Intro to Analysis Homework 4

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1. Given that (a_n) is a Cauchy sequence, use the definition (as an Epsilon argument) to show that (a_n^2) is also a Cauchy sequence. Then provide an example to show that the converse is not true.

   Proof. Since (a_n) is Cauchy iit is also bounded. So $\exists M \in \mathbb{R}$ s.t. $|a_n| \leq M$. Let $\varepsilon > 0$ be given. Since (a_n) is Cauchy $\exists H \in \mathbb{R}$ s.t. $\forall n, m \geq H |a_n - a_m| < \frac{\varepsilon}{2M}$.
   Then $\forall n, m \geq H$

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\begin{aligned}
|a_n^2 - a_m^2| &= |a_n - a_m||a_n + a_m| \\
&\leq \frac{\varepsilon}{2M}|a_n + a_m| \\
&\leq \frac{\varepsilon}{2M}(|a_n| + |a_m|) \\
&= \frac{\varepsilon}{2M}(M + M) \\
&= \frac{\varepsilon}{2M}(2M) \\
&= \varepsilon
\end{aligned}
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   The counterexample to the converse is if $(a_n) = (-1)^n = \{-1, 1, -1, \cdots\}$ is not Cauchy, but $(a_n^2) = (-1)^2 = \{1, 1, \cdots\}$. So (a_n^2) is Cauchy, but (a_n) is not.

2. Given that (a_n) and (b_n) are Cuachy sequences, use the definition (as an Epsilon argument) to show that $(a_n b_n)$ is also a Cauchy sequence.

   Proof. Let $\varepsilon > 0$ be given. Because (a_n) is Cauchy $|a_n| \leq A$. Since (a_n) is Cauchy $\exists H_a \in \mathbb{R}$ s.t. $\forall n, m \geq H_a$.
   $|a_n - a_m| < \frac{\varepsilon}{2B}$
   Because (b_n) is Cauchy $|b_n| \leq B$. Since (b_n) is Cauchy $\exists H_b \in \mathbb{R}$ s.t. $\forall n, m \geq H_b$
   $|b_n - b_m| < \frac{\varepsilon}{2A}$

Consider $H = \max(H_a, H_b)$, then $\forall n, m \in \mathbb{R}$

$$\begin{aligned}
|a_nb_n - a_mb_m| &= |a_nb_n - a_nb_m + a_nb_m - a_mb_m| \\
&\leq |a_nb_n - a_nb_m| + |a_nb_m - a_mb_m| && \text{(Triangle Inequality)} \\
&= |a_n||b_n - b_m| + |b_m||a_n - a_m| \\
&\leq A|b_n - b_m| + B|a_n - a_m| \\
&< A(\frac{\varepsilon}{2A}) + B(\frac{\varepsilon}{2B}) \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

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3. Given that (a_n) nad (b_n) are Cauchy sequences, use the definition (an Epsilon argument) to show that $(6a_n - 2b_n)$ is also a Cauchy sequence.

Proof. Let $\varepsilon > 0$ be given. Since (a_n) is Cauchy $\exists H_a \in \mathbb{R}$ s.t. $\forall n, m \geq H_a$
$|a_n - a_m| < \frac{\varepsilon}{12}$
Since (b_n) is Cauchy $\exists H_b \in \mathbb{R}$ s.t.
$|b_n - b_m| < \frac{\varepsilon}{4}$
Consider $H = \max(H_a, H_b)$, then $\forall n, m \geq H$

$$\begin{aligned}
|(6a_n - 2b_n) - (6a_m - 2b_n)| &= |6a_n - 6a_m - 2b_n + 2b_m| \\
&\leq |6a_n - 6a_m| + |-2b_n + 2b + m| && \text{(Triangle Inequality)} \\
&= 6|a_n - a_m| + 2|b_n - b_m| \\
&< 6\frac{\varepsilon}{12} + 2\frac{\varepsilon}{4} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&= \varepsilon
\end{aligned}$$

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