TANGENT PLANES AND LINEARIZATION

ZACH NORWOOD

Learning Objectives.

1. Connect linear approximation of a function of two variables to that of a function of one variable.
2. Compute the tangent plane or linearization of a function of two variables at a point.
3. Use the linearization to estimate a value of a function.

Motivation and review.

- Recall from Calc I that if we zoom in on the graph of a 'nice' function $y = f(x)$ then it looks like a line.
- But not all functions are *differentiable* or *locally linear*:

  e.g.

- Zoom in on a graph $z = f(x, y)$ and it looks like a *plane*.
- In Calc I we approximated (differentiable) functions by tangent lines;
- In Calc III we approximate (differentiable) functions by tangent planes.

go.unl.edu/ZG-zoom-in-for-line

go.unl.edu/ZG-zoom-in-for-plane

Equation of the tangent plane. We have a function $f(x, y)$ and a point (x_0, y_0). How do we find the tangent plane of f 'anchored' at (x_0, y_0)?

What do we need?

- Slope in the x-direction =
- Slope in the y-direction =
- Point =

Combine it all to get the equation for a tangent plane:

go.unl.edu/ZG-trace-tangents

Date: 13 March 2025.

Example 1. Find an equation for the plane tangent to the graph of

$$f(x, y) = \sqrt{x^2 + y^3}$$

at the point $(1, 2, f(1, 2))$.

Linearization: using the tangent plane to approximate f. The *linearization* of a function f at (x_0, y_0) is defined to be the linear function whose graph is the tangent plane. That is,

$$L(x, y) = \boxed{}$$

The linearization can be used to approximate f near (x_0, y_0); i.e., $L(x, y) \approx f(x, y)$ when $(x, y) \approx (x_0, y_0)$.

The linearization is useful because …

This is just like the situation in one variable: the tangent line at x_0 approximates f near x_0.

Bike shop example. Fixed costs total \$10000; each mountain bike sells for \$1500; and each road bike sells for \$2000. The profit is modeled by the function

$$P(x, y) = 1500x + 2000y - 10000.$$

If the bike shop sells 4 fewer mountain bikes and 5 more road bikes, how will that affect their profit?

Change in profit =

More generally, on a *plane* with x-slope a and y-slope b, shifting the input

$$(x, y) \longmapsto (x + \Delta x, y + \Delta y)$$

results in a change of in the value of z.

go.unl.edu/ZG-1D-linearization

Suppose that we know $f(x_0, y_0)$ but shift the input slightly:

$$(x_0, y_0) \longmapsto (x_0 + \Delta x, y_0 + \Delta y).$$

We can estimate the resulting change Δf in f by the *differential*

$$df = \qquad \approx \Delta f.$$

Example 2. Use differentials to estimate $\sqrt{1.01^2 + 1.97^3}$.