Intro to Analysis Homework 3

Zachary Meyner

1. Let $a_1 = 1$ and define $a_{n+1} = 1 + \frac{a_n}{2}$. Show that the sequence converges and make an educated guess for the limit value.
   Guess: $a_n \to 2$

$$\begin{aligned} a_n &\leq a_{n+1} \\ a_n &\leq 1 + \frac{a_n}{2} \\ \frac{a_n}{2} &\leq 1 \\ a_n &\leq 2 \end{aligned}$$

Proof. Claim 1: (a_n) is bounded above by 2.
Base case: $(n = 2)$

$$\begin{aligned} a_2 &= 1 + \frac{a_1}{2} \\ &= 1 + \frac{1}{2} \\ &= \frac{3}{2} \end{aligned}$$

Inductive Hypothesis: $(n = k)$
Assume that $a_k < 2$
Inductive Step $(n = k + 1)$

$$\begin{aligned} a_{k+1} \leq 2 \Rightarrow 1 + \frac{a_k}{2} &\leq 2 \\ \Rightarrow \frac{a_k}{2} &\leq 1 \\ \Rightarrow a_k &\leq 2 && \text{(Inductive Hypothesis)} \end{aligned}$$

$\therefore a_n$ is bounded above by 2.
Claim 2: (a_n) is increasing.
By claim 1,

$$\begin{aligned} a_n &\leq 2 \\ \Rightarrow \frac{a_n}{2} &\leq 1 \\ \Rightarrow a_n &\leq 1 + \frac{a_n}{2} \\ \Rightarrow a_n &\leq a_{n+1} \end{aligned}$$

Thus (a_n) is decreasing, and by the Monotone Convergence Theorem it converges. $\blacksquare$