Abstract Algebra Homework 1

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1. Which of the following multiplication tables defined on the set $G = \{a, b, c, d\}$ form a group? Support your answer in each case.

   Theorem 1. *Let $a, b, c \in G$. Given a is the Identity element of the set G, then $a \circ (b \circ c) = (a \circ b) \circ c$, $\forall b, c \in G$.*

   Proof. Let $d \in G$ with $b \circ c = d$. Then $a \circ (b \circ c) \implies a \circ d = d$.
   We also have $(a \circ b) \circ c \implies b \circ c = d$.
   $\therefore a \circ (b \circ c) = (a \circ b) \circ c$. ∎

   (a)

$\circ$	a	b	c	d
a	a	c	d	a
b	b	b	c	d
c	c	d	a	b
d	d	a	b	c

   This Cayley Table does not form a group because it is not Associative:
   $a \circ (b \circ c) = d$ and
   $(a \circ b) \circ c = a$, so
   $a \circ (b \circ c) \neq (a \circ b) \circ c$

   (b)

$\circ$	a	b	c	d
a	a	b	c	d
b	b	a	d	c
c	c	d	a	b
d	d	c	b	a

   Closure: Every element in the Cayley Table is in the set G, so it is closed.
   Identity: taking any element and multiplying it by a returns that element. So a is the indentity element.
   Inverse: A diagonal is formed in the table with the identity element a, so every element is its own inverse.
   Associative: Because a is the identity element it is associative with every set of two elements by Theorem 1. Because every element $p_{ij} = p_{ji}$ it is communtative as well, so only one permutation of the elements b, c, d needs to be tested for associativity. We have $(b \circ c) \circ d = a$ and $b \circ (c \circ d) = a$, so $(b \circ c) \circ d = b \circ (c \circ d)$.
   Therefore this Cayley Table is a group.

(c)

$\circ$	a	b	c	d
a	a	b	c	d
b	b	c	d	a
c	c	d	a	b
d	d	a	b	c

This Cayley Table is the same as the Cayley Table for the group $(\mathbb{Z}_4, +)$ where $a = 0,\ b = 1,\ c = 2,\ d = 3$, so This Cayley Table must be a group.

(d)

$\circ$	a	b	c	d
a	a	b	c	d
b	b	a	c	d
c	c	b	a	d
d	d	d	b	c

The identity element of this Cayley Table is a. There is no inverse for d where $d \circ p = a$ in this Cayley Table. Therefore this Cayley Table is not a Group.