Intro to Analysis Homework 9

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1. Prive that $f(x) = x^3$ is NOT uniformly continuous for all real numbers.
   Let $(x_n) = n$ and $(y_n) = n + \frac{1}{n}$. Then $\lim\limits_{n\to\infty}(x_n - y_n) = \lim\limits_{n\to\infty}\left[n - \left(n + \frac{1}{n}\right)\right]$ and $\varepsilon_0 = 1$.
$$\lim_{n\to\infty}\left[n - \left(n + \frac{1}{n}\right)\right] = \lim_{n\to\infty}\left(-\frac{1}{n}\right) = 0$$
   But $f(x_n) = n^3$ and $f(y_n) = \left(n - \frac{1}{n}\right)^3 = n^3 + 3n + \frac{3}{n} + \frac{1}{n^3}$
$$\begin{aligned}|f(x_n) - f(y_n)| &= \left|n^3 - \left(n^3 + 3n + \frac{3}{n} + \frac{1}{n^3}\right)\right| \\ &= \left|-3n - \frac{3}{n} - \frac{1}{n^3}\right| \\ &= 3n + \frac{3}{n} + \frac{1}{n^3} \\ &\geq 3n \\ &\geq 3(1) \\ &= \varepsilon\end{aligned}$$

2. Prove that $f(x) = \frac{-3x}{x-4}$ is uniformly continuous on $(5, \infty)$.

   Proof. Let $\varepsilon > 0$ be given. Consider $\delta = \frac{\varepsilon}{12}$
   Then $\forall x, c \in (5, \infty)$ with $0 < |x - c| < \delta$
$$\begin{aligned}|f(x) - f(c)| &= \left|\frac{-3x}{x-4} - \frac{-3c}{c-4}\right| \\ &= \left|\frac{-3x}{x-4} + \frac{3c}{c-4}\right| \\ &= \left|\frac{-3xc + 12x + 3xc - 12c}{(x-4)(c-4)}\right| \\ &= \frac{12|x-c|}{(x-4)(c-4)} \\ &< \frac{12\delta}{(x-4)(c-4)} \\ &< \frac{12\delta}{(5-4)(5-4)} \\ &< 12\delta \\ &\leq \varepsilon\end{aligned}$$

$\blacksquare$

3. Suppose f and g are both uniformly continuous functions on the set $(0, \infty)$. Prove that $2f + 3g$ is uniformly continuous on $(0, \infty)$.

Proof. Let $\varepsilon > 0$ be given.
Since f is uniformly continuous on $(0, \infty)$ $\exists \delta_f s.t. \forall x, c \in (0, \infty)$ with $0 < |x - c| < \delta_f$,

$$\left|f(x) - f(c)\right| < \frac{\varepsilon}{4}$$

Since g is uniformly continuous on $(0, \infty)$ $\exists \delta_g s.t. \forall x, c \in (0, \infty)$ with $0 < |x - c| < \delta_g$,

$$\left|g(x) - g(c)\right| < \frac{\varepsilon}{6}$$

Consider $\delta = \min(\delta_f, \delta_g)$, then $\forall x, c \in (0, \infty)$ with $0 < |x - c| < \delta$

$$\begin{aligned}
\Big|\big[2f(x) + 3g(x)\big] - \big[2f(c) + 3g(c)\big]\Big| &= \big|2f(x) - 2f(c) + 3g(x) - 3g(c)\big| \\
&\leq 2\big|f(x) - f(c)| + 3|g(x) - g(c)\big| \\
&< \frac{2\varepsilon}{4} + \frac{3\varepsilon}{6} \\
&= \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \\
&\leq \varepsilon
\end{aligned}$$

$\blacksquare$

4. Prove that $f(x) = \frac{2x}{6x+1}$ is differentiable on the interval $(0, \infty)$

$$f'(x) = \frac{12x + 2 - 12x}{(6x + 1)^2} = \frac{2}{(6x + 1)^2}$$

Proof. Let $\varepsilon > 0$ be given and $c \in (0, \infty)$.
Consider $\delta = \min\left(\frac{c}{2}, \frac{\varepsilon(3c+1)(6c+1)^2}{12}\right)$

Then $\forall x \in (0, \infty)$ with $0 < |x - c| < \delta$.

$$
\begin{aligned}
\left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| &= \left|\frac{\frac{2x}{6x+1} - \frac{2c}{6c+1}}{x-c} - \frac{2}{(6c+1)^2}\right| \\
&= \left|\frac{\frac{12xc+2x-2xc-2c}{(6x+1)(6c+1)}}{x-c} - \frac{2}{(6c+1)^2}\right| \\
&= \left|\frac{2(x-c)}{(x-c)(6x+1)(6c+1)} - \frac{2}{(6c+1)^2}\right| \\
&= \left|\frac{2}{(6x+1)(6c+1)} - \frac{2}{(6c+1)^2}\right| \\
&= \left|\frac{12c+2-12x-2}{(6x+1)(6c+1)^2}\right| \\
&= \frac{12|x-c|}{(6x+1)(6c+1)^2} \\
&< \frac{12\delta}{(6x+1)(6c+1)^2} \\
&< \frac{12\delta}{(3c+1)(6c+1)^2} \qquad \text{(Note)} \\
&\leq \varepsilon
\end{aligned}
$$

Note:

$$
\begin{gathered}
|x-c| < \frac{c-0}{2} \\
-\frac{c}{2} < x - c < \frac{c}{2} \\
\frac{c}{2} < x < \frac{3c}{2} \\
3c < 6x < 9c \\
3c+1 < 6x+1 < 9c+1 \\
|6x+1| = 6x+1 > 3c+1
\end{gathered}
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5. Prove that $f(x) = \frac{5}{2x-4}$ is differentiable on $(2, \infty)$

$$
f'(x) = -\frac{10}{(2x-4)^2}
$$

Proof. Let $\varepsilon > 0$ be given, and $c \in (2, \infty)$.
Consider $\delta = \min\left(\frac{c-2}{2}, \frac{\varepsilon(c-2)(2c-4)^2}{10}\right)$

Then $\forall x \in (2, \infty)$ with $0 < |x - c| < \delta$

$$\begin{aligned}
\left| \frac{f(x) - f(c)}{x - c} - f'(c) \right| &= \left| \frac{\frac{5}{2x-4} - \frac{5}{2c-4}}{x - c} + \frac{10}{(2c-4)^2} \right| \\
&= \left| \frac{\frac{10c-20-10x+20}{(2x-4)(2c-4)}}{x - c} + \frac{10}{(2c-4)^2} \right| \\
&= \left| \frac{-10(x-c)}{(2x-4)(2c-4)(x-c)} + \frac{10}{(2c-4)^2} \right| \\
&= \left| -\frac{10}{(2x-4)(2c-4)} + \frac{10}{(2c+4)^2} \right| \\
&= \left| \frac{10x - 40 - 10c + 40}{(2x-4)(2c-4)^2} \right| \\
&< \frac{10|x-c|}{|2x-4|(2c-4)^2} \\
&< \frac{10\delta}{|2x-4|(2c-4)^2} \\
&< \frac{10\delta}{(c-2)(2c-4)^2} && \text{(Note)} \\
&\leq \varepsilon
\end{aligned}$$

Note:

$$\begin{gathered}
|x - c| < \frac{c-2}{2} \\
\frac{2-c}{2} < x - c < \frac{c-2}{2} \\
\frac{c+2}{2} < x < \frac{3c-2}{2} \\
c + 2 < 2x < 3c - 2 \\
c - 2 < 2x - 4 < 3c - 6 \\
|2x - 4| = 2x - 4 > c - 2
\end{gathered}$$

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6. Let $f(x) = 4x^2 + 3x + 10$. Prove that f is differentiable for (a) $x = 2$ and then for (b) any $c \in \mathbb{R}$.

$$f'(x) = 8x + 3$$

(a) *Proof.* Let $\varepsilon > 0$, consider $\delta = \frac{\varepsilon}{4}$

Then $\forall x \in \mathbb{R}$ with $0 < |x-2| < \delta$

$$\begin{aligned}
\left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| &= \left|\frac{4x^2+3x+10-32}{x-2} - 19\right| \\
&= \left|\frac{4x^2+3x-22}{x-2} - 19\right| \\
&= \left|\frac{(x-2)(4x+11)}{x-2} - 19\right| \\
&= |4x+11-19| \\
&= |4x-8| \\
&= |4(x-2)| \\
&< 4\delta \\
&= \varepsilon
\end{aligned}$$

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(b) *Proof.* Let $\varepsilon > 0$ and $c \in \mathbb{R}$.
Consider $\delta = \frac{\varepsilon}{4}$.
Then $\forall x \in \mathbb{R}$ with $0 < |x-c| < \delta$

$$\begin{aligned}
\left|\frac{f(x)-f(c)}{x-c} - f'(c)\right| &= \left|\frac{4x^2+3x+10-4c^2-3c-10}{x-c} - 8c - 3\right| \\
&= \left|\frac{4x^2-4c^2+3x-3c}{x-c} - 8c - 3\right| \\
&= \left|\frac{4(x-c)(x+c)+3(x-c)}{x-c} - 8c - 3\right| \\
&= |4x+4c+3-8c-3| \\
&= |4c-4x| \\
&= 4|x-c| \\
&< 4\delta \\
&= \varepsilon
\end{aligned}$$

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