Abstract Algebra Homework 5

Zachary Meyner

2. Prove that $\mathbb{Z} \cong n\mathbb{Z}$ for $n \neq 0$.

Proof. Let $\varphi : \mathbb{Z} \to n\mathbb{Z}$ with $n \neq 0$, and $\varphi(x) = nx$.
One-to-one:
Let $a, b \in \mathbb{Z}$ with $\varphi(a) = \varphi(b)$. Then

$$\begin{aligned} na &= nb \\ \Rightarrow a &= b \end{aligned}$$

so φ is one-to-one.
Onto:
Let $x \in n\mathbb{Z}$ and consider

$$\begin{aligned} y &= \frac{x}{n} \\ \varphi(y) &= n\frac{x}{n} \\ \varphi(y) &= x \end{aligned}$$

so $y \in \mathbb{Z}$ because $x \in n\mathbb{Z}$, and φ is onto.
Preserves Operation:
Let $a, b \in \mathbb{Z}$, then

$$\begin{aligned} \varphi(a+b) &= n(a+b) \\ &= na + nb \\ &= \varphi(a) + \varphi(b) \end{aligned}$$

so φ preserves operation. ∎

26. Let $\varphi : G \to H$ be a group isomorphism. Show that $\varphi(x) = e_H$ iff $x = e_G$, where e_G and e_H are the identities of G and H, respectively.

Proof. ($\Rightarrow$) Let $x \in G$ s.t. $\varphi(x) = e_H$. Then

$$\begin{aligned} \varphi(x)\varphi^{-1}(x) = e_H(e_H)^{-1} &= e_H \\ \varphi(xx^{-1}) = \varphi(e_G) &= e_H \\ \varphi(x) &= e_H \end{aligned}$$

thus $x = e_G$.
($\Leftarrow$) Let $x = e_G$ and $h = \varphi(g) \in H$, then

$$\begin{aligned} h\varphi(x) &= \varphi(g)\varphi(x) \\ &= \varphi(gx) \\ &= \varphi(ge_G) \\ &= \varphi(g) \\ &= h \\ &= \varphi(e_G g) \\ &= \varphi(x)\varphi(g) \\ &= \varphi(x)h \end{aligned}$$

So by definition of identity $\varphi(x) = e_H$. ■