Homework 3

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9/16/2016

Exercise 3.3

```
y.a = c(12,9,12,14,13,13,15,8,15,6)
y.b = c(11,11,10,9,9,8,7,10,6,8,8,9,7)

a.a = 120; a.b = 10
b.a = 12 ; b.b = 1
n.y.a = length(y.a)
n.y.b = length(y.b)
sumy.a = sum(y.a)
sumy.b = sum(y.b)
post.a.a = a.a + sumy.a
post.a.b = n.y.a + a.b

post.b.a = b.a + sumy.b
post.b.b = n.y.b + b.b
```

The closed form of posterior distribution of a poisson sampling model with a gamma prior is $\theta|Y \sim Ga(a + \sum Y, b + n)$

The posterior distribution of $\theta_A|Y_A \sim Ga(237, 20)$, and the posterior distribution of $\theta_B|Y_B \sim Ga(125, 14)$

```
post.mean.a = post.a.a / post.a.b
post.var.a = post.a.a / post.a.b ^2
post.mean.a
```

```
## [1] 11.85
```

```
post.var.a
```

```
## [1] 0.5925
```

```
qgamma(c(.025,.975), post.a.a,post.a.b)
```

```
## [1] 10.38924 13.40545
```

```
post.mean.b = post.b.b / post.b.b
post.var.b = post.b.b / post.b.b ^2
post.mean.b
```

```
## [1] 1
```

```
post.var.b
```

```
## [1] 0.07142857
```

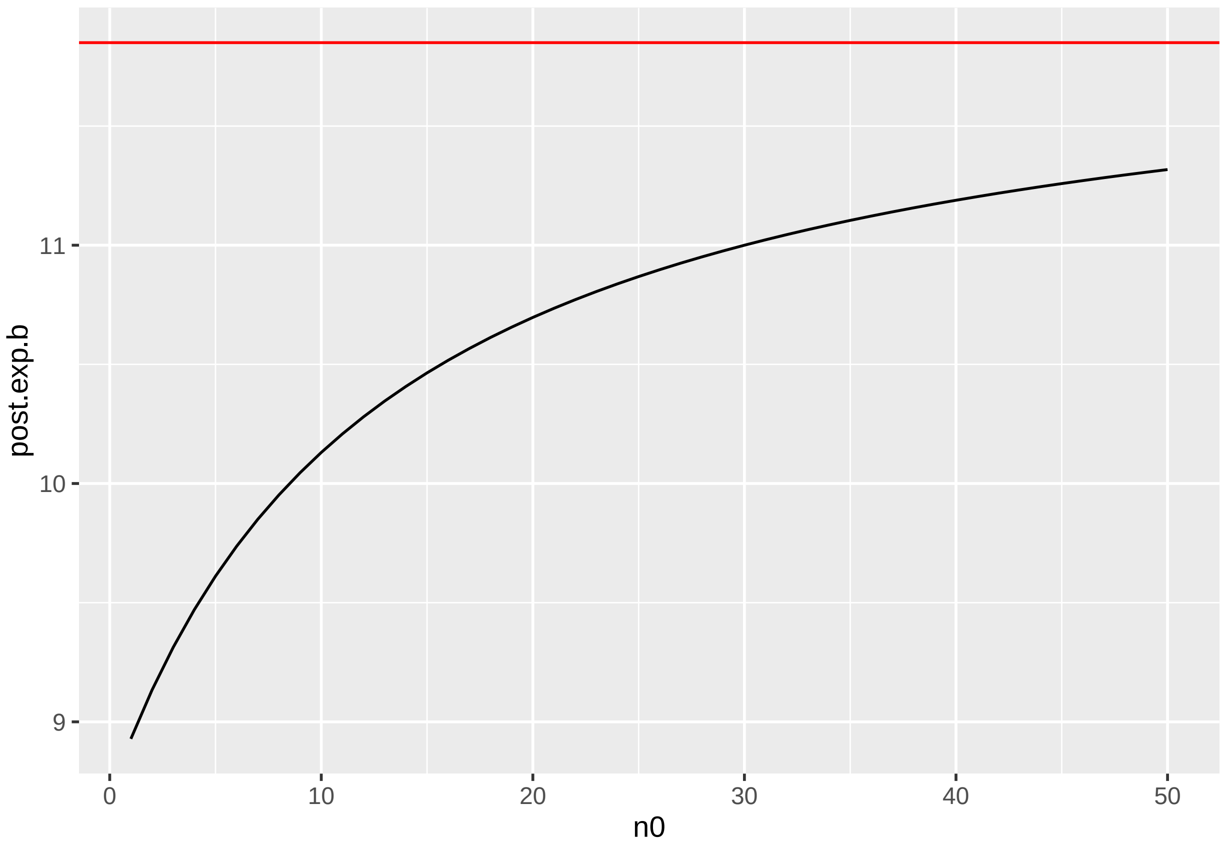
```
qgamma(c(.025,.975), post.b.b,post.b.b)
```

```
## [1] 0.5467093 1.5878854
```

The posterior mean of $\theta = 11.85$. The posterior variance of $\theta = = 0.0714286$.

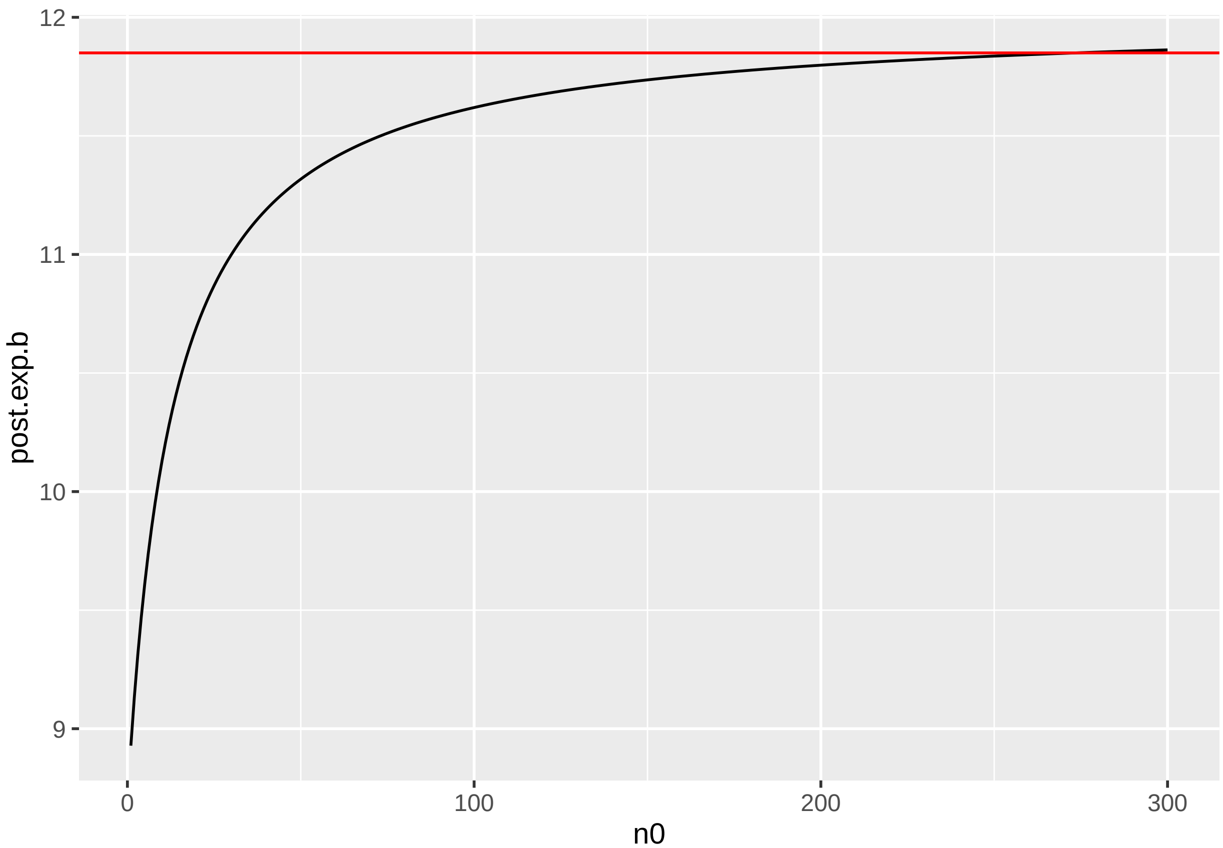
Part B

```
n0 = 1:50
post.b.a.n = 12*n0 + sumy.b
post.b.b.n = n.y.b + n0
post.exp.b = post.b.a.n / post.b.b.n
n0.plot.frame = data.frame(n0,post.exp.b)
plot.exp = ggplot(n0.plot.frame, aes(x= n0, y = post.exp.b))
plot.exp + geom_line() + geom_hline(yintercept = post.mean.a, color = "red")
```



It seems clear that n_0 would need to be much greater than 50 for $\theta_B|Y_B$ to approach $E(\theta_A|Y_A)$. Below shows the conditions for n_0 for $\theta_B|Y_B$ to equal $E(\theta_A|Y_A)$. It appears as if it's around 260-275.

```
n0 = 1:300
post.b.a.n = 12*n0 + sumy.b
post.b.b.n = n.y.b + n0
post.exp.b = post.b.a.n / post.b.b.n
n0.plot.frame = data.frame(n0,post.exp.b)
plot.exp = ggplot(n0.plot.frame, aes(x= n0, y = post.exp.b))
plot.exp + geom_line() + geom_hline(yintercept = post.mean.a, color = "red")
```



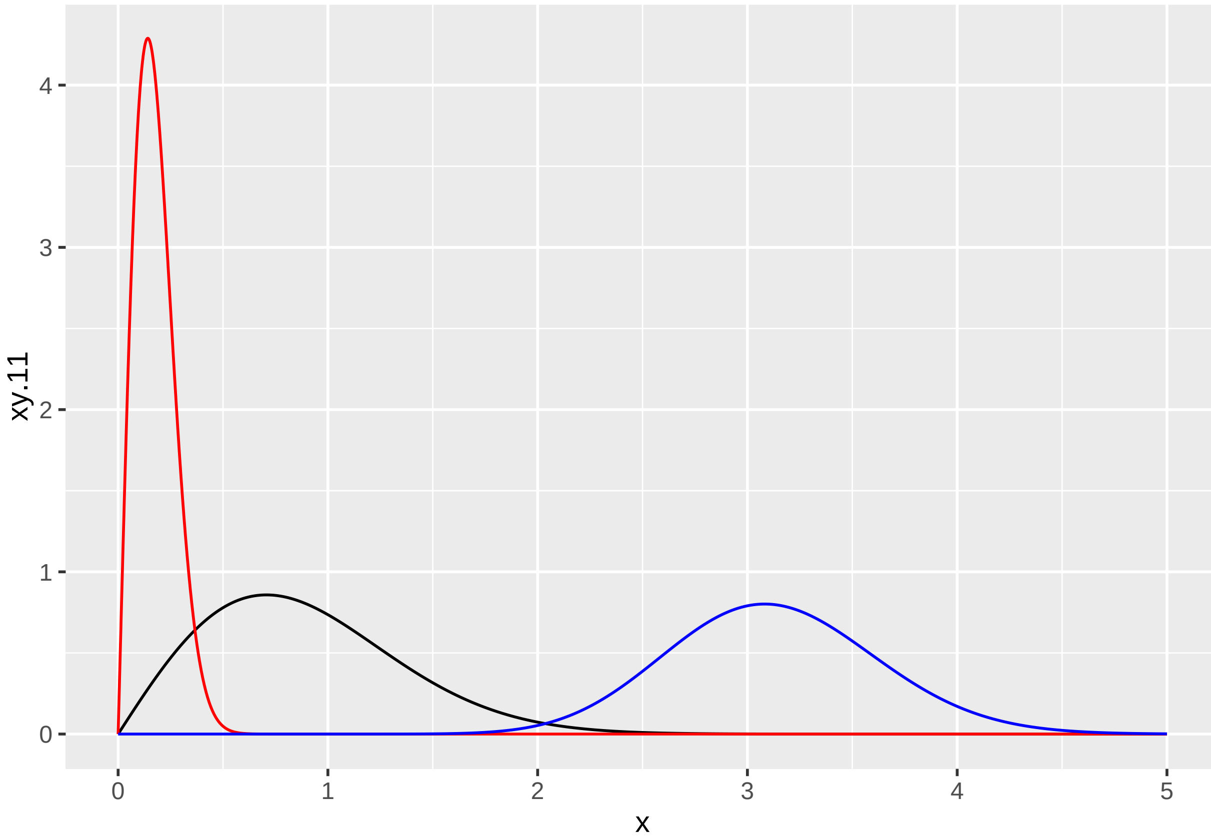
Part C

If knowledge about population A informs us about population B, then it doesn't make sense to have a prior $p(\theta_A, \theta_B) = p(\theta_A) \times p(\theta_B)$ because that prior operates under the assumption that they are independent. A informing B makes that assumption invalid.

Exercise 3.9

```
galenshore = function(y,a,theta){
  (2 / gamma(a)) * theta^(2*a) * y^(2*a - 1) * exp(-theta^2*y^2)
}

x = seq(0,5, by = .001)
xy.11 = galenshore(x,1,1)
xy.15 = galenshore(x,1,5)
xy.101 = galenshore(x,10,1)
galen.data.frame = data.frame(cbind(x,xy.11,xy.15,xy.101))
#curve(galenshore(x,1,1),ylim = c(0,4.5), xlim = c(0,5))
#curve(galenshore(x,1,5), add = TRUE, col = "red")
#curve(galenshore(x,10,5), add = TRUE, col = "blue")

f = ggplot(data = galen.data.frame, aes(x = x), ylim = c(0,5))
f + geom_line(aes(y = xy.11), color = "black") +
```

```
  geom_line(aes(y = xy.15), color = "red") +
  geom_line(aes(y = xy.101), color = "blue")
```



Exercise 4.1

```
n.2 = 50
sum.x.2 = 30
n.1 = 100
sum.x.1 = 57
alpha = 1
beta = 1
post.alpha.1 = sum.x.1 + alpha
post.beta.1 = n.1 - sum.x.1 + beta
post.alpha.2 = sum.x.2 + alpha
post.beta.2 = n.2 - sum.x.2 + beta
```

In this case, we assume a uniform prior distribution, which is equivalent to $\theta_2 \sim Beta(1,1)$. The posterior distribution of $\theta_2 | \sum X \sim Beta(31{,}21)$.

```
theta1 = rbeta(5000, post.alpha.1, post.beta.1)
theta2 = rbeta(5000, post.alpha.2, post.beta.2)

theta.both = mean(theta2 > theta1)
theta.both
```

```
## [1] 0.6316
```

Thus $Pr(\theta_A < \theta_B | y_A, y_B) = 0.6316$ in this case.

Exercise 4.2

Part A

```
y.a = c(12,9,12,14,13,13,15,8,15,6)
y.b = c(11,11,10,9,9,8,7,10,6,8,8,9,7)

a.a = 120; a.b = 10
b.a = 12 ; b.b = 1
n.y.a = length(y.a)
n.y.b = length(y.b)
sumy.a = sum(y.a)
sumy.b = sum(y.b)
post.a.a = a.a + sumy.a
post.a.b = n.y.a + a.b

post.b.a = b.a + sumy.b
post.b.b = n.y.b + b.b

theta.a.draws = rgamma(10000,post.a.a,post.a.b)
theta.b.draws = rgamma(10000,post.b.a,post.b.b)

prob.diff = mean(theta.b.draws < theta.a.draws)
```

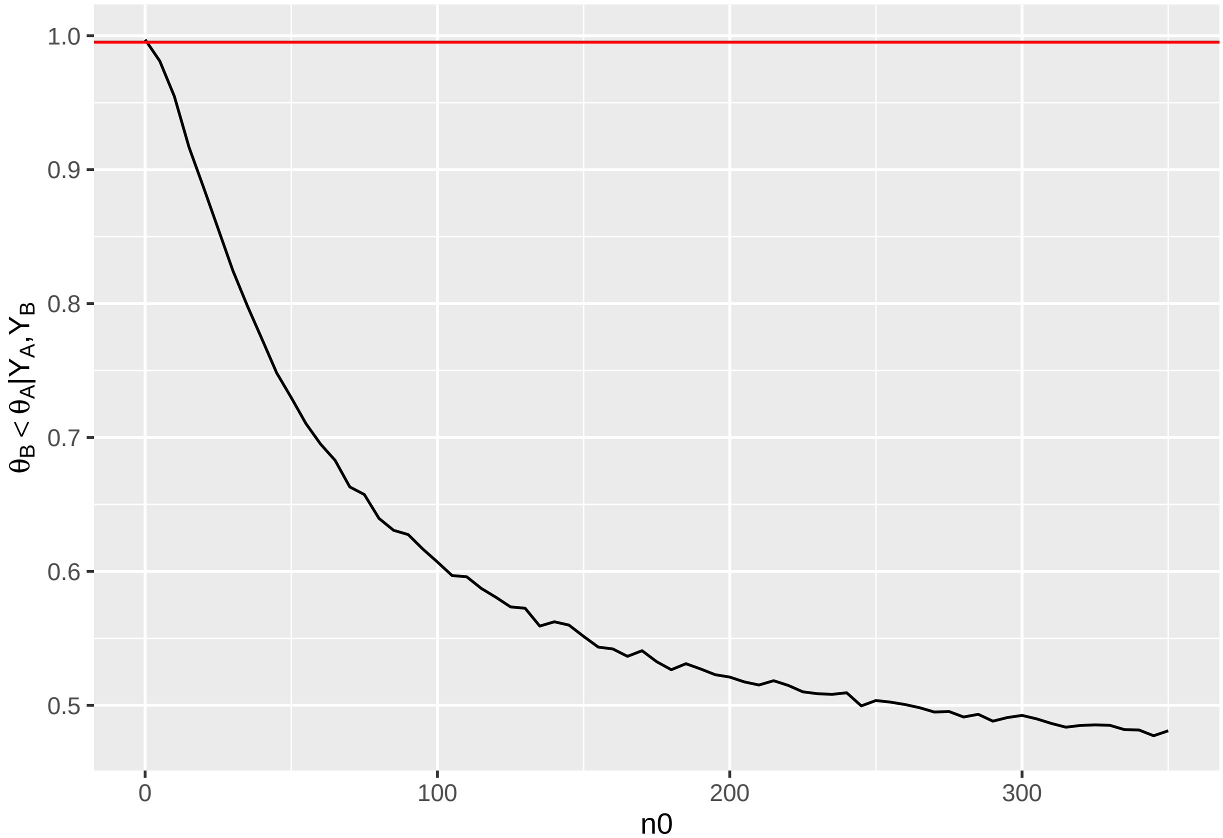
According to this, $P(\theta_B < \theta_A | y_B, y_A) = 0.9952$, which would indicate that we are quite sure that $\theta_B < \theta_A$.

Part B

We perform a sensitivity analysis below.

```
n0 = seq(0,350, by = 5)
post.b.a.n = 12*n0 + sumy.b
post.b.b.n = n.y.b + n0

post.a.a = a.a + sumy.a
post.a.b = n.y.a + a.b

theta.a.draws = rgamma(10000,post.a.a,post.a.b)
theta.b.draws = rgamma(10000,post.b.a,post.b.b)

n.iter = 10000
# This vector is a vector of \theta_B < \theta_A | y_B, y_A for different levels of n_0
sensitivity.analysis.n0 = rep(NA, length(n0))
for(i in 1:length(n0)){
  post.b.a.val = post.b.a.n[i]
  post.b.b.val = post.b.b.n[i]
  theta.b.draws1 = rgamma(n.iter,post.b.a.val,post.b.b.val)
  sensitivity.analysis.n0[i] = mean(theta.b.draws1 < theta.a.draws)
}
```

```
sens.data.frame = as.data.frame(cbind(n0,sensitivity.analysis.n0))

ggplot(sens.data.frame, aes(x = n0)) + geom_line(aes(y = sensitivity.analysis.n0)) +
  geom_hline(yintercept = prob.diff,color = "red") + ylab(expression(theta[B] < theta[A] *"|"* Y[A]*","*
```



Thus as the prior gets stronger, we become less sure that $P(\theta_B < \theta_A | y_B, y_A)$.

Part C.A

Posterior predictive

```
theta.a.draws = rgamma(10000,post.a.a,post.a.b)
theta.b.draws = rgamma(10000,post.b.a,post.b.b)

y.a.draws = rpois(10000,theta.a.draws)
y.b.draws = rpois(10000,theta.b.draws)

post.pred.diff = mean(y.b.draws < y.a.draws)
post.pred.diff
```

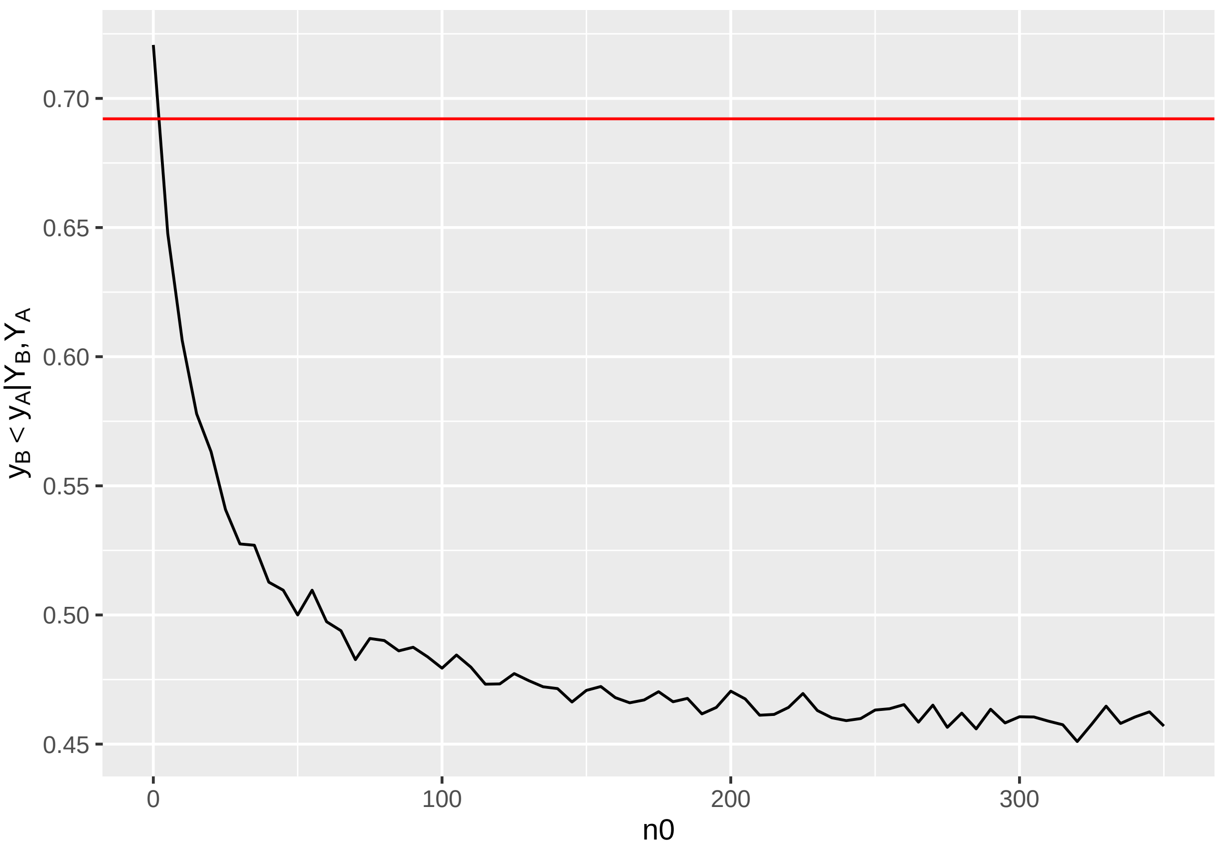
```
## [1] 0.6921
```

Through this, we find that $P(\widetilde{Y}_B < \widetilde{Y}_A | Y_A, Y_B) = 0.6921$.

Part C.B

Posterior predictive

```
n0 = seq(0,350, by = 5)
post.b.a.n = 12*n0 + sumy.b
post.b.b.n = n.y.b + n0

post.a.a = a.a + sumy.a
post.a.b = n.y.a + a.b

theta.a.draws = rgamma(10000,post.a.a,post.a.b)
y.a.draws = rpois(10000,theta.a.draws)

n.iter = 10000
# This vector is a vector of \theta_B < \theta_A | y_B, y_A for different levels of n_0
sensitivity.pred.n0 = rep(NA, length(n0))
for(i in 1:length(n0)){
  post.b.a.val = post.b.a.n[i]
  post.b.b.val = post.b.b.n[i]
  theta.b.draws1 = rgamma(n.iter,post.b.a.val,post.b.b.val)
  y.b.draws1 = rpois(n.iter,theta.b.draws1)
  sensitivity.pred.n0[i] = mean(y.b.draws1 < y.a.draws)
}
sens.pred.data.frame = as.data.frame(cbind(n0,sensitivity.pred.n0))

ggplot(sens.pred.data.frame, aes(x = n0)) + geom_line(aes(y = sensitivity.pred.n0)) +
  geom_hline(yintercept = post.pred.diff,color = "red") + labs(y = expression(y[B] < y[A]*"|"*Y[B]*","*
```



This plot resembles the shape of the previous, except we are less sure with a weaker prior to begin with. It also declines a little bit more rapidly, and it seems like there is more fluctuation and perhaps uncertainty in this case. So instead of it approaching .5, it actually dips past it really pretty early with a prior sample size of 50.