MATH 2418 Linear Algebra. Week 2

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Summary of this Week's Goals

This week we will discuss Sections 1.3 (Matrices) and 2.1 (Vectors and Linear Equations) of our text. By the end of the week, you should be able to interpret a matrix as a collection of vectors, understand whether the columns of a matrix are linearly dependent or independent, multiply a matrix by a vector, describe the row picture and column pictures of matrix multiplication by a vector and describe the row picture and column pictures associated with the linear system $Ax = b$. You should know three possible solution scenarios associated with the system $Ax = b$: (1) a unique solution exists; (2) no solution exists; or (3) an infinite number of solutions exist. For simple examples where a unique solution exists, you should be able to express the solution in the form $x = A^{-1}b$.

Announcements

- Contact Information: James W. Miller (James.Miller3@utdallas.edu or jwm170630@utdallas.edu).
- Office Hours: Monday 5pm - 6pm in FA 2.106, or by appointment.
- Your first homework is due and you will take your first quiz this week during your problem session.

1.3 Matrices

What is a matrix?

- A rectangular array of numbers having m rows and n columns. We can give matrices names, just as we do scalars and vectors.

$$A = \begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}, B = \begin{bmatrix} \sqrt{2} & -1 \\ -1 & 1/2 \\ 0 & -\sqrt{3} \end{bmatrix}, C = \begin{bmatrix} 0 & -1/2 & 3 & 1 & 0 \\ 0 & 1/2 & 0 & 1 & -1 \\ 1 & 0 & -1 & 1 & 1 \\ -1 & -1 & 2/3 & 1 & 0 \end{bmatrix}$$

- The columns of a matrix may be regarded as a collection of n vectors with m components.
- The columns of a matrix are said to be "linearly dependent" if any column can be expressed as a linear combination of other columns. The columns are said to be "linearly independent" if no column can be expressed as a linear combination of other columns.
- Here are some examples matrices having columns which are linearly dependent. How can you tell?

$$A = \begin{bmatrix} 1 & 0 \\ -1 & 0 \end{bmatrix}, B = \begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix}, C = \begin{bmatrix} 1 & 1 & 0 \\ 2 & 1 & 1 \\ 3 & 1 & 2 \end{bmatrix}, D = \begin{bmatrix} 2 & -1 & -1 \\ 0 & 1 & -1 \\ -1 & 0 & 1 \end{bmatrix}, E = \begin{bmatrix} 1 & -1 & 2 \\ 2 & 1 & 4 \\ -1 & 0 & -2 \end{bmatrix}$$

Simple patterns indicating linear dependence among the columns of a matrix:

- A column of all zeros.
- One column is a scalar multiple of another. (The columns are "parallel vectors." Parallel vectors lie on the same line when drawn originating from the origin. They point in the same or opposite direction and may have different lengths.)
- One column is the sum or difference of the other two.
- The sum is zero across all rows.

More complex linear dependencies are harder to recognize at a glance, but we'll learn some more tricks for identifying linearly dependent columns later.

Multiplying a Matrix by a Vector

- An $m \times n$ matrix A can be multiplied by a vector $\boldsymbol{x}$ having n components. The result is a vector having m components.
- One way to perform the multiplication is to calculate the dot product of each row of the matrix with the vector $\boldsymbol{x}$. This is called "multiplying by rows."

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} (1,-1)\cdot(x_1,x_2) \\ (0,2)\cdot(x_1,x_2) \\ (1,1)\cdot(x_1,x_2) \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ 2x_2 \\ x_1 + x_2 \end{bmatrix}$$

- Another way to calculate the product $A\boldsymbol{x}$ is by calculating a linear combination of the columns of A. This is called "combining columns."

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = x_1 \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} + x_2 \begin{bmatrix} -1 \\ 2 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 - x_2 \\ 2x_2 \\ x_1 + x_2 \end{bmatrix}$$

- Examples:

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} 2 \\ 3 \end{bmatrix} = \begin{bmatrix} (1)(2) + (-1)(3) \\ (0)(2) + (2)(3) \end{bmatrix} = \begin{bmatrix} -1 \\ 6 \end{bmatrix}$$

$$\begin{bmatrix} -1 & 1 & 2 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} = \begin{bmatrix} (-1)(1) + (1)(2) + (2)(3) \\ (1)(1) + (0)(2) + (2)(3) \end{bmatrix} = \begin{bmatrix} 7 \\ 7 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_2 - x_1 \\ x_3 - x_2 \end{bmatrix}$$

- A square matrix containing the value "1" in each main diagonal entry (top left to bottom right) and zeros elsewhere is called an "identity matrix." The 3×3 identity matrix is shown below:

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Multiplying a vector $\boldsymbol{x}$ by an identity matrix returns the vector $\boldsymbol{x}$ unchanged. ($I\boldsymbol{x} = \boldsymbol{x}$).

The Equation $\boldsymbol{Ax} = \boldsymbol{b}$

- If A and $\boldsymbol{x}$ are known, $\boldsymbol{b}$ can be calculated via multiplication of a matrix and a vector.
- If A and $\boldsymbol{b}$ are known, the vector $\boldsymbol{x}$ can be found as the solution to a system of linear equations. This is the case we will consider carefully in the next few sections.
- There are various possibilities for solutions: no solution may exist; there may be a unique solution; or there may be an infinite number of solutions.
- There are two ways of viewing the problem of solving $A\boldsymbol{x} = \boldsymbol{b}$:
  - The row picture: Each row of the matrix equation (the inner product of a row of A with the vector $\boldsymbol{x}$ set equal to a component of $\boldsymbol{b}$) is an equation defining a line, plane or hyperplane in $\mathbb{R}^n$. The solution to the matrix equation is the intersection of the solution spaces corresponding to each row.
  - The column picture: The product $A\boldsymbol{x}$ represents a linear combination of the columns of A. The vector $\boldsymbol{x}$ is the set of coefficients defining the linear combination of the columns of A which produces the vector $\boldsymbol{b}$.

The 2×2 Case

- Example 1 (having a unique solution):

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

  - The row picture considers the equations

$$\begin{aligned} x - y &= 1 \\ 2y &= 2 \end{aligned}$$

    Each equation may be graphed as a line in the xy-plane and the solution is where the lines intersect (the point $(2, 1)$).
  - The column picture considers what linear combination of the columns of A produces $\boldsymbol{b}$.

$$x \begin{bmatrix} 1 \\ 0 \end{bmatrix} + y \begin{bmatrix} -1 \\ 2 \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

    The correct result occurs when $x = 2$ and $y = 1$, as we know from the row picture.
  - There is a unique solution which may be expressed in the form $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ as shown below:

$$\boldsymbol{x} = \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix} \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

    You'll learn how to find the matrix A^{-1} (called "A-inverse") later.

- Example 2 (having an infinite number of solutions):

$$\begin{bmatrix} 1 & -1 \\ -2 & 2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 1 \\ -2 \end{bmatrix}$$

  - The row picture considers the equations

$$\begin{aligned} x - y &= 1 \\ -2x + 2y &= -2 \end{aligned}$$

    Each equation may be graphed as a line in the xy-plane and both lines turn out to be identical (the line $y = x - 1$).

- The column picture considers what linear combination of the columns of A produces $\boldsymbol{b}$.

$$x\begin{bmatrix}1\\-2\end{bmatrix}+y\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}1\\-2\end{bmatrix}$$

  There are many combinations possible for x and y which produce the correct result, as we know from the row picture. The vectors (1, 0) and (0, -1) are obvious solutions, but many more are possible.

- All ordered pairs of the form $(x, x-1)$ are solutions to the system. Because the columns of A are linearly dependent, the matrix does not have an inverse.

• Example 3 (having no solutions):

$$\begin{bmatrix}1&-1\\-2&2\end{bmatrix}\begin{bmatrix}x\\y\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$$

- The row picture considers the equations

$$\begin{aligned}x-y&=1\\-2x+2y&=2\end{aligned}$$

  Each equation may be graphed as a line in the xy-plane. The lines are parallel and never intersect.

- The column picture considers what linear combination of the columns of A produces $\boldsymbol{b}$.

$$x\begin{bmatrix}1\\-2\end{bmatrix}+y\begin{bmatrix}-1\\2\end{bmatrix}=\begin{bmatrix}1\\2\end{bmatrix}$$

  The columns of A are scalar multiples of each other. They lie on the same line, so all linear combinations of the columns must also lie on the line. The vector (1, 2) does not lie on the line, so no solution is possible.

- There are no solutions to this system. Because the columns of A are linearly dependent, the matrix does not have an inverse.

The 3×3 Case

• Example 1 (having a unique solution):

$$\begin{bmatrix}1&0&0\\-1&1&0\\0&-1&1\end{bmatrix}\begin{bmatrix}x_1\\x_2\\x_3\end{bmatrix}=\begin{bmatrix}1\\3\\5\end{bmatrix}$$

- The row picture considers the equations

$$\begin{aligned}x_1&=1\\x_2-x_1&=3\\x_3-x_2&=5\end{aligned}$$

  Each equation defines a plane in $\mathbb{R}^3$. The intersection of two of these planes is a line in $\mathbb{R}^3$. The line intersects the third plane at a single point—the solution $(1, 4, 9)$.

- The column picture considers what linear combination of the columns of A produces $\boldsymbol{b}$.

$$x_1\begin{bmatrix}1\\-1\\0\end{bmatrix}+x_2\begin{bmatrix}0\\1\\-1\end{bmatrix}+x_3\begin{bmatrix}0\\0\\1\end{bmatrix}=\begin{bmatrix}1\\3\\5\end{bmatrix}$$

  The correct result occurs when $x_1 = 1$, $x_2 = 4$ and $x_3 = 9$, as we know from the row picture.

- There is a unique solution which may be expressed in the form $\boldsymbol{x} = A^{-1}\boldsymbol{b}$ as shown below:

$$\boldsymbol{x} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix} \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix} = \begin{bmatrix} 1 \\ 4 \\ 9 \end{bmatrix}$$

  You'll learn how to find the matrix A^{-1} (called "A-inverse") later.

- The matrix A in this example is called a "difference matrix" because multiplication by a vector produces a vector containing the differences of the components of the vector, with the first component being the difference between x_1 and 0. It should not be surprising that the inverse of this matrix is a matrix which produces sums. Notice also that both matrices are triangular. This actually makes them very easy to work with and invert.

• Example 2 (having an infinite number of solutions):

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

- The row picture considers the equations

$$\begin{aligned} x_1 - x_3 &= 2 \\ x_2 - x_1 &= -1 \\ x_3 - x_2 &= -1 \end{aligned}$$

  Each equation defines a plane in $\mathbb{R}^3$. The intersection of two of these planes is a line in $\mathbb{R}^3$. The third plane intersects the first two along the entire line. The line consists of all points having the form $(2 + x_3, 1 + x_3, x_3)$.

- The column picture considers what linear combination of the columns of A produces $\boldsymbol{b}$.

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ -1 \\ -1 \end{bmatrix}$$

  There are many combinations possible for x_1, x_2 and x_3 which produce the correct result, as we know from the row picture. The vectors (2, 1, 0), (1, 0, -1) and (0, -1, -2) are obvious solutions, but many more are possible.

- All ordered pairs of the form $(2 + x_3, 1 + x_3, x_3)$ are solutions to the system. The solutions may also be expressed as $(x_2 + 1, x_2, x_2 - 1)$ or $(x_1, x_1 - 1, x_1 - 2)$. Because the columns of A are linearly dependent, the matrix does not have an inverse.

- The matrix A in this example is called a "cyclic difference matrix" because multiplication by a vector produces a vector containing the differences of the components of the vector, with the first component being the difference between x_1 and x_3. Because the columns add to a zero vector, it is not hard to write one column as a linear combination of other columns, thus showing the columns are linearly dependent.

• Example 3 (having no solutions):

$$\begin{bmatrix} 1 & 0 & -1 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

– The row picture considers the equations

$$\begin{aligned} x_1 - x_3 &= 1 \\ x_2 - x_1 &= 3 \\ x_3 - x_2 &= 5 \end{aligned}$$

Each equation defines a plane in $\mathbb{R}^3$. The intersection of two of these planes is a line in $\mathbb{R}^3$. This line runs parallel to the plane defined by the third equation and never intersects it. Therefore, no solution exists.

– The column picture considers what linear combination of the columns of A produces $\boldsymbol{b}$.

$$x_1 \begin{bmatrix} 1 \\ -1 \\ 0 \end{bmatrix} + x_2 \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix} + x_3 \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \\ 5 \end{bmatrix}$$

The columns of A all lie in the plane of vectors for which the sum of the coordinates of the vector is zero. The linear combinations of these columns must also lie in that plane, producing a vector whose cooridinates sum to zero. Since the vector $(1, 3, 5)$ does not have coodinates whose sum is zero, no solution exists.

– There are no solutions to this system. Because the columns of A are linearly dependent, the matrix does not have an inverse.

Finding A^{-1} When a Unique Solution Exists

- In the examples we looked at, we found two examples of inverse matrices:

$$\begin{bmatrix} 1 & -1 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 1/2 \\ 0 & 1/2 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ 0 & -1 & 1 \end{bmatrix}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

- We'll learn other ways later, but here is a way you can find an inverse for yourself in your homework for this week:
  – Write the system $A\mathbf{x} = \mathbf{b}$ as a system of equations using the row picture we have discussed this week. The equations relate (x_1, x_2, x_3) to (b_1, b_2, b_3).
  – Solve the system, expressing (x_1, x_2, x_3) in terms of (b_1, b_2, b_3). Solve the system using any method you learned in an algebra class. For example, you can use the substitution method to solve for one variable in terms of others thus reducing the number of unknown variables, or you can combine equations by adding a multiple of one equation to another to eliminate variables.
  – Once you have (x_1, x_2, x_3) expressed in terms of (b_1, b_2, b_3), write the equations in the form $\mathbf{x} = A^{-1}\mathbf{b}$.
- Example. Solve the following system of the form $A\mathbf{x} = \mathbf{b}$:

$$\begin{bmatrix} 2 & 1 & -1 \\ 1 & -1 & 1 \\ -1 & 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

  – The row picture considers the equations

$$\begin{aligned} 2x_1 + x_2 - x_3 &= b_1 \\ x_1 - x_2 + x_3 &= b_2 \\ -x_1 + x_2 + 2x_3 &= b_3 \end{aligned}$$

– Replacing the third equation with the sum of the second and third equations gives

$$\begin{aligned} 2x_1 + x_2 - x_3 &= b_1 \\ x_1 - x_2 + x_3 &= b_2 \\ 3x_3 &= b_2 + b_3 \end{aligned}$$

– Multiplying the second equation by 2 and subtracting the first equation gives

$$\begin{aligned} 2x_1 + x_2 - x_3 &= b_1 \\ -3x_2 + 3x_3 &= -b_1 + 2b_2 \\ 3x_3 &= b_2 + b_3 \end{aligned}$$

– Next, solve the first equation for x_1, the second for x_2 and the third for x_3.

$$\begin{aligned} x_1 &= \frac{1}{2}b_1 - \frac{1}{2}x_2 + \frac{1}{2}x_3 \\ x_2 &= \frac{1}{3}b_1 - \frac{2}{3}b_2 + x_3 \\ x_3 &= \frac{1}{3}b_2 + \frac{1}{3}b_3 \end{aligned}$$

– Substitute the expression found for x_3 into the second equation and solve for x_2 , then substitute the expressions for x_3 and x_2 into the first equation and solve for x_1.

$$\begin{aligned} x_1 &= \frac{1}{2}b_1 - \frac{1}{2}\left(\frac{1}{3}b_1 - \frac{1}{3}b_2 + \frac{1}{3}b_3\right) + \frac{1}{2}\left(\frac{1}{3}b_2 + \frac{1}{3}b_3\right) = \frac{1}{3}b_1 + \frac{1}{3}b_2 \\ x_2 &= \frac{1}{3}b_1 - \frac{2}{3}b_2 + \left(\frac{1}{3}b_2 + \frac{1}{3}b_3\right) = \frac{1}{3}b_1 - \frac{1}{3}b_2 + \frac{1}{3}b_3 \\ x_3 &= \frac{1}{3}b_2 + \frac{1}{3}b_3 \end{aligned}$$

– Finally, the solution can be expressed in the following way. The matrix A^{-1} can be picked out from the expression below.

$$\mathbf{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 1/3 & 1/3 & 0 \\ 1/3 & -1/3 & 1/3 \\ 0 & 1/3 & 1/3 \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = A^{-1}\mathbf{b}$$

- The matrix A^{-1} reverses the operations that A performs when it is multiplied by $\mathbf{x}$. So if $A\mathbf{x} = \mathbf{b}$, then $A^{-1}\mathbf{b} = \mathbf{x}$. The operations performed by A are only "reversible" if the columns of A are linearly independent.