Intro to Analysis Homework 3

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1. Let $a_1 = 1$ and define $a_{n+1} = 1 + \frac{a_n}{2}$. Show that the sequence converges and make an educated guess for the limit value.
   Guess: $a_n \to 2$

$$\begin{aligned} a_n &\leq a_{n+1} \\ a_n &\leq 1 + \frac{a_n}{2} \\ \frac{a_n}{2} &\leq 1 \\ a_n &\leq 2 \end{aligned}$$

Proof. <u>Claim 1</u>: (a_n) is bounded above by 2.
We prove this using induction.
Base case: $(n = 2)$

$$\begin{aligned} a_2 &= 1 + \frac{a_1}{2} \\ &= 1 + \frac{1}{2} \\ &= \frac{3}{2} \end{aligned}$$

Inductive Hypothesis: $(n = k)$
Assume that $a_k < 2$
Inductive Step $(n = k + 1)$

$$\begin{aligned} a_{k+1} \leq 2 \Rightarrow 1 + \frac{a_k}{2} &\leq 2 \\ \Rightarrow \frac{a_k}{2} &\leq 1 \\ \Rightarrow a_k &\leq 2 && \text{(Inductive Hypothesis)} \end{aligned}$$

$\therefore a_n$ is bounded above by 2.
<u>Claim 2</u>: (a_n) is increasing.

By claim 1,

$$\begin{aligned}
&a_n \leq 2 \\
\Rightarrow &\frac{a_n}{2} \leq 1 \\
\Rightarrow &a_n \leq 1 + \frac{a_n}{2} \\
\Rightarrow &a_n \leq a_{n+1}
\end{aligned}$$

Thus (a_n) is decreasing, and by the Monotone Convergence Theorem it converges. ■

2. Let $(x_n) = \frac{2^n}{n!}$. Show that this sequence converges using the Monotone convergence theorem.
   Guess: $(x_n) \to 0$

   Proof. <u>Clain 1</u>: (x_n) is bounded below by 0.
   This is clearly true because $2^n > 0$ and $n! > 0$.
   <u>Claim 2</u>: (x_n) is decreasing.

$$\begin{aligned}
\frac{2^n}{k!} &\geq \frac{2^{n+1}}{(n+1)!} \\
2^n(n+1)! &\geq 2^{n+1}n! \\
2^n(n+1)n! &\geq 2^n 2n! \\
n+1 &\geq 2 \\
n &\geq 1
\end{aligned}$$

   and $n \in \mathbb{N}$ so this is clearly true. Thus (x_n) is decreasing.
   By the Monotone Convergence Theorem (x_n) converges. ■

3. Let $x_1 = 1$ and $x_{n+1} = \frac{4+3x_n}{3+2x_2}$. Show that this sequence converges by the monotone covergence theorem.
   Guess: $x_n \to \sqrt{2}$.

$$\begin{aligned}
x_n &< x_{n+1} \\
x_n &< \frac{4+3x_n}{3+2x_n} \\
x_n(3+2x_n) &< 4+3x_n \\
2x_n^2 + 3x_n &< 4+3x_n \\
2x_n^2 &< 4 \\
x_n^2 &< 2 \\
x_n &< \sqrt{2}
\end{aligned}$$

Proof. Claim 1: (x_n) is bounded above by $\sqrt{2}$.
Proof by induction:
Base case: $(n = 2)$

$$\begin{aligned} x_2 &= \frac{4+3(1)}{3+2(1)} \\ &= \frac{7}{5} < \sqrt{2} \end{aligned}$$

Inductive Hypothesis $(n = k)$
Assume $x_k \leq \sqrt{2}$
Inductive Step $(n = k+1)$

$$\begin{aligned} x_{k+1} \leq \sqrt{2} \Rightarrow \frac{4+3x_n}{3+2x_n} &\leq \sqrt{2} \\ \Rightarrow 4+3x_n &\leq \sqrt{2}(3+2x_n) \\ \Rightarrow 4+3x_n &\leq 3\sqrt{2}+2x_n\sqrt{2} \\ \Rightarrow 3x_n - 2x_n\sqrt{2} &\leq 3\sqrt{2}-4 \\ \Rightarrow x_n(3-2\sqrt{2}) &\leq 3\sqrt{2}-4 \\ \Rightarrow x_n &\leq \frac{3\sqrt{2}-4}{3-2\sqrt{2}}\left(\frac{3+2\sqrt{2}}{3+2\sqrt{2}}\right) \\ \Rightarrow x_n &\leq \frac{9\sqrt{2}+12-12-8\sqrt{2}}{9-8} \\ \Rightarrow x_n &\leq \frac{\sqrt{2}}{1} \\ \Rightarrow x_n &\leq \sqrt{2} \end{aligned}$$

$\therefore x_n$ is bounded above by $\sqrt{2}$.
Claim 2: (x_n) is decreasing.
By clain 1,

$$\begin{aligned} x_n \leq \sqrt{2} \Rightarrow x_n^2 &\leq 2 \\ 2x_n^2 &\leq 4 \\ 2x_n^2 + 3x_n &\leq 4+3x_n \\ x_n(2x_n+3) &\leq 4+3x_n \\ x_n &\leq \frac{4+3x_n}{3+2x_n} \\ x_n &\leq x_{n+1} \end{aligned}$$

Thus (x_n) is decreasing.
By the Monotone Convergence Theorem (x_n) converges. ∎

4. Establish the divergence of the folowing sequences. You can use any of the divernce criteria in 3.4.

(a) $\frac{2n^2(-1+^n+3n)}{n^2}$
Consider the subsequence

$$\begin{aligned}
x_{2n} &= \frac{2(2n)^2(-1)^{2n}+3(2n)}{2(2n)^2} \\
&= \frac{2(4n^2)+6n}{4n^2} \\
&= \frac{8n^2+6n}{4n^2} \\
\Rightarrow \lim_{n\to\infty} x_{2n} &= 2
\end{aligned}$$

Now also consider the subsequence

$$\begin{aligned}
x_{2n+1} &= \frac{2(2n+1)^2(-1)+3(2n+1)}{(2n+1)^2} \\
&= \frac{-2(4n^2+4n+1)+6n+3}{4n^2+4n+1} \\
&= \frac{-8n^2-8n-2+6n+3}{4n^2+4n+1} \\
&= \frac{-8n^2-2n+1}{4n^2+4n+1} \\
\Rightarrow \lim_{n\to\infty} x_{2n+1} &= -2
\end{aligned}$$

Since $x_{2n} \to 2$ and $x_{2n+1} \to -2$, (x_n) does not converge.

(b) $(-n)^3$

$$\begin{aligned}
(-n)^3 &= (-1)^3 n^3 \\
&= (-1)n^3
\end{aligned}$$

Which goes to $-\infty$, so it is not bounded below and diverges.

5. Suppose (x_n) is a bounded subsequence with $s = \sup(x_n : n \in \mathbb{N})$ and suppose further that $s \notin (x_n)$. Construct a subsequnce which converges to s.
Consider $s - \frac{1}{k}\ \forall k \in \mathbb{N}$. We know that because $s - \frac{1}{k} < s$ so $s - \frac{1}{k}$ cannot be an upper bound on (x_n). Therefore $\exists x_{n_k} \in (x_n)$ where $x_{n_k} \geq s - \frac{1}{k}$. So

$$\begin{aligned}
s \geq x_{n_k} &\geq s - \frac{1}{k} \\
\lim_{k\to\infty} s \geq \lim_{k\to\infty} x_{n_k} &\geq \lim_{k\to\infty}\left(s - \frac{1}{k}\right) \\
s \geq \lim_{k\to\infty} x_{n_k} &\geq s
\end{aligned}$$

So by the Squeeze Theorem $\lim_{k\to\infty} x_{n_k} = s$

6. Show that the sequence $x_n = \frac{(2n^2-3n)\sin(n^2)}{n^2+3n+5}$ has a convergent subsequence.

$$\begin{aligned}
|x_n| &= \frac{|2n^2-3n||\sin(n^2)|}{|n^2+3n+5|} \\
&\leq \frac{|2n^2-3n|}{|n^2+3n+5|} \\
&\leq \frac{|2n^2|+|-3n|}{n^2+3n+5} && \text{(Triangle Inequality)} \\
&\leq \frac{2n^2+3n}{n^2+3n+5} \\
&\leq \frac{2n^2+3n}{n^2} \\
&\leq \frac{6n^2}{n^2} && \text{(See Note*)} \\
&= 6
\end{aligned}$$

So (x_n) is bounded and by the Bolzano-Weirstrass Theorem has a convergent subsequence.
(Note*)

$$\begin{gathered}
2n^2+3n \leq 8n^2 \\
0 \leq 6n^2-3n \\
0 \leq n(6n-3) \\
0 \leq 6n-3 \ \forall n \in \mathbb{N}
\end{gathered}$$