

## Historical Roots of Mathematics Homework 6

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1. It is said that Pythagoras himself generated Pythagorean triples using the following method:  
Let  $n \in \mathbb{Z}^+$  and let

$$\left. \begin{aligned} x &= 2n + 1 \\ y &= 2n^2 + 2n \\ z &= 2n^2 + 2n + 1 \end{aligned} \right\} \quad (*)$$

- (a) Find  $x$ ,  $y$ , and  $z$  for

i.  $n = 1$

$$\begin{aligned} x &= 2(1) + 1 = 3 \\ y &= 2(1)^2 + 2(1) = 4 \\ z &= 2(1)^2 + 2(1) + 1 = 5 \end{aligned}$$

ii.  $n = 2$

$$\begin{aligned} x &= 2(2) + 1 = 5 \\ y &= 2(2)^2 + 2(2) = 12 \\ z &= 2(2)^2 + 2(2) + 1 = 13 \end{aligned}$$

iii.  $n = 3$

$$\begin{aligned} x &= 2(3) + 1 = 7 \\ y &= 2(3)^2 + 2(3) = 24 \\ z &= 2(3)^2 + 2(3) + 1 = 25 \end{aligned}$$

- (b) Prove that  $(*)$  produces a Pythagorean triple for any positive integer  $n$ .

*Proof.* Let  $x, y, z \in \mathbb{Z}^+$  s.t.

$$\left. \begin{aligned} x &= 2n + 1 \\ y &= 2n^2 + 2n \\ z &= 2n^2 + 2n + 1 \end{aligned} \right\}$$

with  $n \in \mathbb{Z}^+$ . Then:

$$\begin{aligned} x^2 + y^2 &= (2n + 1)^2 + 2n^2 + 2n^2 \\ &= (4n^2 + 4n + 1) + (4n^4 + 8n^3 + 4n^2) \\ &= 4n^4 + 8n^3 + 8n^2 + 4n + 1 \end{aligned}$$

Also:

$$\begin{aligned} z^2 &= (2n^2 + 2n + 1)^2 \\ &= 4n^4 + 8n^3 + 8n^2 + 4n + 1 \end{aligned}$$

Thus  $x^2 + y^2 = z^2$  and (\*) produces Pythagorean triples. ■

(c) Will (\*) produce every primitive Pythagorean triple? Explain.

No (\*) will not generate every Pythagorean triple. It only generates Pythagorean triples where  $y$  and  $z$  differ by 1, and there are primitive Pythagorean triples where  $y$  and  $z$  differ by more than 1, eg (8, 15, 17).

2. Write each of the following numbers as the sum of three or fewer triangular numbers:

(a)  $56 = t_{10} + t_1 = 55 + 1$

(b)  $69 = t_{11} + t_2 = 66 + 3$

(c)  $185 = t_{13} + t_{13} + t_2 = 91 + 91 + 3$

(d)  $287 = t_{22} + t_7 + t_3 = 253 + 28 + 6$

3. Use mathematical induction to prove that  $t_n = \frac{n(n+1)}{2}$ .

*Proof.* Let  $t_n = \sum_{i=1}^n i$  be the  $n$ th triangular number.

Base case ( $n = 1$ ):

Let  $n = 1$ , then  $\sum_{i=1}^1 i = 1$  and  $\frac{1(1+1)}{2} = \frac{2}{2} = 1$ .

Hypothesis ( $n = k$ ):

Assume  $t_k = \sum_{i=1}^k i = 1 + 2 + \dots + k = \frac{k(k+1)}{2} \forall k \in \mathbb{Z}^+$

Induction ( $n = k + 1$ ):

Consider  $\sum_{i=1}^{k+1} i$ , then

$$\begin{aligned} \sum_{i=1}^{k+1} i &= 1 + 2 + \dots + k + (k+1) \\ &= \frac{k(k+1)}{2} + (k+1) && \text{(Hypothesis)} \\ &= \frac{k^2 + k + 2(k+1)}{2} \\ &= \frac{k^2 + 3k + 2}{2} \\ &= \frac{(k+1)(k+2)}{2} = \frac{(k+1)((k+1)+1)}{2} \end{aligned}$$

$$\therefore t_n = \sum_{i=1}^n i = \frac{n(n+1)}{2}$$

■

4. Prove that 1184 and 1210 are amicable.

*Proof.* Divisors of 1184 are  $\{1, 2, 4, 8, 16, 32, 37, 74, 148, 296, 592, 1184\}$

$$1 + 2 + 4 + 16 + 32 + 37 + 74 + 148 + 296 + 592 = 1210$$

Divisors of 1210 are  $\{1, 2, 5, 10, 11, 22, 55, 110, 121, 242, 605, 1210\}$

$$1 + 2 + 5 + 10 + 11 + 22 + 55 + 110 + 121 + 242 + 605 = 1184 \text{ Thus 1184 and 1210 are amicable. } \blacksquare$$

5. Verify that  $\left(\frac{m^2-1}{2}\right)^2 + m^2 = \left(\frac{m^2+1}{2}\right)^2$ .

$$\begin{aligned} \left(\frac{m^2-1}{2}\right)^2 + m^2 &= \frac{m^4 - 2m^2 + 1}{4} + m^2 \\ &= \frac{m^4 - 2m^2 + 1 + 4m^2}{4} \\ &= \frac{m^4 + 2m^2 + 1}{4} \\ &= \left(\frac{m^2+1}{2}\right)^2 \end{aligned}$$