# Chapter 3: Discrete Random Variables
# Geometric & Poisson R.V.'s

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# Goal

- Learn 4 discrete probability distributions this week:
  1. Monday: Bernoulli and Binomial ✓
  2. Wednesday: Geometric and Poisson

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# Review: Geometric Sequences

- Examples:

$$1, 2, 4, 8, 16, 32, 64, \ldots \implies b_n = 1(2)^{n-1}$$

$$9, 3, 1, 1/3, 1/9, 1/27, \ldots \implies b_n = 9(1/3)^{n-1}$$

$$a, ar, ar^2, ar^3, ar^4, \ldots \implies b_n = a(r)^{n-1}$$

- Generally, given $a =$ first term and $r =$ common ratio, any term $b_x$ of geometric sequence can be written as:

$$b_x = a(r)^{x-1} \text{ for } x = 1, \ldots, \infty$$

or

$$b_x = a(r)^x \text{ for } x = 0, \ldots, \infty$$

# Review: Sum of an Infinite Geometric Sequence

- If $0 < r < 1$, then the sum of the infinite geometric series is given by

$$g(r) = \sum_{x=0}^{\infty} a(r)^x = a + ar + ar^2 + ar^3 + \ldots = \boxed{\frac{a}{1-r}}$$

$$g(r) = \sum_{x=1}^{\infty} a(r)^{x-1} = a + ar + ar^2 + ar^3 + \ldots = \boxed{\frac{a}{1-r}}$$

- Example: Calculate $g(x) = \sum_{x=0}^{\infty} 3(1/2)^x \implies a = 3, r = 1/2$. Since $0 < r < 1$, the infinite sum exists and its value is

$$\frac{a}{1-r} = \frac{3}{1-1/2} = 6$$

# Introduction to the Concepts

- Examples of a Geometric random variable
  1. Taking a driver's test until you pass.
  2. The number of days a patient has to wait until they get a kidney donor.
  3. The number of tries playing a slot machine or the lottery to hit a jackpot.
  4. The number of miles or flights until a plane is retired.
  5. The number of attempts it takes a salesperson to make a successful sale.
- Any experiment that stops when a particular events occurs has a Geometric distribution.

## Introduction to the Concepts

- Any experiment that stops after the first success has a Geometric distribution.
- Let $F =$ failure probability (not observed)
- Let $S =$ success probability (observed)
- What is the probability that the first success occurs on the $n^{th}$ try?

# Formula for Geometric Probability

## Geometric p.m.f.

The geometric probability mass function is given by

$$P(X = x) = p \cdot (1 - p)^{x-1}$$

which calculates the probability that the first success occurs at the $x^{th}$ trial.

1. $p$ is the success probability.
2. $x$ is the total number of independent trials where $x = \{1, 2, 3, ..., \infty\}$.
3. The r.v. $X$ only depends on $p$

$$X \sim \text{Geometric}(p)$$

# Axiom 2

Show that the sum of the probabilities of the Geometric distribution is 1.

# Formula for Cumulative Geometric Probability

## Geometric c.d.f.

The geometric cumulative distribution function is given by

$$P(X \leq x) = 1 - P(X > x)$$
$$P(X \leq x) = 1 - (1-p)^x$$

which calculates the cumulative probability that the first success occurs within the first $x$ number of trials.

1. Ex: $P(X \leq 4)$ is interpreted as:
   - The probability that the first success occurs within the first 4 trials.
2. Ex: $P(X = 4)$ is interpreted as:
   - The probability that the first success occurs at the 4th trial.

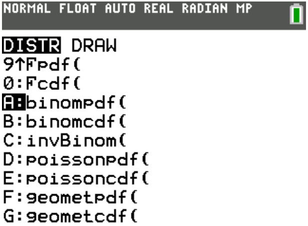
# Complete Summary of the Geometric Distribution

| Term | Notation | Formula |
|---|---|---|
| 1. Probability | $P(X = x)$ or $f_X(x)$ | $p \cdot (1-p)^{x-1}$ |
| 2. Cumulative Probability | $P(X \leq x)$ or $F_X(x)$ | $1 - (1-p)^x$ |
| 3. Mean | $\mu$ or $E(X)$ | $\frac{1}{p}$ |
| 4. Variance | $\sigma^2$ or $Var(X)$ | $\frac{1-p}{p^2}$ |
| 5. Standard Deviation | $\sigma$ or $SD(X)$ | $\sqrt{\frac{1-p}{p^2}}$ |

where $p$ is the success probability and $x$ is the number of trials until success.

# Using the TI84 to calculate geometric probabilities

- Press $\boxed{2nd} \rightarrow \boxed{VARS}$




| Quantity | TI84 |
|---|---|
| $P(X = x)$ or $f_X(x)$ | `geometpdf(p,x)` |
| $P(X \leq x)$ or $F_X(x)$ | `geometcdf(p,x)` |

## Example

A patient is waiting for a blood donor where the probability of a match is 0.2. Numerous donors are tested until a match is found.

(a) What is the expected number of donors tested until a match is found?

## Example

A patient is waiting for a blood donor where the probability of a match is 0.2. Numerous donors are tested until a match is found.

(b) What is the probability that the first match is the third donor?

# Example

A patient is waiting for a blood donor where the probability of a match is 0.2. Numerous donors are tested until a match is found.

(c) What is the probability of at most four donors until a match is found?

## Example - Your Turn

A patient is waiting for a blood donor where the probability of a match is 0.2. Numerous donors are tested until a match is found.

(d) What is the probability of needing less than 4 donors?

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# Introducing Poisson

1. Named after Siméon-Denis Poisson
2. Probability for rare events over a period of time
   - Two events are unlikely to occur simultaneously or within a very short period of time
3. Examples
   - Number of car accidents at an intersection in a year
   - Number of phone calls in a day
   - Number of emails/texts in an hour
   - Number of blackouts, viruses, errors, etc.

# Formula for Poisson Probability

## Poisson p.m.f.

The Poisson probability mass function is given by

$$P(X = x) = \frac{e^{-\lambda} \cdot \lambda^x}{x!}$$

and calculates the probability of $x$ rare events within a period of time.

1. $\lambda$ is the mean number of events in a given time frame.
2. $x$ is the number of events of interest.
3. The r.v. $X$ only depends on $\lambda$

$$X \sim \text{Poisson}(\lambda)$$

# Complete Summary of the Poisson Distribution

| Term | Notation | Formula |
|---|---|---|
| 1. Probability | $P(X = x)$ or $f_X(x)$ | $\frac{e^{-\lambda} \cdot \lambda^x}{x!}$ |
| 2. Mean | $\mu$ or $E(X)$ | $\lambda$ |
| 3. Variance | $\sigma^2$ or $Var(X)$ | $\lambda$ |
| 4. Standard Deviation | $\sigma$ or $SD(X)$ | $\sqrt{\lambda}$ |

- The Poisson mean, $\lambda$, depends on time and is parametrized as

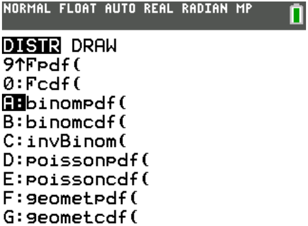
$$\lambda = \theta t$$

where $t =$ time and $\theta =$ rate per unit time.

- Note: The mean and variance are equal for a Poisson r.v.

# Using the TI84 to calculate Poisson probabilities

- Press $\boxed{2nd} \rightarrow \boxed{VARS}$




| Quantity | TI84 |
|---|---|
| $P(X = x)$ or $f_X(x)$ | `poissonpdf(`$\lambda$`,x)` |
| $P(X \leq x)$ or $F_X(x)$ | `poissoncdf(`$\lambda$`,x)` |

# Example

At a certain intersection, there are on average 5 accidents per month.

(a) Calculate the mean and variance of the Poisson r.v. $X$

# Example

At a certain intersection, there are on average 5 accidents per month.

(b) What is the probability of exactly 7 accidents this month?

# Example

At a certain intersection, there are on average 5 accidents per month.

(c) What is the probability of exactly 20 accidents over the next three months?

# Example

At a certain intersection, there are on average 5 accidents per month.

(d) What is the probability of more than 70 accidents in a year at this intersection?

# Example

At a certain intersection, there are on average 5 accidents per month.

(e) What is the probability of having between 30 and 40 accidents in a five month period?

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# Summary of Discrete Probability Distributions

| Distribution | Main Interest | Parameters |
|---|---|---|
| 1. Binomial | Number of successes within a *fixed* number of trials | $n, p$ |
| 2. Geometric | Number of trials until the the first success occurs. Trials can be infinite. | $p$ |
| 3. Poisson | Number of successes within an interval of unit time. | $\lambda$ |