Abstract Algebra Homework 10

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5. Prove or disprove: If D is an integral domain, then every prime element in D is also irreducible in D.

Proof. Let D be an integral domain and let $p \in D$ be prime. Consider $p = ab$ for some $a, b \in D$. Because p is prime $p|a$ or $p|b$. WLOG assume $p|a$. Then $pn = a$ for some $n \in D$. Then we have

$$\begin{aligned} a &= pn \\ &= (ab)n \\ &= a(bn) \end{aligned}$$

Because D is an integral domain we can use the cancellation law and get $1 = bn$. So b is a unit. Thus p is irreducible. ■

14. Let D be a Euclidean domain with Euclidean valuation ν. If u is a unit in D, show $\nu(u) = \nu(1)$.

Proof. Let D be a Euclidean Domain and $u \in D$ be a unit. Since u is a unit we know $ua = 1$ $\forall a \in D$. Since D is a Euclidean Domain we also know

$$\nu(u) \leq \nu(ua) = \nu(1) \leq \nu(1 \cdot u) = \nu(u)$$

And since $\nu(u) \leq \nu(1) \leq \nu(u)$ we know that $\nu(u) = \nu(1)$. ■

15. Let D be a Euclidean Domain with Euclidean valuation ν. If a and b are associates in D, prove that $\nu(a) = \nu(b)$.

Proof. Let D be a Euclidean Domain with $a, b \in D$ being associates. We know $\exists u \in D$ s.t. $a = bu$. Because D is a Euclidean Domain

$$\nu(b) \leq \nu(bu) = \nu(a) \leq \nu(au^{-1}) = \nu(b)$$

Since $\nu(b) \leq \nu(a) \leq \nu(b)$ we know that $\nu(a) = \nu(b)$. ■