Linear Algebra II Quiz 3

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1. Assume A is an $m \times n$ matrix and $CA = I_n$ the $n \times n$ identity matrix. Show that the equation $Ax = 0$ has only the trivial solution. Explain why the matrix A can not have more columns than rows.

   Proof. Let A be an $m \times n$ matrix s.t. $CA = I_n$ where I_n is the $n \times n$ identity matrix. Consider the equation $Ax = 0$

   $$\begin{aligned} Ax &= 0 \\ CAx &= C0 \\ (CA)x &= C0 && \text{(Associative Property)} \\ I_n x &= 0 \\ x &= 0 \end{aligned}$$

   Thus $Ax = 0$ can only have the trivial solution. ■

   If A had more columns than rows then the columns of A would be be linearly dependent, thus $Ax = 0$ wuold have a nontrivial solution.

2. Assume A is an $m \times n$ matrix nad $AD = I_m$ the $m \times m$ identity matrix. Show that for any $b \in \mathbb{R}^m$, the equation $Ax = b$ has a solution. Try explaining why the matrix A can not have more rows than columns.

   Proof. Let A be an $m \times n$ matrix s.t. $AD = I_m$ where I_m is the $m \times m$ identity matrix. Consider $A(Db)$ with $b \in \mathbb{R}^m$ then

   $$\begin{aligned} A(Db) &= (AD)b && \text{(Associative Property)} \\ &= I_m b \\ &= b \end{aligned}$$

   Thus $x = Db$ is a solution $Ax = b$. ■

   Since $Ax = b$ has a sultion $\forall b \in \mathbb{R}^m$ A has a pivot in every row. Since each column has to have a pivot there cannot be more rows than columns.

3. Assume A is an $m \times n$ matrix and there exists $n \times m$ matrices C and D such that $CA = I_n$ and $AD = I_m$. Prove that $m = n$ and $C = D$.

Proof. Let the matrices A being $m \times n$, and C, D being $n \times m$ s.t. $CA = I_n$ and $AD = I_m$. Then by problem 1 since $CA = I_n$ we know that $m \geq n$ and by problem 2 since $AD = I_m$ we know that $n \geq m$. Thus $m = n$. Consider the product CAD, then

$$\begin{aligned} CAD &= (CA)D && \text{(Associative Property)} \\ &= I_n D \\ &= D \end{aligned}$$

But we can also do this as

$$\begin{aligned} CAD &= C(AD) && \text{(Associative Property)} \\ &= C I_m \\ &= C \end{aligned}$$

$\therefore C = D$. ∎