MATH 2418: Linear Algebra

Assignment# 1

Due : 08/30/2022, Tuesday, before 11:59pm   Term `:Fall 2022`

[Last Name]   [First Name]   [Net ID]   [Lab Section]

Recommended Problems:(Do not turn in)
Sec 1.1: 1, 2, 3, 8, 9, 10, 11, 12, 13, 14, 15, 18, 26, 27, 28, 31.

1. Let $\mathbf{v} = (2, 3, 1)$, $\mathbf{w} = (1, -1, -1)$, and $3\mathbf{u} + 2\mathbf{v} - 4\mathbf{w} = (1, -2, 3)$. Find

   (a) the vector $\mathbf{u}$
   Solution: Given

$$\begin{aligned} 3\mathbf{u} + 2\mathbf{v} - 4\mathbf{w} &= (1, -2, 3) \\ \Rightarrow \mathbf{u} &= \tfrac{1}{3}[4\mathbf{w} - 2\mathbf{v} + (1, -2, 3)] \\ &= -\tfrac{1}{3}[4(1, -1, -1) - 2(2, 3, 1) + (1, -2, 3)] \\ &= \tfrac{1}{3}(4 - 4 + 1, -4 - 6 - 2, -4 - 2 + 3) \\ &= \tfrac{1}{3}(1, -12, -3), \end{aligned}$$

   (b) the linear combination: $2\mathbf{u} - 3\mathbf{v} + 5\mathbf{w}$.
   Solution: Plug in $\mathbf{u}, \mathbf{v}, \mathbf{w}$:

$$\begin{aligned} 2\mathbf{u} - 3\mathbf{v} + 5\mathbf{w} &= \tfrac{2}{3}(1, -12, -3) - 3(2, 3, 1) + 5(1, -1, -1) \\ &= \left(-6 + 5 + \tfrac{2}{3}, \ -8 - 9 - 5, \ -2 - 3 - 5\right) \\ &= \left(-\tfrac{1}{3}, \ -22, \ -10\right). \end{aligned}$$

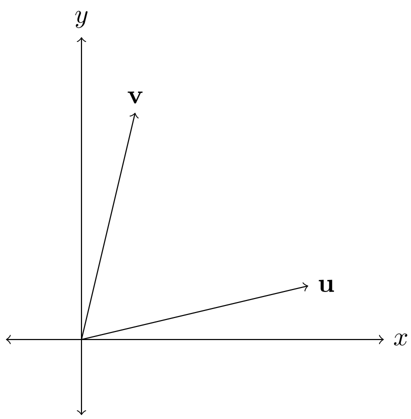
2. Given vectors $\mathbf{u}$ and $\mathbf{v}$ in diagram below, shade in all linear combinations $c\mathbf{u} + d\mathbf{v}$ for

   (a) $0 \le c \le 1$ and $0 \le d \le 1$

   (b) $0 \le c \le 1$ and $d > 1$

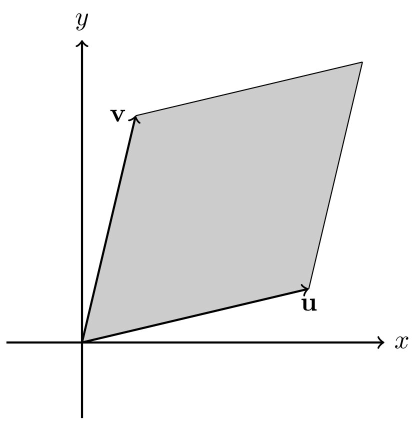
   (c) $0 \le d \le 1$ and $c > 1$.

   (You can use different shading styles in same picture for all three parts or can graph them separately in three different pictures)

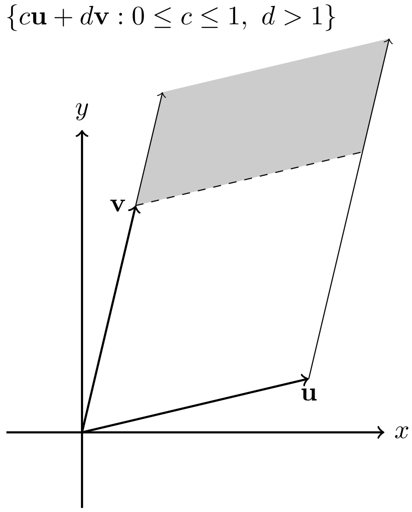


Solution:

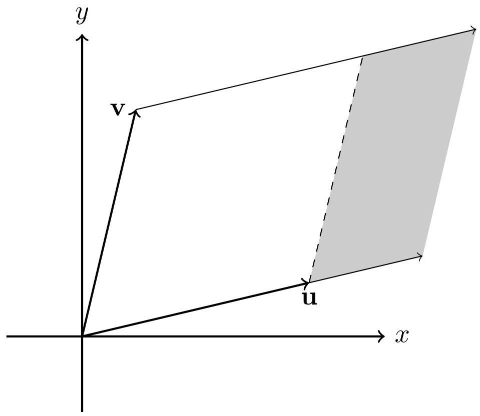
(a) $\{c\mathbf{u} + d\mathbf{v} : 0 \le c \le 1, 0 \le d \le 1\}$



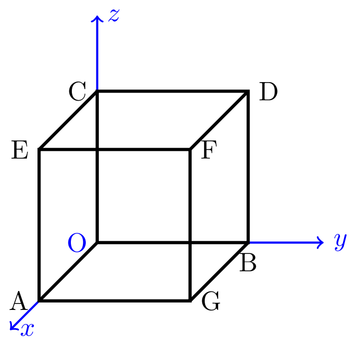
(b) $\{c\mathbf{u} + d\mathbf{v} : 0 \le c \le 1,\ d > 1\}$



(c) $\{c\mathbf{u} + d\mathbf{v} : 0 \leq d \leq 1,\ c > 1\}$



3. Let $\mathbf{0} = (0,0,0)$, $\mathbf{i} = (1,0,0)$, $\mathbf{j} = (0,1,0)$, $\mathbf{k} = (0,0,1)$ be vectors in $\mathbb{R}^3$. Given a cube with an edge 5 inches in the figure below



(a) Write down the vectors $\overrightarrow{OE}$, $\overrightarrow{OD}$, $\overrightarrow{OF}$, $\overrightarrow{OG}$ as linear combinations of $\mathbf{i}, \mathbf{j}, \mathbf{k}$

(i) $\overrightarrow{OE} = (5, 0, 5)$

(ii) $\overrightarrow{OD} = (0, 5, 5)$

(iii) $\overrightarrow{OF} = (5, 5, 5)$

(iv) $\overrightarrow{OG} = (5, 5, 0)$

(b) Let P, Q, R, S, T, U be the centers of the faces $AGFE$, $GBDF$, $DCEF$, $OAEC$, $OAGB$, $OBDC$ respectively, write the vectors: $\overrightarrow{OP}, \overrightarrow{OQ}, \overrightarrow{OR}, \overrightarrow{OS}, \overrightarrow{OT}, \overrightarrow{OU}$ as a linear combinations of $\mathbf{i}, \mathbf{j}, \mathbf{k}$.

(i) $\overrightarrow{OP} = \left(5, \frac{5}{2}, \frac{5}{2}\right)$

(ii) $\overrightarrow{OQ} = \left(\frac{5}{2}, 5, \frac{5}{2}\right)$

(iii) $\overrightarrow{OR} = \left(\frac{5}{2}, \frac{5}{2}, 5\right)$

(iv) $\overrightarrow{OS} = \left(\frac{5}{2}, 0, \frac{5}{2}\right)$

(v) $\overrightarrow{OT} = \left(\frac{5}{2}, \frac{5}{2}, 0\right)$

(vi) $\overrightarrow{OU} = \left(0, \frac{5}{2}, \frac{5}{2}\right)$

4. Let $\mathbf{u}+\mathbf{v}=(3,-3)$ and $\mathbf{u}-\mathbf{v}=(1,1)$.

   (a) Find $\mathbf{u}$ and $\mathbf{v}$

   (b) Draw the vectors $\mathbf{u},\ \mathbf{v},\ \mathbf{u}+\mathbf{v},\ (\mathbf{u}-\mathbf{v}),\ (-\mathbf{u}+\mathbf{v}),\ (-\mathbf{u}-\mathbf{v})$ in a single $xy-$plane.

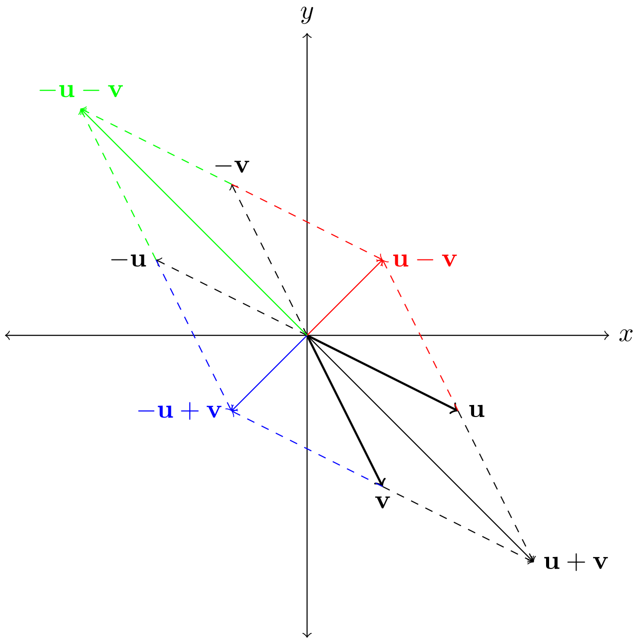
Solution:
(a) We have $\mathbf{u}+\mathbf{v}=(3,-3)$ and $\mathbf{u}-\mathbf{v}=(1,1)$.
Adding these two vectors we get,

$$2\mathbf{u}=(4,-2) \implies \boxed{\mathbf{u}=(2,-1)}$$

Using this information, we get,

$$\mathbf{v}=(3,-3)-\mathbf{u} \implies \boxed{\mathbf{v}=(1,-2)}$$

(b)
Note that $-\mathbf{u}+\mathbf{v}=(-1,-1)$ and $-\mathbf{u}-\mathbf{v}=(-3,3)$.



5. (a) Determine all real values of p such that the set of all linear combination of $\mathbf{u} = (-3, p)$ and $\mathbf{v} = (2, 3)$ is all of $\mathbb{R}^2$. Justify your answer.

Solution. Consider that if the set of all linear combinations of the vectors:

$$\mathbf{u} = (-3, p) \qquad \mathbf{v} = (2, 3)$$

is all of $\mathbb{R}^2$, then it must hold that:

$$c\mathbf{v} = (2c, 3c) \neq (-3, p) = \mathbf{u} \tag{1}$$

for every real number c (for otherwise, $\mathbf{u}$ and $\mathbf{v}$ occupy the same line in $\mathbb{R}^2$, so that any linear combination of these vectors must also occupy that same line, and thus the set of all such linear combinations of $\mathbf{u}$ and $\mathbf{v}$ cannot be all of $\mathbb{R}^2$).

Now condition (1) holds if, and only if when $2c = -3$, hence $c = -3/2$, we have:

$$3c = 3 \cdot -\frac{3}{2} = -\frac{9}{2} \neq p$$

and thus the set of all real values p such that the set of all linear combinations of vectors $\mathbf{u}$ and $\mathbf{v}$ is all $\mathbb{R}^2$ is computed as:

$$\boxed{\{p \in \mathbb{R} : p \neq -9/2\}}$$

(b) Determine all real values of p and q such that the set of all linear combinations of $\mathbf{u} = (1, p, -1)$ and $\mathbf{v} = (3, 2, q)$ is a plane in $\mathbb{R}^3$. Justify your answer.

Solution. Similar to problem (a) above, for the set of all linear combinations of $\mathbf{u}$ and $\mathbf{v}$ to be a plane of $\mathbb{R}^3$ we require:

$$c\mathbf{v} = (3c, 2c, qc) \neq (1, p, -1) = \mathbf{u} \tag{2}$$

for every real c (this conditions assures, similar to problem (a), that $\mathbf{u}$ and $\mathbf{v}$ are not colinear in $\mathbb{R}^3$, hence the set all linear combinations of these two vectors constitutes a plane of $\mathbb{R}^3$).

Now condition (2) holds if, and only if, when $3c = 1$, hence $c = 1/3$, we have either of the two below conditions:

$$2c = 2 \cdot \frac{1}{3} = \frac{2}{3} \neq p \qquad \text{or} \qquad qc = q \cdot \frac{1}{3} = \frac{q}{3} \neq -1$$

Thus we find that condition (2) holds if, and only if, $p \neq 2/3$ or $q \neq -3$. In other words, the set all real pairs (p, q) such that the set all linear combinations of $\mathbf{u}$ and $\mathbf{v}$ is a plane in $\mathbb{R}^3$ is:

$$\boxed{\{(p, q) \in \mathbb{R}^2 : p \neq 2/3 \text{ or } q \neq -3\}}$$

6. Determine whether the set of all linear combinations of the following set of vectors in $\mathbb{R}^3$ is **a line or a plane or all of** $\mathbb{R}^3$. Justify, your answer.

   (a) $\{(-2,5,-3),(6,-15,9),(-10,25,-15)\}$

   (b) $\{(0,0,3),(0,1,2),(1,1,0)\}$

   (c) $\{(1,2,0),(1,1,1),(4,5,3)\}$

   Solution:

   (a) Since $(6,-15,9) = 3(-2,5,-3)$ and $(-10,25,-15) = 5(-2,5,-3)$, the linear combination of the vectors form the set $\{k(-2,5,-3) \in \mathbb{R}^3 | k \in \mathbb{R}\}$. This set is a line.

   (b) Let us see if we can express the third vector as a linear combination of the first two.
   Say we have $p, q \in \mathbb{R}$ such that

   $$p(0,0,3) + q(0,1,2) = (1,1,0)$$

   For any values of p and q, the first coordinate cannot be matched. So the vector $(1,1,0)$ does not belong in the plane of $(0,0,3)$ and $(0,1,2)$. Also, $(0,1,2)$ is not a multiple of $(0,0,3)$.
   The set of linear combinations of the vectors is all of $\mathbb{R}^3$.

   (c) We can express the third vector in the set, $(4,5,3)$, as a linear combination of the other two vectors.

   $$(4,5,3) = (1,2,0) + 3(1,1,1)$$

   So, the third vector belongs to the plane of $(1,2,0)$ and $(1,1,1)$. The linear combination of the three vectors is a plane.