Intro to Analysis Homework 3

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1. Let $a_1 = 1$ and define $a_{n+1} = 1 + \frac{a_n}{2}$. Show that the sequence converges and make an educated guess for the limit value.
   Guess: $a_n \to 2$

$$\begin{aligned} a_n &\leq a_{n+1} \\ a_n &\leq 1 + \frac{a_n}{2} \\ \frac{a_n}{2} &\leq 1 \\ a_n &\leq 2 \end{aligned}$$

   Proof. <u>Claim 1</u>: (a_n) is bounded above by 2.
   We prove this using induction.
   Base case: $(n = 2)$

$$\begin{aligned} a_2 &= 1 + \frac{a_1}{2} \\ &= 1 + \frac{1}{2} \\ &= \frac{3}{2} \end{aligned}$$

   Inductive Hypothesis: $(n = k)$
   Assume that $a_k < 2$
   Inductive Step $(n = k + 1)$

$$\begin{aligned} a_{k+1} \leq 2 \Rightarrow 1 + \frac{a_k}{2} &\leq 2 \\ \Rightarrow \frac{a_k}{2} &\leq 1 \\ \Rightarrow a_k &\leq 2 \qquad \text{(Inductive Hypothesis)} \end{aligned}$$

   $\therefore a_n$ is bounded above by 2.
   <u>Claim 2</u>: (a_n) is increasing.

By claim 1,

$$\begin{aligned} a_n &\leq 2 \\ \Rightarrow \frac{a_n}{2} &\leq 1 \\ \Rightarrow a_n &\leq 1 + \frac{a_n}{2} \\ \Rightarrow a_n &\leq a_{n+1} \end{aligned}$$

Thus (a_n) is decreasing, and by the Monotone Convergence Theorem it converges. ■

2. Let $(x_n) = \frac{2^n}{n!}$. Show that this sequence converges using the Monotone convergence theorem.
   Guess: $(x_n) \to 0$

   Proof. Clain 1: (x_n) is bounded below by 0.
   This is clearly true because $2^n > 0$ and $n! > 0$.
   Claim 2: (x_n) is decreasing.

$$\begin{aligned} \frac{2^n}{k!} &\geq \frac{2^{n+1}}{(n+1)!} \\ 2^n(n+1)! &\geq 2^{n+1}n! \\ 2^n(n+1)n! &\geq 2^n 2n! \\ n+1 &\geq 2 \\ n &\geq 1 \end{aligned}$$

   and $n \in \mathbb{N}$ so this is clearly true. Thus (x_n) is decreasing.
   By the Monotone Convergence Theorem (x_n) converges. ■

3. Let $x_1 = 1$ and $x_{n+1} = \frac{4+3x_n}{3+2x_2}$. Show that this sequence converges by the monotone covergence theorem.
   Guess: $x_n \to \sqrt{2}$.

$$\begin{aligned} x_n &< x_{n+1} \\ x_n &< \frac{4+3x_n}{3+2x_n} \\ x_n(3+2x_n) &< 4+3x_n \\ 2x_n^2 + 3x_n &< 4+3x_n \\ 2x_n^2 &< 4 \\ x_n^2 &< 2 \\ x_n &< \sqrt{2} \end{aligned}$$

Proof. <u>Claim 1</u>: (x_n) is bounded above by $\sqrt{2}$.
Proof by induction:
Base case: $(n = 2)$

$$\begin{aligned} x_2 &= \frac{4+3(1)}{3+2(1)} \\ &= \frac{7}{5} < \sqrt{2} \end{aligned}$$

Inductive Hypothesis $(n = k)$
Assume $x_k \le \sqrt{2}$
Inductive Step $(n = k+1)$

$$\begin{aligned} x_{k+1} \le \sqrt{2} \Rightarrow \frac{4+3x_n}{3+2x_n} &\le \sqrt{2} \\ \Rightarrow 4+3x_n &\le \sqrt{2}(3+2x_n) \\ \Rightarrow 4+3x_n &\le 3\sqrt{2}+2x_n\sqrt{2} \\ \Rightarrow 3x_n - 2x_n\sqrt{2} &\le 3\sqrt{2}-4 \\ \Rightarrow x_n(3-2\sqrt{2}) &\le 3\sqrt{2}-4 \\ \Rightarrow x_n &\le \frac{3\sqrt{2}-4}{3-2\sqrt{2}}\left(\frac{3+2\sqrt{2}}{3+2\sqrt{2}}\right) \\ \Rightarrow x_n &\le \frac{9\sqrt{2}+12-12-8\sqrt{2}}{9-8} \\ \Rightarrow x_n &\le \frac{\sqrt{2}}{1} \\ \Rightarrow x_n &\le \sqrt{2} \end{aligned}$$

$\therefore x_n$ is bounded above by $\sqrt{2}$.
<u>Claim 2</u>: (x_n) is decreasing.
By clain 1,

$$\begin{aligned} x_n \le \sqrt{2} \Rightarrow x_n^2 &\le 2 \\ 2x_n^2 &\le 4 \\ 2x_n^2 + 3x_n &\le 4+3x_n \\ x_n(2x_n+3) &\le 4+3x_n \\ x_n &\le \frac{4+3x_n}{3+2x_n} \\ x_n &\le x_{n+1} \end{aligned}$$

Thus (x_n) is decreasing.
By the Monotone Convergence Theorem (x_n) converges. ■

4. Establish the divergence of the folowing sequences. You can use any of the divernce criteria in 3.4.

(a) $\frac{2n^2(-1+^n+3n)}{n^2}$
Consider the subsequence

$$\begin{aligned}
x_{2n} &= \frac{2(2n)^2(-1)^{2n} + 3(2n)}{2(2n)^2} \\
&= \frac{2(4n^2) + 6n}{4n^2} \\
&= \frac{8n^2 + 6n}{4n^2} \\
\Rightarrow \lim_{n\to\infty} x_{2n} &= 2
\end{aligned}$$

Now also consider the subsequence

$$\begin{aligned}
x_{2n+1} &= \frac{2(2n+1)^2(-1) + 3(2n+1)}{(2n+1)^2} \\
&= \frac{-2(4n^2+4n+1) + 6n + 3}{4n^2+4n+1} \\
&= \frac{-8n^2 - 8n - 2 + 6n + 3}{4n^2+4n+1} \\
&= \frac{-8n^2 - 2n + 1}{4n^2+4n+1} \\
\Rightarrow \lim_{n\to\infty} x_{2n+1} &= -2
\end{aligned}$$

Since $x_{2n} \to 2$ and $x_{2n+1} \to -2, (x_n)$ does not converge.

(b) $(-n)^3$

$$\begin{aligned}
(-n)^3 &= (-1)^3 n^3 \\
&= (-1)n^3
\end{aligned}$$

Which goes to $-\infty$, so it is not bounded below and diverges.

5. Suppose (x_n) is a bounded subsequence with $s = \sup(x_n : n \in \mathbb{N})$ and suppose further that $s \notin (x_n)$. Construct a subsequnce which converges to s.
Consider $s - \frac{1}{k}\ \forall k \in \mathbb{N}$. We know that because $s - \frac{1}{k} < s$ so $s - \frac{1}{k}$ cannot be an upper bound on (x_n). Therefore $\exists x_{n_k} \in (x_n)$ where $x_{n_k} \geq s - \frac{1}{k}$. So

$$\begin{gathered}
s \geq x_{n_k} \geq s - \frac{1}{k} \\
\lim_{k\to\infty} s \geq \lim_{k\to\infty} x_{n_k} \geq \lim_{k\to\infty}\left(s - \frac{1}{k}\right) \\
s \geq \lim_{k\to\infty} x_{n_k} \geq s
\end{gathered}$$

So by the Squeeze Theorem $\lim\limits_{k\to\infty} x_{n_k} = s$

6. Show that the sequence $x_n = \frac{(2n^2 - 3n)\sin(n^2)}{n^2+3n+5}$ has a convergent subsequence.

$$
\begin{aligned}
|x_n| &= \frac{|2n^2 - 3n||\sin(n^2)|}{|n^2 + 3n + 5|} \\
&\leq \frac{|2n^2 - 3n|}{|n^2 + 3n + 5|} \\
&\leq \frac{|2n^2| + |-3n|}{n^2 + 3n + 5} && \text{(Triangle Inequality)} \\
&\leq \frac{2n^2 + 3n}{n^2 + 3n + 5} \\
&\leq \frac{2n^2 + 3n}{n^2} \\
&\leq \frac{6n^2}{n^2} && \text{(See Note*)} \\
&= 6
\end{aligned}
$$

So (x_n) is bounded and by the Bolzano-Weirstrass Theorem has a convergent subsequence. (Note*)

$$
\begin{gathered}
2n^2 + 3n \leq 8n^2 \\
0 \leq 6n^2 - 3n \\
0 \leq n(6n - 3) \\
0 \leq 6n - 3 \ \forall n \in \mathbb{N}
\end{gathered}
$$