Trivial FUNC Lower Bound

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Chapter 1

Main

Lemma 1.1. *Let Y be a set and $x \in Y$ some point in Y. Let $C \subseteq Y$ be a non-empty set satisfying $C \neq x$. Then there is some $y \in Y$ such that $y \neq x$ and $y \in C$.*

Proof. We split into cases depending on whether $x \in C$.

Case 1: $x \notin C$. Since C is non-empty, there is some $y \in C$. We claim such a y satisfies $y \neq x$ and $y \in C$. The latter is by definition. For the former, if $y = x$, then that $y \in C$ implies $x \in C$, contradicting $x \notin C$.

Case 2: $x \in C$. We show that, assuming the conclusion is false, it holds that $C = \{x\}$. To show $\{x\} \subseteq C$, it suffices to show $x \in C$, but we know this already. To show $C \subseteq \{x\}$, we take a $y \in C$, and by assumption, since $y \in C$, we know $y = x$ and hence $y \in \{x\}$. □

Proposition 1.2. *Let $\mathcal{F}$ be a finite union-closed family of sets and $\mathcal{G} \subseteq \mathcal{F}$ a non-empty subfamily. Then , $\bigcup \mathcal{G} \in \mathcal{F}$. Here, $\bigcup G$ is the union of all sets in $\mathcal{G}$.*

Proof. We induct on the size of $\mathcal{G}$, with the base case of $|\mathcal{G}| = 0$ being vacuous, since $\mathcal{G}$ is non-empty. Now assume the result holds for all such $\mathcal{G}$ of size $|\mathcal{G}| = n$, and take such a $\mathcal{G}$ of size $|\mathcal{G}| = n + 1$.

We split into cases based on whether G is a singleton. □