CHAPTER 6
Counting

SECTION 6.1 The Basics of Counting

The secret to solving counting problems is to look at the problem the right way and apply the correct rules (usually the product rule or the sum rule), often with some common sense and cleverness thrown in. This is usually easier said than done, but it gets easier the more problems you do. Sometimes you need to count more than you want and then subtract the overcount. (This notion is made more precise in Section 8.5, where the inclusion–exclusion principle is discussed explicitly.) At other times you compensate by dividing; see Exercise 73, for example. Counting problems are sometimes ambiguous, so it is possible that your answer, although different from the answer we obtain, is the correct answer to a different interpretation of the problem; try to figure out whether that is the case.

If you have trouble with a problem, simplify the parameters to make them more manageable (if necessary) and try to list the set in question explicitly. This will often give you an idea of what is going on and suggest a general method of attack that will solve the problem as given. For example, in Exercise 11 you are asked about bit strings of length 10. *If you are having difficulty, investigate the analogous question about bit strings of length* 2 *or* 3, *where you can write down the entire set, and see if a pattern develops. (Some people define mathematics as the study of patterns.) Sometimes tree diagrams make the analysis in these small cases easier to keep track of.*

See the solution to Exercise 47 for a discussion of the powerful tool of **symmetry**, *which you will often find helpful. Another useful trick is* **gluing**; *see the solution to Exercise 65.*

Finally, do not be misled, if you find these exercises easy, into thinking that combinatorial problems are a piece of cake. It is very easy to ask combinatorial questions that look just like the ones asked here but in fact are extremely difficult, if not impossible. For example, try your hand at the following problem: how many strings are there, using 10 A*'s,* 12 B*'s,* 11 C*'s, and* 15 D*'s, such that no* A *is followed by a* B, *and no* C *is followed by a* D?

1. This problem illustrates the difference between the product rule and the sum rule. If we must make one choice *and* then another choice, the product rule applies, as in part **(a)**. If we must make one choice *or* another choice, the sum rule applies, as in part **(b)**. We assume in this problem that there are no double majors.

a) The product rule applies here, since we want to do two things, one after the other. First, since there are 18 mathematics majors, and we are to choose one of them, there are 18 ways to choose the mathematics major. Then we must choose the computer science major from among the 325 computer science majors, and that can clearly be done in 325 ways. Therefore there are, by the product rule, $18 \cdot 325 = 5850$ ways to pick the two representatives.

b) The sum rule applies here, since we want to do one of two mutually exclusive things. Either we can choose a mathematics major to be the representative, or we can choose a computer science major. There are 18 ways to choose a mathematics major, and there are 325 ways to choose a computer science major. Since these two actions are mutually exclusive (no one is both a mathematics major and a computer science major), and since we want to do one of them or the other, there are $18 + 325 = 343$ ways to pick the representative.

3. **a)** The product rule applies, since the student will perform each of 10 tasks, one after the other. There are 4 ways to do each task. Therefore there are $4 \cdot 4 \cdots 4 = 4^{10} = 1{,}048{,}576$ ways to answer the questions on the test.

b) This is identical to part **(a)**, except that now there are 5 ways to answer each question—give any of the 4 answers or give no answer at all. Therefore there are $5^{10} = 9{,}765{,}625$ ways to answer the questions on the test.

5. The product rule applies here, since a flight is determined by choosing an airline for the flight from New York to Denver (which can be done in 6 ways) and then choosing an airline for the flight from Denver to San Francisco (which can be done in 7 ways). Therefore there are $6 \cdot 7 = 42$ different possibilities for the entire flight.

7. Three-letter initials are determined by specifying the first initial (26 ways), then the second initial (26 ways), and then the third initial (26 ways). Therefore by the product rule there are $26 \cdot 26 \cdot 26 = 26^3 = 17{,}576$ possible three-letter initials.

9. There is only one way to specify the first initial, but as in Exercise 7, there are 26 ways to specify each of the other initials. Therefore there are, by the product rule, $1 \cdot 26 \cdot 26 = 26^2 = 676$ possible three-letter initials beginning with A.

11. A bit string is determined by choosing the bits in the string, one after another, so the product rule applies. We want to count the number of bit strings of length 10, except that we are not free to choose either the first bit or the last bit (they are mandated to be 1's). Therefore there are 8 choices to make, and each choice can be made in 2 ways (the bit can be either a 1 or a 0). Thus the product rule tells us that there are $2^8 = 256$ such strings.

13. This is a trick question, since it is easier than one might expect. Since the string is given to consist entirely of 1's, there is nothing to choose except the length. Since there are $n+1$ possible lengths not exceeding n (if we include the empty string, of length 0), the answer is simply $n+1$. Note that the empty string consists—vacuously—entirely of 1's.

15. By the sum rule we can count the number of strings of length 4 or less by counting the number of strings of length i, for $0 \leq i \leq 4$, and then adding the results. Now there are 26 letters to choose from, and a string of length i is specified by choosing its characters, one after another. Therefore, by the product rule there are 26^i strings of length i. The answer to the question is thus $\sum_{i=0}^{4} 26^i = 1 + 26 + 676 + 17576 + 456976 = 475{,}255$.

17. The easiest way to count this is to find the total number of ASCII strings of length five and then subtract off the number of such strings that do not contain the @ character. Since there are 128 characters to choose from in each location in the string, the answer is $128^5 - 127^5 = 34{,}359{,}738{,}368 - 33{,}038{,}369{,}407 = 1{,}321{,}368{,}961$.

19. Recall that an RNA sequence is a sequence of letters, each of which is one of A, C, G, or U. Thus by the product rule there are 4^6 RNA sequences of length six if we impose no restrictions.

a) If U is excluded, then each position can be chosen from among three letters, rather than four. Therefore the answer is $3^6 = 729$.

b) If the last two letters are specified, then we get to choose only four letters, rather than six, so the answer is $4^4 = 256$.

c) If the first letter is specified, then we get to choose only five letters, rather than six, so the answer is $4^5 = 1024$.

d) If only A or U is allowed in each position, then there are just two choices at each of six stages, so the answer is $2^6 = 64$.

21. Because neither 100 nor 50 is divisible by either 7 or 11, whether the ranges are meant to be inclusive or exclusive of their endpoints is moot.

a) There are $\lfloor 100/7 \rfloor = 14$ integers less than 100 that are divisible by 7, and $\lfloor 50/7 \rfloor = 7$ of them are less than 50 as well. This leaves $14 - 7 = 7$ numbers between 50 and 100 that are divisible by 7. They are 56, 63, 70, 77, 84, 91, and 98.

b) There are $\lfloor 100/11 \rfloor = 9$ integers less than 100 that are divisible by 11, and $\lfloor 50/11 \rfloor = 4$ of them are less than 50 as well. This leaves $9 - 4 = 5$ numbers between 50 and 100 that are divisible by 11. They are 55, 66, 77, 88, and 99.

c) A number is divisible by both 7 and 11 if and only if it is divisible by their least common multiple, which is 77. Obviously there is only one such number between 50 and 100, namely 77. We could also work this out as we did in the previous parts: $\lfloor 100/77 \rfloor - \lfloor 50/77 \rfloor = 1 - 0 = 1$. Note also that the intersection of the sets we found in the previous two parts is precisely what we are looking for here.

23. This problem deals with the set of positive integers between 100 and 999, inclusive. Note that there are exactly $999 - 100 + 1 = 900$ such numbers. A second way to see this is to note that to specify a three-digit number, we need to choose the first digit to be nonzero (which can be done in 9 ways) and then the second and third digits (which can each be done in 10 ways), for a total of $9 \cdot 10 \cdot 10 = 900$ ways, by the product rule. A third way to see this (perhaps most relevant for this problem) is to note that a number of the desired form is a number less than or equal to 999 (and there are 999 such numbers) but not less than or equal to 99 (and there are 99 such numbers); therefore there are $999 - 99 = 900$ numbers in the desired range.

a) Every seventh number—7, 14, and so on—is divisible by 7. Therefore the number of positive integers less than or equal to n and divisible by 7 is $\lfloor n/7 \rfloor$ (the floor function is used—we have to round down—because the first six positive integers are not multiples of 7; for example there are only $\lfloor 20/7 \rfloor = 2$ multiples of 7 less than or equal to 20). So we find that there are $\lfloor 999/7 \rfloor = 142$ multiples of 7 not exceeding 999, of which $\lfloor 99/7 \rfloor = 14$ do not exceed 99. Therefore there are exactly $142 - 14 = 128$ numbers in the desired range divisible by 7.

b) This is similar to part **(a)**, with 7 replaced by 2, but with the added twist that we want to count the numbers *not* divisible by 2. Mimicking the analysis in part **(a)**, we see that there are $\lfloor 999/2 \rfloor = 499$ even numbers not exceeding 999, and therefore $999 - 499 = 500$ odd ones; there are similarly $99 - \lfloor 99/2 \rfloor = 50$ odd numbers less than or equal to 99. Therefore there are $500 - 50 = 450$ odd numbers between 100 and 999 inclusive.

c) There are just 9 possible digits that a three-digit number can start with. If all of its digits are to be the same, then there is no choice after the leading digit has been specified. Therefore there are 9 such numbers.

d) This is similar to part **(b)**, except that 2 is replaced by 4. Following the analysis there, we find that there are $999 - \lfloor 999/4 \rfloor = 750$ positive integers less than or equal to 999 not divisible by 4, and $99 - \lfloor 99/4 \rfloor = 75$ such positive integers less than or equal to 99. Therefore there are $750 - 75 = 675$ three-digit integers not divisible by 4.

e) The method is similar to that used in the earlier parts. There are $\lfloor 999/3 \rfloor - \lfloor 99/3 \rfloor = 300$ three-digit numbers divisible by 3, and $\lfloor 999/4 \rfloor - \lfloor 99/4 \rfloor = 225$ three-digit numbers divisible by 4. Moreover there are $\lfloor 999/12 \rfloor - \lfloor 99/12 \rfloor = 75$ numbers divisible by both 3 and 4, i.e., divisible by 12. In order to count each number divisible by 3 or 4 once and only once, we need to add the number of numbers divisible by 3 to the number of numbers divisible by 4, and then subtract the number of numbers divisible by both 3 and 4 so as not to count them twice. Therefore the answer is $300 + 225 - 75 = 450$.

f) In part **(e)** we found that there were 450 three-digit integers that *are* divisible by either 3 or 4. The others, of course, are not. Therefore there are $900 - 450 = 450$ three-digit integers that are not divisible by either 3 or 4.

g) We saw in part **(e)** that there are 300 three-digit numbers divisible by 3 and that 75 of them are also divisible by 4. The remainder of those 300 numbers, therefore, are not divisible by 4. Thus the answer is $300 - 75 = 225$.

h) We saw in part **(e)** that there are 75 three-digit numbers divisible by both 3 and 4.

25. This problem involves 1000 possible strings, since there is a choice of 10 digits for each of the three positions in the string.

a) This is most easily done by subtracting from the total number of strings the number of strings that violate the condition. Clearly there are 10 strings that consist of the same digit three times (000, 111, ..., 999). Therefore there are $1000 - 10 = 990$ strings that do not.

b) If we must begin our string with an odd digit, then we have only 5 choices for this digit. We still have 10 choices for each of the remaining digits. Therefore there are $5 \cdot 10 \cdot 10 = 500$ such strings. Alternatively, we note that by symmetry exactly half the strings begin with an odd digit (there being the same number of odd digits as even ones). Therefore half of the 1000 strings, or 500 of them, begin with an odd digit.

c) Here we need to choose the position of the digit that is not a 4 (3 ways) and choose that digit (9 ways). Therefore there are $3 \cdot 9 = 27$ such strings.

27. There are 50 choices to make, each of which can be done in 3 ways, namely by choosing the governor, choosing the senior senator, or choosing the junior senator. By the product rule the answer is therefore $3^{50} \approx 7.2 \times 10^{23}$.

29. By the sum rule we need to add the number of license plates of the first type and the number of license plates of the second type. By the product rule there are $26 \cdot 26 \cdot 10 \cdot 10 \cdot 10 \cdot 10 = 6{,}760{,}000$ license plates consisting of 2 letters followed by 4 digits; and there are $10 \cdot 10 \cdot 26 \cdot 26 \cdot 26 \cdot 26 = 45{,}697{,}600$ license plates consisting of 2 digits followed by 4 letters. Therefore the answer is $6{,}760{,}000 + 45{,}697{,}600 = 52{,}457{,}600$.

31. First let us compute the number of ways to choose the letters. By the sum rule this is the sum of the number of ways to use two letters and the number of ways to use three letters. By the product rule there are 26^2 ways to choose two letters and 26^3 ways to choose three letters. Therefore there are $26^2 + 26^3$ ways to choose the letters. By similar reasoning there are $10^2 + 10^3$ ways to choose the digits. Thus the answer to the question is $(26^2 + 26^3)(10^2 + 10^3) = 18252 \cdot 1100 = 20{,}077{,}200$.

33. We take as known that there are 26 letters including 5 vowels in the English alphabet.

a) There are 8 slots, each of which can be filled with one of the $26 - 5 = 21$ nonvowels (consonants), so by the product rule the answer is $21^8 = 37{,}822{,}859{,}361$.

b) There are 21 choices for the first slot in our string, but only 20 choices for the second slot, 19 for the third, and so on. So the answer is $21 \cdot 20 \cdot 19 \cdot 18 \cdot 17 \cdot 16 \cdot 15 \cdot 14 = 8{,}204{,}716{,}800$.

c) There are 26 choices for each slot except the first, for which there are 5 choices, so the answer is $5 \cdot 26^7 = 40{,}159{,}050{,}880$.

d) This is similar to **(b)**, except that there are only five choices in the first slot, and we are free to choose from all the letters not used so far, rather than just the consonants. Thus the answer is $5 \cdot 25 \cdot 24 \cdot 23 \cdot 22 \cdot 21 \cdot 20 \cdot 19 = 12{,}113{,}640{,}000$.

e) We subtract from the total number of strings (26^8) the number that do not contain at least one vowel (21^8, the answer to **(a)**), obtaining the answer $26^8 - 21^8 = 208{,}827{,}064{,}576 - 37{,}822{,}859{,}361 = 171{,}004{,}205{,}215$.

f) The best way to do this is first to decide where the vowel goes (8 choices), then to decide what the vowel is to be (A, E, I, O, or U—5 choices), and then to fill the remaining slots with any consonants (21^7 choices, since one slot has already been filled). Therefore the answer is $8 \cdot 5 \cdot 21^7 = 72{,}043{,}541{,}640$.

g) We can ignore the first slot, since there is no choice. Now the problem is almost identical to **(e)**, except that there are only 7 slots to fill. So the answer is $26^7 - 21^7 = 8{,}031{,}810{,}176 - 1{,}801{,}088{,}541 = 6{,}230{,}721{,}635$.

h) The problem is almost identical to **(g)**, except that there are only 6 slots to fill. So the answer is $26^6 - 21^6 = 308{,}915{,}776 - 85{,}766{,}121 = 223{,}149{,}655$.

35. For each part of this problem, we need to find the number of one-to-one functions from a set with 5 elements to a set with k elements. To specify such a function, we need to make 5 choices, in succession, namely the values of the function at each of the 5 elements in its domain. Therefore the product rule applies. The first choice can be made in k ways, since any element of the codomain can be the image of the first element of the domain. After that choice has been made, there are only $k-1$ elements of the codomain available to be the image of the second element of the domain, since images must be distinct for the function to be one-to-one. Similarly, for the third element of the domain, there are $k-2$ possible choices for a function value. Continuing in this way, and applying the product rule, we see that there are $k(k-1)(k-2)(k-3)(k-4)$ one-to-one functions from a set with 5 elements to a set with k elements.

a) By the analysis above the answer is $4 \cdot 3 \cdot 2 \cdot 1 \cdot 0 = 0$, what we would expect since there are no one-to-one functions from a set to a strictly smaller set.

b) By the analysis above the answer is $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$.

c) By the analysis above the answer is $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 720$.

d) By the analysis above the answer is $7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 = 2520$.

37. a) There can clearly be no one-to-one function from $\{1, 2, \ldots, n\}$ to $\{0, 1\}$ if $n > 2$. If $n = 1$, then there are 2 such functions, the one that sends 1 to 0, and the one that sends 1 to 1. If $n = 2$, then there are again 2 such functions, since once it is determined where 1 is sent, the function must send 2 to the other value in the codomain.

b) If the function assigns 0 to both 1 and n, then there are $n-2$ function values free to be chosen. Each can be chosen in 2 ways. Therefore, by the product rule (since we have to choose values for all the elements of the domain) there are 2^{n-2} such functions, as long as $n > 1$. If $n = 1$, then clearly there is just one such function.

c) If $n = 1$, then there are no such functions, since there are no positive integers less than n. So assume $n > 1$. In order to specify such a function, we have to decide which of the numbers from 1 to $n-1$, inclusive, will get sent to 1. There are $n-1$ ways to make this choice. There is no choice for the remaining numbers from 1 to $n-1$, inclusive, since they all must get sent to 0. Finally, we are free to specify the value of the function at n, and this may be done in 2 ways. Hence, by the product rule the final answer is $2(n-1)$.

39. The easiest way to view a partial function in terms of counting is to add an additional element to the codomain of the function—let's call it u for "undefined"— and then imagine that the function assigns a value to *all* elements of the domain. If the original function f had previously been undefined at x, we now say that $f(x) = u$. Thus all we have done is to increase the size of the codomain from n elements to $n+1$ elements. By Example 6 we conclude that there are $(n+1)^m$ such partial functions.

41. The trick here is to realize that a palindrome of length n is completely determined by its first $\lceil n/2 \rceil$ bits. This is true because once these bits are specified, the remaining bits, read from right to left, must be identical to the first $\lfloor n/2 \rfloor$ bits, read from left to right. Furthermore, these first $\lceil n/2 \rceil$ bits can be specified at will, and by the product rule there are $2^{\lceil n/2 \rceil}$ ways to do so.

43. Recall that an RNA sequence is a sequence of letters, each of which is one of A, C, G, or U. Thus by the product rule there are 4^4 RNA sequences of length four if we impose no restrictions.

a) There are 3^4 RNA sequences that use only letters other than U, so the answer is $4^4 - 3^4 = 175$.

b) We "contain the sequence" to mean that those letters must be consecutive. We compute the number of sequences that *do* contain CUG. There are two cases: these are the first three bases, or these are the last three. In either case, there are four ways to fill in the remaining base, so there are $4 + 4 = 8$ such sequences. Therefore there are $4^4 - 8 = 248$ sequences that do not.

c) We first count the number of sequences that *do* contain all four bases. All that matters is the order of the bases. There are four choices for which base to put first, three choices for which base to put second, then two choices for the third position, and one choice for the last position. By the product rule, the answer to this subproblem is $4 \cdot 3 \cdot 2 \cdot 1 = 24$. Therefore the answer to the given question is $4^4 - 24 = 232$.

d) We need to choose two the two bases to be included, and this can be done in six ways (AU, AC, AG, UC, UG, CG). Once we make this choice (say AU), we can count the sequences that use those two bases. There are $2^4 = 16$ ways to choose which base goes at which position (two choices at each of the four positions), but two of these in fact use only one base, so there are 14 sequences that use exactly those two bases. Therefore the answer is $6 \cdot 14 = 84$.

45. We can use the division rule here; see Example 20. If we imagine the places at the table as fixed, then there are $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 6!$ ways to seat the six people—choose one of the six people for seat #1, choose one of the five remaining people for seat #2, and so on. However, this overcounts by a factor of 6, corresponding to rotating the table (which does not change a person's neighbors), and by a factor of 2, because any given arrangement is the "same" under the stated rules as the arrangement that reverses the order of the people (changing clockwise to counterclockwise). Therefore the answer is $6!/(6 \cdot 2) = 60$. An alternative way to look at it is as follows. Without loss of generality, have person A take a seat. Choose his two neighbors in one of $5 \cdot 4/2 = 10$ ways and have them sit on either side of him. Now place the other three people in one of the 3! ways possible in the remaining seats. The number of ways, then, is $10 \cdot 6 = 60$.

47. a) Here is a good way (but certainly not the only way) to approach this problem. Since the bride and groom must stand next to each other, let us treat them as one unit. Then the question asks for the number of ways to arrange five units in a row (the bride-and-groom unit and the four other people). We can think of filling five positions one at a time, so by the product rule there are $5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 120$ ways to make these choices. This is not quite the answer, however, since there are also two ways to decide on which side of the groom the bride will stand. Therefore the final answer is $120 \cdot 2 = 240$.

b) There are clearly $6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 720$ arrangements in all. We just determined in part **(a)** that 240 of them involve the bride standing next to the groom. Therefore there are $720 - 240 = 480$ ways to arrange the people with the bride not standing next to the groom.

c) Of the 720 arrangements of these people (see part **(b)**), exactly half must have the bride somewhere to the left of the groom. (We are invoking **symmetry** here—a useful tool for solving some combinatorial problems.) Therefore the answer is $720/2 = 360$.

49. There are 2^7 bit strings of length 10 that begin with three 0's, since each of the remaining seven bits has two possible values. Similarly, there are 2^8 bit strings of length 10 that end with two 0's. Furthermore, there are 2^5 bit strings of length 10 that both begin with three 0's and end with two 0's, since each of the five "middle" bits can be specified in two ways. Using the principle of inclusion–exclusion, we conclude that there are $2^7 + 2^8 - 2^5 = 352$ such strings. The idea behind this principle here is that the strings that both begin with three 0's and end with two 0's were counted twice when we added 2^7 and 2^8, so we need to subtract for the overcounting. It is definitely *not* the case that we are subtracting because we do not want to count such strings at all.

51. First let us count the number of 8-bit strings that contain three consecutive 0's. We will organize our count by looking at the leftmost bit that contains a 0 followed by at least two more 0's. If this is the first bit, then the second and third bits are determined (namely, they are both 0), but bits 4 through 8 are free to be specified, so there are $2^5 = 32$ such strings. If it is the second, third, or fourth bits, then the bit preceding it must be a 1 and the two bits after it must be 0's, but the other four bits are free. Therefore there are $2^4 = 16$ such strings in each of these three cases, or 48 in all. Next let us suppose that the substring of three or more 0's starts at bit 5. Then bit 4 must be a 1. Bits 1 through 3 can be anything other than three 0's (if it were three 0's, then we already counted this string); thus there are $2^3 - 1 = 7$ ways to specify them. Bit 8 is free. Therefore there are $7 \cdot 2 = 14$ such strings. Finally, suppose that the substring 000 starts in bit 6, so that bit 5 is a 1. There are $2^4 = 16$ possibilities for the first four bits, but three of them contain three consecutive 0's (0000, 0001, and 1000); therefore there are only 13 such strings. Adding up all the cases we have discussed, we obtain the final answer of $32 + 48 + 14 + 13 = 107$ for the number of 8-bit strings that contain three consecutive 0's.

Next we need to compute the number of 8-bit strings that contain four consecutive 1's. The analysis is very similar to what we have just done. There are $2^4 = 16$ such strings starting with 1111; there are $2^3 = 8$ starting 01111; there are $2^3 = 8$ starting $x01111$ (where x is either 0 or 1); and there are similarly 8 starting each of $xy01111$ and $xyz01111$. This gives us a total of $16 + 4 \cdot 8 = 48$ strings containing four consecutive 1's.

Finally, we need to count the number of strings that contain both three consecutive 0's and four consecutive 1's. It is easy enough to just list them all: 00001111, 00011111, 00011110, 10001111, 11110000, 11111000, 01111000, and 11110001, eight in all. Now applying the principle of inclusion–exclusion to what we have calculated above, we obtain the answer to the entire problem: $107 + 48 - 8 = 147$. There are only 256 bit strings of length eight altogether, so this answer is somewhat surprising, in that more than half of them satisfy the stated condition.

53. We can solve this problem by computing the number of positive integers not exceeding 100 that are divisible by 4, the number of them divisible by 6, and the number divisible by both 4 and 6; and then applying the principle of inclusion–exclusion. There are clearly $100/4 = 25$ multiples of 4 in this range, since every fourth number is divisible by 4. The number of integers in this range divisible by 6 is $\lfloor 100/6 \rfloor = 16$; we needed to round down because the multiples of 6 occur at the end of each consecutive block of 6 integers (6, 12, 18, etc.). Furthermore a number is divisible by both 4 and 6 if and only if it is divisible by their least common multiple, namely 12. Therefore there are $\lfloor 100/12 \rfloor = 8$ numbers divisible by both 4 and 6 in this range. Finally, applying inclusion–exclusion, we see that the number of positive integers not exceeding 100 that are divisible either by 4 or by 6 is $25 + 16 - 8 = 33$.

55. a) We are told that there are $26 + 26 + 10 + 6 = 68$ available characters. A password of length k using these characters can be formed in 68^k ways. Therefore the number of passwords with the specified length restriction is $68^8 + 68^9 + 68^{10} + 68^{11} + 68^{12} = 9{,}920{,}671{,}339{,}261{,}325{,}541{,}376$, which is about 9.9×10^{21} or about ten sextillion.

b) For a password *not* to contain one of the special characters, it must be constructed from the other 62 characters. There are $62^8 + 62^9 + 62^{10} + 62^{11} + 62^{12} = 3{,}279{,}156{,}377{,}874{,}257{,}103{,}616$ of these. Thus there are $6{,}641{,}514{,}961{,}387{,}068{,}437{,}760 \approx 6.6 \times 10^{21}$ (about seven sextillion) passwords that do contain at least one occurrence of one of the special symbols.

c) Assuming no restrictions, it will take one nanosecond (one billionth of a second, or 10^{-9} sec) for each password. We just multiply this by our answer in part **(a)** to find the number of seconds the hacker will require. We can convert this to years by dividing by $60 \cdot 60 \cdot 24 \cdot 365.2425$ (the average number of seconds in a year). It will take about 314,374 years.

57. Suppose the name has length k, where $1 \le k \le 65535$. There are $26 + 26 + 1 + 1 + 10 = 64$ choices for each character, except that the first character cannot be a digit, so there are only 54 choices for the first character.

By the product rule there are $54 \cdot 64^{k-1}$ choices for such a string. To get the final answer, we need to sum this over all lengths, for a total of

$$\sum_{k=1}^{65535} 54 \cdot 64^{k-1}\,.$$

Applying the formula for the sum of a geometric series gives $54(64^{65536}-1)/63$, which is about 5.5×10^{118369}. A programmer will never run out of variable names!

59. By the result of Example 8, there are $C = 6{,}400{,}000{,}000$ possible numbers of the form *NXX-NXX-XXXX*. To determine the number of different telephone numbers worldwide, then, we need to determine how many country codes there are and multiply by C. There are clearly 10 country codes of length 1, 100 country codes of length 2, and 1000 country codes of length 3. Thus there are $10+100+1000 = 1110$ country codes in all, by the sum rule. Our final answer is $1110 \cdot C = 7{,}104{,}000{,}000{,}000$.

61. There are 16 hexadecimal digits. We just apply the sum rule and the product rule to obtain the answer $16^{10} + 16^{26} + 16^{58} \approx 6.9 \times 10^{69}$. Almost all of this comes from the last term (16^{58}).

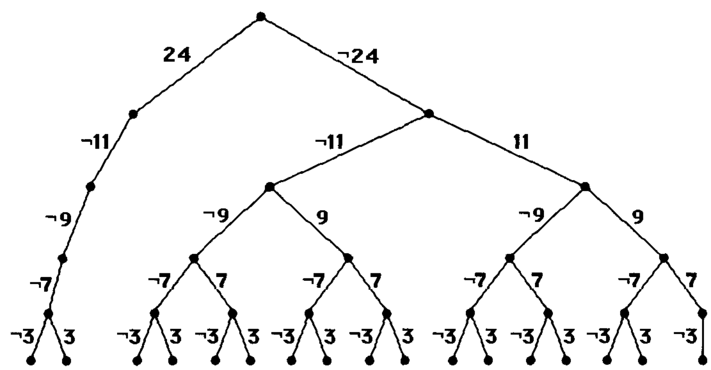
63. Every fourth number is divisible by 4, and every sixth number is divisible by 6. Therefore $\lfloor 999{,}999/4 \rfloor = 249{,}999$ numbers in the given range are divisible by 4, and $\lfloor 999{,}999/6 \rfloor = 166{,}666$ numbers in the given range are divisible by 6. We want to exclude these from the 999,999 numbers we have. However, numbers can be divisible by both 4 and 6; namely those divisible by 12, the least common multiple of 4 and 6. There are $\lfloor 999{,}999/12 \rfloor = 83{,}333$ such numbers. Using the principle of inclusion–exclusion, we must add this number back in to avoid overcounting. Therefore our answer is $999{,}999 - 249{,}999 - 166{,}000 + 83{,}000 = 666{,}667$.

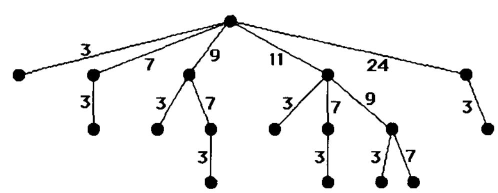
65. We assume that what is intended is that each of the 4 letters is to be used exactly once. There are at least two ways to do this problem. First let us break it into two cases, depending on whether the a comes at the end of the string or not. If a is not at the end, then there are 3 places to put it. After we have placed the a, there are only 2 places to put the b, since it cannot go into the position occupied by the a and it cannot go into the position following the a. Then there are 2 positions in which the c can go, and only 1 position for the d. Therefore, there are by the product rule $3 \cdot 2 \cdot 2 \cdot 1 = 12$ allowable strings in which the a does not come last. Second, if the a comes last, then there are $3 \cdot 2 \cdot 1 = 6$ ways to arrange the letters b, c, and d in the first three positions. The answer, by the sum rule, is therefore $12 + 6 = 18$.

Here is another approach. Ignore for a moment the restriction that a b cannot follow an a. Then we need to choose the letter that comes first (which can be done in 4 ways), then the letter that comes second (which can be done in 3 ways, since one letter has already been used), then the letter that comes third (which can be done in 2 ways, since two of the letters have already been used), and finally the letter that comes last (which can only be done in 1 way, since there is only one unused letter at that point). Therefore there are, by the product rule, $4 \cdot 3 \cdot 2 \cdot 1 = 24$ such strings. Now we need to subtract from this total the number of strings in which the a is immediately followed by the b. To count these, let us imagine the a and b glued together into one superletter, ab. (This **gluing** technique often comes in handy.) Now there are 3 things to arrange. We can choose any of them (the letters c or d or the superletter ab) to come first, and that can be done in 3 ways. We can choose either of the other two to come second (which can be done in 2 ways), and we are forced to choose the remaining one to come third. By the product rule there are $3 \cdot 2 \cdot 1 = 6$ ways to make these choices. Therefore our final answer is $24 - 6 = 18$.

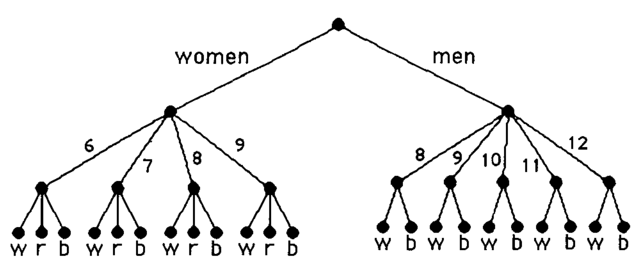
67. There are at least two approaches that are effective here. In our first tree, we let each branching point represent a decision as to whether to include the next element in the set (starting with the largest element). At the top of the tree, for example, we can either choose to include 24 or to exclude it (denoted $\neg 24$). We branch one way for each possibility. In the first figure below, the entire subtree to the right represents those sets that do not include 24, and the subtree to the left represents those that do. At the point below and to the left of the 24, we have only one branch, $\neg 11$, since after we have included 24 in our set, we cannot include 11, because the sum would not be less than 28 if we did. At the point below and to the right of the $\neg 24$, however, we again branch twice, since we can choose either to include 11 or to exclude it. To answer the question, we look at the points in the last row of the tree. Each represents a set whose sum is less than 28. For example, the sixth point from the right represents the set $\{11, 3\}$. Since there are 17 such points, the answer to the problem is 17.

Our other solution is more compact. In the tree below we show branches from a point only for the inclusion of new numbers in the set. The set formed by including no more numbers is represented by the point itself. This time *every* point represents a set. For example, the point at the top represents the empty set, the point below and to the right of the number 11 represents the set $\{11\}$, and the left-most point on the bottom row represents the set $\{3, 7, 9\}$. In general the set that a point represents is the set of numbers found on the path up the tree from that point to the root of the tree (the point at the top). We only included branches when the sum would be less than 28. Since there are 17 points altogether in this figure, the answer to the problem is 17.





69. a) The tree shown here enumerates the possible outcomes. First we branch on gender, then on size, and finally on color. There are 22 ends, so the answer to the question is 22.



b) First we apply the sum rule: the number of shoes is the sum of the number of men's shoes and the number of women's shoes. Next we apply the product rule. For a woman's shoe we need to specify size (4 choices) and then for each choice of size, we need to specify color (3 choices). Therefore there are $4 \cdot 3 = 12$ possible women's models. Similarly, there are $5 \cdot 2 = 10$ men's models. Therefore the answer is $12 + 10 = 22$.

71. We want to prove $P(m)$, the sum rule for m tasks, which says that if tasks T_1, T_2, ..., T_m can be done in n_1, n_2, ..., n_m ways, respectively, and no two of them can be done at the same time, then there are $n_1 + n_2 + \cdots + n_m$ ways to do one of the tasks. The basis step of our proof by mathematical induction is $m = 2$, and that has already been given. Now assume that $P(m)$ is true, and we want to prove $P(m+1)$. There are $m+1$ tasks, no two of which can be done at the same time; we want to do one of them. Either we choose one from among the first m, or we choose the task T_{m+1}. By the sum rule for two tasks, the number of ways we can do this is $n + n_{m+1}$, where n is the number of ways we can do one of the tasks among the first m. But by the inductive hypothesis $n = n_1 + n_2 + \cdots + n_m$. Therefore the number of ways we can do one of the $m+1$ tasks is $n_1 + n_2 + \cdots + n_m + n_{m+1}$, as desired.

73. A diagonal joins two vertices of the polygon, but they must be vertices that are not already joined by a side of the polygon. Thus there are $n-3$ diagonals emanating from each vertex of the polygon (we've excluded two of the $n-1$ other vertices as possible targets for diagonals). If we multiply $n-3$ by n, the number of vertices, we will have counted each diagonal exactly twice—once for each endpoint. We compensate for this overcounting by dividing by 2. Therefore there are $n(n-3)/2$ diagonals. (Note that the convexity of the polygon had nothing to do with the problem—we were counting the diagonals, whether or not we could be sure that they all lay inside the polygon.)

SECTION 6.2 The Pigeonhole Principle

The pigeonhole principle seems so trivial that it is difficult to realize how powerful it is in solving some mathematical problems. As usual with combinatorial problems, the trick is to look at things the right way, which usually means coming up with the clever insight, perhaps after hours of agonizing and frustrating exploration with a problem.

Try to solve these problems by invoking the pigeonhole principle explicitly, even if you can see other ways of doing them; you will gain some insights by formulating the problem and your solution in this way. The trick, of course, is to figure out what should be the pigeons and what should be the pigeonholes. Many of the the hints of Section 6.1 apply here, as well as general problem-solving techniques, especially the willingness to play with a problem for a long time before giving up.

Many of the elegant applications are quite subtle and difficult, and there are even more subtle and difficult applications not touched on here. Not every problem, of course, fits into the model of one of the examples in the text. In particular, Exercise 42 looks deceptively like a problem amenable to the technique discussed in Example 10. Keep in mind that the process of grappling with problems such as these is worthwhile and educational in itself, even if you never find the solution.

1. There are six classes: these are the pigeons. There are five days on which classes may meet (Monday through Friday): these are the pigeonholes. Each class must meet on a day (each pigeon must occupy a pigeonhole). By the pigeonhole principle at least one day must contain at least two classes.

3. **a)** There are two colors: these are the pigeonholes. We want to know the least number of pigeons needed to insure that at least one of the pigeonholes contains two pigeons. By the pigeonhole principle the answer is 3. If three socks are taken from the drawer, at least two must have the same color. On the other hand two socks are not enough, because one might be brown and the other black. Note that the number of socks was irrelevant (assuming that it was at least 3).

b) He needs to take out 14 socks in order to insure at least two black socks. If he does so, then at most 12 of them are brown, so at least two are black. On the other hand, if he removes 13 or fewer socks, then 12 of them could be brown, and he might not get his pair of black socks. This time the number of socks did matter.

5. There are four possible remainders when an integer is divided by 4 (these are the pigeonholes here): 0, 1, 2, or 3. Therefore, by the pigeonhole principle at least two of the five given remainders (these are the pigeons) must the same.

7. Let the n consecutive integers be denoted $x+1$, $x+2$, ..., $x+n$, where x is some integer. We want to show that exactly one of these is divisible by n. There are n possible remainders when an integer is divided by n, namely 0, 1, 2, ..., $n-1$. There are two possibilities for the remainders of our collection of n numbers: either they cover all the possible remainders (in which case exactly one of our numbers has a remainder of 0 and is therefore divisible by n), or they do not. If they do not, then by the pigeonhole principle, since there are then fewer than n pigeonholes (remainders) for n pigeons (the numbers in our collection), at least one remainder must occur twice. In other words, it must be the case that $x+i$ and $x+j$ have the same remainder when divided by n for some pair of numbers i and j with $0 < i < j \leq n$. Since $x+i$ and $x+j$ have the same remainder when divided by n, if we subtract $x+i$ from $x+j$, then we will get a number divisible by n. This means that $j-i$ is divisible by n. But this is impossible, since $j-i$ is a positive integer strictly less than n. Therefore the first possibility must hold, that exactly one of the numbers in our collection is divisible by n.

9. The generalized pigeonhole principle applies here. The pigeons are the students (no slur intended), and the pigeonholes are the states, 50 in number. By the generalized pigeonhole principle if we want there to be at least 100 pigeons in at least one of the pigeonholes, then we need to have a total of N pigeons so that $\lceil N/50 \rceil \geq 100$. This will be the case as long as $N \geq 99 \cdot 50 + 1 = 4951$. Therefore we need at least 4951 students to guarantee that at least 100 come from a single state.

11. We must recall from analytic geometry that the midpoint of the segment whose endpoints are (a,b,c) and (d,e,f) is $((a+d)/2, (b+e)/2, (c+f)/2)$. We are concerned only with integer values of the original coordinates. Clearly the coordinates of these fractions will be integers as well if and only if a and d have the same parity (both odd or both even), b and e have the same parity, and c and f have the same parity. Thus what matters in this problem is the parities of the coordinates. There are eight possible triples of parities: (odd, odd, odd), (odd, odd, even), (odd, even, odd), ..., (even, even, even). Since we are given nine points, the pigeonhole principle guarantees that at least two of them will have the same triple of parities. The midpoint of the segment joining these two points will therefore have integer coordinates.

13. **a)** We can group the first eight positive integers into four subsets of two integers each, each subset adding up to 9: $\{1,8\}$, $\{2,7\}$, $\{3,6\}$, and $\{4,5\}$. If we select five integers from this set, then by the pigeonhole principle (at least) two of them must come from the same subset. These two integers have a sum of 9, as desired.
b) No. If we select one element from each of the subsets specified in part **(a)**, then no sum will be 9. For example, we can select 1, 2, 3, and 4.

15. We can apply the pigeonhole principle by grouping the numbers cleverly into pairs (subsets) that add up to 7, namely $\{1,6\}$, $\{2,5\}$, and $\{3,4\}$. If we select four numbers from the set $\{1,2,3,4,5,6\}$, then at least two of them must fall within the same subset, since there are only three subsets. Two numbers in the same subset are the desired pair that add up to 7. We also need to point out that choosing three numbers is not enough, since we could choose $\{1,2,3\}$, and no pair of them add up to more than 5.

17. The given information tells us that there are $50 \cdot 85 \cdot 5 = 21{,}250$ bins. If we had this many products, then each could be stored in a separate bin. By the pigeonhole principle, however, if there are at least 21,251 products, then at least two of them must be stored in the same bin. This is the number the problem is asking us for.

19. **a)** If this statement were not true, then there would be at most 8 from each class standing, for a total of at most 24 students. This contradicts the fact that there are 25 students in the class.

b) If this statement were not true, then there would be at most 2 freshmen, at most 18 sophomores, and at most 4 juniors, for a total of at most 24 students. This contradicts the fact that there are 25 students in the class.

21. One way to do this is to have the sequence contain four groups of four numbers each, so that the numbers within each group are decreasing, and so that the numbers between groups are increasing. For example, we could take the sequence to be 4, 3, 2, 1; 8, 7, 6, 5; 12, 11, 10, 9; 16, 15, 14, 13. There can be no increasing subsequence of five terms, because any increasing subsequence can have only one element from each of the four groups. There can be no decreasing subsequence of five terms, because any decreasing subsequence cannot have elements from more than one group.

23. The key here is that 25 is an odd number. If there were an even number of boys and the same even number of girls, then we could position them around the table in the order BBGGBBGG... and never have a person both of whose neighbors are boys. But with an odd number of each sex, that cannot happen. Here is one nice way to see why when there are 25 of each. Number the seats around the table from 1 to 50, and think of seat 50 as being adjacent to seat 1. There are 25 seats with odd numbers and 25 seats with even numbers. If no more than 12 boys occupied the odd-numbered seats, then at least 13 boys would occupy the even-numbered seats, and vice versa. Without loss of generality, assume that at least 13 boys occupy the 25 odd-numbered seats. Then at least two of those boys must be in consecutive odd-numbered seats. The person sitting between those two boys will have boys as both of his or her neighbors.

25. This is actually a fairly hard problem, in terms of what we need to keep track of. Call the given sequence $a_1, a_2, \ldots, a_n$. We will keep track of the lengths of long increasing or decreasing subsequences by assigning values i_k and d_k for each k from 1 to n, indicating the length of the longest increasing subsequence ending with a_k and the length of the longest decreasing subsequence ending with a_k, respectively. Thus $i_1 = d_1 = 1$, since a_1 is both an increasing and a decreasing subsequence of length 1. If $a_2 < a_1$, then $i_2 = 1$ and $d_2 = 2$, since the longest increasing subsequence ending at a_2 is just a_2, but the longest decreasing subsequence ending at a_2 is a_1, a_2. If $a_2 > a_1$, then it is the other way around: $i_2 = 2$ and $d_2 = 1$. In general, we can determine i_k in the following manner (the determination of d_k is similar, with the roles of i and d, and the roles of $<$ and $>$, reversed). We look at the numbers $a_1, a_2, \ldots, a_{k-1}$. For each a_j that is less than a_k, we know that the value of i_k is at least $i_j + 1$, since the increasing subsequence of length i_j ending at a_j can be extended by following it by a_k, resulting in an increasing subsequence of length $i_j + 1$, ending at a_k. Furthermore, there is no other way of producing an increasing subsequence ending at a_k, other than the subsequence of length 1. Thus we set i_k equal to either 1 or the maximum of the numbers $i_j + 1$ for those values of $j < k$ for which $a_j < a_k$. Finally, once we have determined all the values i_k and d_k, we choose the largest of these $2n$ numbers as our answer.

The procedure just described, however, does not keep track of what the longest subsequence is, so we need to use two more sets of variables, $iprev_k$ and $dprev_k$. These will point back to the terms in the sequence that caused the values of i_k and d_k to be what they are. To retrieve the longest increasing or decreasing subsequence, once we know which type it is and where it ends, we follow these pointers, thereby exhibiting the subsequence backwards. We will not write the pseudocode for this final phase of the algorithm. (See the answer in the back of the textbook for an alternative procedure, which explicitly computes the sequence.)

```
procedure long subsequence(a_1, a_2, ..., a_n : distinct integers)
for k := 1 to n
        i_k := 1; d_k := 1
        iprev_k := k; dprev_k := k
        for j := 1 to k - 1
                if a_j < a_k and i_j + 1 > i_k then
                        i_k := i_j + 1
                        iprev_k := j
                if a_j > a_k and d_j + 1 > d_k then
                        d_k := d_j + 1
                        dprev_k := j
{at this point correct values of i_k and d_k have all been assigned}
longest := 1
for k := 2 to n
        if i_k > longest then longest := i_k
        if d_k > longest then longest := d_k
{ longest is the length of the longest increasing or decreasing subsequence}
```

27. We can prove these statements using both the result and the method of Example 13. First note that the role of "mutual friend" and "mutual enemy" is symmetric, so it is really enough to prove one of these statements; the other will follow by interchanging the roles. So let us prove that in every group of 10 people, either there are 3 mutual friends or 4 mutual enemies. Consider one person; call this person A. Of the 9 other people, either there must be 6 enemies of A, or there must be 4 friends of A (if there were 5 or fewer enemies and 3 or fewer friends, that would only account for 8 people). We need to consider the two cases separately. First suppose that A has 6 enemies. Apply the result of Example 13 to these 6 people: among them either there are 3 mutual friends or there are 3 mutual enemies. If there are 3 mutual friends, then we are done. If there are 3 mutual enemies, then these 3 people, together with A, form a group of 4 mutual enemies, and again we are done. That finishes the first case. The second case was that A had 4 friends. If some pair of these people are friends, then they, together with A, form the desired group of 3 mutual friends. Otherwise, these 4 people are the desired group of 4 mutual enemies. Thus in either case we have found either 3 mutual friends or 4 mutual enemies.

29. We need to show two things: that if we have a group of n people, then among them we must find either a pair of friends or a subset of n of them all of whom are mutual enemies; and that there exists a group of $n-1$ people for which this is not possible. For the first statement, if there is any pair of friends, then the condition is satisfied, and if not, then every pair of people are enemies, so the second condition is satisfied. For the second statement, if we have a group of $n-1$ people all of whom are enemies of each other, then there is neither a pair of friends nor a subset of n of them all of whom are mutual enemies.

31. First we need to figure out how many distinct combinations of initials and birthdays there are. The product rule tells us that since there are 26 ways to choose each of the 3 initials and 366 ways to choose the birthday, there are $26 \cdot 26 \cdot 26 \cdot 366 = 6{,}432{,}816$ such combinations. By the generalized pigeonhole principle, with these combinations as the pigeonholes and the 37 million people as the pigeons, there must be at least $\lceil 37{,}000{,}000/6{,}432{,}816 \rceil = 6$ people with the same combination.

33. The numbers from 1 to 200,000 are the pigeonholes, and the inhabitants of Paris are the pigeons, which number at least 800,001. Therefore by Theorem 1 there are at least two Parisians with the same number of hairs on their heads; and by Theorem 2 there are at least $\lceil 800{,}001/200{,}000 \rceil = 5$ Parisians with the same number of hairs on their heads.

35. The 38 time periods are the pigeonholes, and the 677 classes are the pigeons. By the generalized pigeonhole principle there is some time period in which at least $\lceil 677/38 \rceil = 18$ classes are meeting. Since each class must meet in a different room, we need 18 rooms.

37. Let c_i be the number of computers that the i^{th} computer is connected to. Each of these integers is between 0 and 5, inclusive. The pigeonhole principle does not allow us to conclude immediately that two of these numbers must be the same, since there are six numbers (pigeons) and six possible values (pigeonholes). However, if not all of the values are used, then the pigeonhole principle would allow us to draw the desired conclusion. Let us therefore show that not all of the numbers can be used. The only way that the value 5 can appear as one of the c_i's is if one computer is connected to each of the others. In that case, the number 0 cannot appear, since no computer could be connected to none of the others. Thus not both 5 and 0 can appear in our list, and the above argument is valid.

39. This is similar to Example 9. Label the computers C_1 through C_{100}, and label the printers P_1 through P_{20}. If we connect C_k to P_k for $k = 1, 2, \ldots, 20$ and connect each of the computers C_{21} though C_{100} to *all* the printers, then we have used a total of $20 + 80 \cdot 20 = 1620$ cables. Clearly this is sufficient, because if computers C_1 through C_{20} need printers, then they can use the printers with the same subscripts, and if any computers with higher subscripts need a printer instead of one or more of these, then they can use the printers that are not being used, since they are connected to all the printers. Now we must show that 1619 cables are not enough. Since there are 1619 cables and 20 printers, the average number of computers per printer is $1619/20$, which is less than 81. Therefore some printer must be connected to fewer than 81 computers (the average of a set of numbers cannot be bigger than each of the numbers in the set). That means it is connected to 80 or fewer computers, so there are at least 20 computers that are not connected to it. If those 20 computers all needed a printer simultaneously, then they would be out of luck, since they are connected to at most the 19 other printers.

41. This problem is similar to Example 10, so we follow the method of solution suggested there. Let a_j be the number of matches held during or before the j^{th} hour. Then a_1, a_2, $\ldots$, a_{75} is an increasing sequence of distinct positive integers, since there was at least one match held every hour. Furthermore $1 \leq a_j \leq 125$, since there were only 125 matches altogether. Moreover, $a_1 + 24$, $a_2 + 24$, $\ldots$, $a_{75} + 24$ is also an increasing sequence of distinct positive integers, with $25 \leq a_i + 24 \leq 149$.

Now the 150 positive integers a_1, a_2, $\ldots$, a_{75}, $a_1 + 24$, $a_2 + 24$, $\ldots$, $a_{75} + 24$ are all less than or equal to 149. Hence by the pigeonhole principle two of these integers are equal. Since the integers a_1, a_2, $\ldots$, a_{75} are all distinct, and the integers $a_1 + 24$, $a_2 + 24$, $\ldots$, $a_{75} + 24$ are all distinct, there must be distinct indices i and j such that $a_j = a_i + 24$. This means that exactly 24 matches were held from the beginning of hour $i + 1$ to the end of hour j, precisely the occurrence we wanted to find.

43. This is exactly a restatement of the generalized pigeonhole principle. The pigeonholes are the elements in the codomain (the elements of the set T), and the pigeons are the elements of the domain (the elements of the set S). To say that a pigeon s is in pigeonhole t is just to say that $f(s) = t$. The elements s_1, s_2, $\ldots$, s_m are just the m pigeons guaranteed by the generalized pigeonhole principle to occupy a common pigeonhole.

45. Let d_j be $jx - N(jx)$, where $N(jx)$ is the integer closest to jx, for $1 \leq j \leq n$. We want to show that $|d_j| < 1/n$ for some j. Note that each d_j is an irrational number strictly between $-1/2$ and $1/2$ (since jx is irrational and every irrational number is closer than $1/2$ to the nearest integer). The proof is slightly messier if n is odd, so let us assume that n is even. Consider the n intervals

$$\left(0, \frac{1}{n}\right), \left(\frac{1}{n}, \frac{2}{n}\right), \ldots, \left(\frac{(n/2)-1}{n}, \frac{1}{2}\right), \left(-\frac{1}{n}, 0\right), \left(-\frac{2}{n}, -\frac{1}{n}\right), \ldots, \left(-\frac{1}{2}, -\frac{(n/2)-1}{n}\right).$$

The intervals are the pigeonholes and the d_j's are the pigeons. If the interval $(0, 1/n)$ or $(-1/n, 0)$ is occupied, then we are done, since the d_j in that interval tells us which j makes $|d_j| < 1/n$. If not, then there are n pigeons for at most $n-2$ pigeonholes, so by the pigeonhole principle there is some interval, say $\big((k-1)/n, k/n\big)$, with two pigeons in it, say d_r and d_s, with $r < s$. Now we will consider $sx - rx$ and show that it is within $1/n$ of its nearest integer; that will complete our proof, since $sx - rx = (s-r)x$, and $s-r$ is a positive integer less than n.

We can write $rx = N(rx) + d_r$ and $sx = N(sx) + d_s$, where $(k-1)/n \le d_r < k/n$ and $(k-1)/n \le d_s < k/n$. Subtracting, we have that $sx - rx = [N(sx) - N(rx)] + [d_s - d_r]$. Now the quantity in the first pair of square brackets is an integer. Furthermore the quantity in the second pair of square brackets is the difference of two numbers in the interval $\big((k-1)/n, k/n\big)$ and hence has absolute value less than $1/n$ (the extreme case would be when one of them is very close to $(k-1)/n$ and the other is very close to k/n). Therefore, by definition of "closest integer" $sx - rx$ is at most a distance $1/n$ from its closest integer, i.e., $|(sx - rx) - N(sx - rx)| < 1/n$, as desired. (The case in which n is odd is similar, but we need to extend our intervals slightly past $\pm 1/2$, using $n+1$ intervals rather than n. This is okay, since when we subtract 2 from $n+1$ we still have more pigeons than pigeonholes.)

47. **a)** Assuming that each $i_k \le n$, there are only n pigeonholes (namely $1, 2, \ldots, n$) for the n^2+1 numbers $i_1, i_2, \ldots, i_{n^2+1}$. Hence, by the generalized pigeonhole principle at least $\lceil (n^2+1)/n \rceil = n+1$ of the numbers are in the same pigeonhole, i.e., equal.
b) If $a_{k_j} < a_{k_{j+1}}$, then the subsequence consisting of a_{k_j} followed by a maximal increasing subsequence of length $i_{k_{j+1}}$ starting at $a_{k_{j+1}}$ contradicts the fact that $i_{k_j} = i_{k_{j+1}}$. Hence $a_{k_j} > a_{k_{j+1}}$.
c) If there is no increasing subsequence of length greater than n, then parts **(a)** and **(b)** apply. Therefore we have $a_{k_{n+1}} > a_{k_n} > \cdots > a_{k_2} > a_{k_1}$, a decreasing subsequence of length $n+1$.

SECTION 6.3 Permutations and Combinations

In this section we look at counting problems more systematically than in Section 6.1. We have some formulae that apply in many instances, and the trick is to recognize the instances. If an ordered arrangement without repetitions is asked for, then the formula for permutations usually applies; if an unordered selection without repetition is asked for, then the formula for combinations usually applies. Of course the product rule and the sum rule (and common sense and cleverness) are still needed to solve some of these problems—having formulae for permutations and combinations definitely does not reduce the solving of counting problems to a mechanical algorithm.

Again the general comments of Section 6.1 apply. Try to solve problems more than one way and come up with the same answer—you will learn from the process of looking at the same problem from two or more different angles, and you will be (almost) sure that your answer is correct.

1. Permutations are ordered arrangements. Thus we need to list all the ordered arrangements of all 3 of these letters. There are 6 such: a, b, c; a, c, b; b, a, c; b, c, a; c, a, b; and c, b, a. Note that we have listed them in alphabetical order. Algorithms for generating permutations and combinations are discussed in Section 6.6.

3. If we want the permutation to end with a, then we may as well forget about the a, and just count the number of permutations of $\{b, c, d, e, f, g\}$. Each permutation of these 6 letters, followed by a, will be a permutation of the desired type, and conversely. Therefore the answer is $P(6,6) = 6! = 720$.

5. We simply plug into the formula $P(n,r) = n(n-1)(n-2)\cdots(n-r+1)$, given in Theorem 1. Note that there are r terms in this product, starting with n. This is the same as $P(n,r) = n!/(n-r)!$, but the latter formula is not as nice for computation, since it ignores the fact that each of the factors in the denominator cancels one factor in the numerator. Thus to compute $n!$ and $(n-r)!$ and then to divide is to do a lot of extra arithmetic. Of course if the denominator is 1, then there is no extra work, so we note that $P(n,n) = P(n,n-1) = n!$.

a) $P(6,3) = 6\cdot 5\cdot 4 = 120$ **b)** $P(6,5) = 6! = 720$ **c)** $P(8,1) = 8$

d) $P(8,5) = 8\cdot 7\cdot 6\cdot 5\cdot 4 = 6720$ **e)** $P(8,8) = 8! = 40{,}320$ **f)** $P(10,9) = 10! = 3{,}628{,}800$

7. This is $P(9,5) = 9\cdot 8\cdot 7\cdot 6\cdot 5 = 15{,}120$ by Theorem 1.

9. We need to pick 3 horses from the 12 horses in the race, and we need to arrange them in order (first, second, and third), in order to specify the win, place, and show. Thus there are $P(12,3) = 12\cdot 11\cdot 10 = 1320$ possibilities.

11. a) To specify a bit string of length 10 that contains exactly four 1's, we simply need to choose the four positions that contain the 1's. There are $C(10,4) = 210$ ways to do that.

b) To contain at most four 1's means to contain four 1's, three 1's, two 1's, one 1, or no 1's. Reasoning as in part **(a)**, we see that there are $C(10,4)+C(10,3)+C(10,2)+C(10,1)+C(10,0) = 210+120+45+10+1 = 386$ such strings.

c) To contain at least four 1's means to contain four 1's, five 1's, six 1's, seven 1's, eight 1's, nine 1's, or ten 1's. Reasoning as in part **(b)**, we see that there are $C(10,4)+C(10,5)+C(10,6)+C(10,7)+C(10,8)+C(10,9)+C(10,10) = 210+252+210+120+45+10+1 = 848$ such strings. A simpler approach would be to figure out the number of ways not to have at least four 1's (i.e., to have three 1's, two 1's, one 1, or no 1's) and then subtract that from 2^{10}, the total number of bit strings of length 10. This way we get $1024-(120+45+10+1) = 848$, fortunately the same answer as before. Solving a combinatorial problem in more than one way is a useful check on the correctness of the answer.

d) To have an equal number of 0's and 1's in this case means to have five 1's. Therefore the answer is $C(10,5) = 252$. Incidentally, this gives us another way to do part **(b)**. If we don't have an equal number of 0's and 1's, then we have either at most four 1's or at least six 1's. By symmetry, having at most four 1's occurs in half of these cases. Therefore the answer to part **(b)** is $(2^{10}-C(10,5))/2 = 386$, as above.

13. We assume that the row has a distinguished head. Consider the order in which the men appear relative to each other. There are n men, and all of the $P(n,n) = n!$ arrangements is allowed. Similarly, there are $n!$ arrangements in which the women can appear. Now the men and women must alternate, and there are the same number of men and women; therefore there are exactly two possibilities: either the row starts with a man and ends with a woman ($MWMW\ldots MW$) or else it starts with a woman and ends with a man ($WMWM\ldots WM$). We have three tasks to perform, then: arrange the men among themselves, arrange the women among themselves, and decide which sex starts the row. By the product rule there are $n!\cdot n!\cdot 2 = 2(n!)^2$ ways in which this can be done.

15. We assume that a combination is called for, not a permutation, since we are told to *select a set*, not *form an arrangement*. We need to choose 5 things from 26, so there are $C(26,5) = 26\cdot 25\cdot 24\cdot 23\cdot 22/5! = 65{,}780$ ways to do so.

17. We know that there are 2^{100} subsets of a set with 100 elements. All of them have more than two elements except the empty set, the 100 subsets consisting of one element each, and the $C(100,2) = 4950$ subsets with two elements. Therefore the answer is $2^{100} - 5051 \approx 1.3\times 10^{30}$.

19. **a)** Each flip can be either heads or tails, so there are $2^{10} = 1024$ possible outcomes.

b) To specify an outcome that has exactly two heads, we simply need to choose the two flips that came up heads. There are $C(10,2) = 45$ such outcomes.

c) To contain at most three tails means to contain three tails, two tails, one tail, or no tails. Reasoning as in part **(b)**, we see that there are $C(10,3) + C(10,2) + C(10,1) + C(10,0) = 120 + 45 + 10 + 1 = 176$ such outcomes.

d) To have an equal number of heads and tails in this case means to have five heads. Therefore the answer is $C(10,5) = 252$.

21. **a)** If BCD is to be a substring, then we can think of that block of letters as one superletter, and the problem is to count permutations of five items—the letters A, E, F, and G, and the superletter BCD. Therefore the answer is $P(5,5) = 5! = 120$.

b) Reasoning as in part **(a)**, we see that the answer is $P(4,4) = 4! = 24$.

c) As in part **(a)**, we glue BA into one item and glue GF into one item. Therefore we need to permute five items, and there are $P(5,5) = 5! = 120$ ways to do it.

d) This is similar to part **(c)**. Glue ABC into one item and glue DE into one item, producing four items, so the answer is $P(4,4) = 4! = 24$.

e) If both ABC and CDE are substrings, then $ABCDE$ has to be a substring. So we are really just permuting three items: $ABCDE$, F, and G. Therefore the answer is $P(3,3) = 3! = 6$.

f) There are no permutations with both of these substrings, since B cannot be followed by both A and E at the same time.

23. First position the men relative to each other. Since there are eight men, there are $P(8,8)$ ways to do this. This creates nine slots where a woman (but not more than one woman) may stand: in front of the first man, between the first and second men, $\ldots$, between the seventh and eight men, and behind the eighth man. We need to choose five of these positions, in order, for the first through fifth woman to occupy (order matters, because the women are distinct people). This can be done is $P(9,5)$ ways. Therefore the answer is $P(8,8) \cdot P(9,5) = 8! \cdot 9!/4! = 609{,}638{,}400$.

25. **a)** Since the prizes are different, we want an ordered arrangement of four numbers from the set of the first 100 positive integers. Thus there are $P(100,4) = 94{,}109{,}400$ ways to award the prizes.

b) If the grand prize winner is specified, then we need to choose an ordered set of three tickets to win the other three prizes. This can be done is $P(99,3) = 941{,}094$ ways.

c) We can first determine which prize the person holding ticket 47 will win (this can be done in 4 ways), and then we can determine the winners of the other three prizes, exactly as in part **(b)**. Therefore the answer is $4P(99,3) = 3{,}764{,}376$.

d) This is the same calculation as in part **(a)**, except that there are only 99 viable tickets. Therefore the answer is $P(99,4) = 90{,}345{,}024$. Note that this answer plus the answer to part **(c)** equals the answer to part **(a)**, since the person holding ticket 47 either wins a prize or does not win a prize.

e) This is similar to part **(c)**. There are $4 \cdot 3 = 12$ ways to determine which prizes these two lucky people will win, after which there are $P(98,2) = 9506$ ways to award the other two prizes. Therefore the answer is $12 \cdot 9506 = 114{,}072$.

f) This is like part **(e)**. There are $P(4,3) = 24$ ways to choose the prizes for the three people mentioned, and then 97 ways to choose the other winner. This gives $24 \cdot 97 = 2328$ ways in all.

g) Here it is just a matter of ordering the prizes for these four people, so the answer is $P(4,4) = 24$.

h) This is similar to part **(d)**, except that this time the pool of viable numbers has only 96 numbers in it. Therefore the answer is $P(96,4) = 79{,}727{,}040$.

i) There are four ways to determine the grand prize winner under these conditions. Then there are $P(99,3)$ ways to award the remaining prizes. This gives an answer of $4P(99,3) = 3{,}764{,}376$.

j) First we need to choose the prizes for the holder of 19 and 47. Since there are four prizes, there are $P(4,2) = 12$ ways to do this. Then there are 96 people who might win the remaining prizes, and there are $P(96,2) = 9120$ ways to award these prizes. Therefore the answer is $12 \cdot 9120 = 109{,}440$.

27. **a)** Since the order of choosing the members is not relevant (the offices are not differentiated), we need to use a combination. The answer is clearly $C(25,4) = 12{,}650$.

b) In contrast, here we need a permutation, since the order matters (we choose first a president, then a vice president, then a secretary, then a treasurer). The answer is clearly $P(25,4) = 303{,}600$.

29. **a)** In this part the permutation $5,6,32,7$, for example, is to be counted, since it contains the consecutive numbers 5, 6, and 7 in their correct order (even though separated by the 32). In order to specify such a 4-permutation, we first need to choose the 3 consecutive integers. They can be anything from $\{1,2,3\}$ to $\{98,99,100\}$; thus there are 98 possibilities. Next we need to decide which slot is to contain a number not in this set; there are 4 possibilities. Finally, we need to decide which of the 97 other positive integers not exceeding 100 is to fill this slot, and there are of course 97 choices. Thus our first attempt at an answer gives us, by the product rule, $98 \cdot 4 \cdot 97$.

Unfortunately, this answer is not correct, because we have counted some 4-permutations more than once. Consider the 4-permutation $4,5,6,7$, for example. We cannot tell whether it arose from choosing 4, 5, and 6 as the consecutive numbers, or from choosing 5, 6, and 7. (These are the only two ways it could have arisen.) In fact, every 4-permutation consisting of 4 consecutive numbers, in order, has been double counted. Therefore to correct our count, we need to subtract the number of such 4-permutations. Clearly there are 97 of them (they can begin with any number from 1 to 97). Further thought shows that every other 4-permutation in our collection arises in a unique way (in other words, there is a unique subsequence of three consecutive integers). Thus our final answer is $98 \cdot 4 \cdot 97 - 97 = 37{,}927$.

b) In this part we are insisting that the consecutive numbers be consecutive in the 4-permutation as well. The analysis in part **(a)** works here, except that there are only 2 places to put the fourth number—in slot 1 or in slot 4. Therefore the answer is $98 \cdot 2 \cdot 97 - 97 = 18{,}915$.

31. We need to be careful here, because strings can have repeated letters.

a) We need to choose the position for the vowel, and this can be done in 6 ways. Next we need to choose the vowel to use, and this can be done in 5 ways. Each of the other five positions in the string can contain any of the 21 consonants, so there are 21^5 ways to fill the rest of the string. Therefore the answer is $6 \cdot 5 \cdot 21^5 = 122{,}523{,}030$.

b) We need to choose the position for the vowels, and this can be done in $C(6,2) = 15$ ways (we need to choose two positions out of six). We need to choose the two vowels (5^2 ways). Each of the other four positions in the string can contain any of the 21 consonants, so there are 21^4 ways to fill the rest of the string. Therefore the answer is $15 \cdot 5^2 \cdot 21^4 = 72{,}930{,}375$.

c) The best way to do this is to count the number of strings with no vowels and subtract this from the total number of strings. We obtain $26^6 - 21^6 = 223{,}149{,}655$.

d) As in part **(c)**, we will do this by subtracting from the total number of strings, the number of strings with no vowels and the number of strings with one vowel (this latter quantity having been computed in part **(a)**). We obtain $26^6 - 21^6 - 6 \cdot 5 \cdot 21^5 = 223149655 - 122523030 = 100{,}626{,}625$.

33. We are told that we must select three of the 10 men and three of the 15 women. This can be done is $C(10,3)C(15,3) = 54{,}600$ ways.

35. To implement the condition that every 0 be immediately followed by a 1, let us think of "gluing" a 1 to the right of each 0. Then the objects we have to work with are eight blocks consisting of the string 01 and two 1's. The question is, then, how many strings are there consisting of these ten objects? This is easy to calculate, for we simply have to choose two of the "positions" in the string to contain the 1's and fill the remaining "positions" with the 01 blocks. Therefore the answer is $C(10,2) = 45$.

37. Perhaps the most straightforward way to do this is to look at the several cases. The string might contain three 1's and seven 0's, four 1's and six 0's, five of each, six 1's and four 0's, or seven 1's and three 0's. In each case we can determine the number of strings by calculating a binomial coefficient, since we simply need to choose the positions for the 1's. Therefore the answer is $C(10,3)+C(10,4)+C(10,5)+C(10,6)+C(10,7) = 120 + 210 + 252 + 210 + 120 = 912$.

39. To specify such a license plate we need to write down a 3-permutation of the set of 26 letters and follow it by a 3-permutation of the set of 10 digits. By the product rule the answer is therefore $P(26,3) \cdot P(10,3) = 26 \cdot 25 \cdot 24 \cdot 10 \cdot 9 \cdot 8 = 11{,}232{,}000$.

41. Designate the head of the table and seat the people clockwise. Clearly there are $P(n,r)$ ways to do this. Because rotations of the table do not make for a "different" seating, this overcounts by a factor of r. Therefore the answer is $P(n,r)/r = n!/(r(n-r)!)$.

43. If there are no ties, then there are $3! = 6$ possible finishes. If two of the horses tie and the third has a different time, then there are 3 ways to decide which horse is not tied and then 2 ways to decide whether that horse finishes first or last. That gives $3 \cdot 2 = 6$ possibilities. Finally, all three horses can tie. So the answer is $6 + 6 + 1 = 13$.

45. We can solve this problem by breaking it down into cases depending on the ties. There are four basic cases. (1) If there are unique gold and silver winners, then we can choose these winners in $6 \cdot 5 = 30$ ways. Any nonempty subset of the remaining four runners can win the bronze medal. There are $2^4 - 1 = 15$ ways to choose these people, giving us $30 \cdot 15 = 450$ ways in all for this case. (2) If there is a 2-way tie for first place, then there are $C(6,2) = 15$ ways to choose the gold medalists. Any nonempty subset of the remaining four runners can win the bronze medal, so there are $2^4 - 1 = 15$ ways to choose these people, giving us $15 \cdot 15 = 225$ ways in all for this case. (3) If there is a k-way tie for first with $k \geq 3$, then there are $C(6,k)$ ways to choose the gold medalists (there are no other medals in this case). This gives us $C(6,3) + C(6,4) + C(6,5) + C(6,6) = 20 + 15 + 6 + 1 = 42$ more possibilities. (4) The only other case is that there is a single gold medal winner and a k-way tie for second with $k \geq 2$. We can choose the winner in 6 ways and the silver medalists in $2^5 - C(5,1) - C(5,0) = 32 - 5 - 1 = 26$ ways. This gives us $6 \cdot 26 = 156$ possibilities. Putting this all together, the answer is $450 + 225 + 42 + 156 = 873$.

SECTION 6.4 Binomial Coefficients and Identities

In this section we usually write the binomial coefficients using the $\binom{n}{r}$ notation, rather than the $C(n,r)$ notation. These numbers tend to come up in many parts of discrete mathematics.

1. **a)** When $(x+y)^4 = (x+y)(x+y)(x+y)(x+y)$ is expanded, all products of a term in the first sum, a term in the second sum, a term in the third sum, and a term in the fourth sum are added. Terms of the form x^4, x^3y, x^2y^2, xy^3 and y^4 arise. To obtain a term of the form x^4, an x must be chosen in each of the sums, and this can be done in only one way. Thus, the x^4 term in the product has a coefficient of 1. (We can think of this coefficient as $\binom{4}{4}$.) To obtain a term of the form x^3y, an x must be chosen in three of the four sums (and consequently a y in the other sum). Hence, the number of such terms is the number of 3-combinations of four objects, namely $\binom{4}{3} = 4$. Similarly, the number of terms of the form x^2y^2 is the number of ways to pick two of the four sums to obtain x's (and consequently take a y from each of the other two factors). This can be done in $\binom{4}{2} = 6$ ways. By the same reasoning there are $\binom{4}{1} = 4$ ways to obtain the xy^3 terms, and only one way (which we can think of as $\binom{4}{0}$) to obtain a y^4 term. Consequently, the product is $x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$.

b) This is explained in Example 2. The expansion is $\binom{4}{0}x^4 + \binom{4}{1}x^3y + \binom{4}{2}x^2y^2 + \binom{4}{3}xy^3 + \binom{4}{4}y^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4$. Note that it does not matter whether we think of the bottom of the binomial coefficient expression as corresponding to the exponent on x, as we did in part **(a)**, or the exponent on y, as we do here.

3. The coefficients are the binomial coefficients $\binom{6}{i}$, as i runs from 0 to 6, namely 1, 6, 15, 20, 15, 6, 1. Therefore $(x+y)^6 = \sum_{i=0}^{6} \binom{6}{i} x^{6-i}y^i = x^6 + 6x^5y + 15x^4y^2 + 20x^3y^3 + 15x^2y^4 + 6xy^5 + y^6$.

5. There is one term for each i from 0 to 100, so there are 101 terms.

7. By the binomial theorem the term involving x^9 in the expansion of $(2+(-x))^{19}$ is $\binom{19}{9}2^{10}(-x)^9$. Therefore the coefficient is $\binom{19}{9}2^{10}(-1)^9 = -2^{10}\binom{19}{9} = -94{,}595{,}072$.

9. Using the binomial theorem, we see that the term involving x^{101} in the expansion of $((2x)+(-3y))^{200}$ is $\binom{200}{99}(2x)^{101}(-3y)^{99}$. Therefore the coefficient is $\binom{200}{99}2^{101}(-3)^{99} = -2^{101}3^{99}C(200,99)$.

11. Let us apply the binomial theorem to the given binomial:

$$\begin{aligned}(x^2 - x^{-1})^{100} &= \sum_{j=0}^{100} \binom{100}{j} (x^2)^{100-j}(-x^{-1})^j \\ &= \sum_{j=0}^{100} \binom{100}{j} (-1)^j x^{200-2j-j} = \sum_{j=0}^{100} \binom{100}{j} (-1)^j x^{200-3j}\end{aligned}$$

Thus the only nonzero coefficients are those of the form $200-3j$ where j is an integer between 0 and 100, inclusive, namely 200, 197, 194, ..., 2, −1, −4, ..., −100. If we denote $200-3j$ by k, then we have $j = (200-k)/3$. This gives us our answer. The coefficient of x^k is zero for k not in the list just given (namely those vales of k between -100 and 200, inclusive, that are congruent to 2 modulo 3), and for those values of k in the list, the coefficient is $(-1)^{(200-k)/3}\binom{100}{(200-k)/3}$.

13. We are asked simply to display these binomial coefficients. Each can be computed from the formula in Theorem 2 in Section 6.3. Alternatively, we can apply Pascal's identity to the last row of Figure 1(b), adding successive numbers in that row to produce the desired row. We thus obtain

$$1 \quad 9 \quad 36 \quad 84 \quad 126 \quad 126 \quad 84 \quad 36 \quad 9 \quad 1.$$

15. There are many ways to see why this is true. By Corollary 1 the sum of *all* the positive numbers $\binom{n}{k}$, as k runs from 0 to n, is 2^n, so certainly each one of them is no bigger than this sum. Another way to see this is to note that $\binom{n}{k}$ counts the number of subsets of an n-set having k elements, and 2^n counts even more—the number of subsets of an n-set with no restriction as to size; so certainly the former is smaller than the latter.

17. We know that

$$\binom{n}{k} = \frac{n(n-1)(n-2)\cdots(n-k+1)}{k(k-1)(k-2)\cdots 2}.$$

Now if we make the numerator of the right-hand side larger by raising each factor up to n, and make the denominator smaller by lowering each factor to 2, then we have certainly not decreased the value, so the left-hand side is less than or equal to this altered expression. But the result is precisely $n^k/2^{k-1}$, as desired.

19. Using the formula (Theorem 2 in Section 6.3) we have

$$\begin{aligned}\binom{n}{k-1} + \binom{n}{k} &= \frac{n!}{(k-1)!\big(n-(k-1)\big)!} + \frac{n!}{k!(n-k)!} \\ &= \frac{n!k + n!(n-k+1)}{k!(n-k+1)!} \quad \text{(having found a common denominator)} \\ &= \frac{(n+1)n!}{k!\big((n+1)-k\big)!} = \frac{(n+1)!}{k!\big((n+1)-k\big)!} = \binom{n+1}{k}.\end{aligned}$$

21. **a)** We show that each side counts the number of ways to choose from a set with n elements a subset with k elements and a distinguished element of that set. For the left-hand side, first choose the k-set (this can be done in $\binom{n}{k}$ ways) and then choose one of the k elements in this subset to be the distinguished element (this can be done in k ways). For the right-hand side, first choose the distinguished element out of the entire n-set (this can be done in n ways), and then choose the remaining $k-1$ elements of the subset from the remaining $n-1$ elements of the set (this can be done in $\binom{n-1}{k-1}$ ways).

b) This is straightforward algebra:

$$k\binom{n}{k} = k \cdot \frac{n!}{k!(n-k)!} = \frac{n \cdot (n-1)!}{(k-1)!(n-k)!} = n\binom{n-1}{k-1}.$$

23. This identity can be proved algebraically or combinatorially. Algebraically, we compute as follows, starting with the right-hand side (we use twice the fact that $(x+1)x! = (x+1)!$):

$$\begin{aligned}\frac{(n+1)\binom{n}{k-1}}{k} &= \frac{(n+1)n!}{(k-1)!\big(n-(k-1)\big)!k} \\ &= \frac{(n+1)!}{k!\big(n-(k-1)\big)!} \\ &= \frac{(n+1)!}{k!\big((n+1)-k\big)!} \\ &= \binom{n+1}{k}\end{aligned}$$

For a combinatorial argument, we need to construct a situation in which both sides count the same thing. Suppose that we have a set of $n+1$ people, and we wish to choose k of them. Clearly there are $\binom{n+1}{k}$ ways to do this. On the other hand, we can choose our set of k people by first choosing one person to be in the set (there are $n+1$ choices), and then choosing $k-1$ additional people to be in the set, from the n people remaining. This can be done in $\binom{n}{k-1}$ ways. Therefore apparently there are $(n+1)\binom{n}{k-1}$ ways to choose the set of k people. However, we have overcounted: there are k ways that every such set can be chosen, since once we have the set, we realize that any of the k people could have been chosen first. Thus we have overcounted by a factor of k, and the real answer is $(n+1)\binom{n}{k-1}/k$ (we correct for the overcounting by dividing by k). Comparing our two approaches, one yielding the answer $\binom{n+1}{k}$, and the other yielding the answer $(n+1)\binom{n}{k-1}/k$, we conclude that $\binom{n+1}{k} = (n+1)\binom{n}{k-1}/k$.

Finally, we are asked to use this identity to give a recursive definition of the $\binom{n}{k}$'s. Note that this identity expresses $\binom{n}{k}$ in terms of $\binom{i}{j}$ for values of i and j less than n and k, respectively (namely $i = n-1$ and $j = k-1$). Thus the identity will be the recursive part of the definition. We need the base cases to handle $n = 0$ or $k = 0$. Our full definition becomes

$$\binom{n}{k} = \begin{cases} 1 & \text{if } k = 0 \\ 0 & \text{if } k > 0 \text{ and } n = 0 \\ n\binom{n-1}{k-1}/k & \text{if } n > 0 \text{ and } k > 0. \end{cases}$$

Actually, if we assume (as we usually do) that $k \le n$, then we do not need the second line of the definition. Note that $\binom{n}{k} = 0$ for $n < k$ under the definition given here, which is consistent with the combinatorial definition, since there are no ways to choose k different elements from a set with fewer than k elements.

25. We use Pascal's identity twice (Theorem 2 of this section) and Corollary 1 of the previous section:

$$\binom{2n}{n+1} + \binom{2n}{n} = \binom{2n+1}{n+1} = \frac{1}{2}\left(\binom{2n+1}{n+1} + \binom{2n+1}{n+1}\right)$$
$$= \frac{1}{2}\left(\binom{2n+1}{n+1} + \binom{2n+1}{n}\right) = \frac{1}{2}\left(\binom{2n+2}{n+1}\right)$$

27. The reason this is called the "hockey stick identity" is that the binomial coefficients being summed lie along a line in Pascal's triangle, and the sum is the entry in the triangle that, together with that line, forms a shape vaguely resembling a hockey stick.

a) We need to find something to count so that the left-hand side of the equation counts it in one way and the right-hand side counts it in a different way. After much thought, we might try the following. We will count the number of bit strings of length $n+r+1$ containing exactly r 0's and $n+1$ 1's. There are $\binom{n+r+1}{r}$ such strings, since a string is completely specified by deciding which r of the bits are to be the 0's. To see that the left-hand side of the identity counts the same thing, let $l+1$ be the position of the last 1 in the string. Since there are $n+1$ 1's, we know that l cannot be less than n. Thus there are disjoint cases for each l from n to $n+r$. For each such l, we completely determine the string by deciding which of the l positions in the string before the last 1 are to be 0's. Since there are n 1's in this range, there are $l-n$ 0's. Thus there are $\binom{l}{l-n}$ ways to choose the positions of the 0's. Now by the sum rule the total number of bit strings will be $\sum_{l=n}^{n+r} \binom{l}{l-n}$. By making the change of variable $k = l - n$, this transforms into the left-hand side, and we are finished.

b) We need to prove this by induction on r; Pascal's identity will enter at the crucial step. We let $P(r)$ be the statement to be proved. The basis step is clear, since the equation reduces to $\binom{n}{0} = \binom{n+1}{0}$, which is the true proposition $1 = 1$. Assuming the inductive hypothesis, we derive $P(r+1)$ in the usual way:

$$\sum_{k=0}^{r+1}\binom{n+k}{k} = \left(\sum_{k=0}^{r}\binom{n+k}{k}\right) + \binom{n+r+1}{r+1}$$
$$= \binom{n+r+1}{r} + \binom{n+r+1}{r+1} \quad \text{(by the inductive hypothesis)}$$
$$= \binom{n+(r+1)+1}{r+1} \quad \text{(by Pascal's identity)}$$

29. We will follow the hint and count the number of ways to choose a committee with leader from a set of n people. Note that the size of the committee is not specified, although it clearly needs to have at least one person (its leader). On the one hand, we can choose the leader first, in any of n ways. We can then choose the rest of the committee, which can be any subset of the remaining $n-1$ people; this can be done in 2^{n-1} ways since there are this many subsets. Therefore the right-hand side of the proposed identity counts this.

On the other hand, we can organize our count by the size of the committee. Let k be the number of people who will serve on the committee. The number of ways to select a committee with k people is clearly $\binom{n}{k}$, and once we have chosen the committee, there are clearly k ways in which to choose its leader. By the sum rule the left-hand side of the proposed identity therefore also counts the number of such committees. Since the two sides count the same quantity, they must be equal.

31. Corollary 2 says that $\binom{n}{0} - \binom{n}{1} + \binom{n}{2} - \cdots \pm \binom{n}{n} = 0$. If we put all the negative terms on the other side, we obtain $\binom{n}{0} + \binom{n}{2} + \binom{n}{4} + \cdots = \binom{n}{1} + \binom{n}{3} + \cdots$ (one side ends at $\binom{n}{n}$ and the other side ends at $\binom{n}{n-1}$—which is which depends on whether n is even or odd). Now the left-hand side counts the number of subsets with even cardinality of a set with n elements, and the right-hand side counts the number of subsets with odd cardinality of the same set. That these two quantities are equal is precisely what we wanted to prove.

33. **a)** Clearly a path of the desired type must consist of m moves to the right and n moves up. Therefore each such path can be represented by a bit string consisting of m 0's and n 1's, with the 0's representing moves to the right and the 1's representing moves up. Note that the total length of this bit string is $m+n$.
b) We know from this section that the number of bit strings of length $m+n$ containing exactly n 1's is $\binom{m+n}{n}$, since one need only specify the positions of the 1's. Note that this is the same as $\binom{m+n}{m}$.

35. We saw in Exercise 33 that the number of paths of length n was the same as the number of bit strings of length n, which we know to be 2^n, the right-hand side of the identity. On the other hand, a path of length n must end up at some point the sum of whose coordinates is n, say at $(n-k,k)$ for some k from 0 to n. We saw in Exercise 33 that the number of paths ending up at $(n-k,k)$ was equal to $\binom{n-k+k}{k} = \binom{n}{k}$. Therefore the left-hand side of the identity counts the number of such paths, too.

37. The right-hand side of the identity we are asked to prove counts, by Exercise 33, the number of paths from $(0,0)$ to $(n+1,r)$. Now let us count these paths by breaking them down into $r+1$ cases, depending on how many steps upward they begin with. Let k be the number of steps upward they begin with before taking a step to the right. Then k can take any value from 0 to r. The number of paths from $(0,0)$ to $(n+1,r)$ that begin with exactly k steps upward before turning to the right is clearly the same as the number of paths from $(1,k)$ to $(n+1,r)$, since after these k upward steps and the move to the right we have reached $(1,k)$. This latter quantity is the same as the number of paths from $(0,0)$ to $(n+1-1, r-k) = (n, r-k)$, since we can relabel our diagram to make $(1,k)$ the origin. From Exercise 33, this latter quantity is $\binom{n+r-k}{r-k}$. Therefore the total number of paths is the desired sum

$$\sum_{k=0}^{r} \binom{n+r-k}{r-k} = \sum_{k=0}^{r} \binom{n+k}{k},$$

where the equality comes from changing the dummy variable from k to $r-k$. Since both sides count the same thing, they are equal.

39. **a)** This looks like the third negatively sloping diagonal of Pascal's triangle, starting with the leftmost entry in the second row and reading down and to the right. In other words, the n^{th} term of this sequence is $\binom{n+1}{n-1} = \binom{n+1}{2}$.
b) This looks like the fourth negatively sloping diagonal of Pascal's triangle, starting with the leftmost entry in the third row and reading down and to the right. In other words, the n^{th} term of this sequence is $\binom{n+2}{n-1} = \binom{n+2}{3}$.
c) These seem to be the entries reading straight down the middle of the Pascal's triangle. Only every other row has a middle element. The first entry in the sequence is $\binom{0}{0}$, the second is $\binom{2}{1}$, the third is $\binom{4}{2}$, the fourth is $\binom{6}{3}$, and so on. In general, then, the n^{th} term is $\binom{2n-2}{n-1}$.

d) These seem to be the entries reading down the middle of the Pascal's triangle. Only every other row has an exact middle element, but in the other rows, there are two elements sharing the middle. The first entry in the sequence is $\binom{0}{0}$, the second is $\binom{1}{0}$, the third is $\binom{2}{1}$, the fourth is $\binom{3}{1}$, the fifth is $\binom{4}{2}$, the fourth is $\binom{5}{2}$, and so on. In general, then, the n^{th} term is $\binom{n-1}{\lfloor (n-1)/2 \rfloor}$.

e) One pattern here is a wandering through Pascal's triangle according to the following rule. The n^{th} term is in the n^{th} row of the triangle. We increase the position in that row as we go down, but as soon as we are about to reach the middle of the row, we jump back to the start of the next row. For example, the fifth term is the first entry in row 5; the sixth term is the second entry in row 6; the seventh term is the third entry in row 7; the eighth term is the fourth entry in row 8. The next entry following that pattern would take us to the middle of the ninth row, so instead we jump back to the beginning, and the ninth term is the first entry of row 9. To come up with a formula here, we see that the n^{th} entry is $\binom{n-1}{k-1}$ for a particular k; let us determine k as a function of n. A little playing around with the pattern reveals that k is n minus the largest power of 2 less than n (where for this purpose we consider 0 to be the largest power of 2 less than 1). For example, the 14^{th} term has $k = 14 - 8 = 6$, so it is $\binom{13}{5} = 1287$.

f) The terms seem to come from every third row, so the n^{th} term is $\binom{3n-3}{k}$ for some k. A little observation indicates that in fact these terms are $\binom{0}{0}$, $\binom{3}{1}$, $\binom{6}{2}$, $\binom{9}{3}$, $\binom{12}{4}$, and so on. Thus the n^{th} term is $\binom{3n-3}{n-1}$.

SECTION 6.5 Generalized Permutations and Combinations

As in Section 6.3, we have formulae that give us the answers to some combinatorial problems, if we can figure out which formula applies to which problem, and in what way it applies. Here, even more than in previous sections, the ability to see a problem from the right perspective is the key to solving it. Expect to spend several minutes staring at each problem before any insight comes. Reread the examples in the section and try to imagine yourself going through the thought processes explained there. Gradually your mind will begin to think in the same terms. In particular, ask yourself what is being selected from what, whether ordered or unordered selections are to be made, and whether repetition is allowed. In most cases, after you have answered these questions, you can find the appropriate formula from Table 1.

1. Since order is important here, and since repetition is allowed, this is a simple application of the product rule. There are 3 ways in which the first element can be selected, 3 ways in which the second element can be selected, and so on, with finally 3 ways in which the fifth element can be selected, so there are $3^5 = 243$ ways in which the 5 elements can be selected. The general formula is that there are n^k ways to select k elements from a set of n elements, in order, with unlimited repetition allowed.

3. Since we are considering strings, clearly order matters. The choice for each position in the string is from the set of 26 letters. Therefore, using the same reasoning as in Exercise 1, we see that there are $26^6 = 308{,}915{,}776$ strings.

5. We assume that the jobs and the employees are distinguishable. For each job, we have to decide which employee gets that job. Thus there are 5 ways in which the first job can be assigned, 5 ways in which the second job can be assigned, and 5 ways in which the third job can be assigned. Therefore, by the multiplication principle (just as in Exercise 1) there are $5^3 = 125$ ways in which the assignments can be made. (Note that we do not require that every employee get at least one job.)

7. Since the selection is to be an unordered one, Theorem 2 applies. We want to choose $r = 3$ items from a set with $n = 5$ elements. Theorem 2 tells us that there are $C(5+3-1, 3) = C(7,3) = 7 \cdot 6 \cdot 5/(3 \cdot 2) = 35$ ways to do so. (Equivalently, this problem is asking us to count the number of nonnegative integer solutions to $x_1 + x_2 + x_3 + x_4 + x_5 = 3$, where x_i represents the number of times that the i^{th} element of the 5-element set gets selected.)

9. Let b_1, b_2, ..., b_8 be the number of bagels of the 8 types listed (in the order listed) that are selected. Order does not matter: we are presumably putting the bagels into a bag to take home, and the order in which we put them there is irrelevant.

a) If we want to choose 6 bagels, then we are asking for the number of nonnegative solutions to the equation $b_1 + b_2 + \cdots + b_8 = 6$. Theorem 2 applies, with $n = 8$ and $r = 6$, giving us the answer $C(8+6-1,6) = C(13,6) = 1716$.

b) This is the same as part **(a)**, except that $r = 12$ rather than 6. Thus there are $C(8+12-1,12) = C(19,12) = C(19,7) = 50{,}388$ ways to make the selection. (Note that $C(19,7)$ was easier to compute than $C(19,12)$, and since they are equal, we chose the latter form.)

c) This is the same as part **(a)**, except that $r = 24$ rather than 6. Thus there are $C(8+24-1,24) = C(31,24) = C(31,7) = 2{,}629{,}575$ ways to make the selection.

d) This one is more complicated. Here we want to solve the equation $b_1 + b_2 + \cdots + b_8 = 12$, subject to the constraint that each $b_i \geq 1$. We reduce this problem to the form in which Theorem 2 is applicable with the following trick. Let $b'_i = b_i - 1$; then b'_i represents the number of bagels of type i, in excess of the required 1, that are selected. If we substitute $b_i = b'_i + 1$ into the original equation, we obtain $(b'_1+1)+(b'_2+1)+\cdots+(b'_8+1) = 12$, which reduces to $b'_1+b'_2+\cdots+b'_8 = 4$. In other words, we are asking how many ways are there to choose the 4 extra bagels (in excess of the required 1 of each type) from among the 8 types, repetitions allowed. By Theorem 2 the number of solutions is $C(8+4-1,4) = C(11,4) = 330$.

e) This final part is even trickier. First let us ignore the restriction that there can be no more than 2 salty bagels (i.e., that $b_4 \leq 2$). We will take into account, however, the restriction that there must be at least 3 egg bagels (i.e., that $b_3 \geq 3$). Thus we want to count the number of solutions to the equation $b_1+b_2+\cdots+b_8 = 12$, subject to the condition that $b_i \geq 0$ for all i and $b_3 \geq 3$. As in part **(d)**, we use the trick of choosing the 3 egg bagels at the outset, leaving only 9 bagels free to be chosen; equivalently, we set $b'_3 = b_3 - 3$, to represent the extra egg bagels, above the required 3, that are chosen. Now Theorem 2 applies to the number of solutions of $b_1 + b_2 + b'_3 + b_4 + \cdots + b_8 = 9$, so there are $C(8+9-1,9) = C(16,9) = C(16,7) = 11{,}440$ ways to make this selection.

Next we need to worry about the restriction that $b_4 \leq 2$. We will impose this restriction by subtracting from our answer so far the number of ways to violate this restriction (while still obeying the restriction that $b_3 \geq 3$). The difference will be the desired answer. To violate the restriction means to have $b_4 \geq 3$. Thus we want to count the number of solutions to $b_1 + b_2 + \cdots + b_8 = 12$, with $b_3 \geq 3$ and $b_4 \geq 3$. Using the same technique as we have just used, this is equal to the number of nonnegative solutions to the equation $b_1 + b_2 + b'_3 + b'_4 + b_5 + \cdots + b_8 = 6$ (the 6 on the right being $12-3-3$). By Theorem 2 there are $C(8+6-1,6) = C(13,6) = 1716$ ways to make this selection. Therefore our final answer is $11440 - 1716 = 9724$.

11. This can be solved by common sense. Since the pennies are all identical and the nickels are all identical, all that matters is the number of each type of coin selected. We can select anywhere from 0 to 8 pennies (and the rest nickels); since there are nine numbers in this range, the answer is 9. (The number of pennies and nickels is irrelevant, as long as each is at least eight.) If we wanted to use a high-powered theorem for this problem, we could observe that Theorem 2 applies, with $n = 2$ (there are two types of coins) and $r = 8$. The formula gives $C(2+8-1,8) = C(9,8) = 9$.

13. Assuming that the warehouses are distinguishable, let w_i be the number of books stored in warehouse i. Then we are asked for the number of solutions to the equation $w_1 + w_2 + w_3 = 3000$. By Theorem 2 there are $C(3+3000-1,3000) = C(3002,3000) = C(3002,2) = 4{,}504{,}501$ of them.

15. **a)** Let $x_1 = x'_1 + 1$; thus x'_1 is the value that x_1 has in excess of its required 1. Then the problem asks for the number of nonnegative solutions to $x_1' + x_2 + x_3 + x_4 + x_5 = 20$. By Theorem 2 there are $C(5+20-1, 20) = C(24, 20) = C(24, 4) = 10{,}626$ of them.

b) Substitute $x_i = x_i' + 2$ into the equation for each i; thus x_i' is the value that x_i has in excess of its required 2. Then the problem asks for the number of nonnegative solutions to $x_1' + x_2' + x_3' + x_4' + x_5' = 11$. By Theorem 2 there are $C(5+11-1, 11) = C(15, 11) = C(15, 4) = 1365$ of them.

c) There are $C(5+21-1, 21) = C(25, 21) = C(25, 4) = 12650$ solutions with no restriction on x_1. The restriction on x_1 will be violated if $x_1 \geq 11$. Following the procedure in part **(a)**, we find that there are $C(5+10-1, 10) = C(14, 10) = C(14, 4) = 1001$ solutions in which the restriction is violated. Therefore there are $12650 - 1001 = 11{,}649$ solutions of the equation with its restriction.

d) First let us impose the restrictions that $x_3 \geq 15$ and $x_2 \geq 1$. Then the problem is equivalent to counting the number of solutions to $x_1 + x_2' + x_3' + x_4 + x_5 = 5$, subject to the constraints that $x_1 \leq 3$ and $x_2' \leq 2$ (the latter coming from the original restriction that $x_2 < 4$). Note that these two restrictions cannot be violated simultaneously. Thus if we count the number of solutions to $x_1 + x_2' + x_3' + x_4 + x_5 = 5$, subtract the number of its solutions in which $x_1 \geq 4$, and subtract the numbers of its solutions in which $x_2' \geq 3$, then we will have the answer. By Theorem 2 there are $C(5+5-1, 5) = C(9, 5) = 126$ solutions of the unrestricted equation. Applying the first restriction reduces the equation to $x_1' + x_2' + x_3' + x_4 + x_5 = 1$, which has $C(5+1-1, 1) = C(5, 1) = 5$ solutions. Applying the second restriction reduces the equation to $x_1 + x_2'' + x_3' + x_4 + x_5 = 2$, which has $C(5+2-1, 2) = C(6, 2) = 15$ solutions. Therefore the answer is $126 - 5 - 15 = 106$.

17. Theorem 3 applies here, with $n = 10$ and $k = 3$. The answer is therefore
$$\frac{10!}{2!3!5!} = 2520\,.$$

19. Theorem 3 applies here, with $n = 14$, $n_1 = n_2 = 3$ (the triplets), $n_3 = n_4 = n_5 = 2$ (the twins), and $n_6 = n_7 = 1$. The answer is therefore
$$\frac{14!}{3!3!2!2!2!1!1!} = 302{,}702{,}400\,.$$

21. If we think of the balls as doing the choosing, then this is asking for the number of ways to choose six bins from the nine given bins, with repetition allowed. (The number of times each bin is chosen is the number of balls in that bin.) By Theorem 2 with $n = 9$ and $r = 6$, this choice can be made in $C(9+6-1, 6) = C(14, 6) = 3003$ ways.

23. There are several ways to count this. We can first choose the two objects to go into box #1 ($C(12, 2)$ ways), then choose the two objects to go into box #2 ($C(10, 2)$ ways, since only 10 objects remain), then choose the two objects to go into box #3 ($C(8, 2)$ ways), and so on. So the answer is $C(12, 2) \cdot C(10, 2) \cdot C(8, 2) \cdot C(6, 2) \cdot C(4, 2) \cdot C(2, 2) = (12 \cdot 11/2)(10 \cdot 9/2)(8 \cdot 7/2)(6 \cdot 5/2)(4 \cdot 3/2)(2 \cdot 1/2) = 12!/2^6 = 7{,}484{,}400$. Alternatively, just line up the 12 objects in a row ($12!$ ways to do that), and put the first two into box #1, the next two into box #2, and so on. This overcounts by a factor of 2^6, since there are that many ways to swap objects in the permutation without affecting the result (swap the first and second objects or not, and swap the third and fourth objects or not, and so on). So this results in the same answer. Here is a third way to get this answer. First think of pairing the objects. Think of the objects as ordered (a first, a second, and so on). There are 11 ways to choose a mate for the first object, then 9 ways to choose a mate for the first unused object, then 7 ways to choose a mate for the first still unused object, and so on. This gives $11 \cdot 9 \cdot 7 \cdot 5 \cdot 3$ ways to do the pairing. Then there are $6!$ ways to choose the boxes for the pairs. So the answer is the product of these two quantities. which is again $7{,}484{,}400$.

25. Let d_1, d_2, ..., d_6 be the digits of a natural number less than 1,000,000; they can each be anything from 0 to 9 (in particular, we may as well assume that there are leading 0's if necessary to make the number exactly 6 digits long). If we want the sum of the digits to equal 19, then we are asking for the number of solutions to the equation $d_1 + d_2 + \cdots + d_6 = 19$ with $0 \le d_i \le 9$ for each i. Ignoring the upper bound restriction, there are, by Theorem 2, $C(6+19-1,19) = C(24,19) = C(24,5) = 42504$ of them. We must subtract the number of solutions in which the restriction is violated. If the digits are to add up to 19 and one or more of them is to exceed 9, then exactly one of them will have to exceed 9, since $10 + 10 > 19$. There are 6 ways to choose the digit that will exceed 9. Once we have made that choice (without loss of generality assume it is d_1 that is to be made greater than or equal to 10), then we count the number of solutions to the equation by counting the number of solutions to $d_1' + d_2 + \cdots + d_6 = 19 - 10 = 9$; by Theorem 2 there are $C(6+9-1,9) = C(14,9) = C(14,5) = 2002$ of them. Thus there are $6 \cdot 2002 = 12012$ solutions that violate the restriction. Subtracting this from the 42504 solutions altogether, we find that $42504 - 12012 = 30{,}492$ is the answer to the problem.

27. We assume that each problem is worth a whole number of points. Then we want to find the number of integer solutions to $x_1 + x_2 + \cdots + x_{10} = 100$, subject to the constraint that each $x_i \ge 5$. Letting x_i' be the number of points assigned to problem i in excess of its required 5, and substituting $x_i = x_i' + 5$ into the equation, we obtain the equivalent equation $x_1' + x_2' + \cdots + x_{10}' = 50$. By Theorem 2 the number of solutions is given by $C(10+50-1,50) = C(59,50) = C(59,9) = 12{,}565{,}671{,}261$.

29. There are at least two good ways to do this problem. First we present a solution in the spirit of this section. Let us place the 1's and some gaps in a row. A 1 will come first, followed by a gap, followed by another 1, another gap, a third 1, a third gap, a fourth 1, and a fourth gap. Into the gaps we must place the 12 0's that are in this string. Let g_1, g_2, g_3, and g_4 be the numbers of 0's placed in gaps 1 through 4, respectively. The only restriction is that each $g_i \ge 2$. Thus we want to count the number of solutions to the equation $g_1+g_2+g_3+g_4 = 12$, with $g_i \ge 2$ for each i. Letting $g_i = g_i'+2$, we want to count, equivalently, the number of nonnegative solutions to $g_1'+g_2'+g_3'+g_4' = 4$. By Theorem 2 there are $C(4+4-1,4) = C(7,4) = C(7,3) = 35$ solutions. Thus our answer is 35.

Here is another way to solve the problem. Since each 1 must be followed by two 0's, suppose we glue 00 to the right end of each 1. This uses up 8 of the 0's, leaving 4 unused 0's. Now we have 8 objects, namely 4 0's and 4 100's. We want to find the number of strings we can form with these 8 objects, starting with a 100. After placing the 100 first, there are 7 places left for objects, 3 of which have to be 100's. Clearly there are $C(7,3) = 35$ ways to choose the positions for the 100's, so our answer is 35.

31. This is a direct application of Theorem 3, with $n = 11$, $n_1 = 5$, $n_2 = 2$, $n_3 = n_4 = 1$, and $n_5 = 2$ (where n_1 represents the number of A's, etc.). Thus the answer is $11!/(5!2!1!1!2!) = 83{,}160$.

33. We need to use the sum rule at the outermost level here, adding the number of strings using each subset of letters. There are quite a few cases. First, there are 3 strings of length 1, namely O, R, and N. There are several strings of length 2. If the string uses no O's, then there are 2; if it uses 1 O, then there are 2 ways to choose the other letter, and 2 ways to permute the letters in the string, so there are 4; and of course there is just 1 string of length 2 using 2 O's. Strings of length 3 can use 1, 2, or 3 O's. A little thought shows that the number of such strings is $3! = 6$, $2 \cdot 3 = 6$, and 1, respectively. There are 3 possibilities of the choice of letters for strings of length 4. If we omit an O, then there are $4!/2! = 12$ strings; if we omit either of the other letters (2 ways to choose the letter), then there are 4 strings. Finally, there are $5!/3! = 20$ strings of length 5. This gives a total of $3+2+4+1+6+6+1+12+2\cdot 4+20 = 63$ strings using some or all of the letters.

35. We need to consider the three cases determined by the number of characters used in the string: 7, 8, or 9. If all nine letters are to be used, then Theorem 3 applies and we get

$$\frac{9!}{4!2!1!1!1!} = 7560$$

strings. If only eight letters are used, then we need to consider which letter is left out. In each of the cases in which the V, G, or N is omitted, Theorem 3 tells us that there are

$$\frac{8!}{4!2!1!1!} = 840$$

strings, for a total of 2520 for these cases. If an R is left out, then Theorem 3 tells us that there are

$$\frac{8!}{4!1!1!1!} = 1680$$

strings, and if an E is left out, then Theorem 3 tells us that there are

$$\frac{8!}{3!2!1!1!} = 3360$$

strings. This gives a total of $2520+1680+3360 = 7560$ strings of length 8. (It was not an accident that there are as many strings of length 8 as there are of length 9, since there is a one-to-one correspondence between these two sets, given by associating with a string of length 9 its first 8 characters.) For strings of length 7 there are even more cases. We tabulate them here:

omitting VG	$7!/(4!2!1!) = 105$ strings
omitting VN	$7!/(4!2!1!) = 105$ strings
omitting GN	$7!/(4!2!1!) = 105$ strings
omitting VR	$7!/(4!1!1!1!) = 210$ strings
omitting GR	$7!/(4!1!1!1!) = 210$ strings
omitting NR	$7!/(4!1!1!1!) = 210$ strings
omitting RR	$7!/(4!1!1!1!) = 210$ strings
omitting EV	$7!/(3!2!1!1!) = 420$ strings
omitting EG	$7!/(3!2!1!1!) = 420$ strings
omitting EN	$7!/(3!2!1!1!) = 420$ strings
omitting ER	$7!/(3!1!1!1!1!) = 840$ strings
omitting EE	$7!/(2!2!1!1!1!) = 1260$ strings

Adding up these numbers we see that there are 4515 strings of length 7. Thus the answer is $7560 + 7560 + 4515 = 19{,}635$.

37. We assume that all the fruit is to be eaten; in other words, this process ends after 7 days. This is a permutation problem since the order in which the fruit is consumed matters (indeed, there is nothing else that matters here). Theorem 3 applies, with $n = 7$, $n_1 = 3$, $n_2 = 2$, and $n_3 = 2$. The answer is therefore $7!/(3!2!2!) = 210$.

39. We can describe any such travel in a unique way by a sequence of 4 x's, 3 y's, and 5 z's. By Theorem 3 there are

$$\frac{12!}{4!3!5!} = 27720$$

such sequences.

41. This is like Example 8. If we approach it as is done there, we see that the answer is

$$C(52,7)C(45,7)C(38,7)C(31,7)C(24,7) = \frac{52!}{7!45!}\cdot\frac{45!}{7!38!}\cdot\frac{38!}{7!31!}\cdot\frac{31!}{7!24!}\cdot\frac{24!}{7!17!} = \frac{52!}{7!7!7!7!7!17!} \approx 7.0\times 10^{34}\,.$$

Applying Theorem 4 will yield the same answer; in this approach we think of the five players and the undealt cards as the six distinguishable boxes.

43. We assume that we are to care about which player gets which cards. For example, a deal in which Laurel gets a royal flush in spades and Blaine gets a royal flush in hearts will be counted as different from a deal in which Laurel gets a royal flush in hearts and Blaine gets a royal flush in spades (and the other four players get the same cards each time). The order in which a player receives his or her cards is not relevant, however, so we are dealing with combinations. We can look at one player at a time. There are $C(48,5)$ ways to choose the cards for the first player, then $C(43,5)$ ways to choose the cards for the second player (because five of the cards are gone), and so on. So the answer, by the multiplication principle, is $C(48,5)\cdot C(43,5)\cdot C(38,5)\cdot C(33,5)\cdot C(28,5)\cdot C(23,5) = 649{,}352{,}163{,}073{,}816{,}339{,}512{,}038{,}979{,}194{,}880 \approx 6.5\times 10^{32}$.

45. a) All that matters is how many copies of the book get placed on each shelf. Letting x_i be the number of copies of the book placed on shelf i, we are asking for the number of solutions to the equation $x_1+x_2+\cdots+x_k=n$, with each x_i a nonnegative integer. By Theorem 2 this is $C(k+n-1,n)$.
b) No generality is lost if we number the books b_1, b_2, $\ldots$, b_n and think of placing book b_1, then placing b_2, and so on. There are clearly k ways to place b_1, since we can put it as the first book (for now) on any of the shelves. After b_1 is placed, there are $k+1$ ways to place b_2, since it can go to the right of b_1 or it can be the first book on any of the shelves. We continue in this way: there are $k+2$ ways to place b_3 (to the right of b_1, to the right of b_2, or as the first book on some shelf), $k+3$ ways to place b_4, $\ldots$, $k+n-1$ ways to place b_n. Therefore the answer is the product of these numbers, which can more easily be expressed as $(k+n-1)!/(k-1)!$.

Another, perhaps easier, way to obtain this answer is to think of first choosing the locations for the books, which is what we counted in part **(a)**, and then choose a permutation of the n books to put into those locations (shelf by shelf, from the top down, and from left to right on each shelf). Thus the answer is $C(k+n-1,n)\cdot n!$, which evaluates to the same thing we obtained with our other analysis.

47. The first box holds n_1 objects, and there are $C(n,n_1)$ ways to choose those objects from among the n objects in the collection. Once these objects are chosen, we can choose the objects to be placed in the second box in $C(n-n_1,n_2)$ ways, since there are $n-n_1$ objects not yet placed, and we need to put n_2 of them into the second box. Similarly, there are then $C(n-n_1-n_2,n_3)$ ways to choose objects for the third box. We continue in this way, until finally there are $C(n-n_1-n_2-\cdots-n_{k-1},n_k)$ ways to choose the objects to put in the last (k^{th}) box. Note that this last expression equals $C(n_k,n_k)=1$, since $n_1+n_2+\cdots+n_k=n$. Now by the product rule the number of ways to make the entire assignment is

$$C(n,n_1)\cdot C(n-n_1,n_2)\cdot C(n-n_1-n_2,n_3)\cdots C(n-n_1-n_2-\cdots-n_{k-1},n_k)\,.$$

We use the formula for combinations to write this as

$$\frac{n!}{n_1!(n-n_1)!}\cdot\frac{(n-n_1)!}{n_2!(n-n_1-n_2)!}\cdot\frac{(n-n_1-n_2)!}{n_3!(n-n_1-n_2-n_3)!}\cdots\frac{(n-n_1-n_2-\cdots-n_{k-1})!}{n_k!(n-n_1-n_2-\cdots-n_{k-1}-n_k)!}\,,$$

which simplifies after the telescoping cancellation to

$$\frac{n!}{n_1!n_2!\cdots n_k!}$$

(we use the fact that $n-n_1-n_2-\cdots-n_{k-1}=n_k$, since $n_1+n_2+\cdots+n_k=n$), as desired.

49. a) The sequence was nondecreasing to begin with. By adding $k-1$ to the k^{th} term, we are adding more to each term than was added to the previous term. Hence even if two successive terms in the sequence were originally equal, the second term must strictly exceed the first after this addition is completed. Therefore the sequence is made up of distinct numbers. The smallest can be no smaller than $1+(1-1)=1$, and the largest can be no larger than $n+(r-1)=n+r-1$; therefore the terms all come from T.
b) If we are given an increasing sequence of r terms from T, then by subtracting $k-1$ from the k^{th} term we have a nondecreasing sequence of r terms from S, repetitions allowed. (The k^{th} term in the original sequence

must be between k and $n+r-1-(r-k)=n+(k-1)$, so subtracting $k-1$ leaves a number between 1 and n, inclusive. Furthermore, only 1 more is subtracted from a term than is subtracted from the previous term; thus no term can become strictly smaller than its predecessor, since it exceeded it by at least 1 to start with.) This operation exactly inverts the operation described in part **(a)**, so the correspondence is one-to-one.

c) The first two parts show that there are exactly as many r-combinations with repetition allowed from S as there are r-combinations (without repetition) from T. Since T has $n+r-1$ elements, this latter quantity is clearly $C(n+r-1,r)$.

51. We use the formula for the Stirling numbers of the second kind stated near the end of this section, which gives the number of ways to distribute n distinguishable objects into j indistinguishable boxes with no box empty:
$$S(n,j)=\frac{1}{j!}\sum_{i=0}^{j-1}(-1)^i\binom{j}{i}(j-i)^n$$
In this case, $n=6$ and $j=4$, so we have
$$S(6,4)=\frac{1}{4!}\left(\binom{4}{0}4^6-\binom{4}{1}3^6+\binom{4}{2}2^6-\binom{4}{3}1^6\right)=\frac{1}{4!}(4096-2916+384-4)=65\,.$$
If we want to work this out from scratch, we can argue as follows. There are two patterns possible. We can put three of the objects into one box and each of the remaining objects into a separate box; there are $C(6,3)=20$ ways to choose the objects for the crowded box. Alternatively, we can choose a pair of objects for one box ($C(6,2)=15$ ways) and a pair of remaining objects for the second box ($C(4,2)=6$ ways) and put the other two objects into separate boxes, but divide by 2 because of the overcounting caused by the indistinguishability of the first two boxes, for a total of 45 ways. Therefore the answer is $20+45=65$.

53. We assume that people are distinguishable, so this problem is identical to Exercise 51. There are 65 ways to place the employees.

55. Since each box has to contain at least one object, we might as well put one object into each box to begin with. This leaves us with just two more objects, and there are only two choices: we can put them both into the same box (so that the partition we end up with is $6=3+1+1+1$), or we can put them into different boxes (so that the partition we end up with is $6=2+2+1+1$). So the answer is 2.

57. Since each box has to contain at least two DVDs, we might as well put two DVDs into each box to begin with. This leaves us with just three more DVDs, and there are only three choices: we can put them all into the same box (so that the partition we end up with is $9=5+2+2$), or we can put two into one box and one into another (so that the partition we end up with is $9=4+3+2$), or we can put them all into different boxes (so that the partition we end up with is $9=3+3+3$). So the answer is 3.

59. To begin, notice that because each box must have at least one ball, there are only two basic arrangements: to put three balls into one box and one ball into each of the other two boxes (denoted 3-1-1), or to put one ball into one box and two balls into each of the other two boxes (denoted 1-2-2).

a) For the 3-1-1 arrangement, there are 3 ways to choose the crowded box, $C(5,3)=10$ ways to choose the balls to be put there, and 2 ways to decide where the other balls go, for a total of $3\cdot10\cdot2=60$ possibilities. For the 1-2-2 arrangement, there are 3 ways to choose the box that will have just one ball, 5 ways to choose which ball goes there, and $C(4,2)=6$ ways to decide which two balls go into the lower-numbered remaining box, for a total of $3\cdot5\cdot6=90$ possibilities. Thus the answer is $60+90=150$.

b) There are $C(5,3)=10$ ways to choose the balls for the crowded box in the 3-1-1 arrangement. For the 1-2-2 arrangement there are 5 ways to choose the lonely ball and 3 ways to choose the partner of the lowest-numbered remaining ball. Therefore the answer is $10+5\cdot3=25$.

c) There are 3 ways to choose the crowded box for the 3-1-1 arrangement, and there are 3 ways to choose the solo box for the 1-2-2 arrangement. Therefore the answer is $3 + 3 = 6$.

d) There are just the 2 possibilities we have been discussing: 3-1-1 and 1-2-2.

61. Without the restriction on site X, we are simply asking for the number of ways to order the ten symbols V, V, W, W, X, X, Y, Y, Z, Z (the ordering will give us the visiting schedule). By Theorem 3 this can be done in $10!/(2!)^5 = 113{,}400$ ways. If the inspector visits site X on consecutive days, then in effect we are ordering nine symbols (including only one X), where now the X means to visit site X twice in a row. There are $9!/(2!)^4 = 22{,}680$ ways to do this. Therefore the answer is $113{,}400 - 22{,}680 = 90{,}720$.

63. When $(x_1 + x_2 + \cdots + x_m)^n$ is expanded, each term will clearly be of the form $Cx_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$, for some constants C that depend on the exponents, where the exponents sum to n. Thus the form of the given formula is correct, and the only question is whether the constants are correct. We need to count the number of ways in which a product of one term from each of the n factors can be $x_1^{n_1}x_2^{n_2}\cdots x_m^{n_m}$. In order for this to happen, we must choose n_1 x_1's, n_2 x_2's, ..., n_m x_m's. By Theorem 3 this can be done in

$$C(n; n_1, n_2, \ldots, n_m) = \frac{n!}{n_1!n_2!\cdots n_m!}$$

ways.

65. By the multinomial theorem, given in Exercise 63, the coefficient is

$$C(10; 3, 2, 5) = \frac{10!}{3!2!5!} = \frac{10 \cdot 9 \cdot 8 \cdot 7 \cdot 6}{12} = 2520\,.$$

SECTION 6.6 Generating Permutations and Combinations

This section is quite different from the rest of this chapter. It is really about algorithms and programming. These algorithms are not easy, and it would be worthwhile to "play computer" with them to get a feeling for how they work. In constructing such algorithms yourself, try assuming that you will list the permutations or combinations in a nice order (such as lexicographic order); then figure out how to find the "next" one in this order.

1. Lexicographic order is the same as numerical order in this case, so the ordering from smallest to largest is 14532, 15432, 21345, 23451, 23514, 31452, 31542, 43521, 45213, 45321.

3. Our list will have $3 \cdot 3 \cdot 3 \cdot 2 = 54$ items in it. Here it is in lexicographic order: AAA1, AAA2, AAB1, AAB2, AAC1, AAC2, ABA1, ABA2. ABB1, ABB2, ABC1, ABC2, ACA1, ACA2, ACB1, ACB2, ACC1, ACC2, BAA1, BAA2, BAB1, BAB2, BAC1, BAC2, BBA1, BBA2, BBB1, BBB2, BBC1, BBC2, BCA1, BCA2, BCB1, BCB2, BCC1, BCC2, AAA1, CAA2, CAB1, CAB2, CAC1, CAC2, CBA1, CBA2, CBB1, CBB2, CBC1, CBC2, CCA1, CCA2, CCB1, CCB2, CCC1, CCC2.

5. We use Algorithm 1 to find the next permutation. Our notation follows that algorithm, with j being the largest subscript such that $a_j < a_{j+1}$ and k being the subscript of the smallest number to the right of a_j that is larger than a_j.

a) Since $4 > 3 > 2$, we know that the 1 is our a_j. The smallest integer to the right of 1 and greater than 1 is 2, so $k = 4$. We interchange a_j and a_k, giving the permutation 2431, and then we reverse the entire substring to the right of the position now occupied by the 2, giving the answer 2134.

b) The first integer from the right that is less than its right neighbor is the 2 in position 4. Therefore $j = 4$ here, and of course k has to be 5. The next permutation is the one that we get by interchanging the fourth and fifth numbers, 54132. (Note that the last phase of the algorithm, reversing the end of the string, was vacuous this time—there was only one element to the right of position 4, so no reversing was necessary.)

c) Since $5 > 3$, we know that the 4 is our a_j. The smallest integer to the right of 4 and greater than 4 is $a_k = 5$. We interchange a_j and a_k, giving the permutation 12543, and then we reverse the entire substring to the right of the position now occupied by the 5, giving the answer 12534.

d) Since $3 > 1$, we know that the 2 is our a_j. The smallest integer to the right of 2 and greater than 2 is $a_k = 3$. We interchange a_j and a_k, giving the permutation 45321, and then we reverse the entire substring to the right of the position now occupied by the 3, giving the answer 45312.

e) The first integer from the right that is less than its right neighbor is the 3 in position 6. Therefore $j = 6$ here, and of course k has to be 7. The next permutation is the one that we get by interchanging the sixth and seventh numbers, 6714253. As in part **(b)**, no reversing was necessary.

f) Since $8 > 7 > 6 > 4$, we know that the 2 is our a_j, so $j = 4$. The smallest integer to the right of 2 and greater than 2 is $a_8 = 4$. We interchange a_4 and a_8, giving the permutation 31548762, and then we reverse the entire substring to the right of the position now occupied by the 4, giving the answer 31542678.

7. We begin with the permutation 1234. Then we apply Algorithm 1 23 times in succession, giving us the other 23 permutations in lexicographic order: 1243, 1324, 1342, 1423, 1432, 2134, 2143, 2314, 2341, 2413, 2431, 3124, 3142, 3214, 3241, 3412, 3421, 4123, 4132, 4213, 4231, 4312, and 4321. The last permutation is the one entirely in decreasing order. Each application of Algorithm 1 follows the pattern in Exercise 5.

9. We begin with the first 3-combination, namely $\{1,2,3\}$. Let us trace through Algorithm 3 to find the next. Note that $n = 5$ and $r = 3$; also $a_1 = 1$, $a_2 = 2$, and $a_3 = 3$. We set i equal to 3 and then decrease i until $a_i \neq 5 - 3 + i$. This inequality is already satisfied for $i = 3$, since $a_3 \neq 5$. At this point we increment a_i by 1 (so that now $a_3 = 4$), and fill the remaining spaces with consecutive integers following a_i (in this case there are no more remaining spaces). Thus our second 3-combination is $\{1,2,4\}$. The next call to Algorithm 3 works the same way, producing the third 3-combination, namely $\{1,2,5\}$. To get the fourth 3-combination, we again call Algorithm 3. This time the i that we end up with is 2, since $5 = a_3 = 5 - 3 + 3$. Therefore the second element in the list is incremented, namely goes from a 2 to a 3, and the third element is the next larger element after 3, namely 4. Thus this 3-combination is $\{1,3,4\}$. Another call to the algorithm gives us $\{1,3,5\}$, and another call gives us $\{1,4,5\}$. Now when we call the algorithm, we find $i = 1$ at the end of the **while** loop, since in this case the last two elements are the two largest elements in the set. Thus a_1 is increased to 2, and the remainder of the list is filled with the next two consecutive integers, giving us $\{2,3,4\}$. Continuing in this manner, we get the rest of the 3-combinations: $\{2,3,5\}$, $\{2,4,5\}$, $\{3,4,5\}$.

11. Clearly the next larger r-combination must differ from the old one in position i, since positions $i+1$, $i+2$, $\ldots$, r are occupied by the largest possible numbers (namely $i+n-r+1$ to n). Also $a_i + 1$ is the smallest possible number that can be put in position i if we want an r-combination greater than the given one, and then similarly $a_i + 2$, $a_i + 3$, $\ldots$, $a_i + r - i + 1$ are the smallest allowable numbers for positions $i+1$ to r. Therefore there is no r-combination between the given one and the one that Algorithm 3 produces, which is exactly what we had to prove.

13. One way to do this problem (and to have done Exercise 12) is to generate the r-combinations using Algorithm 3, and then to find all the permutations of each, using Algorithm 1 (except that now the elements to be permuted are not the integers from 1 to r, but are instead the r elements of the r-combination currently being used). Thus we start with the first 3-combination, $\{1,2,3\}$, and we list all 6 of its permutations: 123, 132, 213, 231, 312, 321. Next we find the next 3-combination, namely $\{1,2,4\}$, and list all of its permutations: 124, 142, 214, 241, 412, 421. We continue in this manner to generate the remaining 48 3-permutations of $\{1,2,3,4,5\}$: 125, 152, 215, 251, 512, 521; 134, 143, 314, 341, 413, 431; 135, 153, 315, 351, 513, 531; 145, 154, 415, 451, 514, 541; 234, 243, 324, 342, 423, 432; 235, 253, 325, 352, 523, 532; 245, 254, 425, 452, 524, 542; 345, 354, 435, 453, 534, 543. There are of course $P(5,3) = 5 \cdot 4 \cdot 3 = 60$ items in our list.

15. One way to show that a function is a bijection is to find its inverse, since only bijections can have inverses. Note that the sizes of the two sets in question are the same, since there are $n!$ nonnegative integers less than $n!$, and there are $n!$ permutations of $\{1,2,\ldots,n\}$. In this case, since Cantor expansions are unique, we need to take the digits a_1, a_2, $\ldots$, a_{n-1} of the Cantor expansion of a nonnegative integer m less than $n!$ (so that $m = a_1 1! + a_2 2! + \cdots + a_{n-1}(n-1)!$), and produce a permutation with these a_k's satisfying the definition given before Exercise 12—indeed the only such permutation.

We will fill the positions in the permutation one at a time. First we put n into position $n - a_{n-1}$; clearly a_{n-1} will be the number of integers less than n that follow n in the permutation, since exactly a_{n-1} positions remain empty to the right of where we put the n. Next we renumber the free positions (the ones other than the one into which we put n), from left to right, as $1,2,\ldots,n-1$. Under this numbering, we put $n-1$ into position $(n-1) - a_{n-2}$. Again it is clear that a_{n-2} will be the number of integers less than $n-1$ that follow $n-1$ in the permutation. We continue in this manner, renumbering the free positions, from left to right, as $1,2,\ldots,n-k+1$, and then placing $n-k+1$ in position $(n-k+1) - a_{n-k}$, for $k = 1,2,\ldots,n-1$. Finally we place 1 in the only remaining position.

17. The algorithm is really given in our solution to Exercise 15. To produce all the permutations, we find the permutation corresponding to i, where $0 \le i < n!$, under the correspondence given in Exercise 15. To do this, we need to find the digits in the Cantor expansion of i, starting with the digit a_{n-1}. In what follows, that digit will be called c. We use k to keep track of which digit we are working on; as k goes from 1 to $n-1$, we will be computing the digit a_{n-k} in the Cantor expansion and inserting $n-k+1$ into the proper location in the permutation. (At the end, we need to insert 1 into the only remaining position.) We will call the positions in the permutation p_1, p_2, $\ldots$, p_n. We write only the procedure that computes the permutation corresponding to i; obviously to get all the permutations we simply call this procedure for each i from 0 to $n! - 1$.

```
procedure Cantor permutation(n, i : integers with n ≥ 1 and 0 ≤ i < n!)
x := n {to help in computing Cantor digits}
for j := 1 to n {initialize permutation to be all 0's}
        p_j := 0
for k := 1 to n − 1 {figure out where to place n − k + 1}
        c := ⌊x/(n − k)!⌋ {the Cantor digit}
        x := x − c(n − k)! {what's left of x}
        h := n {now find the (c + 1)^th free position from the right}
        while p_h ≠ 0
                h := h − 1
        for j := 1 to c
                h := h − 1
                while p_h ≠ 0
                        h := h − 1
        p_h := n − k + 1 {here is the key step}
h := 1 {now find the last free position}
while p_h ≠ 0
        h := h + 1
p_h := 1
{p_1, p_2, ..., p_n is the permutation corresponding to i}
```

GUIDE TO REVIEW QUESTIONS FOR CHAPTER 6

1. $1 + 2 + 2 \cdot 2 + 2 \cdot 2 \cdot 2 + \cdots + 2^{10} = 2047$

2. Subtract 11 from the answer to the previous review question, since λ, 1, 11, ..., 11...1 are the bit strings that do not have at least one 0 bit.

3. **a)** See Example 6 in Section 6.1. **b)** 10^5 **c)** See Example 7 in Section 6.1. **d)** $10 \cdot 9 \cdot 8 \cdot 7 \cdot 6$ **e)** 0

4. with a tree diagram; see Example 22 in Section 6.1 (extended to larger tree)

5. Using the inclusion–exclusion principle, we get $2^7 + 2^7 - 2^4$; see Example 18 in Section 6.1.

6. **a)** See p. 399. **b)** 11 pigeons, 10 holes (digits)

7. **a)** See p. 401. **b)** $N = 91$, $k = 10$

8. **a)** Permutations are ordered arrangements; combinations are unordered (or just arbitrarily ordered for convenience) selections.
b) $P(n,r) = C(n,r) \cdot r!$ (see the proof of Theorem 2 in Section 6.3) **c)** $C(25,6)$ **d)** $P(25,6)$

9. **a)** See pp. 418–419. **b)** by adding the two numbers above each number in the new row

10. A combinatorial proof is a proof of an algebraic identity that shows that both sides count the same thing (in some application). An algebraic proof is totally different—it shows that the two sides are equal by doing formal manipulations with the unknowns, with no reference to what the expressions might mean in an application.

11. See p. 418.

12. **a)** See p. 416. **b)** See p. 416. **c)** $2^{100}5^{101}C(201,101)$

13. **a)** See Theorem 2 in Section 6.5. **b)** $C(5+12-1,12)$ **c)** $C(5+9-1,9)$
d) $C(5+12-1,12) - C(5+7-1,7)$ **e)** $C(5+10-1,10) - C(5+6-1,6)$

14. **a)** See Example 5 in Section 6.5. **b)** $C(4+17-1,17)$
c) $C(4+13-1,13)$ (see Exercise 15a in Section 6.5)

15. **a)** See Theorem 3 in Section 6.5. **b)** $14!/(2!2!1!3!1!1!3!1!)$

16. See pp. 435–436.

17. **a)** $C(52,5)\cdot C(47,5)\cdot C(42,5)\cdot C(37,5)\cdot C(32,5)\cdot C(27,5)$ **b)** See Theorem 4 in Section 6.5.

18. See pp. 437–438.

SUPPLEMENTARY EXERCISES FOR CHAPTER 6

1. In each part of this problem we have $n=10$ and $r=6$.

a) If the items are to be chosen in order with no repetition allowed, then we need a simple permutation. Therefore the answer is $P(10,6)=10\cdot9\cdot8\cdot7\cdot6\cdot5=151{,}200$.

b) If repetition is allowed, then this is just a simple application of the product rule, with 6 tasks, each of which can be done in 10 ways. Therefore the answer is $10^6=1{,}000{,}000$.

c) If the items are to be chosen without regard to order but with no repetition allowed, then we need a simple combination. Therefore the answer is $C(10,6)=C(10,4)=10\cdot9\cdot8\cdot7\cdot/(4\cdot3\cdot2)=210$.

d) Unordered choices with repetition allowed are counted by $C(n+r-1,r)$, which in this case is $C(15,6)=5005$.

3. The student has 3 choices for each question: true, false, and no answer. There are 100 questions, so by the product rule there are $3^{100}\approx5.2\times10^{47}$ ways to answer the test.

5. We will apply the inclusion–exclusion principle from Section 6.1. First let us calculate the number of these strings with exactly three a's. To specify such a string we need to choose the positions for the a's, which can be done in $C(10,3)$ ways. Then we need to choose either a b or a c to fill each of the other 7 positions in the string, which can be done in 2^7 ways. Therefore there are $C(10,3)\cdot2^7=15360$ such strings. Similarly, there are $C(10,4)\cdot2^6=13440$ strings with exactly four b's. Next we need to compute the number of strings satisfying both of these conditions. To specify a string with exactly three a's and exactly four b's, we need to choose the positions for the a's, which can be done in $C(10,3)$ ways, and then choose the positions for the b's, which can be done in $C(7,4)$ ways (only seven slots remain after the a's are placed). Therefore there are $C(10,3)\cdot C(7,4)=4200$ such strings. Finally, by the inclusion–exclusion principle the number of strings having either exactly three a's or exactly four b's is $15360+13440-4200=24{,}600$.

7. **a)** We want a combination with repetition allowed, with $n=28$ and $r=3$. By Theorem 2 of Section 6.5, there are $C(28+3-1,3)=C(30,3)=4060$ possibilities.

b) This is just a simple application of the product rule. There are 28 ways to choose the ice cream, 8 ways to choose the sauce, and 12 ways to choose the topping, so there are $28\cdot8\cdot12=2688$ possible small sundaes.

c) By the product rule we have to multiply together the number of ways to choose the ice cream, the number of ways to choose the sauce, and the number of ways to choose the topping. There are $C(28+3-1,3)$ ways to choose the ice cream, just as in part **(a)**. There are $C(8,2)$ ways to choose the sauce, since repetition is not allowed. There are similarly $C(12,3)$ ways to choose the toppings. Multiplying these numbers together, we find that the answer is $4060\cdot28\cdot220=25{,}009{,}600$ different large sundaes.

9. We can solve this problem by counting the number of numbers that have the given digit in 1, 2, or 3 places.

a) The digit 0 appears in 1 place in some two-digit numbers and in some three-digit numbers. There are clearly 9 two-digit numbers in which 0 appears, namely 10, 20, ..., 90. We can count the number of three-digit numbers in which 0 appears exactly once as follows: first choose the place in which it is to appear (2 ways, since it cannot be the leading digit), then choose the left-most of the remaining digits (9 ways, since it cannot be a 0), then choose the final digit (also 9 ways). Therefore there are $9 + 2 \cdot 9 \cdot 9 = 171$ numbers in which the 0 appears exactly once, accounting for 171 appearances of the digit 0. Finally there are another 9 numbers in which the digit 0 appears twice, namely 100, 200, ..., 900. This accounts for 18 more 0's. And of course the number 1000 contributes 3 0's. Therefore our final answer is $171 + 18 + 3 = 192$.

b) The analysis for the digit 1 is not the same as for the digit 0, since we can have leading 1's but not leading 0's. One 1 appears in the one-digit numbers. Two-digit numbers can have a 1 in the ones place (and there are 9 of these, namely 11, 21, ..., 91), or in the tens place (and there are 10 of these, namely 10 through 19). Of course the number 11 is counted in both places, but that is proper, since we want to count each appearance of a 1. Therefore there are $10 + 9 = 19$ 1's appearing in two-digit numbers. Similarly, the three-digit numbers have 90 1's appearing in the ones place (every tenth number, and there are 900 numbers), 90 1's in the tens place (10 per decade, and there are 9 decades), and 100 1's in the hundreds place (100 through 199); therefore there are 280 ones appearing in three-digit numbers. Finally there is a 1 in 1000, so the final answer is $1 + 19 + 280 + 1 = 301$.

c) The analysis for the digit 2 is exactly the same as for the digit 1, with the exception that we do not get any 2's in 1000. Therefore the answer is $301 - 1 = 300$.

d) The analysis for the digit 9 is exactly the same as for the digit 2, so the answer is again 300.

Let us check all of the answers to this problem simultaneously. There are 300 each of the digits 2 through 9, for a total of 2400 digits. There are 192 0's and 301 1's. Therefore $2400+192+301 = 2893$ digits are used altogether. Let us count this another way. There are 9 one-digit numbers, 90 two-digit numbers, 900 three-digit numbers, and 1 four-digit number, so the total number of digits is $9\cdot 1+90\cdot 2+900\cdot 3+1\cdot 4 = 2893$. This agreement tends to confirm our analysis.

11. This is a negative instance of the generalized pigeonhole principle. The worst case would be if the student gets each fortune 3 times, for a total of $3 \times 213 = 639$ meals. If the student ate 640 or more meals, then the student will get the same fortune at least $\lceil 640/213 \rceil = 4$ times.

13. We have no guarantee ahead of time that this will work, but we will try applying the pigeonhole principle. Let us count the number of different possible sums. If the numbers in the set do not exceed 50, then the largest possible sum of a 5-element subset will be $50 + 49 + 48 + 47 + 46 = 240$. The smallest possible sum will be $1 + 2 + 3 + 4 + 5 = 15$. Therefore the sum has to be a number between 15 and 240, inclusive, and there are $240 - 15 + 1 = 226$ such numbers. Now let us count the number of different subsets. That is of course $C(10,5) = 252$. Since there are more subsets (pigeons) than sums (pigeonholes), we know that there must be two subsets with the same sum.

15. We assume that the drawings of the cards is done without replacement (i.e., no repetition allowed).

a) The worst case would be that we drew 1 ace and the 48 cards that are not aces, a total of 49 cards. Therefore we need to draw 50 cards to guarantee at least 2 aces (and it is clear that 50 is sufficient, since at worst 2 of the 4 aces would then be left in the deck).

b) The same analysis as in part **(a)** applies, so again the answer is 50.

c) In this problem we can use the pigeonhole principle. If we drew 13 cards, then they might all be of different kinds (ranks). If we drew 14 cards, however, then since there are only 13 kinds we would be assured of having at least two of the same kind. (The drawn cards are the pigeons and the kinds are the pigeonholes.)

d) If we drew 16 cards, we might get one of each kind except, say, aces, together with four aces. So 16 is not sufficient. If we drew 17 cards, however, then there must be at least two cards of each of two different kinds.

17. This problem can be solved using the pigeonhole principle if we look at it correctly. Let s_i be the sum of the first i of these numbers, where $1 \leq i \leq m$. Now if $s_i \equiv 0 \pmod{m}$ for some i, then we have our desired consecutive terms whose sum is divisible by m. Otherwise the numbers s_1 **mod** m, s_2 **mod** m, $\ldots$, s_m **mod** m are all integers in the set $\{1, 2, \ldots, m-1\}$. By the pigeonhole principle we know that two of them are the same, say s_i **mod** $m = s_j$ **mod** m with $i < j$. Then $s_j - s_i$ is divisible by m. But $s_j - s_i$ is just the sum of the $(i+1)^{\text{th}}$ through j^{th} terms in the sequence, and we are done.

19. The decimal expansion of a rational number a/b (we can assume that a and b are positive integers) can be obtained by long division of the number b into the number a, where a is written with a decimal point and an arbitrarily long string of 0's following it. The basic step in long division is finding the next digit of the quotient, which is just $\lfloor r/b \rfloor$, where r is the current remainder with the next digit of the dividend brought down. Now in our case, eventually the dividend has nothing but 0's to bring down. Furthermore there are only b possible remainders, namely the numbers from 0 to $b-1$. Thus at some point in our calculation after we have passed the decimal point, we will, by the pigeonhole principle, be looking at exactly the same situation as we had previously. From that point onward, then, the calculation must follow the same pattern as it did previously. In particular, the digits of the quotient will repeat.

For example, to compute the decimal expansion of the rational number $349/11$, we divide 11 into $349.00\ldots$. The first digit of the quotient is 3, and the remainder is 1. The next digit of the quotient is 1 and the remainder is 8. At this point there are only 0's left to bring down. The next digit of the quotient is a 7 with a remainder of 3, and then a quotient digit of 2 with a remainder of 8. We are now in exactly the same situation as at the previous appearance of a remainder of 8, so the quotient digits 72 repeat forever. Thus $349/11 = 31.\overline{72}$.

21. **a)** This is a simple combination, so the answer is $C(20, 12) = 125{,}970$.

b) The only choice is the choice of a variety, so the answer is 20.

c) We assume that order does not matter (all the donuts will go into a bag). Therefore, since repetitions are allowed, Theorem 2 of Section 6.5 applies, and the answer is $C(20 + 12 - 1, 12) = C(31, 12) = 141{,}120{,}525$.

d) We can simply subtract from our answer to part **(c)** our answer to part **(b)**, which asks for the number of ways this restriction can be violated. Therefore the answer is $141{,}120{,}505$.

e) We put the 6 blueberry filled donuts into our bag, and the problem becomes one of choosing 6 donuts with no restrictions. In analogy with part **(c)**, we obtain the answer $C(20 + 6 - 1, 6) = C(25, 6) = 177{,}100$.

f) There are $C(20 + 5 - 1, 5) = C(24, 5) = 42504$ ways to choose at least 7 blueberry donuts among our dozen (the calculation is essentially the same as that in part **(e)**). Our answer is therefore 42504 less than our unrestricted answer to part **(c)**: $141120525 - 42504 = 141{,}078{,}021$.

23. **a)** The given equation is equivalent to $n(n-1)/2 = 45$, which reduces to $n^2 - n - 90 = 0$. The quadratic formula (or factoring) tells us that the roots are $n = 10$ and $n = -9$. Since n is assumed to be nonnegative, the only relevant solution is $n = 10$.

b) The given equation is equivalent to $n(n-1)(n-2)/6 = n(n-1)$. Since $P(n, 2)$ is not defined for $n < 2$, we know that neither n nor $n - 1$ is 0, so we can divide both sides by these factors, obtaining $n - 2 = 6$, whence $n = 8$. (Alternatively, one can think of $P(n, k)$ and $C(n, k)$ as being defined to be 0 if $n < k$, in which case all n less than 2 also satisfy this equation, as well as the equation in part **(c)**.)

c) Recall the identity $C(n, k) = C(n, n-k)$. The given equation fits that model if $n = 7$ and $k = 5$. Hence $n = 7$ is a solution. That there are no more solutions follows from the fact that $C(n, k)$ is an increasing function in k for $0 \le k \le n/2$, and decreasing for $n/2 \le k \le n$, and hence there are no numbers k' other than k and $n-k$ for which $C(n,k') = C(n,k)$.

25. Following the hint, we see that each element of S falls into exactly one of three categories: either it is an element of A, or else it is not an element of A but is an element of B (in other words, is an element of $B-A$), or else it is not an element of B either (in other words, is an element of $S-B$). So the number of ways to choose sets A and B to satisfy these conditions is the same as the number of ways to place each element of S into one of these three categories. Therefore the answer is 3^n. For example, if $n=2$ and $S=\{x,y\}$, then there are 9 pairs: $(\emptyset,\emptyset)$, $(\emptyset,\{x\})$, $(\emptyset,\{y\})$, $(\emptyset,\{x,y\})$, $(\{x\},\{x\})$, $(\{x\},\{x,y\})$, $(\{y\},\{y\})$, $(\{y\},\{x,y\})$, $(\{x,y\},\{x,y\})$.

27. We start with the right-hand side and use Pascal's identity three times to obtain the left-hand side:

$$\begin{aligned} &C(n+2,r+1) - 2C(n+1,r+1) + C(n,r+1) \\ &\qquad = C(n+1,r+1) + C(n+1,r) - 2C(n+1,r+1) + C(n,r+1) \\ &\qquad = C(n+1,r) - C(n+1,r+1) + C(n,r+1) \\ &\qquad = [C(n,r) + C(n,r-1)] - [C(n,r+1) + C(n,r)] + C(n,r+1) \\ &\qquad = C(n,r-1) \end{aligned}$$

29. Substitute $x=1$ and $y=3$ into the binomial theorem (Theorem 1 in Section 6.4) and we obtain exactly this identity.

31. We just have to notice that the summation runs over exactly all the triples (i,j,k) such that $1 \le i < j < k \le n$. Since we are adding 1 for each such triple, the sum simply counts the number of such triples, which is just all the ways of choosing three distinct numbers from $\{1,2,3,\ldots,n\}$. Therefore the sum must equal $C(n,3)$.

33. The trick to the analysis here is to imagine what such a string has to look like. Every string of 0's and 1's can be thought of as consisting of alternating blocks—a block of 1's (possibly empty) followed by a block of 0's followed by a block of 1's followed by a block of 0's, and so on, ending with a block of 0's (again, possibly empty). If we want there to be exactly two occurrences of 01, then in fact there must be exactly six such blocks, the middle four all being nonempty (the transitions from 0's to 1's create the 01's) and the outer two possibly being empty. In other words, the string must look like this:

$$x_1 \text{ 1's} — x_2 \text{ 0's} — x_3 \text{ 1's} — x_4 \text{ 0's} — x_5 \text{ 1's} — x_6 \text{ 0's},$$

where $x_1 + x_2 + \cdots + x_6 = n$ and $x_1 \ge 0$, $x_6 \ge 0$, and $x_i \ge 1$ for $i = 2,3,4,5$. Clearly such a string is totally specified by the values of the x_i's. Therefore we are simply asking for the number of solutions to the equation $x_1+x_2+\cdots+x_6=n$ subject to the stated constraints. This kind of problem is solved in Section 6.5 (Example 5 and several exercises). The stated problem is equivalent to finding the number of solutions to $x_1+x_2'+x_3'+x_4'+x_5'+x_6=n-4$ where each variable here is nonnegative (we let $x_i = x_i'+1$ for $i=2,3,4,5$ in order to insure that these x_i's are strictly positive). The number of such solutions is, by the results just cited, $C(6+n-4-1, n-4)$, which simplifies to $C(n+1,n-4)$ or $C(n+1,5)$.

35. An answer key is just a permutation of 8 a's, 3 b's, 4 c's, and 5 d's. We know from Theorem 3 in Section 6.5 that there are

$$\frac{20!}{8!3!4!5!} = 3{,}491{,}888{,}400$$

such permutations.

37. We assume that each student is to get one advisor, that there are no other restrictions, and that the students and advisors are to be considered distinct. Then there are 5 ways to assign each student, so by the product rule there are $5^{24} \approx 6.0 \times 10^{16}$ ways to assign all of them.

39. For all parts of this problem, Theorem 2 in Section 6.5 is used.

a) We let $x_1 = x_1' + 2$, $x_2 = x_2' + 3$, and $x_3 = x_3' + 4$. Then the restrictions are equivalent to requiring that each of the x_i''s be nonnegative. Therefore we want the number of nonnegative integer solutions to the equation $x_1' + x_2' + x_3' = 8$. There are $C(3+8-1,8) = C(10,8) = C(10,2) = 45$ of them.

b) The number of solutions with $x_3 > 5$ is the same as the number of solutions to $x_1 + x_2 + x_3' = 11$, where $x_3 = x_3' + 6$. There are $C(3+11-1,11) = C(13,11) = C(13,2) = 78$ of these. Now we want to subtract the number of solutions for which also $x_1 \geq 6$. This is equivalent to the number of solutions to $x_1' + x_2 + x_3' = 5$, where $x_1 = x_1' + 6$. There are $C(3+5-1,5) = C(7,5) = C(7,2) = 21$ of these. Therefore the answer to the problem is $78 - 21 = 57$.

c) Arguing as in part **(b)**, we know that there are 78 solutions to the equation $x_1 + x_2 + x_3' = 11$, which is equivalent to the number of solutions to $x_1 + x_2 + x_3 = 17$ with $x_3 > 5$. We now need to subtract the number of these solutions that violate one or both of the restrictions $x_1 < 4$ and $x_2 < 3$. The number of solutions with $x_1 \geq 4$ is the number of solutions to $x_1' + x_2 + x_3' = 7$, namely $C(3+7-1,7) = C(9,7) = C(9,2) = 36$. The number of solutions with $x_2 \geq 3$ is the number of solutions to $x_1 + x_2' + x_3' = 8$, namely $C(3+8-1,7) = C(10,8) = C(10,2) = 45$. However, there are also solutions in which both restrictions are violated, namely the solutions to $x_1' + x_2' + x_3' = 4$. There are $C(3+4-1,4) = C(6,4) = C(6,2) = 15$ of these. Therefore the number of solutions in which one or both conditions are violated is $36 + 45 - 15 = 66$; we needed to subtract the 15 so as not to count these solutions twice. Putting this all together, we see that there are $78 - 66 = 12$ solutions of the given problem.

41. **a)** We want to find the number of r-element subsets for $r = 0,1,2,3,4$ and add. Therefore the answer is $C(10,0) + C(10,1) + C(10,2) + C(10,3) + C(10,4) = 1 + 10 + 45 + 120 + 210 = 386$.

b) This time we want $C(10,8) + C(10,9) + C(10,10) = C(10,2) + C(10,1) + C(10,0) = 45 + 10 + 1 = 56$.

c) This time we want $C(10,1) + C(10,3) + C(10,5) + C(10,7) + C(10,9) = C(10,1) + C(10,3) + C(10,5) + C(10,3) + C(10,1) = 10 + 120 + 252 + 120 + 10 = 512$. We can also solve this problem by using the fact from Exercise 31 in Section 6.4 that a set has the same number of subsets with an even number of elements as it has subsets with an odd number of elements. Since the set has $2^{10} = 1024$ subsets altogether, half of these—512 of them—must have an odd number of elements.

43. Since the objects are identical, all that matters is the number of objects put into each container. If we let x_i be the number of objects put into the i^{th} container, then we are asking for the number of solutions to the equation $x_1 + x_2 + \cdots + x_m = n$ with the restriction that each $x_i \geq 1$. By the usual trick this is equivalent to asking for the number of nonnegative integer solutions to $x_1' + x_2' + \cdots + x_m' = n - m$, where we have set $x_i = x_i' + 1$ to insure that each container gets at least one object. By Theorem 2 in Section 6.5, there are $C(m + (n-m) - 1, n - m) = C(n-1, n-m)$ solutions. This can also be written as $C(n-1, m-1)$, since $(n-1) - (n-m) = m - 1$. (Of course if $n < m$, then there are no solutions, since it would be impossible to put at least one object in each container. Our answer is consistent with this observation if we think of $C(x,y)$ as being 0 if $y > x$.)

45. **a)** This can be done with the multiplication principle. There are five choices for each ball, so the answer is $5^6 = 15{,}625$.

b) This is like Example 10 in Section 6.5, and we can use the formula for the Stirling numbers of the second kind given near the end of that section:

$$\sum_{j=1}^{k} \frac{1}{j!} \sum_{i=0}^{j-1} (-1)^i \binom{j}{i} (j-i)^n$$

with $n = 6$ and $k = 5$. We get

$$\frac{1}{1!}1^6 + \frac{1}{2!}(1\cdot 2^6 - 2\cdot 1^6) + \frac{1}{3!}(1\cdot 3^6 - 3\cdot 2^6 + 3\cdot 1^6) + \frac{1}{4!}(1\cdot 4^6 - 4\cdot 3^6 + 6\cdot 2^6 - 4\cdot 1^6) + \frac{1}{5!}(1\cdot 5^6 - 5\cdot 4^6 + 10\cdot 3^6 - 10\cdot 2^6 + 5\cdot 1^6),$$

which is $1 + 31 + 90 + 65 + 15 = 202$. The command in *Maple* for this, using the "combinat" package, is `sum(stirling2(6,j),j=1..5);`.

c) We saw in the discussion surrounding Example 9 in Section 6.5 that the number of ways to distribute n unlabeled objects into k labeled boxes is $C(n + k - 1, k - 1)$, because this is really the same as the problem of choosing an n-combination from the set of k boxes, with repetitions allowed. In this case we have $n = 6$ and $k = 5$, so the answer is $C(10, 4) = 210$.

d) Since both the boxes and the objects are indistinguishable, what is really being asked is how many different ways there are to write 6 as the sum of five nonnegative integers, with order ignored. We will just enumerate the possibilities and count them. We have $6 = 6+0+0+0+0$; $6 = 5+1+0+0+0$; $6 = 4+2+0+0+0$; $6 = 4+1+1+0+0$; $6 = 3+3+0+0+0$; $6 = 3+2+1+0+0$; $6 = 3+1+1+1+0$; $6 = 2+2+2+0+0$; $6 = 2+2+1+1+0$; and $6 = 2+1+1+1+1$. There are ten ways in all. Notice that we are allowing some of the boxes to be empty.

47. **a)** If there are just two tables, then the only choice involved is the person to sit alone. Therefore $c(3, 2) = 3$.

b) There are two possibilities. If two people sit at each table, then there are three ways to decide who sits with person A. If one person sits alone, there are four ways to choose that person and then two ways to arrange the other three people at the second table. So $c(4, 2) = 3 + 4 \cdot 2 = 11$.

c) There must be two people sitting alone and one pair sitting together. No other choices are involved, so $c(4, 3) = C(4, 2) = 6$.

d) This is similar to the previous case. There must be three people sitting alone and one pair sitting together, so $c(5, 4) = C(5, 2) = 10$.

49. Here is one approach. There are two possibilities for seating n people at $n-2$ tables. We might have three of them at one table with everyone else sitting alone. This can be done in $2C(n, 3)$ ways, because after choosing the table-mates we have to seat them clockwise or counterclockwise. Or we might choose two groups of two people to sit together and have everyone else sit alone. This can be done in $C(n, 2) \cdot C(n - 2, 2)/2$ ways (the division by 2 is to account for overcounting, because the order in which we pick the pairs is irrelevant). Therefore $c(n, n - 2) =$

$$2C(n,3) + \frac{C(n,2)\cdot C(n-2,2)}{2} = \frac{n(n-1)(n-2)}{3} + \frac{1}{2}\cdot\frac{n(n-1)}{2}\cdot\frac{(n-2)(n-3)}{2} = \frac{n^4}{8} - \frac{5n^3}{12} + \frac{3n^2}{8} - \frac{n}{12}.$$

Expanding $(3n - 1)C(n, 3)/4$ gives the same polynomial.

51. Following the hint, we observe that there are $(2n)!/2^n$ permutations of $2n$ objects of n different types, two of each type. Because this must be an integer, the denominator must divide the numerator, which is exactly what we are asked to prove.

53. Because the second list contains GAAAG (which does not end in C or U), those letters must end the string. Because it contains GGU, there must be two G's together followed by a U, and, looking at the first list of fragments, we infer that CCGGUCCG must be a substring. It follows that the original chain was CCGGUC-CGAAAG.

55. For convenience let us assume that the finite set is $\{1, 2, \ldots, n\}$. If we call a permutation $a_1 a_2 \ldots a_r$, then we simply need to allow each of the variables a_i to take on all n of the values from 1 to n. This is essentially just counting in base n, so our algorithm will be similar to Algorithm 2 in Section 6.6. The procedure shown here generates the next permutation. To get all the permutations, we just start with $11 \ldots 1$ and call this procedure $r^n - 1$ times.

procedure *next permutation*(n : positive integer, $a_1, a_2, \ldots, a_r$: positive integers $\leq n$)
{this procedure replaces the input with the next permutation, repetitions allowed,
in lexicographic order; assume that there is a next permutation, i.e., $a_1 a_2 \ldots a_r \neq nn \ldots n$}
$i := r$
while $a_i = n$
    $a_i := 1$
    $i := i - 1$
$a_i := a_i + 1$
{$a_1 a_2 \ldots a_r$ is the next permutation in lexicographic order}

57. We must show that if there are $R(m, n-1) + R(m-1, n)$ people at a party, then there must be at least m mutual friends or n mutual enemies. Consider one person; let's call him Jerry. Then there are $R(m-1, n) + R(m, n-1) - 1$ other people at the party, and by the pigeonhole principle there must be at least $R(m-1, n)$ friends of Jerry or $R(m, n-1)$ enemies of Jerry among these people. First let's suppose there are $R(m-1, n)$ friends of Jerry. By the definition of R, among these people we are guaranteed to find either $m-1$ mutual friends or n mutual enemies. In the former case these $m-1$ mutual friends together with Jerry are a set of m mutual friends; and in the latter case we have the desired set of n mutual enemies. The other situation is similar: Suppose there are $R(m, n-1)$ enemies of Jerry; we are guaranteed to find among them either m mutual friends or $n-1$ mutual enemies. In the former case we have the desired set of m mutual friends, and in the latter case these $n-1$ mutual enemies together with Jerry are a set of n mutual enemies.

WRITING PROJECTS FOR CHAPTER 6

Books and articles indicated by bracketed symbols below are listed near the end of this manual. You should also read the general comments and advice you will find there about researching and writing these essays.

1. You might start with the standard history of mathematics books, such as [Bo4] or [Ev3].

2. To learn about telephone numbers in North America, refer to books on telecommunications, such [Fr]. The term to look for in an index is the North American Numbering Plan.

3. See [Fe].

4. A lot of progress has been made recently by research mathematicians such as Herbert Wilf in finding general methods of proving essentially *all* true combinatorial identities, more or less mechanically. See whether you can find some of this work by looking in *Mathematical Reviews* (MathSciNet on the Web) or the book [PeWi]. There is also some discussion of this in [Wi2], a book on generating functions. Also, a classical book on combinatorial identities is [Ri2].

5. Students who have had an advanced physics course will be at an advantage here. Maybe you have a friend who is a physics major! In any case, it should not be hard to find a fairly elementary textbook on this subject.

6. More advanced combinatorics textbooks usually deal with Stirling numbers, at least in the exercises. See [Ro1], for instance. Other sources here are a chapter in [MiRo] and the amazing [GrKn].

7. See the comments for Writing Project 6.

8. There are entire books devoted to Ramsey theory, dealing not only with the classical Ramsey numbers, but also with applications to number theory, graph theory, geometry, linear algebra, etc. For a fairly advanced such book, see [GrRo]; for a gentler introduction, see the relevant sections of [Ro1] or the chapter in [MiRo]. Up-to-the-minute results can be found with a Web search.

9. Try books with titles such as "combinatorial algorithms"—that's what methods of generating permutations are, after all. See [Ev1] or [ReNi], for example. Another fascinating source (which deals with combinatorial algorithms as well as many other topics relevant to this text) is [GrKn]. Volume 2 of Knuth's classic [Kn] should have some relevant material. There is also an older article you might want to check out, [Le1]. An interesting related problem is to generate a *random* permutation; this is needed, for example, when using a computer to simulate the shuffling of a deck of cards for playing card games.

10. See the comments for Writing Project 9.