MATH 2418 Linear Algebra. Week 6

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Summary of this Week's Goals

This week we will cover Section 3.1 (Spaces of Vectors). We will define what is meant by a "vector space." A vector space is a closed set of mathematical objects for which vector addition and scalar multiplication are defined. The definitions of vector addition and scalar multiplication must satisfy eight fundamental axioms. We will present the axioms and consider examples of vector spaces. One particular vector space which we will study is the column space of a matrix—that is, the set of all possible linear combinations of the columns of a given $m \times n$ matrix.

Announcements

- Your first midterm exam will be next week on Thursday.

MATH 2418 – MIDTERM 1 – INFORMATION

1. Thursday, OCT/06, 7:30 – 8:45 pm
2. Room assignment:

	Room	Sections
1.	**ECSS 2.412**	301, 302, 303
2.	**GR 4.428**	304, 305, 306
3.	**SCI 1.210**	307, 308, 309
4.	**JSOM 1.212**	311, 314, 315
5.	**SCI 1.220**	316, 317, 318, 320

3. Sections covered: 1.1, 1.2, 1.3, 2.1, 2.2, 2.3, 2.4, 2.5, 2.6
4. Only basic calculators can be used (no calculators with matrix functions or access to the internet are allowed)
5. Scratch paper will be provided. Work on scratch paper will NOT be graded
6. How to practice for the exam: do problems in HW1-5 and recommended problems

Figure 1: Midterm 1 Room Assignments

- Peer Tutoring will be doing a review for the upcoming exam in MATH 2418 this Friday, September 30, from 4:30pm-6:30pm. The review will be online, and the link to join is posted on their webpage in the "Exam Reviews" section: https://studentsuccess.utdallas.edu/programs/peer-tutoring/. We will be reviewing concepts and working through some example problems. The review will be recorded and uploaded to our Youtube channel, and we will share the link with everyone afterwards as well.

3.1 Spaces of Vectors

Definition of a Vector Space

- A vector space is a closed set of mathematical objects for which vector addition and scalar multiplication are defined. Here are the keys parts of a vector space:

- Vectors. These are the mathematical objects that make up the set to which the vector space refers. They may be vectors like the ones we have discussed already (n-tuples of real numbers interpreted as arrows with magnitude and direction), or they may be more other mathematical objects like matrices and functions.
- Vector Addition. A vector space requires that addition be defined so that any vector in the vector space may be added to any other vector in the vector space.
- Scalar Multiplication. A vector space requires that scalar multiplication be defined so that any vector in the vector space may be multiplied by a scalar. For our purposes, scalars are usually real numbers, but they could be other sets of numbers such as rational numbers or complex numbers.
- Closure. A vector space must be closed with respect to vector addition and scalar multiplication. This means that the sum of any two vectors in the vector space must also be a vector in the space and any scalar multiple of a vector in the vector space must also be a vector in the space. This implies that every possible linear combination of vectors in the space must also be an element of the space.

- For a vector space S, vector addition and scalar multiplication must satisfy the following axioms, where $\mathbf{x}$, $\mathbf{y}$ and $\mathbf{z}$ are vectors in S and c and d are scalars:

  1. Commutivity. $\mathbf{x} + \mathbf{y} = \mathbf{y} + \mathbf{x}$
  2. Associativity of Vector Addition. $\mathbf{x} + (\mathbf{y} + \mathbf{z}) = (\mathbf{x} + \mathbf{y}) + \mathbf{z}$
  3. Additive Identity. There must exist a unique vector $\mathbf{0}$ in S such that $\mathbf{x} + \mathbf{0} = \mathbf{x}$ for all $\mathbf{x}$ in S.
  4. Additive Inverse. For each $\mathbf{x}$ in S, there must exist a unique vector $-\mathbf{x}$ in S such that $\mathbf{x} + (-\mathbf{x}) = \mathbf{0}$.
  5. Scalar Multiplication Identify. $1\mathbf{x} = \mathbf{x}$ for all $\mathbf{x}$ in S. To be clear, "1" is a scalar in this expression.
  6. Associativity of Scalar Multiplication. $(cd)\mathbf{x} = c(d\mathbf{x})$
  7. Distributivity of Vector Sums. $c(\mathbf{x} + \mathbf{y}) = c\mathbf{x} + c\mathbf{y}$
  8. Distributivity of Scalar Sums. $(c + d)\mathbf{x} = c\mathbf{x} + d\mathbf{x}$

Examples of Vector Spaces

- Single-point Space. The smallest possible vector space is one which contains only a zero vector. $S = \{\mathbf{0}\}$.

- The Real Numbers $\mathbb{R}$. The set of real numbers itself satisfies the axioms of a vector space.

- Euclidean Coordinate Spaces ($\mathbb{R}^n$). The set of column vectors with n real-valued components. These are the vectors we have been studying so far this semester. We tend to visualize them as directed line segments (arrows) in the context of Euclidean Geometry, where lines and planes are infinite, straight and flat.

- Matrices ($\mathbb{R}^{m \times n}$). The set of all real-valued $m \times n$ matrices is a vector space. Can you guess what the zero vector is in the vector space of 3×3 real-valued matrices?

$$\mathbf{0} = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

- Polynomials. The set of all polynomials with degree n or less: $P(x) = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n$

  - Vectors. In this set, polynomials are the vectors. For example, $\mathbf{u} = P(x)$, $\mathbf{v} = Q(x)$. $Q(x)$ is another polynomial of degree n or less, with perhaps a difference set of coefficients.
  - Vector Addition. $\mathbf{u} + \mathbf{v} = P(x) + Q(x)$.
  - Scalar Multiplication. $c\mathbf{u} = cP(x)$.

    - All eight axioms of a vector addition and scalar multiplication hold true.
    - Closure. Linear combinations of polynomials in this set result in another polynomials which are also contained in the set.
- Function Spaces. The set of all real-valued functions $f(x)$.

Subspaces

- Some vector spaces are subsets of larger vector spaces.
- Subspaces inherit their definitions of vector addition and scalar multiplication from the vector spaces they are contained in. The critical point to showing that a subset is actually a vector space itself is to demonstrate closure. That is, in order for a subset of a vector space to be a subspace, all linear combinations of vectors in that subset must also lie in the subset.
- If a subspace contains the vectors $\mathbf{u}$ and $\mathbf{v}$, it must also contain $\mathbf{u}+\mathbf{v}$, $c\mathbf{u}$ (for any scalar c) and all linear combinations $c\mathbf{u}+d\mathbf{v}$ (for any scalars c and d).
- Every vector space contains a subspace equivalent to the single-point vector space $(\{\mathbf{0}\})$.
- In $\mathbb{R}^2$, each line passing through the origin is a one-dimensional subspace of $\mathbb{R}^2$.
    - The x-axis in $\mathbb{R}^2$ is a vector space within $\mathbb{R}^2$.
    - The y-axis in $\mathbb{R}^2$ is a vector space within $\mathbb{R}^2$.
    - The set of all scalar multiples of $\mathbf{u}=(-1,3)$ is a vector space within $\mathbb{R}^2$.
- In $\mathbb{R}^3$, each line passing through the origin is a one-dimensional subspace of $\mathbb{R}^3$ and each plane passing through the origin is a two-dimensional subspace of $\mathbb{R}^3$.
    - The x-axis in $\mathbb{R}^3$ is a vector space within $\mathbb{R}^3$.
    - The yz-plane in $\mathbb{R}^3$ is a vector space within $\mathbb{R}^3$.
    - The set of all scalar multiples of $\mathbf{u}=(-2,1,0)$ is a vector space within $\mathbb{R}^3$.
    - The set of all linear combinations of $\mathbf{u}=(-2,1,0)$ and $\mathbf{v}=(1,0,4)$ is a vector space within $\mathbb{R}^3$.
- The set of polynomials having degree 2 or less is a subspace of the set of polynomials having degree 3 or less.
- The set of functions $y=f(x)$ which are solutions to the differential equation $\alpha\frac{\partial^2 y}{\partial x^2}+\beta\frac{\partial y}{\partial x}+\gamma y=0$ are a subspace of the set of all real functions.

Examples of Sets Which Are Not Vector Spaces

- The set of all unit vectors in $\mathbb{R}^2$ is not a vector space.
    - The set does not contain a zero vector.
    - The set is not closed with respect to vector addition. $(1,0)+(0,1)=(1,1)$ is not contained in the set.
    - The set is not closed with respect to scalar multiplication. $2(1,0)=(2,0)$ is not contained in the set.
- The set of all vectors above the xy-plane in $\mathbb{R}^3$ (all points of the form (x,y,z) where $z>0$) is not a vector space.
    - The set is closed with respect to vector addition, but...
    - The set does not contain a zero vector.

- The set is not closed with respect to scalar multiplication. $-1(1,1,1) = (-1,-1,-1)$ is not contained in the set.

- The set of polynomials having only real roots is not a vector space.
  - The set contains the zero vector, and...
  - The set is closed with respect to scalar multiplication, but...
  - The set is not closed with respect to vector addition. $P(x) = x^2 - x$ (roots are 0 and 1) and $Q(x) = x + 1$ (root is -1) are in the set but $P(x) + Q(x) = x^2 + 1$ is not (roots are i and $-i$).

The Column Space of a Matrix

- The column space of an $m \times n$ matrix A, denoted $C(A)$, is the set of all possible linear combinations of the columns of A. This is the set of all vectors $\mathbf{y}$ which can be written as the product $\mathbf{y} = A\mathbf{x}$ for some choice of $\mathbf{x}$.
- The column space of A contains vectors in $\mathbb{R}^m$. It is a set of "all possible results" from the operation $A\mathbf{x}$.
- The column space of A is a vector space (a subspace of $\mathbb{R}^m$). It is closed with respect to vector addition and scalar multiplication.
  - If $\mathbf{y_1}$ and $\mathbf{y_2}$ are in the column space of A, then there exist vectors $\mathbf{x_1}$ and $\mathbf{x_2}$ in $\mathbb{R}^n$ such that $A\mathbf{x_1} = \mathbf{y_1}$ and $A\mathbf{x_2} = \mathbf{y_2}$. Also $A(\mathbf{x_1} + \mathbf{x_2}) = A\mathbf{x_1} + A\mathbf{x_2} = \mathbf{y_1} + \mathbf{y_2}$, so $\mathbf{y_1} + \mathbf{y_2}$ is in $C(A)$.
  - If $\mathbf{y_1}$ is in the column space of A, then there exists $\mathbf{x_1}$ in $\mathbb{R}^n$ such that $A\mathbf{x_1} = \mathbf{y_1}$. Then $A(c\mathbf{x_1}) = cAc\mathbf{x_1} = c\mathbf{y_1}$, so $c\mathbf{y_1}$ is in $C(A)$.
- The column space of A might be $\mathbb{R}^m$, or it might be a subspace of $\mathbb{R}^m$. If $n < m$, it is certainly a subspace, not all of $\mathbb{R}^m$. If $n \geq m$, it might be a subspace or all of $\mathbb{R}^m$ depending on how many columns are linearly independent.
- The system $A\mathbf{x} = \mathbf{b}$ has a solution only if $\mathbf{b}$ is in the column space of A.
- Examples

$$A = \begin{bmatrix} 1 & -1 \\ 2 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & 0 \\ 2 & -1 \\ 3 & -1 \end{bmatrix}, I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, D = \begin{bmatrix} 1 & 0 & 1 & -1 \\ 0 & 1 & 1 & 1 \end{bmatrix}, E = \begin{bmatrix} 1 & 2 & 3 \\ -1 & -2 & -3 \\ 2 & 4 & 6 \end{bmatrix}$$

  - $C(A)$ is all of $\mathbb{R}^2$.
  - $C(B)$ is a plane in $\mathbb{R}^3$.
  - $C(I)$ is all of $\mathbb{R}^3$.
  - $C(D)$ is all of $\mathbb{R}^2$.
  - $C(E)$ is a line in $\mathbb{R}^3$.

The Nullspace of a Matrix

- The nullspace of an $m \times n$ matrix A, denoted $N(A)$, is the set of all vectors $\mathbf{x}$ which are solutions to the equation $A\mathbf{x} = \mathbf{0}$.
- The nullspace of A contains vectors in $\mathbb{R}^n$. It is a set of "all possible inputs" into the operation $A\mathbf{x}$ which result in the $\mathbb{R}^m$ zero vector output.
- The nullspace of A is a vector space (a subspace of $\mathbb{R}^n$). It is closed with respect to vector addition and scalar multiplication.

  - If $\mathbf{x_1}$ and $\mathbf{x_2}$ are in the nullspace of A, then $A\mathbf{x_1} = \mathbf{0}$ and $A\mathbf{x_2} = \mathbf{0}$. Also $A(\mathbf{x_1} + \mathbf{x_2}) = A\mathbf{x_1} + A\mathbf{x_2} = 0 + 0 = 0$, so $\mathbf{x_1} + \mathbf{x_2}$ is in $N(A)$.
  - If $\mathbf{x_1}$ is in the nullspace of A, then $A(c\mathbf{x_1}) = cA\mathbf{x_1} = c\mathbf{0} = \mathbf{0}$, so $c\mathbf{x_1}$ is in $N(A)$.

- We will discuss how to find and describe the nullspace of A in the next section.