Spanning Tree, Matching, and TSP for Moving Points: Complexity and Regret

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Abstract

We explore the computational complexity, and regret, of some geometric structures under the recently introduced moving point model of Akitaya et al. [3]. Specifically, we want to build a single geometric structure (e.g. spanning tree, matching, or traveling salesman path) whose maximum cost during the motion of the input points is minimized. We call these structures, whose cost (sum of edge lengths) changes with the motion of points but whose topology remains fixed, *minimum moving point spanning tree* (MMST), *minimum moving point matching* (MMM), and *minimum moving point traveling salesman path* (MMTSP), respectively. We focus on linear motion of points in one dimension and prove the following results: (1) each of these problems is (weakly) NP-hard in one dimension, (2) remains NP-hard even under *radial expansion* where all points are moving away from a center, (3) remains NP-hard even if points are all moving with the *same speed*. A fixed topology is attractive in that it avoids expensive and continuous recomputation as input points move but it is inevitably sub-optimal. To quantify this tradeoff, we define the *regret* as the worst-case ratio between the cost of an optimal moving point structure and the maximum cost for the same input when the structure is continuously updated. We show the following results: (4) the regret ratio is $\Omega(\sqrt{n})$ for all three problems even in one dimension, but (5) has a tight bound of 2 for MMST and MMTSP if all points are moving with the same speed. We also point out some simple settings under which optimal moving point structures are easy to compute.

1 Introduction

Suppose we want to interconnect a collection of mobile agents, modeled as points in d-dimensional space, in one or more groups to enable a certain collaborative task. For instance, a *minimum spanning tree* is helpful if we want to form a single connected group, a *matching* where we wish to pair each point with another, or a *traveling salesman path* if we want a tour of the points. These are natural problems arising in applications such as multi-robot systems, mobile sensor networks, or a group of human and robotic agents performing collaborative tasks. It is typically desirable that these agents are able to communicate with others, which can be abstracted as a problem of maintaining certain graph structures among a set of moving points, with edge lengths in the graph serving as a natural optimization criterion. The minimum spanning tree, matching, and TSP are classical graph optimization problems for which polynomial time (exact or approximation) algorithms have been known for several decades [7, 6].

When the underlying points are in motion, however, the optimal (or near-optimal) graph structure must be frequently recomputed, which is both inconvenient and resource expensive. There is extensive research in robotics, sensor networks, and computational geometry on how to efficiently detect when the underlying graph structures must be updated and how to update them [4, 9, 5, 1, 8, 2]. Our work is motivated by the recent work of Akitaya et al [3], which explored an alternative approach to maintaining the spanning tree over data in motion. Specifically, if we wanted to choose a *single* spanning tree T for the *entire* motion, which spanning tree would be the best? In other words, which fixed spanning tree topology minimizes the *maximum* length during the course of the motion? Akitaya et al [3] shows that this problem is NP-Hard in the plane and describe a 2-approximation for it.

Our Results

In this paper, we continue the line of research in [3] and explore moving point versions of three classical problems: minimum spanning tree, matching and traveling salesman tour. We show that all of them are NP-hard *even in one dimension*, and *even under highly constrained linear motions*. Second, we establish bounds on the *regret ratio* of these moving point structures as a way to quantify the tradeoff between convenience of a fixed topology and the maximum cost of the structure. Finally, we also point out two natural instances of the moving point structures for which an optimal is easy to find. In the interest of simpler presentation, we focus on the moving point MST in describing our main results, and summarize their extensions to matching and TSP at the end. Our key results can be summarized as follows.

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- The MMST problem is NP-hard even for *unit speed* linear motion in one dimension.
- The problem remains NP-hard in 1D even if all points linearly move away from the origin (radial expansion).
- MMST regret ratio is $\Omega(\sqrt{n})$ even for linear motion in one dimension, but has a tight bound of 2 for *unit speed* motion.
- The MMST problem can be solved in polynomial time, in any fixed dimension d, if (1) all points are moving away from the origin at uniform velocity (uniform expansion), or (2) we want to minimize the *average* cost of the MST during the motion.
- All the hardness results also hold for MMM and MMTSP.

2 Preliminaries

Throughout this paper, we consider points under linear motion, each moving along a straight line with constant speed; different points can move with different speeds unless otherwise specified. Thus, a *moving point* p is a continuous function $p : [0, 1] \to \mathbb{R}^d$, and the distance between two moving points p and q at time t is $\|p(t) - q(t)\|$.

Given a set $S = \{p_0, \ldots, p_{n-1}\}$ of n moving points, we call a spanning tree T of S a *moving spanning tree* whose weight (or length) at time t is $w_T(t) = \sum_{pq\in T} \|p(t) - q(t)\|$. We use $w(T) = \sup_t w_T(t)$ to denote the maximum weight of T during the motion. A *minimum moving spanning tree* (MMST) of S is one with minimum weight, namely,

$$\operatorname*{arg\,min}_{T\in\mathcal{T}(S)} \; w(T),$$

where $\mathcal{T}(S)$ is the set of all possible moving spanning trees of S. Similar definitions hold for a *minimum moving matching* (MMM), where input is a set of $2n$ points and the goal is to find a matching whose maximum weight during the motion is minimized, or *minimum moving traveling salesman tour* (MMTSP), where the goal is to find a spanning path of the points with minimum maximum weight during the motion.

A useful fact about linear motion, as observed in [3], is that the Euclidean distance function $d(t) = \|p(t)-q(t)\|$ is convex, and so the maximum distance between any two points p and q, denoted $|pq|_{\max}$, occurs at an extreme point of the interval $[0, 1]$. That is, $|pq|_{\max} = \max\{\|p(0) - q(0)\|, \|p(1) - q(1)\|\}$. This in turn implies that the weight function w_T is also convex, and so $w(T) = \max\{w_T(0), w_T(1)\}$. We now argue that the weight of MMST is *invariant* under *scaling, translation, and velocity addition*, a fact that will be crucial to some of our proofs.

Let $S = \{p_0, \ldots, p_{n-1}\}$ be a set of moving points in d dimensions. Pick any constant scaling factor $\alpha \in \mathbb{R}^+$, velocity vector $\beta \in \mathbb{R}^d$ and offset $\gamma \in \mathbb{R}^d$. For each p_i, define the transformed moving point $p'_i : [0, 1] \to \mathbb{R}^d$ as $p'_i(t) = \alpha p_i(t) + \beta t + \gamma$. Denote the transformed set as $S' = \{p'_i \mid p_i \in S\}$. Every spanning tree of S maps to a corresponding spanning tree of S' in the obvious way. We then have the following claim.

Lemma 1 *MMST is invariant under scaling, added velocity, and translation.*

Proof. Let S be a set of moving points, and let S' be the transformed set under scaling factor α, velocity increase β and translation γ. We show that if T is an MMST of S, then it is also an MMST of S'.

The distance between two points $p, q \in S$ at time t is $\|p(t) - q(t)\|$, while the distance between their transformed images p' and q' is $\|(\alpha p(t) + \beta t + \gamma) - (\alpha q(t) + \beta t + \gamma)\| = \alpha\|p(t) - q(t)\|$. Since the weight of each edge is simply scaled by α, for any spanning tree T of S and its corresponding tree T' for S', we have $w_{T'}(t) = \alpha w_T(t)$, which implies $w(T') = \alpha w(T)$, thus proving the claim. □

The transform is invertible and so the previous lemma also implies the following.

Corollary 2 *An MMST of a transformed set S' is also an MMST of the original set S.*

3 Hardness of MMST on the Line

In this section, we show that computing the MMST of n points on a line under linear motion is NP-hard *even if all points have the same speed*. We then show that a number of other variations are also hard.

We adopt the convention that positive x-axis is the rightward direction, and so points with positive (resp. negative) velocity are moving to the right (resp. left). In particular, since all points have the same speed, the velocity of each point is either $+1$ or -1.

Our reduction uses the well-known NP-hard problem Partition, where given n positive integers $a_0, \ldots, a_{n-1}$, we must decide if there is a subset $I \subseteq \{0, \ldots, n-1\}$ such that

$$\sum_{i\in I} a_i \;=\; \frac{1}{2}\sum_{i=0}^{n-1} a_i.$$

Given an instance of Partition, we construct an instance of MMST on the line with unit-speed moving points. We simplify the presentation by assuming the velocity of each point is either 0 or -2, which is then easily transformed into unit speeds by adding +1 to each velocity without changing its MMST by virtue of Corollary 2. We construct a decision version of the unit speed MMST problem as follows.

Construction 1 *Let $\ell = \max a_i$. For each $i \in \{0, \dots, n-1\}$, we add the following five moving points to our set S. See Figure 1 for illustration.*

- *A_i stationary at $10i$.*
- *B_i stationary at $10i + 2 - \frac{a_i}{4\ell}$.*
- *C_i stationary at $10i + 2 + \frac{a_i}{4\ell}$.*
- *D_i stationary at $10i + 4$.*
- *E_i moving from $10i + 3$ to $10i + 1$.*

For this input S, we ask if there is a moving spanning tree T with $w(T) \leq 11n - 6$.

Theorem 3 *The decision version of the MMST problem is NP-Hard for unit speed points on a line.*

Proof. Let $S_i = \{A_i, B_i, C_i, D_i, E_i\}$, for each $i \in \{0, \dots, n-1\}$. Let K_0 be the set of edges D_iA_{i+1} for $i \leq n-2$, and define K_1 as K_0 plus the edges joining pairs of points within each S_i, for each $i \leq n-1$. Observe that all edges in K_1 have length ≤ 6 at all times, and any edge not in K_1 has length > 6 at all times.

We claim that there exists a moving minimum spanning tree T all of whose edges are in K_1. Assume the contrary, and let T' be an MMST containing an edge $e \notin K_1$. Removing this edge from T' disconnects the tree into two components, which can be rejoined by an edge in K_1, contradicting the minimality of T'. Therefore we can assume that there is an MMST T with all edges in K_1.

Each edge in K_0 is a bridge of the graph (S, K_1) and therefore must be included in T. Each of these bridges connects two components, where each component is some moving spanning tree T_i on S_i.

We argue that each T_i has the following form: it contains the path $A_iB_iC_iD_i$, with E_i connected either to B_i (in which case $w_{T_i}(0) = 5 + a_i/4\ell$ and $w_{T_i}(1) = 5 - a_i/4\ell$) or connected to C_i (which flips these weights). The other trees are all easily seen to be sub-optimal; in particular, both $A_iE_iB_iC_iD_i$ and $A_iB_iC_iE_iD_i$ have cost at least $6 + a_i/2\ell$ at either the start or the end, while $A_iB_iE_iC_iD_i$ has cost at least $6 - a_i/2\ell$ at both start and end. All other trees are trivially suboptimal.

Define $I = \{i : E_i \text{ connects to } B_i\}$, and let $I' = \{0, \dots, n-1\} - I$. This allows us to write the weights of T as

$$\begin{aligned} w_T(0) &= w(K_0) + \sum_{i \in I} w_{T_i}(0) + \sum_{i \in I'} w_{T_i}(0) \\ &= 6(n-1) + \sum_{i \in I} \left(5 + \frac{a_i}{4\ell}\right) + \sum_{i \in I'} \left(5 - \frac{a_i}{4\ell}\right) \\ &= 11n - 6 + \sum_{i \in I} \frac{a_i}{4\ell} - \sum_{i \in I'} \frac{a_i}{4\ell} \end{aligned}$$

and by symmetry

$$w_T(1) = 11n - 6 - \sum_{i \in I} \frac{a_i}{4\ell} + \sum_{i \in I'} \frac{a_i}{4\ell}.$$

The maximum weight of T is

$$w(T) = 11n - 6 + \left| \sum_{i \in I} \frac{a_i}{4\ell} - \sum_{i \in I'} \frac{a_i}{4\ell} \right|,$$

which achieves its minimum value of $11n - 6$ when

$$\sum_{i \in I} \frac{a_i}{4\ell} = \sum_{i \in I'} \frac{a_i}{4\ell} \iff \sum_{i \in I} a_i = \sum_{i \in I'} a_i = \frac{1}{2} \sum_{i=0}^{n-1} a_i.$$

Thus, if $w(T) \leq 11n - 6$ holds, then the set I is also a solution to PARTITION. This completes the proof. $\square$

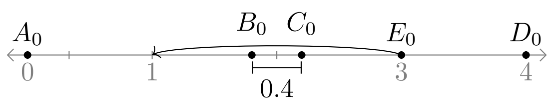

Figure 1: The points in S_0 when $a_0 = 8$ and $\ell = 10$. The optimal trees include the path $A_0B_0C_0D_0$, with E_0 connecting to either B_0 or C_0.

Naturally, the MMST problem is also NP-hard for points moving on the line with *arbitrary* but constant speeds. This can also be shown by a simple modification of the construction in [3] used to show the hardness for points moving in two dimensions. We omit that simple construction, and simply state the result below, which is then used to show additional hardness results.

Theorem 4 *The decision version of the MMST problem is NP-Hard for points moving on a line with constant but arbitrary speeds.*

We next show that the problem remains NP-hard under the bounded speed assumption, where the ratio between the maximum and the minimum speeds is upper bounded by some constant $c > 1$. In particular, let $v_i = p_i(1) - p_i(0)$ be the velocity of p_i, where as before positive velocity means rightward motion. Let $s_i = \|v_i\|$ be the speed of p_i, where we assume that $s_i > 0$ for all i, and that the ratio of max to min speeds is bounded:

$$\frac{\max s_i}{\min s_i} \leq c$$

Lemma 5 *MMST problem on a line under bounded speed linear motion is NP-hard.*

Proof. We reduce the MMST problem on the line with arbitrary speeds to our bounded speed ratio problem. First, let $v_{\min} = \min v_i$ and let $v_{\max} = \max v_i$. We transform the input set of moving points S into S' by first adding the velocity $-v_{\min}$ to each point. This shift makes the new minimum velocity 0 and the new maximum velocity $v_{\max} - v_{\min}$.

Next, we add $(v_{\max} - v_{\min})/(c-1) > 0$ to each point's velocity, which ensures that the ratio between the maximum and the minimum speeds is

$$\frac{\max s_i'}{\min s_i'} = \frac{v_{\max} - v_{\min} + (v_{\max} - v_{\min})/(c-1)}{(v_{\max} - v_{\min})/(c-1)} = c.$$

Thus the transformed input S' has bounded speed ratio, and yet by Corollary 2, the two instances S and S' have the same MMST. This completes the proof. □

Finally, we consider another natural velocity-constraint motion: *radial expansion*, where all points are linearly (with different speeds) moving away from the origin. We show that even under this restricted *big bang* expansion model of motion, the MMST problem remains hard even on the line.

Lemma 6 *MMST problem on a line under radial expansion linear motion is NP-hard.*

Proof. We again reduce the MMST problem on the line with arbitrary speeds to our problem. Given a set of moving points $S = \{p_0, \ldots, p_{n-1}\}$ on the line, let $v_i = p_i(1) - p_i(0)$ be the velocity of p_i. Let $v = \min\{v_i, 0\}$ equal the largest *negative velocity* (leftward speed) or zero. Let $o = \min\{p_i(0)\}$ denote the left-most position in S at the start of the motion. We transform our input instance S into S' by adding velocity $-v$ to all points and translating their positions by $-o$.

We now claim that the moving points in S' satisfy radial expansion. This follows because all points in S' have been shifted to the right of the origin, and none of the points have negative velocity, meaning they are all moving to the right. Because the transformation is addition of velocity and translation, by Corollary 2, MMST remains invariant, which completes the proof. □

4 Regret Ratio of Moving Spanning Trees

The MMST problem is motivated by the attractiveness of keeping a fixed topology throughout the motion, and among all spanning trees, MMST is the one with the smallest maximum weight during the motion. So, how much worse is the MMST compared to *dynamically maintaining the optimal spanning tree* throughout the motion? We quantify this tradeoff using the worst-case ratio between the two.

Given a set S of moving points, let $\mathcal{T}(S)$ be the set of all spanning trees on S. We use $w_{\min}(t) = \min_{T^* \in \mathcal{T}(S)} w_{T^*}(t)$ to denote the weight of a minimum spanning tree at time t. Then $w(\min) = \sup_t w_{\min}(t)$ is the maximum cost of a dynamically updated minimum spanning tree during the motion of S.

For any fixed spanning tree T of S, we define its *regret ratio* as $r(T) = w(T)/w(\min)$. Clearly, among all fixed trees, an MMST has the minimum regret. We now show bounds on MMST's regret ratio for moving points in one dimension.

4.1 Regret Ratio for Arbitrary Speed Linear Motion

Theorem 7 *MMST regret is at least $\Omega(\sqrt{n})$ for n moving points on the line.*

Proof. For any $b \geq 1$, we can construct a set S of $O(b^2)$ moving points on the line where $r(T) \geq b$ for any spanning tree T of S. Setting $b = \sqrt{n}$ establishes the claim. Our construction works as follows.

Let p_{ij} denote a moving point such that $p_{ij}(0) = i$ and $p_{ij}(1) = j$. That is, p_{ij} moves from i to j. Let $k = 2\lceil b \rceil$, and choose S as the set of all p_{ij}, where $0 \leq i, j \leq k$.

We now show that any tree on S has regret at least $k/2 = \lceil b \rceil \geq b$. To aid analysis, we interpret the set S of 1D moving points as a set of 2D stationary points, where a point p is placed at $P = (p(0), p(1))$. Thus, in our construction, each p_{ij} maps to $P_{ij} = (i, j)$. See Figure 2 for illustration.

Now, consider any tree T on these points. At the start, the weight of T is the sum of the *horizontal* components of the edges of T. At the end, the weight is the sum of the *vertical* components. Suppose there are h edges with a non-zero horizontal component, and v edges with a non-zero vertical component. The weight of each of these non-zero components is at least 1, and so $w_T(0) \geq h$ and $w_T(1) \geq v$. Furthermore, $h + v \geq (k+1)^2 - 1$, the number of edges in T. Finally, because the moving points always lie between p_{00} and p_{kk} on the line, we have $w(\min) = k$. This gives that

$$r(T) = \frac{w(T)}{w(\min)} \geq \frac{\max\{v, h\}}{k} \geq \frac{(k+1)^2 - 1}{2k} \geq \frac{k}{2} \geq b.$$

□

4.2 Regret Ratio for Unit Speed

Complementing the previous lower bound on the regret ratio, we show that for unit speed motion, the regret ratio has a tight bound of 2.

Theorem 8 *Regret ratio of an MMST for unit speed moving points on a line is at most 2, and this bound is tight.*

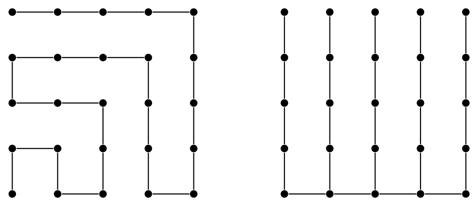

Figure 2: The 2D interpretation of the construction for $b = 2$, along with two sample spanning trees. The tree on the right results by connecting the points in the left-to-right order of their initial positions.

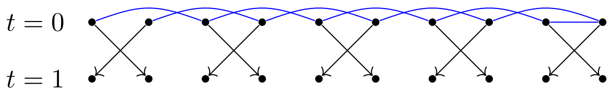

Figure 3: A set of unit speed moving points. Any MMST on this construction will have regret approaching 2 as the number of points increases. One possible MMST is shown in blue.

Proof. Partition S into the set of leftwards-moving points S_l and rightwards-moving points S_r. Let P_l be a path connecting S_l in order and let P_r be a path connecting S_r in order. Let e be an edge between the right-most point of P_l and the right-most point of P_r. Then $T^* = P_l \cup \{e\} \cup P_r$ is a moving spanning tree on S. Let $\ell = w(\min)$ be the maximum distance the points span throughout the motion.

It's easy to see that at all times either $w_{P_l \cup \{e\}}(t) \leq \ell$ and $w_{P_r}(t) \leq \ell$, or $w_{P_r \cup \{e\}}(t) \leq \ell$ and $w_{P_l}(t) \leq \ell$. Thus $w_{T^*}(t) = w_{P_l}(t) + w_{\{e\}}(t) + w_{P_r}(t) \leq 2\ell$, meaning $r(T^*) \leq 2$. The weight of an MMST is no greater, so its regret is also bounded by 2. Figure 3 shows a construction proving that this bound is tight. □

5 Some Tractable Cases of Moving Spanning Trees

In this section, we complement the negative (hardness) results of the previous section with some natural models of motion or cost for which optimal is easy to compute.

5.1 Motion with Uniform Expansion

Our first result complements Lemma 6: if the motion is radial expansion (big bang model) but all points move away from the origin at the *same constant speed* then MMST is easy to compute. In fact, this holds for any fixed dimension $d \geq 1$.

We say that a set $S \in \mathbb{R}^d$ of moving points is under *uniform expansion* if all points in S move away from the origin at a constant speed $c > 0$. That is, if the start position of a point $p \in S$ is at $p(0) \neq 0$, then its end position is at $p(1) = p(0) + c\frac{p(0)}{\|p(0)\|}$. We have the following easy result.

Lemma 9 *Let S be a set of n moving points under uniform expansion in d dimensions. Then, a minimum spanning tree of S at $t = 1$ (end of the motion) is also an MMST of S.*

Proof. Consider any pair of points $p, q \in S$, and let θ, where $0 \leq \theta \leq \pi$, be the angle between $p(0)$ and $q(0)$ with the respect to the origin. During the motion, the distance between the points will increase by $\sqrt{c^2 + c^2 - 2c^2 \cos\theta} = c\sqrt{2}\sin(\theta/2) \geq 0$ (see Figure 4). Since the distance monotonically increases, we must have $|pq|_{\max} = \|p(1) - q(1)\|$.

It follows that, for any moving spanning tree T, the weight of every edge in T achieves its maximum at $t = 1$, and so $w(T)$ also achieves its maximum at that time. Therefore, a minimum weight spanning tree at end of the motion $t = 1$ is also an MMST of the set S. □

Since the minimum spanning tree of n points in any fixed dimension d can be computed in polynomial time, the MMST problem for uniform expansion is tractable.

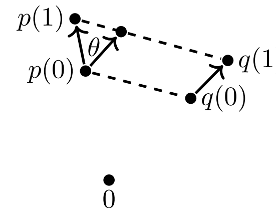

Figure 4: The distance between two points never decreases if they are under uniform expansion.

In fact, Lemma 9 holds for any restriction that causes the distance between any two points to be non-decreasing over the motion. For example, *proportional expansion* requires that each point p move from $p(0)$ to $p(1) = ap(0)$ for some constant $a \geq 1$. An MMST of points under proportional expansion can be found by calculating an MST at the end of their motion, using the same reasoning as above.

5.2 Minimum Average Weight

The MMST problem minimizes the *maximum* weight of the spanning tree at any time during the motion. A different but also sensible objective might be the *total weight* of a moving spanning tree, integrated over the entire motion. For instance, if tree length is a measure of resource consumption, then this reflects the total consumption during the motion. Since we have normalized the motion duration to unit interval $[0, 1]$, this is also equivalent to the *average* weight of the moving spanning tree:

$$\overline{w}(T) = \int_0^1 w_T(t)\, dt.$$

Unlike MMST, computing a moving spanning tree with minimum average weight (MAMST) is easy.

Lemma 10 *For a set S of n moving points in d dimensions, an MAMST of S can be found in $O(n^2)$ time, for any fixed d.*

Proof. Let S be a set of n moving points. For any pair $p, q \in S$, define $|pq|_{\text{avg}} = \int_0^1 \|p(t) - q(t)\| \, dt$. We can then rewrite the average weight of a tree T on S as $\overline{w}(T) = \sum_{pq \in T} |pq|_{\text{avg}}$. Now construct a complete graph G with S as the vertex set, and weight of each edge pq set to $|pq|_{\text{avg}}$. The graph G can be constructed in $O(n^2)$ time because each $|pq|_{\text{avg}}$ can be calculated in constant time.

It is now easy to see that the an MST of G is an MAMST of the moving points, and an MST of G can be found in $O(n^2)$ time using Prim's algorithm. $\square$

In [3], it was shown that the minimum bottleneck moving point spanning tree (MBMST) can be computed efficiently, and our average weight spanning tree adds another natural easy-to-compute variant for moving points. (Recently, Wang et al. [10] presented a subquadratic algorithm for the MBMST, improving the $O(n^2)$ bound of [3]. It is an interesting open question whether a similar time bound is also possible for MAMST.)

6 Moving Point Matching and TSP

In this section, we briefly sketch the constructions for proving NP-hardness of matching and TSP, and state without proof their regret bounds. Technical details are similar to those in MMST, and omitted from this extended abstract.

6.1 Minimum Moving Matching

Let S be a set of $2n$ moving points. A *moving matching* M is a set of n edges such that each point is matched with exactly one other. The weight of M is the *maximum* weight of M throughout the motion.

We show that the problem of finding a minimum moving matching (MMM) is NP-Hard *even for unit speed moving points on the line.* The construction is similar to the one from Theorem 3, and illustrated in Figure 5.

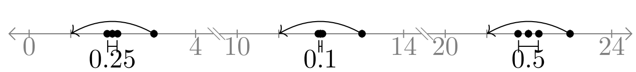

Figure 5: The matching instance corresponding to the Partition input $(a_0, a_1, a_2) = (5, 2, 10)$. The Partition problem has a solution if and only if there is an MMM with weight at most $n + \frac{1}{4\ell}\sum_{i=0}^{n-1} a_i$.

The regret of an MMM is unbounded in general, using a similar construction to Theorem 7. We do not have a non-trivial upper bound for the MMM regret under unit speed motion, but can show that it is *strictly larger than* 2. Specifically, we show a construction (see the Appendix) with regret ratio of 11/5.

6.2 Minimum Moving TSP

Let S be a set of n moving points. A *moving traveling salesman path* P is a path of $n-1$ edges that visits every point in S. The weight of P is the *maximum* weight of the path throughout the motion. The construction shown in Figure 6 is used to prove that the problem of finding the minimum moving traveling salesman path (MMTSP) is NP-Hard even for unit speed moving points on the line.

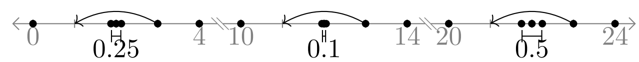

Figure 6: The TSP instance corresponding to the Partition input $(a_0, a_1, a_2) = (5, 2, 10)$. The Partition problem has a solution if and only if there is an MMTSP with weight at most $12n - 6$.

We also can show that regret ratio of an MMTSP is unbounded in general, but is bounded by 2 for unit speed motion, using constructions similar to Theorem 7 and Theorem 8.

7 Concluding Remarks

We explored several classical geometric problems under the moving point model of [3], and showed that they remain NP-hard even in one dimension and even under highly constrained motions. We did not discuss approximation algorithms but a 2-approximation of MMST is easily computed in $O(n \log n)$ time using the approach of [3], namely, map the 1D moving points into 2D stationary points and compute their MST under the L_1 norm.

We also analyzed the regret ratio of these structures, showing that even in one dimension this can be unbounded in general, but is modest for unit speed. Finally, we suggested two simple settings (uniform expansion and minimum average weight) where the optimal structures are easy to compute.

A number of open problems are suggested by our work. First, it will be interesting to derive a non-trivial upper bound on the regret ratio of these structures in higher dimension under *unit speed motion.* (Proving a tight bound for the regret of moving point matching in one dimension is also an interesting problem.) Second, without the unit speed restriction, it will also be interesting to bound the regret ratio if we are allowed to update the structure k times. In particular, how large must k be to guarantee a constant regret ratio?

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Appendix

We provide an example where the regret ratio of a minimum moving point matching (MMM) is strictly larger than 2 under unit speed motion on a line. For clarity, our points are transformed with added velocity +5 using Corollary 2. Construct the following 6 points, which are illustrated in Figure 7:

- p_0, p_1, p_2 moving from $0, 1, 2$ to $10, 11, 12$; and
- q_0, q_1, q_2 static at $3, 6, 9$.

Consider a dynamically updating minimal matching. In the initial positions, the matching (p_0p_1, p_2q_0, q_1q_2) is optimal, having weight 5. With a few edge changes during the motion, it can maintain this cost (or less), eventually ending as the matching (q_0q_1, q_2p_0, p_1p_2). Therefore $w(\min) = 5$.

On the other hand, consider the MMM (p_0q_0, p_1p_2, q_1q_2). It has initial cost 11, and final cost 7. There is a similar MMM (p_0p_1, p_2q_2, q_0q_1) with these costs reversed. But this is as good as we can do. So the MMM regret is $11/5 = 2.2 > 2$.

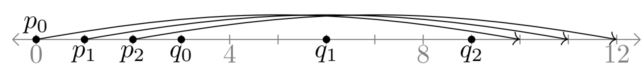

Figure 7: A set of 6 unit speed moving points (transformed for clarity to have velocities 0 and +10). An MMM on these points has regret $11/5 = 2.2$.