Abstract Algebra Homework 8

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16. If R is a field, show that the only two ideals of R are $\{0\}$ and R itself.

Proof. By means of contradiction assume there is an ideal I s.t. $I \neq \{0\}, R$.
Let $a \in I$, we know that $a \neq 0$, and because R is a field a has an inverse a^{-1}, so $a \cdot a^{-1} = 1 \in I$. Since $1 \in I$ the definiton of an ideal tells us that
$\forall a \in I$ and $\forall r \in R$, $ar = ra \in I$ so $1 \cdot r \in I$ $\forall r \in R$. Thus $I = R$, this is a contradiction, therefore the only ideals of a field R are $\{0\}$ and R. ∎

17. Let a be any element in a ring R with identity. Show that $(-1)a = -a$.

Proof. Let R be a unitary ring with multiplicative identity 1. We know $1 + (-1) = 0$, so we know $(1 + (-1))a = 0 \cdot a = 0$. The distributive property tells us
$1a + (-1)a = a + (-1)a = 0$, and we know that $a + (-a) = 0$, so

$$\begin{aligned} a + (-1)a &= a + (-a) \\ (-1)a &= -a \end{aligned}$$

∎

26. Let R be an integral domain. Show that if the only ideals in R are $\{0\}$ and R itself, R must be a field.

Proof. Let R be an integral domain with the only ideals being $\{0\}$ and R, then $\{0\}$ is the maximal ideal of R. Because there are no other proper ideals, so $R/\{0\} = R$ is a field. ∎

27. Let R be a commutative ring. An element a in R is ***nilpotent*** if $a^n = 0$ for some positive integer n Show that the set of all nilpotent elements forms an ideal in R.

Proof. Let $I \subseteq R$ be the set of nilpotent elements in the commutative ring R s.t

$$I = \{r \in R \mid \exists n \in \mathbb{N},\ r^n = 0\}$$

Subring:
Nonempty: We know, $0^n = 0 \forall n \in \mathbb{N}$, so I is nonempty.
Closure -: Let $a, b \in I$ then $\exists n, m \in \mathbb{N}$ s.t. $a^n = b^m = 0$. Consider $(a - b)^{n+m}$, by

the binomial theorem $(a-b)^{n+m} = \sum_{k=0}^{n+m} \binom{n+m}{k} a^k b^{n+m-k}$. If $k \geq n$ then $a^k = 0$, and if $0 < k < n$ then $b^{n+m-k} = 0$ so $(a-b)^{n+m} = 0$, so $(a-b) \in I$.
Absorption: Let $x \in I$ and $y \in R$. Then $xy = (xy)^n = x^n y^n = 0 \cdot y^n = 0$, so the nilpotent is an ideal of a commutative ring. ■

37. An element x in a ring is called an ***idempotent*** if $x^2 = x$. Prove that the only idempotents in an integral domain are 0 and 1. Find a ring with a idempotent x not equal to 0 or 1.

Proof. Let R be an integral domain. Let $r \in R$. Consider $r^2 = r$. Then we know that $r^2 - r = 0$ and by the distributive property $r(r-1) = 0$, so $r = 0, 1$.
Therefore 0 and 1 are the only idempotents of an integral domain. ■

A ring with idempotent not equal to 0 or 1 is $\mathbb{Z}_6$, $3^2 \equiv 3 \mod 6$