Intro to Analysis Homework 1

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1. Using only the field axioms, the order properties (Theorem 2.1.5), and the definition of the inequalities (Defn 2.1.6), prove that

   (a) If $a > b$, then $a + c > b + c$ for any real number c.

   Proof. Let $a > b$, then $a - b \in \mathbb{R}^+$.

   $$\begin{aligned} a - b &= a + 0 - b && \text{(A3)} \\ &= a + (c + -c) - b && \text{(A4)} \\ &= (a + c) + (-c - b) && \text{(A2)} \\ &= (a + c) + -1(c + b) && \text{(D)} \\ &= (a + c) - 1(c + b) && \text{(A1)} \\ &= (a + c) - (c + b) && \text{(M3)} \end{aligned}$$

   So $(a + c) - (c + b) \in \mathbb{R}^+$, and by definition $a + c > b + c$. ∎

   (b) If $a > b$ and $c > 0$, then $ac > bc$.

   Proof. Let $a > b$, then $a - b \in \mathbb{R}^+$, $c > 0$, so $c \in \mathbb{R}^+$. So $c(a - b) \in \mathbb{R}^+$ by the second order property. The distrubitive property can be used to get $ca - cb \in \mathbb{R}^+$. Next the Commutative Property is used to get $ac - bc \in \mathbb{R}^+$.
   Thus by defintiion $ac > bc$. ∎

   (c) If $a > b$ and $c < 0$, then $ac < bc$.

   Proof. If $a > b$, then $a - b \in \mathbb{R}^+$, and $c < 0$ so $-c \in \mathbb{R}^+$.

   $$\begin{aligned} -c(a - b) &\in \mathbb{R}^+ && \text{(Second Order Property)} \\ -ca + cb &\in \mathbb{R}^+ && \text{(D)} \\ cb - ca &\in \mathbb{R}^+ && \text{(A1)} \\ bc - ac &\in \mathbb{R}^+ && \text{(M1)} \end{aligned}$$

   so $bc > ac$ which is the same as $ac < bc$. ∎

2. If $4 < x < 5$ and $f(x) = \frac{x-3}{x^2-9x+14}$, then find a real number M so that $|f(x)| \leq M$.
   $|f(x)| = \frac{|x-3|}{|x^2-9x+14|} = \frac{|x-3|}{|x-2||x-7|}$.
   Let $4 < x < 5$, then $1 < x - 3 < 2$. So $|x - 3| = x - 3 < 2$
   Now let $4 < x < 5$, then $2 < x - 2 < 3$. So $|x - 2| = x - 2 > 2$.
   Finally let $4 < x < 5$, then $-3 < x - 7 < -2 \implies 3 > -(x - 7) > 2$.
   So $|x - 7| = -(x - 7) > 2$.
   $|f(x)| = \frac{|x-3|}{|x^2-9x+14|} = \frac{|x-3|}{|x-2||x-7|} = \frac{2}{(2)(2)} = \frac{2}{4} = \frac{1}{2}$
   $\therefore M = \frac{1}{2}$.

3. Using only the order properties, prove that if $0 < a < b$, then $a^2 < ab < b^2$.

   Proof. If $0 < a < b$, then $a > 0$, and $b > a$. By 1b

$$\begin{aligned} b > a &\implies ba > aa \\ &\implies ba > a^2 \end{aligned}$$

   By Theorem 2.1.7a $a > 0$ and $b > a \implies b > 0$. Again by 1b we show

$$\begin{aligned} b > a &\implies bb > ab \\ &\implies b^2 > ab \end{aligned}$$

   $\therefore a^2 < ab < b^2$. ■

4. Prove that $a \in \mathbb{R}$ satisfies $a^2 = a$ if and only if either $a = 0$ or $a = 1$.

   Proof. Let $a \in \mathbb{R}$ and $a^2 = a$ then we have

$$\begin{aligned} a^2 - a &= a - a && \text{(ii)} \\ a^2 - a &= 0 && \text{(A4)} \\ aa - a1 &= 0 && \text{(M3)} \\ a(a-1) &= 0 && \text{(D)} \\ a = 0 \text{ or } a - 1 &= 0 && \text{(i)} \\ a = 0 \text{ or } (a-1) + 1 &= 0 + 1 && \text{(ii)} \\ a = 0 \text{ or } a + (-1+1) &= 0 + 1 && \text{(A2)} \\ a = 0 \text{ or } a + 0 &= 0 + 1 && \text{(A4)} \\ a = 0 \text{ or } a &= 1 && \text{(A3)} \end{aligned}$$

   Once again suppose that $a^2 = 0$.

   <u>Case 1</u>: $a = 0$

$$\begin{aligned} a^2 &= 0^2 && \text{(Hypothesis)} \\ &= 0 \cdot 0 \\ &= 0 && \text{(iii)} \\ &= a && \text{(Hypothesis)} \end{aligned}$$

   so $a^2 = a$

Case 2: $a = 1$

$$\begin{aligned} a^2 &= 1^2 && \text{(Hypothesis)} \\ &= 1 \cdot 1 \\ &= 1 && \text{(M3)} \\ &= a && \text{(Hypothesis)} \end{aligned}$$

so $a^2 = a$ ∎

5. If B is a bounded set, and A is a subset of B, then show that A is a bounded set as well.

   Proof. B is bounded, so $\exists b_1 \in B$ that is a lower bound, and $\exists b_2 \in B$ that is an upper bound, s.t. $b_1 \leq b \leq b_2 \; \forall b \in B$. Now let $a \in A$, then $a \in B$ too, so $b_1 \leq a \leq b_2$.
   $\therefore A$ is bounded as well. ∎

6. If A and B are nonempty sets of real numbers with $A \subseteq B$ and B is bounded, then show that

$$\sup(A) \leq \sup(B)$$

   Proof. From 5 we know that $A \subseteq B$ with B being bounded, then A is bounded too, so $\sup(A)$ and $\sup(B)$ exist. Let $a \in A$, so $a \in B$ as well, and $a \leq \sup(B)$. Since $\sup(B)$ is an upper bound on A, and $\sup(A)$ is the least upper bound on A, we know $\sup(A) \leq \sup(B)$. ∎

7. If $x, y, z \in \mathbb{R}$ with $x \leq z$, then show that $x \leq y \leq z$ if and only if $|x-y|+|y-z| = |x-z|$.

   Proof. Assume $x \leq y \leq z$.
   Then $x \leq y \implies |x-y| = -(x-y) = -x+y$ and
   $y \leq z \implies |y-z| = -(y-z) = -y+z$
   and by Thm 2.1.7a $x \leq z$, so $|x-z| = -(x-z) = -x+z$. We now have

$$\begin{aligned} |x-y| + |y-z| &= (-x+y) + (-y+z) \\ &= -x + (y + -y) + z && \text{(A2)} \\ &= -x + 0 + z && \text{(A4)} \\ &= -x + z && \text{(A3)} \\ &= |x-z| \end{aligned}$$

   Now assume that $|x-y| + |y-z| = |x-z|$
   BMOC assume that $x > y$ or $y > z$.

Case 1: $(x > y)$ We know $x \le z$, so $y < z$ as well.
Next we know that $|x-y| = x-y, |y-z| = -y+z$, and $|x-z| = -x+z$. Thus

$$\begin{aligned} & |x-y| + |y-z| = |x-z| \\ \implies & x - y + -y + z = -x + z \\ \implies & x - 2y + z = -x + z \\ \implies & x - 2y = -x && \text{(Add -z)} \\ \implies & -2y = -2x && \text{(Add -x)} \\ \implies & y = x && \text{(Multiply } \frac{-1}{2}\text{)} \end{aligned}$$

But $x > y$, so this is a contradiction.

Case 2: $(y > z)$ We know $x \le z$ so $y > x$ as well.
Next we know that $|x-y| = -x+y, |y-z| = y-z$, and $|x-z| = -x+z$. Thus

$$\begin{aligned} & |x-y| + |y-z| = |x-z| \\ \implies & -x + y + y - z = -x + z \\ \implies & -x + 2y - z = -x + z \\ \implies & 2y - z = z && \text{(Add x)} \\ \implies & 2y = 2z && \text{(Add z)} \\ \implies & y = z && \text{(Multiply } \frac{1}{2}\text{)} \end{aligned}$$

But $y > z$, so this is a contradiction. Therefore $x \le y \le z$. ∎

8. If they exist, find the $\inf(S)$ and $\sup(S)$ given the following set.

$$S = \left\{ 1 - \frac{(-1)^n}{n} : n \in \mathbb{N} \right\}$$

$\sup(S) = 2$ and $\inf(S) = \frac{1}{2}$

9. If A is a nonempty subset of real numbers, then $\inf(A) \le \sup(A)$.

Proof. Let $a \in A$. We have $\inf(A) \le a$ and $a \le \sup(A)$. Thus $\inf(A) \le a \le \sup(A)$.
$\therefore \inf(A) \le \sup(A)$. ∎

10. Solve for x:

$$|x+1| + |x-2| = 7$$

When $x < -1$ we know that $|x+1| = -(x+1) = -x-1$ and that $|x-2| = -(x-2) = -x+2$. So we have

$$-x-1-x+2 = 7$$
$$-2x+1 = 7$$
$$-2x = 6$$
$$x = -3$$

Next, when $-1 \leq x \leq 2$ we know that $|x+1| = x+1$ and that $|x-2| = -(x-2) = -x+2$. So we have

$$x+1-x+2 = 7$$
$$3 = 7$$

Which is not true, so there is no solution here.
Finally wheb $x > 2$ we know that $|x+1| = x+1$ and $|x-2| = x-2$. So

$$x+1+x-2 = 7$$
$$2x-1 = 7$$
$$2x = 8$$
$$x = 4$$

So $x = -3, 4$