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**1** $A = \begin{bmatrix} 1 & 0 \\ 9 & 3 \end{bmatrix}$ has $A^{\mathrm{T}} = \begin{bmatrix} 1 & 9 \\ 0 & 3 \end{bmatrix}$, $A^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1/3 \end{bmatrix}$, $(A^{-1})^{\mathrm{T}} = (A^{\mathrm{T}})^{-1} = \begin{bmatrix} 1 & -3 \\ 0 & 1/3 \end{bmatrix}$;

$A = \begin{bmatrix} 1 & c \\ c & 0 \end{bmatrix}$ has $A^{\mathrm{T}} = A$ and $A^{-1} = \frac{1}{c^2}\begin{bmatrix} 0 & c \\ c & -1 \end{bmatrix} = (A^{-1})^{\mathrm{T}}$.

**2** $(AB)^{\mathrm{T}} = \begin{bmatrix} 1 & 2 \\ 3 & 7 \end{bmatrix} = B^{\mathrm{T}}A^{\mathrm{T}}$. This answer is different from $A^{\mathrm{T}}B^{\mathrm{T}}$ (except when $AB = BA$ and transposing gives $B^{\mathrm{T}}A^{\mathrm{T}} = A^{\mathrm{T}}B^{\mathrm{T}}$).

**3** (a) $((AB)^{-1})^{\mathrm{T}} = (B^{-1}A^{-1})^{\mathrm{T}} = (A^{-1})^{\mathrm{T}}(B^{-1})^{\mathrm{T}}$. This is also $(A^{\mathrm{T}})^{-1}(B^{\mathrm{T}})^{-1}$.

(b) If $U$ is upper triangular, so is $U^{-1}$: then $(U^{-1})^{\mathrm{T}}$ is *lower* triangular.

**4** $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ has $A^2 = 0$. But the diagonal of $A^{\mathrm{T}}A$ has dot products of columns of $A$ with *themselves*. If $A^{\mathrm{T}}A = 0$, zero dot products $\Rightarrow$ zero columns $\Rightarrow$ $A =$ zero matrix.

**5** (a) $\boldsymbol{x}^{\mathrm{T}}A\boldsymbol{y} = \begin{bmatrix} 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = 5$

(b) This is the row $\boldsymbol{x}^{\mathrm{T}}A = \begin{bmatrix} 4 & 5 & 6 \end{bmatrix}$ times $\boldsymbol{y}$.

(c) This is also the row $\boldsymbol{x}^{\mathrm{T}}$ times $A\boldsymbol{y} = \begin{bmatrix} 2 \\ 5 \end{bmatrix}$.

**6** $M^{\mathrm{T}} = \begin{bmatrix} A^{\mathrm{T}} & C^{\mathrm{T}} \\ B^{\mathrm{T}} & D^{\mathrm{T}} \end{bmatrix}$; $M^{\mathrm{T}} = M$ needs $A^{\mathrm{T}} = A$ *and* $B^{\mathrm{T}} = C$ *and* $D^{\mathrm{T}} = D$.

**7** (a) False: $\begin{bmatrix} 0 & A \\ A & 0 \end{bmatrix}$ is symmetric only if $A = A^{\mathrm{T}}$.

(b) False: The transpose of $AB$ is $B^{\mathrm{T}}A^{\mathrm{T}} = BA$. So $(AB)^{\mathrm{T}} = AB$ needs $BA = AB$.

(c) True: Invertible symmetric matrices have symmetric inverses! Easiest proof is to transpose $AA^{-1} = I$.

(d) True: $(ABC)^{\mathrm{T}}$ is $C^{\mathrm{T}}B^{\mathrm{T}}A^{\mathrm{T}}(= CBA$ for symmetric matrices $A, B$, and $C$).

**8** The 1 in row 1 has $n$ choices; then the 1 in row 2 has $n-1$ choices ... ($n!$ overall).

**9** $P_1P_2 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}$ but $P_2P_1 = \begin{bmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$.
If $P_3$ and $P_4$ exchange ***different*** pairs of rows, $P_3P_4 = P_4P_3 =$ both exchanges.

**10** $(3,1,2,4)$ and $(2,3,1,4)$ keep 4 in place; 6 more even $P$'s keep 1 or 2 or 3 in place; $(2,1,4,3)$ and $(3,4,1,2)$ and $(4,3,2,1)$ exchange 2 pairs. $(1,2,3,4)$ makes 12.

**11** $PA = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 6 \\ 1 & 2 & 3 \\ 0 & 4 & 5 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 \\ 0 & 4 & 5 \\ 0 & 0 & 6 \end{bmatrix}$ is upper triangular. Multiplying $A$ ***on the right*** by a permutation matrix $P_2$ exchanges the ***columns*** of $A$. To make this $A$ lower triangular, we also need $P_1$ to exchange rows 2 and 3:

$$P_1AP_2 = \begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} A \begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} = \begin{bmatrix} 6 & 0 & 0 \\ 5 & 4 & 0 \\ 3 & 2 & 1 \end{bmatrix}.$$

**12** $(P\boldsymbol{x})^{\mathrm{T}}(P\boldsymbol{y}) = \boldsymbol{x}^{\mathrm{T}}P^{\mathrm{T}}P\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}\boldsymbol{y}$ since $P^{\mathrm{T}}P = I$. In general $P\boldsymbol{x}\cdot\boldsymbol{y} = \boldsymbol{x}\cdot P^{\mathrm{T}}\boldsymbol{y} \neq \boldsymbol{x}\cdot P\boldsymbol{y}$:

Non-equality where $P \neq P^{\mathrm{T}}$: $\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix} \neq \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 2 \end{bmatrix}$.

**13** A cyclic $P = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ or its transpose will have $P^3 = I : (1,2,3) \to (2,3,1) \to (3,1,2) \to (1,2,3)$. The permutation $\widehat{P} = \begin{bmatrix} 1 & 0 \\ 0 & P \end{bmatrix}$ for the same $P$ has $\widehat{P}^4 = \widehat{P} \neq I$.

**14** The "reverse identity" $P$ takes $(1,\ldots,n)$ into $(n,\ldots,1)$. When rows and also columns are reversed, the $1,1$ and $n,n$ entries of $A$ change places in $PAP$. So do the $1,n$ and $n,1$ entries. In general $(PAP)_{ij}$ is $(A)_{n-i+1,n-j+1}$.

**15** (a) If $P$ sends row 1 to row 4, then $P^{\mathrm{T}}$ sends row 4 to row 1 (b) $P = \begin{bmatrix} E & 0 \\ 0 & E \end{bmatrix} = P^{\mathrm{T}}$ with $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ moves all rows: 1 and 2 are exchanged, 3 and 4 are exchanged.

**16** $A^2 - B^2$ and also $ABA$ are symmetric if $A$ and $B$ are symmetric. But $(A+B)(A-B)$ and $ABAB$ are generally ***not*** symmetric.

**17** (a) $S = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} = S^{\mathrm{T}}$ is not invertible (b) $S = \begin{bmatrix} 0 & 1 \\ 1 & 1 \end{bmatrix}$ needs row exchange (c) $S = \begin{bmatrix} 1 & 1 \\ 1 & 0 \end{bmatrix}$ has pivots $D = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$ : no real square root.

**18** (a) $5+4+3+2+1 = 15$ independent entries if $S = S^{\mathrm{T}}$ (b) $L$ has 10 and $D$ has 5; total 15 in $LDL^{\mathrm{T}}$ (c) Zero diagonal if $A^{\mathrm{T}} = -A$, leaving $4+3+2+1 = 10$ choices.

**19** (a) The transpose of $A^{\mathrm{T}}SA$ is $A^{\mathrm{T}}S^{\mathrm{T}}A^{\mathrm{T}\,\mathrm{T}} = A^{\mathrm{T}}SA = n$ by $n$ when $S^{\mathrm{T}} = S$ (any $m$ by $n$ matrix $A$) (b) $(A^{\mathrm{T}}A)_{jj} =$ (column $j$ of $A$)$\cdot$ (column $j$ of $A$) $=$ (length squared of column $j$) $\geq 0$.

**20** $$\begin{bmatrix} 1 & 3 \\ 3 & 2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -7 \end{bmatrix} \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}; \quad \begin{bmatrix} 1 & b \\ b & c \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ b & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & c-b^2 \end{bmatrix} \begin{bmatrix} 1 & b \\ 0 & 1 \end{bmatrix}$$
$$\begin{bmatrix} 2 & -1 & 0 \\ -1 & 2 & -1 \\ 0 & -1 & 2 \end{bmatrix} = \begin{bmatrix} 1 & & \\ -\frac{1}{2} & 1 & \\ 0 & -\frac{2}{3} & 1 \end{bmatrix} \begin{bmatrix} 2 & & \\ & \frac{3}{2} & \\ & & \frac{4}{3} \end{bmatrix} \begin{bmatrix} 1 & -\frac{1}{2} & 0 \\ & 1 & -\frac{2}{3} \\ & & 1 \end{bmatrix} = \boldsymbol{LDL}^{\mathrm{T}}.$$

**21** Elimination on a symmetric 3 by 3 matrix leaves a symmetric lower right 2 by 2 matrix. $\begin{bmatrix} 2 & 4 & 8 \\ 4 & 3 & 9 \\ 8 & 9 & 0 \end{bmatrix}$ and $\begin{bmatrix} 1 & b & c \\ b & d & e \\ c & e & f \end{bmatrix}$ lead to $\begin{bmatrix} -5 & -7 \\ -7 & -32 \end{bmatrix}$ and $\begin{bmatrix} d-b^2 & e-bc \\ e-bc & f-c^2 \end{bmatrix}$ : symmetric!

**22** $\begin{bmatrix} & 1 & \\ 1 & & \\ & & 1 \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 2 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 1 \\ & 1 & 1 \\ & & -1 \end{bmatrix}$; $\begin{bmatrix} 1 & & \\ & & 1 \\ & 1 & \end{bmatrix} A = \begin{bmatrix} 1 & & \\ 1 & 1 & \\ 2 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 & 0 \\ & -1 & 1 \\ & & 1 \end{bmatrix}$

**23** $A = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = P$ and $L = U = I$. Elimination on this $A = P$ exchanges rows 1-2 then rows 2-3 then rows 3-4.

**24** $PA = LU$ is $\begin{bmatrix} & & 1 \\ & 1 & \\ 1 & & \end{bmatrix} \begin{bmatrix} 0 & 1 & 2 \\ 0 & 3 & 8 \\ 2 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & & \\ 0 & 1 & \\ 0 & 1/3 & 1 \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ & 3 & 8 \\ & & -2/3 \end{bmatrix}$. If we wait to exchange and $a_{12}$ is the pivot, $A = L_1 P_1 U_1 = \begin{bmatrix} 1 & & \\ 3 & 1 & \\ & & 1 \end{bmatrix} \begin{bmatrix} & 1 & \\ & & 1 \\ 1 & & \end{bmatrix} \begin{bmatrix} 2 & 1 & 1 \\ 0 & 1 & 2 \\ 0 & 0 & 2 \end{bmatrix}$.

**25** One way to decide even vs. odd is to count all pairs that $P$ has in the wrong order. Then $P$ is even or odd when that count is even or odd. Hard step: Show that an exchange always switches that count! Then 3 or 5 exchanges will leave that count odd.

**26** (a) $E_{21} = \begin{bmatrix} 1 & & \\ -3 & 1 & \\ & & 1 \end{bmatrix}$ puts 0 in the $2, 1$ entry of $E_{21}A$. Then $E_{21}AE_{21}^{\mathrm{T}} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 2 & 4 \\ 0 & 4 & 9 \end{bmatrix}$ is still symmetric, with zero also in its $1, 2$ entry. (b) Now use $E_{32} = \begin{bmatrix} 1 & & \\ & 1 & \\ & -2 & 1 \end{bmatrix}$ to make the $3, 2$ entry zero and $E_{32}E_{21}AE_{21}^{\mathrm{T}}E_{32}^{\mathrm{T}} = D$ also has zero in its $2, 3$ entry. Key point: Elimination from both sides (rows $+$ columns) gives the symmetric $LDL^{\mathrm{T}}$.

**27** $A = \begin{bmatrix} 0 & 1 & 2 & 3 \\ 1 & 2 & 3 & 0 \\ 2 & 3 & 0 & 1 \\ 3 & 0 & 1 & 2 \end{bmatrix} = A^{\mathrm{T}}$ has $0, 1, 2, 3$ in every row. I don't know any rules for a symmetric construction like this "Hankel matrix" with constant antidiagonals.

**28** Reordering the rows and/or the columns of $\left[\begin{smallmatrix}\mathbf{a} & \mathbf{b}\\ \mathbf{c} & \mathbf{d}\end{smallmatrix}\right]$ will move the entry **a**. So the result cannot be the transpose (which doesn't move **a**).

**29** (a) Total currents are $A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} 1 & 0 & 1 \\ -1 & 1 & 0 \\ 0 & -1 & -1 \end{bmatrix} \begin{bmatrix} y_{BC} \\ y_{CS} \\ y_{BS} \end{bmatrix} = \begin{bmatrix} y_{BC} + y_{BS} \\ -y_{BC} + y_{CS} \\ -y_{CS} - y_{BS} \end{bmatrix}$.

(b) Either way $(A\boldsymbol{x})^{\mathrm{T}}\boldsymbol{y} = \boldsymbol{x}^{\mathrm{T}}(A^{\mathrm{T}}\boldsymbol{y}) = x_B y_{BC} + x_B y_{BS} - x_C y_{BC} + x_C y_{CS} - x_S y_{CS} - x_S y_{BS}$. Six terms.

**30** $\begin{bmatrix} 1 & 50 \\ 40 & 1000 \\ 2 & 50 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\boldsymbol{x};\ A^{\mathrm{T}}\boldsymbol{y} = \begin{bmatrix} 1 & 40 & 2 \\ 50 & 1000 & 50 \end{bmatrix} \begin{bmatrix} 700 \\ 3 \\ 3000 \end{bmatrix} = \begin{bmatrix} 6820 \\ 188000 \end{bmatrix} \begin{matrix} \text{1 truck} \\ \text{1 plane} \end{matrix}$

**31** $A\boldsymbol{x} \cdot \boldsymbol{y}$ is the *cost of inputs* while $\boldsymbol{x} \cdot A^{\mathrm{T}}\boldsymbol{y}$ is the *value of outputs*.

**32** $P^3 = I$ so three rotations for $360°$; $P$ rotates every $\boldsymbol{v}$ around the $(1, 1, 1)$ line by $120°$.

**33** $\begin{bmatrix} 1 & 2 \\ 4 & 9 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 2 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 2 & 5 \end{bmatrix} = \boldsymbol{EH} =$ (elementary matrix) times (symmetric matrix).

**34** $L(U^{\mathrm{T}})^{-1}$ is lower triangular times lower triangular, so *lower triangular*. The transpose of $U^{\mathrm{T}}DU$ is $U^{\mathrm{T}}D^{\mathrm{T}}U^{\mathrm{T}\,\mathrm{T}} = U^{\mathrm{T}}DU$ again, so $U^{\mathrm{T}}DU$ is *symmetric*. The factorization multiplies lower triangular by symmetric to get $LDU$ which is $A$.

**35** These are groups: Lower triangular with diagonal 1's, diagonal invertible $D$, permutations $P$, orthogonal matrices with $Q^{\mathrm{T}} = Q^{-1}$.

**36** Certainly $B^{\mathrm{T}}$ is northwest. $B^2$ is a full matrix! $B^{-1}$ is southeast: $\left[\begin{smallmatrix}1 & 1\\ 1 & 0\end{smallmatrix}\right]^{-1} = \left[\begin{smallmatrix}0 & 1\\ 1 & -1\end{smallmatrix}\right]$. The rows of $B$ are in reverse order from a lower triangular $L$, so $B = PL$. Then $B^{-1} = L^{-1}P^{-1}$ has the ***columns*** in reverse order from $L^{-1}$. So $B^{-1}$ is *southeast*. Northwest $B = PL$ times southeast $PU$ is $(PLP)U =$ upper triangular.

**37** There are $n!$ permutation matrices of order $n$. Eventually *two powers of $P$ must be the same permutation*. And if $P^r = P^s$ then $P^{r-s} = I$. Certainly $r - s \le n!$

$P = \begin{bmatrix} P_2 & \\ & P_3 \end{bmatrix}$ is 5 by 5 with $P_2 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$ and $P_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$ and $P^6 = I$.

**38** To split the matrix $M$ into (symmetric $S$) + (anti-symmetric $A$), the only choice is $S = \frac{1}{2}(M + M^{\mathrm{T}})$ and $A = \frac{1}{2}(M - M^{\mathrm{T}})$.

**39** Start from $Q^{\mathrm{T}}Q = I$, as in $\begin{bmatrix} \boldsymbol{q}_1^{\mathrm{T}} \\ \boldsymbol{q}_2^{\mathrm{T}} \end{bmatrix} \begin{bmatrix} \boldsymbol{q}_1 & \boldsymbol{q}_2 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$

(a) The diagonal entries give $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_1 = 1$ and $\boldsymbol{q}_2^{\mathrm{T}}\boldsymbol{q}_2 = 1$: *unit vectors*

(b) The off-diagonal entry is $\boldsymbol{q}_1^{\mathrm{T}}\boldsymbol{q}_2 = 0$ (and in general $\boldsymbol{q}_i^{\mathrm{T}}\boldsymbol{q}_j = 0$)

(c) The leading example for $Q$ is the rotation matrix $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.