Intro to Analysis Homework 8

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1. Consider $f(x) = \frac{x+3}{x^2-9}$

   (a) Show that f is continuous for $x = 4$

   Proof. Let $\varepsilon > 0$ be given, consider $\delta = \min(\frac{1}{2}, \frac{\varepsilon}{2})$
   Then $\forall x \neq 3, -3$ with $0 < |x-4| < \delta$

$$\begin{aligned}
\left| f(x) - f(4) \right| &= \left| \frac{x+3}{x^2-9} - \frac{7}{7} \right| \\
&= \left| \frac{1}{x-3} - \frac{x-3}{x-3} \right| \\
&= \left| \frac{4-x}{x-3} \right| \\
&= \frac{|4-x|}{|x-3|} \\
&< \delta \frac{1}{|x-3|} \\
&< 2\delta && \text{(Note)} \\
&= \varepsilon
\end{aligned}$$

   Note:

$$\begin{aligned}
-\frac{1}{2} < x - 4 < \frac{1}{2} \\
\frac{7}{2} < x < \frac{9}{2} \\
\frac{1}{2} < x - 3 < \frac{3}{2} \\
|x-3| = x - 3 > \frac{1}{2}
\end{aligned}$$

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   (b) Show that f is continuous for $x \in (-3, 3)$

   Proof. Let $\varepsilon > 0$ be given, and $c \in (-3, 3)$
   Consider $\delta = \min(\frac{3-c}{2}, \frac{\varepsilon(3-c)^2}{2})$

Then $\forall x \in (-3,3)$ with $0 < |x-c| < \delta$

$$\begin{aligned}
|f(x)-f(c)| &= \left|\frac{x+3}{x^2-9} - \frac{c+3}{c^2-9}\right| \\
&= \left|\frac{1}{x-3} - \frac{1}{c-3}\right| \\
&= \left|\frac{c-3-x+3}{(x-3)(c-3)}\right| \\
&= \frac{|c-x|}{|x-3||c-3|} \\
&< \delta\frac{1}{|x-3||c-3|} \\
&= \delta\frac{1}{(3-x)(3-c)} && (x \in (-3,3) \text{ and } c \in (-3,3)) \\
&< \frac{2\delta}{(3-c)^2} && \text{(Note)} && = \varepsilon
\end{aligned}$$

Note:

$$\begin{aligned}
|x-c| &< \frac{3-c}{2} \\
\frac{c-3}{2} < x-c &< \frac{3-c}{2} \\
\frac{3c-3}{2} < x &< \frac{c+3}{2} \\
\frac{3c-9}{2} < x-3 &< \frac{c-3}{2} \\
\frac{9-3c}{2} > 3-x &> \frac{3-c}{2} \\
3-x &> \frac{3-c}{2}
\end{aligned}$$

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2. Consider $f(x) = \frac{5x+1}{x-3}$

   (a) how that f is continuous for $x = 4$

   Proof. Let $\varepsilon > 0$ be given, consider $\delta = \min(\frac{1}{2}, \frac{\varepsilon}{32})$

Then $\forall x \neq 3$ with $0 < |x-4| < \delta$

$$\begin{aligned}
|f(x) - f(4)| &= \left| \frac{5x+1}{x-3} - 21 \right| \\
&= \left| \frac{5x+1}{x-3} - \frac{21x-63}{x-3} \right| \\
&= \left| \frac{-16x+64}{x-3} \right| \\
&= 16\frac{|4-x|}{|x-3|} \\
&< 16\delta\frac{1}{|x-3|} \\
&< 32\delta && \text{(Note)} \\
&= \varepsilon
\end{aligned}$$

Note:

$$\begin{aligned}
-\frac{1}{2} < x - 4 < \frac{1}{2} \\
\frac{1}{2} < x - 3 < \frac{3}{2} \\
|x-3| = x - 3 > \frac{1}{2}
\end{aligned}$$

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(b) Show that f is continuous for $x \in (4, \infty)$

Proof. Let $\varepsilon > 0$ be given, and $c \in (4, \infty)$
Consider $\delta = \min(1, \frac{\varepsilon(c-3)(c-4)}{16})$
Then $\forall x \in (4, \infty)$ with $0 < |x-c| < \delta$

$$\begin{aligned}
|f(x) - f(c)| &= \left| \frac{5x+1}{x-3} - \frac{5c+1}{c-3} \right| \\
&= \left| \frac{5xc - 15x + c - 3 - 5xc + 15c - x + 3}{(x-3)(c-3)} \right| \\
&= \left| \frac{16c - 16x}{(x-3)(c-3)} \right| \\
&= 16\frac{|x-c|}{|x-3||c-3|} \\
&< \frac{16\delta}{c-3} \cdot \frac{1}{|x-3|} \\
&< \frac{16\delta}{(c-3)(c-4)} && \text{(Note)} \\
&= \varepsilon
\end{aligned}$$

Note:

$$\begin{aligned}
|x-c| &< 1 \\
-1 < x-c &< 1 \\
c-1 < x &< c+1 \\
c-4 < x-3 &< c-2 \\
|x-3| = x-3 &> c-4 && (c>4)
\end{aligned}$$

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(c) Show that f is continuous for $x \in (3, \infty)$

Proof. Let $\varepsilon > 0$ be given, and $c \in (3, \infty)$
Consider $\delta = \min(\frac{c-3}{2}, \frac{\varepsilon(c-3)^2}{32})$
Then $\forall x \in (3, \infty)$ with $0 < |x-c| < \delta$

$$\begin{aligned}
|f(x) - f(c)| &= \left|\frac{5x+1}{x-3} - \frac{5c+1}{c-3}\right| \\
&= \left|\frac{16c-16x}{(x-3)(c-3)}\right| \\
&= 16\frac{|x-c|}{|x-3||c-3|} \\
&< \frac{16\delta}{c-3} \cdot \frac{1}{|x-3|} \\
&< \frac{32\delta}{(c-3)^2} \\
&= \varepsilon
\end{aligned}$$

Note:

$$\begin{aligned}
|x-c| &< \frac{c-3}{2} \\
\frac{3-c}{2} < x-c &< \frac{c-3}{2} \\
\frac{3+c}{2} < x &< \frac{3c-3}{2} \\
\frac{c-3}{2} < x-3 &< \frac{3c-9}{2} \\
|x-3| = x-3 &> \frac{c-3}{2}
\end{aligned}$$

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(d) Show that f is continuous for $x \in (-\infty, 3) \cup (3, \infty)$

Proof. Let $\varepsilon > 0$ be given, and $c \in (-\infty, 3) \cup (3, \infty)$
Consider $\delta = \min(\frac{|c-3|}{2}, \frac{\varepsilon|c-3|(6-2c-|c-3|)}{32})$
Then $\forall x \in (-\infty, 3) \cup (3, \infty)$ with $0 < |x - c| < \delta$

$$\begin{aligned}
|f(x) - f(c)| &= \left| \frac{5x+1}{x-3} - \frac{5c+1}{c-3} \right| \\
&= \left| \frac{16c - 16x}{(x-3)(c-3)} \right| \\
&= 16 \frac{|x-c|}{|x-3||c-3|} \\
&< \frac{16\delta}{|c-3|} \cdot \frac{1}{|x-3|} \\
&< \frac{32\delta}{|c-3|(6-2c-|c-3|)} \\
&= \varepsilon
\end{aligned}$$

Note:

$$\begin{gathered}
|x-c| < \frac{|c-3|}{2} \\
-\frac{|c-3|}{2} < x - c < \frac{|c-3|}{2} \\
\frac{2c-|c-3|}{2} < x < \frac{2c+|c-3|}{2} \\
\frac{2c-6-|c-3|}{2} < x - 3 < \frac{2c-6+|c-3|}{2}
\end{gathered}$$

If $c > 3$ and $x > 3$ then

$$\begin{gathered}
\frac{2c-6-|c-3|}{2} > 0 \\
2c - 6 - (c-3) > 0 \\
c - 3 > 0 \\
c > 0
\end{gathered}$$

If $c < 3$ and $x < 3$ then

$$\begin{gathered}
\frac{-2c+6+|c-3|}{2} > 3 - x > \frac{6-2c-|c-3|}{2}, \text{ so} \\
\frac{6-2c-|c-3|}{2} > 0 \\
6 - 2c - (3-c) > 0 \\
3 - c > 0 \\
3 > c
\end{gathered}$$

so $|x-3| = 3 - x > \frac{6-2c-|c-3|}{2}$ ∎

3. Prove that $f(x) = \sqrt{x}$ is continuous at a, where $a \in [0, \infty)$

Proof. Let $\varepsilon > 0$ be given, and $a \in [0, \infty)$
Case 1: $a > 0$
Consider $\delta = \min(\frac{a}{2}, \frac{\varepsilon(\sqrt{a}+\sqrt{2a})}{\sqrt{2}})$
Then $\forall x \in [0, \infty)$ where $0 < |x - a| < \delta$

$$\begin{aligned}
|f(x) - f(a)| &= \left|\sqrt{x} - \sqrt{a}\right| \\
&= \left|\sqrt{x} - \sqrt{a}\right| \cdot \frac{\left|\sqrt{x} + \sqrt{a}\right|}{\left|\sqrt{x} + \sqrt{a}\right|} \\
&= \frac{|x - a|}{\left|\sqrt{x} + \sqrt{a}\right|} \\
&= \delta \frac{1}{\sqrt{x} + \sqrt{a}} \\
&< \frac{\delta}{\sqrt{\frac{a}{2}} + \sqrt{a}} \\
&= \frac{\delta\sqrt{2}}{\sqrt{a} + \sqrt{2a}} \\
&= \varepsilon
\end{aligned}$$

Note:

$$\begin{gathered}
|x - a| < \frac{a}{2} \\
-\frac{a}{2} < x - a < \frac{a}{2} \\
\frac{a}{2} < x < \frac{3a}{2} \\
\sqrt{\frac{a}{2}} < \sqrt{x} < \sqrt{\frac{3a}{2}} \\
\sqrt{x} > \sqrt{\frac{a}{2}}
\end{gathered}$$

Case 2: $a = 0$
Consider $\delta = \varepsilon^2$
Then $\forall x$ with $0 < |x| < \delta$

$$\begin{aligned}
|f(x) - f(a)| &= |\sqrt{x}| \\
&< \sqrt{\delta} \\
&= \varepsilon
\end{aligned}$$

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