Mighty Morphin Image Processing

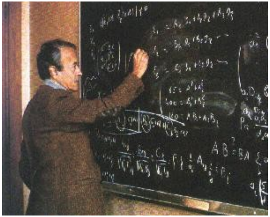
A Taste of Mathematical Morphology

Zach Littrell

McDaniel College

May 9, 2013

History
1964 onwards

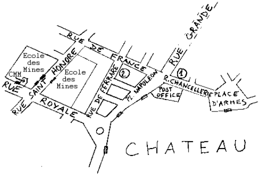


- Primarily developed by Jean Serra, an engineer [1], and his PhD advisor, Georges Matheron.
- Initially developed to detect patterns in images of minerals.

[1] AND he had degrees in psychology/philosophy, Mathematical Geology, and Mathematics

History: The Center of Mathematical Morphology

- In 1968, the Centre de Morphologie Mathématique was founded, for the further development of Mathematical Morphology.



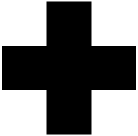
- They have ˜10 patents on MM-related technology, and continue to research the topic.

Mathematical Morphology: Nutshell Edition

When in Mathematical Morphology, we...

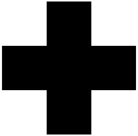
- Want to analyze properties like shapes in an image.
- Want to work with images as sets!

Images $\Rightarrow$ Matrix



$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

Images $\Rightarrow$ Sets



$$\Rightarrow \begin{bmatrix} 1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & 1 \end{bmatrix} \Rightarrow \left\{ \begin{array}{l} \langle 0, 1 \rangle \,, \\ \langle -1, 0 \rangle \,, \langle 0, 0 \rangle \,, \langle 1, 0 \rangle \,, \\ \langle 0, -1 \rangle \end{array} \right\}$$

Introducing Cool Hamblen

For many examples, we'll use simple shapes.

When we don't, I'm going to rely on my assistant: Cool Hamblen.

A Gaggle of Set Operators:Reflection

For a set $A \subseteq \mathbb{Z}^2$,

$$-A = \{\langle -x, -y\rangle \mid \langle x, y\rangle \in A\}$$

A Gaggle of Set Operators: Inversion

For a set $A \subseteq \mathbb{Z}^2$,

$$A^c = \{\langle x, y \rangle \mid \langle x, y \rangle \notin A\}$$

A Gaggle of Set Operators: Translation

For set $A \subseteq \mathbb{Z}^2$, and $z = \langle a, b \rangle \in \mathbb{Z}^2$

$$A + z = \{\langle x + a, y + b \rangle \mid \langle x, y \rangle \in A\}$$

$\oplus$ Dilation and $\ominus$ Erosion

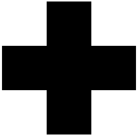
The backbone of morphology boils down to two functions:

- $\oplus$ Dilation
- $\ominus$ Erosion

They are to Mathematical Morphology what Union and Intersection is for Set Theory (they are that deceptively awesome).

Structuring Elements

Many morphological functions use another image that characterizes what the output will look like.
We will call such images *structuring elements*.
More often than not, we'll be using the cross structuring element:



$\oplus$ Dilation: The Intuition Version

We'd like to have a sort of 'smearing' operation. We'll call this dream operation *dilation*. [2]

Black=Original image
Blue=Added regions in dilation
Red=Structuring element at one point during dilation

[2] Fun fact: Matheron called this **dilatation**
More fun fact: Practically no one else does.

$\oplus$ Dilation: The Set Version

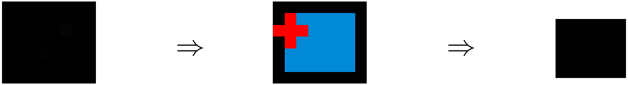
Definition

For $A, B \subseteq \mathbb{Z}^2$, we define the *dilation* of A by B to be:

$$A \oplus B = \left\{ z \in \mathbb{Z}^2 \mid -B + z \cap A \neq \emptyset \right\}$$

$\ominus$ Erosion: The Intuition Version

We'd like to have a sort of 'shrinking' operation. We'll call this dream operation *erosion*.



Black & Blue=Original image
Blue=Remaining regions after erosion
Red=Structuring element at one point during erosion

$\ominus$ Erosion: The Set Version

Definition

For $A, B \subseteq \mathbb{Z}^2$, we define the *erosion* of A by B to be:

$$A \ominus B = \left\{ z \in \mathbb{Z}^2 \mid B + z \subseteq A \right\}$$

Dilating with Funky Structuring Elements (FSEs)

What happens if we try to dilate with structuring elements that aren't centered at the origin? (Let's find out!)

Let $A =$  , $B = \quad = \{\langle 100, 100\rangle\}$

Dilating with FSEs: Results

Well, I'll be a monkey's uncle.

Dilation Redux

Dilation can be rewritten as $A \oplus B = A + B = \{A + b \mid b \in B\}$

Proof.

$$\begin{aligned} \text{Let } z \in A \oplus B &\Rightarrow -B + z \cap A \neq \emptyset \\ &\Rightarrow \exists b \in B, -b + z \in A \\ &\Rightarrow \exists a \in A, -b + z = a \\ &\Rightarrow z = a + b \\ &\Rightarrow z \in A + B \end{aligned}$$

□

Dilation 'n' Erosion Redux

Dilation can be rewritten as

- $A \oplus B = A + B = \{A + b \mid b \in B\}$
- $A \oplus B = \bigcup_{b \in B} A + b$

Erosion can be rewritten as

- $A \ominus B = \bigcap_{b \in B} A + (-b)$

Dilation and Erosion Turn Out To Be Brothers

Erosion and Dilation can actually be written in terms of the other.

$(A \oplus B)^c = A^c \ominus -B$

$(A \ominus B)^c = A^c \oplus -B$

Proof.

$$
\begin{aligned}
(A \oplus B)^c &= \{z \mid -B + z \cap A \neq \emptyset\}^c \\
&= \{z \mid -B + z \cap A = \emptyset\} \\
&= \{z \mid -B + z \subseteq A^c\} \\
&= A^c \ominus -B
\end{aligned}
$$

□

Using Morphology for Pattern Detection

I heard a rumor that Cool Hamblen is now sporting a penguin on his shirt.

Using Erosion for Pattern Detection

Erosion get's us close...

Using Erosion for Pattern Detection

Woopsy daisy.

⊛ Hit or Miss Transform: The Intuition Version

We'd like to specify not only what we want to match, or 'hit,' but also patterns we DON'T want to match, or 'miss.'

$B_1 =$  $B_2 =$ 

$\circledast$ Hit or Miss Transform: The Straightforward Version

- This we can define already as a set.

Definition

For sets $A \subseteq \mathbb{Z}^2$, $B = \langle B_1, B_2 \rangle \in \mathbb{P}\left(\mathbb{Z}^2\right) \times \mathbb{P}\left(\mathbb{Z}^2\right)$, we define the *hit-or-miss transformation* of A by B as

$$A \circledast B = \left\{z \in \mathbb{Z}^2 \mid B_1 + z \subseteq A, B_2 + z \subseteq A^c\right\}$$

- This needs more morphology...

$\circledast$ Hit or Miss Transform: The Morphological Version

- Recall that $A \ominus B_1$ is all points where B_1 is contained in A, and $A^c \ominus B_2$ is all points where B_2 is contained in A^c.
- So let's revise our definition:

Definition

For sets $A \subseteq \mathbb{Z}^2, B = \langle B_1, B_2 \rangle \in \mathbb{P}\left(\mathbb{Z}^2\right) \times \mathbb{P}\left(\mathbb{Z}^2\right)$, we define the *hit-or-miss transformation* of A by B as

$$A \circledast B = A \ominus B_1 \cap A^c \ominus B_2$$

or

$$A \circledast B = A \ominus B_1 \setminus A \oplus (-B_2)$$

Using Hit or Miss Transform for Pattern Detection

$\Rightarrow$

$\Rightarrow$

A Gaggle of Binary Morphological Functions: Boundary Extraction

We can extract the boundary using the equation:

$$A - (A \ominus B)$$

$-$

$=$

A Gaggle of Binary Morpophological Functions: Region Filling:

We can fill up a region using the following recursive equation:

$$X_k = (X_{k-1} \oplus B) \setminus A$$

where $X_0 = \{p\}$, for some point p in the region to be filled.

So, let's fill up the top triangle in this image: 

X_1	X_2	X_3	X_4	$X_4 \cup A$
				

A Gaggle of Binary Morphological Functions: Connected Component Extraction:

We can extract an individual component of an image using the recursive equation:

$$X_k = (X_{k-1} \oplus B) \cap A$$

where $X_0 = \{p\}$, for some point p in the component

Man, I'd sure love to extract that triangle from this image: 

X_1	X_2	X_3	X_4	X_5	X_6
					

What's New With Grayscale?

- Pixels can now be **between** 0 and 1.
- So our pixels now live in $\mathbb{Z}^3$, namely they take the form of $\langle x, y, value \rangle$.
- Can't be nearly as cheatsy as we were with binary images.

Operation Minkowski: Attempt #1 at Dilation

The Minkowski Sum definition of dilation ought to translate pretty well to $\mathbb{Z}^3$

Let $A = \{\langle 0,0,0\rangle, \langle 0,1,0\rangle\}$

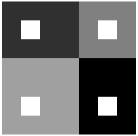
Let $B = \{\langle 1,1,1\rangle, \langle 1,0,0\rangle\}$

$A \oplus B = \{\langle 1,1,1\rangle, \langle 1,2,1\rangle, \langle 1,0,0\rangle, \langle 1,1,0\rangle\}$

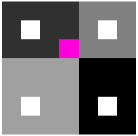
Everything okie dokie artichokie?

Operation Supremum: Attempt #2 at Dilation

Given a grayscale image like this



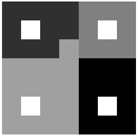
Let's examine what should happen at a point



Operation Supremum: Intuition at Dilation

Using our handy dandy cross structuring element, we'll gather values from our neighborhood.

And we pick the *largest* (i.e. closest to white) value.

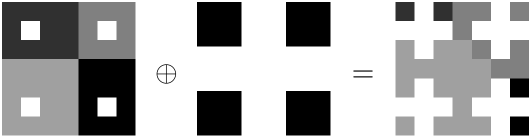


$\oplus$ Dilation

Definition

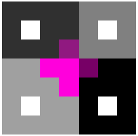
For functions $f, b : \mathbb{Z} \times \mathbb{Z} \to [-\infty, \infty]$, we define the *dilation* of f by b as

$$(f \oplus b)(x, y) = \sup_{\langle x', y' \rangle \in \mathbb{Z}^2} \left\{ f(x - x', y - y') + b(x', y') \right\}$$

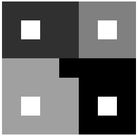


Operation Infimum: Intuition at Erosion

Once again, we observe the neighborhood formed by our structuring element at a point.



This time, however, we take the *smallest* (i.e. closest to black) value.

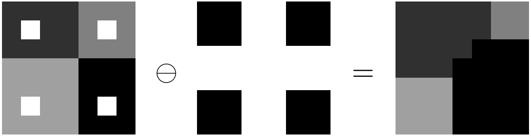


$\ominus$ Erosion

Definition

For functions $f, b : \mathbb{Z} \times \mathbb{Z} \to [-\infty, \infty]$, we define the *erosion* of f by b as

$$(f \ominus b)(x, y) = \inf_{\langle x', y' \rangle \in \mathbb{Z}^2} \left\{ f(x + x', y + y') - b(x', y') \right\}$$



Edge Detection in Grayscale Images

Sure would be nice if we could (morphologically) detect edges in Grayscale.

For this, we'll define edges as being neighborhoods where the intensity changes (boy, I sure wish we had primitive functions that found the light and dark values of neighborhoods).

Edge Detection via Morphological Gradient

Definition

For two grayscale functions, we define the *morphological gradient* to be:

$$g(f, b) = f \oplus b - f \ominus b$$

$-$

$=$

,

Generalizing Morphology

Our current definitions of dilation and erosion are bound to the space we defined them on.

Would be nice to have a general framework for morphology on unfamiliar territory, like how Matrices generalize into Vector Spaces.

What do we want?

Things we need:

- We need generalized forms of dilation and erosion.
- Ought to work for both Binary and Grayscale morphology.

Complete Lattices

Definition

Let L be a partially ordered set such that every subset of L has a supremum and infimum. We call L a *complete lattice*.

Note, we denote the supremum of a set with $\bigvee$ and the infimum as $\bigwedge$.

δ Dilation on Complete Lattices

Let $(L, \leqslant)$ be a complete lattice, and $\{X_i\}$ be a collection of elements from L.

Definition

Let $\delta : L \to L$ be a function such that

$$\bigvee_i \delta(X_i) = \delta\left(\bigvee_i X_i\right)$$

and the value of the least element is the least element.
We call δ a *dilation*.

ϵ Erosion on Complete Lattices

Let $(L, \leqslant)$ be a complete lattice, and $\{X_i\}$ be a collection of elements from L.

Definition

Let $\epsilon : L \to L$ be a function such that

$$\bigwedge_i \epsilon(X_i) = \epsilon\left(\bigwedge_i X_i\right)$$

and the value of the max element is the max element.
We call ϵ an *erosion*.

Old Morphologies (as good as new!)

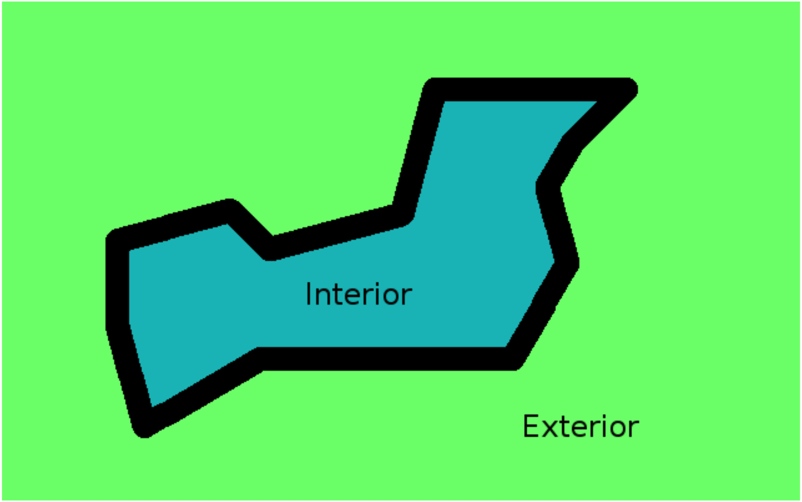
Binary and grayscale morphology are now just special cases of Lattice Morphology.

- Binary Morphology:
  Binary Morphology can be expressed as morphology on $\mathbb{P}\left(\mathbb{Z}^2\right)$, with $\leqslant$ defined by $\subseteq$.
- Grayscale Morphology:
  Grayscale Morphology can be expressed as morphology on grayscale functions, with
  $f \leqslant b \iff \forall \langle x, y\rangle \in \mathbb{Z} \times \mathbb{Z}, f(x, y) \leq g(x, y)$

Formally looking at $\mathbb{Z}^2$

As our original morphological tools operate on $\mathbb{Z}^2$, it'd be nice to figure out whether $\mathbb{Z}^2$ can be trusted to behave like $\mathbb{R}^2$.
Spoiler: Kind of.

Jordan's Curve Theorem



Jordan's Theorem in $\mathbb{R}^2$

Theorem

Let C be a closed, simple path in $\mathbb{R}^2$.
Then $\mathbb{R}^2 \setminus C$ contains exactly 2 connected sets.

- One of the sets is bounded and called the *interior, with respect to C*
- One of the sets is unbounded and called the *exterior, with respect to C*.

Jordan's Theorem in $\mathbb{Z}^2$

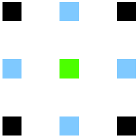
Ok, so let's check this out in $\mathbb{Z}^2$!

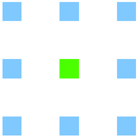
Jordan's Theorem in $\mathbb{Z}^2$

Ok, so let's check this out in $\mathbb{Z}^2$!
Wait...what does "connected" even mean in $\mathbb{Z}^2$?
Ah, poo.

Connectivity in $\mathbb{Z}^2$

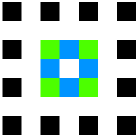
We have at least two ways to define connectivity.

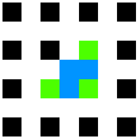
- 4-connectivity: 

- 8-connectivity: 

Shoehorning Jordan's Theorem into $\mathbb{Z}^2$

Alas, Jordan's Theorem doesn't hold in 4-connectivty or 8-connectivity without an adequate amount of jiggery-pokery (which is less than wunderbar).

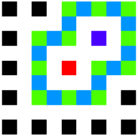
- 4-connectivity: 

- 8-connectivity: 

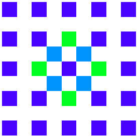
Using Multiple Connectivity

Using a single form of connectivity actually keeps us away from getting Jordan's Theorem back:

- 4-connectivity:



- 8-connectivity:



The Digital Jordan's Theorem

Theorem

Let $k \in \{4, 8\}$, and $\hat{k} \in \{4, 8\} \setminus \{k\}$.
Let C be a simple, closed k-path in $\mathbb{Z}^2$.
Then $\mathbb{Z}^2$ contains exactly 2 $\hat{k}$-connected sets.

- One of the sets is bounded and called the interior, with respect to C.
- One of the sets is unbounded and called the exterior, with respect to C.

Denouement

So here's what we have:

- $\mathbb{Z}^2$ is a surprisingly non-trivial (and mildly scary) place to work.
- Morphology gives us tools to navigate $\mathbb{Z}^2$ and other potentially scary spaces.

Thank you for listening!

Any questions?