Abstract Algebra Homework 9

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2. Compute each of the following.

(a) $(5x^2 + 3x - 4) + (4x^2 - x + 9)$ in $\mathbb{Z}_{12}[x]$
$= (9x^2 + 2x + 5) \mod 12$

(b) $(5x^2 + 4x - 4)(4x^2 - x + 9)$ in $\mathbb{Z}_{12}[x]$
$= 20x^4 + 11x^3 + 25x^2 + 40x - 36 = (8x^4 + 11x^3 + x^2 + 4x) \mod 12$

(c) $(7x^3 + 3x^2 - x) + (6x^2 - 8x + 4)$ in $\mathbb{Z}_9[x]$
$= (7x^3 + 4) \mod 9$

(d) $(3x^2 + 2x - 4) + (4x^2 + 2)$ in $\mathbb{Z}_5[x]$
$= (2x^2 + 2x + 3) \mod 5$

(e) $(3x^2 + 2x - 4)(4x^2 + 2)$ in $\mathbb{Z}_5[x]$
$= 12x^4 + 8x^3 - 10x^2 + 4x - 8 = (2x^4 + 3x^3 + 4x + 2) \mod 5$

(f) $(5x^2 + 3x - 2)^2$ in $\mathbb{Z}_{12}[x]$
$= 25x^4 + 30x^3 - 11x^2 - 12x + 4 = (x^4 + 6x^3 + x^2 + 4) \mod 12$

3. Use the division algorithm to find $q(x)$ and $r(x)$ such that $a(x) = q(x)b(x) + r(x)$ with $\deg r(x) < \deg b(x)$ for each of the following pairs of polynomials.

(a) $a(x) = 5x^3 + 6x^2 - 3x + 4$ and $b(x) = x - 2$ in $\mathbb{Z}_7[x]$

$$
\begin{array}{r}
5x^2 + 16x + 29 \\
x-2\,\overline{)\,5x^3 + 6x^2 - 3x + 4} \\
\underline{-(5x^3 - 10x^2)} \\
0x^3 + 16x^2 - 3x \\
\underline{-(16x^2 - 32x)} \\
29x + 4 \\
\underline{-(29x - 58)} \\
62
\end{array}
$$

$5x^3 + 6x^2 - 3x + 4 = (x - 2)(6x^2 + 16x + 29) + 62$
$= [(x - 2)(5x^2 + 2x + 1) + 6] \mod 7$

(b) $a(x) = 6x^4 - 2x^3 + x^2 - 3x + 1$ and $b(x) = x^2 + x - 2$ in $\mathbb{Z}_7[x]$

$$
\begin{array}{r}
6x^2 - 8x + 21 \\
x^2 + x - 2 \,\big)\, \overline{6x^4 - 2x^3 + \quad x^2 - 3x + 1} \\
\underline{-(6x^4 + 6x^3 - 12x^2)} \\
-\, 8x^3 + 13x^3 - 3x \\
\underline{-(-8x^3 - 8x^2 + 16x)} \\
21x^2 - 19x + 1 \\
\underline{-(21x^2 + 21x - 42)} \\
-\, 40x + 43
\end{array}
$$

$6x^4 - 2x^3 + x^2 - 3x + 1 = (x^2 + x - 2)(6x^2 - 8x + 21) + (-40x + 43)$
$= (x^2 + x - 2)(6x^2 - x) + (2x + 1)$

(c) $a(x) = 4x^5 - x^3 + x^2 + 4$ and $b(x) = x^3 - 2$ in $\mathbb{Z}_5[x]$

$$
\begin{array}{r}
4x^2 - 1 \\
x^3 - 2 \,\big)\, \overline{4x^5 \; - \; x^3 \; + \; x^2 + 4} \\
\underline{-(4x^5 + 0x^3 - 8x^2)} \\
-\, x^3 + 9x^2 + 4 \\
\underline{-(-x^3 + 0x^2 + 2)} \\
9x^2 - 2
\end{array}
$$

$4x^5 - x^3 + x^2 + 4 = (x^3 - 2)(4x^2 - 1) + (9x^2 - 2)$
$= (x^3 - 2)(4x^2 - 1) + (4x^2 - 2) \mod 5$

(d) $a(x) = x^5 + x^3 - x^2 - x$ and $b(x) = x^3 + x$ in $\mathbb{Z}_2[x]$

$$
\begin{array}{r}
x^2 \\
x^3 + x \,\big)\, \overline{x^5 + x^3 - x^2 - x} \\
\underline{-(x^5 + x^3)} \\
0 - x^2 - x
\end{array}
$$

$x^5 + x^3 - x^2 - x = (x^3 + x)(x^2) + (-x^2 - x) \mod 2$

5. Find all of the zeros for each of the following polynomials.

(a) $5x^3 + 4x^2 - x + 9$ in $\mathbb{Z}_{12}[x]$
None

(b) $3x^3 - 4x^2 - x + 4$ in $\mathbb{Z}_5[x]$
$x = 2$

(c) $5x^4 + 2x^2 - 3$ in $\mathbb{Z}_7[x]$
$x = 3, 4$

(d) $x^3 + x + 1$ in $\mathbb{Z}_2[x]$
None

8. Which of the following polynomials are irreducible over $\mathbb{Q}[x]$.

(a) $x^4 - 2x^3 + 2x^2 + x + 4 = (x^2 - 3x + 5)(x^2 + x + 1)$. So it is not irreducible.

(b) $x^4 - 5x^3 + 3x - 2$. It has no roots from the rational roots theorem, so if it's will be so by quadratics. According to Gauss's Lemma it can written be as

$$x^4 - 5x^3 + 3x - 2 = (x^2 + ax + b)(x^2 + cx + d)$$

where $a, b, c, d \in \mathbb{Z}$. We know that

$$\begin{aligned} -5 &= a + c \\ 0 &= b + d + ac \\ 3 &= ad + bc \\ -2 &= bd \end{aligned}$$

WLOG we have either $b = -2, d = 1$ or $b = 2, d = -1$
When $b = -2$ and $d = 1$ then we know

$$\begin{aligned} -5 &= a + c \\ 3 &= a - 2c \end{aligned}$$

Which would mean

$$-8 = 3c$$

Which has no integer solution, so it is irreducible for this case.
When $b = 2$ and $d = -1$ then we have

$$\begin{aligned} -5 &= a + c \\ 3 &= -a + 2c \end{aligned}$$

Which would mean

$$-2 = 3c$$

Which has no integer solution.
Therefore the polynomial is irreducible.

(c) $3x^5 - 4x^3 - 6x^2 + 6$
Let $p = 2$. Then $p|6,\ p|-6,\ p|-4,\ p \nmid 3$ and $p^2 \nmid 6$, so by Einstein's Criterion the function is irreducible over $\mathbb{Q}$.

(d) $5x^5 - 6x^4 - 3x^2 + 9x - 15$
Let $p = 3$. Then $p|15,\ p|9,\ p|-3,\ p|-6, p \nmid 5$ and $p^2 \nmid 15$, so by Einstein's Criterion the function is irreducible over $\mathbb{Q}$.

17. Let F be a field and $a \in F$. If $p(x) \in F[x]$, show that $p(a)$ is the remainder obtained when $p(x)$ is dived by $x - a$.

Proof. According to the division algorithm if $p(x)$ is divided by $x - a$ then $\exists q(x), r(x) \in F[x]$ s.t. $p(x) = (x - a)q(x) + r(x)$ with $\deg(r(x)) < \deg(x - a) = 1$ or $r(x) = 0$. Then

$$\begin{aligned} p(a) &= (a - a)q(a) + r(a) \\ &= 0q(a) + r(a) \\ &= r(a) \end{aligned}$$

Therefore the remaineder $r(a) = p(a)$. ■