Abstract Algebra Homework 4

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2. Suppose that G is a finite group with 60 elements. What are the orders of possible subgroups of G?
By Lagrange's Theorem the orders of the possible subgroups must divide 60, so possible orders are $\{1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60\}$

11. Let H be a subgroup of a group G and suppose that $g_1, g_2\ inG$. Prove that the following conditions are equivalent.

(a) $g_1H = g_2H$

(b) $Hg_1^{-1} = Hg_2^{-1}$

(c) $g_1H \subseteq g_2H$

(d) $g_2 \in g_1H$

(e) $g_1^{-1}g_2 \in H$

Proof. (a) $\Rightarrow$ (c) Done in class
(c) $\Rightarrow$ (d) Because $g_1H \subseteq g_2H$ $\exists h \in H$ s.t. $g_1 = g_2h$. Then $g_1h^{-1} = g_2$. So $g_2 \in g_1H$.
(d) $\Rightarrow$ (e) Done in class
(e) $\Rightarrow$ (b) Because $g_1^{-1}g_2 \in H$ $\exists h \in H$ s.t. $g_1^{-1}g_2 = h \implies g_1^{-1} = hg_2^{-1}$ and $h^{-1}g_1^{-1} = g_2^{-1}$.
Case 1: Suppose $h_1g_1^{-1} \in Hg_1^{-1}$. Then $h_1g_1^{-1} = h_1hg_2^{-1} \in Hg_2^{-1}$.
$\therefore Hg_1^{-1} \subseteq Hg_2^{-1}$.
Case 2: Suppose $h_2g_2^{-1} \in Hg_2^{-1}$. Then $h_2g_2^{-1} = h_2h^{-1}g_1^{-1} \in Hg_1^{-1}$.
$\therefore Hg_2^{-1} \subseteq Hg_1^{-1}$ and $Hg_1^{-1} = Hg_2^{-1}$.
(b) $\Rightarrow$ (a) Since $Hg_1^{-1} = Hg_2^{-1}$ $\exists h \in H$ s.t. $g_1^{-1} = hg_2^{-1} \implies g_2h^{-1} = g_1$ and $g_2 = g_1h$.
Case 1: Let $g_2h_1 \in g_2H$. Then $g_2h_1 = g_1hh_1 \in g_1H$.
$\therefore g_2H \subseteq g_1H$.
Case 2: Let $g_1h_2 \in g_1H$. Then $g_1h_2 = g_2h^{-1}h_2 \in g_2H$.
$\therefore g_1H \subseteq g_2H$ and $g_1H = g_2H$. ■