Intro to Analysis Homework 2

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1. Let S be a nonempty bounded set in $\mathbb{R}$. Let $b < 0$ and consider $bS = \{bs : s \in S\}$. Prove $\sup(bS) = b * \inf(S)$

   Proof. Let $\sup(S) = s$ We know that $\sup(bS) = b\sup(S) = bs$. So $\forall s_0 \in S$ $s_0 \leq s$. Becsue $b < 0$ multiplying it into the inequality give is $bs_0 \geq bs$ $\forall bs_0 \in bS$. So by definition bs is the smallest element in bS when $b < 0$, so $bs = \inf(bS) = b * \inf(S)$. ■

2. Let $I_n = \left[1, 1 + \frac{1}{n}\right]$ $\forall n \in \mathbb{N}$. Prove $\bigcap_{n=1}^{\infty} I_n = \{1\}$.

   Proof. Clearly 1 is in $\left[1, 1 + \frac{1}{n}\right]$. BMOC Let $x \in \bigcap_{n=1}^{\infty} I_n$. Then

$$\begin{aligned} 1 < x &\leq 1 + \frac{1}{n} \\ 0 < x - 1 &\leq \frac{1}{n} \end{aligned}$$

   Since $x - 1 > 0$ by Archimedean Property $\exists m \in \mathbb{N}$ s.t.

$$\begin{aligned} x - 1 &> \frac{1}{m} \\ \implies x &> 1 + \frac{1}{m} \end{aligned}$$

   but $x < 1 + \frac{1}{n}$ $\forall n \in \mathbb{N}$. $\therefore \bigcap_{n=1}^{\infty} I_n = \{1\}$. ■

3. Consider the set $S = \left\{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\right\}$. Find the infimum and supremum of the set. Then, prove your assertions.
   $\inf(S) = -1$ and $\sup(S) = 1$

4. Let $I_n = \left(2 - \frac{1}{n}, 2\right)$ $\forall n \in \mathbb{N}$. Prove $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Proof. BMOC let $x \in \bigcap_{n=1}^{\infty} I_n$. Then

$$\begin{aligned} 2-\frac{1}{n} &< x < 2 \\ 2-\frac{1}{n} &< x-2 < 0 \\ 0 < 2-x &< \frac{1}{n}-2 \end{aligned}$$

Since $2-x>0$ by Archimedean Property $\exists m \in \mathbb{N}$ s.t.

$$\begin{aligned} 2-x &> \frac{1}{m} \\ \implies -x &> \frac{1}{m}-2 \\ \implies x &< 2-\frac{1}{m} \end{aligned}$$

but $x > 2-\frac{1}{n}$ $\forall n \in \mathbb{N}$. $\therefore \bigcap_{n=1}^{\infty} I_n = \emptyset$ ■

5. Find the infimum of the set and prove your result.

$$S = \left\{ \frac{3+n}{n} : n \in \mathbb{N} \right\}$$