Intro to Analysis Homework 10

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1. A number is called a fixed point for a function $f(x)$ if $f(a) = a$. Prove that if $|f'(x)| < 1$ for all real numbers x, then f has at most one fixed point. [f cont and diff on $\mathbb{R}$]

   Proof. BMOC assume $\exists a, b,\ a < b$ s.t. $f(a) = a$ and $f(b) = b$.
   Consider $g(x) = f(x) - x$, g cont on $[a, b]$ and diff on (a, b). Also $g(a) = f(a) - a = 0$ and $g(b) = f(b) - b = 0$. Thus by ROlle's Theorem $\exists c \in (a, b)$ s.t.
   $g'(c) = 0$, $g'(x) = f'(x) - 1$, so $0 = f'(c) - 1 \Rightarrow f'(c) = 1$.
   However $-1 < f'(x) < 1\ \forall x \in \mathbb{R}$. This is a contradiction. ■

2. Suppose f and g are continuous on $[a, b]$ and differentiable on (a, b).
   Suppose $f(a) = g(a)$ and suppose $f'(x) < g'(x)$ for all $x \in (a, b)$.
   Prove that $f(b) < g(b)$.

   Proof. Let $h(x) = f(x) - g(x)$. Because f and g is continuous on $[a, b]$ and differentiable on (a, b), h is also continuous on $[a, b]$ and differentiable on (a, b). By the Mean Value Theorem $\exists c \in [a, b]$ s.t.
   $$h(b) - h(a) = h'(c)(b - a)$$
   Since $h(a) = f(a) - g(a) = 0$ and $h'(c) = f'(c) - g'(c) < 0$,
   $$\begin{aligned} h(b) &= h(a) + h'(c)(b - a) \\ &= 0 + h'(c)(b - a) \\ &< 0 \end{aligned}$$
   Thus $h(b) = f(b) - g(b) < 0 \Rightarrow f(b) < g(b)$. ■

3. Suppose f is differentiable on $(-\infty, \infty)$, that $f(1) = 20$ and that $f'(x) \geq 3$ for $1 \leq x \leq 6$. What is the smallest possible values for $f(6)$?

   Proof. Because f is differentiable on $(-\infty, \infty)$ we know that f is continuous on $[1, 6]$ and differentiable on $(1, 6)$. By the Mean Value Theorem $\exists c \in (1, 6)$ s.t.
   $$\begin{aligned} f(6) - f(1) &= f'(c)(6 - 1) \\ \Rightarrow f(6) &= 5f'(c) + f(1) \end{aligned}$$
   Since $f(1) = 20$ we know that $f(6) = 5f'(c) + 20$, and since $f'(c) \geq 3$ we know $f(6) \geq 5(3) + 20 = 35$. So $f(6)$ has a smallest possible value of 35. ■

4. Prove $e^x \geq x + 1$ for all nonnegative real numbers.

Proof. Let $f(x) = e^x$, clearly f is continuous and differentiable on $\mathbb{R}$. We know that f is also continuous on $[0, x]$ and differentiable on $(0, x)$ where $x \in \mathbb{R}^+$. By the Mean Value Theorem $\exists c \in (0, x)$ s.t.

$$f(x) - f(0) = f'(c)(x - 0)$$

So $e^x - 1 = e^c(x)$, and because $c \in (0, x)$ we can say $e^x \geq xe^0 + 1 = x + 1$.
Thus $e^x \geq x + 1$. ■