11}$$

Via back substitution applied to equations (9)-(11), one computes:

$$\begin{aligned}
x_3 &= \frac{4}{57}\left(b_3 - \frac{3}{2}b_1 - \frac{27}{20}(b_2 - b_1)\right) = \boxed{\frac{4}{57}b_3 - \frac{2}{190}b_1 - \frac{9}{95}b_2} \\
x_2 &= \frac{1}{10}\left(b_2 - b_1 + 5x_3\right) = \frac{1}{10}(b_2 - b_1) + \frac{1}{2}x_3 \\
&= \frac{1}{10}(b_2 - b_1) + \frac{2}{57}b_3 - \frac{1}{190}b_1 - \frac{9}{190}b_2 \\
&= \boxed{-\frac{2}{19}b_1 + \frac{1}{19}b_2 + \frac{2}{57}b_3} \\
x_1 &= \frac{1}{2}(b_1 + 3x_2 - x_3) = \frac{1}{2}b_1 + \frac{3}{2}\left(-\frac{2}{19}b_1 + \frac{1}{19}b_2 + \frac{2}{57}b_3\right) - \frac{1}{2}\left(\frac{4}{57}b_3 - \frac{2}{190}b_1 - \frac{9}{95}b_2\right) \\
&= \left(\frac{1}{2} - \frac{3}{19} + \frac{1}{190}\right)b_1 + \left(\frac{3}{38} + \frac{9}{190}\right)b_2 + \left(\frac{1}{19} - \frac{2}{57}\right)b_3 \\
&= \boxed{\frac{66}{190}b_1 + \frac{12}{95}b_2 + \frac{1}{57}b_3}
\end{aligned}$$

Thus, the solution $\mathbf{x} = (x_1, x_2, x_3)$ to the linear system posed in (a) is:

$$x_1 = \frac{66}{190}b_1 + \frac{12}{95}b_2 + \frac{1}{57}b_3 \tag{12}$$

$$x_2 = -\frac{2}{19}b_1 + \frac{1}{19}b_2 + \frac{2}{57}b_3 \tag{13}$$

$$x_3 = -\frac{2}{190}b_1 - \frac{9}{95}b_2 + \frac{4}{57}b_3 \tag{14}$$

(c) write your answer in the form $\mathbf{x} = A^{-1}\mathbf{b}$. What is A^{-1}?

Solution: The solution of $A\mathbf{x} = \mathbf{b}$ given by equations (12)-(14) yields:

$$\mathbf{x} = \begin{bmatrix} 66/190 & 12/95 & 1/57 \\ -2/19 & 1/19 & 2/57 \\ -2/190 & -9/95 & 4/57 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

and it now follows that: $A^{-1} = \begin{bmatrix} 66/190 & 12/95 & 1/57 \\ -2/19 & 1/19 & 2/57 \\ -2/190 & -9/95 & 4/57 \end{bmatrix}$

(indeed, setting $A\mathbf{x} = \mathbf{b}$, we have shown the relations $\mathbf{x} = A^{-1}(A\mathbf{x})$ and $A(A^{-1}\mathbf{b}) = \mathbf{b}$, confirming that A^{-1} is the unique inverse of the function $\mathbf{x} \mapsto A\mathbf{x}$).

6. (a) Prove that the vectors $\mathbf{u} = (2, 1, 0),\ \mathbf{v} = (1, 0, 3),\ \mathbf{w} = (0, 1, 1)$ are linearly independent.

Assume not. Suppose $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are instead linearly dependent, i.e. suppose that there exists some $a, b \in \mathbb{R}$ such that $\mathbf{u} = a\mathbf{v} + b\mathbf{w}$.

$$\begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} a \\ b \\ 3a + b \end{bmatrix}$$

then $a = 2$ and $b = 1$ but $3 * 2 + 1 \neq 0 \rightarrow$ contradiction. Therefore, the given vectors are indeed LI

(b) Prove that the vectors $\mathbf{u} = (2, 1, 0),\ \mathbf{v} = (4, 2, 0),\ \mathbf{w} = (0, 1, 1)$ are linearly dependent.

We aim to find some $a, b \in \mathbb{R}$ such that $\mathbf{u} = a\mathbf{v} + b\mathbf{w}$. Simply set $a = \frac{1}{2}$ and $b = 0$ and we have the result.

Remark: no set of vectors containing scalar multiples of the same vector can be linearly independent.

(c) Let $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ be three linearly independent vectors, and a, b, c be any three nonzero real numbers. Prove that the vectors $a\mathbf{u}$, $b\mathbf{v}$, $c\mathbf{w}$ are also linearly independent.

Again, lets try to prove using contradiction. Suppose rather that, though $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ are linearly independent, the set $a\mathbf{u}$, $b\mathbf{v}$, $c\mathbf{w}$ is linearly dependent. If so, then we are guaranteed some $d, e \in \mathbb{R}$ such that $a\mathbf{u} = db\mathbf{v} + ec\mathbf{w}$. However, dividing both sides by a we find $u = \frac{db}{a}\mathbf{v} + \frac{ec}{a}\mathbf{w}$ which is a contradiction to our assumption that $\mathbf{u}$, $\mathbf{v}$, $\mathbf{w}$ form a linearly independent set. Therefore, so too must must $a\mathbf{u}$, $b\mathbf{v}$, $c\mathbf{w}$ be LI.

Remark: Notice the importance of a, b, c being nonzero