MATH 2418: Linear Algebra

Assignment# 5

Due: `Tuesday, 09/27/2022, 11:59pm` Term: `Fall 2022`

[First Name] [Last Name] [Net ID]

Recommended Problems (do not turn in): Sec 2.5: 1, 5, 6, 7, 11, 12, 13, 18, 22, 25, 27, 29, 44. Sec 2.6: 1, 3, 4, 6, 8, 9, 10, 17, 22, 23.

1. Consider the matrix

$$A = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 4 \\ 5 & 5 & 9 \end{bmatrix}.$$

(a) Use elementary row operations to reduce A into the identity matrix I.

(b) List all corresponding elementary matrices.

(c) Write A^{-1} as a product of elementary matrices.

Solution:

(a)

$$\begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 4 \\ 5 & 5 & 9 \end{bmatrix} \xrightarrow{R_2-2R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 5 & 5 & 9 \end{bmatrix} \xrightarrow{R_3-5R_1} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 5 & -1 \end{bmatrix}$$
$$\xrightarrow{R_3-5R_2} \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{R_1+2R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix} \xrightarrow{-R_3} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

(b) List all corresponding elementary matrices.

$$E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -5 & 0 & 1 \end{bmatrix} \quad E_{32} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -5 & 1 \end{bmatrix}$$
$$E_{13} = \begin{bmatrix} 1 & 0 & 2 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad E_{33} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{bmatrix}$$

(c) Since these are the elementary matrices that reduced A to I, we have

$$I = E_{33}E_{13}E_{32}E_{31}E_{21}A.$$

Then, multiplying by A^{-1} on the right, we see that

$$A^{-1} = E_{33}E_{13}E_{32}E_{31}E_{21}.$$

2. Determine whether the following matrices are invertible. If they are, find the inverses. If not, justify your answer.

$$A_1 = \begin{bmatrix} 2 & 5 \\ 3 & 5 \end{bmatrix},\ A_2 = \begin{bmatrix} 2 & 5 \\ 0 & 5 \end{bmatrix},\ A_3 = \begin{bmatrix} 2 & 0 \\ 3 & 5 \end{bmatrix},\ A_4 = \begin{bmatrix} 2 & 2 \\ 3 & 3 \end{bmatrix},\ A_5 = \begin{bmatrix} 3 & 5 \\ 3 & 5 \end{bmatrix}.$$

Solution: Using algebra test for invertibility: for 2×2 matrices $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$, if $ad - bc = 0$, A is singular. If $ad - bc \neq 0$, A is invertible.

(a) For A_1,

$$2 \times 5 - 5 \times 3 = -5 \neq 0$$

Thus A_1 is invertible.

$$A_1^{-1} = \frac{1}{-5}\begin{bmatrix} 5 & -5 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} -1 & 1 \\ \frac{3}{5} & -\frac{2}{5} \end{bmatrix}.$$

(b) A_2 is invertible because A_2 has two nonzero pivots, and $2 \times 5 - 0 \times 5 = 10 \neq 0$.

$$A_2^{-1} = \frac{1}{10}\begin{bmatrix} 5 & -5 \\ 0 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & \frac{1}{5} \end{bmatrix}.$$

(c) For A_3,

$$2 \times 5 - 0 \times 3 = 10 \neq 0$$

Thus A_3 is invertible.

$$A_3^{-1} = \frac{1}{10}\begin{bmatrix} 5 & 0 \\ -3 & 2 \end{bmatrix} = \begin{bmatrix} \frac{1}{2} & 0 \\ -\frac{3}{10} & \frac{1}{5} \end{bmatrix}.$$

(d)

$$2 \times 3 - 2 \times 3 = 0$$

Thus A_4 is not invertible.

(e)

$$3 \times 5 - 5 \times 3 = 0$$

Thus A_5 is not invertible.

3. Let

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Find the inverse matrix A^{-1}.

Solution: Let:

$$B_1 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ B_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}, \ B_3 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 3 & 1 \end{bmatrix}, \ B_4 = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Thus, A may be written as the product of matrices $A = B_1B_2B_3B_4$, and A^{-1} may be written as:

$$A^{-1} = B_4^{-1}B_3^{-1}B_2^{-1}B_1^{-1}$$

Since B_1 is an elementary matrix which multiplies row 2 by the scalar -2, B_1^{-1} is that elementary matrix which multiplies row 2 by $-1/2$, namely:

$$B_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Next, B_2 is the elementary matrix which interchanges row 2 with row 3, hence B_2^{-1} is that elementary matrix which performs the same row operation:

$$B_2^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix}$$

Now observing that B_3 is the elementary matrix which adds $3 \times$ row 2 to row 3, it follows that B_3^{-1} is that elementary matrix which subtracts $3 \times$ row 2 from row 3, namely:

$$B_3^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix}$$

Since B_4 is the elementary matrix which adds $2 \times$ row 1 to row 2, it follows B_4^{-1} is the elementary matrix which subtracts $2 \times$ row 1 from row 2:

$$B_4^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

Finally, recalling that $A^{-1} = B_4^{-1}B_3^{-1}B_2^{-1}B_1^{-1}$, we obtain:

$$A^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & -3 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1/2 & 0 \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 0 & 1 \\ 0 & -1/2 & -3 \end{bmatrix}$$

4. For which value of x, is

$$\begin{bmatrix} 5 & 3 & 0 \\ 1 & 0 & 1 \\ -3 & -3 & x \end{bmatrix}$$

not invertible?

Solution: Row operations reduce the matrix to upper triangular form:

$$\begin{bmatrix} 5 & 3 & 0 \\ 1 & 0 & 1 \\ -3 & -3 & x \end{bmatrix} \xrightarrow[l_{21}=\frac{1}{5}]{R_2-\frac{1}{5}R_1} \begin{bmatrix} 5 & 3 & 0 \\ 0 & -\frac{3}{5} & 1 \\ -3 & -3 & x \end{bmatrix} \xrightarrow[l_{31}=-\frac{3}{5}]{R_3+\frac{3}{5}R_1} \begin{bmatrix} 5 & 3 & 0 \\ 0 & -\frac{3}{5} & 1 \\ 0 & -\frac{6}{5} & x \end{bmatrix}. \xrightarrow[l_{32}=2]{R_3-2R_2} \begin{bmatrix} 5 & 3 & 0 \\ 0 & -\frac{3}{5} & 1 \\ 0 & 0 & x-2 \end{bmatrix}$$

From the upper triangular form, we can see that the pivots are 5, $-\frac{3}{5}$, and $x-2$. For a 3×3 matrix to be invertible, it must have 3 pivots. When $x - 2 = 0 \implies x = 2$, the matrix is not invertible.

5. (a) Find the LU decomposition of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 0 & 2 \\ 6 & 3 & 5 \end{bmatrix}.$$

(b) Find the LDU decomposition of the matrix

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 0 & 2 \\ 6 & 3 & 5 \end{bmatrix}.$$

Solution:

(a) We reduce A into an upper triangular U as following:

$$A = \begin{bmatrix} 2 & 1 & 1 \\ 4 & 0 & 2 \\ 6 & 3 & 5 \end{bmatrix} \xRightarrow[E_{21}]{R_2 - 2R_1} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -2 & 0 \\ 6 & 3 & 5 \end{bmatrix} \xRightarrow[E_{31}]{R_3 - 3R_1} \begin{bmatrix} 2 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} = U$$

$$\text{And } E_{21} = \begin{bmatrix} 1 & 0 & 0 \\ -2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ E_{31} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ -3 & 0 & 1 \end{bmatrix},$$

Then the LU decomposition is $A = LU$ with

$$L = E_{21}^{-1} E_{31}^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$$

(b) The LDU decomposition is $A = LDU'$

$$DU' = U = \begin{bmatrix} 2 & 1 & 1 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix},$$

Therefore,

$$D = \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix}, \ U' = \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \text{ and } L = \begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix}.$$

Thus, the LDU decomposition of the Matrix A is

$$\begin{bmatrix} 1 & 0 & 0 \\ 2 & 1 & 0 \\ 3 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & \frac{1}{2} & \frac{1}{2} \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

6. Let $\mathbf{b} = \begin{bmatrix} 8 \\ 30 \\ -48 \end{bmatrix}$ and $A = \begin{bmatrix} 1 & -3 & -6 \\ 3 & -11 & -22 \\ -1 & 14 & 32 \end{bmatrix}$. Use the following steps to solve the system $A\mathbf{x} = \mathbf{b}$ by using the LU-decomposition of A.

(a) Find the LU-decomposition of A, $A = LU$.

(b) Solve $L\mathbf{y} = \mathbf{b}$ by forward substitution.

(c) Solve $U\mathbf{x} = \mathbf{y}$ by back substitution.

(d) Find the solution of $A\mathbf{x} = \mathbf{b}$.

Solution:

(a) Row operations reduce A to upper triangular matrix:

$$\begin{bmatrix} 1 & -3 & -6 \\ 3 & -11 & -22 \\ -1 & 14 & 32 \end{bmatrix} \xrightarrow[l_{21}=3]{R_2-3R_1} \begin{bmatrix} 1 & -3 & -6 \\ 0 & -2 & -4 \\ -1 & 14 & 32 \end{bmatrix} \xrightarrow[l_{31}=-1]{R_3+R_1} \begin{bmatrix} 1 & -3 & -6 \\ 0 & -2 & -4 \\ 0 & 11 & 26 \end{bmatrix} . \xrightarrow[l_{32}=-\frac{11}{2}]{R_3+\frac{11}{2}R_2} \begin{bmatrix} 1 & -3 & -6 \\ 0 & -2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

$$L = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & \frac{-11}{2} & 1 \end{bmatrix}$$

$$U = \begin{bmatrix} 1 & -3 & -6 \\ 0 & -2 & -4 \\ 0 & 0 & 4 \end{bmatrix}$$

The decomposition is

$$\begin{bmatrix} 1 & -3 & -6 \\ 3 & -11 & -22 \\ -1 & 14 & 32 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & \frac{-11}{2} & 1 \end{bmatrix} \begin{bmatrix} 1 & -3 & -6 \\ 0 & -2 & -4 \\ 0 & 0 & 4 \end{bmatrix}.$$

(b) Let $\mathbf{y} = \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix}$.Then we solve the equation

$$\begin{bmatrix} 1 & 0 & 0 \\ 3 & 1 & 0 \\ -1 & -\frac{11}{2} & 1 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \\ y_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 30 \\ -48 \end{bmatrix}.$$

We have $y_1 = 8$. Then $3y_1 - y_2 = 30 \implies 3 \cdot 8 + y_2 = 30 \implies 24 + y_2 = 30 \implies y_2 = 6$ and then $-y_1 - \frac{11}{2}y_2 + y_3 = -48 \implies y_3 = -7$. So the solution is $\mathbf{y} = \begin{bmatrix} 8 \\ 6 \\ -7 \end{bmatrix}$.

(c) We solve the equation

$$\begin{bmatrix} 1 & -3 & -6 \\ 0 & -2 & -4 \\ 0 & 0 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 8 \\ 6 \\ -7 \end{bmatrix}.$$

We get $4x_3 = -7 \implies x_3 = -\frac{7}{4}$. Then

$$-2x_2 - 4x_3 = 6 \implies -2x_2 - 4 \cdot \frac{-7}{4} = 6 \implies -2x_2 + 7 = 6 \implies x_2 = \frac{1}{2}$$

Finally,

$$x_1 - 3x_2 - 6x_3 = 8 \implies x_1 = 8 + \frac{3}{2} - \frac{21}{2} = -1.$$

Thus the solution is $\mathbf{x} = \begin{bmatrix} -1 \\ \frac{1}{2} \\ -\frac{7}{4} \end{bmatrix}$.

(d) Using $\mathbf{x}$ and $\mathbf{y}$ as found in parts (b) and (c), we have

$$A\mathbf{x} = (LU)\mathbf{x} = L(U\mathbf{x}) = L\mathbf{y} = \mathbf{b}.$$

Thus the solution to $A\mathbf{x} = \mathbf{b}$ is $\mathbf{x} = \begin{bmatrix} -1 \\ \frac{1}{2} \\ -\frac{7}{4} \end{bmatrix}$.