Intro to Analysis Homework 8

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1. Suppose f is continuous in $[a, b]$ and $f(c) > 0$ for some $c \in (a, b)$. Then show that there exists an open interval for which $f(x) > 0$ on the interval.

   Proof. Consider $\varepsilon = \frac{f(x)}{2} > 0$. Since f is continuous at $x = c$, $\exists \delta > 0$ s.t. $\forall x \in [a, b]$ with $0 < |x - c| < \delta$ with $|f(x) - f(c)| < \varepsilon = \frac{f(c)}{2}$. Consider the interval $(c - \delta, c + \delta)$. If $x \in (c - \delta, c + \delta)$

$$\begin{aligned} f(x) &= f(c) - f(c) + f(x) \\ &= f(c) - (f(c) + f(x)) \\ &\geq f(c) - |f(c) + f(x)| \\ &\geq f(c) - \frac{f(c)}{2} \\ &= \frac{f(c)}{2} > 0 \end{aligned}$$

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2. Show that $f(x) = x^3 - 2x^2 - 3x + 1$ has at least one zero on $[-1, 1]$
   $f(1) = 3$ and $f(-1) = -5$, so by the Location of Roots Theorem $\exists c$ in the domain s.t. $f(c) = 0$.

3. Let $f : \mathbb{R} \to \mathbb{R}$ be continuous function and suppose $f(x) \leq 0$ for all rational numbers. Prove that $f(x) \leq 0$ for all real numbers.

   Proof. BMOC, suppose $\exists c \in \mathbb{R}$ s.t. $f(c) > 0$. Then by number 1 there is an open interval (a, b) for which $f(x) > 0$ $\forall x \in (a, b)$ with $c \in (a, b)$, but (a, b) is a nonempty, open interval, so $\exists d \in (a, b)$ s.t. $d \in \mathbb{Q}$. Thus $f(d) > 0$, this is a contraction, so c does not exists. ■

4. Consider $f(x) = \begin{cases} 3x^2 + 2 & x \neq 2 \\ 0 & x = 2 \end{cases}$. Prove that f is not continuous at $x = 2$.
   Consider $(x_n) = 2 + \frac{1}{n}$. Clearly $\lim\limits_{n\to\infty} x_n = 2$. We know $(x_n) \neq 2$ $\forall n$
   So $f(x_n) = 3(x_n)^2 + 2$, so $\lim\limits_{n\to\infty} f(x_n) = 3(x_n) + 2 = 3(2)^2 + 2 = 14$.
   But $f(\lim\limits_{n\to\infty} x_n) = f(2) = 0$. $14 \neq 0$ so this function is not continuous at $x = 2$.

5. Suppose $f, g : \mathbb{R} \to \mathbb{R}$ are both continuous functions. Show that $3f - 2g$ is a continuous function too.

Proof. Let $\varepsilon > 0$ be given, and let $c \in \mathbb{R}$.
Since f is continuous at c, $\exists \delta_f > 0$ s.t. $\forall x \in \mathbb{R}$ with $0 < |x - a| < \delta_f$

$$\left|f(x) - f(a)\right| < \frac{\varepsilon}{6}$$

Since g is continuous at c, $\exists \delta_g > 0$ s.t. $\forall x \in \mathbb{R}$ with $0 < |x - a| < \delta_g$

$$\left|g(x) - g(a)\right| < \frac{\varepsilon}{4}$$

Consider $\delta = \min(\delta_f, delta_g)$
Then $\forall x \in \mathbb{R}$ with $0 < |x - a| < \delta$

$$\begin{aligned}
\left|(3f(x) - 2g(x)) - (3f(a) - 2g(a))\right| &= \left|3f(x) - 3f(a) + 2g(a) + 2g(x)\right| \\
&\leq 3|f(x) - f(a)| + 2|g(a) - g(x)| \\
&< \frac{3\varepsilon}{6} + \frac{2\varepsilon}{4} \\
&= \varepsilon
\end{aligned}$$

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6. For each statement, conclude if the statement is true because of Bolzano's Intermediate Value Theorem

   (a) If f is a function satisfying that $f(0) = 10$ and $f(8) = 2$, then it must be that $\exists x \in (0, 8)$ satisfying that $f(x) = 5$.
   We cannot conclude that $\exists x \in (0, 8)$ satisfying $f(x) = 5$ because the function could be discontinuous.

   (b) Suppose revenue can be modeled by a function $R(x)$, where x is measured in thousands of units for units ranging from 0 to 20,000. Revenue for 2,000 units is known to be \$4500 and revenue for 3,000 units is known to be \$7200, but at no point in time was revenue \$5500. We can conclude therefore that the revenue function is not continuous.
   If R is continuous then $\exists c$ where $R(c) = 5500$.

   (c) If f is a continuous function satisfying that $f(20) = 10$ and $f(8) = 2$, then $\exists x \in (8, 20)$ satisfying $f(x) = 15$.
   We can not make a conclustion on this because the known values for $f(x)$ when $x \in (8, 20)$ only guarantee that $f(x)$ touches all the values in a range of at least $(2, 10)$.