Probability of Unions and Intersections
Chapter 2

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Table of Contents

1 Probability of Unions

2 Independent Events

3 Probability of Intersections

4 Application: System Reliability

Probability of a Union

- In general,

$$P(A \cup B \cup C \cup \ldots) = P(A) + P(B) + P(C) + \ldots$$

  is true when the events $A, B, C, \ldots$ have empty intersections.
- If events intersect, their probabilities cannot simply be added.

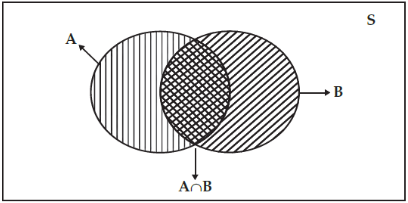


Figure: Union of A and B

- Notice: the intersection is shaded (counted) twice!

Probability of a Union

Probability of the Union of Two Events

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

- $P(A \cup B) = P(A \text{ or } B)$
- This formula is sometimes called the "Or" or "Addition" Rule.
- $P(A \cap B) = 0$ if A and B are mutually exclusive.

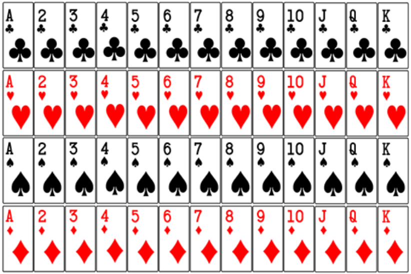
Example 1

During construction, a network blackout occurs on Monday with probability 0.7, on Tuesday with probability 0.5, both Monday and Tuesday with probability 0.45. Find the probability that a blackout occurs on Monday or Tuesday. Solution:

1. Let $A =$ Monday and $B =$ Tuesday
2. Use the "Or" Rule:

$$\begin{aligned} P(A \text{ or } B) &= P(A \cup B) \\ &= P(A) + P(B) - P(A \cap B) \\ &= 0.7 + 0.5 - 0.45 \\ &= \boxed{0.75} \end{aligned}$$

Example 2: Cards



- 52 cards
- 2 colors
- 4 suits (clubs, hearts, spades, diamonds)
- 13 ranks per suit

You draw one card at random. Calculate the probabilities.

1. What is the probability the card is a King (K) or a Queen (Q)?
2. What is the probability that you draw either an Ace (A) or a heart card (H)?

See handwritten solutions.

Example 3: The Titanic

The table below gives the class and survival status of each passenger of the doomed Titanic. Calculate the probabilities.

	1st Class	2nd Class	3rd Class	Total
Survived	203	118	178	499
Died	122	167	528	817
Total	325	285	706	1316

1. Find the probability that a passenger survived or was in 2nd class.
2. Find the probability that a passenger was in first class or second class.

See handwritten solutions.

Table of Contents

1 Probability of Unions

2 Independent Events

3 Probability of Intersections

4 Application: System Reliability

Definition

Independent Events

Events $E_1, E_2, \ldots, E_n$ are independent if they occur independently where the occurrence of one event does not affect the probability of another event.

- Ex: Flip a coin two times. Are the tosses independent of one another?

$$\text{Toss 1: } P(heads) = 0.5$$
$$\text{Toss 2: } P(heads) = 0.5$$

The two tosses are independent because the first toss did not change the probability of the second toss.

Example 4

Suppose we have a bin with 4 red chips and 6 white chips.

1. Let $A =$ "choosing a red chip" $\implies P(A) = 4/10$.
2. Let $B =$ "choosing a white chip" $\implies P(B) = 6/10$.
3. With Replacement: You randomly pick one chip and then you *put it back* in the bin. It is red. What is the probability of white on your second pick? $P(B) = 6/10$.
   - A and B are independent events since the probability for B did not change.
4. Without Replacement: You randomly pick one chip and *do not put it back* in the bin. It is red. What is the probability of white on your second pick? $P(B) = 6/9$.
   - A and B are dependent events since the probability for B changed from $6/10$ to $6/9$.

Table of Contents

1 Probability of Unions

2 Independent Events

3 Probability of Intersections

4 Application: System Reliability

Definition

Probability of the Intersection of Independent Events

If events $E_1, E_2, \ldots, E_n$ are independent events, then
$P(E_1 \cap E_2 \cap \ldots \cap E_n) = P(E_1) \cdot P(E_2) \ldots P(E_n)$

- $A \cap B =$ "A and B"
- $P(A \cap B) = P(A) \cdot P(B)$ if A and B are independent.
- This formula is sometimes called the "And Rule" or the "Multiplication Rule"
- "A and B are independent" is denoted as $A \perp B$.
- The intersection is where A and B are joined. So, $P(A \cap B)$ is commonly called the joint probability of A and B.

Example 5

During construction, a network blackout occurs on Monday with probability 0.7, on Tuesday with probability 0.5, both Monday and Tuesday with probability 0.45. Are blackouts on Monday and Tuesday independent events?

Solution: Check to see if $P(A \cap B) = P(A) \cdot P(B)$:

$$\begin{aligned} P(A \cap B) &= P(A) \cdot P(B) \\ 0.45 &= (0.7)(0.5) \\ 0.45 &\neq 0.35 \end{aligned}$$

$\therefore$ A and B are not independent.

Example 6

Suppose we have a bin with 4 red chips and 6 white chips. You select three chips with replacement. Let $A =$ red and $B =$ white.

1. What is the probability that all three chips are red?
2. What is the probability that at least one chip is white?

See handwritten solutions.

Example 7

There is a 1% probability for a hard drive to crash. Therefore, it has two backups each having 2% probability to crash. All three components are independent of each other. Stored information is lost only when all three crash.

1. What is the probability that all three devices crash?
2. What is the probability that the information saved?

See handwritten solutions.

Table of Contents

1 Probability of Unions

2 Independent Events

3 Probability of Intersections

4 Application: System Reliability

System Reliability I

- Computing devices are connected either in sequel or in parallel. The devices make up one whole system.

  1. Sequel: failure of one component causes failure of the whole system.
     - All components must work.

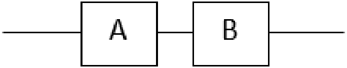


Figure: A and B are connected in sequel.

System Reliability II

2. Parallel: System does not fail if at least one component works.

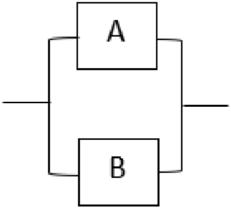


Figure: A and B are connected in parallel.

System Reliability

Definition: Reliability

Reliability is the probability that the system of independent components does not fail.

1. Sequel: Compute the probability that all components work.

$$P(\text{All work}) = P(A \cap B \cap C \cap \ldots)$$

2. Parallel: Compute the probability that at least one component works.

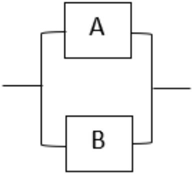
$$P(\text{At least one works}) = 1 - P(\text{none work}) = 1 - P(A^C \cap B^C \cap \ldots)$$

Example 8

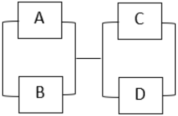
Example: Suppose each component is operable with probability 0.92. Find the reliability of each system. See handwritten solutions.

(a)

(b)



(c)



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