MATH 2418: Linear Algebra

Assignment# 4

Due: Tuesday, 09/20/2022, 11:59pm Term: `Fall 2022`

[First Name] [Last Name] [Net ID]

Recommended Problems (do not turn in): Sec 2.3: 1, 3, 4, 7, 8, 9, 18, 21, 25, 27, 28.
Sec 2.4: 3, 6, 7, 10, 11, 12, 14, 17, 21, 26, 32.

1. Consider the linear system of equations:

$$\begin{cases} x + 6y + 2z = -13 \\ 12x + 6y + 18z = -6 \\ 6x + 37y + 12z = -80 \end{cases}$$

(a) Write down its augmented matrix.

The given linear system can be written as $A\mathbf{x} = \mathbf{b}$, where

$$A = \begin{bmatrix} 1 & 6 & 2 \\ 12 & 6 & 18 \\ 6 & 37 & 12 \end{bmatrix}, \ \mathbf{x} = \begin{bmatrix} x \\ y \\ z \end{bmatrix}, \ \mathbf{b} = \begin{bmatrix} -13 \\ -6 \\ -80 \end{bmatrix}.$$

Then the augmented matrix is

$$[A|\mathbf{b}] = \begin{bmatrix} 1 & 6 & 2 & : & -13 \\ 12 & 6 & 18 & : & -6 \\ 6 & 37 & 12 & : & -80 \end{bmatrix}.$$

(b) Solve the linear system by reducing the coefficient matrix to an upper triangular matrix followed by back substitution.

Now, performing elimination on the augmented matrix $[A|\mathbf{b}]$, gives

$$\begin{bmatrix} 1 & 6 & 2 & : & -13 \\ 12 & 6 & 18 & : & -6 \\ 6 & 37 & 12 & : & -80 \end{bmatrix} \xrightarrow[l_{21}=12]{R_2-12R_1} \begin{bmatrix} 1 & 6 & 2 & : & -13 \\ 0 & -66 & -6 & : & 150 \\ 6 & 37 & 12 & : & -80 \end{bmatrix}$$

$$\xrightarrow[l_{31}=6]{R_3-6R_1} \begin{bmatrix} 1 & 6 & 2 & : & -13 \\ 0 & -66 & -6 & : & 150 \\ 0 & 1 & 0 & : & -2 \end{bmatrix}$$

$$\xrightarrow{R_3 \Longleftrightarrow R_2} \begin{bmatrix} 1 & 6 & 2 & : & -13 \\ 0 & 1 & 0 & : & -2 \\ 0 & -66 & -6 & : & 150 \end{bmatrix}$$

$$\xrightarrow[l_{32}=-66]{R_3+66R_2} \begin{bmatrix} 1 & 6 & 2 & : & -13 \\ 0 & 1 & 0 & : & -2 \\ 0 & 0 & -6 & : & 18 \end{bmatrix}.$$

Back-substitution gives,

$$-6z = 18 \Rightarrow z = -3,$$
$$y = -2,$$
$$x + 6y + 2z = -13 \Rightarrow x - 12 - 6 = -13 \Rightarrow x = 5.$$

Thus, the solution is $\mathbf{x} = (5, -2, -3)$.

(c) Write down each of the elementary matrices E_{ij} used in step(b).

Solution: The elimination matrices are

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -l_{21} & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -12 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix},$$
$$E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -l_{31} & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -6 & 0 & 1 \end{bmatrix}$$
$$P_{23} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$
$$E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -l_{31} & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 66 & 1 \end{bmatrix}.$$

2. Consider the system $A\mathbf{x} = \mathbf{b}$ with

$$A = \begin{bmatrix} 1 & -3 & -5 \\ 10 & -2 & -48 \\ -3 & 10 & 15 \end{bmatrix}, \mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}, \mathbf{b} = \begin{bmatrix} -3 \\ 50 \\ 12 \end{bmatrix}.$$

(a) Write down each of the elementary matrices that reduce A to an upper triangular matrix U.

(b) Write down the system $U\mathbf{x} = \mathbf{c}$ which is equivalent to $A\mathbf{x} = \mathbf{b}$.

(c) Solve the system $U\mathbf{x} = \mathbf{c}$ for $\mathbf{x}$.

Solution:

(a)

$$\begin{bmatrix} 1 & -3 & -5 \\ 10 & -2 & -48 \\ -3 & 10 & 15 \end{bmatrix} \xrightarrow{R_2 - 10R_1} \begin{bmatrix} 1 & -3 & -5 \\ 0 & 28 & 2 \\ -3 & 10 & 15 \end{bmatrix} \qquad E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -10 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 + 3R_1} \begin{bmatrix} 1 & -3 & -5 \\ 0 & 28 & 2 \\ 0 & 1 & 0 \end{bmatrix} \qquad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}$$

$$\xrightarrow{R_3 - \frac{1}{28}R_2} \begin{bmatrix} 1 & -3 & -5 \\ 0 & 28 & 2 \\ 0 & 0 & \frac{-1}{14} \end{bmatrix} \qquad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-1}{28} & 1 \end{bmatrix}$$

(b) We have

$$A\mathbf{x} = \mathbf{b} \implies E_{32}E_{31}E_{21}A\mathbf{x} = E_{32}E_{31}E_{21}\mathbf{b} \implies U\mathbf{x} = \mathbf{c}.$$

Calculating,

$$\begin{aligned} \mathbf{c} &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-1}{28} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ -10 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 50 \\ 12 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-1}{28} & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 80 \\ 12 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & \frac{-1}{28} & 1 \end{bmatrix} \begin{bmatrix} -3 \\ 80 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} -3 \\ 80 \\ \frac{1}{7} \end{bmatrix}. \end{aligned}$$

Thus, $U\mathbf{x} = \mathbf{c}$ is given by

$$\begin{bmatrix} 1 & -3 & -5 \\ 0 & 28 & 2 \\ 0 & 0 & \frac{-1}{14} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} -3 \\ 80 \\ \frac{1}{7} \end{bmatrix}.$$

(c) Solving for x_3,

$$\begin{aligned} \frac{-1}{14} x_3 &= \frac{1}{7} \\ \implies x_3 &= -2. \end{aligned}$$

Solving for x_2,

$$\begin{aligned} 28 x_2 + 2\left(-2\right) &= 80 \\ \implies x_2 &= \frac{84}{28} = 3. \end{aligned}$$

Solving for x_1,

$$\begin{aligned} x_1 - 3\left(3\right) - 5\left(-2\right) &= -3 \\ \implies x_1 &= -4. \end{aligned}$$

Thus

$$\mathbf{x} = \begin{bmatrix} -4 \\ 3 \\ -2 \end{bmatrix}.$$

3. Compute the following products:

(a) $\begin{bmatrix} a & 1 & 2 \\ 0 & b & 3 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} d & 2 & 3 \\ 0 & e & 1 \\ 0 & 0 & f \end{bmatrix}$

(b) $\begin{bmatrix} a & 0 & 0 \\ 1 & b & 0 \\ 3 & 2 & c \end{bmatrix} \begin{bmatrix} d & 0 & 0 \\ 2 & e & 0 \\ 3 & 1 & f \end{bmatrix}$

(c) $\begin{bmatrix} 4 & 3 & -1 \\ 1 & 3 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

(d) $\begin{bmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix}$

Solution:

(a) $$\begin{bmatrix} a & 1 & 2 \\ 0 & b & 3 \\ 0 & 0 & c \end{bmatrix} \begin{bmatrix} d & 2 & 3 \\ 0 & e & 1 \\ 0 & 0 & f \end{bmatrix} = \begin{bmatrix} (a,1,2)\cdot(d,0,0) & (a,1,2)\cdot(2,e,0) & (a,1,2)\cdot(3,1,f) \\ (0,b,3)\cdot(d,0,0) & (0,b,3)\cdot(2,e,0) & (0,b,3)\cdot(3,1,f) \\ (0,0,c)\cdot(d,0,0) & (0,0,c)\cdot(2,e,0) & (0,0,c)\cdot(3,1,f) \end{bmatrix} =$$

$$\begin{bmatrix} ad & 2a+e & 3a+1+2f \\ 0 & be & b+3f \\ 0 & 0 & cf \end{bmatrix}$$

(b) $$\begin{bmatrix} a & 0 & 0 \\ 1 & b & 0 \\ 3 & 2 & c \end{bmatrix} \begin{bmatrix} d & 0 & 0 \\ 2 & e & 0 \\ 3 & 1 & f \end{bmatrix} = \begin{bmatrix} (a,0,0)\cdot(d,2,3) & (a,0,0)\cdot(0,e,1) & (a,0,0)\cdot(0,0,f) \\ (1,b,0)\cdot(d,2,3) & (1,b,0)\cdot(0,e,1) & (1,b,0)\cdot(0,0,f) \\ (3,2,c)\cdot(d,2,3) & (3,2,c)\cdot(0,e,1) & (3,2,c)\cdot(0,0,f) \end{bmatrix} =$$

$$\begin{bmatrix} ad & 0 & 0 \\ 2b+d & be & 0 \\ 3d+4+3c & 2e+c & cf \end{bmatrix}$$

(c) $$\begin{bmatrix} 4 & 3 & -1 \\ 1 & 3 & 2 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} (4,3,-1)\cdot(a,0,0) & (4,3,-1)\cdot(0,b,0) & (4,3,-1)\cdot(0,0,c) \\ (1,3,2)\cdot(a,0,0) & (1,3,2)\cdot(0,b,0) & (1,3,2)\cdot(0,0,c) \\ (1,-1,1)\cdot(a,0,0) & (1,-1,1)\cdot(0,b,1) & (1,-1,1)\cdot(0,0,c) \end{bmatrix} =$$

$$\begin{bmatrix} 4a & 3b & -c \\ a & 3b & 2c \\ a & -b & c \end{bmatrix}$$

(d) $$\begin{bmatrix} d & 0 & 0 \\ 0 & e & 0 \\ 0 & 0 & f \end{bmatrix} \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} = \begin{bmatrix} (d,0,0)\cdot(a,0,0) & (d,0,0)\cdot(0,b,0) & (d,0,0)\cdot(0,0,c) \\ (0,e,0)\cdot(a,0,0) & (0,e,0)\cdot(0,b,0) & (0,e,0)\cdot(0,0,c) \\ (0,0,f)\cdot(a,0,0) & (0,0,f)\cdot(0,b,0) & (0,0,f)\cdot(0,0,c) \end{bmatrix} =$$

$$\begin{bmatrix} ad & 0 & 0 \\ 0 & be & 0 \\ 0 & 0 & cf \end{bmatrix}$$

4. Let $A = \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix}$ and $B = \begin{bmatrix} 7 & 5 \\ 6 & 9 \\ 4 & 3 \end{bmatrix}$. Without calculating the complete matrices AB and BA, compute (if possible) the following:

   (a) The entry $(AB)_{22}$ of AB.
   (b) The entry $(BA)_{22}$ of BA.
   (c) Column 2 of AB.
   (d) Row 3 of BA.

   Solution:

   (a)

$$\begin{aligned} (AB)_{22} \text{ of } AB &= (\text{row 2 of } A) \cdot (\text{column 2 of } B) \\ &= \begin{bmatrix} 1 & 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 9 \\ 3 \end{bmatrix} \\ &= 5 + 27 + 6 \\ &= 38 \end{aligned}$$

   (b)

$$\begin{aligned} (BA)_{22} \text{ of } BA &= (\text{row 2 of } B) \cdot (\text{column 2 of } A) \\ &= \begin{bmatrix} 6 & 9 \end{bmatrix} \cdot \begin{bmatrix} 2 \\ 3 \end{bmatrix} \\ &= 12 + 27 \\ &= 39 \end{aligned}$$

   (c)

$$\begin{aligned} \text{Column 2 of } AB &= A \cdot (\text{column 2 of } B) \\ &= \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \cdot \begin{bmatrix} 5 \\ 9 \\ 3 \end{bmatrix} \\ &= \begin{bmatrix} 31 \\ 38 \end{bmatrix} \end{aligned}$$

   (d)

$$\begin{aligned} \text{Row 3 of } BA &= (\text{row 3 of } B) \cdot A \\ &= \begin{bmatrix} 4 & 3 \end{bmatrix} \cdot \begin{bmatrix} 2 & 2 & 1 \\ 1 & 3 & 2 \end{bmatrix} \\ &= \begin{bmatrix} 11 & 17 & 10 \end{bmatrix} \end{aligned}$$

5. Answer the following (you need to show your work).

(a) Give a 2×2 matrix A such that $A^2 = 0$ but $A \neq 0$.

Solution: A simple example would be $A = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix}$. Then

$$A^2 = \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} -1 & -1 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(b) Give a 2×2 matrix B such that $B^2 = I$ but $B \neq \pm I$ (I is the identity matrix).

Solution: Let $B = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$. Then

$$B^2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} = I$$

(c) Write down a pair of 2×2 non-zero matrices A and B such that $AB = 0$ (0 is the 2×2 zero matrix)

Solution: Consider $A = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 0 \\ 0 & -20 \end{bmatrix}$. Then

$$AB = \begin{bmatrix} 10 & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & -20 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$$

(d) Given $A = \begin{bmatrix} 6 & 2 \\ 3 & 4 \end{bmatrix}$ write down the elementary matrices E_1, E_2 and E_3 such that

$$E_1 A = \begin{bmatrix} 3 & 4 \\ 6 & 2 \end{bmatrix}, \; E_2 A = \begin{bmatrix} 6 & 2 \\ -3 & 2 \end{bmatrix} \text{ and } E_3 A = \begin{bmatrix} 3 & 1 \\ 3 & 4 \end{bmatrix}.$$

Solution:
$E_1 A$ is obtained by interchanging Row 1 and Row 2 of matrix A.So we can have E_1 as the permutation matrix $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$.

$E_2 A$ is obtained by applying the row operation $R_2 = R_2 - R_1$ on matrix A. So we can have E_2 as $\begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$.

$E_3 A$ is obtained by applying the row operation $R_1 = R_1 \cdot \frac{1}{2}$ on matrix A. So we can have E_3 as $\begin{bmatrix} \frac{1}{2} & 0 \\ 0 & 1 \end{bmatrix}$.

6. Let $B = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 0 \\ 3 & 3 & 0 \end{bmatrix}$.

(a) For any 4×3 matrix A, what is the column 3 of AB? Write down all components in the column 3 of AB, and explain your answer.

The third column of AB is $A \cdot [\text{col } 3 \text{ of } B] = A \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$.

(b) For any 3×2 matrix C, what is the row 2 of BC? Write down all components in the row 2 of BC. Explain your answer.

The second row of BC is $(\text{row } 2 \text{ of } B) \cdot C = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} C = \begin{bmatrix} 0 & 0 \end{bmatrix}$.

(c) Is it possible to find a 3×3 matrix D such that $DB = I_3$, the 3×3 identity matrix? Explain your answer.

No, the column 3 of DB is $D \cdot [\text{col } 3 \text{ of } B] = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \neq$ the column 3 of the identity matrix $\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

(d) Is it possible to find a 3×3 matrix F such that $BF = I_3$, the 3×3 identity matrix? Explain your answer.

No, the row 2 of BF is $(\text{row } 2 \text{ of } B) \cdot F = \begin{bmatrix} 0 & 0 & 0 \end{bmatrix} \neq$ the row 2 of the identity matrix $\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$.