Abstract Algebra Homework 6

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5. Show that the intersection of two normal subgroups is a normal subgroup.

Proof. Let G be a group with $X, Y \leq G$. In homeowrk 3 we proved that $(X \cap Y) \leq G$. So we only need to prove that $(X \cap Y) \trianglelefteq G$.
Let $n \in (X \cap Y)$. So $\forall g \in G$ $gng^{-1} \in X$ and $gng^{-1} \in Y$ $\forall n \in (X \cap Y)$.
Thus $gng^{-1} \in (X \cap Y)$ so $(X \cap Y) \trianglelefteq G$. ∎

6. If G is abelian, prove that G/H must also be abelian.

Proof. Let G be abelian and $G/H = \{gH | g \in G\}$. Let $g_1, g_2 \in G$, then

$$\begin{aligned}(g_1H)(g_2H) &= g_1g_2H \\ &= g_2g_1H && G \text{ is abelian} \\ &= (g_2H)(g_1H)\end{aligned}$$

Therefore G/H is abelian. ∎

8. If G is cyclic, prove that G/H must also be cyclic.

Proof. Let G be cyclic and $G/H = \{gH | g \in G\}$. Because G is cyclic $\langle x \rangle = G$. So $gH = x^nH = (xH)^n = \langle xH \rangle$. Therefore G/H is cyclic. ∎

4. Let $\varphi : \mathbb{Z} \to \mathbb{Z}$ be given my $\varphi(n) = 7n$. Prove that φ is a group homomorphism. Find the kernal and the image of φ.

Proof. Let $a, b \in \mathbb{Z}$. Then

$$\begin{aligned}\varphi(a+b) &= 7(a+b) \\ &= 7a + 7b \\ &= \varphi(a) + \varphi(b)\end{aligned}$$

Therefore φ is a homomorphism. ∎

$\ker(\varphi) = \{a \in \mathbb{Z} | \varphi(a) = 0\} = 0$.
The image of φ is $\{7n | n \in \mathbb{Z}\}$.