

# Abstract Algebra Homework 1

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2. Which of the following multiplication tables defined on the set  $G = \{a, b, c, d\}$  form a group? Support your answer in each case.

**Theorem 1.** Let  $a, b, c \in G$ . Given  $a$  is the Identity element of the set  $G$ , then  $a \circ (b \circ c) = (a \circ b) \circ c, \forall b, c \in G$ .

*Proof.* Let  $d \in G$  with  $b \circ c = d$ . Then  $a \circ (b \circ c) \implies a \circ d = d$ .

We also have  $(a \circ b) \circ c \implies b \circ c = d$ .

$\therefore a \circ (b \circ c) = (a \circ b) \circ c$ . ■

(a)

$\circ$	$a$	$b$	$c$	$d$
$a$	$a$	$c$	$d$	$a$
$b$	$b$	$b$	$c$	$d$
$c$	$c$	$d$	$a$	$b$
$d$	$d$	$a$	$b$	$c$

This Cayley Table does not form a group because it is not Associative:

$$\begin{aligned} a \circ (b \circ c) &= d \text{ and} \\ (a \circ b) \circ c &= a \\ \implies a \circ (b \circ c) &\neq (a \circ b) \circ c \end{aligned}$$

(b)

$\circ$	$a$	$b$	$c$	$d$
$a$	$a$	$b$	$c$	$d$
$b$	$b$	$a$	$d$	$c$
$c$	$c$	$d$	$a$	$b$
$d$	$d$	$c$	$b$	$a$

Closure: Every element in the Cayley Table is in the set  $G$ , so it is closed.

Identity: taking any element and multiplying it by  $a$  returns that element. So  $a$  is the identity element.

Inverse: A diagonal is formed in the table with the identity element  $a$ , so every element is its own inverse.

Associative: Because  $a$  is the identity element it is associative with every set of two elements by Theorem 1. Because every

element  $p_{ij} = p_{ji}$  it is commutative as well, so only one permutation of the elements  $b, c, d$  needs to be tested for associativity. We have

$(b \circ c) \circ d = a$ , and  $b \circ (c \circ d) = a$ , so  $(b \circ c) \circ d = b \circ (c \circ d)$ .

Therefore this Cayley Table is a group.

- (c) This Cayley Table is the same as the Cayley Table for the group  $(\mathbb{Z}_4, +)$  where  $a = 0, b = 1, c = 2, d = 3$ , so This Cayley Table must be a group.

$\circ$	$a$	$b$	$c$	$d$
$a$	$a$	$b$	$c$	$d$
$b$	$b$	$c$	$d$	$a$
$c$	$c$	$d$	$a$	$b$
$d$	$d$	$a$	$b$	$c$

- (d) The identity element of this Cayley Table is  $a$ . There is no inverse for  $d$  where  $d \circ p = a$  in this Cayley Table. Therefore this Cayley Table is not a Group.

$\circ$	$a$	$b$	$c$	$d$
$a$	$a$	$b$	$c$	$d$
$b$	$b$	$a$	$c$	$d$
$c$	$c$	$b$	$a$	$d$
$d$	$d$	$d$	$b$	$c$

13. Show that  $\mathbb{R}^* = \mathbb{R} \setminus \{0\}$  is a group under the operation of multiplication.

Let  $a, b, c \in \mathbb{R}^*$

Closure: Multiplying two real numbers will always return result in a real number.

Associativity: The Field Axioms of real numbers state  $a * (b * c) = (a * b) * c$ , so it is Associative as well.

Identity: The multiplicative identity for  $\mathbb{R}^*$  is 1 because  $a \cdot 1 = a \forall a \in \mathbb{R}^*$ .

Inverse: The multiplicative inverse for  $\mathbb{R}^*$  is  $\frac{1}{a}$  because  $a \cdot \frac{1}{a} = \frac{a \cdot 1}{a} = \frac{a}{a} = 1 \forall a \in \mathbb{R}^*$ .

27. Prove that the inverse of  $g_1 g_2 \dots g_n$  is  $g_n^{-1} g_{n-1}^{-1} \dots g_1^{-1}$ .

*Proof by induction.* Base Case ( $n = 1$ ):

$$g_1 g_1^{-1} = e$$

Inductive Hypothesis ( $n = k$ ):

$$(g_1 g_2 \dots g_k)(g_k^{-1} g_{k-1}^{-1} \dots g_1^{-1}) = e$$

Inductive Step ( $n = k + 1$ ) (WTS:  $(g_1 g_2 \dots g_{k+1})(g_{k+1}^{-1} g_k^{-1} \dots g_1^{-1}) = e$ )

$$\begin{aligned}
& (g_1 g_2 \dots g_{k+1})(g_{k+1}^{-1} g_k^{-1} \dots g_1^{-1}) \\
&= (g_1 g_2 \dots g_k)(g_{k+1})(g_{k+1}^{-1})(g_k^{-1} g_{k-1}^{-1} \dots g_1^{-1}) && \text{(Associative Property)} \\
&= (g_1 g_2 \dots g_k)(e)(g_k^{-1} g_{k-1}^{-1} \dots g_1^{-1}) && \text{(Definition)} \\
&= (g_1 g_2 \dots g_k)(g_k^{-1} g_{k-1}^{-1} \dots g_1^{-1}) && \text{(Definition)} \\
&= e && \text{(Inductive Hypothesis)}
\end{aligned}$$

■

**31.** Show that if  $a^2 = e$  for all elements in  $a$  in a group  $G$ , then  $G$  must be abelian.

*Proof.* Let  $a, b \in G$ . Consider  $a \cdot b \cdot b \cdot a$ . Since  $a^2 = a \cdot a = e \forall a \in G$ , and  $G$  is a Group, we have

$$\begin{aligned} a \cdot b \cdot b \cdot a &= a \cdot (b \cdot b) \cdot a && \text{(Commutative Property)} \\ &= a \cdot e \cdot a && \text{(Hypothesis)} \\ &= a \cdot a && \text{(Identity Property)} \\ &= e && \text{(Hypothesis)} \end{aligned}$$

But  $a \cdot b$  has an inverse element of  $a \cdot b$ , and we have proven that  $(a \cdot b) \cdot (b \cdot a) = e$ , so  $a \cdot b = b \cdot a \forall a, b \in G$ . Therefore  $G$  is commutative which makes it an Abelian Group. ■