CHAPTER 9
Relations

SECTION 9.1 Relations and Their Properties

This chapter is one of the most important in the book. Many structures in mathematics and computer science are formulated in terms of relations. Not only is the terminology worth learning, but the experience to be gained by working with various relations will prepare the student for the more advanced structures that he or she is sure to encounter in future work.

This section gives the basic terminology, especially the important notions of reflexivity, symmetry, antisymmetry, and transitivity. If we are given a relation as a set of ordered pairs, then reflexivity is easy to check for: we make sure that each element is related to itself. Symmetry is also fairly easy to test for: we make sure that no pair (a,b) *is in the relation without its opposite* (b,a) *being present as well. To check for antisymmetry we make sure that no pair* (a,b) *with* $a \neq b$ *and its opposite are both in the relation. In other words, at most one of* (a,b) *and* (b,a) *is in the relation if* $a \neq b$. *Transitivity is much harder to verify, since there are many triples of elements to check. A common mistake to try to avoid is forgetting that a transitive relation that has pairs* (a,b) *and* (b,a) *must also include* (a,a) *and* (b,b).

More importantly, we can be given a relation as a rule as to when elements are related. Exercises 4–7 are particularly useful in helping to understand the notions of reflexivity, symmetry, antisymmetry, and transitivity for relations given in this manner. Here you have to ask yourself the appropriate questions in order to determine whether the properties hold. Is every element related to itself? If so, the relation is reflexive. Are the roles of the variables in the definition interchangeable? If so, then the relation is symmetric. Does the definition preclude two different elements from each being related to the other? If so, then the relation is antisymmetric. Does the fact that one element is related to a second, which is in turn related to a third, mean that the first is related to the third? If so, then the relation is transitive.

In general, try to think of a relation in these two ways at the same time: as a set of ordered pairs and as a propositional function describing a relationship among objects.

1. In each case, we need to find all the pairs (a,b) with $a \in A$ and $b \in B$ such that the condition is satisfied. This is straightforward.

a) $\{(0,0),(1,1),(2,2),(3,3)\}$ **b)** $\{(1,3),(2,2),(3,1),(4,0)\}$

c) $\{(1,0),(2,0),(2,1),(3,0),(3,1),(3,2),(4,0),(4,1),(4,2),(4,3)\}$

d) Recall that $a \mid b$ means that b is a multiple of a (a is not allowed to be 0). Thus the answer is $\{(1,0),(1,1),(1,2),(1,3),(2,0),(2,2),(3,0),(3,3),(4,0)\}$.

e) We need to look for pairs whose greatest common divisor is 1—in other words, pairs that are relatively prime. Thus the answer is $\{(0,1),(1,0),(1,1),(1,2),(1,3),(2,1),(2,3),(3,1),(3,2),(4,1),(4,3)\}$.

f) There are not very many pairs of numbers (by definition only positive integers are considered) whose least common multiple is 2: only 1 and 2, and 2 and 2. Thus the answer is $\{(1,2),(2,1),(2,2)\}$.

3. **a)** This relation is not reflexive, since it does not include, for instance $(1,1)$. It is not symmetric, since it includes, for instance, $(2,4)$ but not $(4,2)$. It is not antisymmetric since it includes both $(2,3)$ and $(3,2)$, but $2 \neq 3$. It is transitive. To see this we have to check that *whenever* it includes (a,b) and (b,c), then it also includes (a, c). We can ignore the element 1 since it never appears. If (a, b) is in this relation, then by inspection we see that a must be either 2 or 3. But $(2, c)$ and $(3, c)$ are in the relation for all $c \neq 1$; thus (a, c) has to be in this relation whenever (a, b) and (b, c) are. This proves that the relation is transitive. Note that it is very tedious to prove transitivity for an arbitrary list of ordered pairs.

b) This relation is reflexive, since all the pairs $(1,1)$, $(2,2)$, $(3,3)$, and $(4,4)$ are in it. It is clearly symmetric, the only nontrivial case to note being that both $(1,2)$ and $(2,1)$ are in the relation. It is not antisymmetric because both $(1,2)$ and $(2,1)$ are in the relation. It is transitive; the only nontrivial cases to note are that since both $(1,2)$ and $(2,1)$ are in the relation, we need to have (and do have) both $(1,1)$ and $(2,2)$ included as well.

c) This relation clearly is not reflexive and clearly is symmetric. It is not antisymmetric since both $(2,4)$ and $(4,2)$ are in the relation. It is not transitive, since although $(2,4)$ and $(4,2)$ are in the relation, $(2,2)$ is not.

d) This relation is clearly not reflexive. It is not symmetric, since, for instance, $(1,2)$ is included but $(2,1)$ is not. It is antisymmetric, since there are no cases of (a, b) and (b, a) both being in the relation. It is not transitive, since although $(1,2)$ and $(2,3)$ are in the relation, $(1,3)$ is not.

e) This relation is clearly reflexive and symmetric. It is trivially antisymmetric since there are no pairs (a, b) in the relation with $a \neq b$. It is trivially transitive, since the only time the hypothesis $(a, b) \in R \wedge (b, c) \in R$ is met is when $a = b = c$.

f) This relation is clearly not reflexive. The presence of $(1,4)$ and absence of $(4,1)$ shows that it is not symmetric. The presence of both $(1,3)$ and $(3,1)$ shows that it is not antisymmetric. It is not transitive; both $(2,3)$ and $(3,1)$ are in the relation, but $(2,1)$ is not, for instance.

5. Recall the definitions: R is reflexive if $(a, a) \in R$ for all a; R is symmetric if $(a, b) \in R$ always implies $(b, a) \in R$; R is antisymmetric if $(a, b) \in R$ and $(b, a) \in R$ always implies $a = b$; and R is transitive if $(a, b) \in R$ and $(b, c) \in R$ always implies $(a, c) \in R$.

a) It is tautological that everyone who has visited Web page a has also visited Web page a, so R is reflexive. It is not symmetric, because there surely are Web pages a and b such that the set of people who visited a is a proper subset of the set of people who visited b (for example, the only link to page a may be on page b). Whether R is antisymmetric in truth is hard to say, but it is certainly conceivable that there are two different Web pages a and b that have had exactly the same set of visitors. In this case, $(a, b) \in R$ and $(b, a) \in R$, so R is not antisymmetric. Finally, R is transitive: if everyone who has visited a has also visited b, and everyone who has visited b has also visited c, then clearly everyone who has visited a has also visited c.

b) This relation is not reflexive, because for any page a that has links on it, $(a, a) \notin R$. The definition of R is symmetric in its very statement, so R is clearly symmetric. Also R is certainly not antisymmetric, because there surely are two different Web pages a and b out there that have no common links found on them. Finally, R is not transitive, because the two Web pages just mentioned, assuming they have links at all, give an example of the failure of the definition: $(a, b) \in R$ and $(b, a) \in R$, but $(a, a) \notin R$.

c) This relation is not reflexive, because for any page a that has no links on it, $(a, a) \notin R$. The definition of R is symmetric in its very statement, so R is clearly symmetric. Also R is certainly not antisymmetric, because there surely are two different Web pages a and b out there that have a common link found on them. Finally, R is surely not transitive. Page a might have only one link (say to this textbook), page c might have only one link different from this (say to the Erdős Number Project), and page b may have only the two links mentioned in this sentence. Then $(a, b) \in R$ and $(b, c) \in R$, but $(a, c) \notin R$.

d) This relation is probably not reflexive, because there probably exist Web pages out there with no links at all to them (for example, when they are in the process of being written and tested); for any such page a we have $(a, a) \notin R$. The definition of R is symmetric in its very statement, so R is clearly symmetric. Also R is certainly not antisymmetric, because there surely are two different Web pages a and b out there that are referenced by some third page. Finally, R is surely not transitive. Page a might have only one page that links to it, page c might also have only one page, different from this, that links to it, and page b may be cited on both of these two pages. Then there would be no page that includes links to both pages a and c, so we have $(a,b) \in R$ and $(b,c) \in R$, but $(a,c) \notin R$.

7. a) This relation is not reflexive since it is not the case that $1 \neq 1$, for instance. It is symmetric: if $x \neq y$, then of course $y \neq x$. It is not antisymmetric, since, for instance, $1 \neq 2$ and also $2 \neq 1$. It is not transitive, since $1 \neq 2$ and $2 \neq 1$, for instance, but it is not the case that $1 \neq 1$.

b) This relation is not reflexive, since $(0,0)$ is not included. It is symmetric, because the commutative property of multiplication tells us that $xy = yx$, so that one of these quantities is greater than or equal to 1 if and only if the other is. It is not antisymmetric, since, for instance, $(2,3)$ and $(3,2)$ are both included. It is transitive. To see this, note that the relation holds between x and y if and only if either x and y are both positive or x and y are both negative. So assume that (a,b) and (b,c) are both in the relation. There are two cases, nearly identical. If a is positive, then so is b, since $(a,b) \in R$; therefore so is c, since $(b,c) \in R$, and hence $(a,c) \in R$. If a is negative, then so is b, since $(a,b) \in R$; therefore so is c, since $(b,c) \in R$, and hence $(a,c) \in R$.

c) This relation is not reflexive, since $(1,1)$ is not included, for instance. It is symmetric; the equation $x = y - 1$ is equivalent to the equation $y = x + 1$, which is the same as the equation $x = y + 1$ with the roles of x and y reversed. (A more formal proof of symmetry would be by cases. If x is related to y then either $x = y + 1$ or $x = y - 1$. In the former case, $y = x - 1$, so y is related to x; in the latter case $y = x + 1$, so y is related to x.) It is not antisymmetric, since, for instance, both $(1,2)$ and $(2,1)$ are in the relation. It is not transitive, since, for instance, although both $(1,2)$ and $(2,1)$ are in the relation, $(1,1)$ is not.

d) Recall that $x \equiv y \pmod 7$ means that $x - y$ is a multiple of 7, i.e., that $x - y = 7t$ for some integer t. This relation is reflexive, since $x - x = 7 \cdot 0$ for all x. It is symmetric, since if $x \equiv y \pmod 7$, then $x - y = 7t$ for some t; therefore $y - x = 7(-t)$, so $y \equiv x \pmod 7$. It is not antisymmetric, since, for instance, we have both $2 \equiv 9$ and $9 \equiv 2 \pmod 7$. It is transitive. Suppose $x \equiv y$ and $y \equiv z \pmod 7$. This means that $x - y = 7s$ and $y - z = 7t$ for some integers s and t. The trick is to add these two equations and note that the y disappears; we get $x - z = 7s + 7t = 7(s+t)$. By definition, this means that $x \equiv z \pmod 7$, as desired.

e) Every number is a multiple of itself (namely 1 times itself), so this relation is reflexive. (There is one bit of controversy here; we assume that 0 is to be considered a multiple of 0, even though we do not consider that 0 is a divisor of 0.) It is clearly not symmetric, since, for instance, 6 is a multiple of 2, but 2 is not a multiple of 6. The relation is not antisymmetric either; we have that 2 is a multiple of -2, for instance, and -2 is a multiple of 2, but $2 \neq -2$. The relation is transitive, however. If x is a multiple of y (say $x = ty$), and y is a multiple of z (say $y = sz$), then we have $x = t(sz) = (ts)z$, so we know that x is a multiple of z.

f) This relation is reflexive, since a and a are either both negative or both nonnegative. It is clearly symmetric from its form. It is not antisymmetric, since 5 is related to 6 and 6 is related to 5, but $5 \neq 6$. Finally, it is transitive, since if a is related to b and b is related to c, then all three of them must be negative, or all three must be nonnegative.

g) This relation is not reflexive, since, for instance, $17 \neq 17^2$. It is not symmetric, since although $289 = 17^2$, it is not the case that $17 = 289^2$. To see whether it is antisymmetric, suppose that we have both (x,y) and (y,x) in the relation. Then $x = y^2$ and $y = x^2$. To solve this system of equations, plug the second into the first, to obtain $x = x^4$, which is equivalent to $x - x^4 = 0$. The left-hand side factors as $x(1-x^3) = x(1-x)(1+x+x^2)$, so the solutions for x are 0 and 1 (and a pair of irrelevant complex numbers). The corresponding solutions for y are therefore also 0 and 1. Thus the only time we have both $x = y^2$ and $y = x^2$ is when $x = y$; this means that the relation is antisymmetric. It is not transitive, since, for example, $16 = 4^2$ and $4 = 2^2$, but $16 \neq 2^2$.

h) This relation is not reflexive, since, for instance, $17 \not\geq 17^2$. It is not symmetric, since although $289 \geq 17^2$, it is not the case that $17 \geq 289^2$. To see whether it is antisymmetric, we assume that both (x,y) and (y,x)

are in the relation. Then $x \geq y^2$ and $y \geq x^2$. Since both sides of the second inequality are nonnegative, we can square both sides to get $y^2 \geq x^4$. Combining this with the first inequality, we have $x \geq x^4$, which is equivalent to $x - x^4 \geq 0$. The left-hand side factors as $x(1-x^3) = x(1-x)(1+x+x^2)$. The last factor is always positive, so we can divide the original inequality by it to obtain the equivalent inequality $x(1-x) \geq 0$. Now if $x > 1$ or $x < 0$, then the factors have different signs, so the inequality does not hold. Thus the only solutions are $x = 0$ and $x = 1$. The corresponding solutions for y are therefore also 0 and 1. Thus the only time we have both $x \geq y^2$ and $y \geq x^2$ is when $x = y$; this means that the relation is antisymmetric. It is transitive. Suppose $x \geq y^2$ and $y \geq z^2$. Again the second inequality implies that both sides are nonnegative, so we can square both sides to obtain $y^2 \geq z^4$. Combining these inequalities gives $x \geq z^4$. Now we claim that it is always the case that $z^4 \geq z^2$; if so, then we combine this fact with the last inequality to obtain $x \geq z^2$, so x is related to z. To verify the claim, note that since we are working with integers, it is always the case that $z^2 \geq |z|$ (equality for $z = 0$ and $z = 1$, strict inequality for other z). Squaring both sides gives the desired inequality.

9. Each of the properties is a universally quantified statement. Because the domain is empty, each of them is vacuously true.

11. The relations in parts **(a)**, **(b)**, and **(e)** all have at least one pair of the form (x, x) in them, so they are not irreflexive. The relations in parts **(c)**, **(d)**, and **(f)** do not, so they are irreflexive.

13. According to the preamble to Exercise 11, an irreflexive relation is one for which a is never related to itself; i.e., $\forall a((a,a) \notin R)$.

a) Since we saw in Exercise 5a that $\forall a((a,a) \in R)$, clearly R is not irreflexive.

b) Since there are probably pages a with no links at all, and for such pages it is true that there are no common links found on both page a and page a, this relation is probably not irreflexive.

c) This relation is not irreflexive, because for any page a that has links on it, $(a,a) \in R$.

d) This relation is not irreflexive, because for any page a that has links on it that are ever cited, $(a,a) \in R$.

15. The relation in Exercise 3a is neither reflexive nor irreflexive. It contains some of the pairs (a,a) but not all of them.

17. Of course many answers are possible. The empty relation is always irreflexive (x is never related to y). A less trivial example would be $(a,b) \in R$ if and only if a is taller than b. Since nobody is taller than him/herself, we always have $(a,a) \notin R$.

19. The relation in part **(a)** is asymmetric, since if a is taller than b, then certainly b cannot be taller than a. The relation in part **(b)** is not asymmetric, since there are many instances of a and b born on the same day (both cases in which $a = b$ and cases in which $a \neq b$), and in all such cases, it is also the case that b and a were born on the same day. The relations in part **(c)** and part **(d)** are just like that in part **(b)**, so they, too, are not asymmetric.

21. According to the preamble to Exercise 18, an asymmetric relation is one for which $(a,b) \in R$ and $(b,a) \in R$ can never hold simultaneously, even if $a = b$. Thus R is asymmetric if and only if R is antisymmetric and also irreflexive.

a) not asymmetric since $(-1,1) \in R$ and $(1,-1) \in R$

b) not asymmetric since $(-1,1) \in R$ and $(1,-1) \in R$

c) not asymmetric since $(-1,1) \in R$ and $(1,-1) \in R$

d) not asymmetric since $(0,0) \in R$

e) not asymmetric since $(2,1) \in R$ and $(1,2) \in R$

f) not asymmetric since $(0,1) \in R$ and $(1,0) \in R$

g) not asymmetric since $(1,1) \in R$

h) not asymmetric since $(2,1) \in R$ and $(1,2) \in R$

23. According to the preamble to Exercise 18, an asymmetric relation is one for which $(a,b) \in R$ and $(b,a) \in R$ can never hold simultaneously. In symbols, this is simply $\forall a \forall b \neg((a,b) \in R \wedge (b,a) \in R)$. Alternatively, $\forall a \forall b\,((a,b) \in R \to (b,a) \notin R)$.

25. There are mn elements of the set $A \times B$, if A is a set with m elements and B is a set with n elements. A relation from A to B is a subset of $A \times B$. Thus the question asks for the number of subsets of the set $A \times B$, which has mn elements. By the product rule, it is 2^{mn}.

27. a) By definition the answer is $\{\,(b,a) \mid a \text{ divides } b\,\}$, which, by changing the names of the dummy variables, can also be written $\{\,(a,b) \mid b \text{ divides } a\,\}$. (The universal set is still the set of positive integers.)

b) By definition the answer is $\{\,(a,b) \mid a \text{ does not divide } b\,\}$. (The universal set is still the set of positive integers.)

29. The inverse relation is just the graph of the inverse function. Somewhat more formally, we have $R^{-1} = \{\,(f(a),a) \mid a \in A\,\} = \{\,(b,f^{-1}(b)) \mid b \in B\,\}$, since we can index this collection just as easily by elements of B as by elements of A (using the correspondence $b = f(a)$).

31. This exercise is just a matter of the definitions of the set operations.

a) the set of pairs (a,b) where a is required to read b in a course or has read b

b) the set of pairs (a,b) where a is required to read b in a course and has read b

c) the set of pairs (a,b) where a is required to read b in a course or has read b, but not both; equivalently, the set of pairs (a,b) where a is required to read b in a course but has not done so, or has read b although not required to do so in a course

d) the set of pairs (a,b) where a is required to read b in a course but has not done so

e) the set of pairs (a,b) where a has read b although not required to do so in a course

33. To find $S \circ R$ we want to find the set of pairs (a,c) such that for some person b, a is a parent of b, and b is a sibling of c. Since brothers and sisters have the same parents, this means that a is also the parent of c. Thus $S \circ R$ is contained in the relation R. More specifically, $(a,c) \in S \circ R$ if and only if a is the parent of c, and c has a sibling (who is necessarily also a child of a). To find $R \circ S$ we want to find the set of pairs (a,c) such that for some person b, a is a sibling of b, and b is a parent of c. This is the same as the condition that a is the aunt or uncle of c (by blood, not marriage).

35. a) The union of two relations is the union of these sets. Thus $R_2 \cup R_4$ holds between two real numbers if R_2 holds or R_4 holds (or both, it goes without saying). Since it is always true that $a \le b$ or $b \le a$, $R_2 \cup R_4$ is all of $\mathbf{R}^2$, i.e., the relation that always holds.

b) For (a,b) to be in $R_3 \cup R_6$, we must have $a < b$ or $a \ne b$. Since this happens precisely when $a \ne b$, we see that the answer is R_6.

c) The intersection of two relations is the intersection of these sets. Thus $R_3 \cap R_6$ holds between two real numbers if R_3 holds and R_6 holds as well. Thus for (a,b) to be in $R_3 \cap R_6$, we must have $a < b$ and $a \ne b$. Since this happens precisely when $a < b$, we see that the answer is R_3.

d) For (a,b) to be in $R_4 \cap R_6$, we must have $a \le b$ and $a \ne b$. Since this happens precisely when $a < b$, we see that the answer is R_3.

e) Recall that $R_3 - R_6 = R_3 \cap \overline{R_6}$. But $\overline{R_6} = R_5$, so we are asked for $R_3 \cap R_5$. It is impossible for $a < b$ and $a = b$ to hold at the same time, so the answer is $\emptyset$, i.e., the relation that never holds.

f) Reasoning as in part (**e**), we want $R_6 \cap \overline{R_3} = R_6 \cap R_2$, which is clearly R_1 (since $a \neq b$ and $a \geq b$ precisely when $a > b$).

g) Recall that $R_2 \oplus R_6 = (R_2 \cap \overline{R_6}) \cup (R_6 \cap \overline{R_2})$. We see that $R_2 \cap \overline{R_6} = R_2 \cap R_5 = R_5$, and $R_6 \cap \overline{R_2} = R_6 \cap R_3 = R_3$. Thus our answer is $R_5 \cup R_3 = R_4$.

h) Recall that $R_3 \oplus R_5 = (R_3 \cap \overline{R_5}) \cup (R_5 \cap \overline{R_3})$. We see that $R_3 \cap \overline{R_5} = R_3 \cap R_6 = R_3$, and $R_5 \cap \overline{R_3} = R_5 \cap R_2 = R_5$. Thus our answer is $R_3 \cup R_5 = R_4$.

37. Recall that the composition of two relations all defined on a common set is defined as follows: $(a,c) \in S \circ R$ if and only if there is some element b such that $(a,b) \in R$ and $(b,c) \in S$. We have to apply this in each case.

a) For (a,c) to be in $R_2 \circ R_1$, we must find an element b such that $(a,b) \in R_1$ and $(b,c) \in R_2$. This means that $a > b$ and $b \geq c$. Clearly this can be done if and only if $a > c$ to begin with. But that is precisely the statement that $(a,c) \in R_1$. Therefore we have $R_2 \circ R_1 = R_1$.

b) For (a,c) to be in $R_2 \circ R_2$, we must find an element b such that $(a,b) \in R_2$ and $(b,c) \in R_2$. This means that $a \geq b$ and $b \geq c$. Clearly this can be done if and only if $a \geq c$ to begin with. But that is precisely the statement that $(a,c) \in R_2$. Therefore we have $R_2 \circ R_2 = R_2$. In particular, this shows that R_2 is transitive.

c) For (a,c) to be in $R_3 \circ R_5$, we must find an element b such that $(a,b) \in R_5$ and $(b,c) \in R_3$. This means that $a = b$ and $b < c$. Clearly this can be done if and only if $a < c$ to begin with (choose $b = a$). But that is precisely the statement that $(a,c) \in R_3$. Therefore we have $R_3 \circ R_5 = R_3$. One way to look at this is to say that R_5, the equality relation, acts as an identity for the composition operation (on the right—although it is also an identity on the left as well).

d) For (a,c) to be in $R_4 \circ R_1$, we must find an element b such that $(a,b) \in R_1$ and $(b,c) \in R_4$. This means that $a > b$ and $b \leq c$. Clearly this can always be done simply by choosing b to be small enough. Therefore we have $R_4 \circ R_1 = \mathbf{R}^2$, the relation that always holds.

e) For (a,c) to be in $R_5 \circ R_3$, we must find an element b such that $(a,b) \in R_3$ and $(b,c) \in R_5$. This means that $a < b$ and $b = c$. Clearly this can be done if and only if $a < c$ to begin with (choose $b = c$). But that is precisely the statement that $(a,c) \in R_3$. Therefore we have $R_5 \circ R_3 = R_3$. One way to look at this is to say that R_5, the equality relation, acts as an identity for the composition operation (on the left—although it is also an identity on the right as well).

f) For (a,c) to be in $R_3 \circ R_6$, we must find an element b such that $(a,b) \in R_6$ and $(b,c) \in R_3$. This means that $a \neq b$ and $b < c$. Clearly this can always be done simply by choosing b to be small enough. Therefore we have $R_3 \circ R_6 = \mathbf{R}^2$, the relation that always holds.

g) For (a,c) to be in $R_4 \circ R_6$, we must find an element b such that $(a,b) \in R_6$ and $(b,c) \in R_4$. This means that $a \neq b$ and $b \leq c$. Clearly this can always be done simply by choosing b to be small enough. Therefore we have $R_4 \circ R_6 = \mathbf{R}^2$, the relation that always holds.

h) For (a,c) to be in $R_6 \circ R_6$, we must find an element b such that $(a,b) \in R_6$ and $(b,c) \in R_6$. This means that $a \neq b$ and $b \neq c$. Clearly this can always be done simply by choosing b to be something other than a or c. Therefore we have $R_6 \circ R_6 = \mathbf{R}^2$, the relation that always holds. Note that since the answer is not R_6 itself, we know that R_6 is not transitive.

39. One earns a doctorate by, among other things, writing a thesis under an advisor, so this relation makes sense. (We ignore anomalies like someone having two advisors or someone being awarded a doctorate without having an advisor.) For (a,b) to be in R^2, we must find a c such that $(a,c) \in R$ and $(c,b) \in R$. In our context, this says that b got his/her doctorate under someone who got his/her doctorate under a. Colloquially, a is the academic grandparent of b, or b is the academic grandchild of a. Generalizing, $(a,b) \in R^n$ precisely when there is a sequence of $n+1$ people, starting with a and ending with b, such that each is the advisor of the next person in the sequence. People with doctorates like to look at these sequences (and trace their ancestry) back as far as they can. There is an excellent website for doing so in mathematics (`www.genealogy.math.ndsu.nodak.edu`).

41. **a)** The union of two relations is the union of these sets. Thus $R_1 \cup R_2$ holds between two integers if R_1 holds or R_2 holds (or both, it goes without saying). Thus $(a, b) \in R_1 \cup R_2$ if and only if $a \equiv b \pmod{3}$ or $a \equiv b \pmod{4}$. There is not a good easier way to state this, other than perhaps to say that $a - b$ is a multiple of either 3 or 4, or to work modulo 12 and write $a - b \equiv 0,\ 3,\ 4,\ 6,\ 8,\ \text{or}\ 9 \pmod{12}$.

b) The intersection of two relations is the intersection of these sets. Thus $R_1 \cap R_2$ holds between two integers if R_1 holds and R_2 holds. Thus $(a, b) \in R_1 \cap R_2$ if and only if $a \equiv b \pmod{3}$ and $a \equiv b \pmod{4}$. Since this means that $a - b$ is a multiple of both 3 and 4, and that happens if and only if $a - b$ is a multiple of 12, we can state this more simply as $a \equiv b \pmod{12}$.

c) By definition $R_1 - R_2 = R_1 \cap \overline{R_2}$. Thus this relation holds between two integers if R_1 holds and R_2 does not hold. We can write this in symbols by saying that $(a, b) \in R_1 - R_2$ if and only if $a \equiv b \pmod{3}$ and $a \not\equiv b \pmod{4}$. We could, if we wished, state this working modulo 12: $(a, b) \in R_1 - R_2$ if and only if $a - b \equiv 3,\ 6,\ \text{or}\ 9 \pmod{12}$.

d) By definition $R_2 - R_1 = R_2 \cap \overline{R_1}$. Thus this relation holds between two integers if R_2 holds and R_1 does not hold. We can write this in symbols by saying that $(a, b) \in R_2 - R_1$ if and only if $a \equiv b \pmod{4}$ and $a \not\equiv b \pmod{3}$. We could, if we wished, state this working modulo 12: $(a, b) \in R_2 - R_1$ if and only if $a - b \equiv 4 \text{ or } 8 \pmod{12}$.

e) We know that $R_1 \oplus R_2 = (R_1 - R_2) \cup (R_2 - R_1)$, so we look at our solutions to part **(c)** and part **(d)**. Thus this relation holds between two integers if R_1 holds and R_2 does not hold, or vice versa. We can write this in symbols by saying that $(a, b) \in R_1 \oplus R_2$ if and only if $(a \equiv b \pmod{3}$ and $a \not\equiv b \pmod{4})$ or $(a \equiv b \pmod{4}$ and $a \not\equiv b \pmod{3})$. We could, if we wished, state this working modulo 12: $(a, b) \in R_1 \oplus R_2$ if and only if $a - b \equiv 3,\ 4,\ 6,\ 8 \text{ or } 9 \pmod{12}$. We could also say that $a - b$ is a multiple of 3 or 4 but not both.

43. A relation is just a subset. A subset can either contain a specified element or not; half of them do and half of them do not. Therefore 8 of the 16 relations on $\{0, 1\}$ contain the pair $(0, 1)$. Alternatively, a relation on $\{0, 1\}$ containing the pair $(0, 1)$ is just a set of the form $\{(0, 1)\} \cup X$, where $X \subseteq \{(0, 0), (1, 0), (1, 1)\}$. Since this latter set has 3 elements, it has $2^3 = 8$ subsets.

45. This is similar to Example 16 in this section.

a) A relation on a set S with n elements is a subset of $S \times S$. Since $S \times S$ has n^2 elements, we are asking for the number of subsets of a set with n^2 elements, which is 2^{n^2}. In our case $n = 4$, so the answer is $2^{16} = 65{,}536$.

b) In solving part **(a)**, we had 16 binary choices to make—whether to include a pair (x, y) in the relation or not as x and y ranged over the set $\{a, b, c, d\}$. In this part, one of those choices has been made for us: we *must* include (a, a). We are free to make the other 15 choices. So the answer is $2^{15} = 32{,}768$. See Exercise 47 for more problems of this type.

47. These are combinatorics problems, some harder than others. Let A be the set with n elements on which the relations are defined.

a) To specify a symmetric relation, we need to decide, for each unordered pair $\{a, b\}$ of distinct elements of A, whether to include the pairs (a, b) and (b, a) or leave them out; this can be done in 2 ways for each such unordered pair. Also, for each element $a \in A$, we need to decide whether to include (a, a) or not, again 2 possibilities. We can think of these two parts as one by considering an element to be an unordered pair with repetition allowed. Thus we need to make this 2-fold choice $C(n+1, 2)$ times, since there are $C(n+2-1, 2)$ ways to choose an unordered pair with repetition allowed. Therefore the answer is $2^{C(n+1,2)} = 2^{n(n+1)/2}$.

b) This is somewhat similar to part **(a)**. For each unordered pair $\{a,b\}$ of distinct elements of A, we have a 3-way choice—either include (a,b) only, include (b,a) only, or include neither. For each element of A we have a 2-way choice. Therefore the answer is $3^{C(n,2)}2^n = 3^{n(n-1)/2}2^n$.

c) As in part **(b)** we have a 3-way choice for $a \neq b$. There is no choice about including (a,a) in the relation—the definition prohibits it. Therefore the answer is $3^{C(n,2)} = 3^{n(n-1)/2}$.

d) For each ordered pair (a,b), with $a \neq b$ (and there are $P(n,2)$ such pairs), we can choose to include (a,b) or to leave it out. There is no choice for pairs (a,a). Therefore the answer is $2^{P(n,2)} = 2^{n(n-1)}$.

e) This is just like part **(a)**, except that there is no choice about including (a,a). For each unordered pair of distinct elements of A, we can choose to include neither or both of the corresponding ordered pairs. Therefore the answer is $2^{C(n,2)} = 2^{n(n-1)/2}$.

f) We have complete freedom with the ordered pairs (a,b) with $a \neq b$, so that part of the choice gives us $2^{P(n,2)}$ possibilities, just as in part **(d)**. For the decision as to whether to include (a,a), two of the 2^n possibilities are prohibited: we cannot include all such pairs, and we cannot leave them all out. Therefore the answer is $2^{P(n,2)}(2^n-2) = 2^{n^2-n}(2^n-2) = 2^{n^2} - 2^{n^2-n+1}$.

49. The second sentence of the proof asks us to "take an element $b \in A$ such that $(a,b) \in R$." There is no guarantee that such an element exists for the taking. This is the only mistake in the proof. If one could be guaranteed that each element in A is related to at least one element, then symmetry and transitivity would indeed imply reflexivity. Without this assumption, however, the proof and the proposition are wrong. As a simple example, take the relation $\emptyset$ on any nonempty set. This relation is vacuously symmetric and transitive, but not reflexive. Here is another counterexample: the relation $\{(1,1),(1,2),(2,1),(2,2)\}$ on the set $\{1,2,3\}$.

51. We need to show two things. First, we need to show that if a relation R is symmetric, then $R = R^{-1}$, which means we must show that $R \subseteq R^{-1}$ and $R^{-1} \subseteq R$. To do this, let $(a,b) \in R$. Since R is symmetric, this implies that $(b,a) \in R$. But since R^{-1} consists of all pairs (a,b) such that $(b,a) \in R$, this means that $(a,b) \in R^{-1}$. Thus we have shown that $R \subseteq R^{-1}$. Next let $(a,b) \in R^{-1}$. By definition this means that $(b,a) \in R$. Since R is symmetric, this implies that $(a,b) \in R$ as well. Thus we have shown that $R^{-1} \subseteq R$.

Second we need to show that $R = R^{-1}$ implies that R is symmetric. To this end we let $(a,b) \in R$ and try to show that (b,a) is also necessarily an element of R. Since $(a,b) \in R$, the definition tells us that $(b,a) \in R^{-1}$. But since we are under the hypothesis that $R = R^{-1}$, this tells us that $(b,a) \in R$, exactly as desired.

53. Suppose that R is reflexive. We must show that R^{-1} is reflexive, i.e., that $(a,a) \in R^{-1}$ for each $a \in A$. Now since R is reflexive, we know that $(a,a) \in R$ for each $a \in R$. By definition, this tells us that $(a,a) \in R^{-1}$, as desired. (Interchanging the two a's in the pair (a,a) leaves it as it was.) Conversely, if R^{-1} is reflexive, then $(a,a) \in R^{-1}$ for each $a \in A$. By definition this means that $(a,a) \in R$ (again we interchanged the two a's).

55. We prove this by induction on n. The case $n=1$ is trivial, since it is the statement $R = R$. Assume the inductive hypothesis that $R^n = R$. We must show that $R^{n+1} = R$. By definition $R^{n+1} = R^n \circ R$. Thus our task is to show that $R^n \circ R \subseteq R$ and $R \subseteq R^n \circ R$. The first uses the transitivity of R, as follows. Suppose that $(a,c) \in R^n \circ R$. This means that there is an element b such that $(a,b) \in R$ and $(b,c) \in R^n$. By the inductive hypothesis, the latter statement implies that $(b,c) \in R$. Thus by the transitivity of R, we know that $(a,c) \in R$, as desired.

Next assume that $(a,b) \in R$. We must show that $(a,b) \in R^n \circ R$. By the inductive hypothesis, $R^n = R$, and therefore R^n is reflexive by assumption. Thus $(b,b) \in R^n$. Since we have $(a,b) \in R$ and $(b,b) \in R^n$, we have by definition that (a,b) is an element of $R^n \circ R$, exactly as desired. (The first half of this proof was not really necessary, since Theorem 1 in this section already told us that $R^n \subseteq R$ for all n.)

57. We use induction on n, the result being trivially true for $n = 1$. Assume that R^n is reflexive; we must show that R^{n+1} is reflexive. Let $a \in A$, where A is the set on which R is defined. By definition $R^{n+1} = R^n \circ R$. By the inductive hypothesis, R^n is reflexive, so $(a,a) \in R^n$. Also, since R is reflexive by assumption, $(a,a) \in R$. Therefore by the definition of composition, $(a,a) \in R^n \circ R$, as desired.

59. It is not necessarily true that R^2 is irreflexive when R is. We might have pairs (a,b) and (b,a) both in R, with $a \neq b$; then it would follow that $(a,a) \in R^2$, preventing R^2 from being irreflexive. As the simplest example, let $A = \{1,2\}$ and let $R = \{(1,2),(2,1)\}$. Then R is clearly irreflexive. In this case $R^2 = \{(1,1),(2,2)\}$, which is not irreflexive.

SECTION 9.2 *n*-ary Relations and Their Applications

This section is a brief introduction to relational models for data bases. The exercises are straightforward and similar to the examples. Projections are formed by omitting certain columns, and then eliminating duplicate rows. Joins are analogous to compositions of relations.

1. We simply need to find solutions of the inequality, which we can do by common sense. The set is $\{(1,2,3),$ $(1,2,4),(1,3,4),(2,3,4)\}$.

3. The 5-tuples are just the lines of the table. Thus the relation is
$\{$(Nadir, 122, 34, Detroit, 08: 10), (Acme, 221, 22, Denver, 08: 17), (Acme, 122, 33, Anchorage, 08: 22), (Acme, 323, 34, Honolulu, 08: 30), (Nadir, 199, 13, Detroit, 08: 47), (Acme, 222, 22, Denver, 09: 10), (Nadir, 322, 34, Detroit, 09: 44)$\}$.

5. We need to find a field that, when used along with the *Airline* field uniquely specifies a row of the table. Certainly *Flight_number* is one such field, since there is only one line of the table for each pair (*Airline*, *Flight_number*); no airline has the same flight number for two different flights. *Gate* and *Destination* do not qualify, however, since Nadir has two flights leaving from Gate 34 going to Detroit. Finally, *Departure_time* is a key by itself (no two flights leave at the same time), so it and *Airline* form a composite key as well.

7. **a)** A school would be giving itself a lot of headaches if it didn't make the student ID number different for each student, so student ID is likely to be a primary key.
b) Name could very easily not be a primary key. Two people named Jennifer Johnson might easily both be students.
c) Phone number is unlikely to be a primary key. Two roommates, or two siblings living at home, will likely have the same phone number, and they might both be students at that school.

9. **a)** Everyone has a different Social Security number, so that field will be a primary key.
b) It is unlikely that (name, street address) will be a composite key. Somewhere in the United States there could easily be two people named Jennifer Johnson both living at 123 Washington Street. In order for this to work, there must never be two people with the same name who happen to have the same street address.
c) For this to work, we must never have two people with the same name living together. Given the size of the country, one would doubt that this would work. For example, a husband and wife each named Morgan White would make this not a composite key, as would a mother and daughter living at home with the same name.

11. The selection operator picks out all the tuples that match the criteria. The 5-tuples in Table 8 that have Detroit as their destination are (Nadir, 122, 34, Detroit, 08: 10), (Nadir, 199, 13, Detroit, 08: 47), and (Nadir, 322, 34, Detroit, 09: 44).

13. The selection operator picks out all the tuples that match the criteria. The 5-tuples in Table 8 that have Nadir as their airline are (Nadir, 122, 34, Detroit, 08: 10), (Nadir, 199, 13, Detroit, 08: 47), and (Nadir, 322, 34, Detroit, 09: 44). The 5-tuples in Table 8 that have Denver as their destination are (Acme, 221, 22, Denver, 08: 17) and (Acme, 222, 22, Denver, 09: 10). We need the union of these two lists: (Nadir, 122, 34, Detroit, 08: 10), (Nadir, 199, 13, Detroit, 08: 47), (Nadir, 322, 34, Detroit, 09: 44), (Acme, 221, 22, Denver, 08: 17), and (Acme, 222, 22, Denver, 09: 10).

15. The subscripts on the projection mapping notation indicate which columns are to be retained. Thus if we want to delete columns 1, 2, and 4 from a 6-tuple, we need to use the projection $P_{3,5,6}$.

17. The table uses columns 1 and 4 of Table 8. We start by deleting columns 2, 3, and 5 from Table 8. At this point, rows 5, 6 and 7 are duplicates of earlier rows, so they are omitted (rather than being listed twice). Therefore the answer is as follows.

Airline	*Destination*
Nadir	Detroit
Acme	Denver
Acme	Anchorage
Acme	Honolulu

19. We need to find rows of Table 9 the last two entries of which are identical to the first two entries of rows of Table 10. We combine each such pair of rows into one row of our new table. For instance, the last two entries in the first row of Table 9 are 1092 and 1. The first two entries in the second row of Table 10 are also 1092 and 1. Therefore we combine them into the row 23, 1092, 1, 2, 2 of our new table, whose columns represent *Supplier*, *Part_number*, *Project*, *Quantity*, and *Color_code*. The new table consists of all pairs found in this way.

Supplier	*Part_number*	*Project*	*Quantity*	*Color_code*
23	1092	1	2	2
23	1101	3	1	1
23	9048	4	12	2
31	4975	3	6	2
31	3477	2	25	2
32	6984	4	10	1
32	9191	2	80	4
33	1001	1	14	8

21. Both sides of this equation pick out the subset of R consisting of those n-tuples satisfying both conditions C_1 and C_2. This follows immediately from the definition of the selection operator.

23. Both sides of this equation pick out the set of n-tuples that satisfy three conditions: they are in R, they are in S, and they satisfy condition C. This follows immediately from the definitions of intersection and the selection operator.

25. Both sides of this equation pick out the m-tuples consisting of i_1^{th}, i_2^{th}, …, i_m^{th} components of n-tuples in either R or S (or, of course, both). This follows immediately from the definitions of union and the projection operator.

27. Note that we lose information when we delete columns. Therefore we might be taking something away when we form the second set of m-tuples that might not have been taken away if all the original information is there (forming the first set of m-tuples). A simple example would be to let $R = \{(a,b)\}$ and $S = \{(a,c)\}$, $n = 2$, $m = 1$, and $i_1 = 1$. Then $R - S = R$, so $P_1(R - S) = P_1(R) = \{(a)\}$. On the other hand, $P_1(R) = P_1(S) = \{(a)\}$, so $P_1(R) - P_1(S) = \emptyset$.

29. This is similar to Example 13.

a) Since two databases are listed in the "FROM" field, the first operation is to form the join of these two databases, specifically the join J_2 of these two databases. We then apply the selection operator with the condition "Quantity ≤ 10." This join will have eight 5-tuples in it. Finally we want just the Supplier and Project, so we are forming the projection $P_{1,3}$.

b) Four of the 5-tuples in the joined database have a quantity of no more than 10. The output, then, is the set of the four 2-tuples corresponding to these fields: $(23,1), (23,3), (31,3), (32,4)$.

31. A primary key is a domain whose value determines the values of all the other domains. For this relation, this does not happen. The third domain (the modulus) is not a primary key, because, for example, $1 \equiv 11 \pmod{10}$ and $2 \equiv 12 \pmod{10}$, so the triples $(1,11,10)$ and $(2,12,10)$ are both in the relation. Knowing that the third component of a triple is 10 does not tell us what the other two components are. Similarly, the triples $(1,11,10)$ and $(1,21,10)$ are both in the relation, so the first domain is not a key; and the triples $(1,11,10)$ and $(11,11,10)$ are both in the relation, so the second domain is not a key.

SECTION 9.3 Representing Relations

Matrices and directed graphs provide useful ways for computers and humans to represent relations and manipulate them. Become familiar with working with these representations and the operations on them (especially the matrix operation for forming composition) by working these exercises. Some of these exercises explore how properties of a relation can be found from these representations.

1. In each case we use a 3×3 matrix, putting a 1 in position (i,j) if the pair (i,j) is in the relation and a 0 in position (i,j) if the pair (i,j) is not in the relation. For instance, in part **(a)** there are 1's in the first row, since each of the pairs $(1,1)$, $(1,2)$, and $(1,3)$ are in the relation, and there are 0's elsewhere.

a) $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$ **b)** $\begin{bmatrix} 0 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ **d)** $\begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}$

3. **a)** Since the $(1,1)^{\text{th}}$ entry is a 1, $(1,1)$ is in the relation. Since $(1,2)^{\text{th}}$ entry is a 0, $(1,2)$ is not in the relation. Continuing in this manner, we see that the relation contains $(1,1)$, $(1,3)$, $(2,2)$, $(3,1)$, and $(3,3)$.
b) $(1,2)$, $(2,2)$, and $(3,2)$ **c)** $(1,1)$, $(1,2)$, $(1,3)$, $(2,1)$, $(2,3)$, $(3,1)$, $(3,2)$, and $(3,3)$

5. An irreflexive relation (see the preamble to Exercise 11 in Section 9.1) is one in which no element is related to itself. In the matrix, this means that there are no 1's on the main diagonal (position m_{ii} for some i). Equivalently, the relation is irreflexive if and only if every entry on the main diagonal of the matrix is 0.

7. For reflexivity we want all 1's on the main diagonal; for irreflexivity we want all 0's on the main diagonal; for symmetry, we want the matrix to be symmetric about the main diagonal (equivalently, the matrix equals its transpose); for antisymmetry we want there never to be two 1's symmetrically placed about the main diagonal (equivalently, the meet of the matrix and its transpose has no 1's off the main diagonal); and for transitivity we want the Boolean square of the matrix (the Boolean product of the matrix and itself) to be "less than or equal to" the original matrix in the sense that there is a 1 in the original matrix at every location where there is a 1 in the Boolean square.

a) Since there are all 1's on the main diagonal, this relation is reflexive and not irreflexive. Since the matrix is symmetric, the relation is symmetric. The relation is not antisymmetric—look at positions $(1,3)$ and $(3,1)$. Finally, the Boolean square of this matrix is itself, so the relation is transitive.

b) Since there are both 0's and 1's on the main diagonal, this relation is neither reflexive nor irreflexive. Since the matrix is not symmetric, the relation is not symmetric (look at positions $(1,2)$ and $(2,1)$, for example). The relation is antisymmetric since there are never two 1's symmetrically placed with respect to the main diagonal. Finally, the Boolean square of this matrix is itself, so the relation is transitive.

c) Since there are both 0's and 1's on the main diagonal, this relation is neither reflexive nor irreflexive. Since the matrix is symmetric, the relation is symmetric. The relation is not antisymmetric—look at positions $(1,3)$ and $(3,1)$, for example. Finally, the Boolean square of this matrix is the matrix with all 1's, so the relation is not transitive (1 is related to 2, and 2 is related to 1, but 2 is not related to 2).

9. Note that the total number of entries in the matrix is $100^2 = 10{,}000$.

a) There is a 1 in the matrix for each pair of distinct positive integers not exceeding 100, namely in position (a,b) where $a > b$. Thus the answer is the number of subsets of size 2 from a set of 100 elements, i.e., $C(100,2) = 4950$.

b) There is a 1 in the matrix at each position except the 100 positions on the main diagonal. Therefore the answer is $100^2 - 100 = 9900$.

c) There is a 1 in the matrix at each entry just below the main diagonal (i.e., in positions $(2,1)$, $(3,2)$, ..., $(100,99)$. Therefore the answer is 99.

d) The entire first row of this matrix corresponds to $a = 1$. Therefore the matrix has 100 nonzero entries.

e) This relation has only the one element $(1,1)$ in it, so the matrix has just one nonzero entry.

11. Since the relation $\overline{R}$ is the relation that contains the pair (a,b) (where a and b are elements of the appropriate sets) if and only if R does not contain that pair, we can form the matrix for $\overline{R}$ simply by changing all the 1's to 0's and 0's to 1's in the matrix for R.

13. Exercise 12 tells us how to do part **(a)** (we take the transpose of the given matrix $\mathbf{M}_R$, which in this case happens to be the matrix itself). Exercise 11 tells us how to do part **(b)** (we change 1's to 0's and 0's to 1's in $\mathbf{M}_R$). For part **(c)** we take the Boolean product of $\mathbf{M}_R$ with itself.

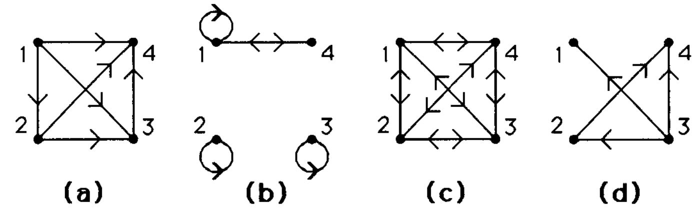
a) $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$ **c)** $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

15. We compute the Boolean powers of $\mathbf{M}_R$; thus $\mathbf{M}_{R^2} = \mathbf{M}_R^{[2]} = \mathbf{M}_R \odot \mathbf{M}_R$, $\mathbf{M}_{R^3} = \mathbf{M}_R^{[3]} = \mathbf{M}_R \odot \mathbf{M}_R^{[2]}$, and $\mathbf{M}_{R^4} = \mathbf{M}_R^{[4]} = \mathbf{M}_R \odot \mathbf{M}_R^{[3]}$.

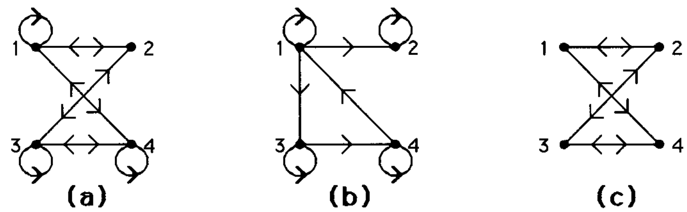
a) $\begin{bmatrix} 0 & 0 & 1 \\ 1 & 1 & 0 \\ 0 & 1 & 1 \end{bmatrix}$ **b)** $\begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$ **c)** $\begin{bmatrix} 0 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

17. The matrix for the complement has a 1 wherever the matrix for the relation has a 0, and vice versa. Therefore the number of nonzero entries in $\mathbf{M}_{\overline{R}}$ is $n^2 - k$, since these matrices have n rows and n columns.

19. In each case we need a vertex for each of the elements, and we put in a directed edge from x to y if there is a 1 in position (x,y) of the matrix. For simplicity we have indicated pairs of edges between the same two vertices in opposite directions by using a double arrowhead, rather than drawing two separate lines.



21. In each case we need a vertex for each of the elements, and we put in a directed edge from x to y if there is a 1 in position (x, y) of the matrix. For simplicity we have indicated pairs of edges between the same two vertices in opposite directions by using a double arrowhead, rather than drawing two separate lines.



23. We list all the pairs (x, y) for which there is an edge from x to y in the directed graph: $\{(a,b),(a,c),(b,c),(c,b)\}$.

25. We list all the pairs (x, y) for which there is an edge from x to y in the directed graph: $\{(a,c),(b,a),(c,d),(d,b)\}$.

27. We list all the pairs (x, y) for which there is an edge from x to y in the directed graph: $\{(a,a),(a,b),(a,c),(b,a),(b,b),(b,c),(c,a),(c,b),(d,d)\}$.

29. An asymmetric relation is one for which it never happens that a is related to b and simultaneously b is related to a, even when $a = b$. In terms of the directed graph, this means that we must see no loops and no closed paths of length 2 (i.e., no pairs of edges between two vertices going in opposite directions).

31. Recall that the relation is reflexive if there is a loop at each vertex; irreflexive if there are no loops at all; symmetric if edges appear only in **antiparallel** pairs (edges from one vertex to a second vertex and from the second back to the first); antisymmetric if there is no pair of antiparallel edges; and transitive if all paths of length 2 (a pair of edges (x, y) and (y, z)) are accompanied by the corresponding path of length 1 (the edge (x, z)). The relation drawn in Exercise 23 is not reflexive but is irreflexive since there are no loops. It is not symmetric, since, for instance, the edge (a, b) is present but not the edge (b, a). It is not antisymmetric, since both edges (b, c) and (c, b) are present. It is not transitive, since the path $(b, c), (c, b)$ from b to b is not accompanied by the edge (b, b). The relation drawn in Exercise 24 is reflexive and not irreflexive since there is a loop at each vertex. It is not symmetric, since, for instance, the edge (b, a) is present but not the edge (a, b). It is antisymmetric, since there are no pairs of antiparallel edges. It is transitive, since the only nontrivial path of length 2 is bac, and the edge (b, c) is present. The relation drawn in Exercise 25 is not reflexive but is irreflexive since there are no loops. It is not symmetric, since, for instance, the edge (b, a) is present but not the edge (a, b). It is antisymmetric, since there are no pairs of antiparallel edges. It is not transitive, since the edges (a, c) and (c, d) are present, but not (a, d).

33. Since the inverse relation consists of all pairs (b, a) for which (a, b) is in the original relation, we just have to take the digraph for R and reverse the direction on every edge.

35. We prove this statement by induction on n. The basis step $n = 1$ is tautologically true, since $\mathbf{M}_R^{[1]} = \mathbf{M}_R$. Assume the inductive hypothesis that $\mathbf{M}_R^{[n]}$ is the matrix representing R^n. Now $\mathbf{M}_R^{[n+1]} = \mathbf{M}_R \odot \mathbf{M}_R^{[n]}$. By the inductive hypothesis and the assertion made before Example 5, that $\mathbf{M}_{S\circ R} = \mathbf{M}_R \odot \mathbf{M}_S$, the right-hand side is the matrix representing $R^n \circ R$. But $R^n \circ R = R^{n+1}$, so our proof is complete.

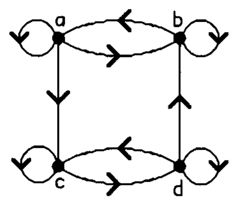
SECTION 9.4 Closures of Relations

This section is harder than the previous ones in this chapter. Warshall's algorithm, in particular, is fairly tricky, and Exercise 27 should be worked carefully, following Example 8. It is easy to forget to include the loops (a,a) when forming transitive closures "by hand."

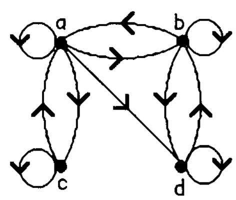
1. **a)** The reflexive closure of R is R together with all the pairs (a,a). Thus in this case the closure of R is $\{(0,0),(0,1),(1,1),(1,2),(2,0),(2,2),(3,0),(3,3)\}$.
b) The symmetric closure of R is R together with all the pairs (b,a) for which (a,b) is in R. For example, since $(1,2)$ is in R, we need to add $(2,1)$. Thus the closure of R is $\{(0,1),(0,2),(0,3),(1,0),(1,1),(1,2),(2,0),(2,1),(2,2),(3,0)\}$.

3. To form the symmetric closure we need to add all the pairs (b,a) such that (a,b) is in R. In this case, that means that we need to include pairs (b,a) such that a divides b, which is equivalent to saying that we need to include all the pairs (a,b) such that b divides a. Thus the closure is $\{\,(a,b) \mid a \text{ divides } b \text{ or } b \text{ divides } a\,\}$.

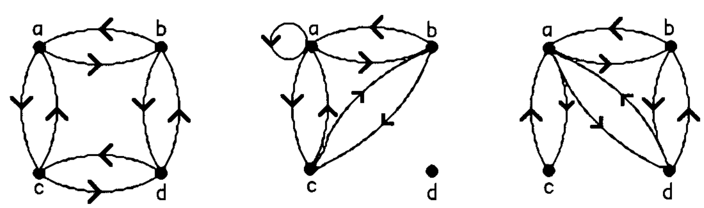
5. We form the reflexive closure by taking the given directed graph and appending loops at all vertices at which there are not already loops.



7. We form the reflexive closure by taking the given directed graph and appending loops at all vertices at which there are not already loops.

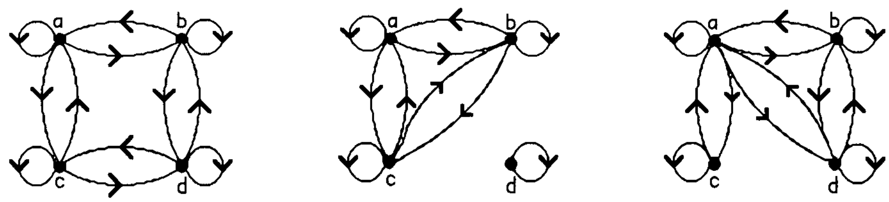


9. We form the symmetric closure by taking the given directed graph and appending an edge pointing in the opposite direction for every edge already in the directed graph (unless it is already there); in other words, we append the edge (b,a) whenever we see the edge (a,b). We have labeled the figures below (a), (b), and (c), corresponding to Exercises 5, 6, and 7, respectively.



(a) (b) (c)

11. We are asked for the symmetric and reflexive closure of the given relation. We form it by taking the given directed graph and appending (1) a loop at each vertex at which there is not already a loop and (2) an edge pointing in the opposite direction for every edge already in the directed graph (unless it is already there). We have labeled the figures below (a), (b), and (c), corresponding to Exercises 5, 6, and 7, respectively.



(a) (b) (c)

13. The symmetric closure of R is $R \cup R^{-1}$. The matrix for R^{-1} is $\mathbf{M}_R^t$, as we saw in Exercise 12 in Section 9.3. The matrix for the union of two relations is the join of the matrices for the two relations, as we saw in Section 9.3. Therefore the matrix representing the symmetric closure of R is indeed $\mathbf{M}_R \vee \mathbf{M}_R^t$.

15. If R is already irreflexive, then it is clearly its own irreflexive closure. On the other hand if R is not irreflexive, then there is no relation containing R that is irreflexive, since the loop or loops in R prevent any such relation from being irreflexive. Thus in this case R has no irreflexive closure. This exercise shows essentially that the concept of "irreflexive closure" is rather useless, since no relation has one unless it is already irreflexive (in which case it is its own "irreflexive closure").

17. A circuit of length 3 can be written as a sequence of 4 vertices, each joined to the next by an edge of the given directed graph, ending at the same vertex at which it began. There are several such circuits here, and we just have to be careful and systematically list them all. There are the circuits formed entirely by the loops: *aaaa*, *cccc*, and *eeee*. The triangles *abea* and *adea* also qualify. Two circuits start at b: *bccb* and *beab*. There are two more circuits starting at c, namely *ccbc* and *cbcc*. Similarly there are the circuits *deed*, *eede* and *edee*, as well as the other trips around the triangle: *eabe*, *dead*, and *eade*.

19. The way to form these powers is first to form the matrix representing R, namely

$$\mathbf{M}_R = \begin{bmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 0 & 1 & 0 \end{bmatrix},$$

and then take successive Boolean powers of it to get the matrices representing R^2, R^3, and so on. Finally, for part **(f)** we take the join of the matrices representing R, R^2, $\ldots$, R^5. Since the matrix is a perfectly good way to express the relation, we will not list the ordered pairs.

a) The matrix for R^2 is the Boolean product of the matrix displayed above with itself, namely

$$\mathbf{M}_{R^2} = \mathbf{M}_R^{[2]} = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 1 & 0 \end{bmatrix}.$$

b) The matrix for R^3 is the Boolean product of the first matrix displayed above with the answer to part **(a)**, namely

$$\mathbf{M}_{R^3} = \mathbf{M}_R^{[3]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 & 1 \end{bmatrix}.$$

c) The matrix for R^4 is the Boolean product of the first matrix displayed above with the answer to part **(b)**, namely

$$\mathbf{M}_{R^4} = \mathbf{M}_R^{[4]} = \begin{bmatrix} 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

d) The matrix for R^5 is the Boolean product of the first matrix displayed above with the answer to part **(c)**, namely

$$\mathbf{M}_{R^5} = \mathbf{M}_R^{[5]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

e) The matrix for R^6 is the Boolean product of the first matrix displayed above with the answer to part **(d)**, namely

$$\mathbf{M}_{R^6} = \mathbf{M}_R^{[6]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

f) The matrix for R^* is the join of the first matrix displayed above and the answers to parts **(a)** through **(d)**, namely

$$\mathbf{M}_{R^*} = \mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \mathbf{M}_R^{[3]} \vee \mathbf{M}_R^{[4]} \vee \mathbf{M}_R^{[5]} = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

21. **a)** The pair (a,b) is in R^2 if there is a person c other than a or b who is in a class with a and a class with b. Note that it is almost certain that (a,a) is in R^2, since as long as a is taking a class that has at least one other person in it, that person serves as the "c."

b) The pair (a,b) is in R^3 if there are persons c (different from a) and d (different from b and c) such that c is in a class with a, c is in a class with d, and d is in a class with b.

c) The pair (a,b) is in R^* if there is a sequence of persons, $c_0, c_1, c_2, \ldots, c_n$, with $n \geq 1$, such that $c_0 = a$, $c_n = b$, and for each i from 1 to n, $c_{i-1} \neq c_i$ and c_{i-1} is in at least one class with c_i.

23. Suppose that $(a,b) \in R^*$; then there is a path from a to b in (the digraph for) R. Given such a path, if R is symmetric, then the reverse of every edge in the path is also in R; therefore there is a path from b to a in R (following the given path backwards). This means that (b,a) is in R^* whenever (a,b) is, exactly what we needed to prove.

25. Algorithm 1 finds the transitive closure by computing the successive powers and taking their join. We exhibit our answers in matrix form as $\mathbf{M}_R \vee \mathbf{M}_R^{[2]} \vee \ldots \vee \mathbf{M}_R^{[n]} = \mathbf{M}_{R^*}$.

a)
$$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

b)
$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$$

c) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Note that the relation was already transitive, so its transitive closure is itself.

d) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 0 & 1 \end{bmatrix} \vee \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

27. In Warshall's algorithm (Algorithm 2 in this section), we compute a sequence of matrices $\mathbf{W}_0$ (the matrix representing R), $\mathbf{W}_1$, $\mathbf{W}_2$, ..., $\mathbf{W}_n$, the last of which represents the transitive closure of R. Each matrix $\mathbf{W}_k$ comes from the matrix $\mathbf{W}_{k-1}$ in the following way. The $(i,j)^{\text{th}}$ entry of $\mathbf{W}_k$ is the "$\vee$" of the $(i,j)^{\text{th}}$ entry of $\mathbf{W}_{k-1}$ with the "$\wedge$" of the $(i,k)^{\text{th}}$ entry and the $(k,j)^{\text{th}}$ entry of $\mathbf{W}_{k-1}$. We will exhibit our solution by listing the matrices $\mathbf{W}_0$, $\mathbf{W}_1$, $\mathbf{W}_2$, $\mathbf{W}_3$, $\mathbf{W}_4$, in that order; $\mathbf{W}_4$ represents the answer. In each case $\mathbf{W}_0$ is the matrix of the given relation. To compute the next matrix in the solution, we need to compute it one entry at a time, using the equation just discussed (the "$\vee$" of the corresponding entry in the previous matrix with the "$\wedge$" of two entries in the old matrix), i.e., as i and j each go from 1 to 4, we need to write down the $(i,j)^{\text{th}}$ entry using this formula. Note that in computing $\mathbf{W}_k$ the k^{th} row and the k^{th} column are unchanged, but some of the entries in other rows and columns may change.

a) $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 0 \\ 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

b) $\begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 0 & 1 & 1 \end{bmatrix}$

c) $\begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 0 & 1 & 1 & 1 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix}$

Note that the relation was already transitive, so each matrix in the sequence was the same.

d) $\begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{bmatrix}$

29. **a)** We need to include at least the transitive closure, which we can compute by Algorithm 1 or Algorithm 2 to be (in matrix form) $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$. All we need in addition is the pair $(2,2)$ in order to make the relation reflexive. Note that the result is still transitive (the addition of a pair (a,a) cannot make a transitive relation no longer transitive), so our answer is $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$.

b) The symmetric closure of the original relation is represented by $\begin{bmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}$. We need at least the transitive closure of this relation, namely $\begin{bmatrix} 1 & 1 & 0 & 1 \\ 1 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 1 \end{bmatrix}$. Since it is also symmetric, we are done. Note that it would not have been correct to find first the transitive closure of the original matrix and then make it symmetric, since the pair $(2,2)$ would be missing. What is going on here is that the transitive closure of a symmetric relation is still symmetric, but the symmetric closure of a transitive relation might not be transitive.

c) Since the answer to part **(b)** was already reflexive, it must be the answer to this part as well.

31. Algorithm 1 has a loop executed $O(n)$ times in which the primary operation is the Boolean product computation (the join operation is fast by comparison). If we can do the product in $O(n^{2.8})$ bit operations, then the number of bit operations in the entire algorithm is $O(n \cdot n^{2.8}) = O(n^{3.8})$. Since Algorithm 2 does not use the Boolean product, a fast Boolean product algorithm is irrelevant, so Algorithm 2 still requires $O(n^3)$ bit operations.

33. There are two ways to go. One approach is to take the output of Algorithm 1 as it stands and then make sure that all the pairs (a,a) are included by forming the join with the identity matrix (specifically set $\mathbf{B} := \mathbf{B} \vee \mathbf{I}_n$). See the discussion in Exercise 29a for the justification. The other approach is to insure the reflexivity at the beginning by initializing $\mathbf{A} := \mathbf{M}_r \vee \mathbf{I}_n$; if we do this, then only paths of length strictly less than n need to be looked at, so we can change the n in the loop to $n-1$.

35. a) No relation that contains R is not reflexive, since R already contains all the pairs $(0,0)$, $(1,1)$, and $(2,2)$. Therefore there is no "nonreflexive" closure of R.

b) Suppose S were the closure of R with respect to this property. Since R does not have an odd number of elements, $S \neq R$, so S must be a proper superset of R. Clearly S cannot have more than 5 elements, for if it did, then any subset of S consisting of R and one element of $S - R$ would be a proper subset of S with the property; this would violate the requirement that S be a subset of every superset of R with the property. Thus S must have exactly 5 elements. Let T be another superset of R with 5 elements (there are $9 - 4 = 5$ such sets in all). Thus T has the property, but S is not a subset of T. This contradicts the definition. Therefore our original assumption was faulty, and the closure does not exist.

SECTION 9.5 Equivalence Relations

This section is extremely important. If you do nothing else, do Exercise 9 and understand it, for it deals with the most common instances of equivalence relations. (See the comments in our solution below for some added insight.) Exercise 16 is interesting—it hints at what fractions really are (if understood properly) and perhaps helps to explain why children (and adults) usually have so much trouble with fractions: they really involve equivalence relations. Spend some time thinking about fractions in this context. (See also Writing Project 4 for this chapter.)

It is usually easier to understand equivalence relations in terms of the associated partition—it's a more concrete visual image. Thus make sure you understand exactly what Theorem 2 says. Look at Exercise 67 for the relationship between equivalence relations and closures.

1. In each case we need to check for reflexivity, symmetry, and transitivity.

a) This is an equivalence relation; it is easily seen to have all three properties. The equivalence classes all have just one element.

b) This relation is not reflexive since the pair $(1,1)$ is missing. It is also not transitive, since the pairs $(0,2)$ and $(2,3)$ are there, but not $(0,3)$.

c) This is an equivalence relation. The elements 1 and 2 are in the same equivalence class; 0 and 3 are each in their own equivalence class.

d) This relation is reflexive and symmetric, but it is not transitive. The pairs $(1,3)$ and $(3,2)$ are present, but not $(1,2)$.

e) This relation would be an equivalence relation were the pair $(2,1)$ present. As it is, its absence makes the relation neither symmetric nor transitive.

3. As in Exercise 1, we need to check for reflexivity, symmetry, and transitivity.

a) This is an equivalence relation, one of the general form that two things are considered equivalent if they have the same "something" (see Exercise 9 for a formalization of this idea). In this case the "something" is the value at 1.

b) This is not an equivalence relation because it is not transitive. Let $f(x)=0$, $g(x)=x$, and $h(x)=1$ for all $x \in \mathbf{Z}$. Then f is related to g since $f(0)=g(0)$, and g is related to h since $g(1)=h(1)$, but f is not related to h since they have no values in common. By inspection we see that this relation is reflexive and symmetric.

c) This relation has none of the three properties. It is not reflexive, since $f(x)-f(x)=0 \neq 1$. It is not symmetric, since if $f(x)-g(x)=1$, then $g(x)-f(x)=-1 \neq 1$. It is not transitive, since if $f(x)-g(x)=1$ and $g(x)-h(x)=1$, then $f(x)-h(x)=2 \neq 1$.

d) This is an equivalence relation. Two functions are related here if they differ by a constant. It is clearly reflexive (the constant is 0). It is symmetric, since if $f(x)-g(x)=C$, then $g(x)-f(x)=-C$. It is transitive, since if $f(x)-g(x)=C_1$ and $g(x)-h(x)=C_2$, then $f(x)-h(x)=C_3$, where $C_3=C_1+C_2$ (add the first two equations).

e) This relation is not reflexive, since there are lots of functions f (for instance, $f(x)=x$) that do not have the property that $f(0)=f(1)$. It is symmetric by inspection (the roles of f and g are the same). It is not transitive. For instance, let $f(0)=g(1)=h(0)=7$, and let $f(1)=g(0)=h(1)=3$; fill in the remaining values arbitrarily. Then f and g are related, as are g and h, but f is not related to h since $7 \neq 3$.

5. Obviously there are many possible answers here. We can say that two buildings are equivalent if they were opened during the same year; an equivalence class consists of the set of buildings opened in a given year (as long as there was at least one building opened that year). For another example, we can define two buildings to be equivalent if they have the same number of stories; the equivalence classes are the set of 1-story buildings, the set of 2-story buildings, and so on (one class for each n for which there is at least one n-story building). In our third example, partition the set of all buildings into two classes—those in which you do have a class this semester and those in which you don't. (We assume that each of these is nonempty.) Every building in which you have a class is equivalent to every building in which you have a class (including itself), and every building in which you don't have a class is equivalent to every building in which you don't have a class (including itself).

7. Two propositions are equivalent if their truth tables are identical. This relation is reflexive, since the truth table of a proposition is identical to itself. It is symmetric, since if p and q have the same truth table, then q and p have the same truth table. There is one technical point about transitivity that should be noted. We need to assume that the truth tables, as we consider them for three propositions p, q, and r, have the same atomic variables in them. If we make this assumption (and it cannot hurt to do so, since adding information about extra variables that do not appear in a pair of propositions does not change the truth value of the propositions), then we argue in the usual way: if p and q have identical truth tables, and if q and r have identical truth tables, then p and r have that same common truth table. The proposition $\mathbf{T}$ is always true; therefore the equivalence class for this proposition consists of all propositions that are always true, no matter what truth values the atomic variables have. Recall that we call such a proposition a tautology. Therefore the equivalence class of $\mathbf{T}$ is the set of all tautologies. Similarly, the equivalence class of $\mathbf{F}$ is the set of all contradictions.

9. This is an important exercise, since very many equivalence relations are of this form. (In fact, all of them are—see Exercise 10. A relation defined by a condition of the form "x and y are equivalent if and only if they have the same ..." is an equivalence relation. The function f here tells what about x and y are "the same.")
a) This relation is reflexive, since obviously $f(x) = f(x)$ for all $x \in A$. It is symmetric, since if $f(x) = f(y)$, then $f(y) = f(x)$ (this is one of the fundamental properties of equality). It is transitive, since if $f(x) = f(y)$ and $f(y) = f(z)$, then $f(x) = f(z)$ (this is another fundamental property of equality).
b) The equivalence class of x is the set of all $y \in A$ such that $f(y) = f(x)$. This is by definition just the inverse image of $f(x)$. Thus the equivalence classes are precisely the sets $f^{-1}(b)$ for every b in the range of f.

11. This follows from Exercise 9, where f is the function that takes a bit string of length 3 or more to its first 3 bits.

13. This follows from Exercise 9, where f is the function that takes a bit string of length 3 or more to the ordered pair (b_1, b_3), where b_1 is the first bit of the string and b_3 is the third bit of the string. Two bit strings agree on their first and third bits if and only if the corresponding ordered pairs for these two strings are equal ordered pairs.

15. By algebra, the given condition is the same as the condition that $f((a,b)) = f((c,d))$, where $f((x,y)) = x - y$. Therefore by Exercise 9 this is an equivalence relation. If we want a more explicit proof, we can argue as follows. For reflexivity, $((a,b),(a,b)) \in R$ because $a + b = b + a$. For symmetry, $((a,b),(c,d)) \in R$ if and only if $a + d = b + c$, which is equivalent to $c + b = d + a$, which is true if and only if $((c,d),(a,b)) \in R$. For transitivity, suppose $((a,b),(c,d)) \in R$ and $((c,d),(e,f)) \in R$. Thus we have $a + d = b + c$ and $c + e = d + f$. Adding, we obtain $a + d + c + e = b + c + d + f$. Simplifying, we have $a + e = b + f$, which tells us that $((a,b),(e,f)) \in R$.

17. a) This follows from Exercise 9, where the function f from the set of differentiable functions (from $\mathbf{R}$ to $\mathbf{R}$) to the set of functions (from $\mathbf{R}$ to $\mathbf{R}$) is the differentiation operator—i.e., f of a function g is the function g'. The best way to think about this is that any relation defined by a statement of the form "a and b are equivalent if they have the same whatever" is an equivalence relation. Here "whatever" is "derivative"; in the general situation of Exercise 9, "whatever" is "function value under f."
b) We are asking for all functions that have the same derivative that the function $f(x) = x^2$ has, i.e., all functions of x whose derivative is $2x$. In other words, we are asking for the general antiderivative of $2x$, and we know that $\int 2x = x^2 + C$, where C is any constant. Therefore the functions in the same equivalence class as $f(x) = x^2$ are all the functions of the form $g(x) = x^2 + C$ for some constant C. Indefinite integrals in calculus, then, give equivalence classes of functions as answers, not just functions.

19. This follows from Exercise 9, where the function f from the set of all URLs to the set of all Web pages is the function that assigns to each URL the Web page for that URL.

21. We need to observe whether the relation is reflexive (there is a loop at each vertex), symmetric (every edge that appears is accompanied by its antiparallel mate—an edge involving the same two vertices but pointing in the opposite direction), and transitive (paths of length 2 are accompanied by the path of length 1—i.e., edge—between the same two vertices in the same direction). We see that this relation is not transitive, since the edges (c,d) and (d,c) are missing.

23. As in Exercise 21, this relation is not transitive, since several required edges are missing (such as (a,c)).

25. This follows from Exercise 9, with f being the function from bit strings to nonnegative integers given by $f(s) =$the number of 1's in s.

27. Only parts **(a)** and **(b)** are relevant here, since the others are not equivalence relations.
a) An equivalence class is the set of all people who are the same age. (To really identify the equivalence class and the equivalence relation itself, one would need to specify exactly what one meant by "the same age." For example, we could define two people to be the same age if their official dates of birth were identical. In that case, everybody born on April 25, 1948, for example, would constitute one equivalence class.)
b) For each pair (m,f) of a man and a woman, the set of offspring of their union, if nonempty, is an equivalence class. In many cases, then, an equivalence class consists of all the children in a nuclear family with children. (In real life, of course, this is complicated by such things as divorce and remarriage.)

29. The equivalence class of 011 is the set of all bit strings that are related to 011, namely the set of all bit strings that have the same number of 1's as 011. In other words, it is the (infinite) set of all bit strings with exactly 2 1's: $\{11, 110, 101, 011, 1100, 1010, 1001, \ldots\}$.

31. Since two strings are related if they agree beyond their first 3 bits, the equivalence class of a bit string $xyzt$, where x, y, and z are bits, and t is a bit string, is the set of all bit strings of the form $x'y'z't$, where x', y', and z' are any bits.
a) the set of all bit strings of length 3 (take $t = \lambda$ in the formulation given above)
b) the set of all bit strings of length 4 that end with a 1
c) the set of all bit strings of length 5 that end 11
d) the set of all bit strings of length 8 that end 10101

33. This is like Example 15. Each bit string of length less than 4 is in an equivalence class by itself ($[\lambda]_{R_4} = \{\lambda\}$, $[0]_{R_4} = \{0\}$, $[1]_{R_4} = \{1\}$, $[00]_{R_4} = \{00\}$, $[01]_{R_4} = \{01\}$, ..., $[111]_{R_4} = \{111\}$). This accounts for $1+2+4+8 = 15$ equivalence classes. The remaining 16 equivalence classes are determined by the bit strings of length 4:

$$[0000]_{R_4} = \{0000, 00000, 00001, 000000, 000001, 000010, 000011, 0000000, \ldots\}$$
$$[0001]_{R_4} = \{0001, 00010, 00011, 000100, 000101, 000110, 000111, 0001000, \ldots\}$$
$$[0010]_{R_4} = \{0010, 00100, 00101, 001000, 001001, 001010, 001011, 0010000, \ldots\}$$
$$\vdots$$
$$[1111]_{R_4} = \{1111, 11110, 11111, 111100, 111101, 111110, 111111, 1111000, \ldots\}$$

35. We have by definition that $[n]_5 = \{\, i \mid i \equiv n \pmod 5 \,\}$.
a) $[2]_5 = \{\, i \mid i \equiv 2 \pmod 5 \,\} = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$
b) $[3]_5 = \{\, i \mid i \equiv 3 \pmod 5 \,\} = \{\ldots, -7, -2, 3, 8, 13, \ldots\}$
c) $[6]_5 = \{\, i \mid i \equiv 6 \pmod 5 \,\} = \{\ldots, -9, -4, 1, 6, 11, \ldots\}$
d) $[-3]_5 = \{\, i \mid i \equiv -3 \pmod 5 \,\} = \{\ldots, -8, -3, 2, 7, 12, \ldots\}$ (the same as $[2]_5$)

37. This is very similar to Example 14. There are 6 equivalence classes, namely

$$\begin{aligned}
[0]_6 &= \{\ldots, -12, -6, 0, 6, 12, \ldots\}, \\
[1]_6 &= \{\ldots, -11, -5, 1, 7, 13, \ldots\}, \\
[2]_6 &= \{\ldots, -10, -4, 2, 8, 14, \ldots\}, \\
[3]_6 &= \{\ldots, -9, -3, 3, 9, 15, \ldots\}, \\
[4]_6 &= \{\ldots, -8, -2, 4, 10, 16, \ldots\}, \\
[5]_6 &= \{\ldots, -7, -1, 5, 11, 17, \ldots\}.
\end{aligned}$$

Another way to describe this collection is to say that it is the collection of sets $\{\,6n + k \mid n \in \mathbf{Z}\,\}$ for $k = 0, 1, 2, 3, 4, 5$.

39. **a)** We observed in the solution to Exercise 15 that (a, b) is equivalent to (c, d) if $a - b = c - d$. Thus because $1 - 2 = -1$, we have $[(1, 2)] = \{\,(a, b) \mid a - b = -1\,\} = \{(1, 2), (3, 4), (4, 5), (5, 6), \ldots\}$.

b) Since the equivalence class of (a, b) is entirely *determined* by the integer $a - b$, which can be negative, positive, or zero, we can interpret the equivalences classes as *being* the integers. This is a standard way to *define* the integers once we have defined the whole numbers.

41. The sets in a partition must be nonempty, pairwise disjoint, and have as their union all of the underlying set.

a) This is not a partition, since the sets are not pairwise disjoint (the elements 2 and 4 each appear in two of the sets).

b) This is a partition. **c)** This is a partition.

d) This is not a partition, since none of the sets includes the element 3.

43. In each case, we need to see that the collection of subsets satisfy three conditions: they are nonempty, they are pairwise disjoint, and their union is the entire set of 256 bit strings of length 8.

a) This is a partition, since strings must begin either 1 or 0, and those that begin 0 must continue with either 0 or 1 in their second position. It is clear that the three subsets satisfy the conditions.

b) This is not a partition, since these subsets are not pairwise disjoint. The string 00000001, for example, contains both 00 and 01.

c) This is clearly a partition. Each of these four subsets contains 64 bit strings, and no two of them overlap.

d) This is not a partition, because the union of these subsets is not the entire set. For example, the string 00000010 is in none of the subsets.

e) This is a partition. Each bit string contains some number of 1's. This number can be identified in exactly one way as of the form $3k$, the form $3k + 1$, or the form $3k + 2$, where k is a nonnegative integer; it really is just looking at the equivalence classes of the number of 1's modulo 3.

45. In each case, we need to see that the collection of subsets satisfy three conditions: they are nonempty, they are pairwise disjoint, and their union is the entire set $\mathbf{Z} \times \mathbf{Z}$.

a) This is not a partition, since the subsets are not pairwise disjoint. The pair $(2, 3)$, for example, is in both of the first two subsets listed.

b) This is a partition. Every pair satisfies exactly one of the conditions listed about the parity of x and y, and clearly these subsets are nonempty.

c) This is not a partition, since the subsets are not pairwise disjoint. The pair $(2, 3)$, for example, is in both of the first two subsets listed. Also, $(0, 0)$ is in none of the subsets.

d) This is a partition. Every pair satisfies exactly one of the conditions listed about the divisibility of x and y by 3, and clearly these subsets are nonempty.

e) This is a partition. Every pair satisfies exactly one of the conditions listed about the positiveness of x and y, and clearly these subsets are nonempty.

f) This is not a partition, because the union of these subsets is not all of $\mathbf{Z} \times \mathbf{Z}$. In particular, $(0,0)$ is in none of the parts.

47. In each case, we need to list all the pairs we can where both coordinates are chosen from the same subset. We should proceed in an organized fashion, listing all the pairs corresponding to each part of the partition.

a) $\{(0,0),(1,1),(1,2),(2,1),(2,2),(3,3),(3,4),(3,5),(4,3),(4,4),(4,5),(5,3),(5,4),(5,5)\}$

b) $\{(0,0),(0,1),(1,0),(1,1),(2,2),(2,3),(3,2),(3,3),(4,4),(4,5),(5,4),(5,5)\}$

c) $\{(0,0), (0,1), (0,2), (1,0), (1,1), (1,2), (2,0), (2,1), (2,2), (3,3), (3,4), (3,5), (4,3), (4,4), (4,5), (5,3), (5,4),(5,5)\}$

d) $\{(0,0),(1,1),(2,2),(3,3),(4,4),(5,5)\}$

49. We need to show that every equivalence class modulo 6 is contained in an equivalence class modulo 3. We claim that in fact, for each $n \in \mathbf{Z}$, $[n]_6 \subseteq [n]_3$. To see this suppose that $m \in [n]_6$. This means that $m \equiv n \pmod 6$, i.e., that $m - n$ is a multiple of 6. Then perforce $m - n$ is a multiple of 3, so $m \equiv n \pmod 3$, which means that $m \in [n]_3$.

51. By the definition given in the preamble to Exercise 49, we need to show that every set in the first partition is a subset of some set in the second partition. Let A be a set in the first partition. So A is the set of all bit strings of length 16 that agree on their last eight bits. Pick a particular element x of A, and suppose that the last four bits of x are $abcd$. Then the set of all bit strings of length 16 whose last four bits are $abcd$ is one of the sets in the second partition, and clearly every string in A is in that set, since every string in A agrees with x on the last eight bits, and therefore perforce agrees on the last four bits.

53. We are asked to show that every equivalence class for R_{31} is a subset of some equivalence class for R_8. Let $[x]_{R_{31}}$ be an arbitrary equivalence class for R_{31}. We claim that $[x]_{R_{31}} \subseteq [x]_{R_8}$; proving this claim finishes the proof. To show that one set is a subset of another set, we choose an arbitrary element y in the first set and show that it is also an element of the second set. In this case since $y \in [x]_{R_{31}}$, we know that y is equivalent to x under R_{31}, that is, that either $y = x$ or y and x are each at least 31 characters long and agree on their first 31 characters. Because strings that are at least 31 characters long and agree on their first 31 characters perforce are at least 8 characters long and on their first 8 characters, we know that either $y = x$ or y and x are each at least 8 characters long and agree on their first 8 characters. This means that y is equivalent to x under R_8, that is, that $y \in [x]_{R_8}$.

55. We need first to make the relation symmetric, so we add the pairs (b,a), (c,a), and (e,d). Then we need to make it transitive, so we add the pairs (b,c), (c,b), (a,a), (b,b), (c,c), (d,d), and (e,e). (In other words, we formed the transitive closure of the symmetric closure of the original relation.) It happens that we have already achieved reflexivity, so we are done; if there had been some pairs (x,x) missing at this point, we would have added them as well. Thus the desired equivalence relation is the one consisting of the original 3 pairs and the 10 we have added. There are two equivalence classes, $\{a,b,c\}$ and $\{d,e\}$.

57. a) The equivalence class of 1 is the set of all real numbers that differ from 1 by an integer. Obviously this is the set of all integers.

b) The equivalence class of $1/2$ is the set of all real numbers that differ from $1/2$ by an integer, namely $1/2$, $3/2$, $5/2$, etc., and $-1/2$, $-3/2$, etc. These are often called **half-integers**. We could write this set as $\{\,(2n+1)/2 \mid n \in \mathbf{Z}\,\}$, among other ways.

59. This problem actually deals with a branch of mathematics called group theory; the object being studied here is related to a certain dihedral group. If this fascinates you, you might want to take a course with a title like Abstract Algebra or Modern Algebra, in which such things are studied in depth.

In order to have a way to talk about specific colorings, let us agree that a sequence of length four, each element of which is either r or b, represents a coloring of the 2×2 checkerboard, where the first letter denotes the color of the upper left square, the second letter denotes the color of the upper right square, the third letter denotes the color of the lower left square, and the fourth letter denotes the color of the lower right square. For example, the board in which every square is red except the upper right would be represented by $rbrr$. There are really only four different rotations, since after the rotation we need to end up with another checkerboard (and we can assume that the edges of the board are horizontal and vertical). If we rotate our sample coloring 90° clockwise, then we obtain the coloring $rrrb$; if we rotate it 180°, then we obtain the coloring $rrbr$; if we rotate it 270° clockwise (or 90° counterclockwise), then we obtain the coloring $brrr$; and if we rotate it 360° clockwise (or 0°—i.e., not at all), then we obtain the coloring $rbrr$ itself back. Note also that some colorings are *invariant* (i.e., unchanged) under rotations in addition to the 360° one; for example, $bbbb$ is invariant under all rotations, and $brrb$ is invariant under a 180° rotation. Similarly there are four reflections: around the center vertical axis of the board, around the center horizontal axis, around the lower-left-to-upper-right diagonal, and around the lower-right-to-upper-left diagonal. For example, applying the vertical axis reflection to $rrbb$ yields itself, while applying the lower-left-to-upper-right diagonal reflection results in $brbr$.

The definition of equivalence for this problem makes the proof rather messy, since both rotations and reflections are involved, and it is required that we reduce everything to just one or two operations. In fact, we claim that there are only eight possible motions of this square: clockwise rotations of 0°, 90°, 180°, or 270°, and reflections through the vertical, horizontal, lower-left-to-upper-right, and lower-right-to-upper-left diagonals. To verify this, we must show that the composition of every two of these operations is again an operation in our list. Below is the "group table" that shows this, where we use the symbols $r0$, $r90$, $r180$, $r270$, fv, fh, fp, and fn for these operations, respectively. (The mnemonic is that r stands for "rotation," f stands for "flip," and v, h, p, and n stand for "vertical," "horizontal", "positive-sloping," and "negative-sloping," respectively.) It is read just like a multiplication table, with the operation $\circ$ meaning "followed by." For example, if we first perform $r90$ and then perform fh, then we get the same result as if we had just performed fp (try it!).

$\circ$	$r0$	$r90$	$r180$	$r270$	fv	fh	fp	fn
$r0$	$r0$	$r90$	$r180$	$r270$	fv	fh	fp	fn
$r90$	$r90$	$r180$	$r270$	$r0$	fn	fp	fv	fh
$r180$	$r180$	$r270$	$r0$	$r90$	fh	fv	fn	fp
$r270$	$r270$	$r0$	$r90$	$r180$	fp	fn	fh	fv
fv	fv	fp	fh	fn	$r0$	$r180$	$r90$	$r270$
fh	fh	fn	fv	fp	$r180$	$r0$	$r270$	$r90$
fp	fp	fh	fn	fv	$r270$	$r90$	$r0$	$r180$
fn	fn	fv	fp	fh	$r90$	$r270$	$r180$	$r0$

So the result of this computation is that we can consider only these eight moves, and not have to worry about combinations of them—every combination of moves equals just one of these eight.

a) To show reflexivity, we note that every coloring can be obtained from itself via a 0° rotation. In technical terms, the 0° rotation is the *identity element* of our group. To show symmetry, we need to observe that rotations and reflections have inverses: If C_1 comes from C_2 via a rotation of n° clockwise, then C_2 comes from C_1 via a rotation of n° counterclockwise (or equivalently, via a rotation of $(360 - n)^\circ$ clockwise); and every reflection applied twice brings us back to the position (and therefore coloring) we began with.

And transitivity follows from the fact that the composition of two of these operations is again one of these operations.

b) The equivalence classes are represented by colorings that are truly distinct, in the sense of not being obtainable from each other via these operations. Let us list them. Clearly there is just one coloring using four red squares, and so just one equivalence class, $[rrrr]$. Similarly there is only one using four blues, $[bbbb]$. There is also just one equivalence class of colorings using three reds and one blue, since no matter which corner the single blue occupies in such a coloring, we can rotate to put the blue in any other corner. Thus our third and fourth equivalence classes are $[rrrb]$ and $[bbbr]$. Note that each of them contains four colorings. (For example, $[rrrb] = \{rrrb, rrbr, rbrr, brrr\}$.) This leaves only the colorings with two reds and two blues to consider. In every such coloring, either the red squares are adjacent (i.e., share a common edge), such as in $bbrr$, or they are not (e.g., $brrb$). Clearly the red squares are adjacent if and only if the blue ones are, since the only pairs of nonadjacent squares are (lower-left,upper-right) and (upper-left,lower-right). It is equally clear that there are only two colorings in which the red squares are not adjacent, namely $rbbr$ and $brrb$, and they are equivalent via a $90°$ rotation (among other transformations). So our fifth equivalence class is $[rbbr] = \{rbbr, brrb\}$. Finally, there is only one more equivalence class, and it contains the remaining four colorings (in which the two red squares are adjacent and the two blue squares are adjacent), namely $\{rrbb, brbr, bbrr, rbrb\}$, since each of these can be obtained from each of the others by a rotation. In summary we have partitioned the set of $2^4 = 16$ colorings (i.e., r-b strings of length four) into six equivalence classes, two of which have cardinality one, three of which have cardinality four, and one of which has cardinality two.

One final comment. We saw in the solution to part **(b)** that only rotations are needed to show the equivalence of every pair of equivalent colorings using just red and blue. This means that we are actually dealing with just part of the dihedral group here. If more colors had been used, then we would have needed to use the reflections as well. A complete discussion would get us into Pólya's theory of enumeration, which is studied in advanced combinatorics classes.

61. It is easier to write down a partition than it is to list the pairs in an equivalence relation, so we will answer the question using this notation. Let the set be $\{1, 2, 3\}$. We want to write down all possible partitions of this set. One partition is just $\{\{1, 2, 3\}\}$, i.e., having just one set (this corresponds to the equivalence relation in which every pair of elements are related). At the other extreme, there is the partition $\{\{1\}, \{2\}, \{3\}\}$, which corresponds to the equality relation (each x is related only to itself). The only other way to split up the elements of this set is into a set with two elements and a set with one element, and there are clearly three ways to do this, depending on which element we decide to put in the set by itself. Thus we get the partitions (pay attention to the punctuation!) $\{\{1, 2\}, \{3\}\}$, $\{\{1, 3\}, \{2\}\}$, and $\{\{2, 3\}, \{1\}\}$. If we wished to list the ordered pairs, we could; for example, the relation corresponding to $\{\{2, 3\}, \{1\}\}$ is $\{(2, 2), (2, 3), (3, 2), (3, 3), (1, 1)\}$. We found five partitions, so the answer to the question is 5.

63. We do get an equivalence relation. The issue is whether the relation formed in this way is reflexive, transitive and symmetric. It is clearly reflexive, since we included all the pairs (a, a) at the outset. It is clearly transitive, since the last thing we did was to form the transitive closure. It is symmetric by Exercise 23 in Section 9.4.

65. We end up with the relation R that we started with. Two elements are related if they are in the same set of the partition, but the partition is made up of the equivalence classes of R, so two elements are related precisely if they are related in R.

67. We make use of Exercise 63. Given the relation R, we first form the reflexive closure R' of R by adding to R each pair (a, a) that is not already there. Next we form the symmetric closure R'' of R', by adding, for each pair $(a, b) \in R'$ the pair (b, a) if it is not already there. Finally we apply Warshall's algorithm (or

Algorithm 1) from Section 9.4 to form the transitive closure of R''. This is the smallest equivalence relation containing R.

69. The exercise asks us to compute $p(n)$ for $n = 0, 1, 2, \ldots, 10$. In doing this we will use the recurrence relation, building on what we have already computed (namely $p(n-j-1)$, noting that $n-j-1 < n$), as well as using the binomial coefficients $C(n-1, j) = \dfrac{(n-1)!}{j!(n-1-j)!}$. We organize our computation in the obvious way, using the formula in Exercise 68.

$$
\begin{aligned}
p(0) &= 1 \quad \text{(the initial condition)}\\
p(1) &= C(0,0)p(0) = 1 \cdot 1 = 1\\
p(2) &= C(1,0)p(1) + C(1,1)p(0) = 1 \cdot 1 + 1 \cdot 1 = 2\\
p(3) &= C(2,0)p(2) + C(2,1)p(1) + C(2,2)p(0) = 1 \cdot 2 + 2 \cdot 1 + 1 \cdot 1 = 5\\
p(4) &= C(3,0)p(3) + C(3,1)p(2) + C(3,2)p(1) + C(3,3)p(0) = 1 \cdot 5 + 3 \cdot 2 + 3 \cdot 1 + 1 \cdot 1 = 15\\
p(5) &= C(4,0)p(4) + C(4,1)p(3) + C(4,2)p(2) + C(4,3)p(1) + C(4,4)p(0)\\
&= 1 \cdot 15 + 4 \cdot 5 + 6 \cdot 2 + 4 \cdot 1 + 1 \cdot 1 = 52\\
p(6) &= C(5,0)p(5) + C(5,1)p(4) + C(5,2)p(3) + C(5,3)p(2) + C(5,4)p(1) + C(5,5)p(0)\\
&= 1 \cdot 52 + 5 \cdot 15 + 10 \cdot 5 + 10 \cdot 2 + 5 \cdot 1 + 1 \cdot 1 = 203\\
p(7) &= C(6,0)p(6) + C(6,1)p(5) + C(6,2)p(4) + C(6,3)p(3) + C(6,4)p(2) + C(6,5)p(1) + C(6,6)p(0)\\
&= 1 \cdot 203 + 6 \cdot 52 + 15 \cdot 15 + 20 \cdot 5 + 15 \cdot 2 + 6 \cdot 1 + 1 \cdot 1 = 877\\
p(8) &= C(7,0)p(7) + C(7,1)p(6) + C(7,2)p(5) + C(7,3)p(4) + C(7,4)p(3) + C(7,5)p(2)\\
&\qquad + C(7,6)p(1) + C(7,7)p(0)\\
&= 1 \cdot 877 + 7 \cdot 203 + 21 \cdot 52 + 35 \cdot 15 + 35 \cdot 5 + 21 \cdot 2 + 7 \cdot 1 + 1 \cdot 1 = 4140\\
p(9) &= C(8,0)p(8) + C(8,1)p(7) + C(8,2)p(6) + C(8,3)p(5) + C(8,4)p(4) + C(8,5)p(3)\\
&\qquad + C(8,6)p(2) + C(8,7)p(1) + C(8,8)p(0)\\
&= 1 \cdot 4140 + 8 \cdot 877 + 28 \cdot 203 + 56 \cdot 52 + 70 \cdot 15 + 56 \cdot 5 + 28 \cdot 2 + 8 \cdot 1 + 1 \cdot 1 = 21147\\
p(10) &= C(9,0)p(9) + C(9,1)p(8) + C(9,2)p(7) + C(9,3)p(6) + C(9,4)p(5) + C(9,5)p(4)\\
&\qquad + C(9,6)p(3) + C(9,7)p(2) + C(9,8)p(1) + C(9,9)p(0)\\
&= 1 \cdot 21147 + 9 \cdot 4140 + 36 \cdot 877 + 84 \cdot 203 + 126 \cdot 52\\
&\qquad + 126 \cdot 15 + 84 \cdot 5 + 36 \cdot 2 + 9 \cdot 1 + 1 \cdot 1 = 115975
\end{aligned}
$$

SECTION 9.6 Partial Orderings

Partial orderings (or "partial orders"—the two phrases are used interchangeably) rival equivalence relations in importance in mathematics and computer science. Again, try to concentrate on the visual image—in this case the Hasse diagram. Play around with different posets to become familiar with the different possibilities; not all posets have to look like the less than or equal relation on the integers. Exercises 32 and 33 are important, and they are not difficult if you pay careful attention to the definitions.

1. The question in each case is whether the relation is reflexive, antisymmetric, and transitive. Suppose the relation is called R.

a) Clearly this relation is reflexive because each of 0, 1, 2, and 3 is related to itself. The relation is also antisymmetric, because the only way for a to be related to b is for a to equal b. Similarly, the relation is transitive, because if a is related to b, and b is related to c, then necessarily $a = b = c$ so a is related to c (because the relation is reflexive). This is just the equality relation on $\{0,1,2,3\}$; more generally, the equality relation on any set satisfies all three conditions and is therefore a partial ordering. (It is the smallest partial ordering; reflexivity insures that every partial ordering contains at least all the pairs (a,a).)

b) This is not a partial ordering, because although the relation is reflexive, it is not antisymmetric (we have $2\,R\,3$ and $3\,R\,2$, but $2 \neq 3$), and not transitive ($3\,R\,2$ and $2\,R\,0$, but 3 is not related to 0).

c) This is a partial ordering, because it is clearly reflexive; is antisymmetric (we just need to note that $(1,2)$ is the only pair in the relation with unequal components); and is transitive (for the same reason).

d) This is a partial ordering because it is the "less than or equal to" relation on $\{1,2,3\}$ together with the isolated point 0.

e) This is not a partial ordering. The relation is clearly reflexive, but it is not antisymmetric ($0\,R\,1$ and $1\,R\,0$, but $0 \neq 1$) and not transitive ($2\,R\,0$ and $0\,R\,1$, but 2 is not related to 1).

3. The question in each case is whether the relation is reflexive, antisymmetric, and transitive.

a) Since nobody is taller than himself, this relation is not reflexive so (S,R) cannot be a poset.

b) To be not taller means to be exactly the same height or shorter. Two different people x and y could have the same height, in which case $x\,R\,y$ and $y\,R\,x$ but $x \neq y$, so R is not antisymmetric and this is not a poset.

c) This is a poset. The equality clause in the definition of R guarantees that R is reflexive. To check antisymmetry and transitivity it suffices to consider unequal elements (these rules hold for equal elements trivially). If a is an ancestor of b, then b cannot be an ancestor of a (for one thing, an ancestor needs to be born before any descendant), so the relation is vacuously antisymmetric. If a is an ancestor of b, and b is an ancestor of c, then by the way "ancestor" is defined, we know that a is an ancestor of b; thus R is transitive.

d) This relation is not antisymmetric. Let a and b be any two distinct friends of yours. Then $a\,R\,b$ and $b\,R\,a$, but $a \neq b$.

5. The question in each case is whether the relation is reflexive, antisymmetric, and transitive.

a) The equality relation on any set satisfies all three conditions and is therefore a partial partial ordering. (It is the smallest partial partial ordering; reflexivity insures that every partial order contains at least all the pairs (a,a).)

b) This is not a poset, since the relation is not reflexive, not antisymmetric, and not transitive (the absence of one of these properties would have been enough to give a negative answer).

c) This is a poset, as explained in Example 1.

d) This is not a poset. The relation is not reflexive, since it is not true, for instance, that $2 \not| 2$. (It also is not antisymmetric and not transitive.)

7. **a)** This relation is $\{(1,1),(1,2),(1,3),(2,1),(2,2),(3,3)\}$. It is not antisymmetric because $(1,2)$ and $(2,1)$ are both in the relation, but $1 \neq 2$. We can see this visually by the pair of 1's symmetrically placed around the main diagonal at positions $(1,2)$ and $(2,1)$. Therefore this matrix does not represent a partial order.

b) This matrix represents a partial order. Reflexivity is clear. The only other pairs in the relation are $(1,2)$ and $(1,3)$, and clearly neither can be part of a counterexample to antisymmetry or transitivity.

c) A little trial and error shows that this relation is not transitive ($(4,1)$ and $(1,3)$ are present, but not $(4,3)$) and therefore not a partial order.

9. This relation is not transitive (there are arrows from a to b and from b to d, but there is no arrow from a to d), so it is not a partial order.

11. This relation is a partial order, since it has all three properties—it is reflexive (there is an arrow at each point), antisymmetric (there are no pairs of arrows going in opposite directions between two different points), and transitive (there is no missing arrow from some x to some z when there were arrows from x to y and y to z).

13. The dual of a poset is the poset with the same underlying set and with the relation defined by declaring a related to b if and only if $b \preceq a$ in the given poset.

a) The dual relation to $\leq$ is $\geq$, so the dual poset is $(\{0,1,2\}, \geq)$. Explicitly it is the set $\{(0,0),(1,0),(1,1),(2,0),(2,1),(2,2)\}$.

b) The dual relation to $\geq$ is $\leq$, so the dual poset is $(\mathbf{Z}, \leq)$.

c) The dual relation to $\supseteq$ is $\subseteq$, so the dual poset is $(P(\mathbf{Z}), \subseteq)$.

d) There is no symbol generally used for the "is a multiple of" relation, which is the dual to the "divides" relation in this part of the exercise. If we let R be the relation such that aRb if and only if $b \mid a$, then the answer can be written $(\mathbf{Z}^+, R)$.

15. We need to find elements such that the relation holds in neither direction between them. The answers we give are not the only ones possible.

a) One such pair is $\{1\}$ and $\{2\}$. These are both subsets of $\{0,1,2\}$, so they are in the poset, but neither is a subset of the other.

b) Neither 6 nor 8 divides the other, so they are incomparable.

17. We find the first coordinate (from left to right) at which the tuples differ and place first the tuple with the smaller value in that coordinate.

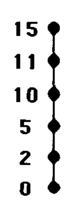
a) Since $1 = 1$ in the first coordinate, but $1 < 2$ in the second coordinate, $(1,1,2) < (1,2,1)$.

b) The first two coordinates agree, but $2 < 3$ in the third, so $(0,1,2,3) < (0,1,3,2)$.

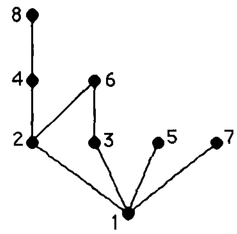
c) Since $0 < 1$ in the first coordinate, $(0,1,1,1,0) < (1,0,1,0,1)$.

19. All the strings that begin with 0 precede all those that begin with 1. The 0 comes first. Next comes 0001, which begins with three 0's, then 001, which begins with two 0's. Among the strings that begin 01, the order is $01 < 010 < 0101 < 011$. Putting this all together, we have $0 < 0001 < 001 < 01 < 010 < 0101 < 011 < 11$.

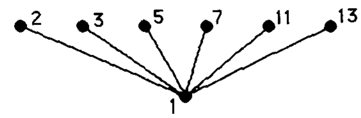
21. This is a totally ordered set, so the Hasse diagram is linear.



23. We put x above y if y divides x. We draw a line between x and y, where y divides x, if there is no number z in our set, other than x or y, such that $y \mid z \wedge z \mid x$. Note that in part **(b)** the numbers other than 1 are all (relatively) prime, so the Hasse diagram is short and wide, whereas in part **(d)** the numbers all divide one another, so the Hasse diagram is tall and narrow.



(a)



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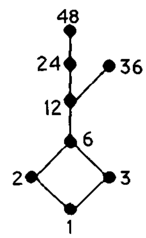

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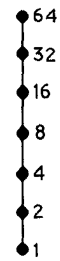


(d)

25. We need to include every pair (x,y) for which we can find a path going upward in the diagram from x to y. We also need to include all the reflexive pairs (x,x). Therefore the relation is the following set of pairs: $\{(a,a),(a,b),(a,c),(a,d),(b,b),(b,c),(b,d),(c,c),(d,d)\}$.

27. The procedure is the same as in Exercise 25: $\{(a,a),(a,d),(a,e),(a,f),(a,g),(b,b),(b,d),(b,e),(b,f),(b,g),(c,c),(c,d),(c,e),(c,f),(c,g),(d,d),(e,e),(f,f),(g,d),(g,e),(g,f),(g,g)\}$.

29. In this problem $X \preceq Y$ when $X \subseteq Y$. For (X,Y) to be in the covering relation, we need X to be a proper subset of Y but we also must have no subset strictly between X and Y. For example, $(\{a\},\{a,b,c\})$ is not in the covering relation, since $\{a\} \subset \{a,b\}$ and $\{a,b\} \subset \{a,b,c\}$. With this understanding it is easy to list the pairs in the covering relation: $(\emptyset,\{a\})$, $(\emptyset,\{b\})$, $(\emptyset,\{c\})$, $(\{a\},\{a,b\})$, $(\{a\},\{a,c\})$, $(\{b\},\{a,b\})$, $(\{b\},\{b,c\})$, $(\{c\},\{a,c\})$, $(\{c\},\{b,c\})$, $(\{a,b\},\{a,b,c\})$, $(\{a,c\},\{a,b,c\})$, and $(\{b,c\},\{a,b,c\})$.

31. Let $(S,\preceq)$ be a finite poset. We claim that this poset is just the reflexive transitive closure of its covering relation. Suppose that (a,b) is in the reflexive transitive closure of the covering relation. Then either $a=b$ or $a \prec b$ (in which cases certainly $a \preceq b$) or else there is a sequence $a \prec a_1 \prec a_2 \prec \cdots \prec a_n \prec b$, in which case again $a \preceq b$, by the transitivity of $\preceq$. Conversely, suppose that $a \preceq b$. If $a=b$, then (a,b) is certainly in the reflexive transitive closure of the covering relation. If $a \prec b$ and there is no z such that $a \prec z \prec b$, then $(a,b,)$ is in the covering relation and again therefore in its reflexive transitive closure. Otherwise, let $a \prec a_1 \prec a_2 \prec \cdots \prec a_n \prec b$ be a longest possible sequence of this form; since the poset is finite, there must be such a longest sequence. Then no intermediate elements can be inserted into this sequence (to do so would lengthen it), so each pair (a,a_1), (a_1,a_2), $\ldots$, (a_n,b) is in the covering relation, so again (a,b) is in its reflexive transitive closure. This completes the proof. Note how the finiteness of the poset was crucial here. If we let S be the set of all subsets of $\mathbf{N}$ (the set of natural numbers) under the subset relation, then we cannot recover S from its covering relation, since nothing in the covering relation allows us to relate a finite set to an infinite one; thus for example we could not recover the relationship $\{1,2\} \subset \mathbf{N}$.

33. It is helpful in this exercise to draw the Hasse diagram.

a) Maximal elements are those that do not divide any other elements of the set. In this case 24 and 45 are the only numbers that meet that requirement.

b) Minimal elements are those that are not divisible by any other elements of the set. In this case 3 and 5 are the only numbers that meet that requirement.

c) A greatest element would be one that all the other elements divide. The only two candidates (maximal elements) are 24 and 45, and since neither divides the other, we conclude that there is no greatest element.

d) A least element would be one that divides all the other elements. The only two candidates (minimal elements) are 3 and 5, and since neither divides the other, we conclude that there is no least element.

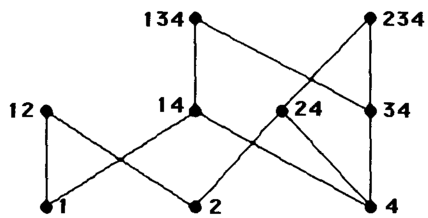
e) We want to find all elements that both 3 and 5 divide. Clearly only 15 and 45 meet this requirement.

f) The least upper bound is 15 since it divides 45 (see part **(e)**).

g) We want to find all elements that divide both 15 and 45. Clearly only 3, 5, and 15 meet this requirement.

h) The number 15 is the greatest lower bound, since both 3 and 5 divide it (see part **(g)**).

35. To help us answer the questions, we will draw the Hasse diagram, with the commas and braces eliminated in the labels, for readability.



a) The maximal elements are the ones without any elements lying above them in the Hasse diagram, namely $\{1,2\}$, $\{1,3,4\}$, and $\{2,3,4\}$.

b) The minimal elements are the ones without any elements lying below them in the Hasse diagram, namely $\{1\}$, $\{2\}$, and $\{4\}$.

c) There is no greatest element, since there is more than one maximal element, none of which is greater than the others.

d) There is no least element, since there is more than one minimal element, none of which is less than the others.

e) The upper bounds are the sets containing both $\{2\}$ and $\{4\}$ as subsets, i.e., the sets containing both 2 and 4 as elements. Pictorially, these are the elements lying above both $\{2\}$ and $\{4\}$ (in the sense of there being a path in the diagram), namely $\{2,4\}$ and $\{2,3,4\}$.

f) The least upper bound is an upper bound that is less than every other upper bound. We found the upper bounds in part **(e)**, and since $\{2,4\}$ is less than (i.e., a subset of) $\{2,3,4\}$, we conclude that $\{2,4\}$ is the least upper bound.

g) To be a lower bound of both $\{1,3,4\}$ and $\{2,3,4\}$, a set must be a subset of each, and so must be a subset of their intersection, $\{3,4\}$. There are only two such subsets in our poset, namely $\{3,4\}$ and $\{4\}$. In the diagram, these are the points which lie below (in the path sense) both $\{1,3,4\}$ and $\{2,3,4\}$.

h) The greatest lower bound is a lower bound that is greater than every other lower bound. We found the lower bounds in part **(g)**, and since $\{3,4\}$ is greater than (i.e., a superset of) $\{4\}$, we conclude that $\{3,4\}$ is the greatest lower bound.

37. First we need to show that lexicographic order is reflexive, i.e., that $(a,b) \preceq (a,b)$; this is true by fiat, since we defined $\preceq$ by adding equality to $\prec$. Next we need to show antisymmetry: if $(a,b) \preceq (c,d)$ and $(a,b) \neq (c,d)$, then $(c,d) \not\preceq (a,b)$. By definition $(a,b) \prec (c,d)$ if and only if either $a \prec c$, or $a = c$ and $b \prec d$. In the first case, by the antisymmetry of the underlying relation, we know that $c \not\prec a$, and similarly in the second case we know that $d \not\prec d$. Thus there is no way that we could have $(c,d) \prec (a,b)$. Finally, for transitivity, let $(a,b) \preceq (c,d) \preceq (e,f)$. We want to show that $(a,b) \preceq (e,f)$. If one of the given inequalities is an equality, then there is nothing to prove, so we may assume that $(a,b) \prec (c,d) \prec (e,f)$. If $a \prec c$, then by the transitivity of the underlying relation, we know that $a \prec e$ and so $(a,b) \prec (e,f)$. Similarly, if $c \prec e$, then again $a \prec e$

and so $(a,b) \prec (e,f)$. The only other way for the given inequalities to hold is if $a = c = e$ and $b \prec d \prec f$. In this case the latter string of inequalities implies that $b \prec f$ and so again by definition $(a,b) \prec (e,f)$.

39. First we must show that $\preceq$ is reflexive. Since $s \preceq_1 s$ and $t \preceq_2 t$ by the reflexivity of these underlying partial orders, $(s,t) \preceq (s,t)$ by definition. For antisymmetry, assume that $(s,t) \preceq (u,v)$ and $(u,v) \preceq (s,t)$. Then by definition $s \preceq_1 u$ and $t \preceq_2 v$, and $u \preceq_1 s$ and $v \preceq_2 t$. By the antisymmetry of the underlying relations, we conclude that $s = u$ and $t = v$, whence $(s,t) = (u,v)$. Finally, for transitivity, suppose that $(s,t) \preceq (u,v) \preceq (w,x)$. This means that $s \preceq_1 u \preceq_1 w$ and $t \preceq_2 v \preceq_2 x$. The transitivity of the underlying partial orders tells us that $s \preceq_1 w$ and $t \preceq_2 x$, whence by definition $(s,t) \preceq (w,x)$.

41. **a)** We argue essentially by contradiction. Suppose that m_1 and m_2 are two maximal elements in a poset that has a greatest element g; we will show that $m_1 = m_2$. Now since g is greatest, we know that $m_1 \preceq g$, and similarly for m_2. But since each m_i is maximal, it cannot be that $m_i \prec g$; hence $m_1 = g = m_2$.

b) The proof is exactly dual to the proof in part **(a)**, so we just copy over that proof, making the appropriate changes in wording. To wit: we argue essentially by contradiction. Suppose that m_1 and m_2 are two minimal elements in a poset that has a least element l; we will show that $m_1 = m_2$. Now since l is least, we know that $l \preceq m_1$, and similarly for m_2. But since each m_i is minimal, it cannot be that $l \prec m_i$; hence $m_1 = l = m_2$.

43. In each case, we need to check whether every pair of elements has both a least upper bound and a greatest lower bound.

a) This is a lattice. If we want to find the l.u.b. or g.l.b. of two elements in the same vertical column of the Hasse diagram, then we simply take the higher or lower (respectively) element. If the elements are in different columns, then to find the g.l.b. we follow the diagonal line upward from the element on the left, and then continue upward on the right, if necessary to reach the element on the right. For example, the l.u.b. of d and c is f; and the l.u.b. of a and e is e. Finding greatest lower bounds in this poset is similar.

b) This is not a lattice. Elements b and c have f, g, and h as upper bounds, but none of them is a l.u.b.

c) This is a lattice. By considering all the pairs of elements, we can verify that every pair of them has a l.u.b. and a g.l.b. For example, b and e have g and a filling these roles, respectively.

45. As usual when trying to extend a theorem from two items to an arbitrary finite number, we will use mathematical induction. The statement we wish to prove is that if S is a subset consisting of n elements from a lattice, where n is a positive integer, then S has a least upper bound and a greatest lower bound. The two proofs are duals of each other, so we will just give the proof for least upper bound here. The basis is $n = 1$, in which case there is really nothing to prove. If $S = \{x\}$, then clearly x is the least upper bound of S. The case $n = 2$ could be singled out for special mention also, since the l.u.b. in that case is guaranteed by the definition of lattice. But there is no need to do so. Instead, we simply assume the inductive hypothesis, that every subset containing n elements has a l.u.b., and prove that every subset S containing $n+1$ elements also has a l.u.b. Pick an arbitrary element $x \in S$, and let $S' = S - \{x\}$. Since S' has only n elements, it has a l.u.b. y, by the inductive hypothesis. Since we are in a lattice, there is an element z that is the l.u.b. of x and y. We will show that in fact z is the least upper bound of S. To do this, we need to show two things: that z is an upper bound, and that every upper bound is greater than or equal to z. For the first statement, let w be an arbitrary element of S; we must show that $w \preceq z$. There are two cases. If $w = x$, then $w \preceq z$ since z is the l.u.b. of x and y. Otherwise, $w \in S'$, and so $w \preceq y$ because y is the l.u.b. of S'. But since z is the l.u.b. of x and y, we also have $y \preceq z$. By transitivity, then, $w \preceq z$. For the second statement, suppose that u is any other upper bound of S; we must show that $z \preceq u$. Since u is an upper bound of S, it is also an upper bound of x and y. But since z is the *least* upper bound of x and y, we know that $z \preceq u$.

47. The needed definitions are in Example 25.

a) No. The authority level of the first pair (1) is less than or equal to (less than, in this case) that of the second (2); but the subset of the first pair is not a subset of that of the second.

b) Yes. The authority level of the first pair (2) is less than or equal to (less than, in this case) that of the second (3); and the subset of the first pair is a subset of that of the second.

c) The classes into which information can flow are those classes whose authority level is at least as high as *Proprietary*, and whose subset is a superset of {*Cheetah, Puma*}. We can list these classes: (*Proprietary*, {*Cheetah*, *Puma*}), (*Restricted*, {*Cheetah*, *Puma*}), (*Registered*, {*Cheetah*, *Puma*}), (*Proprietary*, {*Cheetah*, *Puma*, *Impala*}), (*Restricted*, {*Cheetah*, *Puma*, *Impala*}), and (*Registered*, {*Cheetah*, *Puma*, *Impala*}).

d) The classes from which information can flow are those classes whose authority level is at least as low as *Restricted*, and whose subset is a subset of {*Impala, Puma*}, namely (*Nonproprietary*,{*Impala*,*Puma*}), (*Proprietary*,{*Impala*,*Puma*}), (*Restricted*,{*Impala*,*Puma*}), (*Nonproprietary*,{*Impala*}), (*Proprietary*, {*Impala*}), (*Restricted*,{*Impala*}), (*Nonproprietary*,{*Puma*}), (*Proprietary*,{*Puma*}), (*Restricted*,{*Puma*}), (*Nonproprietary*,∅), (*Proprietary*,∅), and (*Restricted*,∅).

49. Let Π be the set of all partitions of a set S, with a relation $\preceq$ defined on Π according to the referenced preamble: a partition P_1 is a refinement of P_2 if every set in P_1 is a subset of one of the sets in P_2. We need to verify all the properties of a lattice. First we need to show that $(\Pi, \preceq)$ is a poset, that is, that $\preceq$ is reflexive, antisymmetric, and transitive. For reflexivity, we need to show that $P \preceq P$ for every partition P. This means that every set in P is a subset of one of the sets in P, and this is trivially true, since every set is a subset of itself. For antisymmetry, suppose that $P_1 \preceq P_2$ and $P_2 \preceq P_1$. We must show that $P_1 = P_2$. By the equivalent roles played here by P_1 and P_2, it is enough to show that every $T \in P_1$ (where $T \subseteq S$) is also an element of P_2. Suppose we have such a T. Then since $P_1 \preceq P_2$, there is a set $T' \in P_2$ such that $T \subseteq T'$. But then since $P_2 \preceq P_1$, there is a set $T'' \in P_1$ such that $T' \subseteq T''$. Putting these together, we have $T \subseteq T''$. But P_1 is a partition, and so the elements of P_1 are nonempty and pairwise disjoint. The only way for this to happen if one is a subset of the other is for the two subsets T and T'' to be the same. But this implies that T' (which is caught in the middle) is also equal to T. Thus $T \in P_2$, which is what we were trying to show. Finally, for transitivity, suppose that $P_1 \preceq P_2$ and $P_2 \preceq P_3$. We must show that $P_1 \preceq P_3$. To this end, we take an arbitrary element $T \in P_1$. Then there is a set $T' \in P_2$ such that $T \subseteq T'$. But then since $P_2 \preceq P_3$, there is a set $T'' \in P_3$ such that $T' \subseteq T''$. Putting these together, we have $T \subseteq T''$. This demonstrates that $P_1 \preceq P_3$.

Next we have to show that every two partitions P_1 and P_2 have a least upper bound and a greatest lower bound in Π. We will show that their greatest lower bound is their "coarsest common refinement", namely the partition P whose subsets are all the nonempty sets of the form $T_1 \cap T_2$, where $T_1 \in P_1$ and $T_2 \in P_2$. As an example, if $P_1 = \{\{1,2,3\},\{4\},\{5\}\}$ and $P_2 = \{\{1,2\},\{3,4\},\{5\}\}$, then the coarsest common refinement is $P = \{\{1,2\},\{3\},\{4\},\{5\}\}$. First, we need to check that this is a partition. It certainly is a set of nonempty subsets of S. It is pairwise disjoint, because the only way an element could be in $T_1 \cap T_2 \cap T_1' \cap T_2'$ if $T_1 \cap T_2 \neq T_1' \cap T_2'$ is for that element to be in both $T_1 \cap T_1'$ and $T_2 \cap T_2'$, which means that $T_1 = T_1'$ and $T_2 = T_2'$, a contradiction. And it covers all of S, because if $x \in S$, then $x \in T_1$ for some $T_1 \in P_1$, and $x \in T_2$ for some $T_2 \in P_2$, and so $x \in T_1 \cap T_2 \in P$. Second, we need to check that P is a refinement of both P_1 and P_2. So suppose $T \in P$. Then $T = T_1 \cap T_2$, for some $T_1 \in P_1$ and $T_2 \in P_2$. It follows that $T \subseteq T_1$ and $T \subseteq T_2$. But then T_1 and T_2 satisfy the requirements in the definition of refinement. Third, we need to check that if P' is any other common refinement of both P_1 and P_2, then P' is also a refinement of P. To this end, suppose that $T \in P'$. Then by definition of refinement, there are subsets $T_1 \in P_1$ and $T_2 \in P_2$ such that $T \subseteq T_1$ and $T \subseteq T_2$. Therefore $T \subseteq T_1 \cap T_2$. But $T_1 \cap T_2 \in P$, and our proof for greatest lower bounds is complete.

It's a little harder to state the definition of the least upper bound (which again we'll call P) of two given partitions P_1 and P_2. Essentially it is just the set of all minimal nonempty subsets of S that do not "split apart" any element of either P_1 or P_2. (In the example above, it is $\{\{1,2,3,4\},\{5\}\}$.) It will be a little easier if we define it in terms of an equivalence relation rather than a partition. Note that from this point of view, one equivalence relation is a refinement of a second equivalence relation if whenever two elements are related by the first relation, then they are related by the second. The equivalence relation determining P is the relation in which $x \in S$ is related to $y \in S$ if there is a "path" (a sequence) $x = x_0$, x_1, x_2, ..., $x_n = y$, for some $n \geq 0$, such that for each i from 1 to n, x_{i-1} and x_i are in the same element of partition P_1 or of partition P_2 (in other words, x_{i-1} and x_i are related either by the equivalence relation corresponding to P_1 or by that corresponding to P_2). It is clear that this is an equivalence relation: it is reflexive by taking $n = 0$; it is symmetric by following the path backwards; and it is transitive by composing paths. It is also clear that P_1 (and P_2 similarly) is a refinement of this partition, since if two elements of S are in the same equivalence class in P_1, then we can take $n = 1$ in our path definition to see that they are in the same equivalence class in P. Thus P is an upper bound of both P_1 and P_2. Finally, we must show that P is the *least* upper bound, that is, a refinement of every other upper bound. This is clear from our construction: we only forced two elements of S to be related (i.e., in the same class of the partition) when they *had* to be related in order to enable P_1 and P_2 to be refinements. Therefore if two elements are related by P, then they have to be related by every equivalence relation (partition) Q of which both P_1 and P_2 are refinements; so P is a refinement of Q.

51. This follows immediately from Exercise 45. To be more specific, according to Exercise 45, there is a least upper bound (respectively, a greatest lower bound) for the entire finite lattice. This element is by definition a greatest element (respectively, a least element).

53. We need to show that every nonempty subset of $\mathbf{Z}^+ \times \mathbf{Z}^+$ has a least element under lexicographic order. Given such a subset S, look at the set S_1 of positive integers that occur as first coordinates in elements of S. Let m_1 be the least element of S_1, which exists since $\mathbf{Z}^+$ is well-ordered under $\leq$. Let S' be the subset of S consisting of those pairs that have m_1 as their first coordinate. Thus S' is clearly nonempty, and by the definition of lexicographic order, every element of S' is less than every element in $S - S'$. Now let S_2 be the set of positive integers that occur as second coordinates in elements of S', and let m_2 be the least element of S_2. Then clearly the element (m_1, m_2) is the least element of S' and hence is the least element of S.

55. If x is an integer in a decreasing sequence of elements of this poset, then at most $|x|$ elements can follow x in the sequence, namely integers whose absolute values are $|x| - 1$, $|x| - 2$, ..., 1, 0. Therefore there can be no infinite decreasing sequence. This is not a totally ordered set, since 5 and -5, for example, are incomparable; from the definition given here, it is neither true that $5 \prec -5$ nor that $-5 \prec 5$, because neither one of $|5|$ or $|-5|$ is less than the other (they are equal).

57. We know from elementary arithmetic that $\mathbf{Q}$ is totally ordered by $<$, and so perforce it is a partially ordered set. To be precise, to find which of two rational numbers is larger, write them with a positive common denominator and compare numerators. To show that this set is dense, suppose $x < y$ are two rational numbers. Let z be their average, i.e., $(x + y)/2$. Since the set of rational numbers is closed under addition and division, z is also a rational number, and it is easy to show that $x < z < y$.

59. Let $(S, \preceq)$ be a partially ordered set. From the definitions of well-ordered, totally ordered, and well-founded, it is clear that what we have to show is that every nonempty subset of S contains a least element if and only if there is no infinite decreasing sequence of elements $a_1, a_2, a_3, \ldots$ in S (i.e., where $a_{i+1} \prec a_i$ for all i). One direction is clear: An infinite decreasing sequence of elements has no least element. Conversely, let A be any nonempty subset of S that has no least element. Since A is nonempty, let a_1 be any element of A. Since a_1 is not the least element of A, there is some $a_2 \in A$ smaller than a_1, i.e., $a_2 \prec a_1$. Since a_2 is not the least element of A, A must contain an element a_3 with $a_3 \prec a_2$. We continue in this manner, giving us an infinite decreasing sequence in S. Note that this proof is nonconstructive; it uses what set theorists call the Axiom of Choice.

61. We need to peel elements off the bottom of the Hasse diagram. We can begin with a, b, or c. Suppose we decide to start with a. Next we may choose any minimal element of what remains after we have removed a; only b and c meet this requirement. Suppose we choose b next. Then c, d, and e are minimal elements in what remains, so any of those can come next. We continue in this manner until we have listed and removed all the elements. One possible order, then, is $a \prec_t b \prec_t d \prec_t e \prec_t c \prec_t f \prec_t g \prec_t h \prec_t i \prec_t j \prec_t k \prec_t m \prec_t l$.

63. Clearly 1 must come first, and 20 must follow each element except possibly 12. The relative positions of 2, 4, and 12 are fixed. The 5 can go anywhere, as long as it lies between 1 and 20. Following these guidelines, we see that the following seven total orderings are the ones compatible with the given relation: $1 \prec 5 \prec 2 \prec 4 \prec 12 \prec 20$, $1 \prec 2 \prec 5 \prec 4 \prec 12 \prec 20$, $1 \prec 2 \prec 4 \prec 5 \prec 12 \prec 20$, $1 \prec 2 \prec 4 \prec 12 \prec 5 \prec 20$, $1 \prec 5 \prec 2 \prec 4 \prec 20 \prec 12$, $1 \prec 2 \prec 5 \prec 4 \prec 20 \prec 12$, $1 \prec 2 \prec 4 \prec 5 \prec 20 \prec 12$.

65. There are a few restrictions, but there are many choices, so we will get many (18) compatible total orderings. Note that A and C must precede B; B and E must precede F; B must precede D; and G must come last. We can therefore make the following list: $A \prec C \prec E \prec B \prec D \prec F \prec G$, $A \prec E \prec C \prec B \prec D \prec F \prec G$, $C \prec A \prec E \prec B \prec D \prec F \prec G$, $C \prec E \prec A \prec B \prec D \prec F \prec G$, $E \prec A \prec C \prec B \prec D \prec F \prec G$, $E \prec C \prec A \prec B \prec D \prec F \prec G$, $A \prec C \prec B \prec E \prec D \prec F \prec G$, $C \prec A \prec B \prec E \prec D \prec F \prec G$, $A \prec C \prec B \prec D \prec E \prec F \prec G$, $C \prec A \prec B \prec D \prec E \prec F \prec G$, $A \prec C \prec E \prec B \prec F \prec D \prec G$, $A \prec E \prec C \prec B \prec F \prec D \prec G$, $C \prec A \prec E \prec B \prec F \prec D \prec G$, $C \prec E \prec A \prec B \prec F \prec D \prec G$, $E \prec A \prec C \prec B \prec F \prec D \prec G$, $E \prec C \prec A \prec B \prec F \prec D \prec G$, $A \prec C \prec B \prec E \prec F \prec D \prec G$, and $C \prec A \prec B \prec E \prec F \prec D \prec G$.

67. We need to find a total order compatible with this partial order. We work from the bottom up, writing down a task (vertex in the diagram) and removing it from the diagram, so that at each stage we choose a vertex with no vertices below it. One such order is: Determine user needs $\prec$ Write functional requirements $\prec$ Set up test sites $\prec$ Develop system requirements $\prec$ Develop module A $\prec$ Develop module C $\prec$ Develop module B $\prec$ Write documentation $\prec$ Integrate modules $\prec$ α test $\prec$ β test $\prec$ Completion.

GUIDE TO REVIEW QUESTIONS FOR CHAPTER 9

1. a) See p. 575 (which refers to the definition on p. 573). **b)** See Example 6 in Section 9.1.

2. a) See p. 576. **b)** See p. 577. **c)** See p. 577. **d)** See p. 578.

3. a) $\{(1,1),(1,2),(2,1),(2,2),(2,3),(3,2),(3,3),(4,4)\}$
b) $\emptyset$ **c)** $\{(1,1),(1,2),(2,2),(2,3),(3,3),(4,4)\}$
d) $\{(1,1),(2,2),(3,3),(4,4)\}$ **e)** $\{(1,1),(2,2),(3,3),(4,4)\}$

4. a) See Example 16 in Section 9.1. **b)** See Exercise 47a in Section 9.1.
c) See Exercise 47b in Section 9.1.

5. a) See pp. 584–585. **b)** Take the projection $P_{1,4,5}$.
c) First rearrange the order of the fields in the relations, so that the first is in the order address, telephone number, name, major, and the second is in the order name, major, student number, number of credit hours. Then form the join J_2, to get a single relation with the fields in the order address, telephone number, name, major, student number, number of credit hours. Finally, if desired, rearrange the fields to a more natural order.

6. a) See p. 591. b) See p. 592.

7. a) See p. 594. b) See p. 595.

8. a) See p. 598. b) Add all the pairs (a, a).
c) Whenever a pair (a, b) is in the relation, add the pair (b, a).
d) The reflexive closure is $\{(1,1),(1,2),(2,2),(2,3),(2,4),(3,1),(3,3),(4,4)\}$, and the symmetric closure is $\{(1,2),(1,3),(2,1),(2,3),(2,4),(3,1),(3,2),(4,2)\}$.

9. a) the smallest transitive relation containing R b) no c) See Algorithms 1 and 2 in Section 9.4.
d) the relation that always holds between two elements of $\{1,2,3,4\}$ (in symbols, $\{1,2,3,4\} \times \{1,2,3,4\}$)

10. a) See p. 608.
b) One equivalence relation is the one with equivalence classes $\{a, b, d\}$ and $\{c\}$. The only other one meeting this condition is the relation that always holds, i.e., in which there is just one equivalence class, $\{a, b, c, d\}$.

11. a) See Example 3 in Section 9.5.
b) It is easy to see that this relation is
$\{(0,0),(1,1),(1,6),(6,1),(6,6),(2,2),(2,5),(5,2),(5,5),(3,3),(3,4),(4,3),(4,4)\}$.

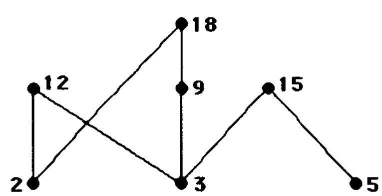
12. a) See p. 610.
b) $[0] = \{\ldots, -5, 0, 5, \ldots\}$, $[1] = \{\ldots, -4, 1, 6, \ldots\}$, $[2] = \{\ldots, -3, 2, 7, \ldots\}$, $[3] = \{\ldots, -2, 3, 8, \ldots\}$, and $[4] = \{\ldots, -1, 4, 9, \ldots\}$
c) $\{0\}$, $\{1,6\}$, $\{2,5\}$, and $\{3,4\}$

13. See Theorem 2 in Section 9.5.

14. a) See p. 618. b) See Example 2 in Section 9.6.

15. See the definition of lexicographic ordering on p. 621.

16. a) See pp. 622–623. b) Here is the Hasse diagram:



17. a) See pp. 624–625. b) $(\{1,2,3,4,5\}, \mid)$ c) $(\{1,2,3,4,5\}, \leq)$

18. a) See p. 626.
b) Extreme examples are the lattice $(\{1,2,3,4,5\}, \leq)$ (every total order is a lattice) and the nonlattice $(\{1,2,3,4,5\}, =)$ (no two distinct elements have an upper bound).

19. a) See Exercise 41 in Section 9.6. b) See Exercise 51 in Section 9.6

20. a) See p. 620. (We assume for the rest of this question that the set is finite, in which case a well-ordered set is the same as a totally ordered set, as defined on p. 619.) b) See topological sorting on pp. 627–628.
c) See Example 27 in Section 9.6.

SUPPLEMENTARY EXERCISES FOR CHAPTER 9

1. **a)** This relation is not reflexive, since most strings have many letters in common with themselves. Whether it is irreflexive depends on whether we mean to include the empty string; the empty string is the only string s such that $(s,s) \in R_1$ (the empty string has no letters in common with itself, since it has no letters). Thus if we mean not to include the empty string in the underlying set, then the relation is irreflexive; otherwise it is not. The relation is symmetric by inspection (the roles of a and b in the sentence are symmetric). It is not antisymmetric, since there are many pairs of strings such that $(a,b) \in R_1 \wedge (b,a) \in R_1$; for instance $a =$ *bullfinch* and $b =$ *parrot*. The relation is not transitive, since, for example, although *bullfinch* and *parrot* are related, and *parrot* and *chicken* are related, *bullfinch* and *chicken* do have letters in common and so are not related.

b) This relation is very similar to the relation R_1. For no string is $(a,a) \in R_2$, so the relation is irreflexive and not reflexive. It is symmetric by inspection, and not antisymmetric (same example as above). It is also not transitive, since, for instance, *finch* is related to *parrot*, which is related to *robin*, but *finch* is not related to *robin*, since they are the same length.

c) No string is longer than itself, so R_3 is irreflexive and not reflexive. It is not symmetric, since, for instance, *robin* is longer than *wren*, but *wren* is not longer than *robin*. It is antisymmetric: there is no way for both a to be longer than b and b to be longer than a. Finally, it is clearly transitive, since if a is longer than b which is longer than c, then a is longer than c.

3. By algebra, the given condition is the same as the condition that $f((a,b)) = f((c,d))$, where $f((x,y)) = x-y$. Therefore by Exercise 9 in Section 9.5, this is an equivalence relation.

5. Suppose that $(a,b) \in R$. We must show that $(a,b) \in R^2$. By reflexivity, we know that $(b,b) \in R$. Therefore by the definition of R^2, we combine the facts that $(a,b) \in R$ and $(b,b) \in R$ to conclude that $(a,b) \in R^2$.

7. Both of these conclusions are valid. Since each pair (a,a) is in both R_1 and R_2, we can conclude that each pair (a,a) is in $R_1 \cap R_2$ and $R_1 \cup R_2$.

9. Both of these conclusions are valid. For the first, suppose that $(a,b) \in R_1 \cap R_2$. This means that $(a,b) \in R_1$ and $(a,b) \in R_2$. By the symmetry of R_1 and R_2, we conclude that $(b,a) \in R_1$ and $(b,a) \in R_2$. Therefore (b,a) is in their intersection, as desired. For the second part, suppose that $(a,b) \in R_1 \cup R_2$. This means that $(a,b) \in R_1$ or $(a,b) \in R_2$. By the symmetry of R_1 and R_2, we conclude either that $(b,a) \in R_1$ or that $(b,a) \in R_2$. Therefore (b,a) is in their union, as desired.

11. A primary key is one for which there are no two different rows with the same value in this field. If there were two different rows with the same value after projection, then there certainly would have been two different rows with the same value before projection.

13. The key point is that $\Delta^{-1} = \Delta$, where Δ consists of all the pairs (a,a). Thus it does not matter whether we add the pairs in Δ before or after we add the reverse of every pair in the original relation.

15. **a)** We observed in Exercise 29b in Section 9.4 that we need to take the symmetric closure first in order to insure that the result is symmetric. The relation given in that exercise provides an example. An even simpler one is the relation $\{(0,1),(2,1)\}$; the symmetric closure of the transitive closure is $\{(0,1),(1,0),(1,2),(2,1)\}$, but the transitive closure of the symmetric closure is all of $\{0,1,2\} \times \{0,1,2\}$.

b) Suppose that (a,b) is in the symmetric closure of the transitive closure of R. We must show that (a,b) is also in the transitive closure of the symmetric closure of R. Now either (a,b) or (b,a) is in the transitive closure of R. This means that either there is a path from a to b or a path from b to a in R. In the former case, there is perforce a path from a to b in the symmetric closure of R. In the latter case, the path from b to a can be followed backwards in the symmetric closure of R, since the symmetric closure adds the reverses of all the edges in R. Therefore in either case (a,b) is in the transitive closure of the symmetric closure of R. (See also the related Exercise 23 in Section 9.4.)

17. The closure of S with respect to $\mathbf{P}$ is a relation S' which contains S as a subset and has property $\mathbf{P}$. Since $R \subseteq S$, we conclude that $R \subseteq S'$. By definition of closure, then, the closure of R must be a subset of S', as desired.

19. We use the basic idea of Warshall's algorithm, except that $w_{ij}^{[k]}$ will be a numerical variable (taking values from 0 to ∞, inclusive) representing the length of the longest path from v_i to v_j all of whose interior vertices are labeled less than or equal to k, rather than simply a Boolean variable indicating whether such a path exists. A value of 0 for $w_{ij}^{[k]}$ will mean that there is no path from v_i to v_j all of whose interior vertices are labeled less than or equal to k. To compute $w_{ij}^{[k]}$ from the matrix $\mathbf{W}_{k-1}$, we determine, for each pair (i,j), whether there are paths from v_i to v_k and from v_k to v_j using no interior vertices labeled greater than $k-1$. If either of $w_{ik}^{[k-1]}$ or $w_{kj}^{[k-1]}$ equals 0, then such a pair of paths does not exist, so we set $w_{ij}^{[k]}$ equal to $w_{ij}^{[k-1]}$. Otherwise (if such a pair of paths does exist), then there are two possibilities. If $w_{kk}^{[k-1]} > 0$, then we now know that there are paths of arbitrary length from v_i to v_j, since we can loop around v_k as long as we please; in this case we set $w_{ij}^{[k]}$ to ∞. If $w_{kk}^{[k-1]} = 0$, then we do not yet have such looping, so we set $w_{ij}^{[k]}$ to the larger of $w_{ij}^{[k-1]}$ and $w_{ik}^{[k-1]} + w_{kj}^{[k-1]}$. (Initially we set $\mathbf{W}_0$ equal to the matrix representing the relation.)

21. There are 52 partitions in all, but that is not the question. If there are to be three equivalence classes, then the classes must have sizes $3,1,1$ or $2,2,1$. There are $C(5,3) = 10$ partitions into one set with 3 elements and the other two sets of 1 element each, since the only choice involved is choosing the 3-set. There are $C(5,2)C(3,2)/2 = 15$ ways to partition our set into sets of size 2, 2, and 1; we need to choose the 2 elements for the first set of size 2, then we need to choose the 2 elements from the 3 remaining for the second set of size 2, except that we have overcounted by a factor of 2, since we could choose these two 2-sets in either order. Therefore there are $10 + 15 = 25$ partitions into three classes.

23. There is no question that the collection defined here is a refinement of each of the given partitions, since each set $A_i \cap B_j$ is a subset of A_i and of B_j. We must show that it is actually a partition. By construction, each of the sets in this collection is nonempty. To see that their union is all of S, let $s \in S$. Since P_1 and P_2 are partitions of S, there are sets A_i and B_j such that $s \in A_i$ and $s \in B_j$. Therefore $s \in A_i \cap B_j$, which shows that s is in one of the sets in our collection. Finally, to see that these sets are pairwise disjoint, simply note that unless $i = i'$ and $j = j'$, then $(A_i \cap B_j) \cap (A_{i'} \cap B_{j'}) = (A_i \cap A_{i'}) \cap (B_j \cap B_{j'})$ is empty, since either $(A_i \cap A_{i'})$ or $(B_j \cap B_{j'})$ is empty.

25. The subset relation is a partial order on *every* collection of sets, since it is reflexive, antisymmetric, and transitive. Here the collection of sets happens to be $\mathbf{R}(S)$.

27. We need to find a total order compatible with this partial order. We work from the bottom up, writing down a task (vertex in the diagram) and removing it from the diagram, so that at each stage we choose a vertex with no vertices below it. One such order is: Find recipe $\prec$ Buy seafood $\prec$ Buy groceries $\prec$ Wash shellfish $\prec$ Cut ginger and garlic $\prec$ Clean fish $\prec$ Steam rice $\prec$ Cut fish $\prec$ Wash vegetables $\prec$ Chop water chestnuts $\prec$ Make garnishes $\prec$ Cook in wok $\prec$ Arrange on platter $\prec$ Serve.

29. Since every subset of an antichain is clearly an antichain, we will list only the maximal antichains; the actual answers will be everything we list together with all the subsets of them.

a) Here every two elements are comparable except c and d. Thus the maximal antichains are $\{c, d\}$, $\{a\}$, and $\{b\}$. (There are three more antichains which are subsets of these: $\{c\}$, $\{d\}$, and $\emptyset$.)

b) Here the maximal antichains are $\{a\}$, $\{b, c\}$, $\{c, e\}$, and $\{d, e\}$.

c) In this case there are only three maximal antichains: $\{a, b, c\}$, $\{d, e, f\}$, and $\{g\}$.

31. Let C be a maximal chain. We must show that C contains a minimal element of S. Since C can itself be viewed as a finite poset (being a subset of a poset), it contains a minimal element m. We need to show that m is also a minimal element of S. If it were not, then there would be another element $a \in S$ such that $a \prec m$. Now we claim that $C \cup \{a\}$ is a chain, which will contradict the maximality of C. We need to show that a is comparable to every element of C. We already know that a is comparable to m. Let x be any other element of C. Since m is minimal in C, it cannot be that $x \prec m$; thus since x and m have to be comparable (they are both in C), it must be that $m \prec x$. Now by transitivity we have $a \prec x$, and we are done.

33. Consider the relation R on the set of $mn + 1$ people given by $(a, b) \in R$ if and only if a is a descendant of or equal to b. This makes the collection into a poset. In the terminology of Exercise 32, if there is not a subset of $n+1$ people none of whom is a descendant of any other, then $k \leq n$, since such a subset is certainly an antichain. Therefore the poset can be partitioned into $k \leq n$ chains. Now by the generalized pigeonhole principle, at least one of these chains must contain at least $m + 1$ elements, and this is the desired list of descendants.

35. Recall the definition of well-founded from the preamble to Exercise 55 in Section 9.6—that there is no infinite decreasing sequence. We must show that under this hypothesis, and if $\forall x((\forall y(y \prec x \rightarrow P(y))) \rightarrow P(x))$, then $P(x)$ is true for all $x \in S$. We give a proof by contradiction. If it does not hold that $P(x)$ is true for all $x \in S$, let x_1 be an element of S such that $P(x_1)$ is not true. Then by the conditional statement given above, it must be the case that $\forall y(y \prec x_1 \rightarrow P(y))$ is not true. This means that there is some y with $y \prec x_1$ such that $P(y)$ is not true. Rename this y as x_2. So we know that $P(x_2)$ is not true. Again invoking the conditional statement, we get an $x_3 \prec x_2$ such that $P(x_3)$ is not true. And so on forever. This contradicts the well-foundedness of our poset. Therefore $P(x)$ is true for all $x \in S$.

37. We assume that R is reflexive and transitive on A, and we must show that $R \cap R^{-1}$ is reflexive, symmetric, and transitive. Reflexivity is easy: if $a \in A$, then we know that $(a, a) \in R$, so by the definition of R^{-1} as the reverses of the pairs in R, we know that $(a, a) \in R^{-1}$ as well, whence it follows that $(a, a) \in R \cap R^{-1}$. *Every* relation of the form $R \cap R^{-1}$ is symmetric, no matter what R is, since if $(a, b) \in R$, then $(b, a) \in R^{-1}$ and vice versa. For transitivity, suppose that $(a, b) \in R \cap R^{-1}$ and $(b, c) \in R \cap R^{-1}$. We must show that $(a, c) \in R \cap R^{-1}$. Since $(a, b) \in R$ and $(b, c) \in R$, and since R is transitive, $(a, c) \in R$. Similarly, since $(a, b) \in R^{-1}$ and $(b, c) \in R^{-1}$, $(b, a) \in R$ and $(c, b) \in R$. Again, since R is transitive, $(c, a) \in R$, and hence $(a, c) \in R^{-1}$. Putting these two parts together, we conclude that $(a, c) \in R \cap R^{-1}$, as desired.

39. There is not much to show in this exercise, since the definitions of greatest lower bound and least upper bound exhibit these properties by their very form.

a) The g.l.b. of x and y was defined to be the greatest element that is a lower bound of both x and y. The roles of x and y in this statement are symmetric, so it follows immediately that $x \wedge y = y \wedge x$. Similarly for least upper bound.

b) By definition, $(x \wedge y) \wedge z$ is a lower bound of x, y, and z that is greater than every other common lower bound (this is how we proceeded in Exercise 45 of Section 9.6). Since x, y, and z play interchangeable roles in this statement, grouping does not matter, so $x \wedge (y \wedge z)$ is the same element. Similarly for l.u.b.

c) The two statements are duals, so we will prove just the first one; the proof of the second can be obtained formally simply by exchanging each symbol and word for its dual. To show that $x \wedge (x \vee y) = x$, we must show that x is the greatest lower bound of x and $x \vee y$. Clearly x is a lower bound for x, and since $x \vee y$ is by definition greater than or equal to x, x is a lower bound for it as well. Therefore x is a lower bound. But every other lower bound for x has to be less than x, so x is the greatest lower bound.

d) Obviously x is a lower (upper) bound for itself and itself, and the greatest (least) such.

41. There is nothing very deep going on here—it's just a matter of applying the definitions.

a) Since 1 is the only element greater than or equal to 1, it is the only upper bound for 1 and therefore the only possible value of the least upper bound of x and 1.

b) Clearly x is a lower bound for both x and 1 (since $x \preceq 1$), and clearly no other lower bound can be greater than x, so $x \wedge 1 = x$.

c) This is the dual to part **(b)**. We formed the following proof on the word processor used to produce this solutions manual by copying the words in the solution to part **(b)** and replacing each word and symbol by its dual: Clearly x is an upper bound for both x and 0 (since $0 \preceq x$), and clearly no other upper bound can be smaller than x, so $x \vee 0 = x$.

d) This is the dual of part **(a)**: Since 0 is the only element less than or equal to 0, it is the only lower bound for 0 and therefore the only possible value of the greatest lower bound of x and 0.

43. One way to solve this problem is to play around with some small examples. Here is one counter-example that the author obtained in this way. The lattice has as its elements $\emptyset$, $\{1\}$, $\{2\}$, $\{3\}$, $\{1,2\}$, $\{2,3\}$, and $\{1,2,3\}$, with, as usual, the relation $\subseteq$. (Draw its Hasse diagram!) It is easy to check that every two elements have both a least upper bound and a greatest lower bound (note that $\emptyset$ is a lower bound for the whole lattice, and $\{1,2,3\}$ is an upper bound for the whole lattice). Take $x = \{1\}$, $y = \{2\}$, and $z = \{3\}$, and compute both sides of the equation $x \vee (y \wedge z) = (x \vee y) \wedge (x \vee z)$. Note that since we do not have the full subset lattice, least upper bounds are not just unions. The left-hand side is x, since $y \wedge z = \emptyset$. The right-hand side is the greatest lower bound of $\{1,2\}$ and $\{1,2,3\}$, which is $\{1,2\}$. Since these are different, the lattice is not distributive.

45. Yes. First, recall from Example 22 in Section 9.6 that $x \wedge y$ is the greatest common divisor (gcd) of x and y, while $x \vee y$ is their least common multiple (lcm). We can analyze this problem by looking at prime factorizations. The power to which a prime p appears in the gcd of two numbers is the minimum of the powers to which it appears in the two numbers. Similarly, the power to which p appears in the lcm is the maximum of the powers to which it appears in the two numbers. Thus if we let a, b, and c represent the powers to which p appears in x, y, and z, respectively, the first identity we need to prove is

$$\max(a, \min(b, c)) = \min(\max(a, b), \max(a, c)).$$

We consider the several cases. If a is the largest of the three numbers, then both sides equal a. If a is the smallest, then both sides equal the smaller of b and c. Otherwise, we can suppose without loss of generality (since the roles of b and c are symmetric) that $b \le a \le c$, in which case we easily compute that both sides equal a. The proof of the other statement is dual to this proof. The result now follows from the fundamental theorem of arithmetic, since numbers are determined by their prime factorizations.

47. As might be expected from the name, the complement of a subset $X \subseteq S$ is its complement $S - X$. To prove this, we need to prove that $X \vee (S - X) = 1$ and $X \wedge (S - X) = 0$, which translated into our particular setting reads: $X \cup (S - X) = S$ and $X \cap (S - X) = \emptyset$. But these are trivially true.

49. Think of the rectangular grid as representing elements in a matrix. Thus we number from top to bottom and within that from left to right. For example, $(2,4)$ is the element in row 2, column 4. The partial order is that $(a,b) \preceq (c,d)$ if $a \le c$ and $b \le d$. Note that $(1,1)$ is the least element under this relation. The rules for Chomp as explained in Chapter 1 coincide with the rules stated in the preamble here. But now we can identify the point (a, b) with the natural number $p^{a-1}q^{b-1}$ for all a and b with $1 \le a \le m$ and $1 \le b \le n$. This identifies the points in the rectangular grid with the set S in this exercise, and the partial order $\preceq$ just described is the same as the divides relation, because $p^{a-1}q^{b-1} \mid p^{c-1}q^{d-1}$ if and only if the exponent on p on the left does not exceed the exponent of p on the right, and similarly for q.

WRITING PROJECTS FOR CHAPTER 9

Books and articles indicated by bracketed symbols below are listed near the end of this manual. You should also read the general comments and advice you will find there about researching and writing these essays.

1. See the same references as suggested for fuzzy logic in Writing Project 2 of Chapter 1, as well as [Zi].

2. There are numerous textbooks on databases. Try to consult one that is fairly recent, because in many areas of computer science, progress is so fast that books soon become out-dated. You will find them in the QA 76.9 area of the library's shelves. Two recommended ones are are [Da1] and [Ma1].

3. Try author or key-word search in an appropriate database (e.g., one provided by *Mathematical Reviews*, which is available on the Web as MathSciNet). Consult the oldest reference you can find that talks about these topics, and it will probably lead you to the original sources. Simultaneous discovery occurs in many branches of intellectual pursuit, not just mathematics and computer science. See Writing Project 17 in Chapter 11 for something along the same lines.

4. The abstraction and difficulty here is part of what makes fractions hard for many children (and some adults) to handle. Be careful to avoid 0 in the denominator! You should be able to figure this out without consulting other sources, and it is a good project to work on with other people.

5. See the hints for Writing Project 3.

6. Entire books have been written on security issues in computer systems ([Pf], for one), and it should not be hard to find a chapter or two on the subject in many more general books (try [De1]).

7. Textbooks on project scheduling should be a good source of information. See [Mo], for example. Scheduling is a topic in a branch of mathematics known as Operations Research. It has its own journals, conferences, subspecialties, software, etc.

8. See the suggestions for Writing Project 7.

9. We have hinted at duality in many of the exercise solutions in this *Guide*. A book on lattice theory (such as [Gr1]) will make the concept more precise.

10. As mentioned in the previous suggestion, you can find entire books on lattice theory. In fact, *Mathematical Reviews* (which is available on the Web as MathSciNet) devotes a whole category (numbered 06) to lattices and other kinds of ordered sets and ordered algebraic structures.