MATH 2418: Linear Algebra

Assignment# 3

Due :Tuesday, 09/13/2020, 11:59pm Term :`Fall 2022`

[Last Name]	[First Name]	[Net ID]	[Lab Section]

Recommended Problems:(Do not turn in)
Sec 2.1: 1, 2, 9, 10, 16, 17, 19, 21, 26, 29. Sec 2.2: 5, 6, 7, 8, 12, 13, 19, 23.
Note: The answers to these problems are available at: `http://math.mit.edu/~gs/linearalgebra/`

1. (a) Find the matrix P that multiplies every vector $(x,\ y,\ z) \in \mathbb{R}^3$ to produce the vector $(3x+2y+z,\ 5y-z,\ 8x)$. Also find P^{-1}.

Solution:

$$P = \begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & -1 \\ 8 & 0 & 0 \end{bmatrix}, X = \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$

Given,

$$\begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & -1 \\ 8 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x+2y+z \\ 5y-z \\ 8x \end{bmatrix} \tag{2}$$

Find D^{-1}: Let: $\begin{bmatrix} 3 & 2 & 1 \\ 0 & 5 & -1 \\ 8 & 0 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$. Then,

$$\begin{cases} x = \dfrac{1}{8} b_3 \\ y = \dfrac{1}{7} b_1 + \dfrac{1}{7} b_2 - \dfrac{3}{56} b_3 \\ z = \dfrac{5}{7} b_1 - \frac{2}{7} b_2 - \frac{15}{56} b_3 \end{cases}$$

Thus, $P^{-1} = \begin{bmatrix} 0 & 0 & \dfrac{1}{8} \\ \dfrac{1}{7} & \dfrac{1}{7} & -\dfrac{15}{56} \\ \dfrac{5}{7} & -\dfrac{2}{7} & -\dfrac{15}{56} \end{bmatrix}$

(b) Find the matrix P that multiplies every vector $(x,\ y) \in \mathbb{R}^2$ to produce $(5x-4y,\ -2x,\ 3y-2x) \in \mathbb{R}^3$.

Solution:

$$P = \begin{bmatrix} 5 & -4 \\ -2 & 0 \\ -2 & 3 \end{bmatrix}, X = \begin{bmatrix} x \\ y \end{bmatrix}$$

Given,

$$\begin{bmatrix} 5 & -4 \\ -2 & 0 \\ -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 5x-4y \\ -2x \\ 3y-2x \end{bmatrix} \tag{4}$$

2. Given linear system $\begin{cases} x - 5y = 2 \\ -x - 3y = -10 \end{cases}$

(a) Write the corresponding matrix equation $A\mathbf{x} = \mathbf{b}$.
Solution:

$$\begin{cases} x - 5y = 2 \\ -x - 3y = -10 \end{cases} \Longrightarrow \begin{bmatrix} 1 & -5 \\ -1 & -3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$$

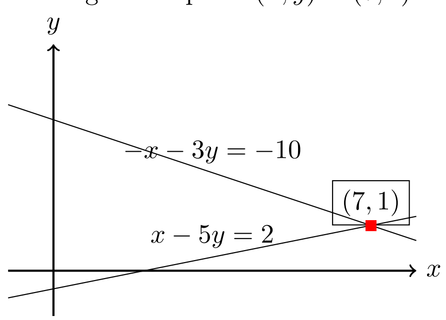
(b) Solve the linear system.
Solution:

$$\begin{cases} x - 5y = 2 \\ -x - 3y = -10 \end{cases} \xrightarrow[l_{21}=-1]{Eq(2)+Eq(1)} \begin{cases} x - 5y = 2 \\ \quad - 8y = -8 \end{cases}$$

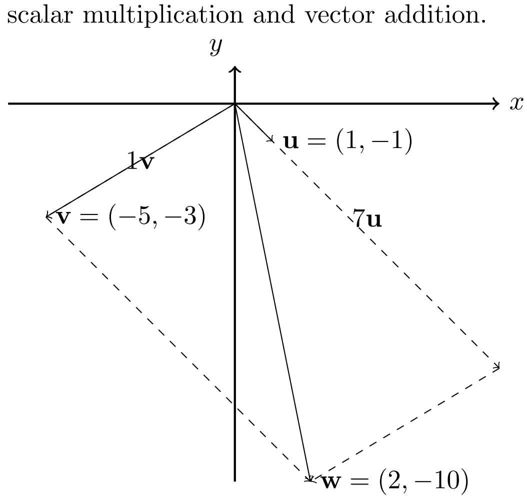
Hence, $y = 1$. Replacing y in the first equation gives $x = 7$. Hence, the solution to the system is $(x, y) = (7, 1)$.

(c) Draw the row picture and the column picture.
Solution:

i) **Row picture**: we have linear system $\begin{cases} x - 5y = 2 \\ -x - 3y = -10 \end{cases}$. Row picture shows two lines which are meeting at the point $(x, y) = (7, 1)$.



ii) **Column picture**: the linear system can be written as a combination of columns: $x \begin{bmatrix} 1 \\ -1 \end{bmatrix} + y \begin{bmatrix} -5 \\ -3 \end{bmatrix} = \begin{bmatrix} 2 \\ -10 \end{bmatrix}$.
Here x is scalar of vector $(1, -1)$ and y is a scalar of vector $(-5, -3)$. Column picture shows the scalar multiplication and vector addition.



3. Consider the function $\mathrm{T} : \mathbb{R}^3 \to \mathbb{R}^2$, defined by $T(x, y, z) = (x + 3y, -2x + 5z)$

   (a) Write the matrix for T.

   (b) For the vectors $\mathbf{u} = (1, 5, -2), \mathbf{v} = (2, 7, 4) \in \mathbb{R}^3$, verify that $\mathrm{T}(2\mathbf{u} + 3\mathbf{v}) = 2\mathrm{T}(\mathbf{u}) + 3\mathrm{T}(\mathbf{v})$.

   (c) For the unit vectors $\mathbf{i} = (1, 0, 0), \mathbf{j} = (0, 1, 0), \mathbf{k} = (0, 0, 1)$, write the matrix $[\mathrm{T}] = \begin{bmatrix}\mathrm{T}(\mathbf{i}) & \mathrm{T}(\mathbf{j}) & \mathrm{T}(\mathbf{k})\end{bmatrix}$ (i.e. write the matrix [T] whose columns are the vectors $\mathrm{T}(\mathbf{i}), \mathrm{T}(\mathbf{j}), \mathrm{T}(\mathbf{k})$)

Solution:

(a) $T(x, y, z) = (x + 3y + 0z, -2x + 0y + 5z)$

$\therefore [\mathrm{T}] \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} x + 3y + 0z \\ -2x + 0y + 5z \end{bmatrix}$

It follows that:

$[\mathrm{T}] = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix}$

(b) Firstly, $2\mathbf{u} + 3\mathbf{v} = (8, 31, 8)$

Hence, $\mathrm{T}(2\mathbf{u} + 3\mathbf{v}) = T(8, 31, 8) = (101, 24)$

Let us now check that we get the same result when we compute $2\mathrm{T}(\mathbf{u}) + 3\mathrm{T}(\mathbf{v})$

$$2\mathrm{T}(\mathbf{u}) + 3\mathrm{T}(\mathbf{v}) = 2 \cdot T(1, 5, -2) + 3 \cdot T(2, 7, 4)$$

But,

$$\mathrm{T}(\mathbf{u}) = T(1, 5, -2) = (16, -12)$$
$$\mathrm{T}(\mathbf{v}) = T(2, 7, 4) = (23, 16)$$

Hence,

$$2\mathrm{T}(\mathbf{u}) + 3\mathrm{T}(\mathbf{v}) = 2 \cdot (16, -12) + 3 \cdot (23, 16) = (101, 24)$$

(c)

$$\mathrm{T}(\mathbf{i}) = T(1, 0, 0) = (1, -2)$$
$$\mathrm{T}(\mathbf{j}) = T(0, 1, 0) = (3, 0)$$
$$\mathrm{T}(\mathbf{k}) = T(0, 0, 1) = (0, 5)$$

Hence,

$$[\mathrm{T}] = \begin{bmatrix}\mathrm{T}(\mathbf{i}) & \mathrm{T}(\mathbf{j}) & \mathrm{T}(\mathbf{k})\end{bmatrix} = \begin{bmatrix} 1 & 3 & 0 \\ -2 & 0 & 5 \end{bmatrix}$$

4. Solve the system $\begin{cases} 2x+y-2z=3 \\ x-y-\ \ z=0 \\ x+y+3z=10 \end{cases}$ by reducing into upper triangular form and using back substitution.

List all multipliers used and circle all pivots.

Solution:

Step 1: Elimination

$$\begin{cases} 2x+y-2z=3 \\ x-y-\ \ z=0 \\ x+y+3z=10 \end{cases} \xrightarrow{Eq(1)\leftrightarrow Eq(2)} \begin{cases} x-y-\ \ z=0 \\ 2x+y-2z=3 \\ x+y+3z=10 \end{cases} \xrightarrow{Eq(2)-2\cdot Eq(1)} \begin{cases} x-\ \ y-\ \ z=0 \\ \quad 3y \qquad =3 \\ x+\ \ y+3z=10 \end{cases}$$

$$\xrightarrow{Eq(3)-Eq(1)} \begin{cases} x-\ \ y-\ \ z=0 \\ \quad 3y \qquad =3 \\ \quad 2y+4z=10 \end{cases} \xrightarrow{Eq(3)-\frac{2}{3}Eq(2)} \begin{cases} x-\ \ y-\ \ z=0 \\ \quad 3y \qquad =3 \\ \qquad\quad 4z=8 \end{cases}$$

Step 2: Back Substitution

Equation(3) $4z=8$ implies that $z=2$

Equation(2) $3y=3$ implies that $y=1$

Substituting $z=2$ and $y=1$ into Equation(1) gives $x-3=0$ which implies $x=3$

Therefore, the system **solution** is $(x,y,z)=(3,1,2)$.

From the upper triangular form we can circle pivots:

$$\begin{aligned} 1x-y-z &= 2 \\ 3y &= 3 \\ 4z &= 8 \end{aligned}$$

And the multipliers are: $l_{21}=2$, $l_{31}=1$ and $l_{32}=\frac{2}{3}$

5. Given linear system $\begin{cases} (3a+1)x + 3y = -3 \\ \qquad\quad 4x - 6y = 6 \end{cases}$

(a) For what value(s) of a does the elimination fail (i) temporarily (ii)permanently?

Solution:

(i) If $a = -\frac{1}{3}$, there will be a 0 in the first pivot position but after a row exchange we will get a pivot in each pivot position. Therefore the elimination breaks down temporarily when $a = -\frac{1}{3}$.

(ii) If $a = -1$, the system represents parallel lines, so we have one equation only with two unknowns and the elimination fails permanently.

(b) Solve the system after fixing the temporary failure of the elimination.

Solution:

When $a = -\frac{1}{3}$, the given system becomes:

$$\begin{cases} 0x + 3y = -3 \\ 4x - 6y = 6 \end{cases}$$

by exchanging equations one and two:

$$\begin{cases} 4x - 6y = 6 \\ \qquad 3y = -3 \end{cases}$$

The second equation gives $y = -1$. From the first equation $4x - 6y = 6$ we have $4x = 6 + 6y \Rightarrow x = 0$. The solution is $(x, y) = (0, -1)$.

(c) Also solve the system in case of permanent failure of elimination.

Solution:

If $a = -1$, the system would permanently break down. In this situation,we have

$$\begin{cases} -2x + 3y = -3 \\ \quad 4x - 6y = 6 \end{cases} \xrightarrow[l_{21}=-2]{R_2+2R_1} \begin{cases} -2x + 3y \quad = -3 \\ \qquad\qquad 0 = 0 \end{cases}$$

The two lines have become one line. Every point on that line satisfies both equations, so there are infinitely many solutions. Assume $y = t$ (free variable), $\quad t \in \mathbb{R}$, then from equation(1), we get $-2x + 3t = -3 \Rightarrow x = \frac{3t+3}{2}$.

Therefore, the solution is $(x, y) = \left(\frac{3t+3}{2}, t\right), \quad t \in \mathbb{R}$.

6. Solve the system $\begin{cases} x \quad + \quad z = 6 \\ \quad -3y + \quad z = 7 \\ 2x + \quad y + 3z = 15 \end{cases}$ by reducing into upper triangular form and using back substitution.

List all multipliers used and circle all pivots.

Solution:

Step 1 (Elimination on R3)

$$\begin{cases} x \quad + \quad z = 6 \\ \quad -3y + \quad z = 7 \\ 2x + \quad y + 3z = 15 \end{cases} \xrightarrow{Eq(3)-2\cdot Eq(1)} \begin{cases} x \quad + z = 6 \\ \quad -3y + z = 7 \\ \quad\quad y + z = 3 \end{cases}$$

Step 2 (Elimination on R3)

$$\begin{cases} x \quad + z = 6 \\ \quad -3y + z = 7 \\ \quad\quad y + z = 3 \end{cases} \xrightarrow{Eq(3)+\frac{1}{3}\cdot Eq(2)} \begin{cases} x \quad + \quad z = 6 \\ \quad -3y + \quad z = 7 \\ \quad\quad \frac{4}{3}z = \frac{16}{3} \end{cases} \qquad (*)$$

Step 3: Back Substitution

From $(*)$ we have that Equation(3) implies that $z = 4$

Substituting $z = 4$ into Equation(2): $-3y + 4 = 7 \Rightarrow y = -1$

substituting $z = 4$ and $y = -1$ into Equation(1) gives $x + 4 = 6 \Rightarrow x = 2$

Therefore, the system solution is $(x, y, z) = (2, -1, 4)$.

From the upper triangular form we get the circled pivots:

$$\begin{cases} 1x + z = 6 \\ \text{-3}y + z = 7 \\ 4/3 z \quad = \frac{16}{3} \end{cases}$$

And the multipliers are: $l_{31} = 2$ and $l_{32} = -\frac{1}{3}$