CHAPTER 1
The Foundations: Logic and Proofs

SECTION 1.1 Propositional Logic

Manipulating propositions and constructing truth tables are straightforward. A truth table is constructed by finding the truth values of compound propositions from the inside out; see the solution to Exercise 31, for instance. This exercise set also introduces **fuzzy logic**.

1. Propositions must have clearly defined truth values, so a proposition must be a declarative sentence with no free variables.

a) This is a true proposition.

b) This is a false proposition (Tallahassee is the capital).

c) This is a true proposition.

d) This is a false proposition.

e) This is not a proposition (it contains a variable; the truth value depends on the value assigned to x).

f) This is not a proposition, since it does not assert anything.

3. **a)** Mei does not have an MP3 player.

b) There is pollution in New Jersey.

c) $2 + 1 \neq 3$

d) It is not the case that the summer in Maine is hot and sunny. In other words, the summer in Maine is not hot and sunny, which means that it is not hot *or* it is not sunny. It is not correct to negate this by saying "The summer in Maine is not hot and not sunny." [For this part (and in a similar vein for part **(b)**) we need to assume that there are well-defined notions of hot and sunny; otherwise this would not be a proposition because of not having a definite truth value.]

5. **a)** Steve does not have more than 100 GB free disk space on his laptop. (Alternatively: Steve has less than or equal to 100 GB free disk space on his laptop.)

b) Zach does not block e-mails and texts from Jennifer. (Alternatively, and more precisely: Zach does not block e-mails from Jennifer, or he does not block texts from Jennifer. Note that negating an "and" statement produces an "or" statement. It would not be correct to say that Zach does not block e-mails from Jennifer, and he does not block texts from Jennifer. That is a stronger statement than just the negation of the given statement.)

c) $7 \cdot 11 \cdot 13 \neq 999$.

d) Diane did not ride her bike 100 miles on Sunday.

7. **a)** This is false, because Acme's revenue was larger.

b) Both parts of this conjunction are true, so the statement is true.

c) The second part of this disjunction is true, so the statement is true.

d) The hypothesis of this conditional statement is false and the conclusion is true, so by the truth-table definition this is a true statement. (Either of those conditions would have been enough to make the statement true.)

e) Both parts of this biconditional statement are true, so by the truth-table definition this is a true statement.

9. This is pretty straightforward, using the normal words for the logical operators.

a) Sharks have not been spotted near the shore.

b) Swimming at the New Jersey shore is allowed, and sharks have been spotted near the shore.

c) Swimming at the New Jersey shore is not allowed, or sharks have been spotted near the shore.

d) If swimming at the New Jersey shore is allowed, then sharks have not been spotted near the shore.

e) If sharks have not been spotted near the shore, then swimming at the New Jersey shore is allowed.

f) If swimming at the New Jersey shore is not allowed, then sharks have not been spotted near the shore.

g) Swimming at the New Jersey shore is allowed if and only if sharks have not been spotted near the shore.

h) Swimming at the New Jersey shore is not allowed, and either swimming at the New Jersey shore is allowed or sharks have not been spotted near the shore. Note that we were able to incorporate the parentheses by using the word "either" in the second half of the sentence.

11. a) Here we have the conjunction $p \wedge q$.

b) Here we have a conjunction of p with the negation of q, namely $p \wedge \neg q$. Note that "but" logically means the same thing as "and."

c) Again this is a conjunction: $\neg p \wedge \neg q$.

d) Here we have a disjunction, $p \vee q$. Note that $\vee$ is the inclusive *or*, so the "(or both)" part of the English sentence is automatically included.

e) This sentence is a conditional statement, $p \to q$.

f) This is a conjunction of propositions, both of which are compound: $(p \vee q) \wedge (p \to \neg q)$.

g) This is the biconditional $p \leftrightarrow q$.

13. a) This is just the negation of p, so we write $\neg p$.

b) This is a conjunction ("but" means "and"): $p \wedge \neg q$.

c) The position of the word "if" tells us which is the antecedent and which is the consequence: $p \to q$.

d) $\neg p \to \neg q$

e) The sufficient condition is the antecedent: $p \to q$.

f) $q \wedge \neg p$

g) "Whenever" means "if": $q \to p$.

15. a) "But" is a logical synonym for "and" (although it often suggests that the second part of the sentence is likely to be unexpected). So this is $r \wedge \neg p$.

b) Because of the agreement about precedence, we do not need parentheses in this expression: $\neg p \wedge q \wedge r$.

c) The outermost structure here is the conditional statement, and the conclusion part of the conditional statement is itself a biconditional: $r \to (q \leftrightarrow \neg p)$.

d) This is similar to part **(b)**: $\neg q \wedge \neg p \wedge r$.

e) This one is a little tricky. The statement that the condition is necessary is a conditional statement in one direction, and the statement that this condition is not sufficient is the negation of the conditional statement in the other direction. Thus we have the structure (safe $\to$ conditions) $\wedge \neg$(conditions $\to$ safe). Fleshing this out gives our answer: $(q \to (\neg r \wedge \neg p)) \wedge \neg((\neg r \wedge \neg p) \to q)$. There are some logically equivalent correct answers as well.

f) We just need to remember that "whenever" means "if" in logic: $(p \wedge r) \to \neg q$.

17. In each case, we simply need to determine the truth value of the hypothesis and the conclusion, and then use the definition of the truth value of the conditional statement. The conditional statement is true in every case except when the hypothesis (the "if" part) is true and the conclusion (the "then" part) is false.

a) Since the hypothesis is true and the conclusion is false, this conditional statement is false.

b) Since the hypothesis is false and the conclusion is true, this conditional statement is true.

c) Since the hypothesis is false and the conclusion is false, this conditional statement is true. Note that the conditional statement is false in both part **(b)** and part **(c)**; as long as the hypothesis is false, we need look no further to conclude that the conditional statement is true.

d) Since the hypothesis is false, this conditional statement is true.

19. a) Presumably the diner gets to choose only one of these beverages, so this is an exclusive *or*.

b) This is probably meant to be inclusive, so that long passwords with many digits are acceptable.

c) This is surely meant to be inclusive. If a student has had both of the prerequisites, so much the better.

d) At first glance one might argue that no one would pay with both currencies simultaneously, so it would seem reasonable to call this an exclusive *or*. There certainly could be cases, however, in which the patron would pay a portion of the bill in dollars and the remainder in euros. Therefore, an inclusive *or* seems better.

21. a) If this is an inclusive *or*, then it is allowable to take discrete mathematics if you have had calculus or computer science or both. If this is an exclusive *or*, then a person who had both courses would not be allowed to take discrete mathematics—only someone who had taken exactly one of the prerequisites would be allowed in. Clearly the former interpretation is intended; if anything, the person who has had both calculus and computer science is even better prepared for discrete mathematics.

b) If this is an inclusive *or*, then you can take the rebate, or you can sign up for the low-interest loan, or you can demand both of these incentives. If this is an exclusive *or*, then you will receive one of the incentives but not both. Since both of these deals are expensive for the dealer or manufacturer, surely the exclusive *or* was intended.

c) If this is an inclusive *or*, you can order two items from column A (and none from B), or three items from column B (and none from A), or five items (two from A and three from B). If this is an exclusive *or*, which it surely is here, then you get your choice of the two A items or the three B items, but not both.

d) If this is an inclusive *or*, then lots of snow, or extreme cold, or a combination of the two will close school. If this is an exclusive *or*, then one form of bad weather would close school but if both of them happened then school would meet. This latter interpretation is clearly absurd, so the inclusive *or* is intended.

23. a) If the wind blows from the northeast, then it snows. ["Whenever" means "if."]

b) If it stays warm for a week, then the apple trees will bloom. [Sometimes word order is flexible in English, as it is here. Other times it is not—"The man bit the dog" does not have the same meaning as "The dog bit the man."]

c) If the Pistons win the championship, then they beat the Lakers.

d) If you get to the top of Long's Peak, then you must have walked eight miles. [The necessary condition is the conclusion.]

e) If you are world famous, then you will get tenure as a professor. [The sufficient condition is the antecedent.]

f) If you drive more than 400 miles, then you will need to buy gasoline. [The word "then" is sometimes omitted in English sentences, but it is still understood.]

g) If your guarantee is good, then you must have bought your CD player less than 90 days ago. [Note that "only if" does not mean "if"; the clause following the "only if" is the conclusion, not the antecedent.]

h) If the water is not too cold, then Jan will go swimming. [Note that "unless" really means "if not." It also can be taken to mean "or."]

25. In each case there will be two statements. It is being asserted that the first one holds true if and only if the second one does. The order doesn't matter, but often one order is more colloquial English.

a) You buy an ice cream cone if and only if it is hot outside.

b) You win the contest if and only if you hold the only winning ticket.

c) You get promoted if and only if you have connections.

d) Your mind will decay if and only if you watch television.

e) The train runs late if and only if it is a day I take the train.

27. Many forms of the answers for this exercise are possible.

a) One form of the converse that reads well in English is "I will ski tomorrow only if it snows today." We could state the contrapositive as "If I don't ski tomorrow, then it will not have snowed today." The inverse is "If it does not snow today, then I will not ski tomorrow."

b) The proposition as stated can be rendered "If there is going to be a quiz, then I will come to class." The converse is "If I come to class, then there will be a quiz." (Or, perhaps even better, "I come to class only if there's going to be a quiz.") The contrapositive is "If I don't come to class, then there won't be a quiz." The inverse is "If there is not going to be a quiz, then I don't come to class."

c) There is a variable ("a positive integer") in this sentence, so technically it is not a proposition. Nevertheless, we can treat sentences such as this in the same way we treat propositions. Its converse is "A positive integer is a prime if it has no divisors other than 1 and itself." (Note that this can be false, since the number 1 satisfies the hypothesis but not the conclusion.) The contrapositive of the original proposition is "If a positive integer has a divisor other than 1 and itself, then it is not prime." (We are simplifying a bit here, replacing "does not have no divisors" by "has a divisor." Note that this is always true, assuming that we are talking about positive divisors.) The inverse is "If a positive integer is not prime, then it has a divisor other than 1 and itself."

29. A truth table will need 2^n rows if there are n variables.

a) $2^1 = 2$ **b)** $2^4 = 16$ **c)** $2^6 = 64$ **d)** $2^4 = 16$

31. To construct the truth table for a compound proposition, we work from the inside out. In each case, we will show the intermediate steps. In part **(d)**, for example, we first construct the truth table for $p \vee q$, then the truth table for $p \wedge q$, and finally combine them to get the truth table for $(p \vee q) \rightarrow (p \wedge q)$. For parts **(a)** and **(b)** we have the following table (column three for part **(a)**, column four for part **(b)**).

p	$\neg p$	$p \wedge \neg p$	$p \vee \neg p$
T	F	F	T
F	T	F	T

For part **(c)** we have the following table.

p	q	$\neg q$	$p \vee \neg q$	$(p \vee \neg q) \rightarrow q$
T	T	F	T	T
T	F	T	T	F
F	T	F	F	T
F	F	T	T	F

For part **(d)** we have the following table.

p	q	$p \vee q$	$p \wedge q$	$(p \vee q) \rightarrow (p \wedge q)$
T	T	T	T	T
T	F	T	F	F
F	T	T	F	F
F	F	F	F	T

For part **(e)** we have the following table. This time we have omitted the column explicitly showing the negations of p and q. Note that this true proposition is telling us that a conditional statement and its contrapositive always have the same truth value.

p	q	$p \to q$	$\neg q \to \neg p$	$(p \to q) \leftrightarrow (\neg q \to \neg p)$
T	T	T	T	T
T	F	F	F	T
F	T	T	T	T
F	F	T	T	T

For part **(f)** we have the following table. The fact that this proposition is not always true tells us that knowing a conditional statement in one direction does not tell us that the conditional statement is true in the other direction.

p	q	$p \to q$	$q \to p$	$(p \to q) \to (q \to p)$
T	T	T	T	T
T	F	F	T	T
F	T	T	F	F
F	F	T	T	T

33. To construct the truth table for a compound proposition, we work from the inside out. In each case, we will show the intermediate steps. In part **(a)**, for example, we first construct the truth table for $p \vee q$, then the truth table for $p \oplus q$, and finally combine them to get the truth table for $(p \vee q) \to (p \oplus q)$. For parts **(a)**, **(b)**, and **(c)** we have the following table (column five for part **(a)**, column seven for part **(b)**, column eight for part **(c)**).

p	q	$p \vee q$	$p \oplus q$	$(p \vee q) \to (p \oplus q)$	$p \wedge q$	$(p \oplus q) \to (p \wedge q)$	$(p \vee q) \oplus (p \wedge q)$
T	T	T	F	F	T	T	F
T	F	T	T	T	F	F	T
F	T	T	T	T	F	F	T
F	F	F	F	T	F	T	F

For part **(d)** we have the following table.

p	q	$\neg p$	$p \leftrightarrow q$	$\neg p \leftrightarrow q$	$(p \leftrightarrow q) \oplus (\neg p \leftrightarrow q)$
T	T	F	T	F	T
T	F	F	F	T	T
F	T	T	F	T	T
F	F	T	T	F	T

For part **(e)** we need eight rows in our truth table, because we have three variables.

p	q	r	$\neg p$	$\neg r$	$p \leftrightarrow q$	$\neg p \leftrightarrow \neg r$	$(p \leftrightarrow q) \oplus (\neg p \leftrightarrow \neg r)$
T	T	T	F	F	T	T	F
T	T	F	F	T	T	F	T
T	F	T	F	F	F	T	T
T	F	F	F	T	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	T	F	T	T
F	F	T	T	F	T	F	T
F	F	F	T	T	T	T	F

For part **(f)** we have the following table.

p	q	$\neg q$	$p \oplus q$	$p \oplus \neg q$	$(p \oplus q) \to (p \oplus \neg q)$
T	T	F	F	T	T
T	F	T	T	F	F
F	T	F	T	F	F
F	F	T	F	T	T

35. The techniques are the same as in Exercises 31–34. For parts **(a)** and **(b)** we have the following table (column four for part **(a)**, column six for part **(b)**).

p	q	$\neg q$	$p \to \neg q$	$\neg p$	$\neg p \leftrightarrow q$
T	T	F	F	F	F
T	F	T	T	F	T
F	T	F	T	T	T
F	F	T	T	T	F

For parts **(c)** and **(d)** we have the following table (columns six and seven, respectively).

p	q	$p \to q$	$\neg p$	$\neg p \to q$	$(p \to q) \vee (\neg p \to q)$	$(p \to q) \wedge (\neg p \to q)$
T	T	T	F	T	T	T
T	F	F	F	T	T	F
F	T	T	T	T	T	T
F	F	T	T	F	T	F

For parts **(e)** and **(f)** we have the following table (this time we have not explicitly shown the columns for negation). Column five shows the answer for part **(e)**, and column seven shows the answer for part **(f)**.

p	q	$p \leftrightarrow q$	$\neg p \leftrightarrow q$	$(p \leftrightarrow q) \vee (\neg p \leftrightarrow q)$	$\neg p \leftrightarrow \neg q$	$(\neg p \leftrightarrow \neg q) \leftrightarrow (p \leftrightarrow q)$
T	T	T	F	T	T	T
T	F	F	T	T	F	T
F	T	F	T	T	F	T
F	F	T	F	T	T	T

37. The techniques are the same as in Exercises 31–36, except that there are now three variables and therefore eight rows. For part **(a)**, we have

p	q	r	$\neg q$	$\neg q \vee r$	$p \to (\neg q \vee r)$
T	T	T	F	T	T
T	T	F	F	F	F
T	F	T	T	T	T
T	F	F	T	T	T
F	T	T	F	T	T
F	T	F	F	F	T
F	F	T	T	T	T
F	F	F	T	T	T

For part **(b)**, we have

p	q	r	$\neg p$	$q \to r$	$\neg p \to (q \to r)$
T	T	T	F	T	T
T	T	F	F	F	T
T	F	T	F	T	T
T	F	F	F	T	T
F	T	T	T	T	T
F	T	F	T	F	F
F	F	T	T	T	T
F	F	F	T	T	T

Parts **(c)** and **(d)** we can combine into a single table.

p	q	r	$p \to q$	$\neg p$	$\neg p \to r$	$(p \to q) \vee (\neg p \to r)$	$(p \to q) \wedge (\neg p \to r)$
T	T	T	T	F	T	T	T
T	T	F	T	F	T	T	T
T	F	T	F	F	T	T	F
T	F	F	F	F	T	T	F
F	T	T	T	T	T	T	T
F	T	F	T	T	F	T	F
F	F	T	T	T	T	T	T
F	F	F	T	T	F	T	F

For part **(e)** we have

p	q	r	$p \leftrightarrow q$	$\neg q$	$\neg q \leftrightarrow r$	$(p \leftrightarrow q) \vee (\neg q \leftrightarrow r)$
T	T	T	T	F	F	T
T	T	F	T	F	T	T
T	F	T	F	T	T	T
T	F	F	F	T	F	F
F	T	T	F	F	F	F
F	T	F	F	F	T	T
F	F	T	T	T	T	T
F	F	F	T	T	F	T

Finally, for part **(f)** we have

p	q	r	$\neg p$	$\neg q$	$\neg p \leftrightarrow \neg q$	$q \leftrightarrow r$	$(\neg p \leftrightarrow \neg q) \leftrightarrow (q \leftrightarrow r)$
T	T	T	F	F	T	T	T
T	T	F	F	F	T	F	F
T	F	T	F	T	F	F	T
T	F	F	F	T	F	T	F
F	T	T	T	F	F	T	F
F	T	F	T	F	F	F	T
F	F	T	T	T	T	F	F
F	F	F	T	T	T	T	T

39. This time the truth table needs $2^4 = 16$ rows. Note the systematic order in which we list the possibilities.

p	q	r	s	$p \leftrightarrow q$	$r \leftrightarrow s$	$(p \leftrightarrow q) \leftrightarrow (r \leftrightarrow s)$
T	T	T	T	T	T	T
T	T	T	F	T	F	F
T	T	F	T	T	F	F
T	T	F	F	T	T	T
T	F	T	T	F	T	F
T	F	T	F	F	F	T
T	F	F	T	F	F	T
T	F	F	F	F	T	F
F	T	T	T	F	T	F
F	T	T	F	F	F	T
F	T	F	T	F	F	T
F	T	F	F	F	T	F
F	F	T	T	T	T	T
F	F	T	F	T	F	F
F	F	F	T	T	F	F
F	F	F	F	T	T	T

41. The first clause $(p \vee q \vee r)$ is true if and only if at least one of p, q, and r is true. The second clause $(\neg p \vee \neg q \vee \neg r)$ is true if and only if at least one of the three variables is false. Therefore both clauses are true, and therefore the entire statement is true, if and only if there is at least one T and one F among the truth values of the variables, in other words, that they don't all have the same truth value.

43. **a)** bitwise $OR = 111\ 1111$; bitwise $AND = 000\ 0000$; bitwise $XOR = 111\ 1111$
b) bitwise $OR = 1111\ 1010$; bitwise $AND = 1010\ 0000$; bitwise $XOR = 0101\ 1010$
c) bitwise $OR = 10\ 0111\ 1001$; bitwise $AND = 00\ 0100\ 0000$; bitwise $XOR = 10\ 0011\ 1001$
d) bitwise $OR = 11\ 1111\ 1111$; bitwise $AND = 00\ 0000\ 0000$; bitwise $XOR = 11\ 1111\ 1111$

45. For "Fred is not happy," the truth value is $1 - 0.8 = 0.2$.
For "John is not happy," the truth value is $1 - 0.4 = 0.6$.

47. For "Fred is happy, or John is happy," the truth value is $\max(0.8, 0.4) = 0.8$.
For "Fred is not happy, or John is not happy," the truth value is $\max(0.2, 0.6) = 0.6$ (using the result of Exercise 45).

49. One great problem-solving strategy to try with problems like this, when the parameter is large (100 statements here) is to lower the parameter. Look at a simpler problem, with just two or three statements, and see if you can figure out what's going on. That was the approach used to discover the solution presented here.

a) Some number of these statements are true, so in fact exactly one of the statements must be true and the other 99 of them must be false. That is what the 99^{th} statement is saying, so it is true and the rest are false.

b) The 100^{th} statement cannot be true, since it is asserting that all the statements are false. Therefore it must be false. That makes the first statement true. Now if the 99^{th} statement were true, then we would conclude that statements 2 through 100 were false, which contradicts the truth of statement 99. So statement 99 must be false. That means that statement 2 is true. We continue in this way and conclude that statements 1 through 50 are all true and statements 51 through 100 are all false.

c) If there are an odd number of statements, then we'd run into a contradiction when we got to the middle. If there were just three statements, for example, then statement 3 would have to be false, making statement 1 true, and now the truth of statement 2 would imply its falsity and its falsity would imply its truth. Therefore this situation cannot occur with three (or any odd number of) statements. It is a logical paradox, showing that in fact these are not statements after all.

SECTION 1.2 Applications of Propositional Logic

Applications of propositional logic abound in computer science, puzzles, and everyday life. For example, much of the operation of our legal system is based on conditional statements. Boolean searches are increasingly important in using the Web (see Exercises 13–14 for example).

1. Recall that "q unless $\neg p$" is another way to state $p \to q$. In this problem, $\neg p$ is a, so p is $\neg a$; and q is $\neg e$. Therefore the statement here is $\neg a \to \neg e$. This could also be stated equivalently as $e \to a$ (if you can edit, then you must be an administrator).

3. Recall that p only if q means $p \to q$. In this case, if you can graduate then you must have fulfilled the three listed requirements. Therefore the statement is $g \to (r \wedge (\neg m) \wedge (\neg b))$. Notice that in everyday life one might actually say "You can graduate if you do these things," but logically that is not what the rules really say.

5. This is similar to Exercise 3. If you are eligible to be President, then you must satisfy the requirements: $e \to (a \wedge (b \vee p) \wedge r)$. Notice that it is only the requirement of being native-born that can be overridden by having parents who were citizens, so $b \vee p$ is grouped as one of the three conditions.

7. a) Since "whenever" means "if," we have $q \to p$.

b) Since "but" means "and," we have $q \wedge \neg p$.

c) This sentence is saying the same thing as the sentence in part **(a)**, so the answer is the same: $q \to p$.

d) Again, we recall that "when" means "if" in logic: $\neg q \to \neg p$.

9. Let m, n, k, and i represent the propositions "The system is in multiuser state," "The system is operating normally," "The kernel is functioning," and "The system is in interrupt mode," respectively. Then we want to make the following expressions simultaneously true by our choice of truth values for m, n, k, and i:

$$m \leftrightarrow n, \quad n \to k, \quad \neg k \vee i, \quad \neg m \to i, \quad \neg i$$

In order for this to happen, clearly i must be false. In order for $\neg m \to i$ to be true when i is false, the hypothesis $\neg m$ must be false, so m must be true. Since we want $m \leftrightarrow n$ to be true, this implies that n must also be true. Since we want $n \to k$ to be true, we must therefore have k true. But now if k is true and i is false, then the third specification, $\neg k \vee i$ is false. Therefore we conclude that this system is not consistent.

11. Let s be "The router can send packets to the edge system"; let a be "The router supports the new address space"; let r be "The latest software release is installed." Then we are told $s \to a$, $a \to r$, $r \to s$, and $\neg a$. Since a is false, the first conditional statement tells us that s must be false. From that we deduce from the third conditional statement that r must be false. If indeed all three propositions are false, then all four specifications are true, so they are consistent.

13. This is similar to Example 6, about universities in New Mexico. To search for beaches in New Jersey, we could enter NEW **AND** JERSEY **AND** BEACHES. If we enter (JERSEY **AND** BEACHES) **NOT** NEW, then we'll get websites about beaches on the isle of Jersey, except for sites that happen to use the word "new" in a different context (e.g., a recently opened beach there). If we were sure that the word "isle" was in the name of the location, then of course we could enter ISLE **AND** JERSEY **AND** BEACHES.

15. There are many correct answers to this problem, but all involve some sort of double layering, or combining a question about the kind of person being addressed with a question about the information being sought. One solution is to ask this question: "If I were to ask you whether the right branch leads to the ruins, would you say 'yes'?" If the villager is a truth-teller, then of course he will reply "yes" if and only if the right branch leads to the ruins. Now let us see what the liar says. If the right branch leads to the ruins, then he would say "no" if asked whether the right branch leads to the ruins. Therefore, the truthful answer to your convoluted question is "no." Since he always lies, he will reply "yes." On the other hand, if the right branch does not lead to the ruins, then he would say "yes" if asked whether the right branch leads to the ruins; and so the truthful answer to your question is "yes"; therefore he will reply "no." Note that in both cases, he gives the same answer to your question as the truth-teller; namely, he says "yes" if and only if the right branch leads to the ruins. A more detailed discussion can be found in Martin Gardner's *Scientific American Book of Mathematical Puzzles and Diversions* (Simon and Schuster, 1959), p. 25; reprinted as *Hexaflexagons and Other Mathematical Diversions: The First Scientific American Book of Puzzles and Games* (University of Chicago Press, 1988).

17. The question was "Does *everyone* want coffee?" If the first professor did not want coffee, then he would know that the answer to the hostess's question was "no." Therefore we—and the hostess and the remaining professors—know that the first professor does want coffee. The same argument applies to the second professor, so she, too, must want coffee. The third professor can now answer the question. Because she said "no," we conclude that she does not want coffee. Therefore the hostess knows to bring coffee to the first two professors but not to the third.

19. If A is a knight, then he is telling the truth, in which case B must be a knave. Since B said nothing, that is certainly possible. If A is a knave, then he is lying, which means that his statement that at least one of them is a knave is false; hence they are both knights. That is a contradiction. So we can conclude that A is a knight and B is a knave.

21. If A is a knight, then he is telling the truth, in which case B must be a knight as well, since A is not a knave. (If $p \vee q$ and $\neg p$ are both true, then q must be true.) Since B said nothing, that is certainly possible. If A is a knave, then his statement is patently true, but that is a contradiction to the behavior of knaves. So we can conclude that A is a knight and B is a knight.

23. If A is a knight, then he should be telling the truth, but he is asserting that he is a knave. So that cannot be. If A is a knave, then in order for his statement to be false, B must be a knight. So we can conclude that A is a knave and B is a knight.

25. Neither the knight nor the knave would say that he is the knave, so B must be the spy. Therefore C is lying and must be the knave, and A is therefore the knight (and told the truth).

27. We know that B is not the knight, because if he were, then his assertion that A is telling the truth would mean that there were two knights. Clearly C is not the knight, because he claims he is the spy. Therefore A is the knight. That means that B was telling the truth, so he must be the spy. And C is the knave, who falsely asserts that he is the spy.

29. We can tell nothing here; each of the six permutations is possible. The knight will always say that he is the knight; the knave will always lie, so he might also say that he is the knight; and the spy may lie and say that he is the knight.

31. If there were a solution, then whoever is the knave here is speaking the truth when he says that he is not the spy. Because knaves always lie, we get a contradiction. Therefore there are no solutions.

33. Because of the first piece of information that Steve has, let's assume first that Fred is not the highest paid. Then Janice is. Therefore Janice is not the lowest paid, so by the second piece of information that Steve has, Maggie is the highest paid. But that is a contradiction. Therefore we know that Fred *is* the highest paid. Next let's assume that Janice is not the lowest paid. Then our second fact implies that Maggie is the highest paid. But that contradicts the fact that Fred is the highest paid. Therefore we know that Janice *is* the lowest paid. So it appears that the only hope of a consistent set of facts is to have Fred paid the most, Maggie next, and Janice the least. (We have just seen that any other assumption leads to a contradiction.) This assumption does not contradict either of our two facts, since in both cases, the hypothesis is false.

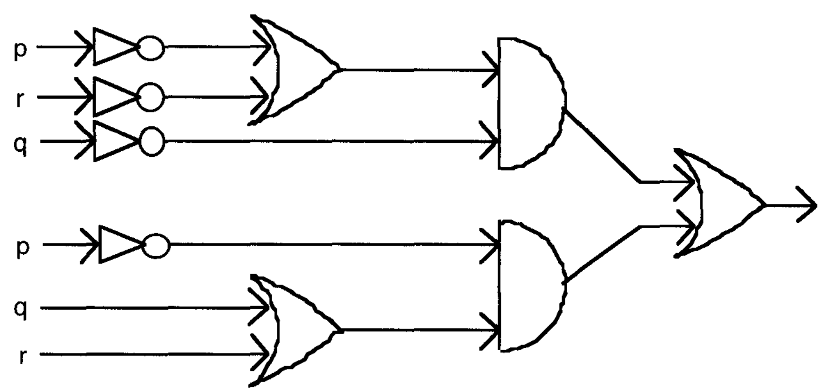
35. Let's use the letters B, C, G, and H for the statements that the butler, cook, gardner, and handyman are telling the truth, respectively. We can then write each fact as a true proposition: $B \to C$; $\neg(C \wedge G)$, which is equivalent to $\neg C \vee \neg G$ (see the discussion of De Morgan's law in Section 1.3); $\neg(\neg G \wedge \neg H)$, which is equivalent to $G \vee H$; and $H \to \neg C$. Suppose that B is true. Then it follows from the first of our propositions that C must also be true. This tells us (using the second proposition) that G must be false, whence the third proposition makes H true. But now the fourth proposition is violated. Therefore we conclude that B cannot be true. If fact, the argument we have just given also proves that C cannot be true. Therefore we know that the butler and the cook are lying. This much already makes the first, second, and fourth propositions true, regardless of the truth of G or H. Thus either the gardner or the handyman could be lying or telling the truth; all we know (from the third proposition) is that at least one of them is telling the truth.

37. If the first sign were true, then the second sign would also be true. In that case, we could not have one true sign and one false sign. Rather, the second sign is true and the first is false; there is a lady in the second room and a tiger in the first room.

39. The given conditions imply that there cannot be two honest senators. Therefore, since we are told that there is at least one honest senator, there must be exactly 49 corrupt senators.

41. **a)** The output of the OR gate is $q \vee \neg r$. Therefore the output of the AND gate is $p \wedge (q \vee \neg r)$. Therefore the output of this circuit is $\neg(p \wedge (q \vee \neg r))$.
b) The output of the top AND gate is $(\neg p) \wedge (\neg q)$. The output of the bottom AND gate is $p \wedge r$. Therefore the output of this circuit is $((\neg p) \wedge (\neg q)) \vee (p \wedge r)$.

43. We have the inputs come in from the left, in some cases passing through an inverter to form their negations. Certain pairs of them enter OR gates, and the outputs of these and other negated inputs enter AND gates. The outputs of these AND gates enter the final OR gate.



SECTION 1.3 Propositional Equivalences

The solutions to Exercises 1–10 are routine; we use truth tables to show that a proposition is a tautology or that two propositions are equivalent. The reader should do more than this, however; think about what the equivalence is saying. See Exercise 11 for this approach. Some important topics not covered in the text are introduced in this exercise set, including the notion of the **dual** *of a proposition,* **disjunctive normal form** *for propositions,* **functional completeness**, **satisfiability**, *and two other logical connectives,* NAND *and* NOR. *Much of this material foreshadows the study of Boolean algebra in Chapter 12.*

1. First we construct the following truth tables, for the propositions we are asked to deal with.

p	$p \wedge \mathbf{T}$	$p \vee \mathbf{F}$	$p \wedge \mathbf{F}$	$p \vee \mathbf{T}$	$p \vee p$	$p \wedge p$
T	T	T	F	T	T	T
F	F	F	F	T	F	F

The first equivalence, $p \wedge \mathbf{T} \equiv p$, is valid because the second column $p \wedge \mathbf{T}$ is identical to the first column p. Similarly, part **(b)** comes from looking at columns three and one. Since column four is a column of F's, and column five is a column of T's, part **(c)** and part **(d)** hold. Finally, the last two parts follow from the fact that the last two columns are identical to the first column.

3. We construct the following truth tables.

p	q	$p \vee q$	$q \vee p$	$p \wedge q$	$q \wedge p$
T	T	T	T	T	T
T	F	T	T	F	F
F	T	T	T	F	F
F	F	F	F	F	F

Part **(a)** follows from the fact that the third and fourth columns are identical; part **(b)** follows from the fact that the fifth and sixth columns are identical.

5. We construct the following truth table and note that the fifth and eighth columns are identical.

p	q	r	$q \vee r$	$p \wedge (q \vee r)$	$p \wedge q$	$p \wedge r$	$(p \wedge q) \vee (p \wedge r)$
T	T	T	T	T	T	T	T
T	T	F	T	T	T	F	T
T	F	T	T	T	F	T	T
T	F	F	F	F	F	F	F
F	T	T	T	F	F	F	F
F	T	F	T	F	F	F	F
F	F	T	T	F	F	F	F
F	F	F	F	F	F	F	F

7. De Morgan's laws tell us that to negate a conjunction we form the disjunction of the negations, and to negate a disjunction we form the conjunction of the negations.

a) This is the conjunction "Jan is rich, and Jan is happy." So the negation is "Jan is not rich, or Jan is not happy."

b) This is the disjunction "Carlos will bicycle tomorrow, or Carlos will run tomorrow." So the negation is "Carlos will not bicycle tomorrow, and Carlos will not run tomorrow." We could also render this as "Carlos will neither bicycle nor run tomorrow."

c) This is the disjunction "Mei walks to class, or Mei takes the bus to class." So the negation is "Mei does not walk to class, and Mei does not take the bus to class." (Maybe she gets a ride with a friend.) We could also render this as "Mei neither walks nor takes the bus to class."

d) This is the conjunction "Ibrahim is smart, and Ibrahim is hard working." So the negation is "Ibrahim is not smart, or Ibrahim is not hard working."

9. We construct a truth table for each conditional statement and note that the relevant column contains only T's. For parts **(a)** and **(b)** we have the following table (column four for part **(a)**, column six for part **(b)**).

p	q	$p \wedge q$	$(p \wedge q) \rightarrow p$	$p \vee q$	$p \rightarrow (p \vee q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	F	T	T	T
F	F	F	T	F	T

For parts **(c)** and **(d)** we have the following table (columns five and seven, respectively).

p	q	$\neg p$	$p \rightarrow q$	$\neg p \rightarrow (p \rightarrow q)$	$p \wedge q$	$(p \wedge q) \rightarrow (p \rightarrow q)$
T	T	F	T	T	T	T
T	F	F	F	T	F	T
F	T	T	T	T	F	T
F	F	T	T	T	F	T

For parts **(e)** and **(f)** we have the following table. Column five shows the answer for part **(e)**, and column seven shows the answer for part **(f)**.

p	q	$p \to q$	$\neg(p \to q)$	$\neg(p \to q) \to p$	$\neg q$	$\neg(p \to q) \to \neg q$
T	T	T	F	T	F	T
T	F	F	T	T	T	T
F	T	T	F	T	F	T
F	F	T	F	T	T	T

11. Here is one approach: Recall that the only way a conditional statement can be false is for the hypothesis to be true and the conclusion to be false; hence it is sufficient to show that the conclusion must be true whenever the hypothesis is true. An alternative approach that works for some of these tautologies is to use the equivalences given in this section and prove these "algebraically." We will demonstrate this second method in some of the solutions.

a) If the hypothesis is true, then by the definition of $\wedge$ we know that p is true. Hence the conclusion is also true. For an algebraic proof, we exhibit the following string of equivalences, each one following from one of the laws in this section: $(p \wedge q) \to p \equiv \neg(p \wedge q) \vee p \equiv (\neg p \vee \neg q) \vee p \equiv (\neg q \vee \neg p) \vee p \equiv \neg q \vee (\neg p \vee p) \equiv \neg q \vee \mathbf{T} \equiv \mathbf{T}$. The first logical equivalence is the first equivalence in Table 7 (with $p \wedge q$ playing the role of p, and p playing the role of q); the second is De Morgan's law; the third is the commutative law; the fourth is the associative law; the fifth is the negation law (with the commutative law); and the sixth is the domination law.

b) If the hypothesis p is true, then by the definition of $\vee$, the conclusion $p \vee q$ must also be true.

c) If the hypothesis is true, then p must be false; hence the conclusion $p \to q$ is true, since *its* hypothesis is false. Symbolically we have $\neg p \to (p \to q) \equiv \neg\neg p \vee (\neg p \vee q) \equiv p \vee (\neg p \vee q) \equiv (p \vee \neg p) \vee q \equiv \mathbf{T} \vee q \equiv \mathbf{T}$.

d) If the hypothesis is true, then by the definition of $\wedge$ we know that q must be true. This makes the conclusion $p \to q$ true, since *its* conclusion is true.

e) If the hypothesis is true, then $p \to q$ must be false. But this can happen only if p is true, which is precisely what we wanted to show.

f) If the hypothesis is true, then $p \to q$ must be false. But this can happen only if q is false, which is precisely what we wanted to show.

13. We first construct truth tables and verify that in each case the two propositions give identical columns. The fact that the fourth column is identical to the first column proves part **(a)**, and the fact that the sixth column is identical to the first column proves part **(b)**.

p	q	$p \wedge q$	$p \vee (p \wedge q)$	$p \vee q$	$p \wedge (p \vee q)$
T	T	T	T	T	T
T	F	F	T	T	T
F	T	F	F	T	F
F	F	F	F	F	F

Alternately, we can argue as follows.

a) If p is true, then $p \vee (p \wedge q)$ is true, since the first proposition in the disjunction is true. On the other hand, if p is false, then both parts of the disjunction are false. Hence $p \vee (p \wedge q)$ always has the same truth value as p does, so the two propositions are logically equivalent.

b) If p is false, then $p \wedge (p \vee q)$ is false, since the first proposition in the conjunction is false. On the other hand, if p is true, then both parts of the conjunction are true. Hence $p \wedge (p \vee q)$ always has the same truth value as p does, so the two propositions are logically equivalent.

15. We need to determine whether we can find an assignment of truth values to p and q to make this proposition false. Let us try to find one. The only way that a conditional statement can be false is for the hypothesis to be true and the conclusion to be false. Hence we must make $\neg p$ false, which means we must make p true. Furthermore, in order for the hypothesis to be true, we will need to make q false, so that the first part of the conjunction will be true. But now with p true and q false, the second part of the conjunction is false.

Therefore the entire hypothesis is false, so this assignment will not yield a false conditional statement. Since we have argued that no assignment of truth values can make this proposition false, we have proved that this proposition is a tautology. (An alternative approach would be to construct a truth table and see that its final column had only T's in it.) This tautology is telling us that if we know that a conditional statement is true, and that its conclusion is false, then we can conclude that its antecedent is also false.

17. The proposition $\neg(p \leftrightarrow q)$ is true when p and q do not have the same truth values, which means that p and q have different truth values (either p is true and q is false, or vice versa). These are exactly the cases in which $p \leftrightarrow \neg q$ is true. Therefore these two expressions are true in exactly the same instances, and therefore are logically equivalent.

19. The proposition $\neg p \leftrightarrow q$ is true when $\neg p$ and q have the same truth values, which means that p and q have different truth values (either p is true and q is false, or vice versa). By the same reasoning, these are exactly the cases in which $p \leftrightarrow \neg q$ is true. Therefore these two expressions are true in exactly the same instances, and therefore are logically equivalent.

21. This is essentially the same as Exercise 17. The proposition $\neg(p \leftrightarrow q)$ is true when $p \leftrightarrow q$ is false. Since $p \leftrightarrow q$ is true when p and q have the same truth value, it is false when p and q have different truth values (either p is true and q is false, or vice versa). These are precisely the cases in which $\neg p \leftrightarrow q$ is true.

23. We'll determine exactly which rows of the truth table will have F as their entries. In order for $(p \rightarrow r) \wedge (q \rightarrow r)$ to be false, we must have at least one of the two conditional statements false, which happens exactly when r is false and at least one of p and q is true. But these are precisely the cases in which $p \vee q$ is true and r is false, which is precisely when $(p \vee q) \rightarrow r$ is false. Since the two propositions are false in exactly the same situations, they are logically equivalent.

25. We'll determine exactly which rows of the truth table will have F as their entries. In order for $(p \rightarrow r) \vee (q \rightarrow r)$ to be false, we must have both of the two conditional statements false, which happens exactly when r is false and both p and q are true. But this is precisely the case in which $p \wedge q$ is true and r is false, which is precisely when $(p \wedge q) \rightarrow r$ is false. Since the two propositions are false in exactly the same situations, they are logically equivalent.

27. This fact was observed in Section 1.1 when the biconditional was first defined. Each of these is true precisely when p and q have the same truth values.

29. We will show that if $p \rightarrow q$ and $q \rightarrow r$ are both true, then $p \rightarrow r$ is true. Thus we want to show that if p is true, then so is r. Given that p and $p \rightarrow q$ are both true, we conclude that q is true; from that and $q \rightarrow r$ we conclude that r is true, as desired. This can also be done with a truth table.

31. To show that these are *not* logically equivalent, we need only find one assignment of truth values to p, q, and r for which the truth values of $(p \rightarrow q) \rightarrow r$ and $p \rightarrow (q \rightarrow r)$ differ. One such assignment is F for all three. Then $(p \rightarrow q) \rightarrow r$ is false and $p \rightarrow (q \rightarrow r)$ is true.

33. To show that these are *not* logically equivalent, we need only find one assignment of truth values to p, q, r, and s for which the truth values of $(p \rightarrow q) \rightarrow (r \rightarrow s)$ and $(p \rightarrow r) \rightarrow (q \rightarrow s)$ differ. Let us try to make the first one false. That means we have to make $r \rightarrow s$ false, so we want r to be true and s to be false. If we let p and q be false, then each of the other three simple conditional statements ($p \rightarrow q$, $p \rightarrow r$, and $q \rightarrow s$) will be true. Then $(p \rightarrow q) \rightarrow (r \rightarrow s)$ will be $\mathrm{T} \rightarrow \mathrm{F}$, which is false; but $(p \rightarrow r) \rightarrow (q \rightarrow s)$ will be $\mathrm{T} \rightarrow \mathrm{T}$, which is true.

35. We apply the rules stated in the preamble.
a) $p \vee \neg q \vee \neg r$ **b)** $(p \vee q \vee r) \wedge s$ **c)** $(p \wedge \mathbf{T}) \vee (q \wedge \mathbf{F})$

37. If we apply the operation for forming the dual twice to a proposition, then every symbol returns to what it originally was. The $\wedge$ changes to the $\vee$, then changes back to the $\wedge$. Similarly the $\vee$ changes to the $\wedge$, then back to the $\vee$. The same thing happens with the **T** and the **F**. Thus the dual of the dual of a proposition s, namely $(s^*)^*$, is equal to the original proposition s.

39. Let p and q be two compound propositions involving only the operators $\wedge$, $\vee$, and $\neg$; we can also allow them to involve the constants **T** and **F**. We want to show that if p and q are logically equivalent, then p^* and q^* are logically equivalent. The trick is to look at $\neg p$ and $\neg q$. They are certainly logically equivalent if p and q are. Now if p is a conjunction, say $r \wedge s$, then $\neg p$ is logically equivalent, by De Morgan's law, to $\neg r \vee \neg s$; a similar statement applies if p is a disjunction. If r and/or s are themselves compound propositions, then we apply De Morgan's laws again to "push" the negation symbol $\neg$ deeper inside the formula, changing $\wedge$ to $\vee$ and $\vee$ to $\wedge$. We repeat this process until all the negation signs have been "pushed in" as far as possible and are now attached to the atomic (i.e., not compound) propositions in the compound propositions p and q. Call these atomic propositions p_1, p_2, etc. Now in this process De Morgan's laws have forced us to change each $\wedge$ to $\vee$ and each $\vee$ to $\wedge$. Furthermore, if there are any constants **T** or **F** in the propositions, then they will be changed to their opposite when the negation operation is applied: $\neg\mathbf{T}$ is the same as **F**, and $\neg\mathbf{F}$ is the same as **T**. In summary, $\neg p$ and $\neg q$ look just like p^* and q^*, except that each atomic proposition p_i within them is replaced by its negation. Now we agreed that $\neg p \equiv \neg q$; this means that for *every* possible assignment of truth values to the atomic propositions p_1, p_2, etc., the truth values of $\neg p$ and $\neg q$ are the same. But assigning T to p_i is the same as assigning F to $\neg p_i$, and assigning F to p_i is the same as assigning T to $\neg p_i$. Thus, for every possible assignment of truth values to the atomic propositions, the truth values of p^* and q^* are the same. This is precisely what we wanted to prove.

41. There are three ways in which exactly two of p, q, and r can be true. We write down these three possibilities as conjunctions and join them by $\vee$ to obtain the answer: $(p \wedge q \wedge \neg r) \vee (p \wedge \neg q \wedge r) \vee (\neg p \wedge q \wedge r)$. See Exercise 42 for a more general result.

43. Given a compound proposition p, we can construct its truth table and then, by Exercise 42, write down a proposition q in disjunctive normal form that is logically equivalent to p. Since q involves only $\neg$, $\wedge$, and $\vee$, this shows that $\neg$, $\wedge$, and $\vee$ form a functionally complete collection of logical operators.

45. Given a compound proposition p, we can, by Exercise 43, write down a proposition q that is logically equivalent to p and uses only $\neg$, $\wedge$, and $\vee$. Now by De Morgan's law we can get rid of all the $\wedge$'s by replacing each occurrence of $p_1 \wedge p_2 \wedge \cdots \wedge p_n$ with the equivalent proposition $\neg(\neg p_1 \vee \neg p_2 \vee \cdots \vee \neg p_n)$.

47. The proposition $\neg(p \wedge q)$ is true when either p or q, or both, are false, and is false when both p and q are true; since this was the definition of $p \mid q$, the two are logically equivalent.

49. The proposition $\neg(p \vee q)$ is true when both p and q are false, and is false otherwise; since this was the definition of $p \downarrow q$, the two are logically equivalent.

51. A straightforward approach, using the results of Exercise 50, parts **(a)** and **(b)**, is as follows: $(p \to q) \equiv (\neg p \vee q) \equiv ((p \downarrow p) \vee q) \equiv (((p \downarrow p) \downarrow q) \downarrow ((p \downarrow p) \downarrow q))$. If we allow the constant **F** in our expression, then a simpler answer is $\mathbf{F} \downarrow ((\mathbf{F} \downarrow p) \downarrow q)$.

53. This is clear from the definition, in which p and q play a symmetric role.

55. A truth table for a compound proposition involving p and q has four lines, one for each of the following combinations of truth values for p and q: TT, TF, FT, and FF. Now each line of the truth table for the compound proposition can be either T or F. Thus there are two possibilities for the first line; for each of those there are two possibilities for the second line, giving $2 \cdot 2 = 4$ possibilities for the first two lines; for each of those there are two possibilities for the third line, giving $4 \cdot 2 = 8$ possibilities for the first three lines; and finally for each of those, there are two possibilities for the fourth line, giving $8 \cdot 2 = 16$ possibilities altogether. This sort of counting will be studied extensively in Chapter 6.

57. Let do, mc, and in stand for the propositions "The directory database is opened," "The monitor is put in a closed state," and "The system is in its initial state," respectively. Then the given statement reads $\neg in \rightarrow (do \rightarrow mc)$. By the third line of Table 7 (twice), this is equivalent to $in \vee (\neg do \vee mc)$. In words, this says that it must always be true that either the system is in its initial state, or the data base is not opened, or the monitor is put in a closed state. Another way to render this would be to say that if the database is open, then either the system is in its initial state or the monitor is put in a closed state.

59. Disjunctions are easy to make true, since we just have to make sure that at least one of the things being "or-ed" is true. In this problem, we notice that $\neg p$ occurs in four of the disjunctions, so we can satisfy all of them by making p false. Three of the remaining disjunctions contain r, so if we let r be true, those will be taken care of. That leaves only $p \vee \neg q \vee s$ and $q \vee \neg r \vee \neg s$, and we can satisfy both of those by making q and s both true. This assignment, then, makes all nine of the disjunctions true.

61. **a)** With a little trial and error we discover that setting $p = \mathbf{F}$ and $q = \mathbf{F}$ produces $(\mathbf{F} \vee \mathbf{T}) \wedge (\mathbf{T} \vee \mathbf{F}) \wedge (\mathbf{T} \vee \mathbf{T})$, which has the value $\mathbf{T}$. So this compound proposition is satisfiable. (Note that this is the only satisfying truth assignment.)

b) We claim that there is no satisfying truth assignment here. No matter what the truth values of p and q might be, the four implications become $\mathbf{T} \rightarrow \mathbf{T}$, $\mathbf{T} \rightarrow \mathbf{F}$, $\mathbf{F} \rightarrow \mathbf{T}$, and $\mathbf{F} \rightarrow \mathbf{F}$, in some order. Exactly one of these is false, so their conjunction is false.

c) This compound proposition is not satisfiable. In order for the first clause, $p \leftrightarrow q$, to be true, p and q must have the same truth value. In order for the second clause, $(\neg p) \leftrightarrow q$, to be true, p and q must have opposite truth values. These two conditions are incompatible, so there is no satisfying truth assignment.

63. This is done in exactly the same manner as was described in the text for a 9×9 Sudoku puzzle, with the variables indexed from 1 to 4, instead of from 1 to 9, and with a similar change for the propositions for the 2×2 blocks: $\bigwedge_{r=0}^{1} \bigwedge_{s=0}^{1} \bigwedge_{n=1}^{4} \bigvee_{i=1}^{2} \bigvee_{j=1}^{2} p(2r+i, 2s+j, n)$.

65. We just repeat the discussion in the text, with the roles of the rows and columns interchanged: To assert that column j contains the number n, we form $\bigvee_{i=1}^{9} p(i,j,n)$. To assert that column j contains all 9 numbers, we form the conjunction of these disjunctions over all nine possible values of n, giving us $\bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i,j,n)$. To assert that every column contains every number, we take the conjunction of $\bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i,j,n)$ over all nine columns. This gives us $\bigwedge_{j=1}^{9} \bigwedge_{n=1}^{9} \bigvee_{i=1}^{9} p(i,j,n)$.

SECTION 1.4 Predicates and Quantifiers

The reader may find quantifiers hard to understand at first. Predicate logic (the study of propositions with quantifiers) is one level of abstraction higher than propositional logic (the study of propositions without quantifiers). Careful attention to this material will aid you in thinking more clearly, not only in mathematics but in other areas as well, from computer science to politics. Keep in mind exactly what the quantifiers mean: $\forall x$ *means "for all* x*" or "for every* x*," and* $\exists x$ *means "there exists an* x *such that" or "for some* x*." It is good practice to read every such sentence aloud, paying attention to English grammar as well as meaning. It is very important to understand how the negations of quantified statements are formed, and why this method is correct; it is just common sense, really.*

The word "any" in mathematical statements can be ambiguous, so it is best to avoid using it. In negative contexts it almost always means "some" (existential quantifier), as in the statement "You will be suspended from school if you are found guilty of violating any of the plagiarism rules" (you don't have to violate all the rules to get into trouble—breaking one is sufficient). In positive contexts, however, it can mean either "some" (existential quantifier) or "every" (universal quantifier), depending on context. For example, in the sentence "The fraternity will be put on probation if any of its members is found intoxicated," the use is existential (one drunk brother is enough to cause the sanction); but in the sentence "Any member of the sorority will be happy to lead you on a tour of the house," the use is universal (every member is able to be the guide). Another interesting example is an exercise in a mathematics textbook that asks you to show that "the sum of any two odd numbers is even." The author clearly intends the universal interpretation here—you need to show that the sum of two odd numbers is always even. If you interpreted the question existentially, you might say, "Look, $3+5=8$*, so I've shown it is true—you said I could do it for any numbers, and those are the ones I chose."*

1. **a)** T, since $0 \le 4$ **b)** T, since $4 \le 4$ **c)** F, since $6 \not\le 4$

3. **a)** This is true.
b) This is false, since Lansing, not Detroit, is the capital.
c) This is false (but $Q(\text{Boston}, \text{Massachusetts})$ is true).
d) This is false, since Albany, not New York, is the capital.

5. **a)** There is a student who spends more than five hours every weekday in class.
b) Every student spends more than five hours every weekday in class.
c) There is a student who does not spend more than five hours every weekday in class.
d) No student spends more than five hours every weekday in class. (Or, equivalently, every student spends less than or equal to five hours every weekday in class.)

7. **a)** This statement is that for every x, if x is a comedian, then x is funny. In English, this is most simply stated, "Every comedian is funny."
b) This statement is that for every x in the domain (universe of discourse), x is a comedian *and* x is funny. In English, this is most simply stated, "Every person is a funny comedian." Note that this is not the sort of thing one wants to say. It really makes no sense and doesn't say anything about the existence of boring comedians; it's surely false, because there exist lots of x for which $C(x)$ is false. This illustrates the fact that you rarely want to use conjunctions with universal quantifiers.
c) This statement is that there exists an x in the domain such that if x is a comedian then x is funny. In English, this might be rendered, "There exists a person such that if s/he is a comedian, then s/he is funny." Note that this is not the sort of thing one wants to say. It really makes no sense and doesn't say anything about the existence of funny comedians; it's surely true, because there exist lots of x for which $C(x)$ is false (recall

the definition of the truth value of $p \to q$). This illustrates the fact that you rarely want to use conditional statements with existential quantifiers.

d) This statement is that there exists an x in the domain such that x is a comedian and x is funny. In English, this might be rendered, "There exists a funny comedian" or "Some comedians are funny" or "Some funny people are comedians."

9. **a)** We assume that this sentence is asserting that the same person has both talents. Therefore we can write $\exists x(P(x) \land Q(x))$.

b) Since "but" really means the same thing as "and" logically, this is $\exists x(P(x) \land \neg Q(x))$

c) This time we are making a universal statement: $\forall x(P(x) \lor Q(x))$

d) This sentence is asserting the nonexistence of anyone with either talent, so we could write it as $\neg\exists x(P(x) \lor Q(x))$. Alternatively, we can think of this as asserting that everyone fails to have either of these talents, and we obtain the logically equivalent answer $\forall x \neg(P(x) \lor Q(x))$. Failing to have either talent is equivalent to having neither talent (by De Morgan's law), so we can also write this as $\forall x((\neg P(x)) \land (\neg Q(x))$. Note that it would *not* be correct to write $\forall x((\neg P(x)) \lor (\neg Q(x))$ nor to write $\forall x \neg(P(x) \land Q(x))$.

11. **a)** T, since $0 = 0^2$ **b)** T, since $1 = 1^2$ **c)** F, since $2 \neq 2^2$
d) F, since $-1 \neq (-1)^2$ **e)** T (let $x = 1$) **f)** F (let $x = 2$)

13. **a)** Since adding 1 to a number makes it larger, this is true.

b) Since $2 \cdot 0 = 3 \cdot 0$, this is true.

c) This statement is true, since $0 = -0$.

d) This is true for the nonnegative integers but not for the negative integers. For example, $3(-2) \not\leq 4(-2)$. Therefore the universally quantified statement is false.

15. Recall that the integers include the positive and negative integers and 0.

a) This is the well-known true fact that the square of a real number cannot be negative.

b) There are two *real* numbers that satisfy $n^2 = 2$, namely $\pm\sqrt{2}$, but there do not exist any *integers* with this property, so the statement is false.

c) If n is a positive integer, then $n^2 \geq n$ is certainly true; it's also true for $n = 0$; and it's trivially true if n is negative. Therefore the universally quantified statement is true.

d) Squares can never be negative; therefore this statement is false.

17. Existential quantifiers are like disjunctions, and universal quantifiers are like conjunctions. See Examples 11 and 16.

a) We want to assert that $P(x)$ is true for some x in the universe, so either $P(0)$ is true or $P(1)$ is true or $P(2)$ is true or $P(3)$ is true or $P(4)$ is true. Thus the answer is $P(0) \lor P(1) \lor P(2) \lor P(3) \lor P(4)$. The other parts of this exercise are similar. Note that by De Morgan's laws, the expression in part **(c)** is logically equivalent to the expression in part **(f)**, and the expression in part **(d)** is logically equivalent to the expression in part **(e)**.

b) $P(0) \land P(1) \land P(2) \land P(3) \land P(4)$

c) $\neg P(0) \lor \neg P(1) \lor \neg P(2) \lor \neg P(3) \lor \neg P(4)$

d) $\neg P(0) \land \neg P(1) \land \neg P(2) \land \neg P(3) \land \neg P(4)$

e) This is just the negation of part **(a)**: $\neg(P(0) \lor P(1) \lor P(2) \lor P(3) \lor P(4))$

f) This is just the negation of part **(b)**: $\neg(P(0) \land P(1) \land P(2) \land P(3) \land P(4))$

19. Existential quantifiers are like disjunctions, and universal quantifiers are like conjunctions. See Examples 11 and 16.

a) We want to assert that $P(x)$ is true for some x in the universe, so either $P(1)$ is true or $P(2)$ is true or $P(3)$ is true or $P(4)$ is true or $P(5)$ is true. Thus the answer is $P(1) \vee P(2) \vee P(3) \vee P(4) \vee P(5)$.

b) $P(1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge P(5)$

c) This is just the negation of part **(a)**: $\neg(P(1) \vee P(2) \vee P(3) \vee P(4) \vee P(5))$

d) This is just the negation of part **(b)**: $\neg(P(1) \wedge P(2) \wedge P(3) \wedge P(4) \wedge P(5))$

e) The formal translation is as follows: $(((1 \neq 3) \rightarrow P(1)) \wedge ((2 \neq 3) \rightarrow P(2)) \wedge ((3 \neq 3) \rightarrow P(3)) \wedge ((4 \neq 3) \rightarrow P(4)) \wedge ((5 \neq 3) \rightarrow P(5))) \vee (\neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4) \vee \neg P(5))$. However, since the hypothesis $x \neq 3$ is false when x is 3 and true when x is anything other than 3, we have more simply $(P(1) \wedge P(2) \wedge P(4) \wedge P(5)) \vee (\neg P(1) \vee \neg P(2) \vee \neg P(3) \vee \neg P(4) \vee \neg P(5))$. Thinking about it a little more, we note that this statement is always true, since if the first part is not true, then the second part must be true.

21. **a)** One would hope that if we take the domain to be the students in your class, then the statement is true. If we take the domain to be all students in the world, then the statement is clearly false, because some of them are studying only other subjects.

b) If we take the domain to be United States Senators, then the statement is true. If we take the domain to be college football players, then the statement is false, because some of them are younger than 21.

c) If the domain consists of just Princes William and Harry of Great Britain (sons of the late Princess Diana), then the statement is true. It is also true if the domain consists of just one person (everyone has the same mother as him- or herself). If the domain consists of all the grandchildren of Queen Elizabeth II of Great Britain (of whom William and Harry are just two), then the statement is false.

d) If the domain consists of Bill Clinton and George W. Bush, then this statement is true because they do not have the same grandmother. If the domain consists of all residents of the United States, then the statement is false, because there are many instances of siblings and first cousins, who have at least one grandmother in common.

23. In order to do the translation the second way, we let $C(x)$ be the propositional function "x is in your class." Note that for the second way, we always want to use conditional statements with universal quantifiers and conjunctions with existential quantifiers.

a) Let $H(x)$ be "x can speak Hindi." Then we have $\exists x\, H(x)$ the first way, or $\exists x(C(x) \wedge H(x))$ the second way.

b) Let $F(x)$ be "x is friendly." Then we have $\forall x\, F(x)$ the first way, or $\forall x(C(x) \rightarrow F(x))$ the second way.

c) Let $B(x)$ be "x was born in California." Then we have $\exists x\, \neg B(x)$ the first way, or $\exists x(C(x) \wedge \neg B(x))$ the second way.

d) Let $M(x)$ be "x has been in a movie." Then we have $\exists x\, M(x)$ the first way, or $\exists x(C(x) \wedge M(x))$ the second way.

e) This is saying that everyone has failed to take the course. So the answer here is $\forall x\, \neg L(x)$ the first way, or $\forall x(C(x) \rightarrow \neg L(x))$ the second way, where $L(x)$ is "x has taken a course in logic programming."

25. Let $P(x)$ be "x is perfect"; let $F(x)$ be "x is your friend"; and let the domain (universe of discourse) be all people.

a) This means that everyone has the property of being not perfect: $\forall x\, \neg P(x)$. Alternatively, we can write this as $\neg\exists x\, P(x)$, which says that there does not exist a perfect person.

b) This is just the negation of "Everyone is perfect": $\neg\forall x\, P(x)$.

c) If someone is your friend, then that person is perfect: $\forall x(F(x) \rightarrow P(x))$. Note the use of conditional statement with universal quantifiers.

d) We do not have to rule out your having more than one perfect friend. Thus we have simply $\exists x(F(x) \wedge P(x))$. Note the use of conjunction with existential quantifiers.

e) The expression is $\forall x(F(x) \wedge P(x))$. Note that here we did use a conjunction with the universal quantifier, but the sentence is not natural (who could claim this?). We could also have split this up into two quantified statements and written $(\forall x\, F(x)) \wedge (\forall x\, P(x))$.

f) This is a disjunction. The expression is $(\neg\forall x\, F(x)) \vee (\exists x\, \neg P(x))$.

27. In all of these, we will let $Y(x)$ be the propositional function that x is in your school or class, as appropriate.

a) If we let $V(x)$ be "x has lived in Vietnam," then we have $\exists x\, V(x)$ if the universe is just your schoolmates, or $\exists x(Y(x) \wedge V(x))$ if the universe is all people. If we let $D(x,y)$ mean that person x has lived in country y, then we can rewrite this last one as $\exists x(Y(x) \wedge D(x, \text{Vietnam}))$.

b) If we let $H(x)$ be "x can speak Hindi," then we have $\exists x\, \neg H(x)$ if the universe is just your schoolmates, or $\exists x(Y(x) \wedge \neg H(x))$ if the universe is all people. If we let $S(x,y)$ mean that person x can speak language y, then we can rewrite this last one as $\exists x(Y(x) \wedge \neg S(x, \text{Hindi}))$.

c) If we let $J(x)$, $P(x)$, and $C(x)$ be the propositional functions asserting x's knowledge of Java, Prolog, and C++, respectively, then we have $\exists x(J(x) \wedge P(x) \wedge C(x))$ if the universe is just your schoolmates, or $\exists x(Y(x) \wedge J(x) \wedge P(x) \wedge C(x))$ if the universe is all people. If we let $K(x,y)$ mean that person x knows programming language y, then we can rewrite this last one as $\exists x(Y(x) \wedge K(x, \text{Java}) \wedge K(x, \text{Prolog}) \wedge K(x, \text{C++}))$.

d) If we let $T(x)$ be "x enjoys Thai food," then we have $\forall x\, T(x)$ if the universe is just your classmates, or $\forall x(Y(x) \to T(x))$ if the universe is all people. If we let $E(x,y)$ mean that person x enjoys food of type y, then we can rewrite this last one as $\forall x(Y(x) \to E(x, \text{Thai}))$.

e) If we let $H(x)$ be "x plays hockey," then we have $\exists x\, \neg H(x)$ if the universe is just your classmates, or $\exists x(Y(x) \wedge \neg H(x))$ if the universe is all people. If we let $P(x,y)$ mean that person x plays game y, then we can rewrite this last one as $\exists x(Y(x) \wedge \neg P(x, \text{hockey}))$.

29. Our domain (universe of discourse) here is all propositions. Let $T(x)$ mean that x is a tautology and $C(x)$ mean that x is a contradiction. Since a contingency is just a proposition that is neither a tautology nor a contradiction, we do not need a separate predicate for being a contingency.

a) This one is just the assertion that tautologies exist: $\exists x\, T(x)$.

b) Although the word "all" or "every" does not appear here, this sentence is really expressing a universal meaning, that the negation of a contradiction is always a tautology. So we want to say that if x is a contradiction, then $\neg x$ is a tautology. Thus we have $\forall x(C(x) \to T(\neg x))$. Note the rare use of a logical symbol (negation) applied to a variable (x); this is purely a coincidence in this exercise because the universe happens itself to be propositions.

c) The words "can be" are expressing an existential idea—that there exist two contingencies whose disjunction is a tautology. Thus we have $\exists x \exists y(\neg T(x) \wedge \neg C(x) \wedge \neg T(y) \wedge \neg C(y) \wedge T(x \vee y))$. The same final comment as in part **(b)** applies here. Also note the explanation about contingencies in the preamble.

d) As in part **(b)**, this is the universal assertion that whenever x and y are tautologies, then so is $x \wedge y$; thus we have $\forall x \forall y((T(x) \wedge T(y)) \to T(x \wedge y))$.

31. In each case we just have to list all the possibilities, joining them with $\vee$ if the quantifier is $\exists$, and joining them with $\wedge$ if the quantifier is $\forall$.

a) $Q(0,0,0) \wedge Q(0,1,0)$ **b)** $Q(0,1,1) \vee Q(1,1,1) \vee Q(2,1,1)$

c) $\neg Q(0,0,0) \vee \neg Q(0,0,1)$ **d)** $\neg Q(0,0,1) \vee \neg Q(1,0,1) \vee \neg Q(2,0,1)$

33. In each case we need to specify some predicates and identify the domain (universe of discourse).

a) Let $T(x)$ be the predicate that x can learn new tricks, and let the domain be old dogs. Our original statement is $\exists x\, T(x)$. Its negation is $\neg\exists x\, T(x)$, which we must to rewrite in the required manner as $\forall x\, \neg T(x)$.

In English this reads "Every old dog is unable to learn new tricks" or "All old dogs can't learn new tricks." (Note that this does *not* say that not all old dogs can learn new tricks—it is saying something stronger than that.) More colloquially, we can say "No old dogs can learn new tricks."

b) Let $C(x)$ be the predicate that x knows calculus, and let the domain be rabbits. Our original statement is $\neg\exists x\, C(x)$. Its negation is, of course, simply $\exists x\, C(x)$. In English this reads "There is a rabbit that knows calculus."

c) Let $F(x)$ be the predicate that x can fly, and let the domain be birds. Our original statement is $\forall x\, F(x)$. Its negation is $\neg\forall x\, F(x)$ (i.e., not all birds can fly), which we must to rewrite in the required manner as $\exists x\, \neg F(x)$. In English this reads "There is a bird who cannot fly."

d) Let $T(x)$ be the predicate that x can talk, and let the domain be dogs. Our original statement is $\neg\exists x\, T(x)$. Its negation is, of course, simply $\exists x\, T(x)$. In English this reads "There is a dog that talks."

e) Let $F(x)$ and $R(x)$ be the predicates that x knows French and knows Russian, respectively, and let the domain be people in this class. Our original statement is $\neg\exists x(F(x) \wedge R(x))$. Its negation is, of course, simply $\exists x(F(x) \wedge R(x))$. In English this reads "There is someone in this class who knows French and Russian."

35. **a)** As we saw in Example 13, this is true, so there is no counterexample.

b) Since 0 is neither greater than nor less than 0, this is a counterexample.

c) This proposition says that 1 is the only integer—that every integer equals 1. If is obviously false, and any other integer, such as -111749, provides a counterexample.

37. In each case we need to make up predicates. The answers are certainly not unique and depend on the choice of predicate, among other things.

a) $\forall x((F(x, 25000) \vee S(x, 25)) \rightarrow E(x))$, where $E(x)$ is "Person x qualifies as an elite flyer in a given year," $F(x, y)$ is "Person x flies more than y miles in a given year," and $S(x, y)$ is "Person x takes more than y flights in a given year"

b) $\forall x(((M(x)\wedge T(x, 3))\vee(\neg M(x)\wedge T(x, 3.5))) \rightarrow Q(x))$, where $Q(x)$ is "Person x qualifies for the marathon," $M(x)$ is "Person x is a man," and $T(x, y)$ is "Person x has run the marathon in less than y hours"

c) $M \rightarrow ((H(60) \vee (H(45) \wedge T)) \wedge \forall y\, G(\mathrm{B}, y))$, where M is the proposition "The student received a masters degree," $H(x)$ is "The student took at least x course hours," T is the proposition "The student wrote a thesis," and $G(x, y)$ is "The person got grade x or higher in his course y"

d) $\exists x\, ((T(x, 21) \wedge G(x, 4.0))$, where $T(x, y)$ is "Person x took more than y credit hours" and $G(x, p)$ is "Person x earned grade point average p" (we assume that we are talking about one given semester)

39. In each case we pretty much just write what we see.

a) If there is a printer that is both out of service and busy, then some job has been lost.

b) If every printer is busy, then there is a job in the queue.

c) If there is a job that is both queued and lost, then some printer is out of service.

d) If every printer is busy and every job is queued, then some job is lost.

41. In each case we need to make up predicates. The answers are certainly not unique and depend on the choice of predicate, among other things.

a) $(\exists x\, F(x, 10)) \rightarrow \exists x\, S(x)$, where $F(x, y)$ is "Disk x has more than y kilobytes of free space," and $S(x)$ is "Mail message x can be saved"

b) $(\exists x\, A(x)) \rightarrow \forall x(Q(x) \rightarrow T(x))$, where $A(x)$ is "Alert x is active," $Q(x)$ is "Message x is queued," and $T(x)$ is "Message x is transmitted"

c) $\forall x((x \neq \text{main console}) \rightarrow T(x))$, where $T(x)$ is "The diagnostic monitor tracks the status of system x"

d) $\forall x(\neg L(x) \rightarrow B(x))$, where $L(x)$ is "The host of the conference call put participant x on a special list" and $B(x)$ is "Participant x was billed"

43. A conditional statement is true if the hypothesis is false. Thus it is very easy for the second of these propositions to be true—just have $P(x)$ be something that is not always true, such as "The integer x is a multiple of 2." On the other hand, it is certainly not always true that if a number is a multiple of 2, then it is also a multiple of 4, so if we let $Q(x)$ be "The integer x is a multiple of 4," then $\forall x(P(x) \to Q(x))$ will be false. Thus these two propositions can have different truth values. Of course, for some choices of P and Q, they will have the same truth values, such as when P and Q are true all the time.

45. Both are true precisely when at least one of $P(x)$ and $Q(x)$ is true for at least one value of x in the domain (universe of discourse).

47. We can establish these equivalences by arguing that one side is true if and only if the other side is true. For both parts, we will look at the two cases: either A is true or A is false.

a) Suppose that A is true. Then the left-hand side is logically equivalent to $\forall xP(x)$, since the conjunction of any proposition with a true proposition has the same truth value as that proposition. By similar reasoning the right-hand side is equivalent to $\forall xP(x)$. Therefore the two propositions are logically equivalent in this case; each one is true precisely when $P(x)$ is true for every x. On the other hand, suppose that A is false. Then the left-hand side is certainly false. Furthermore, for every x, $P(x) \wedge A$ is false, so the right-hand side is false as well. Thus in all cases, the two propositions have the same truth value.

b) This problem is similar to part **(a)**. If A is true, then both sides are logically equivalent to $\exists xP(x)$. If A is false, then both sides are false.

49. We can establish these equivalences by arguing that one side is true if and only if the other side is true. For both parts, we will look at the two cases: either A is true or A is false.

a) Suppose that A is true. Then for each x, $P(x) \to A$ is true, because a conditional statement with a true conclusion is always true; therefore the left-hand side is always true in this case. By similar reasoning the right-hand side is always true in this case (here we used the fact that the domain is nonempty). Therefore the two propositions are logically equivalent when A is true. On the other hand, suppose that A is false. There are two subcases. If $P(x)$ is false for every x, then $P(x) \to A$ is vacuously true (a conditional statement with a false hypothesis is true), so the left-hand side is vacuously true. The same reasoning shows that the right-hand side is also true, because in this subcase $\exists xP(x)$ is false. For the second subcase, suppose that $P(x)$ is true for some x. Then for that x, $P(x) \to A$ is false (a conditional statement with a true hypothesis and false conclusion is false), so the left-hand side is false. The right-hand side is also false, because in this subcase $\exists xP(x)$ is true but A is false. Thus in all cases, the two propositions have the same truth value.

b) This problem is similar to part **(a)**. If A is true, then both sides are trivially true, because the conditional statements have true conclusions. If A is false, then there are two subcases. If $P(x)$ is false for some x, then $P(x) \to A$ is vacuously true for that x (a conditional statement with a false hypothesis is true), so the left-hand side is true. The same reasoning shows that the right-hand side is true, because in this subcase $\forall xP(x)$ is false. For the second subcase, suppose that $P(x)$ is true for every x. Then for every x, $P(x) \to A$ is false (a conditional statement with a true hypothesis and false conclusion is false), so the left-hand side is false (there is no x making the conditional statement true). The right-hand side is also false, because it is a conditional statement with a true hypothesis and a false conclusion. Thus in all cases, the two propositions have the same truth value.

51. We can show that these are not logically equivalent by giving an example in which one is true and the other is false. Let $P(x)$ be the statement "x is odd" applied to positive integers. Similarly let $Q(x)$ be "x is even." Then since there exist odd numbers and there exist even numbers, the statement $\exists xP(x) \wedge \exists xQ(x)$ is true. On the other hand, no number is both odd and even, so $\exists x(P(x) \wedge Q(x))$ is false.

53. **a)** This is certainly true: if there is a unique x satisfying $P(x)$, then there certainly *is an* x satisfying $P(x)$.

b) Unless the domain (universe of discourse) has fewer than two items in it, the truth of the hypothesis implies that there is more than one x such that $P(x)$ holds. Therefore this proposition need not be true. (For example, let $P(x)$ be the proposition $x^2 \geq 0$ in the context of the real numbers. The hypothesis is true, but there is not a unique x for which $x^2 \geq 0$.)

c) This is true: if there is an x (unique or not) such that $P(x)$ is false, then we can conclude that it is not the case that $P(x)$ holds for all x.

55. A Prolog query returns a yes/no answer if there are no variables in the query, and it returns all values that make the query true if there are.

a) One of the facts was that Chan was the instructor of Math 273, so the response is `yes`.

b) None of the facts was that Patel was the instructor of CS 301, so the response is `no`.

c) Prolog returns the names of the people enrolled in CS 301, namely `juana` and `kiko`.

d) Prolog returns the names of the courses Kiko is enrolled in, namely `math273` and `cs301`.

e) Prolog returns the names of the students enrolled in courses which Grossman is the instructor for (which is just CS 301), namely `juana` and `kiko`.

57. Following the idea and syntax of Example 28, we have the following rule: `sibling(X,Y) :- mother(M,X), mother(M,Y), father(F,X), father(F,Y)`. Note that we used the comma to mean "and"; `X` and `Y` must have the same mother and the same father in order to be (full) siblings.

59. **a)** This is the statement that every person who is a professor is not ignorant. In other words, for every person, if that person is a professor, then that person is not ignorant. In symbols: $\forall x(P(x) \to \neg Q(x))$. This is not the only possible answer. We could equivalently think of the statement as asserting that there does not exist an ignorant professor: $\neg\exists x(P(x) \wedge Q(x))$.

b) Every person who is ignorant is vain: $\forall x(Q(x) \to R(x))$.

c) This is similar to part **(a)**: $\forall x(P(x) \to \neg R(x))$.

d) The conclusion (part **(c)**) does not follow. There may well be vain professors, since the premises do not rule out the possibility that there are vain people besides the ignorant ones.

61. **a)** This is asserting that every person who is a baby is necessarily not logical: $\forall x(P(x) \to \neg Q(x))$.

b) If a person can manage a crocodile, then that person is not despised: $\forall x(R(x) \to \neg S(x))$.

c) Every person who is not logical is necessarily despised: $\forall x(\neg Q(x) \to S(x))$.

d) Every person who is a baby cannot manage a crocodile: $\forall x(P(x) \to \neg R(x))$.

e) The conclusion follows. Suppose that x is a baby. Then by the first premise, x is illogical, and hence, by the third premise, x is despised. But the second premise says that if x could manage a crocodile, then x would not be despised. Therefore x cannot manage a crocodile. Thus we have proved that babies cannot manage crocodiles.

SECTION 1.5 Nested Quantifiers

Nested quantifiers are one of the most difficult things for students to understand. The theoretical definition of limit in calculus, for example, is hard to comprehend because it has three levels of nested quantifiers. Study the examples in this section carefully before attempting the exercises, and make sure that you understand the solutions to the exercises you have difficulty with. Practice enough of these until you feel comfortable. The effort will be rewarded in such areas as computer programming and advanced mathematics courses.

1. **a)** For every real number x there exists a real number y such that x is less than y. Basically, this is asserting that there is no largest real number—for any real number you care to name, there is a larger one.

b) For every real number x and real number y, if x and y are both nonnegative, then their product is nonnegative. Or, more simply, the product of nonnegative real numbers is nonnegative.

c) For every real number x and real number y, there exists a real number z such that $xy = z$. Or, more simply, the real numbers are closed under multiplication. (Some authors would include the uniqueness of z as part of the meaning of the word *closed*.)

3. It is useful to keep in mind that x and y can be the same person, so sending messages to oneself counts in this problem.

a) Formally, this says that there exist students x and y such that x has sent a message to y. In other words, there is some student in your class who has sent a message to some student in your class.

b) This is similar to part **(a)** except that x has sent a message to everyone, not just to at least one person. So this says there is some student in your class who has sent a message to every student in your class.

c) Note that this is not the same as part **(b)**. Here we have that for every x there exists a y such that x has sent a message to y. In other words, every student in your class has sent a message to at least one student in your class.

d) Note that this is not the same as part **(c)**, since the order of quantifiers has changed. In part **(c)**, y could depend on x; in other words, the recipient of the messages could vary from sender to sender. Here the existential quantification on y comes first, so it's the same recipient for all the messages. The meaning is that there is a student in your class who has been sent a message by every student in your class.

e) This is similar to part **(c)**, with the role of sender and recipient reversed: every student in your class has been sent a message from at least one student in your class. Again, note that the sender can depend on the recipient.

f) Every student in the class has sent a message to every student in the class.

5. **a)** This simply says that Sarah Smith has visited `www.att.com`.

b) To say that an x exists such that x has visited `www.imdb.org` is just to say that someone (i.e., at least one person) has visited `www.imdb.org`.

c) This is similar to part **(b)**. Jose Orez has visited some website.

d) This is asserting that a y exists that both of these students has visited. In other words, Ashok Puri and Cindy Yoon have both visited the same website.

e) When there are two quantifiers of opposite types, the sentence gets more complicated. This is saying that there is a person (y) other than David Belcher who has visited all the websites that David has visited (i.e., for every website z, if David has visited z, then so has this person). Note that it is not saying that this person has visited only websites that David has visited (that would be the converse conditional statement)—this person may have visited other sites as well.

f) Here the existence of two people is being asserted; they are said to be unequal, and for every website z, one of these people has visited z if and only if the other one has. In plain English, there are two different people who have visited exactly the same websites.

7. **a)** Abdallah Hussein does not like Japanese cuisine.

b) Note that this is the conjunction of two separate quantified statements. Some student at your school likes Korean cuisine, and everyone at your school likes Mexican cuisine.

c) There is some cuisine that either Monique Arsenault or Jay Johnson likes.

d) Formally this says that for every x and z, there exists a y such that if x and z are not equal, then it is not the case that both x and z like y. In simple English, this says that for every pair of distinct students at your school, there is some cuisine that at least one them does not like.

e) There are two students at your school who have exactly the same tastes (i.e., they like exactly the same cuisines).

f) For every pair of students at your school, there is some cuisine about which they have the same opinion (either they both like it or they both do not like it).

9. We need to be careful to put the lover first and the lovee second as arguments in the propositional function L.

a) $\forall x L(x, \text{Jerry})$

b) Note that the "somebody" being loved depends on the person doing the loving, so we have to put the universal quantifier first: $\forall x \exists y L(x,y)$.

c) In this case, one lovee works for all lovers, so we have to put the existential quantifier first: $\exists y \forall x L(x,y)$.

d) We could think of this as saying that there does not exist anyone who loves everybody ($\neg \exists x \forall y L(x,y)$), or we could think of it as saying that for each person, we can find a person whom he or she does not love ($\forall x \exists y \neg L(x,y)$. These two expressions are logically equivalent.

e) $\exists x \neg L(\text{Lydia}, x)$

f) We are asserting the existence of an individual such that everybody fails to love that person: $\exists x \forall y \neg L(y,x)$.

g) In Exercises 52–54 of Section 1.4, we worked with a notation for the existence of a unique object satisfying a certain condition. Employing that device, we could write this as $\exists! x \forall y L(y,x)$. In Exercise 52 of the present section we will discover a way to avoid this notation in general. What we have to say is that the x asserted here exists, and that every z satisfying this condition (of being loved by everybody) must equal x. Thus we obtain $\exists x\big(\forall y L(y,x) \wedge \forall z((\forall w L(w,z)) \rightarrow z = x)\big)$. Note that we could have used y as the bound variable where we used w; since the scope of the first use of y had ended before we came to this point in the formula, reusing y as the bound variable would cause no ambiguity.

h) We want to assert the existence of two distinct people, whom we will call x and y, whom Lynn loves, as well as make the statement that everyone whom Lynn loves must be either x or y: $\exists x \exists y\big(x \neq y \wedge L(\text{Lynn}, x) \wedge L(\text{Lynn}, y) \wedge \forall z(L(\text{Lynn}, z) \rightarrow (z = x \vee z = y))\big)$.

i) $\forall x L(x,x)$ (Note that nothing in our earlier answers ruled out the possibility that variables or constants with different names might be equal to each other. For example, in part **(a)**, x could equal Jerry, so that statement includes as a special case the assertion that Jerry loves himself. Similarly, in part **(h)**, the two people whom Lynn loves either could be two people other than Lynn (in which case we know that Lynn does not love herself), or could be Lynn herself and one other person.)

j) This is asserting that the one and only one person who is loved by the person being discussed is in fact that person: $\exists x \forall y (L(x,y) \leftrightarrow x = y)$.

11. a) We might want to assert that Lois is a student and Michaels is a faculty member, but the sentence doesn't really say that, so the simple answer is just $A(\text{Lois}, \text{Professor Michaels})$.

b) To say that every student (as opposed to every person) has done this, we need to restrict our universally quantified variable to being a student. The easiest way to do this is to make the assertion being quantified a conditional statement. *As a general rule of thumb, use conditional statements with universal quantifiers and conjunctions with existential quantifiers (see part* **(d)***, for example).* Thus our answer is $\forall x(S(x) \rightarrow A(x, \text{Professor Gross}))$.

c) This is similar to part **(b)**: $\forall x(F(x) \rightarrow (A(x, \text{Professor Miller}) \vee A(\text{Professor Miller}, x)))$. Note the need for parentheses in these answers.

d) There is a student such that for every faculty member, that student has not asked that faculty member a question. Note how we need to include the S and F predicates: $\exists x(S(x) \wedge \forall y(F(y) \rightarrow \neg A(x,y)))$. We could also write this as $\exists x(S(x) \wedge \neg \exists y(F(y) \wedge A(x,y)))$.

e) This is very similar to part **(d)**, with the role of the players reversed: $\exists x(F(x) \wedge \forall y(S(y) \rightarrow \neg A(y,x)))$.

f) This is a little ambiguous in English. If the statement is that there is a very inquisitive student, one who has gone around and asked a question of every professor, then this is similar to part **(d)**, without the negation: $\exists x(S(x) \wedge \forall y(F(y) \to A(x,y)))$. On the other hand, the statement might be intended as asserting simply that for every professor, there exists some student who has asked that professor a question. In other words, the questioner might depend on the questionee. Note how the meaning changes with the change in order of quantifiers. Under the second interpretation the answer is $\forall y(F(y) \to \exists x(S(x) \wedge A(x,y)))$. The first interpretation is probably the intended one.

g) This is pretty straightforward, except that we have to rule out the possibility that the askee is the same as the asker. Our sentence needs to say that there exists a faculty member such that for every other faculty member, the first has asked the second a question: $\exists x(F(x) \wedge \forall y((F(y) \wedge y \neq x) \to A(x,y)))$.

h) There is a student such that every faculty member has failed to ask him a question: $\exists x(S(x) \wedge \forall y(F(y) \to \neg A(y,x)))$.

13. Be careful to put in parentheses where needed; otherwise your answer can be either ambiguous or wrong.

a) Clearly this is simply $\neg M(\text{Chou}, \text{Koko})$.

b) We can give two answers, which are equivalent by De Morgan's law: $\neg(M(\text{Arlene}, \text{Sarah}) \vee T(\text{Arlene}, \text{Sarah}))$ or $\neg M(\text{Arlene}, \text{Sarah}) \wedge \neg T(\text{Arlene}, \text{Sarah})$.

c) Clearly this is simply $\neg M(\text{Deborah}, \text{Jose})$.

d) Note that this statement includes the assertion that Ken has sent himself a message: $\forall x\, M(x, \text{Ken})$.

e) We can write this in two equivalent ways, depending on whether we want to say that everyone has failed to phone Nina or to say that there does not exist someone who has phoned her: $\forall x\, \neg T(x, \text{Nina})$ or $\neg \exists x\, T(x, \text{Nina})$.

f) This is almost identical to part **(d)**: $\forall x(T(x, \text{Avi}) \vee M(x, \text{Avi}))$.

g) To get the "else" in there, we have to make sure that y is different from x in our answer: $\exists x \forall y(y \neq x \to M(x,y))$.

h) This is almost identical to part **(g)**: $\exists x \forall y(y \neq x \to (M(x,y) \vee T(x,y)))$.

i) We need to assert the existence of two distinct people who have sent e-mail both ways: $\exists x \exists y(x \neq y \wedge M(x,y) \wedge M(y,x))$.

j) Only one variable is needed: $\exists x\, M(x,x)$.

k) This poor soul (x in our expression) did not receive a message or a phone call (i.e., did not receive a message and did not receive a phone call) from any person y other than possibly himself: $\exists x \forall y(x \neq y \to (\neg M(y,x) \wedge \neg T(y,x)))$.

l) Here y is "another student": $\forall x \exists y(x \neq y \wedge (M(y,x) \vee T(y,x)))$.

m) This is almost identical to part **(i)**: $\exists x \exists y(x \neq y \wedge M(x,y) \wedge T(y,x))$.

n) Note how the "everyone else" means someone different from both x and y in our expression (and note that there are four possibilities for how each such person z might be contacted): $\exists x \exists y(x \neq y \wedge \forall z((z \neq x \wedge z \neq y) \to (M(x,z) \vee M(y,z) \vee T(x,z) \vee T(y,z))))$.

15. The answers presented here are not the only ones possible; other answers can be obtained using different predicates and different variables, or by varying the domain (universe of discourse).

a) $\forall x N(x, \text{discrete mathematics})$, where $N(x,y)$ is "x needs a course in y" and the domain for x is computer science students and the domain for y is academic subjects

b) $\exists x O(x, \text{personal computer})$, where $O(x,y)$ is "x owns y," and the domain for x is students in this class

c) $\forall x \exists y P(x,y)$, where $P(x,y)$ is "x has taken y"; x ranges over students in this class, and y ranges over computer science courses

d) $\exists x \exists y P(x,y)$, with the environment of part **(c)** (i.e., the same definition of P and the same domain)

e) $\forall x \forall y P(x,y)$, where $P(x,y)$ is "x has been in y"; x ranges over students in this class, and y ranges over buildings on campus

f) $\exists x \exists y \forall z (P(z,y) \to Q(x,z))$, where $P(z,y)$ is "z is in y" and $Q(x,z)$ is "x has been in z"; x ranges over students in this class, y ranges over buildings on campus, and z ranges over rooms

g) $\forall x \forall y \exists z (P(z,y) \wedge Q(x,z))$, with the environment of part **(f)**

17. **a)** We need to rule out the possibility that the user has access to another mailbox different from the one that is guaranteed: $\forall u \exists m (A(u,m) \wedge \forall n (n \neq m \to \neg A(u,n)))$, where $A(u,m)$ means that user u has access to mailbox m.

b) $\exists p \forall e (H(e) \to S(p, \text{running})) \to S(\text{kernel}, \text{working correctly})$, where $H(e)$ means that error condition e is in effect and $S(x,y)$ means that the status of x is y. Obviously there are other ways to express this with different choices of predicates. Note that "only if" is the converse of "if," so the kernel's working properly is the conclusion, not the hypothesis.

c) $\forall u \forall s (E(s, \text{.edu}) \to A(u,s))$, where $E(s,x)$ means that website s has extension x, and $A(u,s)$ means that user u can access website s

d) This is tricky, because we have to interpret the English sentence first, and different interpretations would lead to different answers. We will assume that the specification is that there exist two distinct systems such that they monitor every remote server, and no other system has the property of monitoring every remote system. Thus our answer is $\exists x \exists y (x \neq y \wedge \forall z ((\forall s\, M(z,s)) \leftrightarrow (z = x \vee z = y)))$, where $M(a,b)$ means that system a monitors remote server b. Note that the last part of our expression serves two purposes—it says that x and y do monitor all servers, and it says that no other system does. There are at least two other interpretations of this sentence, which would lead to different legitimate answers.

19. **a)** $\forall x \forall y ((x < 0) \wedge (y < 0) \to (x + y < 0))$

b) What does "necessarily" mean in this context? The best explanation is to assert that a certain universal conditional statement is not true. So we have $\neg \forall x \forall y ((x > 0) \wedge (y > 0) \to (x - y > 0))$. Note that we do not want to put the negation symbol inside (it is not true that the difference of two positive integers is never positive), nor do we want to negate just the conclusion (it is not true that the sum is always nonpositive). We could rewrite our solution by passing the negation inside, obtaining $\exists x \exists y ((x > 0) \wedge (y > 0) \wedge (x - y \leq 0))$.

c) $\forall x \forall y\, (x^2 + y^2 \geq (x+y)^2)$

d) $\forall x \forall y\, (|xy| = |x||y|)$

21. $\forall x \exists a \exists b \exists c \exists d\, ((x > 0) \to x = a^2 + b^2 + c^2 + d^2)$, where the domain (universe of discourse) consists of all integers

23. **a)** $\forall x \forall y ((x < 0) \wedge (y < 0) \to (xy > 0))$ **b)** $\forall x\, (x - x = 0)$

c) To say that there are exactly two objects that meet some condition, we must have two existentially quantified variables to represent the two objects, we must say that they are different, and then we must say that an object meets the conditions if and only if it is one of those two. In this case we have $\forall x \exists a \exists b\, (a \neq b \wedge \forall c (c^2 = x \leftrightarrow (c = a \vee c = b)))$.

d) $\forall x\, ((x < 0) \to \neg \exists y\, (x = y^2))$ where the domain (universe of discourse) consists of all real numbers

25. **a)** This says that there exists a real number x such that for every real number y, the product xy equals y. That is, there is a multiplicative identity for the real numbers. This is a true statement, since $x = 1$ is the identity.

b) The product of two negative real numbers is always a positive real number.

c) There exist real numbers x and y such that x^2 exceeds y but x is less than y. This is true, since we can take $x = 2$ and $y = 3$, for instance.

d) This says that for every pair of real numbers x and y, there exists a real number z that is their sum. In other words, the real numbers are closed under the operation of addition, another true fact. (Some authors would include the uniqueness of z as part of the meaning of the word *closed*.)

27. Recall that the integers include the positive and negative integers and 0.

a) The import of this statement is that no matter how large n might be, we can always find an integer m bigger than n^2. This is certainly true; for example, we could always take $m = n^2 + 1$.

b) This statement is asserting that there is an n that is smaller than the square of *every* integer; note that n is not allowed to depend on m, since the existential quantifier comes first. This statement is true, since we could take, for instance, $n = -3$, and then n would be less than every square, since squares are always greater than or equal to 0.

c) Note the order of quantifiers: m here is allowed to depend on n. Since we can take $m = -n$, this statement is true (additive inverses exist for the integers).

d) Here one n must work for all m. Clearly $n = 1$ does the trick, so the statement is true.

e) The statement is that the equation $n^2 + m^2 = 5$ has a solution over the integers. This is true; in fact there are eight solutions, namely $n = \pm 1$, $m = \pm 2$, and vice versa.

f) The statement is that the equation $n^2 + m^2 = 6$ has a solution over the integers. There are only a small finite number of cases to try, since if $|m|$ or $|n|$ were bigger than 2 then the left-hand side would be bigger than 6. A few minutes reflection shows that in fact there is no solution, so the existential statement is false.

g) The statement is that the system of equations $\{n + m = 4,\ n - m = 1\}$ has a solution over the integers. By algebra we see that there is a unique solution to this system, namely $n = 2\frac{1}{2}$, $m = 1\frac{1}{2}$. Since there do not exist *integers* that make the equations true, the statement is false.

h) The statement is that the system of equations $\{n + m = 4,\ n - m = 2\}$ has a solution over the integers. By algebra we see that there is indeed an integral solution to this system, namely $n = 3$, $m = 1$. Therefore the statement is true.

i) This statement says that the average of two integers is always an integer. If we take $m = 1$ and $n = 2$, for example, then the only p for which $p = (m + n)/2$ is $p = 1\frac{1}{2}$, which is not an integer. Therefore the statement is false.

29. a) $P(1,1) \wedge P(1,2) \wedge P(1,3) \wedge P(2,1) \wedge P(2,2) \wedge P(2,3) \wedge P(3,1) \wedge P(3,2) \wedge P(3,3)$

b) $P(1,1) \vee P(1,2) \vee P(1,3) \vee P(2,1) \vee P(2,2) \vee P(2,3) \vee P(3,1) \vee P(3,2) \vee P(3,3)$

c) $(P(1,1) \wedge P(1,2) \wedge P(1,3)) \vee (P(2,1) \wedge P(2,2) \wedge P(2,3)) \vee (P(3,1) \wedge P(3,2) \wedge P(3,3))$

d) $(P(1,1) \vee P(2,1) \vee P(3,1)) \wedge (P(1,2) \vee P(2,2) \vee P(3,2)) \wedge (P(1,3) \vee P(2,3) \vee P(3,3))$

Note the crucial difference between parts **(c)** and **(d)**.

31. As we push the negation symbol toward the inside, each quantifier it passes must change its type. For logical connectives we either use De Morgan's laws or recall that $\neg(p \to q) \equiv p \wedge \neg q$.

a)

$$\begin{aligned} \neg\forall x\exists y\forall z\, T(x,y,z) &\equiv \exists x\neg\exists y\forall z\, T(x,y,z) \\ &\equiv \exists x\forall y\neg\forall z\, T(x,y,z) \\ &\equiv \exists x\forall y\exists z\, \neg T(x,y,z) \end{aligned}$$

b)

$$\begin{aligned} \neg(\forall x\exists y\, P(x,y) \vee \forall x\exists y\, Q(x,y)) &\equiv \neg\forall x\exists y\, P(x,y) \wedge \neg\forall x\exists y\, Q(x,y) \\ &\equiv \exists x\neg\exists y\, P(x,y) \wedge \exists x\neg\exists y\, Q(x,y) \\ &\equiv \exists x\forall y\, \neg P(x,y) \wedge \exists x\forall y\, \neg Q(x,y) \end{aligned}$$

c)

$$\begin{aligned}\neg\forall x\exists y(P(x,y)\wedge\exists z\,R(x,y,z)) &\equiv \exists x\neg\exists y(P(x,y)\wedge\exists z\,R(x,y,z))\\ &\equiv \exists x\forall y\,\neg(P(x,y)\wedge\exists z\,R(x,y,z))\\ &\equiv \exists x\forall y(\neg P(x,y)\vee\neg\exists z\,R(x,y,z))\\ &\equiv \exists x\forall y(\neg P(x,y)\vee\forall z\,\neg R(x,y,z))\end{aligned}$$

d)
$$\begin{aligned}\neg\forall x\exists y(P(x,y)\to Q(x,y)) &\equiv \exists x\,\neg\exists y(P(x,y)\to Q(x,y))\\ &\equiv \exists x\forall y\,\neg(P(x,y)\to Q(x,y))\\ &\equiv \exists x\forall y(P(x,y)\wedge\neg Q(x,y))\end{aligned}$$

33. We need to use the transformations shown in Table 2 of Section 1.4, replacing $\neg\forall$ by $\exists\neg$, and replacing $\neg\exists$ by $\forall\neg$. In other words, we push all the negation symbols inside the quantifiers, changing the sense of the quantifiers as we do so, because of the equivalences in Table 2 of Section 1.4. In addition, we need to use De Morgan's laws (Section 1.3) to change the negation of a conjunction to the disjunction of the negations and to change the negation of a disjunction to the conjunction of the negations. We also use the double negation law.

a) $\exists x\exists y\,\neg P(x,y)$ **b)** $\exists y\forall x\,\neg P(x,y)$

c) We can think of this in two steps. First we transform the expression into the equivalent expression $\exists y\exists x\,\neg(P(x,y)\vee Q(x,y))$, and then we use De Morgan's law to rewrite this as $\exists y\exists x(\neg P(x,y)\wedge\neg Q(x,y))$.

d) First we apply De Morgan's law to write this as a disjunction: $(\neg\exists x\exists y\,\neg P(x,y))\vee(\neg\forall x\forall y\,Q(x,y))$. Then we push the negation inside the quantifiers, and note that the two negations in front of P then cancel out ($\neg\neg P(x,y)\equiv P(x,y)$). So our final answer is $(\forall x\forall y\,P(x,y))\vee(\exists x\exists y\,\neg Q(x,y))$.

e) First we push the negation inside the outer universal quantifier, then apply De Morgan's law, and finally push it inside the inner quantifiers: $\exists x\,\neg(\exists y\forall z\,P(x,y,z)\wedge\exists z\forall y\,P(x,y,z))\equiv\exists x(\neg\exists y\forall z\,P(x,y,z)\vee\neg\exists z\forall y\,P(x,y,z))\equiv\exists x(\forall y\exists z\,\neg P(x,y,z)\vee\forall z\exists y\,\neg P(x,y,z))$.

35. If the domain (universe of discourse) has at least four members, then no matter what values are assigned to x, y, and z, there will always be another member of the domain, different from those three, that we can assign to w to make the statement true. Thus we can use a domain such as United States Senators. On the other hand, for any domain with three or fewer members, if we assign all the members to x, y, and z (repeating some if necessary), then there will be nothing left to assign to w to make the statement true. For this we can use a domain such as your biological parents.

37. In each case we need to specify some predicates and identify the domain (universe of discourse).

a) To get into the spirit of the problem, we should let $T(x,y)$ be the predicate that x has taken y, where x ranges over students in this class and y ranges over mathematics classes at this school. Then our original statement is $\forall x\exists y\exists z(y\neq z\wedge T(x,y)\wedge T(x,z)\wedge\forall w(T(x,w)\to(w=y\vee w=z)))$. Here y and z are the two math classes that x has taken, and our statement says that these are different and that if x has taken any math class w, then w is one of these two. We form the negation by using Table 2 of Section 1.4 and De Morgan's law to push the negation symbol that we place before the entire expression inwards, to achieve $\exists x\forall y\forall z(y=z\vee\neg T(x,y)\vee\neg T(x,z)\vee\exists w(T(x,w)\wedge w\neq y\wedge w\neq z))$. This can also be expressed as $\exists x\forall y\forall z(y\neq z\to(\neg T(x,y)\vee\neg T(x,z)\vee\exists w(T(x,w)\wedge w\neq y\wedge w\neq z)))$. Note that we formed the negation of a conditional statement by asserting that the hypothesis was true and the conclusion was false. In simple English, this last statement reads "There is someone in this class for whom no matter which two distinct math courses you consider, these are not the two and only two math courses this person has taken."

b) Let $V(x,y)$ be the predicate that x has visited y, where x ranges over people and y ranges over countries. The statement seems to be asserting that the person identified here has visited country y if and only if

y is not Libya. So we can write this symbolically as $\exists x \forall y (V(x,y) \leftrightarrow y \neq \text{Libya})$. One way to form the negation of $P \leftrightarrow Q$ is to write $P \leftrightarrow \neg Q$; this can be seen by looking at truth tables. Thus the negation is $\forall x \exists y (V(x,y) \leftrightarrow y = \text{Libya})$. Note that there are two ways for a biconditional to be true; therefore in English this reads "For every person there is a country such that either that country is Libya and the person has visited it, or else that country is not Libya and the person has not visited it." More simply, "For every person, either that person has visited Libya or else that person has failed to visit some country other than Libya." If we are willing to keep the negation in front of the quantifier in English, then of course we could just say "There is nobody who has visited every country except Libya," but that would not be in the spirit of the exercise.

c) Let $C(x,y)$ be the predicate that x has climbed y, where x ranges over people and y ranges over mountains in the Himalayas. Our statement is $\neg \exists x \forall y\, C(x,y)$. Its negation is, of course, simply $\exists x \forall y\, C(x,y)$. In English this reads "Someone has climbed every mountain in the Himalayas."

d) There are different ways to approach this, depending on how many variables we want to introduce. Let $M(x,y,z)$ be the predicate that x has been in movie z with y, where the domains for x and y are movie actors, and for z is movies. The statement then reads: $\forall x((\exists z\, M(x, \text{Kevin Bacon}, z)) \vee (\exists y \exists z_1 \exists z_2 (M(x,y,z_1) \wedge M(y, \text{Kevin Bacon}, z_2))))$. The negation is formed in the usual manner: $\exists x((\forall z\, \neg M(x, \text{Kevin Bacon}, z)) \wedge (\forall y \forall z_1 \forall z_2 (\neg M(x,y,z_1) \vee \neg M(y, \text{Kevin Bacon}, z_2))))$. In simple English this means that there is someone who has neither been in a movie with Kevin Bacon nor been in a movie with someone who has been in a movie with Kevin Bacon.

39. **a)** Since the square of a number and its additive inverse are the same, we have many counterexamples, such as $x = 2$ and $y = -2$.

b) This statement is saying that every number has a square root. If x is negative (like $x = -4$), or, since we are working in the domain of the integers, x is not a perfect square (like $x = 6$), then the equation $y^2 = x$ has no solution.

c) Since negative numbers are not larger than positive numbers, we can take something like $x = 17$ and $y = -1$ for our counterexample.

41. We simply want to say that a certain equation holds for all real numbers: $\forall x \forall y \forall z ((x \cdot y) \cdot z = x \cdot (y \cdot z))$.

43. We want to say that for each pair of coefficients (the m and the b in the expression $mx + b$), as long as m is not 0, there is a unique x making that expression equal to 0. So we write $\forall m \forall b (m \neq 0 \rightarrow \exists x (mx + b = 0 \wedge \forall w (mw + b = 0 \rightarrow w = x)))$. Notice that the uniqueness is expressed by the last part of our proposition.

45. This statement says that every number has a multiplicative inverse.

a) In the universe of nonzero real numbers, this is certainly true. In each case we let $y = 1/x$.

b) Integers usually don't have inverses that are integers. If we let $x = 3$, then no integer y satisfies $xy = 1$. Thus in this setting, the statement is false.

c) As in part **(a)** this is true, since $1/x$ is positive when x is positive.

47. We use the equivalences explained in Table 2 of Section 1.4, twice:
$\neg \exists x \forall y P(x,y) \equiv \forall x \neg \forall y P(x,y) \equiv \forall x \exists y \neg P(x,y)$

49. **a)** We prove this by arguing that whenever the first proposition is true, so is the second; and that whenever the second proposition is true, so is the first. So suppose that $\forall x P(x) \wedge \exists x Q(x)$ is true. In particular, P always holds, and there is some object, call it y, in the domain (universe of discourse) that makes Q true. Now to show that the second proposition is true, suppose that x is any object in the domain. By our assumptions, $P(x)$ is true. Furthermore, $Q(y)$ is true for the particular y we mentioned above. Therefore $P(x) \wedge Q(y)$ is true for this x and y. Since x was arbitrary, we have showed that $\forall x \exists y(P(x) \wedge Q(y))$ is true, as desired. Conversely, suppose that the second proposition is true. Letting x be any member of the domain allows us to assert that there exists a y such that $P(x) \wedge Q(y)$ is true, and therefore $Q(y)$ is true. Thus by the definition of existential quantifiers, $\exists x Q(x)$ is true. Furthermore, our hypothesis tells us in particular that $\forall x P(x)$ is true. Therefore the first proposition, $\forall x P(x) \wedge \exists x Q(x)$ is true.

b) This is similar to part **(a)**. Suppose that $\forall x P(x) \vee \exists x Q(x)$ is true. Thus either P always holds, or there is some object, call it y, in the domain that makes Q true. In the first case it follows that $P(x) \vee Q(y)$ is true for all x, and so we can conclude that $\forall x \exists y(P(x) \vee Q(y))$ is true (it does not matter in this case whether $Q(y)$ is true or not). In the second case, $Q(y)$ is true for this particular y, and so $P(x) \vee Q(y)$ is true regardless of what x is. Again, it follows that $\forall x \exists y(P(x) \vee Q(y))$ is true. Conversely, suppose that the second proposition is true. If $P(x)$ is true for all x, then the first proposition must be true. If not, then $P(x)$ fails for some x, but for this x there must be a y such that $P(x) \vee Q(y)$ is true; hence $Q(y)$ must be true. Therefore $\exists y Q(y)$ holds, and thus the first proposition is true.

51. This will essentially be a proof by (structural) mathematical induction (see Sections 5.1–5.3), where we show how a long expression can be put into prenex normal form if the subexpressions in it can be put into prenex normal form. First we invoke the result of Exercise 45 from Section 1.3 to assume without loss of generality that our given proposition uses only the logical connectives $\vee$ and $\neg$. Then every proposition must either be a single propositional variable (like P), the disjunction of two propositions, the negation of a proposition, or the universal or existential quantification of a predicate. (There is a small technical point that we are sliding over here; disjunction and negation need to be defined for predicates as well as for propositions, since otherwise we would not be able to write down such things as $\forall x(P(x) \wedge Q(x))$. We assume that all that we have done for propositions applies to predicates as well.)

Certainly every proposition that involves no quantifiers is already in prenex normal form; this is the base case of our induction. Next suppose that our proposition is of the form $\mathrm{Q}xP(x)$, where Q is a quantifier. Then $P(x)$ is a shorter expression than the given proposition, so (by the inductive hypothesis) we can put it into prenex form, with all of its quantifiers coming at the beginning. Then $\mathrm{Q}x$ followed by this prenex form is again in prenex form and is equivalent to the original proposition. Next suppose that our proposition is of the form $\neg P$. Again, we can invoke the inductive hypothesis and assume that P is already in prenex form, with all of its quantifiers coming at its front. We now slide the negation symbol past all the quantifiers, using the equivalences in Table 2 of Section 1.4. For example, $\neg\forall x \exists y R(x,y)$ becomes $\exists x \forall y \neg R(x,y)$, which is in prenex form.

Finally, suppose that our given proposition is a disjunction of two propositions, $P \vee Q$, each of which can (again by the inductive hypothesis) be assumed to be in prenex normal form, with their quantifiers at the front. There are several cases. If only one of P and Q has quantifiers, then we invoke the result of Exercise 46 of Section 1.4 to bring the quantifier in front of both. We then apply our process to what remains. For example, $P \vee \forall x Q(x)$ is equivalent to $\forall x(P \vee Q(x))$, and then $P \vee Q(x)$ is put into prenex form. Another case is that the proposition might look like $\exists x R(x) \vee \exists x S(x)$. In this case, by Exercise 45 of Section 1.4, the proposition is equivalent to $\exists x(R(x) \vee S(x))$. Once again, by the inductive hypothesis we can then put $R(x) \vee S(x)$ into prenex form, and so the entire proposition can be put into prenex form. Similarly, using Exercise 48 of the present section we can transform $\forall x R(x) \vee \forall x S(x)$ into the equivalent $\forall x \forall y(R(x) \vee S(y))$; putting $R(x) \vee S(y)$ into prenex form then brings the entire proposition into prenex form. Finally, if the proposition is of the form $\forall x R(x) \vee \exists x Q(x)$, then we invoke Exercise 49b of the present section and apply the same construction.

Note that this proof actually gives us the process for finding the proposition in prenex form equivalent to the given proposition—we just work from the inside out, dealing with one logical operation or quantifier at a time. Here is an example:

$$\begin{aligned}\forall x P(x) \vee \neg\exists x(Q(x) \vee \forall y R(x,y)) &\equiv \forall x P(x) \vee \neg\exists x \forall y(Q(x) \vee R(x,y)) \\ &\equiv \forall x P(x) \vee \forall x \exists y \neg(Q(x) \vee R(x,y)) \\ &\equiv \forall x \forall z(P(x) \vee \exists y \neg(Q(z) \vee R(z,y))) \\ &\equiv \forall x \forall z \exists y(P(x) \vee \neg(Q(z) \vee R(z,y)))\end{aligned}$$

SECTION 1.6 Rules of Inference

This section lays the groundwork for understanding proofs. You are asked to understand the logical rules of inference behind valid arguments, and you are asked to construct some highly stylized proofs using these rules. The proofs will become more informal in the next section and throughout the remainder of this book (and your mathematical studies).

1. This is modus ponens. The first statement is $p \to q$, where p is "Socrates is human" and q is "Socrates is mortal." The second statement is p. The third is q. Modus ponens is valid. We can therefore conclude that the conclusion of the argument (third statement) is true, because the hypotheses (the first two statements) are true.

3. **a)** This is the addition rule. We are concluding from p that $p \vee q$ must be true, where p is "Alice is a mathematics major" and q is "Alice is a computer science major."

b) This is the simplification rule. We are concluding from $p \wedge q$ that p must be true, where p is "Jerry is a mathematics major" and q is "Jerry is a computer science major."

c) This is modus ponens. We are concluding from $p \to q$ and p that q must be true, where p is "it is rainy" and q is "the pool will be closed."

d) This is modus tollens. We are concluding from $p \to q$ and $\neg q$ that $\neg p$ must be true, where p is "it will snow today" and q is "the university will close today."

e) This is hypothetical syllogism. We are concluding from $p \to q$ and $q \to r$ that $p \to r$ must be true, where p is "I will go swimming," q is "I will stay in the sun too long," and r is "I will sunburn."

5. Let w be the proposition "Randy works hard," let d be the proposition "Randy is a dull boy," and let j be the proposition "Randy will get the job." We are given premises w, $w \to d$, and $d \to \neg j$. We want to conclude $\neg j$. We set up the proof in two columns, with reasons, as in Example 6.

Step	**Reason**
1. w	Hypothesis
2. $w \to d$	Hypothesis
3. d	Modus ponens using (2) and (3)
4. $d \to \neg j$	Hypothesis
5. $\neg j$	Modus ponens using (3) and (4)

7. First we use universal instantiation to conclude from "For all x, if x is a man, then x is mortal" the special case of interest, "If Socrates is a man, then Socrates is mortal." Then we use modus ponens to conclude that Socrates is mortal.

9. **a)** Because it was sunny on Tuesday, we assume that it did not rain or snow on Tuesday (otherwise we cannot do anything with this problem). If we use modus tollens on the universal instantiation of the given conditional statement applied to Tuesday, then we conclude that I did not take Tuesday off. If we now apply disjunctive syllogism to the disjunction in light of this conclusion, we see that I took Thursday off. Now use modus ponens on the universal instantiation of the given conditional statement applied to Thursday; we conclude that it rained or snowed on Thursday. One more application of disjunctive syllogism tells us that it rained on Thursday.

b) Using modus tollens we conclude two things—that I did not eat spicy food and that it did not thunder. Therefore by the conjunction rule of inference (Table 1), we conclude "I did not eat spicy food and it did not thunder."

c) By disjunctive syllogism from the first two hypotheses we conclude that I am clever. The third hypothesis gives us no useful information.

d) We can apply universal instantiation to the conditional statement and conclude that if Ralph (respectively, Ann) is a CS major, then he (she) has a PC. Now modus tollens tells us that Ralph is not a CS major. There are no conclusions to be drawn about Ann.

e) The first two conditional statements can be phrased as "If x is good for corporations, then x is good for the U.S." and "If x is good for the U.S., then x is good for you." If we now apply universal instantiation with x being "for you to buy lots of stuff," then we can conclude using modus ponens twice that for you to buy lots of stuff is good for the U.S. and is good for you.

f) The given conditional statement is "For all x, if x is a rodent, then x gnaws its food." We can form the universal instantiation of this with x being a mouse, a rabbit, and a bat. Then modus ponens allows us to conclude that mice gnaw their food; and modus tollens allows us to conclude that rabbits are not rodents. We can conclude nothing about bats.

11. We are asked to show that whenever $p_1, p_2, \ldots, p_n$ are true, then $q \to r$ must be true, given that we know that whenever $p_1, p_2, \ldots, p_n$ and q are true, then r must be true. So suppose that $p_1, p_2, \ldots, p_n$ are true. We want to establish that $q \to r$ is true. If q is false, then we are done, vacuously. Otherwise, q is true, so by the validity of the given argument form, we know that r is true.

13. In each case we set up the proof in two columns, with reasons, as in Example 6.

a) Let $c(x)$ be "x is in this class," let $j(x)$ be "x knows how to write programs in JAVA," and let $h(x)$ be "x can get a high paying job." We are given premises $c(\text{Doug})$, $j(\text{Doug})$, and $\forall x(j(x) \to h(x))$, and we want to conclude $\exists x(c(x) \wedge h(x))$.

Step	**Reason**
1. $\forall x(j(x) \to h(x))$	Hypothesis
2. $j(\text{Doug}) \to h(\text{Doug})$	Universal instantiation using (1)
3. $j(\text{Doug})$	Hypothesis
4. $h(\text{Doug})$	Modus ponens using (2) and (3)
5. $c(\text{Doug})$	Hypothesis
6. $c(\text{Doug}) \wedge h(\text{Doug})$	Conjunction using (4) and (5)
7. $\exists x(c(x) \wedge h(x))$	Existential generalization using (6)

b) Let $c(x)$ be "x is in this class," let $w(x)$ be "x enjoys whale watching," and let $p(x)$ be "x cares about ocean pollution." We are given premises $\exists x(c(x) \wedge w(x))$ and $\forall x(w(x) \to p(x))$, and we want to conclude $\exists x(c(x) \wedge p(x))$. In our proof, y represents an unspecified particular person.

Step	Reason
1. $\exists x(c(x) \land w(x))$	Hypothesis
2. $c(y) \land w(y)$	Existential instantiation using (1)
3. $w(y)$	Simplification using (2)
4. $c(y)$	Simplification using (2)
5. $\forall x(w(x) \to p(x))$	Hypothesis
6. $w(y) \to p(y)$	Universal instantiation using (5)
7. $p(y)$	Modus ponens using (3) and (6)
8. $c(y) \land p(y)$	Conjunction using (4) and (7)
9. $\exists x(c(x) \land p(x))$	Existential generalization using (8)

c) Let $c(x)$ be "x is in this class," let $p(x)$ be "x owns a PC," and let $w(x)$ be "x can use a word processing program." We are given premises $c(\text{Zeke})$, $\forall x(c(x) \to p(x))$, and $\forall x(p(x) \to w(x))$, and we want to conclude $w(\text{Zeke})$.

Step	Reason
1. $\forall x(c(x) \to p(x))$	Hypothesis
2. $c(\text{Zeke}) \to p(\text{Zeke})$	Universal instantiation using (1)
3. $c(\text{Zeke})$	Hypothesis
4. $p(\text{Zeke})$	Modus ponens using (2) and (3)
5. $\forall x(p(x) \to w(x))$	Hypothesis
6. $p(\text{Zeke}) \to w(\text{Zeke})$	Universal instantiation using (5)
7. $w(\text{Zeke})$	Modus ponens using (4) and (6)

d) Let $j(x)$ be "x is in New Jersey," let $f(x)$ be "x lives within fifty miles of the ocean," and let $s(x)$ be "x has seen the ocean." We are given premises $\forall x(j(x) \to f(x))$ and $\exists x(j(x) \land \neg s(x))$, and we want to conclude $\exists x(f(x) \land \neg s(x))$. In our proof, y represents an unspecified particular person.

Step	Reason
1. $\exists x(j(x) \land \neg s(x))$	Hypothesis
2. $j(y) \land \neg s(y)$	Existential instantiation using (1)
3. $j(y)$	Simplification using (2)
4. $\forall x(j(x) \to f(x))$	Hypothesis
5. $j(y) \to f(y)$	Universal instantiation using (4)
6. $f(y)$	Modus ponens using (3) and (5)
7. $\neg s(y)$	Simplification using (2)
8. $f(y) \land \neg s(y)$	Conjunction using (6) and (7)
9. $\exists x(f(x) \land \neg s(x))$	Existential generalization using (8)

15. **a)** This is correct, using universal instantiation and modus ponens.
b) This is invalid. After applying universal instantiation, it contains the fallacy of affirming the conclusion.
c) This is invalid. After applying universal instantiation, it contains the fallacy of denying the hypothesis.
d) This is valid by universal instantiation and modus tollens.

17. We know that *some* x exists that makes $H(x)$ true, but we cannot conclude that Lola is one such x. Maybe only Suzanne is happy and everyone else is not happy. Then $\exists x\, H(x)$ is true, but $H(\text{Lola})$ is false.

19. **a)** This is the fallacy of affirming the conclusion, since it has the form "$p \to q$ and q implies p."
b) This reasoning is valid; it is modus tollens.
c) This is the fallacy of denying the hypothesis, since it has the form "$p \to q$ and $\neg p$ implies $\neg q$."

21. Let us give an argument justifying the conclusion. By the second premise, there is some lion that does not drink coffee. Let us call him Leo. Thus we know that Leo is a lion and that Leo does not drink coffee. By simplification this allows us to assert each of these statements separately. The first premise says that all lions are fierce; in particular, if Leo is a lion, then Leo is fierce. By modus ponens, we can conclude that Leo is fierce. Thus we conclude that Leo is fierce and Leo does not drink coffee. By the definition of the existential quantifier, this tells us that there exist fierce creatures that do not drink coffee; in other words, that some fierce creatures do not drink coffee.

23. The error occurs in step (5), because we cannot assume, as is being done here, that the c that makes P true is the same as the c that makes Q true.

25. We are given the premises $\forall x(P(x) \to Q(x))$ and $\neg Q(a)$. We want to show $\neg P(a)$. Suppose, to the contrary, that $\neg P(a)$ is not true. Then $P(a)$ is true. Therefore by universal modus ponens, we have $Q(a)$. But this contradicts the given premise $\neg Q(a)$. Therefore our supposition must have been wrong, and so $\neg P(a)$ is true, as desired.

27. We can set this up in two-column format.

Step	**Reason**
1. $\forall x(P(x) \wedge R(x))$	Premise
2. $P(a) \wedge R(a)$	Universal instantiation using (1)
3. $P(a)$	Simplification using (2)
4. $\forall x(P(x) \to (Q(x) \wedge S(x)))$	Premise
5. $Q(a) \wedge S(a)$	Universal modus ponens using (3) and (4)
6. $S(a)$	Simplification using (5)
7. $R(a)$	Simplification using (2)
8. $R(a) \wedge S(a)$	Conjunction using (7) and (6)
9. $\forall x(R(x) \wedge S(x))$	Universal generalization using (5)

29. We can set this up in two-column format. The proof is rather long but straightforward if we go one step at a time.

Step	**Reason**
1. $\exists x \neg P(x)$	Premise
2. $\neg P(c)$	Existential instantiation using (1)
3. $\forall x(P(x) \vee Q(x))$	Premise
4. $P(c) \vee Q(c)$	Universal instantiation using (3)
5. $Q(c)$	Disjunctive syllogism using (4) and (2)
6. $\forall x(\neg Q(x) \vee S(x))$	Premise
7. $\neg Q(c) \vee S(c)$	Universal instantiation using (6)
8. $S(c)$	Disjunctive syllogism using (5) and (7), since $\neg\neg Q(c) \equiv Q(c)$
9. $\forall x(R(x) \to \neg S(x))$	Premise
10. $R(c) \to \neg S(c)$	Universal instantiation using (9)
11. $\neg R(c)$	Modus tollens using (8) and (10), since $\neg\neg S(c) \equiv S(c)$
12. $\exists x \neg R(x)$	Existential generalization using (11)

31. Let p be "It is raining"; let q be "Yvette has her umbrella"; let r be "Yvette gets wet." Then our assumptions are $\neg p \vee q$, $\neg q \vee \neg r$, and $p \vee \neg r$. Using resolution on the first two assumptions gives us $\neg p \vee \neg r$. Using resolution on this and the third assumption gives us $\neg r$, so Yvette does not get wet.

33. Assume that this proposition is satisfiable. Using resolution on the first two clauses allows us to conclude $q \vee q$; in other words, we know that q has to be true. Using resolution on the last two clauses allows us to conclude $\neg q \vee \neg q$; in other words, we know that $\neg q$ has to be true. This is a contradiction. So this proposition is not satisfiable.

35. This argument is valid. We argue by contradiction. Assume that Superman does exist. Then he is not impotent, and he is not malevolent (this follows from the fourth sentence). Therefore by (the contrapositives of) the two parts of the second sentence, we conclude that he is able to prevent evil, and he is willing to prevent evil. By the first sentence, we therefore know that Superman does prevent evil. This contradicts the third sentence. Since we have arrived at a contradiction, our original assumption must have been false, so we conclude finally that Superman does not exist.

SECTION 1.7 Introduction to Proofs

This introduction applies jointly to this section and the next (1.8).

Learning to construct good mathematical proofs takes years. There is no algorithm for constructing the proof of a true proposition (there is actually a deep theorem in mathematical logic that says this). Instead, the construction of a valid proof is an art, honed after much practice. There are two problems for the beginning student—figuring out the key ideas in a problem (what is it that really makes the proposition true?) and writing down the proof in acceptable mathematical language.

Here are some general things to keep in mind in constructing proofs. First, of course, you need to find out exactly what is going on—why the proposition is true. This can take anywhere from ten seconds (for a really simple proposition) to a lifetime (some mathematicians have spent their entire careers trying to prove certain conjectures). For a typical student at this level, tackling a typical problem, the median might be somewhere around 15 minutes. This time should be spent looking at examples, making tentative assumptions, breaking the problem down into cases, perhaps looking at analogous but simpler problems, and in general bringing all of your mathematical intuition and training to bear.

It is often easiest to give a proof by contradiction, since you get to assume the most (all the hypotheses as well as the negation of the conclusion), and all you have to do is to derive a contradiction. Another thing to try early in attacking a problem is to separate the proposition into several cases; proof by cases is a valid technique, if you make sure to include all the possibilities. In proving propositions, all the rules of inference are at your disposal, as well as axioms and previously proved results. Ask yourself what definitions, axioms, or other theorems might be relevant to the problem at hand. The importance of constantly returning to the definitions cannot be overstated!

Once you think you see what is involved, you need to write down the proof. In doing so, pay attention both to content (does each statement follow logically? are you making any fallacious arguments? are you leaving out any cases or using hidden assumptions?) and to style. There are certain conventions in mathematical proofs, and you need to follow them. For example, you must use complete sentences and say exactly what you mean. (An equation is a complete sentence, with "equals" as the verb; however, a good proof will usually have more English words than mathematical symbols in it.) The point of a proof is to convince the reader that your line of argument is sound, and that therefore the proposition under discussion is true; put yourself in the reader's shoes, and ask yourself whether you are convinced.

Most of the proofs called for in this exercise set are not extremely difficult. Nevertheless, expect to have a fairly rough time constructing proofs that look like those presented in this solutions manual, the textbook, or other mathematics textbooks. The more proofs you write, utilizing the different methods discussed in this *section, the better you will become at it. As a bonus, your ability to construct and respond to nonmathematical arguments (politics, religion, or whatever) will be enhanced. Good luck!*

1. We must show that whenever we have two odd integers, their sum is even. Suppose that a and b are two odd integers. Then there exist integers s and t such that $a = 2s + 1$ and $b = 2t + 1$. Adding, we obtain $a + b = (2s + 1) + (2t + 1) = 2(s + t + 1)$. Since this represents $a + b$ as 2 times the integer $s + t + 1$, we conclude that $a + b$ is even, as desired.

3. We need to prove the following assertion for an arbitrary integer n: "If n is even, then n^2 is even." Suppose that n is even. Then $n = 2k$ for some integer k. Therefore $n^2 = (2k)^2 = 4k^2 = 2(2k^2)$. Since we have written n^2 as 2 times an integer, we conclude that n^2 is even.

5. We can give a direct proof. Suppose that $m + n$ is even. Then $m + n = 2s$ for some integer s. Suppose that $n + p$ is even. Then $n + p = 2t$ for some integer t. If we add these [this step is inspired by the fact that we want to look at $m + p$], we get $m + p + 2n = 2s + 2t$. Subtracting $2n$ from both sides and factoring, we have $m + p = 2s + 2t - 2n = 2(s + t - n)$. Since we have written $m + p$ as 2 times an integer, we conclude that $m + p$ is even, as desired.

7. The difference of two squares can be factored: $a^2 - b^2 = (a + b)(a - b)$. If we can arrange for our given odd integer to equal $a + b$ and for $a - b$ to equal 1, then we will be done. But we can do this by letting a and b be the integers that straddle $n/2$. For example, if $n = 11$, then we take $a = 6$ and $b = 5$. Specifically, if $n = 2k+1$, then we let $a = k+1$ and $b = k$. Here, then, is our proof. Since n is odd, we can write $n = 2k+1$ for some integer k. Then $(k + 1)^2 - k^2 = k^2 + 2k + 1 - k^2 = 2k + 1 = n$. This expresses n as the difference of two squares.

9. The proposition to be proved here is as follows: If r is a rational number and i is an irrational number, then $s = r + i$ is an irrational number. So suppose that r is rational, i is irrational, and s is rational. Then by Example 7 the sum of the rational numbers s and $-r$ must be rational. (Indeed, if $s = a/b$ and $r = c/d$, where a, b, c, and d are integers, with $b \neq 0$ and $d \neq 0$, then by algebra we see that $s+(-r) = (ad-bc)/(bd)$, so that patently $s + (-r)$ is a rational number.) But $s + (-r) = r + i - r = i$, forcing us to the conclusion that i is rational. This contradicts our hypothesis that i is irrational. Therefore the assumption that s was rational was incorrect, and we conclude, as desired, that s is irrational.

11. To disprove this proposition it is enough to find a counterexample, since the proposition has an implied universal quantification. We know from Example 10 that $\sqrt{2}$ is irrational. If we take the product of the irrational number $\sqrt{2}$ and the irrational number $\sqrt{2}$, then we obtain the rational number 2. This counterexample refutes the proposition.

13. We give an proof by contraposition. The contrapositive of this statement is "If $1/x$ is rational, then x is rational" so we give a direct proof of this contrapositive. Note that since $1/x$ exists, we know that $x \neq 0$. If $1/x$ is rational, then by definition $1/x = p/q$ for some integers p and q with $q \neq 0$. Since $1/x$ cannot be 0 (if it were, then we'd have the contradiction $1 = x \cdot 0$ by multiplying both sides by x), we know that $p \neq 0$. Now $x = 1/(1/x) = 1/(p/q) = q/p$ by the usual rules of algebra and arithmetic. Hence x can be written as the quotient of two integers with the denominator nonzero. Thus by definition, x is rational.

15. We will prove the contrapositive (that if it is not true that $x \geq 1$ or $y \geq 1$, then it is not true that $x+y \geq 2$), using a direct argument. Assume that it is not true that $x \geq 1$ or $y \geq 1$. Then (by De Morgan's law) $x < 1$ and $y < 1$. Adding these two inequalities, we obtain $x + y < 2$. This is the negation of $x + y \geq 2$, and our proof is complete.

17. **a)** We must prove the contrapositive: If n is odd, then n^3+5 is even. Assume that n is odd. Then we can write $n=2k+1$ for some integer k. Then $n^3+5=(2k+1)^3+5=8k^3+12k^2+6k+6=2(4k^3+6k^2+3k+3)$. Thus n^3+5 is two times some integer, so it is even.

b) Suppose that n^3+5 is odd and that n is odd. Since n is odd, and the product of odd numbers is odd, in two steps we see that n^3 is odd. But then subtracting we conclude that 5, being the difference of the two odd numbers n^3+5 and n^3, is even. This is not true. Therefore our supposition was wrong, and the proof by contradiction is complete.

19. The proposition we are trying to prove is "If 0 is a positive integer greater than 1, then $0^2>0$." Our proof is a vacuous one, exactly as in Example 5. Since the hypothesis is false, the conditional statement is automatically true.

21. The proposition we are trying to prove is "If a and b are positive real numbers, then $(a+b)^1 \geq a^1+b^1$." Our proof is a direct one. By the definition of exponentiation, any real number to the power 1 is itself. Hence $(a+b)^1=a+b=a^1+b^1$. Finally, by the addition rule, we can conclude from $(a+b)^1=a^1+b^1$ that $(a+b)^1 \geq a^1+b^1$ (the latter being the disjunction of $(a+b)^1=a^1+b^1$ and $(a+b)^1>a^1+b^1$). One might also say that this is a trivial proof, since we did not use the hypothesis that a and b are positive (although of course we used the hypothesis that they are numbers).

23. We give a proof by contradiction. If there were nine or fewer days on each day of the week, this would account for at most $9\cdot 7=63$ days. But we chose 64 days. This contradiction shows that at least ten of the days must be on the same day of the week.

25. One way to prove this is to use the rational root test from high school algebra: Every rational number that satisfies a polynomial with integer coefficients is of the form p/q, where p is a factor of the constant term of the polynomial, and q is a factor of the coefficient of the leading term. In this case, both the constant and the leading coefficient are 1, so the only possible values for p and q are ± 1. Therefore the only possible rational roots are $\pm 1/(\pm 1)$, which means that 1 and -1 are the only possible rational roots. Clearly neither of them is a root, so there are no rational roots.

Alternatively, we can follow the hint. Suppose by way of contradiction that a/b is a rational root, where a and b are integers and this fraction is in lowest terms (that is, a and b have no common divisor greater than 1). Plug this proposed root into the equation to obtain $a^3/b^3+a/b+1=0$. Multiply through by b^3 to obtain $a^3+ab^2+b^3=0$. If a and b are both odd, then the left-hand side is the sum of three odd numbers and therefore must be odd. If a is odd and b is even, then the left-hand side is odd + even + even, which is again odd. Similarly, if a is even and b is odd, then the left-hand side is even + even + odd, which is again odd. Because the fraction a/b is in simplest terms, it cannot happen that both a and b are even. Thus in all cases, the left-hand side is odd, and therefore cannot equal 0. This contradiction shows that no such root exists.

27. We must prove two conditional statements. First, we assume that n is odd and show that $5n+6$ is odd (this is a direct proof). By assumption, $n=2k+1$ for some integer k. Then $5n+6=5(2k+1)+6=10k+11=2(5k+5)+1$. Since we have written $5n+6$ as 2 times an integer plus 1, we have showed that $5n+6$ is odd, as desired. Now we give an proof by contraposition of the converse. Suppose that n is not odd—in other words, that n is even. Then $n=2k$ for some integer k. Then $5n+6=10k+6=2(5k+3)$. Since we have written $5n+6$ as 2 times an integer, we have showed that $5n+6$ is even. This completes the proof by contraposition of this conditional statement.

29. This proposition is true. We give a proof by contradiction. Suppose that m is neither 1 nor -1. Then mn has a factor (namely $|m|$) larger than 1. On the other hand, $mn = 1$, and 1 clearly has no such factor. Therefore we conclude that $m = 1$ or $m = -1$. It is then immediate that $n = 1$ in the first case and $n = -1$ in the second case, since $mn = 1$ implies that $n = 1/m$.

31. Perhaps the best way to do this is to prove that all of them are equivalent to x being even, which one can discover easily enough by trying a few small values of x. If x is even, then $x = 2k$ for some integer k. Therefore $3x + 2 = 3 \cdot 2k + 2 = 6k + 2 = 2(3k + 1)$, which is even, since it has been written in the form $2t$, where $t = 3k+1$. Similarly, $x+5 = 2k+5 = 2k+4+1 = 2(k+2)+1$, so $x+5$ is odd; and $x^2 = (2k)^2 = 2(2k^2)$, so x^2 is even. For the converses, we will use a proof by contraposition. So assume that x is not even; thus x is odd and we can write $x = 2k + 1$ for some integer k. Then $3x + 2 = 3(2k+1) + 2 = 6k + 5 = 2(3k+2) + 1$, which is odd (i.e., not even), since it has been written in the form $2t + 1$, where $t = 3k + 2$. Similarly, $x+5 = 2k+1+5 = 2(k+3)$, so $x+5$ is even (i.e., not odd). That x^2 is odd was already proved in Example 1. This completes the proof.

33. It is easiest to give proofs by contraposition of $(i) \to (ii)$, $(ii) \to (i)$, $(i) \to (iii)$, and $(iii) \to (i)$. For the first of these, suppose that $3x + 2$ is rational, namely equal to p/q for some integers p and q with $q \neq 0$. Then we can write $x = ((p/q) - 2)/3 = (p - 2q)/(3q)$, where $3q \neq 0$. This shows that x is rational. For the second conditional statement, suppose that x is rational, namely equal to p/q for some integers p and q with $q \neq 0$. Then we can write $3x + 2 = (3p + 2q)/q$, where $q \neq 0$. This shows that $3x + 2$ is rational. For the third conditional statement, suppose that $x/2$ is rational, namely equal to p/q for some integers p and q with $q \neq 0$. Then we can write $x = 2p/q$, where $q \neq 0$. This shows that x is rational. And for the fourth conditional statement, suppose that x is rational, namely equal to p/q for some integers p and q with $q \neq 0$. Then we can write $x/2 = p/(2q)$, where $2q \neq 0$. This shows that $x/2$ is rational.

35. The steps are valid for obtaining possible solutions to the equations. *If* the given equation is true, then we can conclude that $x = 1$ or $x = 6$, since the truth of each equation implies the truth of the next equation. However, the steps are not all reversible; in particular, the squaring step is not reversible. Therefore the possible answers must be checked in the original equation. We know that no other solutions are possible, but we do not know that these two numbers are in fact solutions. If we plug in $x = 1$ we get the true statement $2 = 2$; but if we plug in $x = 6$ we get the false statement $3 = -3$. Therefore $x = 1$ is the one and only solution of $\sqrt{x+3} = 3 - x$.

37. Suppose that we have proved $p_1 \to p_4 \to p_2 \to p_5 \to p_3 \to p_1$. Imagine these conditional statements arranged around a circle. Then to prove that each one of these propositions (say p_i) implies each of the others (say p_j), we just have to follow the circle, starting at p_i, until we come to p_j, using hypothetical syllogism repeatedly.

39. We can give a very satisfying proof by contradiction here. Suppose instead that all of the numbers a_1, a_2, $\ldots$, a_n are less than their average, which we can denote by A. In symbols, we have $a_i < A$ for all i. If we add these n inequalities, we see that

$$a_1 + a_2 + \cdots + a_n < nA\,.$$

By definition,

$$A = \frac{a_1 + a_2 + \cdots + a_n}{n}\,.$$

The two displayed formulae clearly contradict each other, however: they imply that $nA < nA$. Thus our assumption must have been incorrect, and we conclude that at least one of the numbers a_i is greater than or equal to their average.

41. We can prove that these four statements are equivalent in a circular way: $(i) \to (ii) \to (iii) \to (iv) \to (i)$. For the first, we want to show that if n is even, then $n+1$ is odd. Assume that n is even. Then $n = 2k$ for some integer k. Thus $n+1 = 2k+1$, so by definition $n+1$ is odd. This completes the first proof. Next we give a direct proof of $(ii) \to (iii)$. Suppose that $n+1$ is odd, say $n+1 = 2k+1$. Then $3n+1 = 2n+(n+1) = 2n+2k+1 = 2(n+k)+1$. Since this shows that $3n+1$ is 2 times an integer plus 1, we conclude that $3n+1$ is odd, as desired. For the next proof, suppose that $3n+1$ is odd, say $3n+1 = 2k+1$. Then $3n = (3n+1)-1 = (2k+1)-1 = 2k$. Therefore by definition $3n$ is even. Finally, we must prove that if $3n$ is even, then n is even. We will do this using a proof by contraposition. Suppose that n is not even. Then n is odd, so we can write $n = 2k+1$ for some integer k. Then $3n = 3(2k+1) = 6k+3 = 2(3k+1)+1$. This exhibits $3n$ as 2 times an integer plus 1, so $3n$ is odd, completing the proof by contraposition.

SECTION 1.8 Proof Methods and Strategy

The preamble to the solutions for Section 1.7 applies here as well, so you might want to reread it at this time. In addition, the section near the back of this Guide, *entitled "A Guide to Proof-Writing," provides an excellent tutorial, with many additional examples. Don't forget to take advantage of the many additional resources on the website for this text, as well.*

If you are majoring in mathematics, then proofs are the bread and butter of your field. Most likely you will take a course devoted entirely to learning how to read and write proofs, using one of the many textbooks available on this subject. For a review of many of them (as well as reviews of hundreds of mathematics books), see this site provided by the Mathematical Association of America: `mathdl.maa.org/MathDL/19/`.

1. We give an exhaustive proof—just check the entire domain. For $n = 1$ we have $1^2+1 = 2 \geq 2 = 2^1$. For $n = 2$ we have $2^2+1 = 5 \geq 4 = 2^2$. For $n = 3$ we have $3^2+1 = 10 \geq 8 = 2^3$. For $n = 4$ we have $4^2+1 = 17 \geq 16 = 2^4$. Notice that for $n \geq 5$, the inequality is no longer true.

3. Following the hint, we consider the two cases determined by the relative sizes of x and y. First suppose that $x \geq y$. Then by definition $\max(x,y) = x$ and $\min(x,y) = y$. Therefore in this case $\max(x,y)+\min(x,y) = x+y$, exactly as desired. For the second (and final) case, suppose that $x < y$. Then $\max(x,y) = y$ and $\min(x,y) = x$. Therefore in this case $\max(x,y)+\min(x,y) = y+x = x+y$, again the desired conclusion. Hence in all cases, the equality holds.

5. Because $|x-y| = |y-x|$, the values of x and y are interchangeable. Therefore, without loss of generality, we can assume that $x \geq y$. In this case, $|x-y| = x-y$, so the first expression gives us

$$\frac{x+y-(x-y)}{2} = \frac{x+y-x+y}{2} = \frac{2y}{2} = y\,,$$

and indeed y is the smaller. Similarly, the second expression gives us

$$\frac{x+y+(x-y)}{2} = \frac{x+y+x-y}{2} = \frac{2x}{2} = x\,,$$

and indeed x is the larger.

7. There are several cases to consider. If x and y are both nonnegative, then $|x|+|y| = x+y = |x+y|$. Similarly, if both are negative, then $|x|+|y| = (-x)+(-y) = -(x+y) = |x+y|$, since $x+y$ is negative in this case. The complication (and strict inequality) comes if one of the variables is nonnegative and the other is negative. By the symmetry of the roles of x and y here (strictly speaking, by the commutativity of addition), we can assume without loss of generality that it is x that is nonnegative and y that is negative. So we have $x \geq 0$ and $y < 0$.

Now there are two subcases to consider within this case, depending on the relative sizes of the nonnegative numbers x and $-y$. First suppose that $x \geq -y$. Then $x + y \geq 0$. Therefore $|x + y| = x + y$, and this quantity is a nonnegative number smaller than x (since y is negative). On the other hand $|x| + |y| = x + |y|$ is a positive number bigger than x. Therefore we have $|x + y| < x < |x| + |y|$, as desired.

Finally, consider the possibility that $x < -y$. Then $|x + y| = -(x + y) = (-x) + (-y)$ is a positive number less than or equal to $-y$ (since $-x$ is nonpositive). On the other hand $|x| + |y| = |x| + (-y)$ is a positive number greater than or equal to $-y$. Therefore we have $|x + y| \leq -y \leq |x| + |y|$, as desired.

9. We want to find consecutive squares that are far apart. If n is large enough, then $(n + 1)^2$ will be much bigger than n^2, and that will do it. Let's take $n = 100$. Then $100^2 = 10000$ and $101^2 = 10201$, so the 201 consecutive numbers 10001, 10002, ..., 10200 are not perfect squares. The first 100 of these will satisfy the requirements of this exercise. Our proof was constructive, since we actually exhibited the numbers.

11. We try some small numbers and discover that $8 = 2^3$ and $9 = 3^2$. In fact, this is the only solution, but the proof of this fact is not trivial.

13. One way to solve this is the following nonconstructive proof. Let $x = 2$ (rational) and $y = \sqrt{2}$ (irrational). If $x^y = 2^{\sqrt{2}}$ is irrational, we are done. If not, then let $x = 2^{\sqrt{2}}$ and $y = \sqrt{2}/4$; x is rational by assumption, and y is irrational (if it were rational, then $\sqrt{2}$ would be rational). But now $x^y = (2^{\sqrt{2}})^{\sqrt{2}/4} = 2^{\sqrt{2}\cdot(\sqrt{2})/4} = 2^{1/2} = \sqrt{2}$, which is irrational, as desired.

15. **a)** This statement asserts the existence of x with a certain property. If we let $y = x$, then we see that $P(x)$ is true. If y is anything other than x, then $P(y)$ is not true. Thus x is the unique element that makes P true.

b) The first clause here says that there is an element that makes P true. The second clause says that whenever two elements both make P true, they are in fact the same element. Together this says that P is satisfied by exactly one element.

c) This statement asserts the existence of an x that makes P true and has the further property that whenever we find an element that makes P true, that element is x. In other words, x is the unique element that makes P true. Note that this is essentially the same as the definition given in the text, except that the final conditional statement has been replaced by its contrapositive.

17. The equation $|a - c| = |b - c|$ is equivalent to the disjunction of two equations: $a - c = b - c$ or $a - c = -b + c$. The first of these is equivalent to $a = b$, which contradicts the assumptions made in this problem, so the original equation is equivalent to $a - c = -b + c$. By adding $b + c$ to both sides and dividing by 2, we see that this equation is equivalent to $c = (a + b)/2$. Thus there is a unique solution. Furthermore, this c is an integer, because the sum of the odd integers a and b is even.

19. We are being asked to solve $n = (k - 2) + (k + 3)$ for k. Using the usual, reversible, rules of algebra, we see that this equation is equivalent to $k = (n - 1)/2$. In other words, this is the one and only value of k that makes our equation true. Since n is odd, $n - 1$ is even, so k is an integer.

21. If x is itself an integer, then we can take $n = x$ and $\epsilon = 0$. No other solution is possible in this case, since if the integer n is greater than x, then n is at least $x + 1$, which would make $\epsilon \geq 1$. If x is not an integer, then round it up to the next integer, and call that integer n. We let $\epsilon = n - x$. Clearly $0 \leq \epsilon < 1$, this is the only ϵ that will work with this n, and n cannot be any larger, since ϵ is constrained to be less than 1.

23. If $x = 5$ and $y = 8$, then the harmonic mean is $2 \cdot 5 \cdot 8/(5+8) \approx 6.15$, and the geometric mean is $\sqrt{5 \cdot 8} \approx 6.32$. If $x = 10$ and $y = 100$, then the harmonic mean is $2 \cdot 10 \cdot 100/(10 + 100) \approx 18.18$, and the geometric mean is $\sqrt{10 \cdot 100} \approx 31.62$. We conjecture that the harmonic mean of x and y is always less than their geometric mean if x and y are distinct positive real numbers (clearly if $x = y$ then both means are this common value). So we want to verify the inequality $2xy/(x + y) < \sqrt{xy}$. Multiplying both sides by $(x + y)/(2\sqrt{xy})$ gives us the equivalent inequality $\sqrt{xy} < (x + y)/2$, which is proved in Example 14.

25. The key point here is that *the parity (oddness or evenness) of the sum of the numbers written on the board never changes.* If j and k are both even or both odd, then their sum and their difference are both even, and we are replacing the even sum $j+k$ by the even difference $|j-k|$, leaving the parity of the total unchanged. If j and k have different parities, then erasing them changes the parity of the total, but their difference $|j-k|$ is odd, so adding this difference restores the parity of the total. Therefore the integer we end up with at the end of the process must have the same parity as $1 + 2 + \cdots + (2n)$. It is easy to compute this sum. If we add the first and last terms we get $2n + 1$; if we add the second and next-to-last terms we get $2 + (2n - 1) = 2n + 1$; and so on. In all we get n sums of $2n + 1$, so the total sum is $n(2n + 1)$. If n is odd, this is the product of two odd numbers and therefore is odd, as desired.

27. Without loss of generality we can assume that n is nonnegative, since the fourth power of an integer and the fourth power of its negative are the same. To get a handle on the last digit of n, we can divide n by 10, obtaining a quotient k and remainder l, whence $n = 10k + l$, and l is an integer between 0 and 9, inclusive. Then we compute n^4 in each of these ten cases. We get the following values, where ?? is some integer that is a multiple of 10, whose exact value we do not care about.

$$\begin{aligned}
(10k+0)^4 &= 10000k^4 = 10000k^4 + 0\\
(10k+1)^4 &= 10000k^4 + ?? \cdot k^3 + ?? \cdot k^2 + ?? \cdot k + 1\\
(10k+2)^4 &= 10000k^4 + ?? \cdot k^3 + ?? \cdot k^2 + ?? \cdot k + 16\\
(10k+3)^4 &= 10000k^4 + ?? \cdot k^3 + ?? \cdot k^2 + ?? \cdot k + 81\\
(10k+4)^4 &= 10000k^4 + ?? \cdot k^3 + ?? \cdot k^2 + ?? \cdot k + 256\\
(10k+5)^4 &= 10000k^4 + ?? \cdot k^3 + ?? \cdot k^2 + ?? \cdot k + 625\\
(10k+6)^4 &= 10000k^4 + ?? \cdot k^3 + ?? \cdot k^2 + ?? \cdot k + 1296\\
(10k+7)^4 &= 10000k^4 + ?? \cdot k^3 + ?? \cdot k^2 + ?? \cdot k + 2401\\
(10k+8)^4 &= 10000k^4 + ?? \cdot k^3 + ?? \cdot k^2 + ?? \cdot k + 4096\\
(10k+9)^4 &= 10000k^4 + ?? \cdot k^3 + ?? \cdot k^2 + ?? \cdot k + 6561
\end{aligned}$$

Since each coefficient indicated by ?? is a multiple of 10, the corresponding term has no effect on the ones digit of the answer. Therefore the ones digits are 0, 1, 6, 1, 6, 5, 6, 1, 6, 1, respectively, so it is always a 0, 1, 5, or 6.

29. Because $n^3 > 100$ for all $n > 4$, we need only note that $n = 1$, $n = 2$, $n = 3$, and $n = 4$ do not satisfy $n^2 + n^3 = 100$.

31. Since $5^4 = 625$, for there to be positive integer solutions to this equation both x and y must be less than 5. This means that each of x^4 and y^4 is at most $4^4 = 256$, so their sum is at most 512 and cannot be 625.

33. We give a proof by contraposition. Assume that it is not the case that $a \le \sqrt[3]{n}$ or $b \le \sqrt[3]{n}$ or $c \le \sqrt[3]{n}$. Then it must be true that $a > \sqrt[3]{n}$ and $b > \sqrt[3]{n}$ and $c > \sqrt[3]{n}$. Multiplying these inequalities of positive numbers together we obtain $abc < (\sqrt[3]{n})^3 = n$, which implies the negation of our hypothesis that $n = abc$.

35. The idea is to find a small irrational number to add to the smaller of the two given rational numbers. Because we know that $\sqrt{2}$ is irrational, we can use a small multiple of $\sqrt{2}$. Here is our proof: By finding a common denominator, we can assume that the given rational numbers are a/b and c/b, where b is a positive integer and a and c are integers with $a < c$. In particular, $(a+1)/b \le c/b$. Thus $x = (a + \frac{1}{2}\sqrt{2})/b$ is between the two given rational numbers, because $0 < \sqrt{2} < 2$. Furthermore, x is irrational, because if x were rational, then $2(bx - a) = \sqrt{2}$ would be as well, in violation of Example 10 in Section 1.7.

37. **a)** Without loss of generality, we may assume that the x sequence is already sorted into nondecreasing order, since we can relabel the indices. There are only a finite number of possible orderings for the y sequence, so if we can show that we can increase the sum (or at least keep it the same) whenever we find y_i and y_j that are out of order (i.e., $i < j$ but $y_i > y_j$) by switching them, then we will have shown that the sum is largest when the y sequence is in nondecreasing order. Indeed, if we perform the swap, then we have added $x_i y_j + x_j y_i$ to the sum and subtracted $x_i y_i + x_j y_j$. The net effect, then, is to have added $x_i y_j + x_j y_i - x_i y_i - x_j y_j = (x_j - x_i)(y_i - y_j)$, which is nonnegative by our ordering assumptions.

b) This is similar to part **(a)**. Again we assume that the x sequence is already sorted into nondecreasing order. If the y sequence is not in nonincreasing order, then $y_i < y_j$ for some $i < j$. By swapping y_i and y_j we increase the sum by $x_i y_j + x_j y_i - x_i y_i - x_j y_j = (x_j - x_i)(y_i - y_j)$, which is nonpositive by our ordering assumptions.

39. In each case we just have to keep applying the function f until we reach 1, where $f(x) = 3x + 1$ if x is odd and $f(x) = x/2$ if x is even.

a) $f(6) = 3$, $f(3) = 10$, $f(10) = 5$, $f(5) = 16$, $f(16) = 8$, $f(8) = 4$, $f(4) = 2$, $f(2) = 1$. We abbreviate this to $6 \to 3 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$.

b) $7 \to 22 \to 11 \to 34 \to 17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$

c) $17 \to 52 \to 26 \to 13 \to 40 \to 20 \to 10 \to 5 \to 16 \to 8 \to 4 \to 2 \to 1$

d) $21 \to 64 \to 32 \to 16 \to 8 \to 4 \to 2 \to 1$

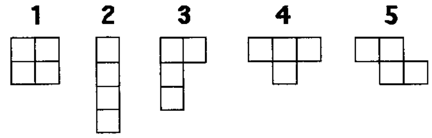
41. We give a constructive proof. Without loss of generality, we can assume that the upper left and upper right corners of the board are removed. We can place three dominoes horizontally to fill the remaining portion of the first row, and we can place four dominoes horizontally in each of the other seven rows to fill them.

43. The number of squares in a rectangular board is the product of the number of squares in each row and the number of squares in each column. We are given that this number is even, so there is either an even number of squares in each row or an even number of squares in each column. In the former case, we can tile the board in the obvious way by placing the dominoes horizontally, and in the latter case, we can tile the board in the obvious way by placing the dominoes vertically.

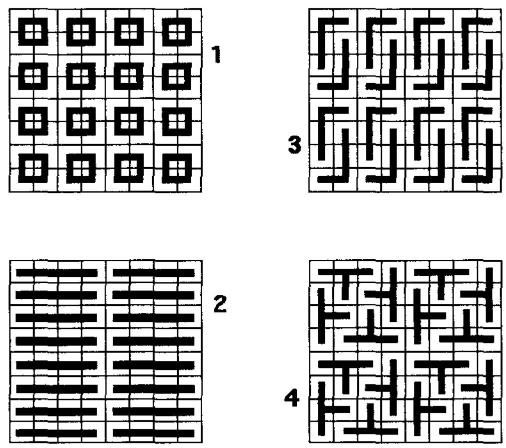
45. We follow the suggested labeling scheme. Clearly we can rotate the board if necessary to make the removed squares be 1 and 16. Square 2 must be covered by a domino. If that domino is placed to cover squares 2 and 6, then the following domino placements are forced in succession: 5-9, 13-14, and 10-11, at which point there is no way to cover square 15. Otherwise, square 2 must be covered by a domino placed at 2-3. Then the following domino placements are forced: 4-8, 11-12, 6-7, 5-9, and 10-14, and again there is no way to cover square 15.

47. Remove the two black squares adjacent to one of the white corners, and remove two white squares other than that corner. Then no domino can cover that white corner, because neither of the squares adjacent to it remains.

49. **a)** It is not hard to find the five patterns:



b) It is clear that the pattern labeled 1 and the pattern labeled 2 will tile the checkerboard. It is harder to find the tiling for patterns 3 and 4, but a little experimentation shows that it is possible.



It remains to argue that pattern 5 cannot tile the checkerboard. Label the squares from 1 to 64, one row at a time from the top, from left to right in each row. Thus square 1 is the upper left corner, and square 64 is the lower right. Suppose we did have a tiling. By symmetry and without loss of generality, we may suppose that the tile is positioned in the upper left corner, covering squares 1, 2, 10, and 11. This forces a tile to be adjacent to it on the right, covering squares 3, 4, 12, and 13. Continue in this manner and we are forced to have a tile covering squares 6, 7, 15, and 16. This makes it impossible to cover square 8. Thus no tiling is possible.

GUIDE TO REVIEW QUESTIONS FOR CHAPTER 1

1. **a)** See p. 3. **b)** This is not a boring course.

2. **a)** See pp. 4, 6, and 9.
b) Disjunction: "I'll go to the movies tonight or I'll finish my discrete mathematics homework." Conjunction: "I'll go to the movies tonight and I'll finish my discrete mathematics homework." Exclusive or: "I'll go to the movies tonight or I'll finish my discrete mathematics homework, but not both." Conditional statement: "If I'll go to the movies tonight, then I'll finish my discrete mathematics homework." Biconditional: "I'll go to the movies tonight if and only if I'll finish my discrete mathematics homework."

3. **a)** See p. 6. **b)** See p. 8.
c) Converse: "If I go for a walk in the woods tomorrow, then it will be sunny." Contrapositive: "If I don't go for a walk in the woods tomorrow, then it will not be sunny."

4. **a)** See p. 25.
b) using truth tables; symbolically, using identities in Tables 6–8 in Section 1.3; by giving a valid argument about the possible truth values of the propositional variables involved
c) Use the fact that $r \to \neg q \equiv \neg r \vee \neg q$, or use truth tables.

5. **a)** Each line of the truth table corresponds to exactly one combination of truth values for the n atomic propositions involved. We can write down a conjunction that is true precisely in this case, namely the conjunction of all the atomic propositions that are true and the negations of all the atomic propositions that are false. If we do this for *each* line of the truth table for which the value of the compound proposition is to be true, and take the disjunction of the resulting propositions, then we have the desired proposition in its disjunctive normal form. See Exercise 42 in Section 1.3.
b) See Exercise 43 in Section 1.3.
c) See Exercises 50 and 52 in Section 1.3.

6. See pp. 40 and 42. The negation of $\forall x P(x)$ is $\exists x \neg P(x)$, and the negation of $\exists x P(x)$ is $\forall x \neg P(x)$.

7. **a)** In the second, x can depend on y. In the first, the same x must "work" for every y.
b) See Example 4 in Section 1.5.

8. See pp. 69–70. This is a valid argument because it uses the valid rule of inference called modus tollens.

9. This is a valid argument because it uses the universal modus ponens rule of inference. Therefore if the premises are true, the conclusion must be true.

10. **a)** See pp. 82, 83, and 86.
b) For a direct proof, the hypothesis implies that $n = 2k$ for some k, whence $n + 4 = 2(k + 2)$, so $n + 4$ is even. For a proof by contraposition, suppose that $n + 4$ is odd; hence $n + 4 = 2k + 1$ for some k. Then $n = 2(k - 2) + 1$, so n is odd, hence not even. For a proof by contradiction, assume that $n = 2k$ and $n + 4 = 2l + 1$ for some k and l. Subtracting gives $4 = 2(l - k) + 1$, which means that 4 is odd, a contradiction.

11. **a)** See p. 87.
b) Suppose that $3n+2$ is odd, so that $3n+2 = 2k+1$ for some k. Multiply both sides by 3 and subtract 1, obtaining $9n+5 = 6k+2 = 2(3k+1)$. This shows that $9n+5$ is even. We prove the converse by contraposition. Suppose that $3n + 2$ is not odd, i.e., that it is even. Then $3n + 2 = 2k$ for some k. Multiply both sides by 3 and subtract 1, obtaining $9n + 5 = 6k - 1 = 2(3k - 1) + 1$. This shows that $9n + 5$ is odd.

12. No—we could add to these $p_2 \rightarrow p_3$ and $p_1 \rightarrow p_4$, for example.

13. **a)** Find a counterexample, i.e., an object c such that $P(c)$ is false. **b)** $n = 1$ is a counterexample.

14. See p. 96.

15. See p. 99.

16. See Example 4 in Section 1.8.

SUPPLEMENTARY EXERCISES FOR CHAPTER 1

1. **a)** $q \rightarrow p$ (note that "only if" does not mean "if")
b) $q \wedge p$ **c)** $\neg q \vee \neg p$ (assuming inclusive use of the English word "or" is intended by the speaker)
d) $q \leftrightarrow p$ (this is another way to say "if and only if" in English words)

3. We could use truth tables, but we can also argue as follows.
a) Since q is false but the conditional statement $p \rightarrow q$ is true, we must conclude that p is also false.
b) The disjunction says that either p or q is true. Since p is given to be false, it follows that q must be true.

5. The inverse of $p \to q$ is $\neg p \to \neg q$. Therefore the converse of the inverse is $\neg q \to \neg p$. Note that this is the same as the contrapositive of the original statement. The converse of $p \to q$ is $q \to p$. Therefore the converse of the converse is $p \to q$, which was the original statement. The contrapositive of $p \to q$ is $\neg q \to \neg p$. Therefore the converse of the contrapositive is $\neg p \to \neg q$, which is the same as the inverse of the original statement.

7. The straightforward approach is to use disjunctive normal form. There are four cases in which exactly three of the variables are true. The desired proposition is $(p \wedge q \wedge r \wedge \neg s) \vee (p \wedge q \wedge \neg r \wedge s) \vee (p \wedge \neg q \wedge r \wedge s) \vee (\neg p \wedge q \wedge r \wedge s)$.

9. Translating these statements into symbols, using the obvious letters, we have $\neg t \to \neg g$, $\neg g \to \neg q$, $r \to q$, and $\neg t \wedge r$. Assume the statements are consistent. The fourth statement tells us that $\neg t$ must be true. Therefore by modus ponens with the first statement, we know that $\neg g$ is true, hence (from the second statement) that $\neg q$ is true. Also, the fourth statement tells us that r must be true, and so again modus ponens (third statement) makes q true. This is a contraction: $q \wedge \neg q$. Thus the statements are inconsistent.

11. We make a table of the eight possibilities for p, q, and r, showing the truth values of the four propositions.

p	q	r	$\neg(p \to (q \wedge r))$	$p \vee \neg q$	$\neg r$	$(p \wedge r) \vee (q \to p)$
T	T	T	F	T	F	T
T	T	F	T	T	T	T
T	F	T	T	T	F	T
T	F	F	T	T	T	T
F	T	T	F	F	F	F
F	T	F	F	F	T	F
F	F	T	F	T	F	T
F	F	F	F	T	T	T

If we look at the first row of the table, we see that if the student rejects the first proposition, accepts the second, rejects the third, and accepts the fourth, then the resulting commitments are consistent, because the second and fourth propositions and the negations of the first and third propositions are all true in this case in which p, q, and r are all true. Similarly, looking at the sixth row of the table, where p and r are false but q is true, we see that a student who accepts the third proposition and rejects the other three also wins. Scanning the entire table, we see that the winning answers are reject-accept-reject-accept, accept-accept-accept-accept, accept-accept-reject-accept, reject-reject-reject-reject, reject-reject-accept-reject, and reject-accept-accept-accept.

13. Aaron must be a knave, because a knight would never make the false statement that all of them are knaves. If Bohan is a knight, then he would be speaking the truth if Crystal is a knight, so that is one possibility. On the other hand, Bohan might be a knave, in which case his statement is already false, regardless of Crystal's identity. In this case, if Crystal were also a knave, then Aaron would have told the truth, which is impossible. So there are two possibilities for the ordered triple (Aaron, Bohan, Crystal), namely (knave, knight, knight) and (knave, knave, knight).

15. We are told that exactly one of these people committed the crime, and exactly one (the guilty party) is a knight. We look at the three cases to determine who the knight is. If Amy were the knight, then her protestations of innocence would be true, but that cannot be, since we know that the knight is guilty. If Claire were the knight, then her statement that Brenda is not a normal is true; and since Brenda cannot be the knight in this situation, Brenda must be a knave. That means that Brenda is lying when she says that Amy was telling the truth; therefore Amy is lying. This means that Amy is guilty, but that cannot be, since Amy isn't the knight. So Brenda must be the knight. Amy is an innocent normal who is telling the truth when she says she is innocent; Brenda is telling the truth when she says that Amy is telling the truth; and Claire is a normal who is telling the truth when she says that Brenda is not a normal. So Brenda committed the crime.

17. The definition of valid argument is an argument in which the truth of all the premises forces the truth of conclusion. In this example, the two premises can never be true simultaneously, because they are contradictory, irrespective of the true status of the tooth fairy. Therefore it is (vacuously) true that whenever both of the premises are true, the conclusion is also true (irrespective of your luck at finding gold at the end of the rainbow). Because the premises are not both true, we cannot conclude that the conclusion is true.

19. This is done in exactly the same manner as was described in the text for a 9×9 Sudoku puzzle (Section 1.3), with the variables indexed from 1 to 16, instead of from 1 to 9, and with a similar change for the propositions for the 4×4 blocks: $\bigwedge_{r=0}^{3}\bigwedge_{s=0}^{3}\bigwedge_{n=1}^{16}\bigvee_{i=1}^{4}\bigvee_{j=1}^{4}p(4r+i,4s+j,n)$.

21. **a)** F, since 4 does not divide 5 **b)** T, since 2 divides 4
c) F, by the counterexample in part **(a)** **d)** T, since 1 divides every positive integer
e) F, since no number is a multiple of all positive integers (No matter what positive integer n one chooses, if we take $m=n+1$, then $P(m,n)$ is false, since $n+1$ does not divide n.)
f) T, since 1 divides every positive integer

23. The given statement tells us that there are exactly two elements in the domain. Therefore if we let the domain be anything with size other than 2 the statement will be false.

25. For each person we want to assert the existence of two different people who are that person's parents. The most elegant way to do so is $\forall x\exists y\exists z(y\neq z\wedge\forall w(P(w,x)\leftrightarrow(w=y\vee w=z)))$. Note that we are saying that w is a parent of x if and only if w is one of the two people whose existence we asserted.

27. To express the statement that exactly n members of the domain satisfy P, we need to use n existential quantifiers, express the fact that these n variables all satisfy P and are all different, and express the fact that every other member of the domain that satisfies P must be one of these.
a) This is a special case, however. To say that there are no values of x that make P true we can simply write $\neg\exists xP(x)$ or $\forall x\neg P(x)$.
b) This is the same as Exercise 52 in Section 1.5, because $\exists_1 xP(x)$ is the same as $\exists! xP(x)$. Thus we can write $\exists x(P(x)\wedge\forall y(P(y)\rightarrow y=x))$.
c) Following the discussion above, we write $\exists x_1\exists x_2(P(x_1)\wedge P(x_2)\wedge x_1\neq x_2\wedge\forall y(P(y)\rightarrow(y=x_1\vee y=x_2)))$.
d) We expand the previous answer to one more variable: $\exists x_1\exists x_2\exists x_3(P(x_1)\wedge P(x_2)\wedge P(x_3)\wedge x_1\neq x_2\wedge x_1\neq x_3\wedge x_2\neq x_3\wedge\forall y(P(y)\rightarrow(y=x_1\vee y=x_2\vee y=x_3)))$.

29. Suppose that $\exists x(P(x)\rightarrow Q(x))$ is true. Then for some x, either $Q(x)$ is true or $P(x)$ is false. If $Q(x)$ is true for some x, then the conditional statement $\forall xP(x)\rightarrow\exists xQ(x)$ is true (having true conclusion). If $P(x)$ is false for some x, then again the conditional statement $\forall xP(x)\rightarrow\exists xQ(x)$ is true (having false hypothesis). Conversely, suppose that $\exists x(P(x)\rightarrow Q(x))$ is false. That means that for every x, the conditional statement $P(x)\rightarrow Q(x)$ is false, or, in other words, $P(x)$ is true and $Q(x)$ is false. The latter statement implies that $\exists xQ(x)$ is false. Thus $\forall xP(x)\rightarrow\exists xQ(x)$ has a true hypothesis and a false conclusion and is therefore false.

31. No. For each x there may be just one y making $P(x,y)$ true, so that the second proposition will not be true. For example, let $P(x,y)$ be $x+y=0$, where the domain (universe of discourse) is the integers. Then the first proposition is true, since for each x there exists a y, namely $-x$, such that $P(x,y)$ holds. On the other hand, there is no one x such that $x+y=0$ for *every* y.

33. Let $T(s,c,d)$ be the statement that student s has taken class c in department d. Then, with the domains (universes of discourse) being the students in this class, the courses at this university, and the departments in the school of mathematical sciences, the given statement is $\forall s\forall d\exists c\,T(s,c,d)$.

35. Let $T(x,y)$ mean that student x has taken class y, where the domain is all students in this class. We want to say that there exists exactly one student for whom there exists exactly one class that this student has taken. So we can write simply $\exists! x \exists! y\, T(x,y)$. To do this without quantifiers, we need to expand the uniqueness quantifier using Exercise 52 in Section 1.5. Doing so, we have $\exists x \forall z((\exists y \forall w(T(z,w) \leftrightarrow w = y)) \leftrightarrow z = x)$.

37. By universal instantiation we have $P(a) \rightarrow Q(a)$ and $Q(a) \rightarrow R(a)$. By modus tollens we then conclude $\neg Q(a)$, and again by modus tollens we conclude $\neg P(a)$.

39. We give a proof by contraposition that if $\sqrt{x}$ is rational, then x is rational, assuming throughout that $x \geq 0$. Suppose that $\sqrt{x} = p/q$ is rational, $q \neq 0$. Then $x = (\sqrt{x})^2 = p^2/q^2$ is also rational (q^2 is again nonzero).

41. We can give a constructive proof by letting $m = 10^{500} + 1$. Then $m^2 = (10^{500} + 1)^2 > (10^{500})^2 = 10^{1000}$.

43. The first three positive cubes are 1, 8, and 27. If we want to find a number that cannot be written as the sum of eight cubes, we would look for a number that is 7 more than a small multiple of 8. Indeed, 23 will do. We can use two 8's but then would have to use seven 1's to reach 23, a total of nine numbers. Clearly no smaller collection will do. This counterexample disproves the statement.

45. The first three positive fifth powers are 1, 32, and 243. If we want to find a number that cannot be written as the sum of 36 fifth powers, we would look for a number that is 31 more than a small multiple of 32. Indeed, $7 \cdot 32 - 1 = 223$ will do. We can use six 32's but then would have to use 31 1's to reach 223, a total of 37 numbers. Clearly no smaller collection will do. This counterexample disproves the statement.

WRITING PROJECTS FOR CHAPTER 1

Books and articles indicated by bracketed symbols below are listed near the end of this manual. You should also read the general comments and advice you will find there about researching and writing these essays.

1. An excellent website for this is `www.paradoxes.co.uk`. It includes a bibliography.

2. Search your library's on-line catalog for a book with the word *fuzzy* in the title. You might find [BaGo], [DuPr], [Ka], [Ko3], or [McFr], for example.

3. Look for this article (available on the Web): Marques-Silva, J. (2008) Practical Applications of Boolean Satisfiability. Also try a book published by the American Mathematical Society, the proceedings of a workshop on this topic: *Satisfiability Problem: Theory and Applications*.

4. A Web search for "solving sudoku" should get you more than enough sources.

5. Even if you can't find a set, you may find some articles about it in materials for high school students and teachers, such as old issues of *Mathematics Teacher*, published by the National Council of Teachers of Mathematics. This journal, and possibly even copies of the game, may exist in the education library at your school (if there is one). The company that currently produces it has a website: `wffnproof.com`.

6. Martin Gardner and others have written some books that annotate Carroll's writings quite extensively. Lewis Carroll has become a cult figure in certain circles. See also [Ca1], [Ca2], and [Ca3], for original material.

7. A textbook on logic programming and/or the language PROLOG, such as [Ho2] or [Sa1], would be a logical place to start. Many bookstores have huge computer science sections, so that source should not be ignored.

8. A course on computational logic at Stanford in 2005–2006 had a Web page with class notes: `logic.stanford.edu/classes/cs157/2005fall/cs157.html`. Enderton's book on logic [En] would be a possible choice for background information.

9. There are books on this subject, such as [Du].

10. A place to start might be a recent article on this topic in *Science* [Re]. As always, a Web search will also turn up more information.

11. There is an excellent article on this by Keith Devlin, writing for the Mathematical Association of America (MAA); see `www.maa.org/devlin/devlin_05_03.html`.

12. The well-known on-line encyclopedia made up of articles by contributors is usually quite good, with accurate information and useful links and cross-references. See their article on Chomp: `en.wikipedia.org/wiki/Chomp`.

13. The references given in the text are the obvious place to start. The mathematics education field has bought into Pólya's ideas, especially as they relate to problem-solving. See what the National Council of Teachers of Mathematics (`www.nctm.org`) has to say about it.

14. The classic work in this field is [GrSh].