Abstract Algebra Homework 2

Zachary Meyner

41. Prove that

$$G = \{a + b\sqrt{2} : a, b \in \mathbb{Q} \text{ and } a \text{ and } b \text{ are not both zero}\}$$

is a subgroup if $\mathbb{R}^*$ under the group operation of multiplication.

Proof. Let $n, m \in G$ *s.t.* $n = a + b\sqrt{2}$ and $m = c + d\sqrt{2}$.
(WTS: $n \cdot m \in G$ and $n^{-1} \in G$)
Multiplying $m \cdot n$ we have

$$\begin{aligned}
(a + b\sqrt{2}) \cdot (c + d\sqrt{2}) &= ac + ad\sqrt{2} + bc\sqrt{2} + bd\sqrt{2}^2 && \text{(Distributive Property)} \\
&= ac + ad\sqrt{2} + bc\sqrt{2} + 2bd \\
&= ac + \sqrt{2}(ad + bc) + 2bd && \text{(Distributive Property)} \\
&= (ac + 2bd) + (ad + bc)\sqrt{2} && \text{(Commutative and Associative Property)}
\end{aligned}$$

and $(ac + 2bd) + (ad + bc)\sqrt{2}$ is clearly an in G, so $n \cdot m$ must be in G. Now if we take n^{-1} we get

$$\begin{aligned}
\frac{1}{a + b\sqrt{2}} &= \frac{1}{a + b\sqrt{2}} \cdot \frac{(a - b\sqrt{2})}{(a - b\sqrt{2})} && \text{(Multiplying by 1)} \\
&= \frac{a - b\sqrt{2}}{(a + b\sqrt{2}) \cdot (a - b\sqrt{2})} \\
&= \frac{a - b\sqrt{2}}{a^2 - b\sqrt{2}^2} && \text{(Distributive Property)} \\
&= \frac{a + (-b)\sqrt{2}}{a^2 - 2b} && \text{(Simplifying)} \\
&= \frac{a}{a^2 - 2b} + \frac{-b}{a^2 - 2b}\sqrt{2} && \text{(Commutative and Associative Propety)}
\end{aligned}$$

Because a and b are in $\mathbb{Q}$ we know $\frac{a}{a^2-2b}$ and $\frac{-b}{a^2-2b}$ must also be in $\mathbb{Q}$, so n^{-1} must be in G.
$\therefore$ by the 2 step test G is a subgroup of $\mathbb{R}^*$ under the operation of multiplication. ■

45. Prove that the intersection of two subgroups of a group G is also a subgroup of G.

Proof. Let $P \leq G$ and $H \leq G$. We know that at least the identity element $e \in P \cap H$ Let $a, b \in P \cap H$ (WTS: $ab^{-1} \in P \cap H$). Because $a, b \in P \cap H$ we know

$$\begin{aligned} & a \in P \cap H \\ \implies & a \in P \text{ and } a \in H \\ & b \in P \cap H \\ \implies & b \in P \text{ and } b \in H \end{aligned}$$

Since P and H are subgroups of G we have

$$\begin{aligned} & ab^{-1} \in P \text{ and } ab^{-1} \in H \\ \implies & ab^{-1} \in P \cap H \end{aligned}$$

Thus $P \cap H$ is also a subgroup of G. ∎