CHAPTER 2
Basic Structures: Sets, Functions, Sequences, Sums, and Matrices

SECTION 2.1 Sets

This exercise set (note that this is a "set" in the mathematical sense!) reinforces the concepts introduced in this section—set description, subset and containment, cardinality, power set, and Cartesian product. A few of the exercises (mostly some of the even-numbered ones) are a bit subtle. Keep in mind the distinction between "is an element of" and "is a subset of." Similarly, there is a big difference between $\emptyset$ and $\{\emptyset\}$. In dealing with sets, as in most of mathematics, it is extremely important to say exactly *what you mean.*

1. a) $\{1,-1\}$ **b)** $\{1,2,3,4,5,6,7,8,9,10,11\}$
c) $\{0,1,4,9,16,25,36,49,64,81\}$ **d)** $\emptyset$ ($\sqrt{2}$ is not an integer)

3. Recall that one set is a subset of another set if every element of the first set is also an element of the second. We just have to apply this definition here.

a) Because every nonstop flight is a flight, every element in the first set is also an element of the second, so the first set is a subset of the second. Because there are flights that do have intermediate stops (say, from New York to Atlanta with a stop in Detroit), the second set is not a subset of the first.

b) Because there is certainly at least one person who speaks English but not Chinese, and vice versa, neither set is a subset of the other.

c) Every flying squirrel is a living creature that can fly, so the first set is a subset of the second. Because birds, for instance, are leaving creatures that can fly but are not flying squirrels, the second set is not a subset of the first.

5. a) Yes; order and repetition do not matter.
b) No; the first set has one element, and the second has two elements.
c) No; the first set has no elements, and the second has one element (namely the empty set).

7. a) Since 2 is an integer greater than 1, 2 is an element of this set.
b) Since 2 is not a perfect square ($1^2 < 2$, but $n^2 > 2$ for $n > 1$), 2 is not an element of this set.
c) This set has two elements, and as we can clearly see, one of those elements is 2.
d) This set has two elements, and as we can clearly see, neither of those elements is 2. Both of the elements of this set are sets; 2 is a number, not a set.
e) This set has two elements, and as we can clearly see, neither of those elements is 2. Both of the elements of this set are sets; 2 is a number, not a set.
f) This set has just one element, namely the set $\{\{2\}\}$. So 2 is not an element of this set. Note that $\{2\}$ is not an element either, since $\{2\} \neq \{\{2\}\}$.

9. a) This is false, since the empty set has no elements.
b) This is false. The set on the right has only one element, namely the number 0, not the empty set.
c) This is false. In fact, the empty set has *no* proper subsets.

d) This is true. Every element of the set on the left is, vacuously, an element of the set on the right; and the set on the right contains an element, namely 0, that is not in the set on the left.

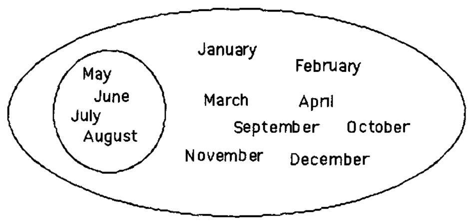
e) This is false. The set on the right has only one element, namely the number 0, not the set containing the number 0.

f) This is false. For one set to be a proper subset of another, the two sets cannot be equal.

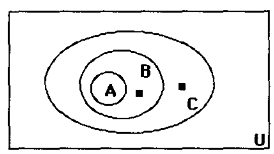
g) This is true. Every set is a subset of itself.

11. a) T (in fact x is the only element) b) T (every set is a subset of itself)
c) F (the only element of $\{x\}$ is a letter, not a set) d) T (in fact, $\{x\}$ is the only element)
e) T (the empty set is a subset of every set) f) F (the only element of $\{x\}$ is a letter, not a set)

13. The four months whose names don't contain the letter R form a subset of the set of twelve months, as shown here.



15. We put the subsets inside the supersets. We also put dots in certain regions to indicate that those regions are not empty (required by the fact that these are proper subset relations). Thus the answer is as shown.



17. We need to show that every element of A is also an element of C. Let $x \in A$. Then since $A \subseteq B$, we can conclude that $x \in B$. Furthermore, since $B \subseteq C$, the fact that $x \in B$ implies that $x \in C$, as we wished to show.

19. The cardinality of a set is the number of elements it has. The number of elements in its elements is irrelevant.
a) 1 b) 1 c) 2 d) 3

21. a) $\{\emptyset, \{a\}\}$ b) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ c) $\{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

23. a) Since the set we are working with has 3 elements, the power set has $2^3 = 8$ elements.
b) Since the set we are working with has 4 elements, the power set has $2^4 = 16$ elements.
c) The power set of the empty set has $2^0 = 1$ element. The power set of this set therefore has $2^1 = 2$ elements. In particular, it is $\{\emptyset, \{\emptyset\}\}$. (See Example 14.)

25. We need to prove two things, because this is an "if and only if" statement. First, let us prove the "if" part. We are given that $A \subseteq B$. We want to prove that the power set of A is a subset of the power set of B, which means that if $C \subseteq A$ then $C \subseteq B$. But this follows directly from Exercise 17. For the "only if" part, we are given that the power set of A is a subset of the power set of B. We must show that every element of A is also an element of B. So suppose a is an arbitrary element of A. Then $\{a\}$ is a subset of A, so it is an element of the power set of A. Since the power set of A is a subset of the power set of B, it follows that $\{a\}$ is an element of the power set of B, which means that $\{a\}$ is a subset of B. But this means that the element of $\{a\}$, namely a, is an element of B, as desired.

27. In each case we need to list all the ordered pairs, and there are $4 \times 2 = 8$ of them.
a) $\{(a,y),(a,z),(b,y),(b,z),(c,y),(c,z),(d,y),(d,z)\}$
b) $\{(y,a),(y,b),(y,c),(y,d),(z,a),(z,b),(z,c),(z,d)\}$

29. This is the set of triples (a,b,c), where a is an airline and b and c are cities. For example, (American, Rochester Hills Michigan, Middletown New Jersey) is an element of this Cartesian product. A useful subset of this set is the set of triples (a,b,c) for which a flies between b and c. For example, (Delta, Detroit, New York) is in this subset, but the triple mentioned earlier is not.

31. By definition, $\emptyset \times A$ consists of all pairs (x,a) such that $x \in \emptyset$ and $a \in A$. Since there are no elements $x \in \emptyset$, there are no such pairs, so $\emptyset \times A = \emptyset$. Similar reasoning shows that $A \times \emptyset = \emptyset$.

33. Recall that $A^2 = A \times A$. Therefore here we must list all the ordered pairs of elements of A.
a) $\{(0,0),(0,1),(0,3),(1,0),(1,1),(1,3),(3,0),(3,1),(3,3)\}$
b) $\{(1,1),(1,2),(1,a),(1,b),(2,1),(2,2),(2,a),(2,b),(a,1),(a,2),(a,a),(a,b),(b,1),(b,2),(b,a),(b,b)\}$

35. The Cartesian product $A \times B$ has mn elements. (This problem foreshadows the general discussion of counting in Chapter 5.) To see that this answer is correct, note that for each $a \in A$ there are n different elements $b \in B$ with which to form the pair (a,b). Since there are m different elements of A, each leading to n different pairs, there must be mn pairs altogether.

37. The Cartesian product $A \times A$ has m^2 elements. (This problem foreshadows the general discussion of counting in Chapter 6.) To see that this answer is correct, note that for each $a \in A$ there are m different elements $b \in A$ (including a itself) with which to form the pair (a,b). Since there are m different elements of A, each leading to m different pairs, there must be m^2 pairs altogether. Similarly, for each of the m^2 choices of a and b, there are m choices of c with which to form the triple (a,b,c) in A^3, so A^3 has m^3 elements. Continuing in this way, we see that A^n has m^n elements.

39. The only difference between $A \times B \times C$ and $(A \times B) \times C$ is parentheses, so for all practical purposes one can think of them as essentially the same thing. By Definition 9, the elements of $A \times B \times C$ consist of 3-tuples (a,b,c), where $a \in A$, $b \in B$, and $c \in C$. By Definition 8, the elements of $(A \times B) \times C$ consist of ordered pairs (p,c), where $p \in A \times B$ and $c \in C$, so the typical element of $(A \times B) \times C$ looks like $((a,b),c)$. A 3-tuple is a different creature from a 2-tuple, even if the 3-tuple and the 2-tuple in this case convey exactly the same information. To be more precise, there is a natural one-to-one correspondence between $A \times B \times C$ and $(A \times B) \times C$ given by $(a,b,c) \leftrightarrow ((a,b),c)$.

41. **a)** Every real number has its square not equal to -1. Alternatively, the square of a real number is never -1. This is true, since squares of real numbers are always nonnegative.
b) There exists an integer whose square is 2. This is false, since the only two numbers whose squares are 2, namely $\sqrt{2}$ and $-\sqrt{2}$, are not integers.
c) The square of every integer is positive. This is almost true, but not quite, since $0^2 \not> 0$.
d) There is a real number equal to its own square. This is true, since $x = 1$ (as well as $x = 0$) fits the bill.

43. In each case we want the set of all values of x in the domain (the set of integers) that satisfy the given equation or inequality.
a) The only integers whose squares are less than 3 are the integers whose absolute values are less than 2. So the truth set is $\{x \in \mathbf{Z} \mid x^2 < 3\} = \{-1,0,1\}$.

b) All negative integers satisfy this inequality, as do all nonnegative integers other than 0 and 1. So the truth set is $\{x \in \mathbf{Z} \mid x^2 > x\} = \mathbf{Z} - \{0,1\} = \{\ldots,-2,-1,2,3,4,\ldots\}$.
c) The only real number satisfying this equation is $x = -1/2$. Because this value is not in our domain, the truth set is empty: $\{x \in \mathbf{Z} \mid 2x+1=0\} = \emptyset$.

45. First we prove the statement mentioned in the hint. The "if" part is immediate from the definition of equality. The "only if" part is rather subtle. We want to show that if $\{\{a\},\{a,b\}\} = \{\{c\},\{c,d\}\}$, then $a=c$ and $b=d$. First consider the case in which $a \neq b$. Then $\{\{a\},\{a,b\}\}$ has exactly two elements, both of which are sets; exactly one of them contains one element, and exactly one of them contains two elements. Thus $\{\{c\},\{c,d\}\}$ must have the same property; hence c cannot equal d, and so $\{c\}$ is the element containing one element. Hence $\{a\}=\{c\}$, and so $a=c$. Also in this case the two-element elements $\{a,b\}$ and $\{c,d\}$ must be equal, and since $b \neq a = c$, we must have $b=d$. The other possibility is that $a=b$. Then $\{\{a\},\{a,b\}\} = \{\{a\}\}$, a set with one element. Hence $\{\{c\},\{c,d\}\}$ must also have only one element, which can only happen when $c=d$ and the set is $\{\{c\}\}$. It then follows that $a=c$, and hence $b=d$, as well.

Now there is really nothing else to prove. The property that we want ordered pairs to have is precisely the one that we just proved is satisfied by this definition. Furthermore, if we look at the proof, then it is clear how to "recover" both a and b from $\{\{a\},\{a,b\}\}$. If this set has two elements, then a is the unique element in the one-element element of this set, and b is the unique member of the two-element element of this set other than a. If this set has only one element, then a and b are both equal to the unique element of the unique element of this set.

47. We can do this recursively, using the idea from Section 5.4 of reducing a problem to a smaller instance of the same problem. Suppose that the elements of the set in question are listed: $A = \{a_1,a_2,a_3,\ldots,a_n\}$. First we will write down all the subsets that do not involve a_n. This is just the same problem we are talking about all over again, but with a smaller set—one with just $n-1$ elements. We do this by the process we are currently describing. Then we write these same subsets down again, but this time adjoin a_n to each one. Each subset of A will have been written down, then—first all those that do not include a_n, and then all those that do.

For example, using this procedure the subsets of $\{p,d,q\}$ would be listed in the order $\emptyset$, $\{p\}$, $\{d\}$, $\{p,d\}$, $\{q\}$, $\{p,q\}$, $\{d,q\}$, $\{p,d,q\}$.

An alternative solution is given in the answer key in the back of the textbook.

SECTION 2.2 Set Operations

Most of the exercises involving operations on sets can be done fairly routinely by following the definitions. It is important to understand what it means for two sets to be equal and how to prove that two given sets are equal—using membership tables, using the definition to reduce the problem to logic, or showing that each is a subset of the other; see, for example, Exercises 5–24. It is often helpful when looking at operations on sets to draw the Venn diagram, even if you are not asked to do so. The **symmetric difference** *is a fairly important set operation not discussed in the section; it is developed in Exercises 32–43. Two other new concepts,* **multisets** *and* **fuzzy sets***, are also introduced in this set of exercises.*

1. **a)** the set of students who live within one mile of school and walk to class (only students who do both of these things are in the intersection)

b) the set of students who either live within one mile of school or walk to class (or, it goes without saying, both)

c) the set of students who live within one mile of school but do not walk to class

d) the set of students who live more than a mile from school but nevertheless walk to class

3. **a)** We include all numbers that are in one or both of the sets, obtaining $\{0,1,2,3,4,5,6\}$.

b) There is only one number in both of these sets, so the answer is $\{3\}$.

c) The set of numbers in A but not in B is $\{1,2,4,5\}$.

d) The set of numbers in B but not in A is $\{0,6\}$.

5. By definition $\overline{\overline{A}}$ is the set of elements of the universal set that are not in $\overline{A}$. Not being in $\overline{A}$ means being in A. Thus $\overline{\overline{A}}$ is the same set as A. We can give this proof in symbols as follows: $\overline{\overline{A}} = \{\, x \mid \neg x \in \overline{A}\,\} = \{\, x \mid \neg\neg x \in A\,\} = \{\, x \mid x \in A\,\} = A$.

7. These identities are straightforward applications of the definitions and are most easily stated using set-builder notation. Recall that **T** means the proposition that is always true, and **F** means the proposition that is always false.

a) $A \cup U = \{\, x \mid x \in A \vee x \in U\,\} = \{\, x \mid x \in A \vee \mathbf{T}\,\} = \{\, x \mid \mathbf{T}\,\} = U$

b) $A \cap \emptyset = \{\, x \mid x \in A \wedge x \in \emptyset\,\} = \{\, x \mid x \in A \wedge \mathbf{F}\,\} = \{\, x \mid \mathbf{F}\,\} = \emptyset$

9. **a)** We must show that every element (of the universal set) is in $A \cup \overline{A}$. This is clear, since every element is either in A (and hence in that union) or else not in A (and hence in that union).

b) We must show that no element is in $A \cap \overline{A}$. This is clear, since $A \cap \overline{A}$ consists of elements that are in A and not in A at the same time, obviously an impossibility.

11. These follow directly from the corresponding properties for the logical operations *or* and *and*.

a) $A \cup B = \{\, x \mid x \in A \vee x \in B\,\} = \{\, x \mid x \in B \vee x \in A\,\} = B \cup A$

b) $A \cap B = \{\, x \mid x \in A \wedge x \in B\,\} = \{\, x \mid x \in B \wedge x \in A\,\} = B \cap A$

13. We will show that these two sets are equal by showing that each is a subset of the other. Suppose $x \in A \cap (A \cup B)$. Then $x \in A$ and $x \in A \cup B$ by the definition of intersection. Since $x \in A$, we have proved that the left-hand side is a subset of the right-hand side. Conversely, let $x \in A$. Then by the definition of union, $x \in A \cup B$ as well. Since both of these are true, $x \in A \cap (A \cup B)$ by the definition of intersection, and we have shown that the right-hand side is a subset of the left-hand side.

15. This exercise asks for a proof of one of De Morgan's laws for sets. The primary way to show that two sets are equal is to show that each is a subset of the other. In other words, to show that $X = Y$, we must show that whenever $x \in X$, it follows that $x \in Y$, and that whenever $x \in Y$, it follows that $x \in X$. Exercises 5–7 could also have been done this way, but it was easier in those cases to reduce the problems to the corresponding problems of logic. Here, too, we can reduce the problem to logic and invoke De Morgan's law for logic, but this problem requests specific proof techniques.

a) This proof is similar to the proof of the dual property, given in Example 10. Suppose $x \in \overline{A \cup B}$. Then $x \notin A \cup B$, which means that x is in neither A nor B. In other words, $x \notin A$ and $x \notin B$. This is equivalent to saying that $x \in \overline{A}$ and $x \in \overline{B}$. Therefore $x \in \overline{A} \cap \overline{B}$, as desired. Conversely, if $x \in \overline{A} \cap \overline{B}$, then $x \in \overline{A}$ and $x \in \overline{B}$. This means $x \notin A$ and $x \notin B$, so x cannot be in the union of A and B. Since $x \notin A \cup B$, we conclude that $x \in \overline{A \cup B}$, as desired.

b) The following membership table gives the desired equality, since columns four and seven are identical.

A	B	$A \cup B$	$\overline{A \cup B}$	$\overline{A}$	$\overline{B}$	$\overline{A} \cap \overline{B}$
1	1	1	0	0	0	0
1	0	1	0	0	1	0
0	1	1	0	1	0	0
0	0	0	1	1	1	1

17. This exercise asks for a proof of a generalization of one of De Morgan's laws for sets from two sets to three.

a) This proof is similar to the proof of the two-set property, given in Example 10. Suppose $x \in \overline{A \cap B \cap C}$. Then $x \notin A \cap B \cap C$, which means that x fails to be in at least one of these three sets. In other words, $x \notin A$ or $x \notin B$ or $x \notin C$. This is equivalent to saying that $x \in \overline{A}$ or $x \in \overline{B}$ or $x \in \overline{C}$. Therefore $x \in \overline{A} \cup \overline{B} \cup \overline{C}$, as desired. Conversely, if $x \in \overline{A} \cup \overline{B} \cup \overline{C}$, then $x \in \overline{A}$ or $x \in \overline{B}$ or $x \in \overline{C}$. This means $x \notin A$ or $x \notin B$ or $x \notin C$, so x cannot be in the intersection of A, B and C. Since $x \notin A \cap B \cap C$, we conclude that $x \in \overline{A \cap B \cap C}$, as desired.

b) The following membership table gives the desired equality, since columns five and nine are identical.

A	B	C	$A \cap B \cap C$	$\overline{A \cap B \cap C}$	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A} \cup \overline{B} \cup \overline{C}$
1	1	1	1	0	0	0	0	0
1	1	0	0	1	0	0	1	1
1	0	1	0	1	0	1	0	1
1	0	0	0	1	0	1	1	1
0	1	1	0	1	1	0	0	1
0	1	0	0	1	1	0	1	1
0	0	1	0	1	1	1	0	1
0	0	0	0	1	1	1	1	1

19. **a)** This is clear, since both of these sets are precisely $\{x \mid x \in A \wedge x \notin B\}$.

b) One approach here is to use the distributive law; see the answer section for that approach. Alternatively, we can argue directly as follows. Suppose $x \in (A \cap B) \cup (A \cap \overline{B})$. Then we know that either $x \in A \cap B$ or $x \in A \cap \overline{B}$ (or both). If either case, this forces $x \in A$. Thus we have shown that the left-hand side is a subset of the right-hand side. For the opposite direction, suppose $x \in A$. There are two cases: $x \in B$ and $x \notin B$. In the former case, x is then an element of $A \cap B$ and therefore also an element of $(A \cap B) \cup (A \cap \overline{B})$. In the latter cases, $x \in \overline{B}$ and therefore x is an element of $A \cap \overline{B}$ and therefore also an element of $(A \cap B) \cup (A \cap \overline{B})$.

21. There are many ways to prove identities such as the one given here. One way is to reduce them to logical identities (some of the associative and distributive laws for $\vee$ and $\wedge$). Alternately, we could argue in each case that the left-hand side is a subset of the right-hand side and vice versa. Another method would be to construct membership tables (they will have eight rows in order to cover all the possibilities). Here we choose the second method. First we show that every element of the left-hand side must be in the right-hand side as well. If $x \in A \cup (B \cup C)$, then x must be either in A or in $B \cup C$ (or both). In the former case, we can conclude that $x \in A \cup B$ and thus $x \in (A \cup B) \cup C$, by the definition of union. In the latter case, x must be either in B or in C (or both). In the former subcase, we can conclude that $x \in A \cup B$ and thus $x \in (A \cup B) \cup C$, by the definition of union; in the latter subcase, we can conclude that $x \in (A \cup B) \cup C$, again using the definition of union. The argument in the other direction is practically identical, with the roles of A, B, and C switched around.

23. There are many ways to prove identities such as the one given here. One way is to reduce them to logical identities (some of the associative and distributive laws for $\vee$ and $\wedge$). Alternately, we could argue in each case that the left-hand side is a subset of the right-hand side and vice versa. Another method would be to construct membership tables (they will have eight rows in order to cover all the possibilities). Here we choose the third method. We construct the following membership table and note that the fifth and eighth columns are identical.

A	B	C	$B \cap C$	$A \cup (B \cap C)$	$A \cup B$	$A \cup C$	$(A \cup B) \cap (A \cup C)$
1	1	1	1	1	1	1	1
1	1	0	0	1	1	1	1
1	0	1	0	1	1	1	1
1	0	0	0	1	1	1	1
0	1	1	1	1	1	1	1
0	1	0	0	0	1	0	0
0	0	1	0	0	0	1	0
0	0	0	0	0	0	0	0

25. These are straightforward applications of the definitions.

a) The set of elements common to all three sets is $\{4, 6\}$.

b) The set of elements in at least one of the three sets is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$.

c) The set of elements in C and at the same time in at least one of A and B is $\{4, 5, 6, 8, 10\}$.

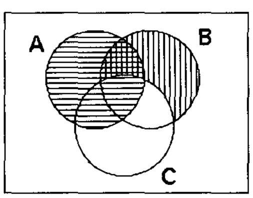
d) The set of elements either in C or in both A and B (or in both of these) is $\{0, 2, 4, 5, 6, 7, 8, 9, 10\}$.

27. a) In the figure we have shaded the A set with horizontal bars (including the double-shaded portion, which includes both horizontal and vertical bars), and we have shaded the set $B - C$ with vertical bars (that portion inside B but outside C. The intersection is where these overlap—the double-shaded portion (shaped like an arrowhead).

b) In the figure we have shaded the set $A \cap B$ with horizontal bars (including the double-shaded portion, which includes both horizontal and vertical bars), and we have shaded the set $A \cap C$ with vertical bars. The union is the entire region that has any shading at all (shaped like a tilted mustache).

c) In the figure we have shaded the set $A \cap \overline{B}$ with horizontal bars (including the double-shaded portion, which includes both horizontal and vertical bars), and we have shaded the set $A \cap \overline{C}$ with vertical bars. The union is the entire region that has any shading at all (everything inside A except the triangular middle portion where all three sets overlap) portion (shaped like an arrowhead).

(a)



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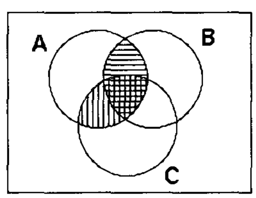

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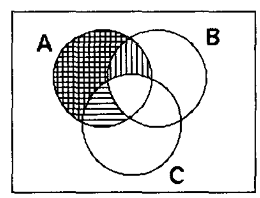


29. a) If B adds nothing new to A, then we can conclude that all the elements of B were already in A. In other words, $B \subseteq A$.

b) In this case, all the elements of A are forced to be in B as well, so we conclude that $A \subseteq B$.

c) This equality holds precisely when none of the elements of A are in B (if there were any such elements, then $A - B$ would not contain all the elements of A). Thus we conclude that A and B are disjoint ($A \cap B = \emptyset$).

d) We can conclude nothing about A and B in this case, since this equality always holds.

e) Every element in $A - B$ must be in A, and every element in $B - A$ must not be in A. Since no item can be in A and not be in A at the same time, there are no elements in both $A - B$ and $B - A$. Thus the only way for these two sets to be equal is if both of them are the empty set. This means that every element of A must be in B, and every element of B must be in A. Thus we conclude that $A = B$.

31. This is the set-theoretic version of the contrapositive law for logic, which says that $p \to q$ is logically equivalent to $\neg q \to \neg p$. We argue as follows.

$A \subseteq B \equiv \forall x(x \in A \to x \in B) \equiv \forall x(x \notin B \to x \notin A) \equiv \forall x(x \in \overline{B} \to x \in \overline{A}) \equiv \overline{B} \subseteq \overline{A}$

33. Clearly this will be the set of students majoring in computer science or mathematics but not both.

35. This is just a restatement of the definition. An element is in $(A \cup B) - (A \cap B)$ if it is in the union (i.e., in either A or B), but not in the intersection (i.e., not in both A and B).

37. We will use the result of Exercise 36 as well as some obvious identities (some of which are in Exercises 6–10).
a) $A \oplus A = (A - A) \cup (A - A) = \emptyset \cup \emptyset = \emptyset$ **b)** $A \oplus \emptyset = (A - \emptyset) \cup (\emptyset - A) = A \cup \emptyset = A$
c) $A \oplus U = (A - U) \cup (U - A) = \emptyset \cup \overline{A} = \overline{A}$ **d)** $A \oplus \overline{A} = (A - \overline{A}) \cup (\overline{A} - A) = A \cup \overline{A} = U$

39. We can conclude that $B = \emptyset$. To see this, suppose that B contains some element b. If $b \in A$, then b is excluded from $A \oplus B$, so $A \oplus B$ cannot equal A. On the other hand, if $b \notin A$, then b must be in $A \oplus B$, so again $A \oplus B$ cannot equal A. Thus in either case, $A \oplus B \neq A$. We conclude that B cannot have any elements.

41. Yes. To show that $A = B$, we need to show that $x \in A$ implies $x \in B$ and conversely. By symmetry, it will be enough to show one direction of this. So assume that $A \oplus C = B \oplus C$, and let $x \in A$ be given. There are two cases to consider, depending on whether $x \in C$. If $x \in C$, then by definition we can conclude that $x \notin A \oplus C$. Therefore $x \notin B \oplus C$. Now if x were *not* in B, then x *would* be in $B \oplus C$ (since $x \in C$ by assumption). Since this is not true, we conclude that $x \in B$, as desired. For the other case, assume that $x \notin C$. Then $x \in A \oplus C$. Therefore $x \in B \oplus C$ as well. Again, if x were *not* in B, then it could not be in $B \oplus C$ (since $x \notin C$ by assumption). Once again we conclude that $x \in B$, and the proof is complete.

43. Yes. Both sides equal the set of elements that occur in an odd number of the sets A, B, C, and D.

45. We give a proof by contradiction. If $A \cup B$ is finite, then it has n elements for some natural number n. But A already has more than n elements, because it is infinite, and $A \cup B$ has all the elements that A has, so $A \cup B$ has more than n elements. This contradiction shows that $A \cup B$ must be infinite.

47. a) The union of these sets is the set of elements that appear in at least one of them. In this case the sets are "increasing": $A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n$. Therefore every element in any of the sets is in A_n, so the union is $A_n = \{1, 2, \ldots, n\}$.
b) The intersection of these sets is the set of elements that appear in all of them. Since $A_1 = \{1\}$, only the number 1 has a chance to be in the intersection. In fact 1 is in the intersection, since it is in all of the sets. Therefore the intersection is $A_1 = \{1\}$.

49. a) Here the sets are increasing. A bit string of length not exceeding 1 is also a bit string of length not exceeding 2, so $A_1 \subseteq A_2$. Similarly, $A_2 \subseteq A_3 \subseteq A_4 \subseteq \cdots \subseteq A_n$. Therefore the union of the sets A_1, A_2, ..., A_n is just A_n itself.
b) Since A_1 is a subset of all the A_i's, the intersection is A_1, the set of all nonempty bit strings of length not exceeding 1, namely $\{0, 1\}$.

51. a) As i increases, the sets get larger: $A_1 \subset A_2 \subset A_3 \cdots$. All the sets are subsets of the set of integers, and every integer is included eventually, so $\bigcup_{i=1}^{\infty} A_i = \mathbf{Z}$. Because A_1 is a subset of each of the others, $\bigcap_{i=1}^{\infty} A_i = A_1 = \{-1, 0, 1\}$.
b) All the sets are subsets of the set of integers, and every nonzero integer is in exactly one of the sets, so $\bigcup_{i=1}^{\infty} A_i = \mathbf{Z} - \{0\}$. Each pair of these sets are disjoint, so no element is common to all of the sets. Therefore $\bigcap_{i=1}^{\infty} A_i = \emptyset$.

c) This is similar to part **(a)**, the only difference being that here we are working with real numbers. Therefore $\bigcup_{i=1}^{\infty} A_i = \mathbf{R}$ (the set of all real numbers), and $\bigcap_{i=1}^{\infty} A_i = A_1 = [-1, 1]$ (the interval of all real numbers between -1 and 1, inclusive).

d) This time the sets are getting smaller as i increases: $\cdots \subset A_3 \subset A_2 \subset A_1$. Because A_1 includes all the others, $\bigcup_{i=1}^{\infty} A_i = A_1 = [1, \infty)$ (all real numbers greater than or equal to 1). Every number eventually gets excluded as i increases, so $\bigcap_{i=1}^{\infty} A_i = \emptyset$. Notice that ∞ is not a real number, so we cannot write $\bigcap_{i=1}^{\infty} A_i = \{\infty\}$.

53. The i^{th} digit in the string indicates whether the i^{th} number in the universal set (in this case the number i) is in the set in question.

a) $\{1, 2, 3, 4, 7, 8, 9, 10\}$ **b)** $\{2, 4, 5, 6, 7\}$ **c)** $\{1, 10\}$

55. We are given two bit strings, representing two sets. We want to represent the set of elements that are in the first set but not the second. Thus the bit in the i^{th} position of the bit string for the difference is 1 if the i^{th} bit of the first string is 1 and the i^{th} bit of the second string is 0, and is 0 otherwise.

57. We represent the sets by bit strings of length 26, using alphabetical order. Thus

A is represented by 11 1110 0000 0000 0000 0000 0000,

B is represented by 01 1100 1000 0000 0100 0101 0000,

C is represented by 00 1010 0010 0000 1000 0010 0111, and

D is represented by 00 0110 0110 0001 1000 0110 0110.

To find the desired sets, we apply the indicated bitwise operations to these strings.

a) 11 1110 0000 0000 0000 0000 0000 $\vee$ 01 1100 1000 0000 0100 0101 0000 $=$

11 1110 1000 0000 0100 0101 0000, which represents the set $\{a, b, c, d, e, g, p, t, v\}$

b) 11 1110 0000 0000 0000 0000 0000 $\wedge$ 01 1100 1000 0000 0100 0101 0000 $=$

01 1100 0000 0000 0000 0000 0000, which represents the set $\{b, c, d\}$

c) (11 1110 0000 0000 0000 0000 0000 $\vee$ 00 0110 0110 0001 1000 0110 0110) $\wedge$

(01 1100 1000 0000 0100 0101 0000 $\vee$ 00 1010 0010 0000 1000 0010 0111) $=$

11 1110 0110 0001 1000 0110 0110 $\wedge$ 01 1110 1010 0000 1100 0111 0111 $=$

01 1110 0010 0000 1000 0110 0110, which represents the set $\{b, c, d, e, i, o, t, u, x, y\}$

d) 11 1110 0000 0000 0000 0000 0000 $\vee$ 01 1100 1000 0000 0100 0101 0000 $\vee$

00 1010 0010 0000 1000 0010 0111 $\vee$ 00 0110 0110 0001 1000 0110 0110 $=$

11 1110 1110 0001 1100 0111 0111, which represents the set

$\{a, b, c, d, e, g, h, i, n, o, p, t, u, v, x, y, z\}$

59. We simply adjoin the set itself to the list of its elements.

a) $\{1, 2, 3, \{1, 2, 3\}\}$ **b)** $\{\emptyset\}$ **c)** $\{\emptyset, \{\emptyset\}\}$ **d)** $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$

61. a) The multiplicity of a in the union is the maximum of 3 and 2, the multiplicities of a in A and B. Since the maximum is 3, we find that a occurs with multiplicity 3 in the union. Working similarly with b, c (which appears with multiplicity 0 in B), and d (which appears with multiplicity 0 in A), we find that $A \cup B = \{3 \cdot a, 3 \cdot b, 1 \cdot c, 4 \cdot d\}$.

b) This is similar to part **(a)**, with "maximum" replaced by "minimum." Thus $A \cap B = \{2 \cdot a, 2 \cdot b\}$. (In particular, c and d appear with multiplicity 0—i.e., do not appear—in the intersection.)

c) In this case we subtract multiplicities, but never go below 0. Thus the answer is $\{1 \cdot a, 1 \cdot c\}$.

d) Similar to part **(c)** (subtraction in the opposite order); the answer is $\{1 \cdot b, 4 \cdot d\}$.

e) We add multiplicities here, to get $\{5 \cdot a, 5 \cdot b, 1 \cdot c, 4 \cdot d\}$.

63. Assume that the universal set contains just Alice, Brian, Fred, Oscar, and Rita. We subtract the degrees of membership from 1 to obtain the complement. Thus $\overline{F}$ is $\{0.4 \text{ Alice}, 0.1 \text{ Brian}, 0.6 \text{ Fred}, 0.9 \text{ Oscar}, 0.5 \text{ Rita}\}$, and $\overline{R}$ is $\{0.6 \text{ Alice}, 0.2 \text{ Brian}, 0.8 \text{ Fred}, 0.1 \text{ Oscar}, 0.3 \text{ Rita}\}$.

65. Taking the minimums, we obtain $\{0.4 \text{ Alice}, 0.8 \text{ Brian}, 0.2 \text{ Fred}, 0.1 \text{ Oscar}, 0.5 \text{ Rita}\}$ for $F \cap R$.

SECTION 2.3 Functions

The importance of understanding what a function is cannot be overemphasized—functions permeate all of mathematics and computer science. This exercise set enables you to make sure you understand functions and their properties. Exercise 33 is a particularly good benchmark to test your full comprehension of the abstractions involved. The definitions play a crucial role in doing proofs about functions. To prove that a function $f: A \to B$ is one-to-one, you need to show that $x_1 \neq x_2 \to f(x_1) \neq f(x_2)$ for all $x_1, x_2 \in A$. To prove that such a function is onto, you need to show that $\forall y \in B\, \exists x \in A\, (f(x) = y)$.

1. a) The expression $1/x$ is meaningless for $x = 0$, which is one of the elements in the domain; thus the "rule" is no rule at all. In other words, $f(0)$ is not defined.

b) Things like $\sqrt{-3}$ are undefined (or, at best, are complex numbers).

c) The "rule" for f is ambiguous. We must have $f(x)$ defined uniquely, but here there are two values associated with every x, the positive square root and the negative square root of $x^2 + 1$.

3. a) This is not a function, because there may be no zero bit in S, or there may be more than one zero bit in S. Thus there may be no value for $f(S)$ or more than one. In either case this violates the definition of a function, since $f(S)$ must have a unique value.

b) This is a function from the set of bit strings to the set of integers, since the number of 1 bits is always a clearly defined nonnegative integer.

c) This definition does not tell what to do with a nonempty string consisting of all 0's. Thus, for example, $f(000)$ is undefined. Therefore this is not a function.

5. In each case we want to find the domain (the set on which the function operates, which is implicitly stated in the problem) and the range (the set of possible output values).

a) Clearly the domain is the set of all bit strings. The range is $\mathbf{Z}$; the function evaluated at a string with n 1's and no 0's is n, and the function evaluated at a string with n 0's and no 1's is $-n$.

b) Again the domain is clearly the set of all bit strings. Since there can be any natural number of 0's in a bit string, the value of the function can be $0, 2, 4, \ldots$. Therefore the range is the set of even natural numbers.

c) Again the domain is the set of all bit strings. Since the number of leftover bits can be any whole number from 0 to 7 (if it were more, then we could form another byte), the range is $\{0, 1, 2, 3, 4, 5, 6, 7\}$.

d) As the problem states, the domain is the set of positive integers. Only perfect squares can be function values, and clearly every positive perfect square is possible. Therefore the range is $\{1, 4, 9, 16, \ldots\}$.

7. In each case, the domain is the set of possible inputs for which the function is defined, and the range is the set of all possible outputs on these inputs.

a) The domain is $\mathbf{Z}^+ \times \mathbf{Z}^+$, since we are told that the function operates on pairs of positive integers (the word "pair" in mathematics is usually understood to mean ordered pair). Since the maximum is again a positive integer, and all positive integers are possible maximums (by letting the two elements of the pair be the same), the range is $\mathbf{Z}^+$.

b) We are told that the domain is $\mathbf{Z}^+$. Since the decimal representation of an integer has to have at least one digit, at most nine digits do not appear, and of course the number of missing digits could be any number less than 9. Thus the range is $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$.

c) We are told that the domain is the set of bit strings. The block 11 could appear no times, or it could appear any positive number of times, so the range is $\mathbf{N}$.

d) We are told that the domain is the set of bit strings. Since the first 1 can be anywhere in the string, its position can be $1, 2, 3, \ldots$. If the bit string contains no 1's, the value is 0 by definition. Therefore the range is $\mathbf{N}$.

9. The floor function rounds down and the ceiling function rounds up.
a) 1 **b)** 0 **c)** 0 **d)** -1 **e)** 3 **f)** -1 **g)** $\lfloor \frac{1}{2} + \lceil \frac{3}{2} \rceil \rfloor = \lfloor \frac{1}{2} + 2 \rfloor = \lfloor 2\frac{1}{2} \rfloor = 2$
h) $\lfloor \frac{1}{2} \lfloor \frac{5}{2} \rfloor \rfloor = \lfloor \frac{1}{2} \cdot 2 \rfloor = \lfloor 1 \rfloor = 1$

11. We need to determine whether the range is all of $\{a, b, c, d\}$. It is for the function in part **(a)**, but not for the other two functions.

13. a) This function is onto, since every integer is 1 less than some integer. In particular, $f(x + 1) = x$.
b) This function is not onto. Since $n^2 + 1$ is always positive, the range cannot include any negative integers.
c) This function is not onto, since the integer 2, for example, is not in the range. In other words, 2 is not the cube of any integer.
d) This function is onto. If we want to obtain the value x, then we simply need to start with $2x$, since $f(2x) = \lceil 2x/2 \rceil = \lceil x \rceil = x$ for all $x \in \mathbf{Z}$.

15. An onto function is one whose range is the entire codomain. Thus we must determine whether we can write every integer in the form given by the rule for f in each case.
a) Given any integer n, we have $f(0, n) = n$, so the function is onto.
b) Clearly the range contains no negative integers, so the function is not onto.
c) Given any integer m, we have $f(m, 25) = m$, so the function is onto. (We could have used any constant in place of 25 in this argument.)
d) Clearly the range contains no negative integers, so the function is not onto.
e) Given any integer m, we have $f(m, 0) = m$, so the function is onto.

17. a) This may be one-to-one (if there are no shared offices) or not (if there are shared offices).
b) Again, this depends on the policy of the school. Assuming that no bus gets more than one teacher chaperone, this function will be one-to-one.
c) This is most likely not one-to-one, because two teachers might have the same salary. On the other hand, it may happen that everyone's salary is different, in which case it is one-to-one.
d) This is clearly a one-to-one function, because social security numbers are unique—no two people can have the same social security number.

19. a) The codomain could be the set of all offices in the school. This is probably not onto; for example, the secretaries probably have offices, and those offices are not in the range of this function (because the secretaries are not in the domain).
b) The codomain could be the set of busses going on the field trip. This is probably an onto function, as long as school policy is that every bus has to have a teacher chaperone.
c) The codomain could be all positive real numbers (with the understanding that the number represents the salary in dollars). The function is clearly not onto; no teacher has a salary of $\sqrt{2}$ dollars or 10^{10} dollars.

d) The codomain could be all strings of nine digits. This function is not onto, because the social security number of everyone who is not a teacher at the school is not in the range of the function.

21. Obviously there are an infinite number of correct answers to each part. The problem asked for a "formula." Parts **(a)** and **(c)** seem harder here, since we somehow have to fold the negative integers into the positive ones without overlap. Therefore we probably want to treat the negative integers differently from the positive integers. One way to do this with a formula is to make it a two-part formula. If one objects that this is not "a formula," we can counter as follows. Consider the function $g(x) = \lfloor 2^x \rfloor / 2^x$. Clearly if $x \geq 0$, then 2^x is a positive integer, so $g(x) = 2^x/2^x = 1$. If $x < 0$, then 2^x is a number between 0 and 1, so $g(x) = 0/2^x = 0$. If we want to define a function that has the value $f_1(x)$ when $x \geq 0$ and $f_2(x)$ when $x < 0$, then we can use the formula $g(x) \cdot f_1(x) + (1 - g(x)) \cdot f_2(x)$.

a) We could map the positive integers (and 0) into the positive multiples of 3, say, and the negative integers into numbers that are 1 greater than a multiple of 3, in a one-to-one manner. This will give us a function that leaves some elements out of the range. So let us define our function as follows:

$$f(x) = \begin{cases} 3x + 3 & \text{if } x \geq 0 \\ 3|x| + 1 & \text{if } x < 0 \end{cases}$$

The values of f on the inputs 0 through 4 are then $3, 6, 9, 12, 15$; and the values on the inputs -1 to -4 are $4, 7, 10, 13$. Clearly this function is one-to-one, but it is not onto since, for example, 2 is not in the range.

b) This is easier. We can just take $f(x) = |x| + 1$. It is clearly onto, but $f(n)$ and $f(-n)$ have the same value for every positive integer n, so f is not one-to-one.

c) This is similar to part **(a)**, except that we have to be careful to hit all values. Mapping the nonnegative integers to the odds and the negative integers to the evens will do the trick:

$$f(x) = \begin{cases} 2x + 1 & \text{if } x \geq 0 \\ 2|x| & \text{if } x < 0 \end{cases}$$

d) Here we can use a trivial example like $f(x) = 17$ or a simple nontrivial one like $f(x) = x^2 + 1$. Clearly these are neither one-to-one nor onto.

23. a) One way to determine whether a function is a bijection is to try to construct its inverse. This function is a bijection, since its inverse (obtained by solving $y = 2x + 1$ for x) is the function $g(y) = (y - 1)/2$. Alternatively, we can argue directly. To show that the function is one-to-one, note that if $2x + 1 = 2x' + 1$, then $x = x'$. To show that the function is onto, note that $2((y - 1)/2) + 1 = y$, so every number is in the range.

b) This function is not a bijection, since its range is the set of real numbers greater than or equal to 1 (which is sometimes written $[1, \infty)$), not all of $\mathbf{R}$. (It is not injective either.)

c) This function is a bijection, since it has an inverse function, namely the function $f(y) = y^{1/3}$ (obtained by solving $y = x^3$ for x).

d) This function is not a bijection. It is easy to see that it is not injective, since x and $-x$ have the same image, for all real numbers x. A little work shows that the range is only $\{\, y \mid 0.5 \leq y < 1 \,\} = [0.5, 1)$.

25. The key here is that larger denominators make smaller fractions, and smaller denominators make larger fractions. We have two things to prove, since this is an "if and only if" statement. First, suppose that f is strictly decreasing. This means that $f(x) > f(y)$ whenever $x < y$. To show that g is strictly increasing, suppose that $x < y$. Then $g(x) = 1/f(x) < 1/f(y) = g(y)$. Conversely, suppose that g is strictly increasing. This means that $g(x) < g(y)$ whenever $x < y$. To show that f is strictly decreasing, suppose that $x < y$. Then $f(x) = 1/g(x) > 1/g(y) = f(y)$.

27. **a)** Let f be the given strictly decreasing function from $\mathbf{R}$ to itself. We need to show that $f(a) = f(b)$ implies $a = b$ for all $a, b \in \mathbf{R}$. We give an indirect proof by proving the contrapositive: if $a \neq b$, then $f(a) \neq f(b)$. There are two cases. Suppose $a < b$; then because f is strictly decreasing, it follows that $f(a) > f(b)$. Similarly, if $a > b$, then $f(a) < f(b)$. Thus in either case, $f(a) \neq f(b)$.

b) We need to make the function decreasing, but not *strictly* decreasing, so, for example, we could take the trivial function $f(x) = 17$. If we want the range to be all of $\mathbf{R}$, we could define f in parts this way: $f(x) = -x - 1$ for $x < -1$; $f(x) = 0$ for $-1 \le x \le 1$; and $f(x) = -x + 1$ for $x > 1$.

29. The function is not one-to-one (for example, $f(2) = 2 = f(-2)$), so it is not invertible. On the restricted domain, the function is the identity function from the set of nonnegative real numbers to itself, $f(x) = x$, so it is one-to-one and onto and therefore invertible; in fact, it is its own inverse.

31. In each case, we need to compute the values of $f(x)$ for each $x \in S$.

a) Note that $f(\pm 2) = \lfloor (\pm 2)^2/3 \rfloor = \lfloor 4/3 \rfloor = 1$, $f(\pm 1) = \lfloor (\pm 1)^2/3 \rfloor = \lfloor 1/3 \rfloor = 0$, $f(0) = \lfloor 0^2/3 \rfloor = \lfloor 0 \rfloor = 0$, and $f(3) = \lfloor 3^2/3 \rfloor = \lfloor 3 \rfloor = 3$. Therefore $f(S) = \{0, 1, 3\}$.

b) In addition to the values we computed above, we note that $f(4) = 5$ and $f(5) = 8$. Therefore $f(S) = \{0, 1, 3, 5, 8\}$.

c) Note this time also that $f(7) = 16$ and $f(11) = 40$, so $f(S) = \{0, 8, 16, 40\}$.

d) $\{f(2), f(6), f(10), f(14)\} = \{1, 12, 33, 65\}$

33. In both cases, we can argue directly from the definitions.

a) Assume that both f and g are one-to-one. We need to show that $f \circ g$ is one-to-one. This means that we need to show that if x and y are two distinct elements of A, then $f(g(x)) \neq f(g(y))$. First, since g is one-to-one, the definition tells us that $g(x) \neq g(y)$. Second, since now $g(x)$ and $g(y)$ are distinct elements of B, and since f is one-to-one, we conclude that $f(g(x)) \neq f(g(y))$, as desired.

b) Assume that both f and g are onto. We need to show that $f \circ g$ is onto. This means that we need to show that if z is any element of C, then there is some element $x \in A$ such that $f(g(x)) = z$. First, since f is onto, we can conclude that there is an element $y \in B$ such that $f(y) = z$. Second, since g is onto and $y \in B$, we can conclude that there is an element $x \in A$ such that $g(x) = y$. Putting these together, we have $z = f(y) = f(g(x))$, as desired.

35. To establish the setting here, let us suppose that $g : A \to B$ and $f : B \to C$. Then $f \circ g : A \to C$. We are told that f and $f \circ g$ are onto. Thus all of C gets "hit" by the images of elements of B; in fact, each element in C gets hit by an element from A under the composition $f \circ g$. But this does not seem to tell us anything about the elements of B getting hit by the images of elements of A. Indeed, there is no reason that they must. For a simple counterexample, suppose that $A = \{a\}$, $B = \{b_1, b_2\}$, and $C = \{c\}$. Let $g(a) = b_1$, and let $f(b_1) = c$ and $f(b_2) = c$. Then clearly f and $f \circ g$ are onto, but g is not, since b_2 is not in its range.

37. We just perform the indicated operations on the defining expressions. Thus $f + g$ is the function whose value at x is $(x^2 + 1) + (x + 2)$, or, more simply, $(f + g)(x) = x^2 + x + 3$. Similarly fg is the function whose value at x is $(x^2 + 1)(x + 2)$; in other words, $(fg)(x) = x^3 + 2x^2 + x + 2$.

39. We simply solve the equation $y = ax + b$ for x. This gives $x = (y - b)/a$, which is well-defined since $a \neq 0$. Thus the inverse is $f^{-1}(y) = (y - b)/a$. To check that our work is correct, we must show that $f \circ f^{-1}(y) = y$ for all $y \in \mathbf{R}$ and that $f^{-1} \circ f(x) = x$ for all $x \in \mathbf{R}$. Both of these are straightforward algebraic manipulations. For the first, we have $f \circ f^{-1}(y) = f(f^{-1}(y)) = f((y - b)/a) = a\big((y - b)/a\big) + b = y$. The second is similar.

41. **a)** Let us arrange for S and T to be nonempty sets that have empty intersection. Then the left-hand side will be $f(\emptyset)$, which is the empty set. If we can make the right-hand side nonempty, then we will be done. We can make the right-hand side nonempty by making the codomain consist of just one element, so that $f(S)$ and $f(T)$ will both be the set consisting of that one element. The simplest example is as follows. Let $A=\{1,2\}$ and $B=\{3\}$. Let f be the unique function from A to B (namely $f(1)=f(2)=3$). Let $S=\{1\}$ and $T=\{2\}$. Then $f(S\cap T)=f(\emptyset)=\emptyset$, which is a proper subset of $f(S)\cap f(T)=\{3\}\cap\{3\}=\{3\}$.

b) Assume that f is one-to-one. We must show that every element of $f(S)\cap f(T)$ is an element of $f(S\cap T)$. Let $y\in B$ be an element of $f(S)\cap f(T)$. Then $y\in f(S)$, so $y=f(x_1)$ for some $x_1\in S$. Similarly, $y\in f(T)$, so $y=f(x_2)$ for some $x_2\in T$. Because f is one-to-one, it follows that $x_1=x_2$. This element is therefore in $S\cap T$, so $y\in f(S\cap T)$.

43. **a)** We want to find the set of all numbers whose floor is 0. Since all numbers from 0 to 1 (including 0 but not 1) round down to 0, we conclude that $g^{-1}(\{0\})=\{\,x\mid 0\le x<1\,\}=[0,1)$.

b) This is similar to part **(a)**. All numbers from -1 to 2 (including -1 but not 2) round down to -1, 0, or 1; we conclude that $g^{-1}(\{-1,0,1\})=\{\,x\mid -1\le x<2\,\}=[-1,2)$.

c) Since $g(x)$ is always an integer, there are no values of x such that $g(x)$ is strictly between 0 and 1. Thus the inverse image in this case is the empty set.

45. Note that the complementation here is with respect to the relevant universal set. Thus $\overline{S}=B-S$ and $\overline{f^{-1}(S)}=A-f^{-1}(S)$. There are two things to prove in order to show that these two sets are equal: that the left-hand side of the equation is a subset of the right-hand side, and that the right-hand side is a subset of the left-hand side. First let $x\in f^{-1}(\overline{S})$. This means that $f(x)\in\overline{S}$, or equivalently that $f(x)\notin S$. Therefore by definition of inverse image, $x\notin f^{-1}(S)$, so $x\in\overline{f^{-1}(S)}$. For the other direction, assume that $x\in\overline{f^{-1}(S)}$. Then $x\notin f^{-1}(S)$. By definition this means that $f(x)\notin S$, which means that $f(x)\in\overline{S}$. Therefore by definition, $x\in f^{-1}(\overline{S})$.

47. There are three cases. Define the "fractional part" of x to be $f(x)=x-\lfloor x\rfloor$. Clearly $f(x)$ is always between 0 and 1 (inclusive at 0, exclusive at 1), and $x=\lfloor x\rfloor+f(x)$. If $f(x)$ is less than $\frac{1}{2}$, then $x-\frac{1}{2}$ will have a value slightly less than $\lfloor x\rfloor$, so when we round up, we get $\lfloor x\rfloor$. In other words, in this case $\lceil x-\frac{1}{2}\rceil=\lfloor x\rfloor$, and indeed that is the integer closest to x. If $f(x)$ is greater than $\frac{1}{2}$, then $x-\frac{1}{2}$ will have a value slightly greater than $\lfloor x\rfloor$, so when we round up, we get $\lfloor x\rfloor+1$. In other words, in this case $\lceil x-\frac{1}{2}\rceil=\lfloor x\rfloor+1$, and indeed that is the integer closest to x in this case. Finally, if the fractional part is exactly $\frac{1}{2}$, then x is midway between two integers, and $\lceil x-\frac{1}{2}\rceil=\lfloor x\rfloor$, which is the smaller of these two integers.

49. We can write the real number x as $\lfloor x\rfloor+\epsilon$, where ϵ is a real number satisfying $0\le\epsilon<1$. Since $\epsilon=x-\lfloor x\rfloor$, we have $0\le x-\lfloor x\rfloor<1$. The first two inequalities, $x-1<\lfloor x\rfloor$ and $\lfloor x\rfloor\le x$, follow algebraically. For the other two inequalities, we can write $x=\lceil x\rceil-\epsilon$, where again $0\le\epsilon<1$. Then $0\le\lceil x\rceil-x<1$, and again the desired inequalities follow by easy algebra.

51. **a)** One direction (the "only if" part) is obvious: If $x<n$, then since $\lfloor x\rfloor\le x$ it follows that $\lfloor x\rfloor<n$. We will prove the other direction (the "if" part) indirectly (we will prove its contrapositive). Suppose that $x\ge n$. Then "the greatest integer not exceeding x" must be at least n, since n is an integer not exceeding x. That is, $\lfloor x\rfloor\ge n$.

b) One direction (the "only if" part) is obvious: If $n<x$, then since $x\le\lceil x\rceil$ it follows that $n<\lceil x\rceil$. We will prove the other direction (the "if" part) indirectly (we will prove its contrapositive). Suppose that $n\ge x$. Then "the smallest integer not less than x" must be no greater than n, since n is an integer not less than x. That is, $\lceil x\rceil\le n$.

53. If n is even, then $n = 2k$ for some integer k. Thus $\lfloor n/2 \rfloor = \lfloor k \rfloor = k = n/2$. If n is odd, then $n = 2k+1$ for some integer k. Thus $\lfloor n/2 \rfloor = \lfloor k + \frac{1}{2} \rfloor = k = (n-1)/2$.

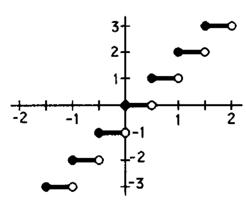
55. Without loss of generality we can assume that $x \geq 0$, since the equation to be proved is equivalent to the same equation with $-x$ substituted for x. Then the left-hand side is $\lceil -x \rceil$ by definition, and the right-hand side is $-\lfloor x \rfloor$. Thus this problem reduces to Exercise 54. Its proof is straightforward. Write x as $n+\epsilon$, where n is a natural number and ϵ is a real number satisfying $0 \leq \epsilon < 1$. Then clearly $\lceil -x \rceil = \lceil -n - \epsilon \rceil = -n$ and $-\lfloor x \rfloor = -\lfloor n + \epsilon \rfloor = -n$ as well.

57. In some sense this question is its own answer—the number of integers strictly between a and b is the number of integers strictly between a and b. Presumably we seek an expression involving a, b, and the floor and/or ceiling function to answer this question. If we round a down and round b up to integers, then we will be looking at the smallest and largest integers just outside the range of integers we want to count, respectively. These values are of course $\lfloor a \rfloor$ and $\lceil b \rceil$, respectively. Then the answer is $\lceil b \rceil - \lfloor a \rfloor - 1$ (just think of counting all the integers between these two values, excluding both ends—if a row of fenceposts one foot apart extends for k feet, then there are $k-1$ fenceposts not counting the end posts). Note that this even works when, for example, $a = 0.3$ and $b = 0.7$.

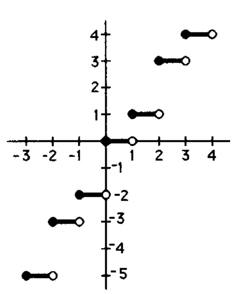
59. Since a byte is eight bits, all we are asking for in each case is $\lceil n/8 \rceil$, where n is the number of bits.
a) $\lceil 7/8 \rceil = 1$ **b)** $\lceil 17/8 \rceil = 3$ **c)** $\lceil 1001/8 \rceil = 126$ **d)** $\lceil 28800/8 \rceil = 3600$

61. In each case we need to divide the number of bytes (octets) by 1500 and round up. In other words, the answer is $\lceil n/1500 \rceil$, where n is the number of bytes.
a) $\lceil 150{,}000/1500 \rceil = 100$ **b)** $\lceil 384{,}000/1500 \rceil = 256$ **c)** $\lceil 1{,}544{,}000/1500 \rceil = 1030$
d) $\lceil 45{,}300{,}000/1500 \rceil = 30{,}200$

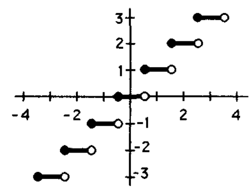
63. The graph will look exactly like the graph of the function $f(x) = \lfloor x \rfloor$, shown in Figure 10a, except that the picture will be compressed by a factor of 2 in the horizontal direction, since x has been replaced by $2x$.



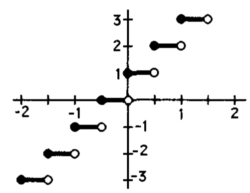
65. This is a step function, with values changing only at the integers. We note the pattern that $f(x)$ jumps by 1 when x passes through an odd integer (because of the $\lfloor x \rfloor$ term), and by 2 when x passes through an even integer (an additional jump caused by the $\lfloor x/2 \rfloor$ term). The result is as shown.



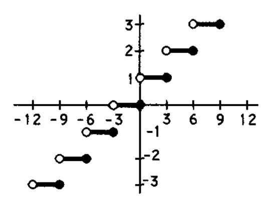
67. **a)** The graph will look exactly like the graph of the function $f(x) = \lfloor x \rfloor$, shown in Figure 10a, except that the picture will be shifted to the left by $\frac{1}{2}$ unit, since x has been replaced by $x + \frac{1}{2} = x - (-\frac{1}{2})$.



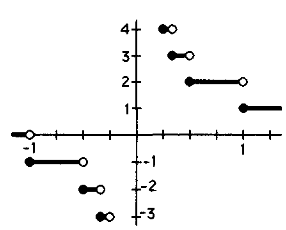
b) The graph will look exactly like the graph of the function $f(x) = \lfloor 2x \rfloor$, shown in the solution to Exercise 63, except that the picture will be shifted to the left by $\frac{1}{2}$ unit, since x has been replaced by $x + \frac{1}{2}$. Alternatively, we can note that $f(x)$ can be rewritten as $\lfloor 2x \rfloor + 1$, so the graph is the graph shown in the solution to Exercise 63 shifted up one unit.



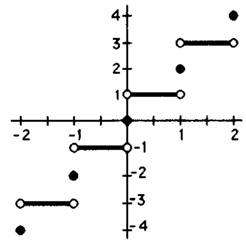
c) The graph will look exactly like the graph of the function $f(x) = \lceil x \rceil$, shown in Figure 10b, except that the x-axis is stretched by a factor of 3. Thus we can use the same picture and just relabel the x-axis.



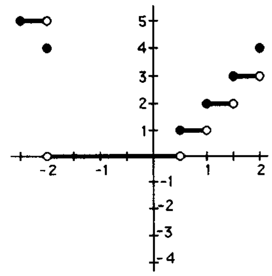
d) The graph is a step version of the usual hyperbola $y = 1/x$. Note that $x = 0$ is not in the domain. The graph can be drawn by first plotting the points at which $1/x$ is an integer ($x = 1, \pm\frac{1}{2}, \pm\frac{1}{3}, \ldots$) and then filling in the horizontal segments, making sure to note that they go to the right (for example, if x is a little bigger than $\frac{1}{2}$, then $1/x$ is a little less than 2, so $f(x) = 2$, since we are rounding up here). Note that $f(x) = 1$ for $x \geq 1$, and $f(x) = 0$ for $x < -1$.



e) The key thing to note is that since we can pull integers outside the floor and ceiling function (identity (4) in Table 1), we can write $f(x)$ more simply as $\lfloor x \rfloor + \lceil x \rceil$. When x is an integer, this is just $2x$. When x is between two integers, however, this has the value of the integer between the two integers $2\lfloor x \rfloor$ and $2\lceil x \rceil$. The graph is therefore as shown here.



f) The basic shape is the parabola, $y = x^2$. In particular, for x an even integer, $f(x) = x^2$, since the terms inside the floor and ceiling function symbols are integers. However, because of these step functions, the curve is broken into steps. At even integers other than $x = 0$ there are isolated points in the graph. Also, the graph takes jumps at all the integer and half-integer values outside the range $-2 < x < \frac{1}{2}$ (where in fact $f(x) = 0$). The portion of the graph near the origin is shown here.



g) Despite the complicated-looking formula, this is really quite similar to part **(a)**; in fact, we'll see that it's identical! First note that the expression inside the outer ceiling function symbols is always going to be a half-integer; therefore we can tell exactly what its rounded-up value will be, namely $\lfloor x - \frac{1}{2} \rfloor + 1$. Furthermore, since identity (4) of Table 1 allows us to move the 1 inside the floor function symbols, we have $f(x) = \lfloor x + \frac{1}{2} \rfloor$. Therefore this is the same function as in part **(a)**.

69. We simply need to solve the equation $y = x^3 + 1$ for x. This is easily done by algebra: $x = (y-1)^{1/3}$. Therefore the inverse function is given by the rule $f^{-1}(y) = (y-1)^{1/3}$ (or, equivalently, by the rule $f^{-1}(x) = (x-1)^{1/3}$, since the variable in the definition is just a dummy variable).

71. We can prove all of these identities by showing that the left-hand side is equal to the right-hand side for all possible values of x. In each instance (except part **(c)**, in which there are only two cases), there are four cases to consider, depending on whether x is in A and/or B.

a) If x is in both A and B, then $f_{A\cap B}(x) = 1$; and the right-hand side is $1 \cdot 1 = 1$ as well. Otherwise $x \notin A \cap B$, so the left-hand side is 0, and the right-hand side is either $0 \cdot 1$ or $1 \cdot 0$ or $0 \cdot 0$, all of which are also 0.

b) If x is in both A and B, then $f_{A\cup B}(x) = 1$; and the right-hand side is $1 + 1 - 1 \cdot 1 = 1$ as well. If x is in A but not B, then $x \in A \cup B$, so the left-hand side is still 1, and the right-hand side is $1 + 0 - 1 \cdot 0 = 1$, as desired. The case in which x is in B but not A is similar. Finally, if x is in neither A nor B, then the left-hand side is 0, and the right-hand side is $0 + 0 - 0 \cdot 0 = 0$ as well.

c) If $x \in A$, then $x \notin \overline{A}$, so $f_{\overline{A}}(x) = 0$. The right-hand side equals $1 - 1 = 0$ in this case, as well. On the other hand, if $x \notin A$, then $x \in \overline{A}$, so the left-hand side is 1, and the right-hand side is $1 - 0 = 1$ as well.

d) If x is in both A and B, then $x \notin A \oplus B$, so $f_{A\oplus B}(x) = 0$. The right-hand side is $1 + 1 - 2 \cdot 1 \cdot 1 = 0$ as well. Next, if $x \in A$ but $x \notin B$, then $x \in A \oplus B$, so the left-hand side is 1. The right-hand side is $1 + 0 - 2 \cdot 1 \cdot 0 = 1$ as well. The case $x \in B \wedge x \notin A$ is similar. Finally, if x is in neither A nor B, then $x \notin A \oplus B$, so the left-hand side is 0; and the right-hand side is also $0 + 0 - 2 \cdot 0 \cdot 0 = 0$.

73. **a)** This is true. Since $\lfloor x \rfloor$ is already an integer, $\lceil \lfloor x \rfloor \rceil = \lfloor x \rfloor$.

b) A little experimentation shows that this is not always true. To disprove it we need only produce a counterexample, such as $x = \frac{1}{2}$. In that case the left-hand side is $\lfloor 1 \rfloor = 1$, while the right-hand side is $2 \cdot 0 = 0$.

c) This is true. We prove it by cases. If x is an integer, then by identity (4b) in Table 1, we know that $\lceil x + y \rceil = x + \lceil y \rceil$, and it follows that the difference is 0. Similarly, if y is an integer. The remaining case is that $x = n + \epsilon$ and $y = m + \delta$, where n and m are integers and ϵ and δ are positive real numbers less than 1. Then $x + y$ will be greater than $m + n$ but less than $m + n + 2$, so $\lceil x + y \rceil$ will be either $m+n+1$ or $m+n+2$. Therefore the given expression will be either $(n+1)+(m+1)-(m+n+1) = 1$ or $(n+1)+(m+1)-(m+n+2) = 0$, as desired.

d) This is clearly false, as we can find with a little experimentation. Take, for example, $x = 1/10$ and $y = 3$. Then the left-hand side is $\lceil 3/10 \rceil = 1$, but the right-hand side is $1 \cdot 3 = 3$.

e) Again a little trial and error will produce a counterexample. Take $x = 1/2$. Then the left-hand side is 1 while the right-hand side is 0.

75. **a)** If x is a positive integer, then the two sides are identical. So suppose that $x = n^2 + m + \epsilon$, where n is the largest perfect square integer less than x, m is a nonnegative integer, and $0 < \epsilon < 1$. For example, if $x = 13.2$, then $n = 3$, $m = 4$, and $\epsilon = 0.2$. Then both $\sqrt{x}$ and $\sqrt{\lfloor x \rfloor} = \sqrt{n^2 + m}$ are between n and $n+1$. Therefore both sides of the equation equal n.

b) If x is a positive integer, then the two sides are identical. So suppose that $x = n^2 - m - \epsilon$, where n is the smallest perfect square integer greater than x, m is a nonnegative integer, and ϵ is a real number with $0 < \epsilon < 1$. For example, if $x = 13.2$, then $n = 4$, $m = 2$, and $\epsilon = 0.8$. Then both $\sqrt{x}$ and $\sqrt{\lceil x \rceil} = \sqrt{n^2 - m}$ are between $n - 1$ and n. Therefore both sides of the equation equal n.

77. In each case we easily read the domain and codomain from the notation. The domain of definition is obtained by determining for which values the definition makes sense. The function is total if the domain of definition is the entire domain (so that there are no values for which the partial function is undefined).

a) The domain is $\mathbf{Z}$ and the codomain is $\mathbf{R}$. Since division is possible by every nonzero number, the domain of definition is all the nonzero integers; $\{0\}$ is the set of values for which f is undefined. (It is not total.)

b) The domain and codomain are both given to be $\mathbf{Z}$. Since the definition makes sense for all integers, this is a total function, whose domain of definition is also $\mathbf{Z}$; the set of values for which f is undefined is $\emptyset$.

c) By inspection, the domain is the Cartesian product $\mathbf{Z} \times \mathbf{Z}$, and the codomain is $\mathbf{Q}$. Since fractions cannot have a 0 in the denominator, we must exclude the "slice" of $\mathbf{Z} \times \mathbf{Z}$ in which the second coordinate is 0. Thus the domain of definition is $\mathbf{Z} \times (\mathbf{Z} - \{0\})$, and the function is undefined for all values in $\mathbf{Z} \times \{0\}$. It is not a total function.

d) The domain is given to be $\mathbf{Z} \times \mathbf{Z}$ and the codomain is given to be $\mathbf{Z}$. Since the definition makes sense for all pairs of integers, this is a total function, whose domain of definition is also $\mathbf{Z} \times \mathbf{Z}$; the set of values for which f is undefined is $\emptyset$.

e) Again the domain and codomain are $\mathbf{Z} \times \mathbf{Z}$ and $\mathbf{Z}$, respectively. Since the definition is only stated for those pairs in which the first coordinate exceeds the second, the domain of definition is $\{\, (m,n) \mid m > n \,\}$, and therefore the set of values for which the function is undefined is $\{\, (m,n) \mid m \leq n \,\}$. It is not a total function.

79. **a)** By definition, to say that S has cardinality m is to say that S has exactly m distinct elements. Therefore we can imagine enumerating them, as a child counts objects: assign the first object to 1, the second to 2, and so on. This provides the one-to-one correspondence.

b) By part **(a)**, there is a bijection f from S to $\{1, 2, \ldots, m\}$ and a bijection g from T to $\{1, 2, \ldots, m\}$. This tells us that g^{-1} is a bijection from $\{1, 2, \ldots, m\}$ to T. Then the composition $g^{-1} \circ f$ is the desired bijection from S to T.

SECTION 2.4 Sequences and Summations

This exercise set contains a lot of routine practice with the concept of and notation for sequences. It also discusses **telescoping sums**; *the* **product notation**, *corresponding to the summation notation discussed in the section; and the* **factorial function**, *which occurs repeatedly in subsequent chapters. There are also a few challenging exercises on more complicated sequences and sums.*

1. **a)** $a_0 = 2\cdot(-3)^0 + 5^0 = 2\cdot 1 + 1 = 3$ **b)** $a_1 = 2\cdot(-3)^1 + 5^1 = 2\cdot(-3) + 5 = -1$
c) $a_4 = 2\cdot(-3)^4 + 5^4 = 2\cdot 81 + 625 = 787$ **d)** $a_5 = 2\cdot(-3)^5 + 5^5 = 2\cdot(-243) + 3125 = 2639$

3. In each case we simply evaluate the given function at $n = 0, 1, 2, 3$.
a) $a_0 = 2^0 + 1 = 2$, $a_1 = 2^1 + 1 = 3$, $a_2 = 2^2 + 1 = 5$, $a_3 = 2^3 + 1 = 9$
b) $a_0 = 1^1 = 1$, $a_1 = 2^2 = 4$, $a_2 = 3^3 = 27$, $a_3 = 4^4 = 256$
c) $a_0 = \lfloor 0/2\rfloor = 0$, $a_1 = \lfloor 1/2\rfloor = 0$, $a_2 = \lfloor 2/2\rfloor = 1$, $a_3 = \lfloor 3/2\rfloor = 1$
d) $a_0 = \lfloor 0/2\rfloor + \lceil 0/2\rceil = 0 + 0 = 0$, $a_1 = \lfloor 1/2\rfloor + \lceil 1/2\rceil = 0 + 1 = 1$, $a_2 = \lfloor 2/2\rfloor + \lceil 2/2\rceil = 1 + 1 = 2$, $a_3 = \lfloor 3/2\rfloor + \lceil 3/2\rceil = 1 + 2 = 3$. Note that $\lfloor n/2\rfloor + \lceil n/2\rceil$ always equals n.

5. In each case we just follow the instructions.
a) $2, 5, 8, 11, 14, 17, 20, 23, 26, 29$ **b)** $1, 1, 1, 2, 2, 2, 3, 3, 3, 4$ **c)** $1, 1, 3, 3, 5, 5, 7, 7, 9, 9$
d) This requires a bit of routine calculation. For example, the fifth term is $5! - 2^5 = 120 - 32 = 88$. The first ten terms are $-1, -2, -2, 8, 88, 656, 4912, 40064, 362368, 3627776$.
e) $3, 6, 12, 24, 48, 96, 192, 384, 768, 1536$ **f)** $2, 4, 6, 10, 16, 26, 42, 68, 110, 178$
g) For $n = 1$, the binary expansion is 1, which has one bit, so the first term of the sequence is 1. For $n = 2$, the binary expansion is 10, which has two bits, so the second term of the sequence is 2. Continuing in this way we see that the first ten terms are $1, 2, 2, 3, 3, 3, 3, 4, 4, 4$. Note that the sequence has one 1, two 2's, four 3's, eight 4's, as so on, with 2^{k-1} copies of k.
h) The English word for 1 is "one" which has three letters, so the first term is 3. This makes a good brain-teaser; give someone the sequence and ask her or him to find the pattern. The first ten terms are $3, 3, 5, 4, 4, 3, 5, 5, 4, 3$.

7. One pattern is that each term is twice the preceding term. A formula for this would be that the n^{th} term is 2^{n-1}. Another pattern is that we obtain the next term by adding increasing values to the previous term. Thus to move from the first term to the second we add 1; to move from the second to the third we add 2; then add 3, and so on. So the sequence would start out $1, 2, 4, 7, 11, 16, 22, \ldots$. We could also have trivial answers such as the rule that the first three terms are $1, 2, 4$ and all the rest are 17 (so the sequence is $1, 2, 4, 17, 17, 17, \ldots$), or that the terms simply repeat $1, 2, 4, 1, 2, 4, 1, 2, 4, \ldots$. Here is another pattern: Take n points on the unit circle, and connect each of them to all the others by line segments. The inside of the circle will be divided into a number of regions. What is the largest this number can be? Call that value a_n. If there is one point, then there are no lines and therefore just the one original region inside the circle; thus $a_1 = 1$. If $n = 2$, then the one chord divides the interior into two parts, so $a_2 = 2$. Three points give us a triangle, and that makes four regions (the inside of the triangle and the three pieces outside the triangle), so $a_3 = 4$. Careful drawing shows that the sequence starts out $1, 2, 4, 8, 16, 31$. That's right: 31, not 32.

Creative students may well find other rules or patterns with various degrees of appeal.

9. We need to compute the terms of the sequence one at a time, since each term is dependent upon one or more of the previous terms.

a) We are given $a_0 = 2$. Then by the recurrence relation $a_n = 6a_{n-1}$ we see (by letting $n = 1$) that $a_1 = 6a_0 = 6 \cdot 2 = 12$. Similarly $a_2 = 6a_1 = 6 \cdot 12 = 72$, then $a_3 = 6a_2 = 6 \cdot 72 = 432$, and $a_4 = 6a_3 = 6 \cdot 432 = 2592$.
b) $a_1 = 2$ (given), $a_2 = a_1^2 = 2^2 = 4$, $a_3 = a_2^2 = 4^2 = 16$, $a_4 = a_3^2 = 16^2 = 256$, $a_5 = a_4^2 = 256^2 = 65536$
c) This time each term depends on the two previous terms. We are given $a_0 = 1$ and $a_1 = 2$. To compute a_2 we let $n = 2$ in the recurrence relation, obtaining $a_2 = a_1 + 3a_0 = 2 + 3 \cdot 1 = 5$. Then we have $a_3 = a_2 + 3a_1 = 5 + 3 \cdot 2 = 11$ and $a_4 = a_3 + 3a_2 = 11 + 3 \cdot 5 = 26$.
d) $a_0 = 1$ (given), $a_1 = 1$ (given), $a_2 = 2a_1 + 2^2 a_0 = 2 \cdot 1 + 4 \cdot 1 = 6$, $a_3 = 3a_2 + 3^2 a_1 = 3 \cdot 6 + 9 \cdot 1 = 27$, $a_4 = 4a_3 + 4^2 a_2 = 4 \cdot 27 + 16 \cdot 6 = 204$
e) We are given $a_0 = 1$, $a_1 = 2$, and $a_2 = 0$. Then $a_3 = a_2 + a_0 = 0 + 1 = 1$ and $a_4 = a_3 + a_1 = 1 + 2 = 3$.

11. a) We simply plug in $n = 0$, $n = 1$, $n = 2$, $n = 3$, and $n = 4$. Thus we have $a_0 = 2^0 + 5 \cdot 3^0 = 1 + 5 \cdot 1 = 6$, $a_1 = 2^1 + 5 \cdot 3^1 = 2 + 5 \cdot 3 = 17$, $a_2 = 2^2 + 5 \cdot 3^2 = 4 + 5 \cdot 9 = 49$, $a_3 = 2^3 + 5 \cdot 3^3 = 8 + 5 \cdot 27 = 143$, and $a_4 = 2^4 + 5 \cdot 3^4 = 16 + 5 \cdot 81 = 421$.
b) Using our data from part **(a)**, we see that $49 = 5 \cdot 17 - 6 \cdot 6$, $143 = 5 \cdot 49 - 6 \cdot 17$, and $421 = 5 \cdot 143 - 6 \cdot 49$.
c) This is algebra. The messiest part is factoring out a large power of 2 and a large power of 3. If we substitute $n - 1$ for n in the definition we have $a_{n-1} = 2^{n-1} + 5 \cdot 3^{n-1}$; similarly $a_{n-2} = 2^{n-2} + 5 \cdot 3^{n-2}$. We start with the right-hand side of our desired identity:

$$\begin{aligned} 5a_{n-1} - 6a_{n-2} &= 5(2^{n-1} + 5 \cdot 3^{n-1}) - 6(2^{n-2} + 5 \cdot 3^{n-2}) \\ &= 2^{n-2}(10 - 6) + 3^{n-2}(75 - 30) \\ &= 2^{n-2} \cdot 4 + 3^{n-2} \cdot 9 \cdot 5 \\ &= 2^n + 3^n \cdot 5 = a_n \end{aligned}$$

13. In each case we have to substitute the given equation for a_n into the recurrence relation $a_n = 8a_{n-1} - 16a_{n-2}$ and see if we get a true statement. Remember to make the appropriate substitutions for n (either $n - 1$ or $n - 2$) on the right-hand side. What we are really doing here is performing the inductive step in a proof by mathematical induction: if the formula is correct for a_{n-1} (and also for a_{n-2}, etc., in some cases), then the formula is also correct for a_n.

a) Plugging $a_n = 0$ into the equation $a_n = 8a_{n-1} - 16a_{n-2}$, we obtain the true statement that $0 = 0$. Therefore $a_n = 0$ is a solution of the recurrence relation.
b) Plugging $a_n = 1$ into the equation $a_n = 8a_{n-1} - 16a_{n-2}$, we obtain the false statement $1 = 8 \cdot 1 - 16 \cdot 1 = -8$. Therefore $a_n = 1$ is not a solution.
c) Plugging $a_n = 2^n$ into the equation $a_n = 8a_{n-1} - 16a_{n-2}$, we obtain the statement $2^n = 8 \cdot 2^{n-1} - 16 \cdot 2^{n-2}$. By algebra, the right-hand side equals $2^{n-2}(8 \cdot 2 - 16) = 0$. Since this is not equal to the left-hand side, we conclude that $a_n = 2^n$ is not a solution.
d) Plugging $a_n = 4^n$ into the equation $a_n = 8a_{n-1} - 16a_{n-2}$, we obtain the statement $4^n = 8 \cdot 4^{n-1} - 16 \cdot 4^{n-2}$. By algebra, the right-hand side equals $4^{n-2}(8 \cdot 4 - 16) = 4^{n-2} \cdot 16 = 4^{n-2} \cdot 4^2 = 4^n$. Since this is the left-hand side, we conclude that $a_n = 4^n$ is a solution.
e) Plugging $a_n = n4^n$ into the equation $a_n = 8a_{n-1} - 16a_{n-2}$, we obtain the statement $n4^n = 8(n-1)4^{n-1} - 16(n-2)4^{n-2}$. By algebra, the right-hand side equals $4^{n-2}\big(8(n-1) \cdot 4 - 16(n-2)\big) = 4^{n-2}(32n - 32 - 16n + 32) = 4^{n-2}(16n) = 4^{n-2} \cdot 4^2 n = n4^n$. Since this is the left-hand side, we conclude that $a_n = n4^n$ is a solution.

f) Plugging $a_n = 2\cdot 4^n + 3n4^n$ into the equation $a_n = 8a_{n-1} - 16a_{n-2}$, we obtain the statement $2\cdot 4^n + 3n4^n = 8\big(2\cdot 4^{n-1} + 3(n-1)4^{n-1}\big) - 16\big(2\cdot 4^{n-2} + 3(n-2)4^{n-2}\big)$. By algebra, the right-hand side equals $4^{n-2}\big(8\cdot 2\cdot 4 + 8\cdot 3(n-1)\cdot 4 - 16\cdot 2 - 16\cdot 3(n-2)\big) = 4^{n-2}(64 + 96n - 96 - 32 - 48n + 96) = 4^{n-2}(48n+32) = 4^{n-2}\cdot 4^2(3n+2) = (2+3n)4^n$. Since this is the same as the left-hand side, we conclude that $a_n = 2\cdot 4^n + 3n4^n$ is a solution.

g) Plugging $a_n = (-4)^n$ into the equation $a_n = 8a_{n-1} - 16a_{n-2}$, we obtain the statement $(-4)^n = 8\cdot(-4)^{n-1} - 16\cdot(-4)^{n-2}$. By algebra the right-hand side equals $(-4)^{n-2}(8\cdot(-4) - 16) = (-4)^{n-2}(-48) = -3(-4)^n$. Since this is not equal to the left-hand side, we conclude that $a_n = (-4)^n$ is not a solution.

h) Plugging $a_n = n^2 4^n$ into the equation $a_n = 8a_{n-1} - 16a_{n-2}$, we obtain the statement $n^2 4^n = 8(n-1)^2 4^{n-1} - 16(n-2)^2 4^{n-2}$. By algebra, the right-hand side equals $4^{n-2}\big(8(n-1)^2\cdot 4 - 16(n-2)^2\big) = 4^{n-2}\big(32(n^2 - 2n + 1) - 16(n^2 - 4n + 4)\big) = 4^{n-2}(32n^2 - 64n + 32 - 16n^2 + 64n - 64) = 4^{n-2}(16n^2 - 32) = 4^{n-2}\cdot 4^2(n^2-2) = 4^n(n^2-2)$. Since this is not equal to the left-hand side, we conclude that $a_n = n^2 4^n$ is not a solution.

15. In each case we have to plug the purported solution into the right-hand side of the recurrence relation and see if it simplifies to the left-hand side. The algebra can get tedious, and it is easy to make a mistake.

a) We have

$$\begin{aligned} a_{n-1} + 2a_{n-2} + 2n - 9 &= -(n-1) + 2 + 2(-(n-2) + 2) + 2n - 9 \\ &= -n + 2 = a_n. \end{aligned}$$

b) We have

$$\begin{aligned} a_{n-1} + 2a_{n-2} + 2n - 9 &= 5(-1)^{n-1} - (n-1) + 2 + 2(5(-1)^{n-2} - (n-2) + 2) + 2n - 9 \\ &= 5(-1)^{n-2}(-1+2) - n + 2 = a_n. \end{aligned}$$

Note that we had to factor out $(-1)^{n-2}$ and that this is the same as $(-1)^n$ since $(-1)^2 = 1$.

c) We have

$$\begin{aligned} a_{n-1} + 2a_{n-2} + 2n - 9 &= 3(-1)^{n-1} + 2^{n-1} - (n-1) + 2 + 2(3(-1)^{n-2} + 2^{n-2} - (n-2) + 2) + 2n - 9 \\ &= 3(-1)^{n-2}(-1+2) + 2^{n-2}(2+2) - n + 2 = a_n. \end{aligned}$$

Note that we had to factor out 2^{n-2} and that $2^{n-2}\cdot 4 = 2^n$.

d) We have

$$\begin{aligned} a_{n-1} + 2a_{n-2} + 2n - 9 &= 7\cdot 2^{n-1} - (n-1) + 2 + 2(7\cdot 2^{n-2} - (n-2) + 2) + 2n - 9 \\ &= 2^{n-2}(7\cdot 2 + 2\cdot 7) - n + 2 = a_n. \end{aligned}$$

17. In the iterative approach, we write a_n in terms of a_{n-1}, then write a_{n-1} in terms of a_{n-2} (using the recurrence relation with $n-1$ plugged in for n), and so on. When we reach the end of this procedure, we use the given initial value of a_0. This will give us an explicit formula for the answer or it will give us a finite series, which we then sum to obtain an explicit formula for the answer.

a) We write

$$\begin{aligned} a_n &= 3a_{n-1} \\ &= 3(3a_{n-2}) = 3^2 a_{n-2} \\ &= 3^2(3a_{n-3}) = 3^3 a_{n-3} \\ &\;\;\vdots \\ &= 3^n a_{n-n} = 3^n a_0 = 3^n\cdot 2. \end{aligned}$$

Note that we figured out the last line by following the pattern that had developed in the first few lines. Therefore the answer is $a_n = 2\cdot 3^n$.

b) We write

$$\begin{aligned}a_n &= 2 + a_{n-1}\\ &= 2 + (2 + a_{n-2}) = (2+2) + a_{n-2} = (2\cdot 2) + a_{n-2}\\ &= (2\cdot 2) + (2 + a_{n-3}) = (3\cdot 2) + a_{n-3}\\ &\ \vdots\\ &= (n\cdot 2) + a_{n-n} = (n\cdot 2) + a_0 = (n\cdot 2) + 3 = 2n+3\,.\end{aligned}$$

Again we figured out the last line by following the pattern that had developed in the first few lines. Therefore the answer is $a_n = 2n+3$.

c) We write (note that it is more convenient to put the a_{n-1} at the end)

$$\begin{aligned}a_n &= n + a_{n-1}\\ &= n + \big((n-1) + a_{n-2}\big) = \big(n + (n-1)\big) + a_{n-2}\\ &= \big(n + (n-1)\big) + \big((n-2) + a_{n-3}\big) = \big(n + (n-1) + (n-2)\big) + a_{n-3}\\ &\ \vdots\\ &= \big(n + (n-1) + (n-2) + \cdots + (n-(n-1))\big) + a_{n-n}\\ &= \big(n + (n-1) + (n-2) + \cdots + 1\big) + a_0\\ &= \frac{n(n+1)}{2} + 1 = \frac{n^2+n+2}{2}\,.\end{aligned}$$

Therefore the answer is $a_n = (n^2+n+2)/2$. The formula used to obtain the last line—for the sum of the first n positive integers—is given in Table 2.

d) We write

$$\begin{aligned}a_n &= 3 + 2n + a_{n-1}\\ &= 3 + 2n + \big(3 + 2(n-1) + a_{n-2}\big) = \big(2\cdot 3 + 2n + 2(n-1)\big) + a_{n-2}\\ &= \big(2\cdot 3 + 2n + 2(n-1)\big) + \big(3 + 2(n-2) + a_{n-3}\big)\\ &= \big(3\cdot 3 + 2n + 2(n-1) + 2(n-2)\big) + a_{n-3}\\ &\ \vdots\\ &= \big(n\cdot 3 + 2n + 2(n-1) + 2(n-2) + \cdots + 2(n-(n-1))\big) + a_{n-n}\\ &= \big(n\cdot 3 + 2n + 2(n-1) + 2(n-2) + \cdots + 2\cdot 1\big) + a_0\\ &= 3n + 2\cdot\frac{n(n+1)}{2} + 4 = n^2 + 4n + 4\,.\end{aligned}$$

Therefore the answer is $a_n = n^2 + 4n + 4$. Again we used the formula for the sum of the first n positive integers from Table 2.

e) We write

$$\begin{aligned}a_n &= -1 + 2a_{n-1}\\ &= -1 + 2(-1 + 2a_{n-2}) = -3 + 4a_{n-2}\\ &= -3 + 4(-1 + 2a_{n-3}) = -7 + 8a_{n-3}\\ &= -7 + 8(-1 + 2a_{n-4}) = -15 + 16a_{n-4}\\ &= -15 + 16(-1 + 2a_{n-5}) = -31 + 32a_{n-5}\\ &\ \vdots\\ &= -(2^n - 1) + 2^n a_{n-n} = -2^n + 1 + 2^n\cdot 1 = 1\,.\end{aligned}$$

This time it was somewhat harder to figure out the pattern developing in the coefficients, but it became clear after we carried out the computation far enough. The answer, namely that $a_n = 1$ for all n, it is clear in retrospect, after we found it, since $2\cdot 1 - 1 = 1$.

f) We write

$$\begin{aligned}
a_n &= 1 + 3a_{n-1} \\
&= 1 + 3(1 + 3a_{n-2}) = (1+3) + 3^2 a_{n-2} \\
&= (1+3) + 3^2(1 + 3a_{n-3}) = (1+3+3^2) + 3^3 a_{n-3} \\
&\ \vdots \\
&= (1 + 3 + 3^2 + \cdots + 3^{n-1}) + 3^n a_{n-n} \\
&= 1 + 3 + 3^2 + \cdots + 3^{n-1} + 3^n \\
&= \frac{3^{n+1} - 1}{3 - 1} \quad \text{(a geometric series)} \\
&= \frac{3^{n+1} - 1}{2}.
\end{aligned}$$

Thus the answer is $a_n = (3^{n+1} - 1)/2$.

g) We write

$$\begin{aligned}
a_n &= na_{n-1} = n(n-1)a_{n-2} \\
&= n(n-1)(n-2)a_{n-3} = n(n-1)(n-2)(n-3)a_{n-4} \\
&\ \vdots \\
&= n(n-1)(n-2)(n-3)\cdots(n-(n-1))\,a_{n-n} \\
&= n(n-1)(n-2)(n-3)\cdots 1 \cdot a_0 \\
&= n! \cdot 5 = 5n!\,.
\end{aligned}$$

h) We write

$$\begin{aligned}
a_n &= 2na_{n-1} \\
&= 2n\big(2(n-1)a_{n-2}\big) = 2^2\big(n(n-1)\big)a_{n-2} \\
&= 2^2\big(n(n-1)\big)\big(2(n-2)a_{n-3}\big) = 2^3\big(n(n-1)(n-2)\big)a_{n-3} \\
&\ \vdots \\
&= 2^n n(n-1)(n-2)(n-3)\cdots\big(n-(n-1)\big)a_{n-n} \\
&= 2^n n(n-1)(n-2)(n-3)\cdots 1 \cdot a_0 \\
&= 2^n n!\,.
\end{aligned}$$

19. **a)** Since the number of bacteria triples every hour, the recurrence relation should say that the number of bacteria after n hours is 3 times the number of bacteria after $n-1$ hours. Letting b_n denote the number of bacteria after n hours, this statement translates into the recurrence relation $b_n = 3b_{n-1}$.
b) The given statement is the initial condition $b_0 = 100$ (the number of bacteria at the beginning is the number of bacteria after no hours have elapsed). We solve the recurrence relation by iteration: $b_n = 3b_{n-1} = 3^2 b_{n-2} = \cdots = 3^n b_{n-n} = 3^n b_0$. Letting $n = 10$ and knowing that $b_0 = 100$, we see that $b_{10} = 3^{10} \cdot 100 = 5{,}904{,}900$.

21. **a)** Let c_n be the number of cars produced in the first n months. The initial condition could be taken to be $c_0 = 0$ (no cars are made in the first 0 months). Since n cars are made in the n^{th} month, and since c_{n-1} cars are made in the first $n-1$ months, we see that $c_n = c_{n-1} + n$.
b) The number of cars produced in the first year is c_{12}. To compute this we will solve the recurrence relation and initial condition, then plug in $n = 12$ (alternately, we could just compute the terms $c_1, c_2, \ldots, c_{12}$

directly from the definition). We proceed by iteration exactly as we did in Exercise 17c:

$$\begin{aligned}
c_n &= n + c_{n-1} \\
&= n + \big((n-1) + c_{n-2}\big) = \big(n + (n-1)\big) + c_{n-2} \\
&= \big(n + (n-1)\big) + \big((n-2) + c_{n-3}\big) = \big(n + (n-1) + (n-2)\big) + c_{n-3} \\
&\ \ \vdots \\
&= \big(n + (n-1) + (n-2) + \cdots + (n-(n-1))\big) + c_{n-n} \\
&= \big(n + (n-1) + (n-2) + \cdots + 1\big) + c_0 \\
&= \frac{n(n+1)}{2} + 0 = \frac{n^2+n}{2}
\end{aligned}$$

Therefore the number of cars produced in the first year is $(12^2 + 12)/2 = 78$.

c) We found the formula in our solution to part **(b)**.

23. Each month our account accrues some interest that must be paid. Since the balance the previous month is $B(k-1)$, the amount of interest we owe is $(0.07/12)B(k-1)$. After paying this interest, the rest of the \$100 payment we make each month goes toward reducing the principal. Therefore we have $B(k) = B(k-1) - (100 - (0.07/12)B(k-1))$. This can be simplified to $B(k) = (1 + (0.07/12))B(k-1) - 100$. The initial condition is $B(0) = 5000$. If one calculates this as k goes from 0 to 60, we see the balance gradually decrease and finally become negative when $k = 60$ (i.e., after five years).

25. In some sense there are no right answers here. The solutions stated are the most appealing patterns that the author has found.

a) It looks as if we have one 1 and one 0, then two of each, then three of each, and so on, increasing the number of repetitions by one each time. Thus we need three more 1's (and then four 0's) to continue the sequence.

b) A pattern here is that the positive integers are listed in increasing order, with each even number repeated. Thus the next three terms are $9, 10, 10$.

c) The terms in the odd locations (first, third, fifth, etc.) are just the successive terms in the geometric sequence that starts with 1 and has ratio 2, and the terms in the even locations are all 0. The n^{th} term is 0 if n is even and is $2^{(n-1)/2}$ if n is odd. Thus the next three terms are $32, 0, 64$.

d) The first term is 3 and each successive term is twice its predecessor. This is a geometric sequence. The n^{th} term is $3 \cdot 2^{n-1}$. Thus the next three terms are $384, 768, 1536$.

e) The first term is 15 and each successive term is 7 less than its predecessor. This is an arithmetic sequence. The n^{th} term is $22 - 7n$. Thus the next three terms are $-34, -41, -48$.

f) The rule is that the first term is 3 and the n^{th} term is obtained by adding n to the $(n-1)^{\text{th}}$ term. One can actually find a quadratic expression for a sequence in which the successive differences form an arithmetic sequence; here it is $(n^2 + n + 4)/2$. The easiest way to see this is to note that the n^{th} term is $3 + 2 + 3 + 4 + 5 + 6 + \cdots + n$. Except for the initial 3 instead of a 1, the n^{th} term is the sum of the first n positive integers, which is $n(n+1)/2$ by a formula in Table 2. Therefore the n^{th} term is $(n(n+1)/2) + 2$, as claimed. We see that the next three terms are $57, 68, 80$.

g) One should play around with the sequence if nothing is apparent at first. Here we note that all the terms are even, so if we divide by 2 we obtain the sequence $1, 8, 27, 64, 125, 216, 343, \ldots$. This sequence appears in Table 1; it is the cubes. So the n^{th} term is $2n^3$. Thus the next three terms are $1024, 1458, 2000$.

h) These terms look close to the terms of the sequence whose n^{th} term is $n!$ (see Table 1). In fact, we see that the n^{th} term here is $n! + 1$. Thus the next three terms are $362881, 3628801, 39916801$.

27. It is pretty clear that a_n should be approximately equal to $n+\sqrt{n}$, since the sequence is just the sequence of positive integers with perfect squares left out. There are about $\sqrt{n}$ perfect squares up to n, so the count needs to get ahead by about this amount. Proving that this plausibility argument gives the correct formula involves some careful counting.

The sequence begins $2,3,5,6,7,8,10,11,\ldots$. We can write it as the sequence $a_n = n$ plus a sequence b_n that jumps every time a perfect square is encountered. Thus $\{b_n\}$ begins $1,1,2,2,2,2,3,3,3,3,3,3,4,\ldots$; there are two 1's, four 2's, six 3's, eight 4's, and so on. So we must show that $b_n = \{\sqrt{n}\}$, where $\{\sqrt{n}\}$ means the integer closest to $\sqrt{n}$ (note that there is never an ambiguity here, since this will never be a half-integer). Because of the way the sequence is formed, $b_n \le k$ if and only if $2+4+6+\cdots+2k \ge n$. This is equivalent to $k(k+1)\ge n$. Applying the quadratic formula and recalling that k is an integer, we obtain $b_n = \lceil(-1+\sqrt{1+4n})/2\rceil$. Simplifying, we have $b_n = \left\lceil -\frac{1}{2}+\sqrt{n+\frac{1}{4}}\right\rceil$. Subtracting $\frac{1}{2}$ and then rounding up is the same as rounding to the nearest integer (the smaller one if $\sqrt{n+\frac{1}{4}}$ is a half-integer—see Exercise 47 in Section 2.3), so (with this understanding) $b_n = \left\{\sqrt{n+\frac{1}{4}}\right\}$. But it can never happen that $\sqrt{n}\le m+\frac{1}{2}$ while $\sqrt{n+\frac{1}{4}} > m+\frac{1}{2}$ for some positive integer m—this would imply that $n \le m^2+m+\frac{1}{4}$ and $n > m^2+m$, an impossibility. Therefore $\{\sqrt{n}\} = \left\{\sqrt{n+\frac{1}{4}}\right\}$, and we are done.

An alternative solution is provided in the answer section of the text.

29. **a)** $2+3+4+5+6=20$ **b)** $1-2+4-8+16=11$ **c)** $3+3+\cdots+3=10\cdot 3=30$
d) This series "telescopes": each term cancels part of the term before it (see also Exercise 35). The sum is $(2-1)+(4-2)+(8-4)+\cdots+(512-256)=-1+512=511$.

31. We use the formula for the sum of a geometric progression: $\sum_{j=0}^{n} ar^j = a(r^{n+1}-1)/(r-1)$.
a) Here $a=3$, $r=2$, and $n=8$, so the sum is $3(2^9-1)/(2-1)=1533$.
b) Here $a=1$, $r=2$, and $n=8$. The sum taken over all the values of j from 0 to n is, by the formula, $(2^9-1)/(2-1)=511$. However, our sum starts at $j=1$, so we must subtract out the term that isn't there, namely 2^0. Hence the answer is $511-1=510$.
c) Again we have to subtract the missing terms, so the sum is $((-3)^9-1)/((-3)-1)-(-3)^0-(-3)^1 = 4921-1-(-3)=4923$.
d) $2((-3)^9-1)/((-3)-1)=9842$

33. The easiest way to do these sums, since the number of terms is reasonably small, is just to write out the summands explicitly. Note that the inside index (j) runs through all of its values for each value of the outside index (i).
a) $(1+1)+(1+2)+(1+3)+(2+1)+(2+2)+(2+3)=21$
b) $(0+3+6+9)+(2+5+8+11)+(4+7+10+13)=78$
c) $(1+1+1)+(2+2+2)+(3+3+3)=18$
d) $(0+0+0)+(1+2+3)+(2+4+6)=18$

35. If we just write out what the sum means, we see that parts of successive terms cancel, leaving only two terms:

$$\sum_{j=1}^{n}(a_j-a_{j-1}) = a_1-a_0+a_2-a_1+a_3-a_2+\cdots+a_{n-1}-a_{n-2}+a_n-a_{n-1}=a_n-a_0$$

37. **a)** We use the hint, where $a_k=k^2$:

$$\sum_{k=1}^{n}(2k-1)=\sum_{k=1}^{n}(k^2-(k-1)^2)=n^2-0^2=n^2$$

b) We can use the distributive law to rewrite $\sum_{k=1}^{n}(2k-1)$ (which we know from part **(a)** equals n^2) in terms of the sum we want, $S = \sum_{k=1}^{n} k$:

$$n^2 = \sum_{k=1}^{n}(2k-1) = 2\sum_{k=1}^{n} k - \sum_{k=1}^{n} 1 = 2S - n\,.$$

Now we solve for S, obtaining $S = (n^2+n)/2$, which is usually expressed as $n(n+1)/2$.

39. This exercise is like Example 23. From Table 2 we know that $\sum_{k=1}^{200} k = 200 \cdot 201/2 = 20100$, and $\sum_{k=1}^{99} k = 99 \cdot 100/2 = 4950$. Therefore the desired sum is $20100 - 4950 = 15150$.

41. If we write down the first few terms of this sum we notice a pattern. It starts $(1+1+1)+(2+2+2+2+2)+(3+3+3+3+3+3+3)+\cdots$. There are three 1's, then five 2'ss, then seven 3'ss, and so on; in general there are $(i+1)^2 - i^2 = 2i+1$ copies of i. So we need to sum $i(2i+1)$ for an appropriate range of values for i. We must find this range. It gets a little messy at the end if m is such that the sequence stops before a complete range of the last value is present. Let $n = \lfloor\sqrt{m}\rfloor - 1$. Then there are $n+1$ blocks, and $(n+1)^2 - 1$ is where the next-to-last block ends. The sum of those complete blocks is $\sum_{i=1}^{n} i(2i+1) = \sum_{i=1}^{n} 2i^2 + i = n(n+1)(2n+1)/3 + n(n+1)/2$. The remaining terms in our summation all have the value $n+1$ and the number of them present is $m - ((n+1)^2 - 1)$. Our final answer is therefore $n(n+1)(2n+1)/3 + n(n+1)/2 + (n+1)(m-(n+1)^2+1)$.

43. **a)** 0 (anything times 0 is 0) **b)** $5 \cdot 6 \cdot 7 \cdot 8 = 1680$
c) Each factor is either 1 or -1, so the product is either 1 or -1. To see which it is, we need to determine how many of the factors are -1. Clearly there are 50 such factors, namely when $i = 1, 3, 5, \ldots, 99$. Since $(-1)^{50} = 1$, the product is 1.
d) $2 \cdot 2 \cdots 2 = 2^{10} = 1024$

45. $0! + 1! + 2! + 3! + 4! = 1 + 1 + 1 \cdot 2 + 1 \cdot 2 \cdot 3 + 1 \cdot 2 \cdot 3 \cdot 4 = 1 + 1 + 2 + 6 + 24 = 34$

SECTION 2.5 Cardinality of Sets

Don't feel bad if you find this section confusing. When Cantor started talking about sizes of infinity in the nineteenth century, many mathematicians thought he made no sense. The basic rule to keep in mind is that if an infinite set can be given in a list, then it is countable. It is not always easy to find the right list. Various indirect means are also available for showing that an infinite set is countable, such as showing that the set is a subset of a countable set, or showing that it is the union of a countable collection of countable sets. Proving sets uncountable usually requires some sort of diagonal argument, although in fact Cantor's first proof of the uncountability of the real numbers used a different approach (a nice summary can be found in an article in the American Mathematical Monthly, **117:7** (2010), 633–637*).*

1. **a)** The negative integers are countably infinite. Each negative integer can be paired with its absolute value to give the desired one-to-one correspondence: $1 \leftrightarrow -1$, $2 \leftrightarrow -2$, $3 \leftrightarrow -3$, and so on.

b) The even integers are countably infinite. We can list the set of even integers in the order $0, 2, -2, 4, -4, 6, -6, \ldots$ and pair them with the positive integers listed in their natural order. Thus $1 \leftrightarrow 0$, $2 \leftrightarrow 2$, $3 \leftrightarrow -2$, $4 \leftrightarrow 4$, and so on. There is no need to give a formula for this correspondence—the discussion given is quite sufficient; but it is not hard to see that we are pairing the positive integer n with the even integer $f(n)$, where $f(n) = n$ if n is even and $f(n) = 1 - n$ if n is odd.

c) This set is again countably infinite. We can list its elements in the order $99, 98, 97, \ldots$. A formula for a correspondence with the set of positive integers is given by $f(n) = 100 - n$. For example, the positive integer 117 is paired with -17.

d) The proof that the set of real numbers between 0 and 1 is not countable (Example 5) can easily be modified to show that the set of real numbers between 0 and $1/2$ is not countable. We need to let the digit d_i be something like 2 if $d_{ii} \neq 2$ and 3 otherwise. The number thus constructed will be a real number between 0 and $1/2$ that is not in the list.

e) This set is finite; it has cardinality 999,999,999.

f) This set is countably infinite, exactly as in part **(b)**; the only difference is that there we are looking at the multiples of 2 and here we are looking at the multiples of 7. The correspondence is given by pairing the positive integer n with $7n/2$ if n is even and $-7(n-1)/2$ if n is odd: $0, 7, -7, 14, -14, 21, -21, \ldots$.

3. **a)** The bit strings not containing 0 are just the bit strings consisting of all 1's, so this set is $\{\lambda, 1, 11, 111, 1111, \ldots\}$, where λ denotes the empty string (the string of length 0). Thus this set is countably infinite, where the correspondence matches the positive integer n with the string of $n-1$ 1's.

b) This is a subset of the set of rational numbers, so it is countable (see Exercise 16). To find a correspondence, we can just follow the path in Example 4, but omit fractions in the top three rows (as well as continuing to omit those fractions that are duplicates of rational numbers already encountered).

c) This set is uncountable, as can be shown by applying the diagonal argument of Example 5.

d) This set is uncountable, as can be shown by applying the diagonal argument of Example 5.

5. Suppose m new guests arrive at the fully occupied hotel. If we move the guest in Room 1 to Room $m+1$, the guest in Room 2 to Room $m+2$, and so on, then rooms with numbers from 1 to m become vacant. The new guests can then occupy these rooms.

7. We can use the guests in the even-numbered rooms to occupy the original rooms, and the guests in the odd-numbered rooms to occupy the rooms in the second building. Specifically, for each positive integer n, put the guest currently in Room $2n$ into Room n, and the guest currently in Room $2n-1$ into Room n of the new building.

9. There is more than one way to do this. Here is one method. First spread out the original guests so that the gaps between occupied rooms get larger and larger. Specifically, keep the first guest (i.e., the one currently in Room 1) in Room 1; leave Room 2 vacant; put the second guest (the one currently in Room 2) into Room 3; leave Rooms 4 and 5 vacant; put the third guest into Room 6; leave Rooms 7, 8, and 9 vacant, and so on. Have the guests from the first bus fill the first free room in each gap (Rooms 2, 4, 7, 11, and so on). After this is done, once again the gaps between occupied rooms get larger and larger. (The unoccupied rooms are now 5, 8, 9, 12, 13, 14, and so on.) So we can repeat the process with the second busload. We continue in this manner for the countable infinity of busloads. An alternative approach is given in the answer key.

11. In each case, we can make the intersection what we want it to be, and then put additional elements into A and into B (with no overlap) to make them uncountable.

a) The simplest solution would be to make $A \cap B = \emptyset$. So, for example, take A to be the interval $(1,2)$ of real numbers, and take B to be the interval $(3,4)$.
b) Take the example from part **(a)** and adjoin the positive integers. Thus, let $A = (1,2) \cup \mathbf{Z}^+$ and let $B = (3,4) \cup \mathbf{Z}^+$.
c) Let $A = (1,3)$ and $B = (2,4)$.

13. Suppose that A is countable. This means either that A is finite or that there exists a one-to-one correspondence f from A to $\mathbf{Z}^+$. In the former case, there is a one-to-one function g from A to a subset of $\mathbf{Z}^+$ (the range of g is the first n positive integers, where $|A| = n$). In either case, we have met the requirements of Definition 2, so $|A| \leq |\mathbf{Z}^+|$. Conversely, suppose that $|A| \leq |\mathbf{Z}^+|$. By definition, this means that there is a one-to-one function g from A to $\mathbf{Z}^+$, so A has the same cardinality as a subset of $\mathbf{Z}^+$ (namely, the range of g). Now by Exercise 16 we conclude that A is countable.

15. This is just the contrapositive of Exercise 16 and so follows directly from it. In more detail, suppose that B were countable, say with elements $b_1, b_2, \ldots$. Then since $A \subseteq B$, we can list the elements of A using the order in which they appear in this listing of B. Therefore A is countable, contradicting the hypothesis. Thus B is not countable.

17. Yes. We need to look at this from the other direction, by noting that $A = (A \cap B) \cup (A - B)$. We are given that B is countable, so its subset $A \cap B$ is also countable (Exercise 16). If $A - B$ were also countable, then, since the union of two countable sets is countable (Theorem 1), we would conclude that A is countable. But we are given that A is not countable. Therefore our assumption that $A - B$ is countable is wrong, and we conclude that $A - B$ is uncountable. (This is an example of a proof by contradiction.)

19. By what we are given, we know that there are bijections f from A to B and g from C to D. Then we can define a bijection from $A \times C$ to $B \times D$ by sending (a,c) to $(f(a), g(c))$. This is clearly one-to-one and onto, so we have shown that $A \times C$ and $B \times D$ have the same cardinality.

21. The definition of $|A| \leq |B|$ is that there is a one-to-one function $f : A \to B$. Similarly, we are given a one-to-one function $g : B \to C$. By Exercise 33 in Section 2.3, the composition $g \circ f : A \to C$ is one-to-one. Therefore by definition $|A| \leq |C|$.

23. This proof implicitly uses an assumption called the Axiom of Choice. Define a sequence a_1, a_2, a_3, $\ldots$ of elements of A as follows. First, a_1 is any element of A. Once we have selected a_1, a_2, a_3, $\ldots$, a_k, let a_{k+1} be any element of $A - \{a_1, a_2, \ldots, a_k\}$. Such an element must exist because A is infinite. The resulting set $\{a_1, a_2, a_3, \ldots\}$ is the desired countably infinite subset of A.

25. The set of finite strings of characters over a finite alphabet is countably infinite, because we can list these strings in alphabetical order by length. For example, if the alphabet is $\{a, b, c\}$, then our list is λ, a, b, c, aa, ab, ac, ba, bb, bc, ca, cb, cc, aaa, aab, ... (See also Exercise 29.) Therefore the infinite set S can be identified with an infinite subset of this countable set, which by Exercise 16 is also countably infinite.

27. Since empty sets do not contribute any elements to unions, we can assume that none of the sets in our given countable collection of countable sets is the empty set. If there are no sets in the collection, then the union is empty and therefore countable. Otherwise let the countable sets be A_1, A_2, ... (If there are only a finite number k of them, then we can still assume that they form an infinite sequence by taking $A_{k+1} = A_{k+2} = \cdots = A_1$.) Since each set A_i is countable and nonempty, we can list its elements in a sequence as a_{i1}, a_{i2}, ...; again, if the set is finite we can list its elements and then list a_{i1} repeatedly to assure an infinite sequence. Now we just need a systematic way to put all the elements a_{ij} into a sequence. We do this by listing first all the elements a_{ij} in which $i+j=2$ (there is only one such pair, $(1,1)$), then all the elements in which $i+j=3$ (there are only two such pairs, $(1,2)$ and $(2,1)$), and so on; except that we do not list any element that we have already listed. So, assuming that these elements are distinct, our list starts a_{11}, a_{12}, a_{21}, a_{13}, a_{22}, a_{31}, a_{14}, ... (If any of these terms duplicates a previous term, then it is simply omitted.) The result of this process will be either an infinite sequence or a finite sequence containing all the elements of the union of the sets A_i. Thus that union is countable.

29. There are only a finite number of bit strings of each finite length, so we can list all the bit strings by listing first those of length 0, then those of length 1, etc. The listing might be $\lambda, 0, 1, 00, 01, 10, 11, 000, 001, \ldots$. (Recall that λ denotes the empty string.) Actually this is a special case of Exercise 27: the set of all bit strings is the union of a countable number of countable (actually finite) sets, namely the sets of bit strings of length n for $n = 0, 1, 2, \ldots$.

31. A little experimentation with this function shows the pattern:

$f(1,1)=1$	$f(2,1)=3$	$f(3,1)=6$	$f(4,1)=10$	$f(5,1)=15$	$f(6,1)=21$
$f(1,2)=2$	$f(2,2)=5$	$f(3,2)=9$	$f(4,2)=14$	$f(5,2)=20$	$f(6,2)=27$
$f(1,3)=4$	$f(2,3)=8$	$f(3,3)=13$	$f(4,3)=19$	$f(5,3)=26$	
$f(1,4)=7$	$f(2,4)=12$	$f(3,4)=18$	$f(4,4)=25$		
$f(1,5)=11$	$f(2,5)=17$	$f(3,5)=24$			
$f(1,6)=16$	$f(2,6)=23$				
$f(1,7)=22$					

We see by looking at the diagonals of this table that the function takes on successive values as $m+n$ increases. When $m+n=2$, $f(m,n)=1$. When $m+n=3$, $f(m,n)$ takes on the values 2 and 3. When $m+n=4$, $f(m,n)$ takes on the values 4, 5, and 6. And so on. It is clear from the formula that the range of values the function takes on for a fixed value of $m+n$, say $m+n=x$, is $\frac{(x-2)(x-1)}{2}+1$ through $\frac{(x-2)(x-1)}{2}+(x-1)$, since m can assume the values $1, 2, 3, \ldots, (x-1)$ under these conditions, and the first term in the formula is a fixed positive integer when $m+n$ is fixed. To show that this function is one-to-one and onto, we merely need to show that the range of values for $x+1$ picks up precisely where the range of values for x left off, i.e., that $f(x-1,1)+1=f(1,x)$. We compute:

$$f(x-1,1)+1 = \frac{(x-2)(x-1)}{2}+(x-1)+1 = \frac{x^2-x+2}{2} = \frac{(x-1)x}{2}+1 = f(1,x)$$

33. It suffices to find one-to-one functions $f : (0,1) \to [0,1]$ and $g : [0,1] \to (0,1)$. We can obviously use the function $f(x) = x$ in the first case. For the second, we can just compress $[0,1]$ into, say, $[\frac{1}{3}, \frac{2}{3}]$; the increasing linear function $g(x) = (x+1)/3$ will do that. It then follows from the Schröder-Bernstein theorem that $|(0,1)| = |[0,1]|$.

35. We can follow the hint or argue as follows, which really amounts to the same thing. (See the answer key for a proof using bit strings.) Suppose there were such a one-to-one correspondence f from $\mathbf{Z}^+$ to the power set of $\mathbf{Z}^+$ (the set of all subsets of $\mathbf{Z}^+$). Thus, for each $x \in \mathbf{Z}^+$, $f(x)$ is a subset of $\mathbf{Z}^+$. We will derive a contradiction by showing that f is not onto; we do this by finding an element not in its range. To this end, let $A = \{x \mid x \notin f(x)\}$. We claim that A is not in the range of f. If it were, then $A = f(x_0)$ for some $x_0 \in \mathbf{Z}^+$. Let us look at whether $x_0 \in A$ or not. On the one hand, if $x_0 \in A$, then by the definition of A, it must be true that $x_0 \notin f(x_0)$, which means that $x_0 \notin A$; that is a contradiction. On the other hand, if if $x_0 \notin A$, then by the definition of A, it must be true that $x_0 \in f(x_0)$, which means that $x_0 \in A$, again a contradiction. Therefore no such one-to-one correspondence exists.

37. We argued in the solution to Exercise 29 that the set of all strings of symbols from the alphabet $\{0, 1\}$ is countable, since there are only a finite number of bit strings of each length. There was nothing special about the alphabet $\{0, 1\}$ in that argument. For any finite alphabet (for example, the alphabet consisting of all upper and lower case letters, numerals, and punctuation and other mathematical marks typically used in a programming language), there are only a finite number of strings of length 1 (namely the number of symbols in the alphabet), only a finite number of strings of length 2 (namely, the square of this number), and so on. Therefore, using the result of Exercise 27, we conclude that there are only countably many strings from any given finite alphabet. Now the set of all computer programs in a particular language is just a subset of the set of all strings over that alphabet (some strings are meaningless jumbles of symbols that are not valid programs), so by Exercise 16, this set, too, is countable.

39. In Exercise 37 we saw that there are only a countable number of computer programs, so there are only a countable number of computable functions. In Exercise 38 we saw that there are an uncountable number of functions. Hence not all functions are computable. Indeed, in some sense, since uncountable sets are so much bigger than countable sets, *almost all* functions are not computable! This is not really so surprising; in real life we deal with only a small handful of useful functions, and these are computable. Note that this is a nonconstructive proof—we have not exhibited even one noncomputable function, merely argued that they have to exist. Actually finding one is much harder, but it can be done. For example, the following function is not computable. Let T be the function from the set of positive integers to $\{0, 1\}$ defined by letting $T(n)$ be 0 if the number 0 is in the range of the function computed by the n^{th} computer program (where we list them in alphabetical order by length) and letting $T(n) = 1$ otherwise.

SECTION 2.6 Matrices

In addition to routine exercises with matrix calculations, there are several exercises here asking for proofs of various properties of matrix operations. In most cases the proofs follow immediately from the definitions of the matrix operations and properties of operations on the set from which the entries in the matrices are drawn. Also, the important notion of the (multiplicative) **inverse** *of a matrix is examined in Exercises 18–21. Keep in mind that some matrix operations are performed "entrywise," whereas others operate on whole rows or columns at a time. Exercise 29 foreshadows material in Section 9.4.*

1. **a)** Since $\mathbf{A}$ has 3 rows and 4 columns, its size is 3×4.

b) The third column of $\mathbf{A}$ is the 3×1 matrix $\begin{bmatrix} 1 \\ 4 \\ 3 \end{bmatrix}$.

c) The second row of $\mathbf{A}$ is the 1×4 matrix $[2\ 0\ 4\ 6]$.

d) This is the element in the third row, second column, namely 1.

e) The transpose of $\mathbf{A}$ is the 4×3 matrix $\begin{bmatrix} 1 & 2 & 1 \\ 1 & 0 & 1 \\ 1 & 4 & 3 \\ 3 & 6 & 7 \end{bmatrix}$.

3. **a)** We use the definition of matrix multiplication to obtain the four entries in the product $\mathbf{AB}$. The $(1, 1)^{\text{th}}$ entry is the sum $a_{11}b_{11} + a_{12}b_{21} = 2 \cdot 0 + 1 \cdot 1 = 1$. Similarly, the $(1, 2)^{\text{th}}$ entry is the sum $a_{11}b_{12} + a_{12}b_{22} = 2 \cdot 4 + 1 \cdot 3 = 11$; $(2, 1)^{\text{th}}$ entry is the sum $a_{21}b_{11} + a_{22}b_{21} = 3 \cdot 0 + 2 \cdot 1 = 2$; and $(2, 2)^{\text{th}}$ entry is the sum $a_{21}b_{12} + a_{22}b_{22} = 3 \cdot 4 + 2 \cdot 3 = 18$. Therefore the answer is $\begin{bmatrix} 1 & 11 \\ 2 & 18 \end{bmatrix}$.

b) The calculation is similar. Again, to get the $(i, j)^{\text{th}}$ entry of the product, we need to add up all the products $a_{ik}b_{kj}$. You can visualize "lifting" the i^{th} row from the first factor ($\mathbf{A}$) and placing it on top of the

j^{th} column from the second factor ($\mathbf{B}$), multiplying the pairs of numbers that lie on top of each other, and taking the sum. Here we have

$$\begin{bmatrix} 1 & -1 \\ 0 & 1 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} 3 & -2 & -1 \\ 1 & 0 & 2 \end{bmatrix} = \begin{bmatrix} 1\cdot 3+(-1)\cdot 1 & 1\cdot(-2)+(-1)\cdot 0 & 1\cdot(-1)+(-1)\cdot 2 \\ 0\cdot 3+1\cdot 1 & 0\cdot(-2)+1\cdot 0 & 0\cdot(-1)+1\cdot 2 \\ 2\cdot 3+3\cdot 1 & 2\cdot(-2)+3\cdot 0 & 2\cdot(-1)+3\cdot 2 \end{bmatrix}$$
$$= \begin{bmatrix} 2 & -2 & -3 \\ 1 & 0 & 2 \\ 9 & -4 & 4 \end{bmatrix}.$$

c) The calculation is similar to the previous parts:

$$\begin{bmatrix} 4 & -3 \\ 3 & -1 \\ 0 & -2 \\ -1 & 5 \end{bmatrix} \begin{bmatrix} -1 & 3 & 2 & -2 \\ 0 & -1 & 4 & -3 \end{bmatrix}$$
$$= \begin{bmatrix} 4\cdot(-1)+(-3)\cdot 0 & 4\cdot 3+(-3)\cdot(-1) & 4\cdot 2+(-3)\cdot 4 & 4\cdot(-2)+(-3)\cdot(-3) \\ 3\cdot(-1)+(-1)\cdot 0 & 3\cdot 3+(-1)\cdot(-1) & 3\cdot 2+(-1)\cdot 4 & 3\cdot(-2)+(-1)\cdot(-3) \\ 0\cdot(-1)+(-2)\cdot 0 & 0\cdot 3+(-2)\cdot(-1) & 0\cdot 2+(-2)\cdot 4 & 0\cdot(-2)+(-2)\cdot(-3) \\ (-1)\cdot(-1)+5\cdot 0 & (-1)\cdot 3+5\cdot(-1) & (-1)\cdot 2+5\cdot 4 & (-1)\cdot(-2)+5\cdot(-3) \end{bmatrix}$$
$$= \begin{bmatrix} -4 & 15 & -4 & 1 \\ -3 & 10 & 2 & -3 \\ 0 & 2 & -8 & 6 \\ 1 & -8 & 18 & -13 \end{bmatrix}$$

5. First we need to observe that $\mathbf{A} = [a_{ij}]$ must be a 2×2 matrix; it must have two rows since the matrix it is being multiplied by on the left has two columns, and it must have two columns since the answer obtained has two columns. If we write out what the matrix multiplication means, then we obtain the following system of linear equations:

$$2a_{11} + 3a_{21} = 3$$
$$2a_{12} + 3a_{22} = 0$$
$$1a_{11} + 4a_{21} = 1$$
$$1a_{12} + 4a_{22} = 2$$

Solving these equations by elimination of variables (or other means—it's really two systems of two equations each in two unknowns), we obtain $a_{11} = 9/5$, $a_{12} = -6/5$, $a_{21} = -1/5$, $a_{22} = 4/5$. As a check we compute that, indeed,

$$\begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} \begin{bmatrix} 9/5 & -6/5 \\ -1/5 & 4/5 \end{bmatrix} = \begin{bmatrix} 3 & 0 \\ 1 & 2 \end{bmatrix}.$$

7. Since the $(i,j)^{\text{th}}$ entry of $\mathbf{0} + \mathbf{A}$ is the sum of the $(i,j)^{\text{th}}$ entry of $\mathbf{0}$ (namely 0) and the $(i,j)^{\text{th}}$ entry of $\mathbf{A}$, this entry is the same as the $(i,j)^{\text{th}}$ entry of $\mathbf{A}$. Therefore by the definition of matrix equality, $\mathbf{0} + \mathbf{A} = \mathbf{A}$. A similar argument shows that $\mathbf{A} + \mathbf{0} = \mathbf{A}$.

9. We simply look at the $(i,j)^{\text{th}}$ entries of each side. The $(i,j)^{\text{th}}$ entry of the left-hand side is $a_{ij} + (b_{ij} + c_{ij})$. The $(i,j)^{\text{th}}$ entry of the right-hand side is $(a_{ij} + b_{ij}) + c_{ij}$. By the associativity law for real number addition, these are equal. The conclusion follows.

11. In order for $\mathbf{AB}$ to be defined, the number of columns of $\mathbf{A}$ must equal the number of rows of $\mathbf{B}$. In order for $\mathbf{BA}$ to be defined, the number of columns of $\mathbf{B}$ must equal the number of rows of $\mathbf{A}$. Thus for some positive integers m and n, it must be the case that $\mathbf{A}$ is an $m \times n$ matrix and $\mathbf{B}$ is an $n \times m$ matrix. Another way to say this is to say that $\mathbf{A}$ must have the same size as $\mathbf{B}^t$ (and/or vice versa).

13. Let us begin with the left-hand side and find its $(i,j)^{\text{th}}$ entry. First we need to find the entries of $\mathbf{BC}$. By definition, the $(q,j)^{\text{th}}$ entry of $\mathbf{BC}$ is $\sum_{r=1}^{k} b_{qr}c_{rj}$. (See Section 2.4 for the meaning of summation notation. This symbolism is a shorthand way of writing $b_{q1}c_{1j} + b_{q2}c_{2j} + \cdots + b_{qk}c_{kj}$.) Therefore the $(i,j)^{\text{th}}$ entry of $\mathbf{A}(\mathbf{BC})$ is $\sum_{q=1}^{p} a_{iq}\left(\sum_{r=1}^{k} b_{qr}c_{rj}\right)$. By distributing multiplication over addition (for real numbers), we can move the term a_{iq} inside the inner summation, to obtain $\sum_{q=1}^{p}\sum_{r=1}^{k} a_{iq}b_{qr}c_{rj}$. (We are also implicitly using associativity of multiplication of real numbers here, to avoid putting parentheses in the product $a_{iq}b_{qr}c_{rj}$.)

A similar analysis with the right-hand side shows that the $(i,j)^{\text{th}}$ entry there is equal to $\sum_{r=1}^{k}\left(\sum_{q=1}^{p} a_{iq}b_{qr}\right)c_{rj}$ $= \sum_{r=1}^{k}\sum_{q=1}^{p} a_{iq}b_{qr}c_{rj}$. Now by the commutativity of addition, the order of summation (whether we sum over r first and then q, or over q first and then r) does not matter, so these two expressions are equal, and the proof is complete.

15. Let us begin by computing $\mathbf{A}^n$ for the first few values of n.

$$\mathbf{A}^1 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}^2 = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}^3 = \begin{bmatrix} 1 & 3 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}^4 = \begin{bmatrix} 1 & 4 \\ 0 & 1 \end{bmatrix}, \quad \mathbf{A}^5 = \begin{bmatrix} 1 & 5 \\ 0 & 1 \end{bmatrix}.$$

It seems clear from this pattern, then, that $\mathbf{A}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$. (A proof of this fact could be given using mathematical induction, discussed in Section 5.1.)

17. **a)** The $(i,j)^{\text{th}}$ entry of $(\mathbf{A}+\mathbf{B})^t$ is the $(j,i)^{\text{th}}$ entry of $\mathbf{A}+\mathbf{B}$, namely $a_{ji}+b_{ji}$. On the other hand, the $(i,j)^{\text{th}}$ entry of $\mathbf{A}^t+\mathbf{B}^t$ is the sum of the $(i,j)^{\text{th}}$ entries of $\mathbf{A}^t$ and $\mathbf{B}^t$, which are the $(j,i)^{\text{th}}$ entries of $\mathbf{A}$ and $\mathbf{B}$, again $a_{ji}+b_{ji}$. Hence $(\mathbf{A}+\mathbf{B})^t = \mathbf{A}^t+\mathbf{B}^t$.

b) The $(i,j)^{\text{th}}$ entry of $(\mathbf{AB})^t$ is the $(j,i)^{\text{th}}$ entry of $\mathbf{AB}$, namely $\sum_{k=1}^{n} a_{jk}b_{ki}$. (See Section 2.4 for the meaning of summation notation. This symbolism is a shorthand way of writing $ba_{j1}b_{1i} + a_{j2}b_{2i} + \cdots + a_{jn}b_{ni}$.) On the other hand, the $(i,j)^{\text{th}}$ entry of $\mathbf{B}^t\mathbf{A}^t$ is $\sum_{k=1}^{n} b_{ki}a_{jk}$ (since the $(i,k)^{\text{th}}$ entry of $\mathbf{B}^t$ is b_{ki} and the $(k,j)^{\text{th}}$ entry of $\mathbf{A}^t$ is a_{jk}). By the commutativity of multiplication of real numbers, these two values are the same, so the matrices are equal.

19. All we have to do is form the products $\mathbf{AA}^{-1}$ and $\mathbf{A}^{-1}\mathbf{A}$, using the purported $\mathbf{A}^{-1}$, and see that both of them are the 2×2 identity matrix. It is easy to see that the upper left and lower right entries in each case are $(ad-bc)/(ad-bc)=1$, and the upper right and lower left entries are all 0.

21. We must show that $\mathbf{A}^n(\mathbf{A}^{-1})^n = \mathbf{I}$, where $\mathbf{I}$ is the $n\times n$ identity matrix. Since matrix multiplication is associative, we can write this product as

$$\mathbf{A}^n\left((\mathbf{A}^{-1})^n\right) = \mathbf{A}(\mathbf{A}\ldots(\mathbf{A}(\mathbf{AA}^{-1})\mathbf{A}^{-1})\ldots\mathbf{A}^{-1})\mathbf{A}^{-1}.$$

By dropping each $\mathbf{AA}^{-1} = \mathbf{I}$ from the center as it is obtained, this product reduces to $\mathbf{I}$. Similarly $\left((\mathbf{A}^{-1})^n\right)\mathbf{A}^n = \mathbf{I}$. Therefore by definition $(\mathbf{A}^n)^{-1} = (\mathbf{A}^{-1})^n$. (A more formal proof requires mathematical induction; see Section 5.1.)

23. By definition, the $(i,j)^{\text{th}}$ entry of $\mathbf{A}+\mathbf{A}^t$ is $a_{ij}+a_{ji}$. Similarly, the $(j,i)^{\text{th}}$ entry of $\mathbf{A}+\mathbf{A}^t$ is $a_{ji}+a_{ij}$. By the commutativity of addition, these are equal, so $\mathbf{A}+\mathbf{A}^t$ is symmetric by the definition of symmetric matrices.

25. Using the idea in Exercise 24, we see that the given system can be expressed as $\mathbf{AX} = \mathbf{B}$, where $\mathbf{A}$ is the coefficient matrix, $\mathbf{X}$ is an $n \times 1$ matrix with x_i the entry in its i^{th} row, and $\mathbf{B}$ is the $n \times 1$ matrix of right-hand sides. Specifically we have

$$\begin{bmatrix} 7 & -8 & 5 \\ -4 & 5 & -3 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix}.$$

If we can find the inverse $\mathbf{A}^{-1}$, then we can find $\mathbf{X}$ simply by computing $\mathbf{A}^{-1}\mathbf{B}$. But Exercise 18 tells us that $\mathbf{A}^{-1} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{bmatrix}$. Therefore

$$\mathbf{X} = \begin{bmatrix} 2 & 3 & -1 \\ 1 & 2 & 1 \\ -1 & -1 & 3 \end{bmatrix} \begin{bmatrix} 5 \\ -3 \\ 0 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ -2 \end{bmatrix}.$$

We should plug in $x_1 = 1$, $x_2 = -1$, and $x_3 = -2$ to see that these do indeed form the solution.

27. These routine exercises simply require application of the appropriate definitions. Parts **(a)** and **(b)** are entry-wise operations, whereas the operation $\odot$ in part **(c)** is similar to matrix multiplication (the $(i,j)^{\text{th}}$ entry of $\mathbf{A} \odot \mathbf{B}$ depends on the i^{th} row of $\mathbf{A}$ and the j^{th} column of $\mathbf{B}$).

a) $\mathbf{A} \vee \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ **b)** $\mathbf{A} \wedge \mathbf{B} = \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}$ **c)** $\mathbf{A} \odot \mathbf{B} = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$

29. Note that $\mathbf{A}^{[2]}$ means $\mathbf{A} \odot \mathbf{A}$, and $\mathbf{A}^{[3]}$ means $\mathbf{A} \odot \mathbf{A} \odot \mathbf{A}$. We just apply the definition.

a) $\mathbf{A}^{[2]} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$ **b)** $\mathbf{A}^{[3]} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ **c)** $\mathbf{A} \vee \mathbf{A}^{[2]} \vee \mathbf{A}^{[3]} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$

31. These are immediate from the commutativity of the corresponding logical operations on variables.

a) $\mathbf{A} \vee \mathbf{B} = [a_{ij} \vee b_{ij}] = [b_{ij} \vee a_{ij}] = \mathbf{B} \vee \mathbf{A}$

b) $\mathbf{B} \wedge \mathbf{A} = [b_{ij} \wedge a_{ij}] = [a_{ij} \wedge b_{ij}] = \mathbf{A} \wedge \mathbf{B}$

33. These are immediate from the distributivity of the corresponding logical operations on variables.

a) $\mathbf{A} \vee (\mathbf{B} \wedge \mathbf{C}) = [a_{ij} \vee (b_{ij} \wedge c_{ij})] = [(a_{ij} \vee b_{ij}) \wedge (a_{ij} \vee c_{ij})] = (\mathbf{A} \vee \mathbf{B}) \wedge (\mathbf{A} \vee \mathbf{C})$

b) $\mathbf{A} \wedge (\mathbf{B} \vee \mathbf{C}) = [a_{ij} \wedge (b_{ij} \vee c_{ij})] = [(a_{ij} \wedge b_{ij}) \vee (a_{ij} \wedge c_{ij})] = (\mathbf{A} \wedge \mathbf{B}) \vee (\mathbf{A} \wedge \mathbf{C})$

35. The proof is identical to the proof in Exercise 13, except that real number multiplication is replaced by $\wedge$, and real number addition is replaced by $\vee$. Briefly, in symbols, $\mathbf{A} \odot (\mathbf{B} \odot \mathbf{C}) = \big[\bigvee_{q=1}^{p} a_{iq} \wedge \big(\bigvee_{r=1}^{k} b_{qr} \wedge c_{rj}\big)\big] = \big[\bigvee_{q=1}^{p} \bigvee_{r=1}^{k} a_{iq} \wedge b_{qr} \wedge c_{rj}\big] = \big[\bigvee_{r=1}^{k} \bigvee_{q=1}^{p} a_{iq} \wedge b_{qr} \wedge c_{rj}\big] = \big[\bigvee_{r=1}^{k} \big(\bigvee_{q=1}^{p} a_{iq} \wedge b_{qr}\big) \wedge c_{rj}\big] = (\mathbf{A} \odot \mathbf{B}) \odot \mathbf{C}$.

GUIDE TO REVIEW QUESTIONS FOR CHAPTER 2

1. See p. 119. To prove that A is a subset of B we need to show that an arbitrarily chosen element x of A must also be an element of B.

2. The empty set is the set with no elements. It satisfies the definition of being a subset of every set vacuously.

3. **a)** See p. 121. **b)** See p. 128.

4. **a)** See p. 121. **b)** always **c)** 2^n

5. **a)** See pp. 127 and 128 and the preamble to Exercise 32 in Section 2.2.
b) union: integers that are odd or positive; intersection: odd positive integers; difference: even positive integers; symmetric difference: even positive integers together with odd negative integers

6. **a)** $A = B \equiv (A \subseteq B \land B \subseteq A) \equiv \forall x(x \in A \leftrightarrow x \in B)$ **b)** See pp. 129–132.
c) $A \cap \overline{B \cap C} = A \cap (\overline{B} \cup \overline{C}) = (A \cap \overline{B}) \cup (A \cap \overline{C}) = (A - B) \cup (A - C)$; use Venn diagrams

7. Underlying each set identity is a logical equivalence. See, for instance, Example 11 in Section 2.2.

8. **a)** See p. 139. **b)** $\mathbf{Z}$, $\mathbf{Z}$, $\mathbf{Z}^+ = \mathbf{N} - \{0\}$

9. **a)** See p. 141. **b)** See p. 143. **c)** $f(n) = n$ **d)** $f(n) = 2n$ **e)** $f(n) = \lceil n/2 \rceil$
f) $f(n) = 42548$

10. **a)** See p. 145: $f^{-1}(b) = a \equiv f(a) = b$
b) when it is one-to-one and onto
c) yes—itself

11. **a)** See p. 149. **b)** integers

12. Hint: subtract 5 from each term and look at the resulting sequence.

13. The formula depends on the initial condition, namely the value of a_0. After that, each term is 5 less than the preceding term, so $a_n = a_0 - 5n$.

14. See p. 164.

15. Set up a one-to-one correspondence between the set of positive integers and the set of all odd integers, such as $1 \leftrightarrow 1$, $2 \leftrightarrow -1$, $3 \leftrightarrow 3$, $4 \leftrightarrow -3$, $5 \leftrightarrow 5$, $6 \leftrightarrow -5$, and so on.

16. See Example 5 in Section 2.5.

17. See page 179. For $\mathbf{AB}$ to be defined, the number of columns of $\mathbf{A}$ must equal the number of rows of $\mathbf{B}$.

18. See Example 4 in Section 2.6.

SUPPLEMENTARY EXERCISES FOR CHAPTER 2

1. a) $\overline{A} =$ the set of words that are not in A

b) $A \cap B =$ the set of words that are in both A and B

c) $A - B =$ the set of words that are in A but not B

d) $\overline{A} \cap \overline{B} = \overline{(A \cup B)} =$ the set of words that are in neither A nor B

e) $A \oplus B =$ the set of words that are in A or B but not both (can also be written as $(A - B) \cup (B - A)$ or as $(A \cup B) - (A \cap B)$)

3. Yes. We must show that every element of A is also an element of B. So suppose a is an arbitrary element of A. Then $\{a\}$ is a subset of A, so it is an element of the power set of A. Since the power set of A is a subset of the power set of B, it follows that $\{a\}$ is an element of the power set of B, which means that $\{a\}$ is a subset of B. But this means that the element of $\{a\}$, namely a, is an element of B, as desired.

5. We will show that each side is a subset of the other. First suppose $x \in A - (A - B)$. Then $x \in A$ and $x \notin A - B$. Now the only way for x not to be in $A - B$, given that it is in A, is for it to be in B. Thus we have that x is in both A and B, so $x \in A \cap B$. For the other direction, let $x \in A \cap B$. Then $x \in A$ and $x \in B$. It follows that $x \notin A - B$, and so x is in $A - (A - B)$.

7. We need only provide a counterexample to show that $(A - B) - C$ is not necessarily equal to $A - (B - C)$. Let $A = C = \{1\}$, and let $B = \emptyset$. Then $(A - B) - C = (\{1\} - \emptyset) - \{1\} = \{1\} - \{1\} = \emptyset$, whereas $A - (B - C) = \{1\} - (\emptyset - \{1\}) = \{1\} - \emptyset = \{1\}$.

9. This is not necessarily true. For a counterexample, let $A = B = \{1, 2\}$, let $C = \emptyset$, and let $D = \{1\}$. Then $(A - B) - (C - D) = \emptyset - \emptyset = \emptyset$, but $(A - C) - (B - D) = \{1, 2\} - \{2\} = \{1\}$.

11. a) Since $\emptyset \subseteq A \cap B \subseteq A \subseteq A \cup B \subseteq U$, we have the order $|\emptyset| \le |A \cap B| \le |A| \le |A \cup B| \le |U|$.

b) Note that $A - B \subseteq A \oplus B \subseteq A \cup B$. Also recall that $|A \cup B| = |A| + |B| - |A \cap B|$, so that $|A \cup B|$ is always less than or equal to $|A| + |B|$. Putting this all together, we have $|\emptyset| \le |A - B| \le |A \oplus B| \le |A \cup B| \le |A| + |B|$.

13. a) Yes, f is one-to-one, since each element of the domain $\{1, 2, 3, 4\}$ is sent by f to a different element of the codomain. No, g is not one-to-one, since g sends the two different elements a and d of the domain to the same element, 2.

b) Yes, f is onto, since every element in the codomain $\{a, b, c, d\}$ is the image under f of some element in the domain $\{1, 2, 3, 4\}$. In other words, the range of f is the entire codomain. No, g is not onto, since the element 4 in the codomain is not in the range of g (is not the image under g of any element of the domain $\{a, b, c, d\}$).

c) Certainly f has an inverse, since it is one-to-one and onto. Its inverse is the function from $\{a, b, c, d\}$ to $\{1, 2, 3, 4\}$ that sends a to 3, sends b to 4, sends c to 2, and sends d to 1. (Each element in $\{a, b, c, d\}$ gets sent by f^{-1} to the element in $\{1, 2, 3, 4\}$ that gets sent to it by f.) Since g is not one-to-one and onto, it has no inverse.

15. If f is one-to-one, then f provides a bijection between S and $f(S)$, so they have the same cardinality by definition. If f is not one-to-one, then there exist elements x and y in S such that $f(x) = f(y)$. Let $S = \{x, y\}$. Then $|S| = 2$ but $|f(S)| = 1$. Note that we do not need the hypothesis that A and B are finite.

17. The key is to look at sets with just one element. On these sets, the induced functions act just like the original functions. So let x be an arbitrary element of A. Then $\{x\} \in \mathcal{P}(A)$, and $S_f(\{x\}) = \{\, f(y) \mid y \in \{x\}\,\} = \{f(x)\}$. By the same reasoning, $S_g(\{x\}) = \{g(x)\}$. Since $S_f = S_g$, we can conclude that $\{f(x)\} = \{g(x)\}$, and so necessarily $f(x) = g(x)$.

19. This is certainly true if either x or y is an integer, since then this equation is equivalent to the identity (4a) in Table 1 of Section 2.3. Otherwise, write x and y in terms of their integer and fractional parts: $x = n + \epsilon$ and $y = m + \delta$, where $n = \lfloor x \rfloor$, $0 < \epsilon < 1$, $m = \lfloor y \rfloor$, and $0 < \delta < 1$. If $\delta + \epsilon < 1$, then the equation is true, since both sides equal $m + n$; if $\delta + \epsilon \geq 1$, then the equation is false, since the left-hand side equals $m + n$, but the right-hand side equals $m + n + 1$. In summary, the equation is true if and only if either at least one of x and y is an integer or the sum of the fractional parts of x and y is less than 1. (Note that the second condition in the disjunction subsumes the first.)

21. Write x and y in terms of their integer and fractional parts: $x = n + \epsilon$ and $y = m + \delta$, where $n = \lfloor x \rfloor$, $0 \leq \epsilon < 1$, $m = \lfloor y \rfloor$, and $0 \leq \delta < 1$. If $\delta = \epsilon = 0$, then both sides equal $n + m$. If $\epsilon = 0$ but $\delta > 0$, then the left-hand side equals $n + m + 1$, but the right-hand side equals $n + m$. If $\epsilon > 0$, then the right-hand side equals $n + m + 1$, so the two sides will be equal if and only if $\epsilon + \delta \leq 1$ (otherwise the left-hand side would be $n + m + 2$). In summary, the equation is true if and only if either both x and y are integers, or x is not an integer but the sum of the fractional parts of x and y is less than or equal to 1.

23. If x is an integer, then clearly $\lfloor x \rfloor + \lfloor m - x \rfloor = x + m - x = m$. Otherwise, write x in terms of its integer and fractional parts: $x = n + \epsilon$, where $n = \lfloor x \rfloor$ and $0 < \epsilon < 1$. In this case $\lfloor x \rfloor + \lfloor m - x \rfloor = \lfloor n + \epsilon \rfloor + \lfloor m - n - \epsilon \rfloor = n + m - n - 1 = m - 1$, because we had to round $m - n - \epsilon$ down to the next smaller integer.

25. Write $n = 2k + 1$ for some integer k. Then $n^2 = 4k^2 + 4k + 1$, so $n^2/4 = k^2 + k + \frac{1}{4}$. Therefore $\lceil n^2/4 \rceil = k^2 + k + 1$. But we also have $(n^2 + 3)/4 = (4k^2 + 4k + 1 + 3)/4 = (4k^2 + 4k + 4)/4 = k^2 + k + 1$.

27. Let us write $x = n + (r/m) + \epsilon$, where n is an integer, r is an nonnegative integer less than m, and ϵ is a real number with $0 \leq \epsilon < 1/m$. In other words, we are peeling off the integer part of x (i.e., $n = \lfloor x \rfloor$) and the whole multiples of $1/m$ beyond that. Then the left-hand side is $\lfloor nm + r + m\epsilon \rfloor = nm + r$. On the right-hand side, the terms $\lfloor x \rfloor$ through $\lfloor x + (m - r - 1)/m \rfloor$ are all just n, and the remaining terms, if any, from $\lfloor x + (m - r)/m \rfloor$ through $\lfloor x + (m - 1)/m \rfloor$, are all $n + 1$. Therefore the right-hand side is $(m - r)n + r(n + 1) = nm + r$ as well.

29. This product telescopes. The numerator in the fraction for k cancels the denominator in the fraction for $k+1$. So all that remains of the product is the numerator for $k = 100$ and the denominator for $k = 1$, namely $101/1 = 101$.

31. There is no good way to determine a nice rule for this kind of problem. One just has to look at the sequence and see what seems to be happening. In this sequence, we notice that $10 = 2 \cdot 5$, $39 = 3 \cdot 13$, $172 = 4 \cdot 43$, and $885 = 5 \cdot 177$. We then also notice that $3 = 1 \cdot 3$ for the second and third terms. So each odd-indexed term (assuming that we call the first term a_1) comes from the term before it, by multiplying by successively larger integers. In symbols, this says that $a_{2n+1} = n \cdot a_{2n}$ for all $n > 0$. Then we notice that the even-indexed terms are obtained in a similar way by adding: $a_{2n} = n + a_{2n-1}$ for all $n > 0$. So the next four terms are $a_{13} = 6 \cdot 891 = 5346$, $a_{14} = 7 + 5346 = 5353$, $a_{15} = 7 \cdot 5353 = 37471$, and $a_{14} = 8 + 37471 = 37479$.

33. If such a function f exists, then S equals the union of a countable number of countable sets, namely $f^{-1}(1) \cup f^{-1}(2) \cup \cdots$. It follows from Exercise 27 in Section 2.5 that S is countable.

35. Because there is a one-to-one correspondence between $\mathbf{R}$ and the open interval $(0,1)$ (given by $f(x) = 2\arctan(x)/\pi$, it suffices to shows that $|(0,1)\times(0,1)| = |(0,1)|$. We use the Schröder-Bernstein theorem; it suffices to find injective functions f from $(0,1)$ to $(0,1)\times(0,1)$ and g from $(0,1)\times(0,1)$ to $(0,1)$. For f we can just use $f(x) = (x, \frac{1}{2})$; it is clearly injective. For g we follow the hint. Suppose $(x,y) \in (0,1)\times(0,1)$, and represent x and y with their decimal expansions, never choosing the expansion of any number that ends in an infinite string of 9s (we can avoid that by having all finite decimals end in an infinite string of 0s). Let $x = 0.x_1x_2x_3\ldots$ and $y = 0.y_1y_2y_3\ldots$ be these expansions. Let $g(x,y)$ be the decimal expansion obtained by interweaving these two strings, namely $0.x_1y_1x_2y_2x_3y_3\ldots$. Because we can recover x and y from $g(x,y)$ (namely by taking every other digit starting with the first or second decimal digit, respectively), it follows that g is one-to-one, and our proof is complete.

37. Let us begin by computing $\mathbf{A}^n$ for the first few values of n.

$$\mathbf{A}^1 = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{A}^2 = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{A}^3 = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{A}^4 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Since $\mathbf{A}^4 = \mathbf{I}$, the pattern will repeat from here: $\mathbf{A}^5 = \mathbf{A}^4\mathbf{A} = \mathbf{I}\mathbf{A} = \mathbf{A}$, $\mathbf{A}^6 = \mathbf{A}^2$, $\mathbf{A}^7 = \mathbf{A}^3$, and so on. Thus for all $n \geq 0$ we have

$$\mathbf{A}^{4n+1} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}, \quad \mathbf{A}^{4n+2} = \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix}, \quad \mathbf{A}^{4n+3} = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, \quad \mathbf{A}^{4n+4} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

39. (The notation $c\mathbf{I}$ means the identity matrix $\mathbf{I}$ with each entry multiplied by the real number c; thus this matrix consists of c's along the main diagonal and 0's elsewhere.) Let $\mathbf{A} = \begin{bmatrix} u & v \\ w & x \end{bmatrix}$. We will determine what these entries have to be by using the fact that $\mathbf{AB} = \mathbf{BA}$ for a few judiciously chosen matrices $\mathbf{B}$. First let $\mathbf{B} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$. Then $\mathbf{AB} = \begin{bmatrix} 0 & u \\ 0 & w \end{bmatrix}$, and $\mathbf{BA} = \begin{bmatrix} w & x \\ 0 & 0 \end{bmatrix}$. Since these two must be equal, we know that $0 = w$ and $u = x$. Next choose $\mathbf{B} = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}$. Then we get $\begin{bmatrix} v & 0 \\ x & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ u & v \end{bmatrix}$, whence $v = 0$. Therefore the matrix $\mathbf{A}$ must be in the form $\begin{bmatrix} u & 0 \\ 0 & u \end{bmatrix}$, which is just u times the identity matrix, as desired.

41. **a)** The $(i,j)^{\text{th}}$ entry of $\mathbf{A} \odot \mathbf{0}$ is by definition the Boolean sum ($\vee$) of some Boolean products ($\wedge$) of the form $a_{ik} \wedge 0$. Since the latter always equals 0, every entry is 0, so $\mathbf{A} \odot \mathbf{0} = \mathbf{0}$. Similarly $\mathbf{0} \odot \mathbf{A}$ consists of entries that are all 0, so it, too, equals $\mathbf{0}$.

b) Since $\vee$ operates entrywise, the statements that $\mathbf{A} \vee \mathbf{0} = \mathbf{A}$ and $\mathbf{0} \vee \mathbf{A} = \mathbf{A}$ follow from the facts that $a_{ij} \vee 0 = a_{ij}$ and $0 \vee a_{ij} = a_{ij}$.

c) Since $\wedge$ operates entrywise, the statements that $\mathbf{A} \wedge \mathbf{0} = \mathbf{0}$ and $\mathbf{0} \wedge \mathbf{A} = \mathbf{0}$ follow from the facts that $a_{ij} \wedge 0 = 0$ and $0 \wedge a_{ij} = 0$.

WRITING PROJECTS FOR CHAPTER 2

Books and articles indicated by bracketed symbols below are listed near the end of this manual. You should also read the general comments and advice you will find there about researching and writing these essays.

1. A classic source here is [Wil]. It gives a very readable account of many philosophical issues in the foundations of mathematics, including the topic for this essay.

2. Our list of references mentions several history of mathematics books, such as [Bo4] and [Ev3]. You should also browse the shelves in your library, around QA 21.

3. Go to the Encyclopedia's website, `oeis.org`.

4. A Web search should turn up some useful references here, including an article in *Science News Online.* It gets its name from the fact that a graph describing it looks like the output of an electrocardiogram.

5. A Web search for this phrase will turn up much information.

6. A classic source here is [Wil]. It gives a very readable account of many mathematical and philosophical issues in the foundations of mathematics, including the topic for this essay. Of course a Web search will turn up lots of useful material, as well. Your essay should delve into the generalized continuum hypothesis for the higher orders of infinity: $2^{\aleph_n} = \aleph_{n+1}$.