CSc 225 Assignment 2:
Runtime Analysis and Proofs

Due date:

The submission deadline is 11:55pm on Monday, June 1st, 2020.

How to hand it in:

Submit your assignment2.pdf file through the Assignment 1 link on the CSC225 conneX page.
IMPORTANT: the file submitted **must** have a **.pdf** extension.

Exercises:

1. Determine the number of assignment (A) and comparison (C) operations in the following:

a. **Algorithm** Q1a(n)

$result \leftarrow 1$ 1A
for $i \leftarrow 1$ to n^2 **do** $1A + n^2(1A+1C)+1C$
for $j \leftarrow 1$ to i **do** $n^2(1A+\frac{n(n+1)}{2}(1A+1C+2A)+1C)$
$n \leftarrow i + j$
$result \leftarrow result + n$
end
end
return $result$ 1A

$$T(n) = 2n^4 + 2n^3 + 4n^2 + 4$$

b. **Algorithm** Q1b(n)

$result \leftarrow 1$ 1A
for $i \leftarrow 1$ to n **do** $1A + n(1A+1C)+1C$
$j \leftarrow 1$ $n(1A)$
while $j \le n$ **do** $n(\log n(2A)+1C)$
$result \leftarrow result + 1$
$j \leftarrow j * 2$
end
end
return $result$ 1A

HINT: Assume n is a power of 2 - $\log n = \log_2 n$

$$T(n) = 3n\log n + 4n + 4$$

2. Solve the following recurrence relations, given an integer $n \geq 1$, through substitution:

a. $T(n) = \begin{cases} 1, & n = 1 \\ 2T(n-1) + 1, & n \geq 2 \end{cases}$

$T(1) = 1$

$T(2) = 2T(2-1)+1 = 2T(1)+1 = 2(1)+1 = 3$

$T(3) = 2T(3-1)+1 = 2T(2)+1 = 2(3)+1 = 7$

$T(4) = 2T(4-1)+1 = 2T(3)+1 = 2(7)+1 = 15$

$$\boxed{T(n) = 2^n - 1 \in n \geq 1}$$

b. $T(n) = \begin{cases} 4, & n = 1 \\ T(n-1) + 2n, & n \geq 2 \end{cases}$

$T(1) = 4$

$T(2) = T(2-1)+2(2) = T(1)+4 = 8$

$T(3) = T(3-1)+2(3) = T(2)+6 = 14$

$T(4) = T(4-1)+2(4) = T(3)+8 = 22$

$T(5) = T(5-1)+2(5) = T(4)+10 = 32$

$$\boxed{T(n) = n^2 + n + 2}$$

3. Proof by induction

$5! = 5\cdot4\cdot3\cdot2\cdot1$
$5+1! = 5+1\cdot5\cdot4\cdot3\cdot2\cdot1$

$k! = k, \dots, 3, 2, 1$
$(k+1)! = (k+1), k, \dots 3\cdot2\cdot1$

a. Prove the following statement using induction:

$$\sum_{i=1}^{n}(i) * (i!) = (n+1)! - 1, \text{ for all } n \geq 1.$$

$= k!(k+1)(k+2)$

$k!(k+1)-1$

Basis: $\sum_{i=1}^{1} 1\cdot1! = 1 = (1+1)! - 1 = 1$ ✓

IH: Suppose that $\sum_{i=1}^{n}(i)(i!) = (n+1)! - 1$, for $n = 1, 2, \dots, k$.

IS: want $\sum_{i=1}^{k+1}(i)(i!) = ((k+1)+1)! - 1$

$$\sum_{i=1}^{k}(i)(i!) + (k+1) = k!(k+1)(k+2) - 1$$

$$(k+1) = (k+2) - 1$$

$$k+1 = k+1$$

$\therefore$ By PMI $\sum_{i=1}^{n}(i)\cdot(i!) = (n+1)! - 1$ for all $n \geq 1$

b. Prove the following statement using induction:

$= \sum_{i=1}^{n}(5i+4)$

$$1 + 6 + 11 + 16 + \dots + (5n - 4) = n(5n - 3)/2, \text{ for any positive integer } n \geq 1.$$

Basis: $5(1) - 4 = 1$ $\quad \frac{1(5(1)-3)}{2} = 1$ ✓

IH: Suppose that $\sum_{i=1}^{n}(5i-4) = \frac{n(5n-3)}{2}$ for $n = 1, 2, \dots, k$

IS: want $\sum_{i=1}^{k+1}(5i-4) = \frac{(k+1)(5(k+1)-3)}{2} = 5k+5-3 = \frac{(5k+2)(k+1)}{2} = \frac{5k^2+7k+2}{2}$

$$\boxed{\frac{k(5k+7)}{2} + 1}$$

RHS

$$\sum_{i=1}^{k}(5i-4) + (5(k+1)-4) = \frac{k(5k+7)}{2} + 1$$

LHS: $\frac{k(5k-3)}{2} + 5k + 1$

LHS

$\frac{k(5k+7)}{2} + 1$ as wanted

$\therefore$ By PMI $\sum_{i=1}^{n}(5i+4) = \frac{n(5n-3)}{2}$ for all $n \geq 1$

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4. Use a *loop invariant* to prove that the algorithm below **returns** a^n.

Algorithm Q4(a, n)
Input: Positive integers $a \geq 0$ and $n \geq 0$
Output: a^n

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i ← 1
pow ← 1
while i ≤ n do
    pow ← pow * a
    i ← i + 1
end
return pow
```

Basis: $pow = 1, i = 1, pow = a$

IH: Assume $pow = a^K$

Show $pow = a^{k+1} = a^K \cdot a$

Given $i = k$, $pow = a^K$, in the while loop, i is incremented so $i = k+1$

$\therefore pow = pow \cdot (k+1) \cdot a = a^{k+1} = a^k \cdot a$

Termination:

loop ends after the $(n-1)^{th}$ iteration $(i = n)$

from IH step, $a^{n+1-1} = a^n$

$\therefore a^n$ is returned $\square$

5. Prove each of the following using the definition of Big-Oh. You must provide constants c and n_0 to satisfy the definition of Big-Oh as defined in class.

a. $(n+2)^4$ is $O(n^4)$

$(n+2)^4 \leq 16n^4$ for $n \geq 2$

$c=16,\ n_0=2$

b. $\frac{n^4+n^2+1}{n^3+1}$ is $O(n)$

$\frac{n^4+n^2+1}{n^3+1} \leq 2n$ for $n \geq 1$

$c=2,\ n_0=1$

c. Find the smallest integer x such that $5n^3 + 3n^2 logn$ is $O(n^x)$

$5n^3 + 3n^2 \log n \leq 5n^3 + 3n^3$ for all $n \geq 1$

$\leq 8n^3$

$c=8,\ n_0=1$

6. Order the following functions by their big-Oh notation. Group together (for example, by underlining) functions that are big-Theta of one another (no justification needed).
Note: $\log n = \log_2 n$ unless otherwise stated.

$\cancel{5n\log n}$ $O(n\log n)$	$\cancel{2^{40}}$ $O(c)$	$O(\cancel{\log\log n})$	4^n $O(c^n)$	$\cancel{14n}$ $O(n)$	$\cancel{2^{\log n}}$ $O(\log n)$
$\cancel{(n\log n)^2}$ $O(n^2)$	$\cancel{\sqrt{\log n}}$ $O(\log n)$	$\cancel{n\log_5 n}$ $O(n\log n)$	$\cancel{n^3}$ $O(n^3)$	$\cancel{25n^{0.5}}$ $O(n)$	$7n^{5/2}$ $O(n^c)$
$\cancel{1/n}$ $O(1)$	5^{n^2} $O(c^n)$	$\cancel{n^2\log n}$ $O(n^2)$	$\cancel{\sqrt{n}}$ $O(n^c)$	$\cancel{n^{0.01}}$ $O(n^c)$	2^{2^n} $O(c^n)$

$\frac{1}{n} \in O(1)$

$\sqrt{\log n},\ 2^{\log n} \in O(\log n)$

$2^{40} \in O(c)$

$14n,\ 25n^{0.5},\ n^{0.01},\ \sqrt{n} \in O(n)$ $\boxed{\sqrt{n} \in \Theta(25n^{0.5})}$

$\log\log n \in O(\log\log n)$

$5n\log n,\ n\log_5 n \in O(n\log n)$

$n^2\log n,\ (n\log n)^2 \in O(n^2)$ $\boxed{(n\log n)^2 \in \Theta(n^2\log n)}$

$n^3 \in O(n^3)$

$7n^{5/2} \in O(n^c)$

$2^{2^n},\ 5^{n^2},\ 4^n \in O(c^n)$