30 (a) If $\boldsymbol{u}$ and $\boldsymbol{v}$ are both in $\boldsymbol{S}+\boldsymbol{T}$, then $\boldsymbol{u}=\boldsymbol{s}_1+\boldsymbol{t}_1$ and $\boldsymbol{v}=\boldsymbol{s}_2+\boldsymbol{t}_2$. So $\boldsymbol{u}+\boldsymbol{v}=(\boldsymbol{s}_1+\boldsymbol{s}_2)+(\boldsymbol{t}_1+\boldsymbol{t}_2)$ is also in $\boldsymbol{S}+\boldsymbol{T}$. And so is $c\boldsymbol{u}=c\boldsymbol{s}_1+c\boldsymbol{t}_1$: $\boldsymbol{S}+\boldsymbol{T}=$ *subspace*.

(b) If $\boldsymbol{S}$ and $\boldsymbol{T}$ are different lines, then $\boldsymbol{S}\cup\boldsymbol{T}$ is just the two lines (*not a subspace*) but $\boldsymbol{S}+\boldsymbol{T}$ is the whole plane that they span.

31 If $\boldsymbol{S}=\boldsymbol{C}(A)$ and $\boldsymbol{T}=\boldsymbol{C}(B)$ then $\boldsymbol{S}+\boldsymbol{T}$ is the column space of $M=[\,A\quad B\,]$.

32 The columns of AB are combinations of the columns of A. So all columns of $[\,A\quad AB\,]$ are already in $\boldsymbol{C}(A)$. But $A=\begin{bmatrix}0&1\\0&0\end{bmatrix}$ has a larger column space than $A^2=\begin{bmatrix}0&0\\0&0\end{bmatrix}$. For square matrices, the column space is $\mathbf{R}^n$ exactly when A is *invertible*.

Problem Set 3.2, page 142

1 (a) $U=\begin{bmatrix}1&2&2&4&6\\0&0&1&2&3\\0&0&0&0&0\end{bmatrix}$ Free variables x_2,x_4,x_5 Pivot variables x_1,x_3 (b) $U=\begin{bmatrix}2&4&2\\0&4&4\\0&0&0\end{bmatrix}$ Free x_3 Pivot x_1,x_2

2 (a) Free variables x_2,x_4,x_5 and solutions $(-2,1,0,0,0)$, $(0,0,-2,1,0)$, $(0,0,-3,0,1)$

(b) Free variable x_3: solution $(1,-1,1)$. Special solution for each free variable.

3 $R=\begin{bmatrix}1&2&0&0&0\\0&0&1&2&3\\0&0&0&0&0\end{bmatrix}$, $R=\begin{bmatrix}1&0&-1\\0&1&1\\0&0&0\end{bmatrix}$, R has the same nullspace as U and A.

4 (a) Special solutions $(3,1,0)$ and $(5,0,1)$ (b) $(3,1,0)$. **Total of pivot and free is $\boldsymbol{n}$.**

5 (a) *False*: Any singular square matrix would have free variables (b) *True*: An invertible square matrix has ***no*** free variables. (c) *True* (only n columns to hold pivots) (d) *True* (only m rows to hold pivots)

6 $\begin{bmatrix}0&\mathbf{1}&1&1&1&1&1\\0&0&0&\mathbf{1}&1&1&1\\0&0&0&0&\mathbf{1}&1&1\\0&0&0&0&0&0&0\end{bmatrix}\begin{bmatrix}\mathbf{1}&1&1&1&1&1&1\\0&0&\mathbf{1}&1&1&1&1\\0&0&0&0&0&\mathbf{1}&1\\0&0&0&0&0&0&\mathbf{1}\end{bmatrix}\begin{bmatrix}0&0&0&\mathbf{1}&1&1&1\\0&0&0&0&0&\mathbf{1}&1\\0&0&0&0&0&0&0\\0&0&0&0&0&0&0\end{bmatrix}$

7 $\begin{bmatrix} \mathbf{1} & 1 & 0 & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & \mathbf{1} & 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \mathbf{1} \end{bmatrix}, \begin{bmatrix} 0 & \mathbf{1} & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & \mathbf{1} & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & \mathbf{1} & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}$. Notice the identity matrix in the pivot columns of these *reduced* row echelon forms R.

8 If column 4 of a 3 by 5 matrix is all zero then x_4 is a *free* variable. Its special solution is $\boldsymbol{x} = (0, 0, 0, 1, 0)$, because 1 will multiply that zero column to give $A\boldsymbol{x} = \mathbf{0}$.

9 If column $1 =$ column 5 then x_5 is a free variable. Its special solution is $(-1, 0, 0, 0, 1)$.

10 If a matrix has n columns and r pivots, there are $\boldsymbol{n} - \boldsymbol{r}$ special solutions. The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when $\boldsymbol{r} = \boldsymbol{n}$. The column space is all of $\mathbf{R}^m$ when $r = m$. All those statements are important!

11 The nullspace contains only $\boldsymbol{x} = \mathbf{0}$ when A has 5 pivots. Also the column space is $\mathbf{R}^5$, because we can solve $A\boldsymbol{x} = \boldsymbol{b}$ and every $\boldsymbol{b}$ is in the column space.

12 $A = [1 \ \ -3 \ \ -1]$ gives the plane $x - 3y - z = 0$; y and z are free variables. The special solutions are $(3, 1, 0)$ and $(1, 0, 1)$.

13 Fill in $\mathbf{12}$ then $\mathbf{3}$ then $\mathbf{1}$ to get the complete solution in $\mathbf{R}^3$ to $x - 3y - z = 12$: $\begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} \mathbf{12} \\ 0 \\ 0 \end{bmatrix} + y \begin{bmatrix} \mathbf{3} \\ 1 \\ 0 \end{bmatrix} + z \begin{bmatrix} \mathbf{1} \\ 0 \\ 1 \end{bmatrix}$ = one particular solution + all nullspace solutions.

14 Column 5 is sure to have no pivot since it is a combination of earlier columns. With 4 pivots in the other columns, the special solution is $\boldsymbol{s} = (1, 0, 1, 0, 1)$. The nullspace contains all multiples of this vector $\boldsymbol{s}$ (this nullspace is a line in $\mathbf{R}^5$).

15 To produce special solutions $(2, 2, 1, 0)$ and $(3, 1, 0, 1)$ with free variables x_3, x_4: $R = \begin{bmatrix} 1 & 0 & -2 & -3 \\ 0 & 1 & -2 & -1 \end{bmatrix}$ and A can be any invertible 2 by 2 matrix times this R.

16 The nullspace of $A = \begin{bmatrix} 1 & 0 & 0 & -4 \\ 0 & 1 & 0 & -3 \\ 0 & 0 & 1 & -2 \end{bmatrix}$ is the line through the special solution $\begin{bmatrix} 4 \\ 3 \\ 2 \\ 1 \end{bmatrix}$.

17 $A = \begin{bmatrix} 1 & 0 & -1/2 \\ 1 & 3 & -2 \\ 5 & 1 & -3 \end{bmatrix}$ has $\begin{bmatrix} 1 \\ 1 \\ 5 \end{bmatrix}$ and $\begin{bmatrix} 0 \\ 3 \\ 1 \end{bmatrix}$ in $\boldsymbol{C}(A)$ and $\begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$ in $\boldsymbol{N}(A)$. ***Which other*** A's?

18 This construction is impossible for 3 by 3 ! 2 pivot columns and 2 free variables.

19 $A = \begin{bmatrix} 1 & -1 & 0 & 0 \\ 1 & 0 & -1 & 0 \\ 1 & 0 & 0 & -1 \end{bmatrix}$ has $(1, 1, 1)$ in $\boldsymbol{C}(A)$ and only the line (c, c, c, c) in $\boldsymbol{N}(A)$.

20 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $\boldsymbol{N}(A) = \boldsymbol{C}(A)$. Notice that $\mathsf{rref}(A^{\mathrm{T}}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ is not A^{T}.

21 If nullspace $=$ column space (with r pivots) then $n - r = r$. If $n = 3$ then $3 = 2r$ is impossible.

22 If A times every column of B is zero, the column space of B is contained in the ***nullspace*** of A. An example is $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$. Here $\boldsymbol{C}(B)$ ***equals*** $\boldsymbol{N}(A)$. For $B = 0, \boldsymbol{C}(B)$ is smaller than $\boldsymbol{N}(A)$.

23 For $A =$ random 3 by 3 matrix, R is almost sure to be I. For 4 by 3, R is most likely to be I with a fourth row of zeros. What is R for a random 3 by 4 matrix?

24 $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ shows that (a)(b)(c) are all false. Notice $\mathsf{rref}(A^{\mathrm{T}}) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$.

25 If $\boldsymbol{N}(A) =$ line through $\boldsymbol{x} = (2, 1, 0, 1)$, A has ***three pivots*** (4 columns and 1 special solution). Its reduced echelon form can be $R = \begin{bmatrix} 1 & 0 & 0 & -2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$ (add any zero rows).

26 $R = [\,1 \;\; -2 \;\; -3\,]$, $R = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix}$, $R = I$. Any zero rows come after those rows.

27 (a) $\begin{bmatrix}1&0\\0&1\end{bmatrix}, \begin{bmatrix}1&0\\0&0\end{bmatrix}, \begin{bmatrix}1&1\\0&0\end{bmatrix}, \begin{bmatrix}0&1\\0&0\end{bmatrix}, \begin{bmatrix}0&0\\0&0\end{bmatrix}$ (b) All 8 matrices are R's !

28 One reason that R is the same for A and $-A$: They have the same nullspace. (They also have the same row space. They also have the same column space, but that is not required for two matrices to share the same R. R tells us the nullspace and row space.)

29 The nullspace of $B = [\,A \quad A\,]$ contains all vectors $\boldsymbol{x} = \begin{bmatrix}\boldsymbol{y}\\-\boldsymbol{y}\end{bmatrix}$ for $\boldsymbol{y}$ in $\mathbf{R}^4$.

30 If $C\boldsymbol{x} = \mathbf{0}$ then $A\boldsymbol{x} = \mathbf{0}$ and $B\boldsymbol{x} = \mathbf{0}$. So $\boldsymbol{N}(C) = \boldsymbol{N}(A) \cap \boldsymbol{N}(B) =$ *intersection.*

31 (a) $R = \begin{bmatrix}1&1&1&1\\0&0&0&0\\0&0&0&0\end{bmatrix}$ rank 1 (b) $R = \begin{bmatrix}1&0&-1&-2\\0&1&2&3\\0&0&0&0\end{bmatrix}$ rank 2

(c) $R = \begin{bmatrix}1&-1&1&-1\\0&0&0&0\\0&0&0&0\end{bmatrix}$ rank 1

32 $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$: $y_1 - y_3 + y_4 = -y_1 + y_2 + +y_5 = -y_2 + y_4 + y_6 = -y_4 - y_5 - y_6 = 0$. These equations add to $0 = 0$. Free variables y_3, y_5, y_6: watch for flows around loops. The solutions to $A^{\mathrm{T}}\boldsymbol{y} = \mathbf{0}$ are combinations of $(-1, 0, 0, 1, -1, 0)$ and $(0, 0, -1, -1, 0, 1)$ and $(0, -1, 0, 0, 1, -1)$. Those are flows around the 3 small loops.

33 (a) and (c) are correct; (b) is completely false; (d) is false because R might have 1's in nonpivot columns.

34 $R_A = \begin{bmatrix}1&2&0\\0&0&1\\0&0&0\end{bmatrix}$ $R_B = \begin{bmatrix}R_A & R_A\end{bmatrix}$ $R_C \longrightarrow \begin{bmatrix}R_A & 0\\0 & R_A\end{bmatrix} \longrightarrow$ Zero rows go to the bottom

35 If all pivot variables come last then $R = \begin{bmatrix}0&I\\0&0\end{bmatrix}$. The nullspace matrix is $N = \begin{bmatrix}I\\0\end{bmatrix}$.

36 I think $R_1 = A_1, R_2 = A_2$ is true. But $R_1 - R_2$ may have -1's in some pivots.

37 A and A^{T} have the same rank r = number of pivots. But *pivcol* (the column number) is 2 for this matrix A and 1 for A^{T}: $A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$.

38 Special solutions in $N = [\,-2 \;\; -4 \;\; 1 \;\; 0;\, -3 \;\; -5 \;\; 0 \;\; 1\,]$ and $[\,1 \;\; 0 \;\; 0;\, 0 \;\; -2 \;\; 1\,]$.

39 The new entries keep rank 1: $A = \begin{bmatrix} 1 & 2 & 4 \\ 2 & \mathbf{4} & \mathbf{8} \\ 4 & \mathbf{8} & \mathbf{16} \end{bmatrix}$, $B = \begin{bmatrix} \mathbf{3} & 9 & \mathbf{-4.5} \\ 1 & \mathbf{3} & \mathbf{-1.5} \\ 2 & 6 & -3 \end{bmatrix}$, $M = \begin{bmatrix} a & b \\ c & \boldsymbol{bc/a} \end{bmatrix}$.

40 If A has rank 1, the column space is a *line* in $\mathbf{R}^m$. The nullspace is a *plane* in $\mathbf{R}^n$ (given by one equation). The nullspace matrix N is n by $n-1$ (with $n-1$ special solutions in its columns). The column space of A^{T} is a *line* in $\mathbf{R}^n$.

41 $\begin{bmatrix} 3 & 6 & 6 \\ 1 & 2 & 2 \\ 4 & 8 & 8 \end{bmatrix} = \begin{bmatrix} 3 \\ 1 \\ 4 \end{bmatrix} \begin{bmatrix} 1 & 2 & 2 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 & 6 & 4 \\ -1 & -1 & -3 & -2 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \end{bmatrix} \begin{bmatrix} 1 & 1 & 3 & 2 \end{bmatrix}$

42 With rank 1, the second row of R is a zero row.

43 Invertible r by r submatrices
Use pivot rows and columns $S = \begin{bmatrix} 1 & 3 \\ 1 & 4 \end{bmatrix}$ and $S = [\,1\,]$ and $S = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$.

44 P has rank r (the same as A) because elimination produces the same pivot columns.

45 The rank of R^{T} is also r. The example matrix A has rank 2 with invertible S:

$$P = \begin{bmatrix} 1 & 3 \\ 2 & 6 \\ 2 & 7 \end{bmatrix} \qquad P^{\mathrm{T}} = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 6 & 7 \end{bmatrix} \qquad S^{\mathrm{T}} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} \qquad S = \begin{bmatrix} 1 & 3 \\ 2 & 7 \end{bmatrix}.$$

46 The product of rank one matrices has rank one or zero. These particular matrices have rank$(AB) = 1$; rank$(AC) = 1$ except $AC = 0$ if $c = -1/2$.

47 $(\boldsymbol{u}\boldsymbol{v}^{\mathrm{T}})(\boldsymbol{w}\boldsymbol{z}^{\mathrm{T}}) = \boldsymbol{u}(\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w})\boldsymbol{z}^{\mathrm{T}}$ has rank one unless the inner product is $\boldsymbol{v}^{\mathrm{T}}\boldsymbol{w} = 0$.

48 (a) By matrix multiplication, each column of AB is A times the corresponding column of B. So if column j of B is a combination of earlier columns, then column j of AB is the same combination of earlier columns of AB. Then rank $(AB) \le$ rank (B). No new pivot columns! (b) The rank of B is $r = 1$. Multiplying by A cannot increase this rank. The rank of AB stays the same for $A_1 = I$ and $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. It drops to zero for $A_2 = \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix}$.

49 If we know that $\operatorname{rank}(B^{\mathrm{T}}A^{\mathrm{T}}) \le \operatorname{rank}(A^{\mathrm{T}})$, then since rank stays the same for transposes, (apologies that this fact is not yet proved), we have $\operatorname{rank}(AB) \le \operatorname{rank}(A)$.

50 We are given $AB = I$ which has rank n. Then rank$(AB) \le$ rank(A) forces rank$(A) = n$. This means that A is invertible. The right-inverse B is also a left-inverse: $BA = I$ and $B = A^{-1}$.

51 Certainly A and B have at most $\operatorname{rank} 2$. Then their product AB has at most $\operatorname{rank} 2$. Since BA is 3 by 3, it cannot be I even if $AB = I$.

52 (a) A and B will both have the same nullspace and row space as the R they share.

(b) A equals an ***invertible*** matrix times B, when they share the same R. A key fact!

53 $A =$ (pivot columns)(nonzero rows of R) $= \begin{bmatrix} 1 & 0 \\ 1 & 4 \\ 1 & 8 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 0 \end{bmatrix} +$ $\begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 8 \end{bmatrix}$. $B = \begin{bmatrix} 2 & 2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{matrix} \text{columns} \\ \text{times rows} \end{matrix} = \begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 2 \\ 0 & 3 \end{bmatrix}$

54 If $c = 1, R = \begin{bmatrix} 1 & 1 & 2 & 2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has x_2, x_3, x_4 free. If $c \ne 1, R = \begin{bmatrix} 1 & 0 & 2 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}$ has x_3, x_4 free. Special solutions in $N = \begin{bmatrix} -1 & -2 & -2 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ (for $c = 1$) and $N =$

$\begin{bmatrix}-2 & -2\\ 0 & 0\\ 1 & 0\\ 0 & 1\end{bmatrix}$ (for $c \neq 1$). If $c = 1, R = \begin{bmatrix}0 & 1\\ 0 & 0\end{bmatrix}$ and x_1 free; if $c = 2, R = \begin{bmatrix}1 & -2\\ 0 & 0\end{bmatrix}$ and x_2 free; $R = I$ if $c \neq 1, 2$. Special solutions in $N = \begin{bmatrix}1\\ 0\end{bmatrix}$ $(c = 1)$ or $N = \begin{bmatrix}2\\ 1\end{bmatrix}$ $(c = 2)$ or $N = 2$ by 0 empty matrix.

55 $A = \begin{bmatrix}I & I\end{bmatrix}$ has $N = \begin{bmatrix}I\\ -I\end{bmatrix}$; $B = \begin{bmatrix}I & I\\ 0 & 0\end{bmatrix}$ has the same N; $C = \begin{bmatrix}I & I & I\end{bmatrix}$ has $N = \begin{bmatrix}-I & -I\\ I & 0\\ 0 & I\end{bmatrix}$.

56 $\begin{bmatrix}1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\\ 1 & 1 & 1 & 1\end{bmatrix} =$ (pivot column) (first row) $= \begin{bmatrix}1\\ 1\\ 1\end{bmatrix}\begin{bmatrix}1 & 1 & 1 & 1\end{bmatrix}$

57 The m by n matrix Z has r ones to start its main diagonal. Otherwise Z is all zeros.

58 $R = \begin{bmatrix}I & F\\ 0 & 0\end{bmatrix} = \begin{bmatrix}r \text{ by } r & r \text{ by } n-r\\ m-r \text{ by } r & m-r \text{ by } n-r\end{bmatrix}$; **rref**$(R^{\mathrm{T}}) = \begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}$; **rref**$(R^{\mathrm{T}}R) =$ same R

59 $R = \begin{bmatrix}1 & 2 & 0\\ 0 & 0 & 1\end{bmatrix}$ has $R^{\mathrm{T}}R = \begin{bmatrix}1 & 2 & 0\\ 2 & 4 & 0\\ 0 & 0 & 1\end{bmatrix}$ and this matrix row reduces to $\begin{bmatrix}1 & 2 & 0\\ 0 & 0 & 1\\ 0 & 0 & 0\end{bmatrix} = \begin{bmatrix}R\\ \text{zero row}\end{bmatrix}$. Always $R^{\mathrm{T}}R$ has the same nullspace as R, so its row reduced form must be R with $n - m$ extra zero rows. R is determined by its nullspace and shape !

60 The *row-column reduced echelon form* is always $\begin{bmatrix}I & 0\\ 0 & 0\end{bmatrix}$; I is r by r.