Branching Processes and Fractals

The Death of a Name

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A Little About Probability

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2. An integer valued r.v. takes on a random integer value
3. All probabilities in the sample space sum to 1
4. Define the expected value of ξ as follows:
   $\mathbb{E}[\xi] = \sum_{i=1}^{\infty} i\mathbb{P}(\xi = i)$

The Probability Generating Function

Let X be an integer valued r.v. The P.G.F. is defined as such on $[0, 1]$:

$$G_X(s) = \mathbb{E}[s^X] = \sum_{i=0}^{\infty} \mathbb{P}(X = i)s^i$$

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4. G_X is convex (that is, $G''(s) \geq 0 \ \forall s \in [0, 1]$)

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4. G_X is convex (that is, $G''(s) \geq 0 \; \forall s \in [0,1]$)
5. The convexity of G_X tells us $G'(s)$ is increasing

The Probability Generating Function

Corollary

Let $X_1, X_2, \ldots$ be i.i.d. copies of X with P.G.F. G_X, and let N be a r.v. taking on an integer value independent from X. Then $G_{X_1+X_2+\ldots+X_N}(s) = G_N(G_X(s))$.

The Probability Generating Function

Proof.

$$
\begin{aligned}
G_{X_1+\ldots+X_N}(s) &= \mathbb{E}[s^{X_1+\ldots+X_N}] \\
&= \sum_{i=0}^{\infty} \mathbb{E}[s^{X_1+\ldots+X_N} | N = i]\mathbb{P}(N = i) \\
&= \sum_{i=0}^{\infty} \mathbb{E}[s^{X_1+\ldots+X_i} | N = i]\mathbb{P}(N = i) \\
&= \sum_{i=0}^{\infty} \mathbb{E}[s^{X_1+\ldots+X_i}]\mathbb{P}(N = i) \\
&= \sum_{i=0}^{\infty} (G_X(s))^i \mathbb{P}(N = i) \\
&= \mathbb{E}[(G_X(s))^N] = G_N(G_X(s))
\end{aligned}
$$

A Direct Consequence

Define $G_{X_1+\ldots+X_n}(s) = G_n(s)$

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Corollary

$G_n(s) = G_{n-1}(G(s))$

Binomial Random Variables

Coins, Coins, Coins

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Also note that we call $\xi \sim Binom(1, p)$ Bernoulli.

The Galton-Watson Process

Let X be an integer valued r.v. and assume that $0 < \mathbb{P}(X = 0) < 1$. Let $\zeta_i^k, i, k = 1, 2, ...$ be i.i.d. copies of X. Define a sequence $Z = (Z_n)_{n\geq 1}$ of r.v. by:

$$Z_0 = 1, Z_{n+1} = \zeta_1^{n+1} + ... + \zeta_{Z_n}^{n+1}, n \geq 0$$

This is often called an Z-process.

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Consider a situation in which an individual has some number of offspring (which we will call the individuals of generation 1), and those offspring also produce some number of offspring (generation 2). This process repeats until all the individuals produce no children.

Fixed Points and Extinction

Theorem

Let $Z = (Z_n)_{n \geq 0}$ be an Z-process. Let $\eta = \mathbb{P}(Z_n = 0 \text{ for some } n)$ be the probability of ultimate extinction of said process. Then η is the minimal fixed point of G in $[0, 1]$.

Note $\eta = \lim_{n \to \infty} \mathbb{P}(Z_n = 0)$

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$\eta = 1 \iff \mathbb{E}[X] \leq 1$

Some Examples

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2. If $Y = 0, 2$ and $\mathbb{P}(Y = 0) = 1 - p$ and $\mathbb{P}(Y = 2) = p$, when does this process become extinct?

A Restatement

Theorem

$\mathbb{P}(Z \textit{ contains an infinite tree with root } Z_0) = 0 \iff \mathbb{E}[X] \leq 1$

Binary Splittings

Definition

The process $Z = (Z_n)$ has an infinite binary splitting if it contains a binary tree as a subgraph of Z with the same root.

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Theorem

Let τ be the probability that the Z-process has a binary splitting. Then $1 - \tau$ is the smallest fixed point of $G(s) + (1 - s)G'(s)$ in $[0, 1]$.

A Short Example

Let $\tau(p)$ be the binary splitting probability for an Z-process with offspring distribution given by $\mathbb{P}(\xi = 1) = 1 - p$ and $\mathbb{P}(\xi = 3) = p$.

What can we say about the probability of an infinite binary tree existing?

A Cool Theorem

Theorem

Let $\tau(N)$ be the probability that the Z-process Z has an N-ary splitting. Then $1 - \tau(N)$ is the smallest fixed point in [0,1] of $\sum_{j=0}^{N-1}(1-s)^j \frac{G^{(j)}(s)}{j!}$.

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Proof.

The proof is left as an exercise for the audience. $\square$

Fix $N \in \mathbb{N}$ and some $p \in [0,1]$. Let $A_0 = [0,1]^2$ and let $S_{i,j} = [\frac{i-1}{N}, \frac{i}{N}] \times [\frac{j-1}{N}, \frac{j}{N}]$, for some $1 \leq i$ and $j \leq N$. Let $\varepsilon_{i,j} \sim Binom(1,p)$. When $\varepsilon_{i,j} = 1$ then it is said to be kept and write $A_1 = \bigcup_{\varepsilon_{i,j}=1} S_{i,j}$.

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Let $S^n_{i,j} = [\frac{i-1}{N^n}, \frac{i}{N^n}] \times [\frac{j-1}{N^n}, \frac{j}{N^n}]$ for some $1 \leq i$ and $j \leq N^n$. Let $\varepsilon^n_{i,j} \sim Binom(1,p)$. Define $A_n = A_{n-1} \cap (\bigcup_{\varepsilon^n_{i,j}=1} S^n_{i,j})$.

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Note that A_n is a decreasing sequence of closed, bounded sets. Define $A_\infty = \lim_{n\to\infty} \bigcap_n A_n$.

Survival

Theorem

$\mathbb{P}(A_\infty \neq \emptyset) > 0 \iff p > \frac{1}{N^2}$

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What does this actually mean? The fractal is nonempty if and only if the probability of each square being kept is at least $\frac{1}{N^2}$.

Some Notation and a Theorem

Let $B_n = \{x \in A_n \mid x \text{ can be connected to both } \{0\} \times [0,1] \text{ and } \{1\} \times [0,1]\}$ and let $B_\infty = \lim_{n\to\infty} \bigcap_n B_n$.

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If $B_\infty \neq \emptyset$ then there is a left-to-right crossing of $[0,1]^2$.

One Last Theorem - A Curiosity

Let $p_c(N) = inf\{p \geq 0 \mid \mathbb{P}(B_\infty \neq \emptyset) > 0\}$.

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Theorem

$p_c(N) < 1$ *for all* $N > 1$

Our Proof

Sketch of Proof.

First note that if $N^2 - 1$ of the N^2 boxes are kept then any two adjacent squares in the n^{th} level must have adjacent kept boundary offspring squares, telling us $B_\infty \neq \emptyset$.

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It suffices to show that in order to achieve a left-to-right crossing Z must have an $(N^2 - 1)$-ary splitting. By our previous theorem, $1 - \tau(N^2 - 1)$ is the smallest fixed point of $\sum_{j=0}^{N^2-2}(1-s)^j \frac{G^{(j)}(s)}{j!}$, where $G(s) = (1 - p + ps)^{N^2}$ is the moment generating function.

Our Proof

$$
\begin{aligned}
f(s) &= \sum_{j=0}^{N^2-2} (1-s)^j \frac{N^2}{(N^2-j)!j!} p^j (1-p+ps)^{N^2-j} \\
&= \sum_{j=0}^{N^2-2} \binom{N^2}{j} (p(1-s))^j (1-p+ps)^{N^2-j} \\
&= 1-(p(1-s))^{N^2} - N^2 (p(1-s))^{N^2-1} (1-(p(1-s)) \\
&= \mathbb{P}(Binom(N^2,(1-s)p) < N^2-1)
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&= \mathbb{P}(Binom(N^2,(1-s)p) < N^2-1)
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$$\therefore 1-f(s) = (p(1-s))^{N^2} + N^2(p(1-s))^{N^2-1}(1-(p(1-s))$$

Our Proof

First note that in order to find $p < 1$ such that there is an $s < 1$ with $1 - f(s) = 1 - s$ we can find $s < 1$ for which $1 - f(s) > 1 - s$.

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Next, note that using Taylor Expansions around $x > 0$ we have:

1. $(1 - x)^M \geq 1 - Mx$
2. $(1 - x)^M \leq 1 - Mx + \frac{M(M-1)}{2}x^2$

Our Proof

$$
\begin{aligned}
1 - f(s) &= (p(1-s))^{N^2} + N^2(p(1-s))^{N^2-1}(1-(p(1-s)) \\
&= p^{N^2}(1-s)^{N^2} + N^2 p^{N^2-1}(1-s)^{N^2-1} - N^2 p^{N^2}(1-s)^{N^2} \\
&= N^2 p^{N^2-1}(1-s)^{N^2-1} - (N^2-1)p^{N^2}(1-s)^{N^2} \\
&\geq N^2 p^{N^2-1}(1-(N^2-1)s) - (N^2-1)p^{N^2}(1-N^2 s + \frac{N^2(N^2-1)}{2}s^2) \\
&= (p^{N^2-1}(N^2(1-p)+p)) - (N^2(N^2-1)p^{N^2-1}(1-p))s - (\frac{N^2(N^2-1)^2}{2}p^{N^2})s^2 \\
&\equiv a - bs - cs^2
\end{aligned}
$$

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Note as $p \to 1$ we can conclude $a \to 1$, $b \to 0$, and $c \to \frac{N^2(N^2-1)^2}{2}$.

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Hence for $p = 1$, $\forall s \in (0, \frac{1}{N^2(N^2-1)^2})$, we have that
$1 - \frac{N^2(N^2-1)^2}{2}s^2 > 1 - s$.

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Hence for $p = 1$, $\forall s \in (0, \frac{1}{N^2(N^2-1)^2})$, we have that $1 - \frac{N^2(N^2-1)^2}{2}s^2 > 1 - s$.

As f is continuous there exist δ', $\delta > 0$ small enough such that if $1 - \delta < p < 1$ then $\forall s \in (\delta', \frac{2}{N^2(N^2-1)^2} - \delta') \neq \emptyset$, $1 - f(s) > 1 - s$. In particular, there is some $p < 1$ for which there is an $s < 1$ such that $1 - f(s) > 1 - s$.

$\square$

The End

Questions?