Intro to Analysis Homework 2

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1. Let S be a nonempty bounded set in $\mathbb{R}$. Let $b < 0$ and consider $bS = \{bs : s \in S\}$. Prove $\sup(bS) = b * \inf(S)$

   Proof. Let $\sup(S) = s$ We know that $\sup(bS) = b\sup(S) = bs$. So $\forall s_0 \in S$ $s_0 \leq s$. Becsue $b < 0$ multiplying it into the inequality give is $bs_0 \geq bs$ $\forall bs_0 \in bS$. So by definition bs is the smallest element in bS when $b < 0$, so $bs = \inf(bS) = b * \inf(S)$. ■

2. Let $I_n = \left[1, 1 + \frac{1}{n}\right]$ $\forall n \in \mathbb{N}$. Prove $\bigcap_{n=1}^{\infty} I_n = \{1\}$.

   Proof. Clearly 1 is in $\left[1, 1 + \frac{1}{n}\right]$. BMOC Let $x \in \bigcap_{n=1}^{\infty} I_n$. Then

$$1 < x \leq 1 + \frac{1}{n}$$
$$0 < x - 1 \leq \frac{1}{n}$$

   Since $x - 1 > 0$ by Archimedean Property $\exists m \in \mathbb{N}$ s.t.

$$x - 1 > \frac{1}{m}$$
$$\implies x > 1 + \frac{1}{m}$$

   but $x < 1 + \frac{1}{n}$ $\forall n \in \mathbb{N}$. $\therefore \bigcap_{n=1}^{\infty} I_n = \{1\}$. ■

3. Consider the set $S = \left\{\frac{1}{n} - \frac{1}{m} : n, m \in \mathbb{N}\right\}$. Find the infimum and supremum of the set. Then, prove your assertions.
   $\inf(S) = -1$

   Proof. (i) First we'll show -1 is a lower bound on S. Let $\frac{1}{n_0} - \frac{1}{m_0} \in S$ Then when $n \geq 1$ $\frac{1}{m_0} \leq 1 \implies \frac{-1}{m_0} \geq -1$ Then $\frac{1}{n_0} - \frac{1}{m_0} \geq \frac{1}{n_0} - 1$.
   Also $n > 0$ so $\frac{1}{n_0} > 0$, which means $\frac{1}{n_0} - 1 > 0 - 1$.
   Together this means $\frac{1}{n_0} - \frac{1}{m_0} \geq \frac{1}{n_0} - 1 > 0 - 1 = -1$. So $\frac{1}{n_0} - \frac{1}{m_0} > -1$. So -1 is a lower bound on S.
   (ii) BMOC suppose $x \in \mathbb{R}$ s.t. x is the greatest lower bound on S.
   So $x > -1$ and $x \leq s$ $\forall s \in S$. Then $x + 1 > 0$. and by Archimedean Property $\exists m_0 \in \mathbb{N}$ s.t. $\frac{1}{m_0} < x + 1 \implies \frac{1}{m_0} - 1 < x$ but $\frac{1}{m_0} - 1 \in S$, and $x \leq s$ $\forall s \in S$, a contradiction. Thus there is no x that exists. ■

$\sup(S) = 1$

Proof. (i) First we'll show 1 is an upper bound on S. Let $\frac{1}{n_0} - \frac{1}{m_0} \in S$. Since $n \geq 1$ and $\frac{1}{n} \leq 1$ we know $\frac{1}{n_0} \leq 1$ and $\frac{1}{n_0} - \frac{1}{m_0} \leq 1 - \frac{1}{m_0}$.
Now with $n > 0$ we have $\frac{1}{n} > 0$. This means $\frac{1}{m_0} > 0 \implies \frac{-1}{m_0} < 0$.
With both of these we have $\frac{1}{n_0} - \frac{1}{m_0} \leq 1 - \frac{1}{m_0} < 1 - 0 = 1$.
(ii) BMOC suppose $x \in \mathbb{R}$ s.t. x is the least upper bound on S.
So $x < 1 \implies 1 - x > 0$. By the Archimedean Property $\exists n_0 \in \mathbb{N}$ s.t. $\frac{1}{n_0} < 1 - x \implies \frac{1}{n_0} - 1 < -x \implies 1 - \frac{1}{n_0} > x$, but $1 - \frac{1}{n_0} \in S$, so x is not an upper bound on S, a contradiction.
$\therefore \sup(S) = 1$. ■

4. Let $I_n = \left(2 - \frac{1}{n}, 2\right)\ \forall n \in \mathbb{N}$. Prove $\bigcap_{n=1}^{\infty} I_n = \emptyset$.

Proof. BMOC let $x \in \bigcap_{n=1}^{\infty} I_n$. Then

$$2 - \frac{1}{n} < x < 2$$
$$2 - \frac{1}{n} < x - 2 < 0$$
$$0 < 2 - x < \frac{1}{n} - 2$$

Since $2 - x > 0$ by Archimedean Property $\exists m \in \mathbb{N}$ s.t.

$$2 - x > \frac{1}{m}$$
$$\implies -x > \frac{1}{m} - 2$$
$$\implies x < 2 - \frac{1}{m}$$

but $x > 2 - \frac{1}{n}\ \forall n \in \mathbb{N}$. $\therefore \bigcap_{n=1}^{\infty} I_n = \emptyset$ ■

5. Find the infimum of the set and prove your result.

$$S = \left\{ \frac{3+n}{n} : n \in \mathbb{N} \right\}$$

$\inf(S) = 1$

Proof. (i) $3 + n \geq n \implies \frac{3+n}{n} \geq 1$ so 1 is clearly a lower bound on S.
(ii) BMOC suppose $x \in \mathbb{R}$ s.t. $x > 1$. So $x - 1 > 0 \implies \frac{x-1}{3} > 0$.
By Archimedean Property $\exists m \in \mathbb{N}$ s.t. $\frac{x-1}{3} > \frac{1}{m} \implies x - 1 > \frac{3}{m} \implies x > 1 + \frac{3}{m}$ $\implies x > \frac{3+m}{m}$ but x is a lower bound on $\frac{3+n}{n}$ a contradiction.
$\therefore \inf(S) = 1$ ■

6. Find the limits of the following sequences. Prove your assertions.

(a)

$$\lim_{n\to\infty} \frac{2n-1}{n^2+2}. = 0$$

Proof. Let $\varepsilon > 0$ be given, consider $k = \frac{2}{\varepsilon}$
Then $\forall n \geq k$

$$\begin{aligned}\left|\frac{2n-1}{n^2+2} - 0\right| &= \left|\frac{2n-1}{n^2+2}\right| \\ &= \frac{2n-1}{n^2+2} \\ &< \frac{2n}{n^2} \\ &= \frac{2}{n} \\ &\leq \varepsilon\end{aligned}$$

$(\frac{2}{k} = \varepsilon \implies \frac{2}{\varepsilon} = k)$ ■

(b)

$$\lim_{n\to\infty} \frac{n^2-n}{n^2+2} = 1$$

Proof. Let $\varepsilon > 0$ be given, consider $k = \frac{3}{\varepsilon}$
Then $\forall n \geq k$

$$\begin{aligned}\left|\frac{n^2-n}{n^2+2} - 1\right| &= \left|\frac{n^2-n}{n^2+2} - \frac{n^2+2}{n^2+2}\right| \\ &= \left|\frac{-n-2}{n^2+2}\right| \\ &= \left|\frac{-(n+2)}{n^2+2}\right| \\ &= \frac{n+2}{n^2+2} \\ &< \frac{n+2n}{n^2} \\ &\leq \frac{3n}{n^2} \\ &= \frac{3}{n} \\ &\leq \varepsilon\end{aligned}$$

$(\frac{3}{k} = \varepsilon \implies k = \frac{3}{\varepsilon})$ ■

(c)

$$\lim_{n\to\infty} \frac{(-1)^n\sqrt{n}}{2n-1} = 0$$

Proof. Let $\varepsilon > 0$ be given, consider $k = \frac{1}{\varepsilon^2}$
Then $\forall n \geq k$

$$\begin{aligned}
\left|\frac{(-1)^n\sqrt{n}}{2n-1} - 0\right| &= \left|\frac{(-1)^n\sqrt{n}}{2n-1}\right| \\
&= \frac{\sqrt{n}}{2n-1} \\
&< \frac{\sqrt{n}}{2n-n} \\
&= \frac{1}{\sqrt{n}} \\
&\leq \varepsilon
\end{aligned}$$

$(\frac{1}{\sqrt{k}} = \varepsilon \implies \sqrt{k} = \frac{1}{\varepsilon} \implies k = \frac{1}{\varepsilon^2})$ ∎

(d)

$$\lim_{n\to\infty} \frac{n^2-n}{n^3-2n-4} = 0$$

Proof. Let $\varepsilon > 0$ be given, consider $k = \max(3, \frac{3}{\varepsilon})$
Then $\forall n \geq k$

$$\begin{aligned}
\left|\frac{n^2-n}{n^3-2n-4} - 0\right| &= \left|\frac{n^2-n}{n^3-2n-4}\right| \\
&= \frac{n^2-n}{|n^3-2n-4|} \\
&= \frac{n^2-n}{n^3-2n-4} && (n \geq 3) \\
&= \frac{n^2}{n^3-2n-4} \\
&< \frac{n^2}{\frac{n^3}{3}} && (n^3-2n-4 > \frac{n^3}{3}) \\
&= \frac{1}{\frac{n}{3}} \\
&= \frac{3}{n} \\
&\leq \varepsilon
\end{aligned}$$

$(\frac{3}{k} = \varepsilon \implies k = \frac{3}{\varepsilon})$ ∎