TITLE

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1. Prove (using the definition) $\lim\limits_{x\to 2} \frac{5x-3}{x-1} = 7$.

Proof. Let $\varepsilon > 0$ be given. Consider $\delta = \min(\frac{1}{2}, \frac{\varepsilon}{4})$
$\forall x$ s.t. $0 < |x-2| < \delta$

$$\begin{aligned}
\left|\frac{5x-3}{x-1} - 7\right| &= \left|\frac{5x-3-7x+7}{x-1}\right| \\
&= \left|\frac{-2x+4}{x-1}\right| \\
&= \left|\frac{-2(x-2)}{x-1}\right| \\
&< 2\delta\left|\frac{1}{x-1}\right| \\
&< \frac{2\delta}{\frac{1}{2}} \qquad (*) \\
&= 4\delta \\
&= \varepsilon
\end{aligned}$$

(*)

$$\begin{aligned}
\frac{-1}{2} < x-2 < \frac{1}{2} \\
\frac{1}{2} < x-1 < \frac{3}{2} \\
\therefore |x-1| = x-1 > \frac{1}{2}
\end{aligned}$$

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2. Prove (using the definition) that $\lim\limits_{n\to\infty} \frac{3n-2n^2}{5-n} = \infty$.

Proof. Let $\alpha \in \mathbb{R}$, consider $K = \frac{\alpha+3}{2}$

Then $\forall n \geq K$

$$\begin{aligned}
\frac{3n-2n^2}{5-n} &= \frac{2n^2-3n}{n-5} \\
&> \frac{2n^2-3n}{n} \\
&= 2n-3 \\
&\geq 2K-3 \\
&= \alpha
\end{aligned}$$

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3. Prove (using the definition) that $\lim\limits_{n\to\infty} \frac{n^3-5}{n-n^2} = -\infty$.

Proof. Let $\alpha \in \mathbb{R}$, consider $K = \max(5, -\alpha+1)$
Then $\forall n \geq K$

$$\begin{aligned}
\frac{n^3-5}{n-n^2} &\leq \frac{n^3-5n}{n-n^2} && (n \geq 2) \\
&= \frac{n^2-5}{1-n} \\
&< \frac{n^2-5}{-n} && (n \geq 3) \\
&= -n + \frac{5}{n} \\
&\leq -n+1 && (n \geq 5) \\
&\leq -K+1 \\
&= \alpha
\end{aligned}$$

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4. Establish the proper divergence of the following sequence: $\left(\frac{n^2}{\sqrt{n^3+1}}\right)$.

$$\lim_{n\to\infty} \frac{n^2}{\sqrt{n^3+1}} = \infty$$

Proof. Let $\alpha \in \mathbb{R}$, consider $K = 4\alpha^2$.
Then $\forall n \geq K$

$$\begin{aligned}
\frac{n^2}{\sqrt{n^3+1}} &\geq \frac{n^2}{\sqrt{n^3+n^3}} \\
&= \frac{\sqrt{n}}{2} \\
&\geq \frac{\sqrt{K}}{2} \\
&= \alpha
\end{aligned}$$

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5. Prove the following limits.

(a) $\lim_{x \to 3} \frac{2}{1-x} = -1$

Proof. Let $\varepsilon > 0$ be given, consider $\delta = \min(1, \varepsilon)$,
Then $\forall x$ s.t. $0 < |x-3| < \delta$

$$\begin{aligned}
\left|\frac{2}{1-x} + 1\right| &= \left|\frac{2+1-x}{1-x}\right| \\
&= \left|\frac{x-3}{1-x}\right| \\
&< \delta \left|\frac{1}{x-1}\right| \\
&< \frac{\delta}{1} && (*) \\
&= \varepsilon
\end{aligned}$$

(*)

$$\begin{aligned}
-1 < x - 3 < 1 \\
1 < x - 1 < 3 \\
\therefore |x-1| = x - 1 > 1
\end{aligned}$$

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(b) $\lim_{x \to 1} \frac{x^2-2x+1}{x-2} = 0$

Proof. Let $\varepsilon > 0$ be given, consider $\delta = \min(\frac{1}{2}, \sqrt{\frac{\varepsilon}{2}})$,
then $\forall x$ s.t. $0 < |x-1| < \delta$

$$\begin{aligned}
\left|\frac{x^2-2x+1}{x-2}\right| &= \left|\frac{(x-1)^2}{x-2}\right| \\
&< \delta^2 \left|\frac{1}{x-2}\right| \\
&< 2\delta^2 && (*) \\
&= \varepsilon
\end{aligned}$$

(*)

$$\begin{aligned}
\frac{-1}{2} < x - 1 < \frac{1}{2} \\
\frac{-3}{2} < x - 2 < \frac{-1}{2} \\
\frac{3}{2} > 2 - x > \frac{1}{2} \\
\therefore |x-2| = 2 - x > \frac{1}{2}
\end{aligned}$$

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