Historical Roots of Mathematics Homework 6

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1. It is said that Pythagoras himself genereated Pythagorean triples using the following method: Let $n \in \mathbb{Z}^+$ and let

$$\left.\begin{aligned} x &= 2n + 1 \\ y &= 2n^2 + 2n \\ z &= 2n^2 + 2n + 1 \end{aligned}\right\} \qquad (*)$$

   (a) Find x, y, and z for

      i. $n = 1$

$$x = 2(1) + 1 = 3$$
$$y = 2(1)^2 + 2(1) = 4$$
$$z = 2(1)^2 + 2(1) + 1 = 5$$

      ii. $n = 2$

$$x = 2(2) + 1 = 5$$
$$y = 2(2)^2 + 2(2) = 12$$
$$z = 2(2)^2 + 2(2) + 1 = 13$$

      iii. $n = 3$

$$x = 2(1) + 1 = 7$$
$$y = 2(1)^2 + 2(1) = 24$$
$$z = 2(1)^2 + 2(1) + 1 = 25$$

   (b) Prove that (*) produces a Pythagorean triple for any positive integer n.

   Proof. Let $x, y, z \in \mathbb{Z}^+$ s.t.

$$\left.\begin{aligned} x &= 2n + 1 \\ y &= 2n^2 + 2n \\ z &= 2n^2 + 2n + 1 \end{aligned}\right\}$$

   with $n \in \mathbb{Z}^+$. Then:

$$\begin{aligned} x^2 + y^2 &= (2n + 1)^2 + 2n^2 + 2n^2 \\ &= (4n^2 + 4n + 1) + (4n^4 + 8n^3 + 4n^2) \\ &= 4n^4 + 8n^3 + 8n^2 + 4n + 1 \end{aligned}$$

Also:

$$z^2 = (2n^2 + 2n + 1)^2$$
$$= 4n^4 + 8n^3 + 8n^2 + 4n + 1$$

Thus $x^2 + y^2 = z^2$ and (*) produces Pythagorean triples. ∎

(c) Will (*) produce every primitive Pythagorean triple? Explain.
No (*) will not generate every Pythagorean triple. It only generates Pythagorean triples where y and z differ by 1, and there are primitive Pythagorean triples where y and z differ by more than 1, eg $(8, 15, 17)$.

2. Write each of the following numbers as the sum of three of fewer triangular numbers:

   (a) $56 = t_{10} + t_1 = 55 + 1$
   (b) $69 = t_{11} + t2 = 66 + 3$
   (c) $185 = t_{13} + t_{13} + t_2 = 91 + 91 + 3$
   (d) $287 = t_{22} + t_7 + t_3 = 253 + 28 + 6$

3. Use mathematical induction to prove that $t_n = \frac{n(n+1)}{2}$.

Proof. Let $t_n = \sum_{i=1}^{n} i$ be the nth triangular number.
Base case ($n = 1$):
Let $n = 1$, then $\sum_{i=1}^{1} i = 1$ and $\frac{1(1+1)}{2} = \frac{2}{2} = 1$.
Hypothesis ($n = k$):
Assume $t_k = \sum_{i=1}^{k} i = 1 + 2 + \cdots + k = \frac{k(k+1)}{2}$ $\forall k \in \mathbb{Z}^+$
Induction ($n = k + 1$):
Consider $\sum_{i=1}^{k+1} i$, then

$$\begin{aligned}
\sum_{i=1}^{k+1} i &= 1 + 2 + \cdots + k + (k+1) \\
&= \frac{k(k+1)}{2} + (k+1) && \text{(Hypothesis)} \\
&= \frac{k^2 + k + 2(k+1)}{2} \\
&= \frac{k^2 + 3k + 2}{2} \\
&= \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2}
\end{aligned}$$

$$\therefore t_n = \sum_{i=1}^{n} i = \frac{n(n+1)}{2}$$

∎

4. Prive that 1184 and 1210 are amicable.

   Proof. Divisors of 1184 are $\{1, 2, 4, 8, 16, 32, 37, 74, 148, 296, 592, 1184\}$
   $1 + 2 + 4 + 16 + 32 + 37 + 74 + 148 + 296 + 592 = 1210$
   Divisors of 1210 are $\{1, 2, 5, 10, 11, 22, 55, 110, 121, 242, 605, 1210\}$
   $1 + 2 + 5 + 10 + 11 + 22 + 55 + 110 + 121 + 242 + 605 = 1184$
   Thus 1184 and 1210 are amicable. ■

5. Verify that $\left(\frac{m^2-1}{2}\right)^2 + m^2 = \left(\frac{m^2+1}{2}\right)^2$.

$$\begin{aligned}
\left(\frac{m^2-1}{2}\right)^2 + m^2 &= \frac{m^4 - 2m^2 + 1}{4} + m^2 \\
&= \frac{m^4 - 2m^2 + 1 + 4m^2}{4} \\
&= \frac{m^4 + 2m^2 + 1}{4} \\
&= \left(\frac{m^2+1}{2}\right)^2
\end{aligned}$$