CHAPTER 12
Boolean Algebra

SECTION 12.1 Boolean Functions

The first 28 of these exercises are extremely straightforward and should pose no difficulty. The next four relate to the interconnection between duality and De Morgan's laws; they are a bit subtle. Boolean functions can be proved equal by tables of values (as we illustrate in Exercise 13), by using known identities (as we illustrate in Exercise 11), or by taking duals of equal Boolean functions (Exercise 28 justifies this). To show that two Boolean functions are not equal, we need to find a counterexample, i.e., values of the variables that give the two functions different values. Exercises 35–43 deal with abstract Boolean algebras, and the proofs are of the formal, "symbol-pushing" variety.

1. **a)** $1 \cdot \overline{0} = 1 \cdot 1 = 1$ **b)** $1 + \overline{1} = 1 + 0 = 1$ **c)** $\overline{0} \cdot 0 = 1 \cdot 0 = 0$ **d)** $\overline{(1+0)} = \overline{1} = 0$

3. **a)** We compute $(1 \cdot 1) + (\overline{0 \cdot 1} + 0) = 1 + (\overline{0} + 0) = 1 + (1 + 0) = 1 + 1 = 1$.
b) Following the instructions, we have $(\mathbf{T} \wedge \mathbf{T}) \vee (\neg(\mathbf{F} \wedge \mathbf{T}) \vee \mathbf{F}) \equiv \mathbf{T}$.

5. In each case, we compute the various components of the final expression and put them together as indicated. For part **(a)** we have

x	y	z	$\overline{x}$	$\overline{x}y$
1	1	1	0	0
1	1	0	0	0
1	0	1	0	0
1	0	0	0	0
0	1	1	1	1
0	1	0	1	1
0	0	1	1	0
0	0	0	1	0

For part **(b)** we have

x	y	z	yz	$x + yz$
1	1	1	1	1
1	1	0	0	1
1	0	1	0	1
1	0	0	0	1
0	1	1	1	1
0	1	0	0	0
0	0	1	0	0
0	0	0	0	0

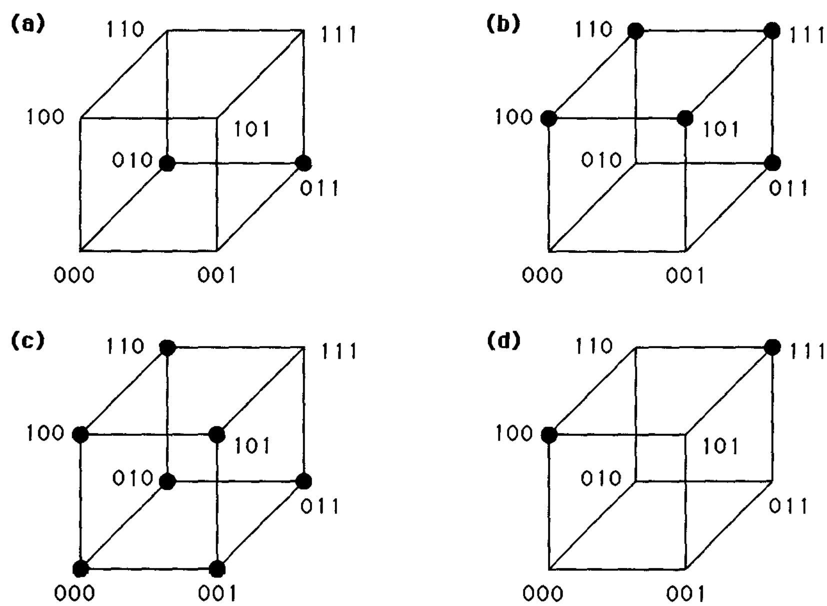
For part **(c)** we have

x	y	z	$\overline{y}$	$x\overline{y}$	xyz	$\overline{xyz}$	$x\overline{y}+\overline{xyz}$
1	1	1	0	0	1	0	0
1	1	0	0	0	0	1	1
1	0	1	1	1	0	1	1
1	0	0	1	1	0	1	1
0	1	1	0	0	0	1	1
0	1	0	0	0	0	1	1
0	0	1	1	0	0	1	1
0	0	0	1	0	0	1	1

For part **(d)** we have

x	y	z	$\overline{y}$	$\overline{z}$	yz	$\overline{y}\,\overline{z}$	$yz+\overline{y}\,\overline{z}$	$x(yz+\overline{y}\,\overline{z})$
1	1	1	0	0	1	0	1	1
1	1	0	0	1	0	0	0	0
1	0	1	1	0	0	0	0	0
1	0	0	1	1	0	1	1	1
0	1	1	0	0	1	0	1	0
0	1	0	0	1	0	0	0	0
0	0	1	1	0	0	0	0	0
0	0	0	1	1	0	1	1	0

7. In each case, we note from our solution to Exercise 5 which vertices need to be blackened in the cube, as in Figure 1.



9. By looking at the definitions, we see that this equation is satisfied if and only if $x = y$, i.e., $x = y = 0$ or $x = y = 1$.

11. First we "factor" out an x by using the identity and distributive laws: $x + xy = x \cdot 1 + x \cdot y = x \cdot (1 + y)$. Then we use the commutative law, the domination law, and finally the identity law again to write this as $x \cdot (y + 1) = x \cdot 1 = x$.

13. Probably the simplest way to do this is by use of a table, as in Example 8. We list all the possible values for the triple (x, y, z) (there being eight such), and for each compute both sides of this equation. For example, for $x = y = z = 0$ we have $x\overline{y} + y\overline{z} + \overline{x}z = 0 \cdot 1 + 0 \cdot 1 + 1 \cdot 0 = 0 + 0 + 0 = 0$, and similarly $\overline{x}y + \overline{y}z + x\overline{z} = 0$. We do this for all eight lines of the table to conclude that the two functions are equal.

x	y	z	$x\overline{y}+y\overline{z}+\overline{x}z$	$\overline{x}y+\overline{y}z+x\overline{z}$
1	1	1	0	0
1	1	0	1	1
1	0	1	1	1
1	0	0	1	1
0	1	1	1	1
0	1	0	1	1
0	0	1	1	1
0	0	0	0	0

15. The idempotent laws state that $x \cdot x = x$ and $x + x = x$. There are only four things to check: $0 \cdot 0 = 0$, $0 + 0 = 0$, $1 \cdot 1 = 1$, and $1 + 1 = 1$, all of which are part of the definitions. The relevant tables, exhibiting these calculations, have only two rows.

17. The domination laws state that $x + 1 = 1$ and $x \cdot 0 = 0$. There are only four things to check: $0 + 1 = 1$, $0 \cdot 0 = 0$, $1 + 1 = 1$, and $1 \cdot 0 = 0$, all of which are part of the definitions. The relevant tables, exhibiting these calculations, have only two rows.

19. We can verify each associative law by constructing the relevant table, which will have eight rows, since there are eight combinations of values for x, y, and z in the equations $x+(y+z) = (x+y)+z$ and $x(yz) = (xy)z$. Rather than write down these tables, let us observe that in the first case, both sides are equal to 1 unless $x = y = z = 0$ (in which case both sides equal 0), and, dually, in the second case, both sides are equal to 0 unless $x = y = z = 1$ (in which case both sides equal 1).

21. We construct the relevant tables (as in Exercise 13) and compute the quantities shown. Since the fourth and seventh columns are equal, we conclude that $\overline{(xy)} = \overline{x} + \overline{y}$; since the ninth and tenth columns are equal, we conclude that $\overline{(x+y)} = \overline{x}\,\overline{y}$.

x	y	xy	$\overline{(xy)}$	$\overline{x}$	$\overline{y}$	$\overline{x}+\overline{y}$	$x+y$	$\overline{(x+y)}$	$\overline{x}\,\overline{y}$
1	1	1	0	0	0	0	1	0	0
1	0	0	1	0	1	1	1	0	0
0	1	0	1	1	0	1	1	0	0
0	0	0	1	1	1	1	0	1	1

23. The zero property states that $x \cdot \overline{x} = 0$. There are only two things to check: $0 \cdot \overline{0} = 0 \cdot 1 = 0$ and $1 \cdot \overline{1} = 1 \cdot 0 = 0$. The relevant table, exhibiting this calculation, has only two rows.

25. We could prove these by constructing tables, as in Exercise 21. Instead we will argue directly.

a) The left-hand side is equal to 1 if $x \neq y$. In this case the right-hand side is necessarily $1 \cdot \overline{0} = 1$, as well. On the other hand if $x = y = 1$, then the left-hand side is by definition equal to 0, and the right-hand side equals $1 \cdot 0 = 0$; similarly if $x = y = 0$, then the left-hand side is by definition equal to 0, and the right-hand side equals $0 \cdot 1 = 0$.

b) The left-hand side is equal to 1 if $x \neq y$. In this case the right-hand side is necessarily $1 + 0 = 1$ or $0 + 1 = 1$, as well. On the other hand if $x = y$, then the left-hand side is by definition equal to 0, and the right-hand side equals $0 + 0 = 0$.

27. **a)** We can prove this by constructing the appropriate table, as in Exercise 21. What we will find is that each side equals 1 if and only if an odd number of the variables are equal to 1. Thus the two functions are equal.

b) This is not an identity. If we let $x = y = z = 1$, then the left-hand side is $1 + 0 = 1$, while the right-hand side is $1 \oplus 1 = 0$.

c) This is not an identity. If we let $x = y = 1$ and $z = 0$, then the left-hand side is $1 \oplus 1 = 0$, while the right-hand side is $0 + 1 = 1$.

29. Let B be a Boolean expression representing F, and let D be the dual of B. We want to show that for every set of values assigned to the variables $x_1, x_2, \ldots, x_n$, the value D equals the opposite of the value of B with the opposites of these values assigned to $x_1, x_2, \ldots, x_n$. The trick is to look at $\overline{B}$. Then by repeatedly applying De Morgan's laws from the outside in, we see that $\overline{B}$ is the same as the expression obtained by replacing each occurrence of x_i in D by $\overline{x}_i$. Thus for any values of $x_1, x_2, \ldots, x_n$, the value of D is the same as the value of $\overline{B}$ for the corresponding value of $\overline{x}_1, \overline{x}_2, \ldots, \overline{x}_n$. This tells us that $\overline{B}$ represents the function whose values are exactly those of the function represented by D when the opposites of each of the values of the variables x_i are used, and that is exactly what we wanted to prove.

31. Because of the stated condition, we are free to specify $F(1, y, z)$ for all pairs (y, z), but then all the values of $F(0, y, z)$ are thereby determined. There are 4 such pairs (y, z) (each one can be either 0 or 1), and for each such pair we have 2 choices as to the value of $F(1, y, z)$. Therefore the answer is $2^4 = 16$.

33. We need to replace each 0 by **F**, 1 by **T**, $+$ by $\vee$, $\cdot$ (or Boolean product implied by juxtaposition) by $\wedge$, and $\bar{}$ by $\neg$. We also replace x by p and y by q so that the variables look like they represent propositions, and we replace the equals sign by the logical equivalence symbol. Thus for the first De Morgan law in Table 5, $\overline{xy} = \overline{x} + \overline{y}$ becomes $\neg(p \wedge q) \equiv \neg p \vee \neg q$, which is the first De Morgan law in Table 6 of Section 1.3. Dually, $\overline{x+y} = \overline{x}\,\overline{y}$ becomes $\neg(p \vee q) \equiv \neg p \wedge \neg q$ for the other De Morgan law.

35. We need to play around with the symbols until the desired results fall out. To prove that $x \vee x = x$, let us compute $x \vee (x \wedge \overline{x})$ in two ways. On the one hand,

$$x \vee (x \wedge \overline{x}) = x \vee 0 = x$$

by the complement law and the identity law. On the other hand, by using the distributive law followed by the complement and identity laws we have

$$\begin{aligned} x \vee (x \wedge \overline{x}) &= (x \vee x) \wedge (x \vee \overline{x}) \\ &= (x \vee x) \wedge 1 = x \vee x. \end{aligned}$$

By transitivity of equality, $x = x \vee x$. The other property is the dual of this one, and its proof can be obtained by formally replacing every $\vee$ with $\wedge$ and replacing every 0 with 1, and vice versa. Thus our proof, shortened to one line, becomes

$$x = x \wedge 1 = x \wedge (x \vee \overline{x}) = (x \wedge x) \vee (x \wedge \overline{x}) = (x \wedge x) \vee 0 = x \vee x.$$

37. By Exercise 36 we know that the complement of an element is that unique element that obeys the complement laws. Therefore to show that $\overline{0} = 1$ we just need to prove that $0 \vee 1 = 1$ and $0 \wedge 1 = 0$. But these follow immediately from the identity laws (and, for the first, the commutative law). The other half follows in the same manner.

39. Before we prove this, we will find the following lemma useful: $x \wedge 0 = 0$ and $x \vee 1 = 1$ for all x. To prove the first half of the lemma, we invoke the results of Exercises 35 and 36 and compute as follows:

$$\begin{aligned} x \wedge 0 &= x \wedge (x \wedge \overline{x}) \\ &= (x \wedge x) \wedge \overline{x} \\ &= x \wedge \overline{x} = 0. \end{aligned}$$

The second half is similar, using the duality replacements mentioned in the solution to Exercise 35.

Now for the exercise at hand, by Exercise 36 it is enough to show that the claimed complements behave correctly. That is, we must show that $(x \vee y) \vee (\overline{x} \wedge \overline{y}) = 1$ and $(x \vee y) \wedge (\overline{x} \wedge \overline{y}) = 0$. For the second of these we compute as follows, using the lemma at the end and using the various defining properties freely (in particular, we use distributivity, associativity and commutativity a lot).

$$\begin{aligned}(x \vee y) \wedge (\overline{x} \wedge \overline{y}) &= \overline{y} \wedge [\overline{x} \wedge (x \vee y)] \\ &= \overline{y} \wedge [(\overline{x} \wedge x) \vee (\overline{x} \wedge y)] \\ &= \overline{y} \wedge [0 \vee (\overline{x} \wedge y)] \\ &= \overline{y} \wedge \overline{x} \wedge y = \overline{x} \wedge (y \wedge \overline{y}) \\ &= \overline{x} \wedge 0 = 0\end{aligned}$$

The first statement is proved in a similar way:

$$\begin{aligned}(x \vee y) \vee (\overline{x} \wedge \overline{y}) &= y \vee [x \vee (\overline{x} \wedge \overline{y})] \\ &= y \vee [(x \vee \overline{x}) \wedge (x \vee \overline{y})] \\ &= y \vee [1 \wedge (x \vee \overline{y})] \\ &= y \vee x \vee \overline{y} = (y \vee \overline{y}) \vee x \\ &= 1 \vee x = 1\end{aligned}$$

And of course the other half of this problem is proved in a manner completely dual to this.

41. Using the hypothesis, we compute as follows. $x = x \vee 0 = x \vee (x \vee y) = (x \vee x) \vee y = x \vee y = 0$. Similarly for y, by the commutative law. Note that we used the result of Exercise 35. The other statement is proved in the dual manner: $x = x \wedge 1 = x \wedge (x \wedge y) = (x \wedge x) \wedge y = x \wedge y = 1$ and similarly for y.

43. Most of this work was done in the Supplementary Exercises for Chapter 9 (or else is immediate from the definition). We need to verify the five laws (each one consisting of a dual pair, of course). The identity laws are Exercise 41. The complement laws are part of the definition of "complemented". The associative and commutative laws are Exercise 39. Finally, the distributive laws are again part of the definition.

SECTION 12.2 Representing Boolean Functions

The first six exercises are straightforward practice dealing with sum-of-products expansions. These are obtained by writing down one product term for each combination of values of the variables that makes the function have the value 1, and taking the sum of these terms. The dual to the sum-of-products expansion is discussed in Exercises 7–11, and these should be looked at. Since the concept of complete sets of operators plays an important role in the logical circuit design in sections to follow, Exercises 12–20 are also important.

1. a) We want $\overline{x}$, $\overline{y}$, and z all to have the value 1; therefore we take the product $\overline{x}\,\overline{y}\,z$. The other parts are similar, so we present only the answers.
b) $\overline{x}\,y\,\overline{z}$ **c)** $\overline{x}\,y\,z$ **d)** $\overline{x}\,\overline{y}\,\overline{z}$

3. a) We want the function to have the value 1 whenever at least one of the variables has the value 1. There are seven minterms that achieve this, so the sum has seven summands: $x\,y\,z+x\,y\,\overline{z}+x\,\overline{y}\,z+\overline{x}\,y\,z+x\,\overline{y}\,\overline{z}+\overline{x}\,y\,\overline{z}+\overline{x}\,\overline{y}\,z$.
b) Here is another way to think about this problem (rather than just making a table and reading off the minterms that make the value equal to 1). If we expand the expression by the distributive law (and use the commutative law), we get $xy+yz$. Now invoking the identity laws, the law that $s+\overline{s}=1$, and the distributive and commutative laws again, we write this as $xy1 + 1yz = xy(z+\overline{z}) + (x+\overline{x})yz = xyz + xy\overline{z} + xyz + \overline{x}yz$. Finally, we use the idempotent law to collapse the first and third term, to obtain our answer: $xyz+xy\overline{z}+\overline{x}yz$.

c) We can use either the straightforward approach or the idea used in part **(b)**. The answer is $xyz + xy\overline{z} + x\overline{y}z + x\overline{y}\,\overline{z}$.

d) The method discussed in part **(b)** works well here, to obtain the answer $x\overline{y}z + x\overline{y}\,\overline{z}$.

5. We need to list all minterms that have an odd number of the variables without bars (and hence an odd number with bars). There are $C(4,1)+C(4,3) = 8$ terms. The answer is $wxy\overline{z} + wx\overline{y}z + w\overline{x}yz + \overline{w}xyz + w\overline{x}\,\overline{y}\,\overline{z} + \overline{w}x\overline{y}\,\overline{z} + \overline{w}\,\overline{x}y\overline{z} + \overline{w}\,\overline{x}\,\overline{y}z$.

7. This exercise is dual to Exercise 1.

a) Note that $\overline{x}$ will have the value 0 if and only if $x = 1$. Similarly, $\overline{y}$ will have the value 0 if and only if $y = 1$. Therefore the expression $\overline{x} + \overline{y} + z$ will have the value 0 precisely in the desired case. The remaining parts are similar, so we list only the answers.

b) $x + y + z$ **c)** $x + \overline{y} + z$

9. By the definition of "+," the sum $y_1 + \cdots + y_n$ has the value 0 if and only if each $y_i = 0$. This happens precisely when $x_i = 0$ for those cases in which $y_i = x_i$ and $x_i = 1$ in those cases in which $y_i = \overline{x}_i$.

11. a) This function is already written in its product-of-sums form (with one factor).

b) This function has the value 0 in case $y = 0$ or both x and z equal 0. Therefore we need maxterms $x + y + z$, $x + y + \overline{z}$, $\overline{x} + y + z$, and $\overline{x} + y + \overline{z}$ (to take care of $y = 0$), and also $x + \overline{y} + z$. Therefore the answer is the product of these five maxterms.

c) This function has the value 0 in case $x = 0$. Therefore we need four maxterms, and the answer is $(x + y + z)(x + y + \overline{z})(x + \overline{y} + z)(x + \overline{y} + \overline{z})$.

d) Let us indicate another way to solve problems like this. In Exercise 3d we found the sum-of-products expansion of this function. Suppose that we take take sum-of-products expansion of the function that is the opposite of this one. It will have all the minterms other than the ones in the answer to Exercise 3d, so it will be $xyz + xy\overline{z} + \overline{x}yz + \overline{x}y\overline{z} + \overline{x}\,\overline{y}z + \overline{x}\,\overline{y}\,\overline{z}$. If we now take the complement of this (put a big bar over it), then we will have an expression for the function we want. Then we push the complementations inside, using De Morgan's laws and the fact that $\overline{\overline{s}} = s$. This will give us the desired product-of-sums expansion. Formally, all we do is put parentheses around the minterms, erase all the plus signs, put plus signs between all the variables (where there used to be implied products), and change every complemented variable to its uncomplemented version and vice versa. The answer is thus $(\overline{x} + \overline{y} + \overline{z})(\overline{x} + \overline{y} + z)(x + \overline{y} + \overline{z})(x + \overline{y} + z)(x + y + \overline{z})(x + y + z)$.

13. To do this exercise we need to use De Morgan's law to replace st by $\overline{(\overline{s} + \overline{t})}$. Thus we just do this formally in the expressions in Exercise 12, and we obtain the answers. It is also good to simplify double complements, of course.

a) This is already in the desired form, having no products.

b) $x + \overline{y}(\overline{x} + z) = x + \overline{\left(\overline{\overline{y}} + \overline{(\overline{x} + z)}\right)} = x + \overline{\left(y + \overline{(\overline{x} + z)}\right)}$

c) This is already in the desired form, having no products.

d) $\overline{x}(x + \overline{y} + \overline{z}) = \overline{\left(\overline{\overline{x}} + \overline{(x + \overline{y} + \overline{z})}\right)} = \overline{\left(x + \overline{(x + \overline{y} + \overline{z})}\right)}$

15. a) We use the definition of $\downarrow$. If $x = 1$, then $x \downarrow x = 0$; and if $x = 0$, then $x \downarrow x = 1$. These are precisely the corresponding values of $\overline{x}$.

b) We can construct a table to look at all four cases, as follows. Since the fifth and sixth columns are equal, the expressions are equivalent.

x	y	$x \downarrow x$	$y \downarrow y$	$(x \downarrow x) \downarrow (y \downarrow y)$	xy
1	1	0	0	1	1
1	0	0	1	0	0
0	1	1	0	0	0
0	0	1	1	0	0

c) We can construct a table to look at all four cases, as follows. Since the fourth and fifth columns are equal, the expressions are equivalent.

x	y	$x \downarrow y$	$(x \downarrow y) \downarrow (x \downarrow y)$	$x + y$
1	1	0	1	1
1	0	0	1	1
0	1	0	1	1
0	0	1	0	0

17. **a)** Since $x + y + z = (x + y) + z$, we first write this as $((x + y) \mid (x + y)) \mid (z \mid z)$, using the identity in Exercise 14c. Then we rewrite $x + y$ as $(x \mid x) \mid (y \mid y)$, using the same identity. This gives us

$$(((x \mid x) \mid (y \mid y)) \mid ((x \mid x) \mid (y \mid y))) \mid (z \mid z).$$

b) First we write this as $((x + z) \mid y) \mid ((x + z) \mid y)$, using the identity in Exercise 14b. Then we rewrite $x + z$ as $(x \mid x) \mid (z \mid z)$, using the identity in Exercise 14c. This gives us

$$(((x \mid x) \mid (z \mid z)) \mid y) \mid (((x \mid x) \mid (z \mid z)) \mid y).$$

c) There are no operators, so nothing needs to be done; the expression as given is the answer.

d) First we write this as $(x \mid \overline{y}) \mid (x \mid \overline{y})$, using the identity in Exercise 14b. Then we rewrite $\overline{y}$ as $y \mid y$, using the identity in Exercise 14a. This gives us $(x \mid (y \mid y)) \mid (x \mid (y \mid y))$.

19. We claim that it is impossible to write a Boolean expression for $\overline{x}$ involving x, $+$, and $\cdot$. The reason is that if $x = 1$, then any combination of these two operators applied to x (and the results of previous calculations) can yield only the value 1. But $\overline{1} = 0$. (This problem is somewhat harder if we allow the use of the constants 0 and 1. The argument given here is then no longer valid, since $x \cdot 0 = 0$. Nevertheless it is possible to show that even with these constants allowed, the set $\{+, \cdot\}$ is not functionally complete—i.e., it is impossible to write down a Boolean expression using only the operators $+$ and $\cdot$ (together with x, 0, and 1) that is equivalent to $\overline{x}$. What one can do is to prove by induction on the length of the expression that any such Boolean expression that has the value 1 when $x = 0$ must also have the value 1 when $x = 1$.)

SECTION 12.3 Logic Gates

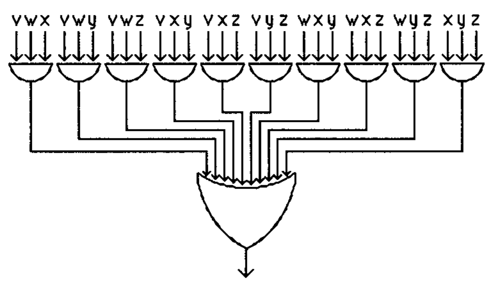
This section is a brief introduction to circuit design using AND and OR gates and inverters. In real life, circuits will have thousands of these components, but you will get some of the flavor in these exercises. Notice particularly Exercise 9, which shows how circuits already constructed can be further combined to give more complex, useful circuits. If we want to get by with fewer types of gates (but more gates), then we can use NOR or NAND gates, as illustrated in Exercises 15–18. Exercise 20 introduces the notion of the **depth** *of a circuit.*

1. The output of the OR gate at the top is $x+y$. This and $\overline{y}$ are the inputs to the final AND gate. Therefore the output of the circuit is $(x+y)\overline{y}$.

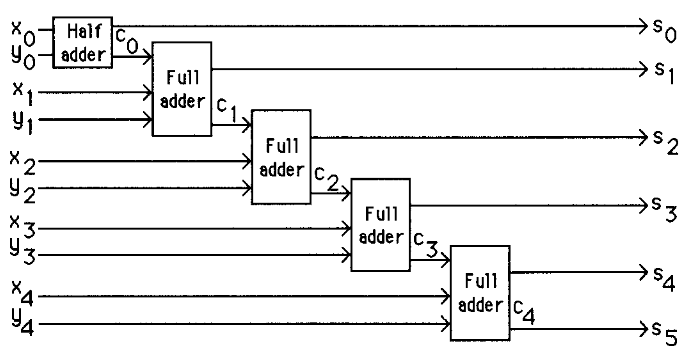
3. The idea is the same as in the previous two exercises. The final output is an *OR* with two inputs. The first of these inputs is the result of inverting xy, and the second is $\overline{z}+x$. Therefore the answer is $\overline{(xy)}+(\overline{z}+x)$.

5. The outputs from the three OR gates are $x+y+z$, $\overline{x}+y+z$, and $\overline{x}+\overline{y}+\overline{z}$. Therefore the output from the final OR gate is $(x+y+z)+(\overline{x}+y+z)+(\overline{x}+\overline{y}+\overline{z})$. It is not hard to see that this is always 1, since if $x=1$ then the output of the top initial OR gate is 1, whereas if $x=0$ then the output of the middle initial OR gate is 1.

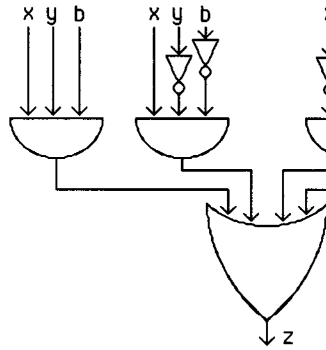
7. Let v, w, x, y, and z be the votes of the five individuals, with a 1 representing a yes vote and a 0 representing a no vote. Then the majority will be a yes vote (represented by an output of 1) if and only if there are at least three yes votes. Thus we make an AND gate for each of the $C(5,3)=10$ triples of voters, and combine the outputs from these 10 gates with an OR. We turn the picture on its side for convenience.

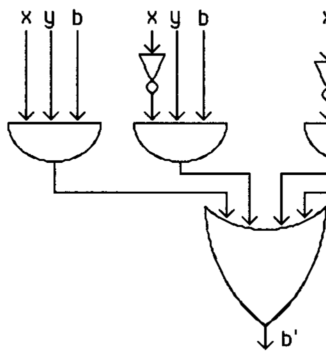


9. The circuit is identical to Figure 10, expanded by two more units to accommodate the two additional bits. To get the computation started, x_0 and y_0 are the inputs to the half adder. Thereafter, the carry bit from each column is input, together with the next pair (x_i, y_i) to a full adder to find the output and carry for the next column. The final carry bit (c_4) is the final answer bit (s_5).

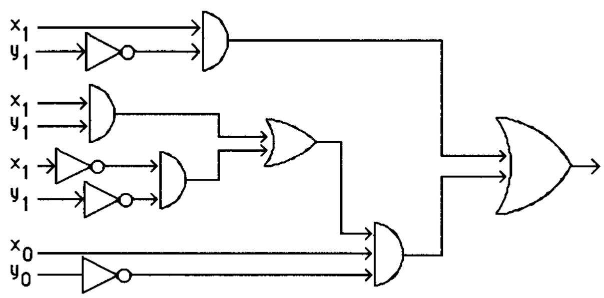


11. We will construct the full subtractor directly. Suppose the input bits are x, y, and b, where we are computing $x-y$ with borrow b. The output is a bit, z, and a borrow from the next column, b'. Then by looking at the eight possibilities for the bits x, y, and b, we see that $z = x\,y\,b + x\,\overline{y}\,\overline{b} + \overline{x}\,y\,\overline{b} + \overline{x}\,\overline{y}\,b$; and that $b' = x\,y\,b + \overline{x}\,y\,b + \overline{x}\,y\,\overline{b} + \overline{x}\,\overline{y}\,b$. Therefore a full subtractor can be formed by using AND gates, OR gates, and inverters to represent these expressions. We obtain the circuits shown below.



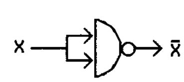


13. The first number is larger than the second if $x_1 > y_1$ (which means $x_1 = 1$ and $y_1 = 0$), or if $x_1 = y_1$ (which means either that $x_1 = 1$ and $y_1 = 1$, or that $x_1 = 0$ and $y_1 = 0$) and also $x_0 > y_0$ (which means $x_0 = 1$ and $y_0 = 0$). We translate this sentence into a circuit in the obvious manner, obtaining the picture shown.

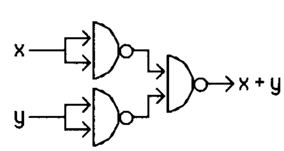


15. Note that in this exercise the usual operation symbol | is used for the *NAND* operation.

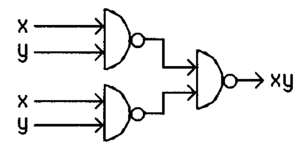
a) By Exercise 14a in Section 12.2, $\overline{x} = x \mid x$. Therefore the gate for $\overline{x}$ is as shown below.



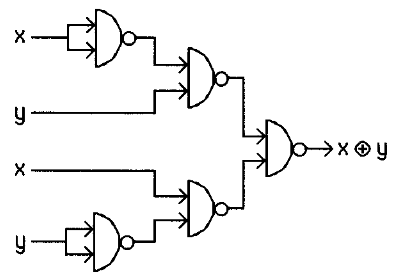
b) By Exercise 14c in Section 12.2, $x + y = (x \mid x) \mid (y \mid y)$. Therefore the gate for $x + y$ is as shown below.



c) By Exercise 14b in Section 12.2, $xy = (x \mid y) \mid (x \mid y)$. Therefore the gate for xy is as shown below.

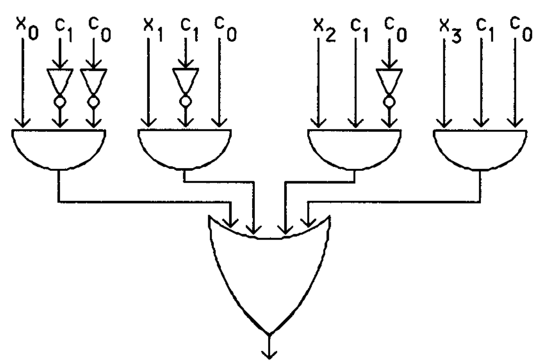


d) First we note that $x \oplus y = x\overline{y} + \overline{x}\,y = \overline{\left(\overline{(x\overline{y})}\,\overline{(\overline{x}\,y)}\right)} = \overline{(x\overline{y})} \mid \overline{(\overline{x}\,y)} = (x \mid \overline{y}) \mid (\overline{x} \mid y)$. We constructed the gate for inverting in part **(a)**. Therefore the gate for xy is as shown below.



17. We know that the sum bit in the half adder is $s = x\overline{y} + \overline{x}\,y$. The answer to Exercise 15d shows precisely this gate constructed from NAND gates, so it gives us this part of the answer. Also, the carry bit in the half adder is $c = xy$. The answer to Exercise 15c shows precisely this gate constructed from NAND gates, so it gives us this part of the answer.

19. We can set this up so that the value of x_i "gets through" to a final OR gate if and only if $(c_1c_0)_2 = i$. For example, we want x_2 to get through if and only if $c_1 = 1$ and $c_0 = 0$, since the Base 2 numeral for 2 is 10. We can do this by combining the x_i input with either c_0 or its inversion and either c_1 or its inversion, using an AND gate with three inputs. Thus the output of each of these gates is either 0 (if $(c_1c_0)_2 \neq i$) or is the value of x_i (if $(c_1c_0)_2 = i$). So if we combine the result of these four outputs, using an OR gate, we will get the desired result (since at most one of the four outputs can possibly be nonzero). Here is the circuit.



SECTION 12.4 Minimization of Circuits

The two methods presented here for minimizing circuits are really the same, just looked at from two different points of view—one geometric (K-maps) and one algebraic (the Quine–McCluskey method). In each case the idea is to get larger blocks, which represent simpler terms, to cover several minterms. The calculations can get very messy, but if you follow the examples and organize your work carefully, you should not have trouble with them. The hard part in these algorithms from a theoretical point of view is finding the set of products that cover all the minterms. In these small examples, this tends not to be a problem.

1. a) The K-map we draw here has the variable x down the side and variable y across the top.

	y	$\bar{y}$
x		
$\bar{x}$	1	

b) The upper left-hand corner cell (whose minterm is xy) and the lower right-hand corner cell (whose minterm is $\bar{x}\bar{y}$) are adjacent to this cell.

3. The 2×2 square is used in each case. We put a 1 in those cells whose minterms are listed.

	y	$\bar{y}$
x		1
$\bar{x}$		

(a)

	y	$\bar{y}$
x	1	
$\bar{x}$		1

(b)

	y	$\bar{y}$
x	1	1
$\bar{x}$	1	1

(c)

5. a) We can draw a K-map for three variables in the manner shown here, with the x variable down the side and the y and z variables across the top, in the order shown. We have placed a 1 in the requested position.

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x				
$\bar{x}$		1		

b) There are three cells adjacent to every cell (since there are three variables). The minterms of the adjacent cells can be read off the picture: $\bar{x}yz$, $xy\bar{z}$, and $\bar{x}\bar{y}\bar{z}$.

7. The 2×4 square is used in each case. We put a 1 in those cells whose minterms are listed.

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x			1	
$\bar{x}$				

(a)

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x				
$\bar{x}$	1		1	

(b)

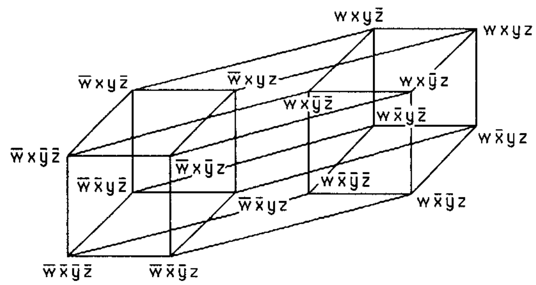
	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1		
$\bar{x}$		1		1

(c)

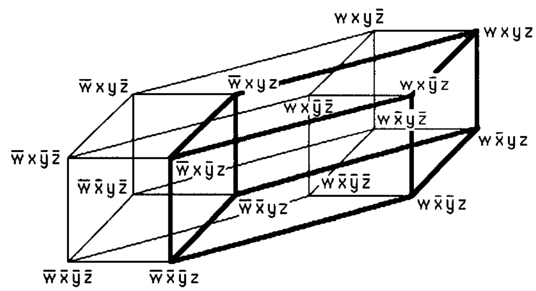
9. In the figure below we have drawn the K-map. For example, since one of the terms was $x\bar{z}$, we put a 1 in each cell whose address contained x and $\bar{z}$. Note that this meant two cells, one for y and one for $\bar{y}$. Each cell with a 1 in it is an implicant, as are the pairs of cells that form blocks, namely xy, $x\bar{z}$, and $y\bar{z}$. Since each cell by itself is contained in a block with two cells, none of them is prime. Each of the mentioned blocks with two cells is prime, since none is contained in a larger block. Furthermore, each of these blocks is essential, since each contains a cell that no other prime implicant contains: xy contains xyz, $x\bar{z}$ contains $x\bar{y}\bar{z}$, and $y\bar{z}$ contains $\bar{x}y\bar{z}$.

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
x	1	1	1	
$\bar{x}$		1		

11. The figure below shows the 4-cube Q_4, labeled as requested. Compare with Figure 1 in Section 12.1. A complemented Boolean variable corresponds to 0, and an uncomplemented Boolean variable corresponds to 1. For example, the lower right corner of the sub-3-cube on the left corresponds to 0001.



The 3-cube on the right corresponds to w, since all of its vertices are labeled w. Similarly, the 3-cube given by the top surface of the whole figure represents x; the 3-cube given by the back surface of the whole figure represents y; and the 3-cube given by the right surfaces of both the left and the right 3-cube represents z. We show the last of these with heavy lines below. The "opposite 3-face" in each case represents the complemented literal.



The 2-cube (i.e., square) that represents wz, in the same way, is the set of vertices that have w and z as part of their labels, rather than $\overline{w}$ and/or $\overline{z}$. This is the right face of the 3-cube on the right. Similarly, the 2-cube that represents $\overline{x}y$ is bottom rear, and the 2-cube that represents $\overline{y}\,\overline{z}$ is front left.

13. **a)** We can draw a K-map for four variables in the manner shown here, with the w and x variables down the side and the y and z variables across the top, in the order shown. We have placed a 1 in the requested position.

	yz	$y\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}z$
wx				
$w\overline{x}$				
$\overline{w}\,\overline{x}$				
$\overline{w}x$		1		

b) There are four cells adjacent to every cell (since there are four variables). The minterms of the adjacent cells can be read off the picture: $\overline{w}xyz$, $\overline{w}\,\overline{x}y\overline{z}$, $\overline{w}x\overline{y}\,\overline{z}$, and (recalling that the top row is adjacent to the bottom row) $wxy\overline{z}$.

15. A K-map for five variables needs 32 cells in all. We arrange them as shown below, following the discussion in Example 6. Reread that example to understand what rows and columns have to be considered adjacent. (We use rectangles rather than squares to save vertical space.)

a) We want to put a 1 in all cells the correspond to x_1, x_2, x_3, and x_4 all being uncomplemented. Looking at the diagram, we see that we need 1's in precisely the two cells shown.

	$x_3x_4x_5$	$x_3x_4\bar{x}_5$	$x_3\bar{x}_4\bar{x}_5$	$x_3\bar{x}_4x_5$	$\bar{x}_3\bar{x}_4x_5$	$\bar{x}_3\bar{x}_4\bar{x}_5$	$\bar{x}_3x_4\bar{x}_5$	$\bar{x}_3x_4x_5$
x_1x_2	1	1						
$x_1\bar{x}_2$								
$\bar{x}_1\bar{x}_2$								
$\bar{x}_1x_2$								

b) We want to put a 1 in all cells the correspond to x_1 being complemented and x_3 and x_5 both being uncomplemented. Looking at the diagram, we see that we need 1's in precisely the four cells shown. Note that these are all adjacent, even though they don't look adjacent, since the first and fourth columns are considered adjacent.

	$x_3x_4x_5$	$x_3x_4\bar{x}_5$	$x_3\bar{x}_4\bar{x}_5$	$x_3\bar{x}_4x_5$	$\bar{x}_3\bar{x}_4x_5$	$\bar{x}_3\bar{x}_4\bar{x}_5$	$\bar{x}_3x_4\bar{x}_5$	$\bar{x}_3x_4x_5$
x_1x_2								
$x_1\bar{x}_2$								
$\bar{x}_1\bar{x}_2$	1			1				
$\bar{x}_1x_2$	1			1				

c) We want to put a 1 in all cells the correspond to x_2 and x_4 both being uncomplemented. Looking at the diagram, we see that we need 1's in precisely the eight cells shown.

	$x_3x_4x_5$	$x_3x_4\bar{x}_5$	$x_3\bar{x}_4\bar{x}_5$	$x_3\bar{x}_4x_5$	$\bar{x}_3\bar{x}_4x_5$	$\bar{x}_3\bar{x}_4\bar{x}_5$	$\bar{x}_3x_4\bar{x}_5$	$\bar{x}_3x_4x_5$
x_1x_2	1	1					1	1
$x_1\bar{x}_2$								
$\bar{x}_1\bar{x}_2$								
$\bar{x}_1x_2$	1	1					1	1

d) We want to put a 1 in all cells the correspond to x_3 and x_4 both being complemented. Looking at the diagram, we see that we need 1's in precisely the eight cells shown.

	$x_3x_4x_5$	$x_3x_4\bar{x}_5$	$x_3\bar{x}_4\bar{x}_5$	$x_3\bar{x}_4x_5$	$\bar{x}_3\bar{x}_4x_5$	$\bar{x}_3\bar{x}_4\bar{x}_5$	$\bar{x}_3x_4\bar{x}_5$	$\bar{x}_3x_4x_5$
x_1x_2					1	1		
$x_1\bar{x}_2$					1	1		
$\bar{x}_1\bar{x}_2$					1	1		
$\bar{x}_1x_2$					1	1		

e) We want to put a 1 in all cells the correspond to x_3 being uncomplemented. Looking at the diagram, we see that we need 1's in precisely the 16 cells shown.

	$x_3x_4x_5$	$x_3x_4\bar{x}_5$	$x_3\bar{x}_4\bar{x}_5$	$x_3\bar{x}_4x_5$	$\bar{x}_3\bar{x}_4x_5$	$\bar{x}_3\bar{x}_4\bar{x}_5$	$\bar{x}_3x_4\bar{x}_5$	$\bar{x}_3x_4x_5$
x_1x_2	1	1	1	1				
$x_1\bar{x}_2$	1	1	1	1				
$\bar{x}_1\bar{x}_2$	1	1	1	1				
$\bar{x}_1x_2$	1	1	1	1				

f) We want to put a 1 in all cells the correspond to x_5 being complemented. Looking at the diagram, we see that we need 1's in precisely the 16 cells shown.

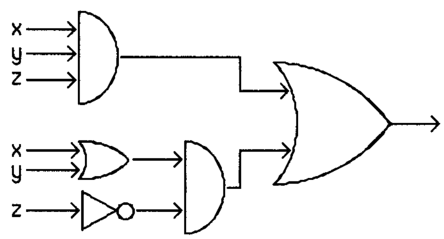
	$x_3x_4x_5$	$x_3x_4\bar{x}_5$	$x_3\bar{x}_4\bar{x}_5$	$x_3\bar{x}_4x_5$	$\bar{x}_3\bar{x}_4x_5$	$\bar{x}_3\bar{x}_4\bar{x}_5$	$\bar{x}_3x_4\bar{x}_5$	$\bar{x}_3x_4x_5$
x_1x_2		1	1			1	1	
$x_1\bar{x}_2$		1	1			1	1	
$\bar{x}_1\bar{x}_2$		1	1			1	1	
$\bar{x}_1x_2$		1	1			1	1	

17. a) There are clearly 2^n cells in a K-map for n variables, since in specifying a minterm each variable can appear either complemented or uncomplemented. Thus the answer is $2^6 = 64$.

b) There are n cells adjacent to each cell in the K-map for n variables, because each of the variables can be changed (from complemented to uncomplemented or vice versa) to produce an adjacent cell. Thus the answer is 6.

19. Clearly we need to consider the first and fourth rows adjacent. The columns are more complicated, since each column needs to be adjacent to three others (those that differ from it in one bit). Thus we must declare the following columns adjacent (using the obvious notation): 1–4, 1–12, 1–16, 2–11, 2–15, 3–6, 3–10, 4–9, 5–8, 5–16, 6–15, 7–10, 7–14, 8–13, 9–12, 11–14, and 13–16. The complexity of this makes it virtually impossible for a human to use this visual aid.

21. We could ignore the stated condition, but then our circuit would be more complex than need be. If we use the condition, then there are really only three inputs; let us call them x, y, and z, where z represents Marcus's vote and x and y represent the votes of the unnamed people. Since Smith and Jones always vote against Marcus, a majority will occur if either Marcus assents with both of the other two (i.e., the minterm xyz), or else if Marcus votes no but at least one of the other two vote yes (which we can represent by $(x+y)\bar{z}$). Thus we design our circuit to be the *OR* of these two expressions.



23. We organize our work as in the text.

a)

	Term	String	Step 1 Term	Step 1 String
1	$\bar{x}yz$	011	$(1,2)\,\bar{x}z$	0–1
2	$\bar{x}\bar{y}z$	001		

In this case we have one product that covers both of the minterms, so this product ($\bar{x}z$) is our answer.

b)

	Term	String	Step 1 Term	Step 1 String	Step 2 Term	Step 2 String
1	xyz	111	$(1,2)\,xy$	11–	$(1,2,3,4)\,y$	–1–
2	$xy\bar{z}$	110	$(1,3)\,yz$	–11		
3	$\bar{x}yz$	011	$(2,4)\,y\bar{z}$	–10		
4	$\bar{x}y\bar{z}$	010	$(3,4)\,\bar{x}y$	01–		

Again we have one product that covers all the minterms, so that is our answer (y).

c)

	Term	String	Step 1 Term	String
1	$x\,y\,\overline{z}$	110	$(1,4)\,x\,\overline{z}$	1–0
2	$x\,\overline{y}\,z$	101	$(2,4)\,x\,\overline{y}$	10–
3	$\overline{x}\,y\,z$	011	$(2,5)\,\overline{y}\,z$	–01
4	$x\,\overline{y}\,\overline{z}$	100	$(3,5)\,\overline{x}\,z$	0–1
5	$\overline{x}\,\overline{y}\,z$	001		

(Note that we reordered the minterms so that the number of 1's decreased as we went down the list.) No further combinations are possible at this point, so there are four products that must be used to cover our five minterms, each product covering two minterms. Clearly, then, two of these are not enough, but it is easy to find three whose sum covers all the minterms. One possible answer is to choose the first, third, and fourth of the products, namely $x\,\overline{z} + \overline{y}\,z + \overline{x}\,z$.

d)

	Term	String	Step 1 Term	String
1	$x\,y\,z$	111	$(1,2)\,x\,z$	1–1
2	$x\,\overline{y}\,z$	101	$(1,3)\,y\,z$	–11
3	$\overline{x}\,y\,z$	011	$(2,4)\,x\,\overline{y}$	10–
4	$x\,\overline{y}\,\overline{z}$	100	$(3,5)\,\overline{x}\,y$	01–
5	$\overline{x}\,y\,\overline{z}$	010	$(4,6)\,\overline{y}\,\overline{z}$	–00
6	$\overline{x}\,\overline{y}\,\overline{z}$	000	$(5,6)\,\overline{x}\,\overline{z}$	0–0

(Note that we reordered the minterms so that the number of 1's decreased as we went down the list.) No further combinations are possible at this point, so there are six products that must be used to cover our six minterms, each product covering two minterms. Clearly, then, two of these are not enough, but it is not hard to find three whose sum covers all the minterms. One possible answer is to choose the first, fourth, and fifth of the products, namely $x\,z + \overline{x}\,y + \overline{y}\,\overline{z}$.

25. We follow the procedure and notation given in the text.

a)

	Term	String	Step 1 Term	String
1	$w\,x\,y\,z$	1111	$(1,2)\,w\,x\,z$	11–1
2	$w\,x\,\overline{y}\,z$	1101	$(2,3)\,w\,x\,\overline{y}$	110–
3	$w\,x\,\overline{y}\,\overline{z}$	1100	$(2,5)\,w\,\overline{y}\,z$	1–01
4	$w\,\overline{x}\,y\,\overline{z}$	1010		
5	$w\,\overline{x}\,\overline{y}\,z$	1001		

The three products in the last column as well as minterm #4 are possible products in the desired expansion, since they are not contained in any other product. We make a table of which products cover which of the original minterms.

	1	2	3	4	5
$w\,x\,z$	X	X			
$w\,x\,\overline{y}$		X	X		
$w\,\overline{y}\,z$		X			X
$w\,\overline{x}\,y\,\overline{z}$				X	

Since only the first of our products covers minterm #1, it must be included. Similarly, the other three products must be included since they are the only ones that cover minterms #3, #4, and #5. If we do include them all, then of course all the minterms are covered. Therefore our answer is $w\,x\,z + w\,x\,\overline{y} + w\,\overline{y}\,z + w\,\overline{x}\,y\,\overline{z}$.

b)

	Term	String	Step 1 Term	Step 1 String
1	$w\,x\,y\,\overline{z}$	1110	$(2,4)\ x\,\overline{y}\,z$	-101
2	$w\,x\,\overline{y}\,z$	1101	$(4,6)\,\overline{w}\,\overline{y}\,z$	$0-01$
3	$w\,\overline{x}\,y\,z$	1011		
4	$\overline{w}\,x\,\overline{y}\,z$	0101		
5	$\overline{w}\,\overline{x}\,y\,\overline{z}$	0010		
6	$\overline{w}\,\overline{x}\,\overline{y}\,z$	0001		

Since minterms #1, #3, and #5 are not contained in any others, these, along with the two products in the last column, are the products that we look at to cover the original minterms. It is not hard to see that all five are needed to cover all the original minterms. Therefore the answer is $x\,\overline{y}\,z+\overline{w}\,\overline{y}\,z+w\,x\,y\,\overline{z}+w\,\overline{x}\,y\,z+\overline{w}\,\overline{x}\,y\,\overline{z}$.

c)

	Term	String	Step 1 Term	Step 1 String	Step 2 Term	Step 2 String
1	$w\,x\,y\,z$	1111	$(1,2)\,w\,x\,y$	$111-\ \Leftarrow$	$(3,4,5,8)\,\overline{y}\,z$	$--01$
2	$w\,x\,y\,\overline{z}$	1110	$(1,3)\,w\,x\,z$	$11-1\ \Leftarrow$		
3	$w\,x\,\overline{y}\,z$	1101	$(3,4)\,w\,\overline{y}\,z$	$1-01$		
4	$w\,\overline{x}\,\overline{y}\,z$	1001	$(3,5)\,x\,\overline{y}\,z$	-101		
5	$\overline{w}\,x\,\overline{y}\,z$	0101	$(4,6)\,w\,\overline{x}\,\overline{y}$	$100-\ \Leftarrow$		
6	$w\,\overline{x}\,\overline{y}\,\overline{z}$	1000	$(4,8)\,\overline{x}\,\overline{y}\,z$	-001		
7	$\overline{w}\,\overline{x}\,y\,\overline{z}$	0010	$(5,8)\,\overline{w}\,\overline{y}\,z$	$0-01$		
8	$\overline{w}\,\overline{x}\,\overline{y}\,z$	0001				

The product in the last column, as well as the products in Step 1 that are marked with an arrow, as well as minterm #7, are possible products in the desired expansion, since they are not contained in any other product. We make a table of which products cover which of the original minterms.

	1	2	3	4	5	6	7	8
$\overline{y}\,z$			X	X	X			X
$w\,x\,y$	X	X						
$w\,x\,z$	X		X					
$w\,\overline{x}\,\overline{y}$				X		X		
$\overline{w}\,\overline{x}\,y\,\overline{z}$							X	

In order to cover minterms #5, #2, #6, and #7, we need the first, second, fourth, and fifth of the products in this table. If we include these four, then all the minterms are covered, and we do not need the third one. Therefore our answer is $\overline{y}\,z + w\,x\,y + w\,\overline{x}\,\overline{y} + \overline{w}\,\overline{x}\,y\,\overline{z}$.

d)

	Term	String	Step 1 Term	Step 1 String	Step 2 Term	Step 2 String
1	$w\,x\,y\,z$	1111	$(1,2)\,w\,x\,y$	$111-$	$(1,2,4,6)\,w\,y$	$1-1-$
2	$w\,x\,y\,\overline{z}$	1110	$(1,3)\,w\,x\,z$	$11-1\ \Leftarrow$	$(1,4,5,7)\,y\,z$	$--11$
3	$w\,x\,\overline{y}\,z$	1101	$(1,4)\,w\,y\,z$	$1-11$	$(4,6,7,8)\,\overline{x}\,y$	$-01-$
4	$w\,\overline{x}\,y\,z$	1011	$(1,5)\,x\,y\,z$	-111		
5	$\overline{w}\,x\,y\,z$	0111	$(2,6)\,w\,y\,\overline{z}$	$1-10$		
6	$w\,\overline{x}\,y\,\overline{z}$	1010	$(4,6)\,w\,\overline{x}\,y$	$101-$		
7	$\overline{w}\,\overline{x}\,y\,z$	0011	$(4,7)\,\overline{x}\,y\,z$	-011		
8	$\overline{w}\,\overline{x}\,y\,\overline{z}$	0010	$(5,7)\,\overline{w}\,y\,z$	$0-11$		
9	$\overline{w}\,\overline{x}\,\overline{y}\,z$	0001	$(6,8)\,\overline{x}\,y\,\overline{z}$	-010		
			$(7,8)\,\overline{w}\,\overline{x}\,y$	$001-$		
			$(7,9)\,\overline{w}\,\overline{x}\,z$	$00-1\ \Leftarrow$		

The products in the last column, as well as the products in Step 1 that are marked with an arrow, are possible products in the desired expansion, since they are not contained in any other product. We make a table of which products cover which of the original minterms.

	1	2	3	4	5	6	7	8	9
wy	X	X		X		X			
yz	X			X	X		X		
$\overline{x}y$				X		X	X	X	
wxz	X		X						
$\overline{w}\,\overline{x}z$							X		X

In order to cover minterms #2, #5, #8, #3, and #9, we need the first, second, third, fourth, and fifth of the products in this table, respectively—i.e., all of them. Therefore our answer is $wy + yz + \overline{x}y + wxz + \overline{w}\,\overline{x}z$.

27. Using the method of Exercise 26, we draw the following picture, putting a 0 in each cell that represents a maxterm in our product-of-sums expansion.

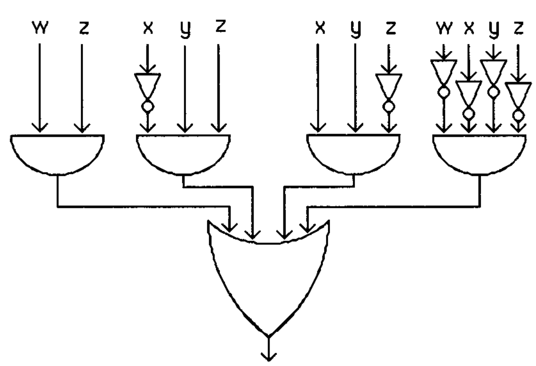
	$y+z$	$y+\overline{z}$	$\overline{y}+\overline{z}$	$\overline{y}+z$
x	0	0	0	0
$\overline{x}$	0			

We then combine them into larger blocks, as shown, obtaining two large blocks: the entire first row, which represents the factor x, and the entire first column, which represents the factor $(y+z)$. Since neither of these blocks alone covers all the maxterms, we need to use both. Therefore the simplified product is $x(y+z)$.

29. We need to review Example 8 and note that the various positions in the 4×4 square correspond to the various decimal digits. For example, the digit 6 corresponds to the box labeled $\overline{w}xy\overline{z}$. There are "don't care" positions as well, for 4-digit binary numerals that exceed 9. (We do not care what output results for such inputs.) Using the techniques of this section, we obtain the following maximal blocks.

	yz	$y\overline{z}$	$\overline{y}\,\overline{z}$	$\overline{y}z$
wx	d	d	d	d
$w\overline{x}$	d	d		1
$\overline{w}\,\overline{x}$	1		1	
$\overline{w}x$		1		

Each one corresponds to a minterm, and our minimal sum-of-products expansion is therefore $wz + \overline{x}yz + xy\overline{z} + \overline{w}\,\overline{x}\,\overline{y}\,\overline{z}$. Since we were asked for a circuit, we turn the products into AND gates and the sum into one big OR gate.



31. We can cover all the 1's with two large blocks here, one consisting of the middle four cells, and the other consisting of the four corners (it is a block because of the wrap-around nature of the figure). In doing so, we happened to have covered all the d's as well, but that is irrelevant. The point is that we obtained the largest possible blocks in this manner, and if we had chosen not to cover the d's, then the blocks would have been smaller. It is clear from this covering that the minimal sum-of-products expansion is just $\overline{x}\,\overline{z} + x\,z$.

33. A formal proof here proceeds by induction on n. If $n = 1$, then we are looking at the 1-cube, which is a line segment, labeled 0 at one end and 1 at the other end. The only possible value of k is also 1, and if the literal is x_1, then the subcube we have is the 0-dimensional subcube consisting of the endpoint labeled 1, and if the literal is $\overline{x}_1$, then the subcube we have is the 0-dimensional subcube consisting of the endpoint labeled 0. Now assume that the statement is true for n; we must show that it is true for $n+1$. If the literal x_{n+1} (or its complement) is not part of the product, then by the inductive hypothesis, the product when viewed in the setting of n variables corresponds to an $(n-k)$-dimensional subcube of the n-dimensional cube, and the Cartesian product of that subcube with the line segment $[0,1]$ gives us a subcube one dimension higher in our given $(n+1)$-dimensional cube, namely having dimension $(n+1)-k$, as desired. On the other hand, if the literal x_{n+1} (or its complement) is part of the product, then the product of the remaining $k-1$ literals corresponds to a subcube of dimension $n-(k-1) = (n+1)-k$ in the n-dimensional cube, and that slice, at either the 1-end or the 0-end in the last variable, is the desired subcube.

GUIDE TO REVIEW QUESTIONS FOR CHAPTER 12

1. See p. 812.

2. 16 (see Table 3 in Section 12.1)

3. See p. 813.

4. **a)** See p. 816. **b)** See p. 816.

5. See pp. 819–820.

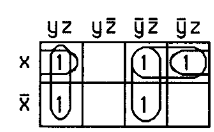
6. **a)** See p. 821. **b)** This set is not functionally complete; see Exercise 19 in Section 12.2. **c)** yes, such as $\{\downarrow\}$ (see Exercise 16 in Section 12.2)

7. See Example 3 in Section 12.3.

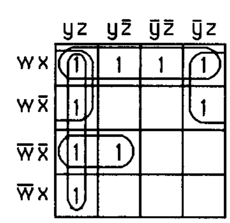
8. See Figure 8 in Section 12.3.

9. yes—NAND gates or NOR gates

10. **a)** See pp. 830–833. **b)** The following figure shows that the simplification is $x\,\overline{y} + y\,z + \overline{y}\,\overline{z}$.



11. **a)** See pp. 833–834. **b)** The following figure shows that the simplification is $w\,x + y\,z + w\,z + \overline{w}\,\overline{x}\,y$.



12. a) See p. 836.

b) The following figure shows that the simplification is $w + x\,y$. The resulting circuit needs just one AND gate, one OR gate, and no inverters.

	yz	$y\bar{z}$	$\bar{y}\bar{z}$	$\bar{y}z$
wx	d	d	d	d
$w\bar{x}$	d	d	1	1
$\bar{w}\bar{x}$				
$\bar{w}x$	1	1		

13. a) See pp. 837–841.

b)

			Step 1		Step 2	
	Term	String	Term	String	Term	String
1	$x\,y\,\bar{z}$	110	$(1,2)\,y\,\bar{z}$	-10	$(1,2,3,5)\,\bar{z}$	$--0$
2	$\bar{x}\,y\,\bar{z}$	010	$(1,3)\,x\,\bar{z}$	$1-0$		
3	$x\,\bar{y}\,\bar{z}$	100	$(2,5)\,\bar{x}\,\bar{z}$	$0-0$		
4	$\bar{x}\,\bar{y}\,z$	001	$(3,5)\,\bar{y}\,\bar{z}$	-00		
5	$\bar{x}\,\bar{y}\,\bar{z}$	000	$(4,5)\,\bar{x}\,\bar{y}$	$00-$		

The product $\bar{z}$ in the last column covers all the minterms except #4, and the fifth product in Step 1 ($\bar{x}\,\bar{y}$) covers it. Thus the answer is $\bar{z} + \bar{x}\,\bar{y}$.

SUPPLEMENTARY EXERCISES FOR CHAPTER 12

1. a) Suppose that the equation holds. If $x = 1$, then the left-hand side is 1; therefore the right-hand side must be 1, as well, and that forces y and z to be 1. Similarly, if $x = 0$, then equality holds if and only if $y = z = 0$. Therefore the only solutions are $(1,1,1)$ and $(0,0,0)$.

b) If $x = 1$, then the right-hand side equals 1, so in order for there to be equality we need $y + z = 1$, whence $y = 1$ or $z = 1$. Thus $(1,1,0)$, $(1,0,1)$, and $(1,1,1)$ are all solutions. If $x = 0$, then the left-hand side is 0, so we need $yz = 0$ in order for equality to hold, whence $y = 0$ or $z = 0$. Thus $(0,0,0)$, $(0,0,1)$, and $(0,1,0)$ are also solutions.

c) There are no solutions here. If any of the variables equals 1, then the left-hand side is 0 and the right-hand side is 1; if all the variables are 0, then the left-hand side is 1 and the right-hand side is 0.

3. In each case we form $\overline{F(\bar{x}_1, \ldots, \bar{x}_n)}$ and simplify. If we get back to what we originally started with, then the function is self-dual; if what we obtain is not equivalent to what we began with, then the function is not self-dual. The simplification is done using the identities for Boolean algebra, especially De Morgan's laws.

a) $\bar{\bar{x}} = x$, so the function is self-dual.

b) $\overline{(\bar{x}\,\bar{y} + \bar{\bar{x}}\bar{\bar{y}})} = \overline{(\bar{x}\,\bar{y} + x\,y)}$, which is the complement of what we originally had. Thus this is as far from being self-dual as it can possibly be.

c) $\overline{(\bar{x} + \bar{y})} = \bar{\bar{x}}\bar{\bar{y}} = x\,y$, which is certainly not equivalent to $x + y$. Therefore this is not self-dual.

d) We first simplify the expression, using the distributive law and the fact that $x + \bar{x} = 1$ to rewrite our function as $F(x,y) = y$. Now, as in part **(a)**, we see that it is indeed self-dual.

5. The reasoning here is essentially the same as in Exercise 31 in Section 12.1. To specify all the values of a self-dual function, we are free to specify the values of $F(1, x_2, x_3, \ldots, x_n)$, and we can do this in $2^{2^{n-1}}$ ways, since there are 2^{n-1} different elements at which we can choose to make the function value either 0 or 1. Once we have specified these values, the values of $F(0, x_2, x_3, \ldots, x_n)$ are all determined by the definition of self-duality, so no further choices are possible. Therefore the answer is $2^{2^{n-1}}$.

7. **a)** At every point in the domain, it is certainly the case that if $F(x_1,\ldots,x_n)=1$, then $(F+G)(x_1,\ldots,x_n)=F(x_1,\ldots,x_n)+G(x_1,\ldots,x_n)=1+G(x_1,\ldots,x_n)=1$, no matter what value G has at that point. Thus by definition $F\leq F+G$.

b) This is dual to the first part. At every point in the domain, it is certainly the case that if $F(x_1,\ldots,x_n)=0$, then $(FG)(x_1,\ldots,x_n)=F(x_1,\ldots,x_n)G(x_1,\ldots,x_n)=0\cdot G(x_1,\ldots,x_n)=0$, no matter what value G has at that point. The contrapositive of this statement is that if $(FG)(x_1,\ldots,x_n)=1$, then $F(x_1,\ldots,x_n)=1$. Thus by definition $FG\leq F$.

9. We need to show that this relation is reflexive, antisymmetric, and transitive. That $F\leq F$ (reflexivity) is simply the tautology "if $F(x_1,\ldots,x_n)=1$, then $F(x_1,\ldots,x_n)=1$." For antisymmetry, suppose that $F\leq G$ and $G\leq F$. Then the definition of the relation says that $F(x_1,\ldots,x_n)=1$ if and only if $G(x_1,\ldots,x_n)=1$, which is the definition of equality between functions, so $F=G$. Finally, for transitivity, suppose that $F\leq G$ and $G\leq H$. We want to show that $F\leq H$. So suppose that $F(x_1,\ldots,x_n)=1$. Then by the first inequality $G(x_1,\ldots,x_n)=1$, whence by the second inequality $H(x_1,\ldots,x_n)=1$, as desired.

11. None of these are identities. The counterexample $x=1$, $y=z=0$ works for all three.

a) We have $1\mid(0\mid 0)=1\mid 1=0$, whereas $(1\mid 0)\mid 0=1\mid 0=1$.

b) We have $1\downarrow(0\downarrow 0)=1\downarrow 1=0$, whereas $(1\downarrow 0)\downarrow(1\downarrow 0)=0\downarrow 0=1$.

c) We have $1\downarrow(0\mid 0)=1\downarrow 1=0$, whereas $(1\downarrow 0)\mid(1\downarrow 0)=0\mid 0=1$.

13. This is clear from the definitions. The given operation applied to x and y is defined to be 1 if and only if $x=y$, while *XOR* applied to x and y is defined (preamble to Exercise 24 in Section 12.1) to be 1 if and only if $x\neq y$.

15. We show this to be true with a table. Since the fifth and seventh columns are equal, the equation is an identity.

x	y	z	$x\odot y$	$(x\odot y)\odot z$	$y\odot z$	$x\odot(y\odot z)$
1	1	1	1	1	1	1
1	1	0	1	0	0	0
1	0	1	0	0	0	0
1	0	0	0	1	1	1
0	1	1	0	0	1	0
0	1	0	0	1	0	1
0	0	1	1	1	0	1
0	0	0	1	0	1	0

17. In each case we can actually list all the functions.

a) The only function values we can get are x, y, 0, 1, $\overline{x}$, and $\overline{y}$, since applying complementation twice does not give us anything new. Therefore the answer is 6.

b) Since $s\cdot s=s$, $s\cdot 1=s$, and $s\cdot 0=0$ for all s, the only functions we can get are x, y, 0, 1, and xy. Therefore the answer is 5.

c) By duality the answer here has to be the same as the answer to part **(b)**, namely 5.

d) We can get the 6 distinct functions x, y, 0, 1, xy and $x+y$. Any further applications of these operations, however, returns us to one of these functions. For example, $xy+x=x$.

19. The sum bit is the exclusive *OR* of the inputs, and the carry bit is their product. Therefore we need only two gates to form the half adder if we allow an XOR gate and an AND gate.

21. We need to figure out which combinations of values for x_1, x_2, and x_3 cause the inequality $-x_1+x_2+2x_3 \geq 1/2$ to be satisfied. Clearly this will be true if $x_3 = 1$. If $x_3 = 0$, then it will be true if and only if $x_2 = 1$ and $x_1 = 0$. Thus a Boolean expression for this function is $x_3 + \overline{x}_1 x_2$.

23. We prove this by contradiction. Suppose that a, b, and T are such that $ax + by \geq T$ if and only if $x \oplus y = 1$, i.e., if and only if either $x = 1$ and $y = 0$, or else $x = 0$ and $y = 1$. Thus for the first case we need $a \geq T$, and for the second we need $b \geq T$. Since we need $ax + by < T$ for $x = y = 0$, we know that $T > 0$. Hence in particular b is positive. Therefore we have $a + b > a \geq T$, which contradicts the fact that $1 \oplus 1 = 0$ (requiring $a + b \leq T$).

WRITING PROJECTS FOR CHAPTER 12

Books and articles indicated by bracketed symbols below are listed near the end of this manual. You should also read the general comments and advice you will find there about researching and writing these essays.

1. Martin Gardner is probably the best writer of mathematical material for the general public. For this project, look at [Ga1].

2. Try a good book on circuits, such as [Ch].

3. Try a good book on circuits, such as [Ch].

4. Try a good book on circuits, such as [Ch].

5. Try a good book on circuits, such as [Ch].

6. Try a good book on circuits, such as [Ch].

7. See Section 9.5 of [HiPe1].

8. Try a good book on switching theory, such as [Ko1].

9. Try a good book on switching theory, such as [Ko1]. Also see [HiPe1].

10. Try a good book on switching theory, such as [Ko1].

11. The classic paper on this algorithm is [RuSa], but good books on circuits should mention it (the second author of that paper has written some—check his Web page).

12. Try a good book on switching theory, such as [Ko1].