CSc 225 Assignment 2:
Runtime Analysis and Proofs

Due date:

The submission deadline is 11:55pm on Monday, June 1st, 2020.

How to hand it in:

Submit your assignment2.pdf file through the Assignment 1 link on the CSC225 conneX page.
IMPORTANT: the file submitted **must** have a **.pdf** extension.

Exercises:

1. Determine the number of assignment (A) and comparison (C) operations in the following:

   a. **Algorithm** Q1a(n)

      result ← 1   1A
      for $i \leftarrow 1$ to n^2 **do** $1A + n^2(1A + 1C) + 1C$
          for $j \leftarrow 1$ to i **do** $n^2\left(1A + \frac{n(n+1)}{2}(1A + 1C + 2A) + 1C\right)$
              $n \leftarrow i + j$
              result ← *result* + n
          end
      end
      return *result*   1A

      $$T(n) = 2n^4 + 2n^3 + 4n^2 + 4$$

   b. **Algorithm** Q1b(n)

      result ← 1   1A
      for $i \leftarrow 1$ to n **do** $1A + n(1A + 1C) + 1C$
          $j \leftarrow 1$ $n(1A)$
          while $j \le n$ **do** $n(\log n(2A) + 1C)$
              result ← *result* + 1
              $j \leftarrow j * 2$
          end
      end
      return *result*   1A

      HINT: Assume n is a power of 2 - $\log n = \log_2 n$

      $$T(n) = 3n\log n + 3n + 4$$

2. Solve the following recurrence relations, given an integer $n \geq 1$, through substitution:

a. $T(n) = \begin{cases} 1, & n = 1 \\ 2T(n-1) + 1, & n \geq 2 \end{cases}$

$T(1) = 1$
$T(2) = 2T(2-1)+1 = 2T(1)+1 = 2(1)+1 = 3$
$T(3) = 2T(3-1)+1 = 2T(2)+1 = 2(3)+1 = 7$
$T(4) = 2T(4-1)+1 = 2T(3)+1 = 2(7)+1 = 15$

$\boxed{T(n) = 2^n - 1 \in n \geq 1}$

b. $T(n) = \begin{cases} 4, & n = 1 \\ T(n-1) + 2n, & n \geq 2 \end{cases}$

$T(1) = 4$
$T(2) = T(2-1)+2(2) = T(1) + 4 = 8$
$T(3) = T(3-1)+2(3) = T(2)+6 = 14$
$T(4) = T(4-1)+2(4) = T(3) + 8 = 22$
$T(5) = T(5-1)+2(5) = T(4)+10 = 32$

$\boxed{T(n) = n^2 + n + 2}$

$5! = 5\cdot4\cdot3\cdot2\cdot1$ $\quad k! = k,\dots,3,2,1$

$5+1! = 5+1\cdot5\cdot4\cdot3\cdot2\cdot1$ $\quad (k+1)! = k+1, k\cdot\dots3\cdot2\cdot1$

3. **Proof by induction**

a. **Prove the following statement using induction:**

$$\sum_{i=1}^{n}(i) * (i!) = (n+1)! - 1, \text{ for all } n \geq 1.$$

$k!(k+1)(k+2)$

$k!(k+1)-1$

Basis: $\sum_{i=1}^{1} 1\cdot1! = 1 = (1+1)! - 1 = 1$ ✓

IH: Suppose that $\sum_{i=1}^{n}(i)(i!) = (n+1)! - 1$, for $n = 1, 2, \dots, k$.

IS: want $\sum_{i=1}^{k+1}(i)(i!) = ((k+1)+1)! - 1$

$\sum_{i=1}^{k}(i)(i!) + (k+1) = k!(k+1)(k+2) - 1$

$(k+1) = (k+2) - 1$

$k+1 = k+1$

$\therefore$ By PMI $\sum_{i=1}^{n}(i)\cdot(i!) = (n+1)! - 1$ for all $n \geq 1$

b. **Prove the following statement using induction:**

$= \sum_{i=1}^{n}(5i+4)$

$$1 + 6 + 11 + 16 + \dots + (5n-4) = n(5n-3)/2, \text{ for any positive integer } n \geq 1.$$

Basis: $5(1) - 4 = 1 \qquad \frac{1(5(1)-3)}{2} = 1$ ✓

IH: Suppose that $\sum_{i=1}^{n}(5i-4) = \frac{n(5n-3)}{2}$ for $n = 1, 2, \dots, k$

IS: want $\sum_{i=1}^{k+1}(5i+4)$ $\frac{(k+1)(5(k+1)-3)}{2} = 5k+5-3 = \frac{(5k+2)(k+1)}{2} = \frac{5k^2+7k+2}{2}$

$\boxed{\frac{k(5k+7)}{2} + 1}$

RHS

$\sum_{i=1}^{k}(5i-4) + (5(k+1)-4) = \frac{k(5k+7)}{2} + 1$

LHS: $\frac{k(5k-3)}{2} + 5k+1$

LHS

$\frac{k(5k+7)}{2} + 1$ as wanted

$\therefore$ By PMI $\sum_{i=1}^{n}(5i+4) = \frac{n(5n-3)}{2}$ for all $n \geq 1$

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4. Use a *loop invariant* to prove that the algorithm below **returns** a^n.

Algorithm Q4(a, n)
Input: Positive integers $a \geq 0$ and $n \geq 0$
Output: a^n

$i \leftarrow 1$
$pow \leftarrow 1$
while $i \leq n$ **do**
$pow \leftarrow pow * a$
$i \leftarrow i + 1$
end
return pow

Basis: pow $= 1$, $i = 1$, Pow $= a$

IH: Assume pow $= a^k$

Show pow $= a^{k+1} = a^k \cdot a$

given $i = k$, Pow $= a^k$, in the while loop, i is incremented so $i = k+1$

$\therefore$ Pow $=$ pow $\cdot (k+1) \cdot a = a^{k+1} = a^k \cdot a$

Termination:

loop ends after the $(n-1)^{th}$ iteration $(i = n)$

from IH step, $a^{n+1-1} = a^n$

$\therefore$ a^n is returned. $\square$

5. Prove each of the following using the definition of Big-Oh. You must provide constants c and n_0 to satisfy the definition of Big-Oh as defined in class.

a. $(n+2)^4$ is $O(n^4)$

$(n+2)^4 \leq 16n^4$ for $n \geq 2$

$c = 16, \; n_0 = 2$

b. $\frac{n^4+n^2+1}{n^3+1}$ is $O(n)$

$\frac{n^4+n^2+1}{n^3+1} \leq 2n$ for $n \geq 1$

$c = 2, \; n_0 = 1$

c. Find the smallest integer x such that $5n^3 + 3n^2 logn$ is $O(n^x)$

$5n^3 + 3n^2 \log n \leq 5n^3 + 3n^3$ for all $n \geq 1$

$\leq 8n^3$

$c = 8, \; n_0 = 1$

6. **Order the following functions by their big-Oh notation. Group together (for example, by underlining) functions that are big-Theta of one another (no justification needed). Note: $\log n = \log_2 n$ unless otherwise stated.**

$\cancel{5n\log n \; O(n\log n)}$	2^{40} $\cancel{O(c)}$	$\cancel{O(\log\log n)}$	4^n $O(c^n)$	$\cancel{14n \; O(n)}$	$\cancel{2^{\log n} \; O(\log n)}$
$\cancel{(n\log n)^2 \; O(n^2)}$	$\sqrt{\cancel{\log n}}$ $O(\log n)$	$\cancel{n\log_5 n \; O(n\log n)}$	$\cancel{n^3 \; O(n^3)}$	$\cancel{25n^{0.5} \; O(n)}$	$7n^{5/2}$ $O(n^c)$
$\cancel{1/n \; O(1)}$	5^{n^2} $O(c^n)$	$\cancel{n^2\log n \; O(n^2)}$	$\cancel{\sqrt{n} \; O(n^c)}$	$\cancel{n^{0.01} \; O(n^c)}$	2^{2^n} $O(c^n)$

$\frac{1}{n} \in O(1)$

$\sqrt{\log n}, 2^{\log n} \in O(\log n)$

$2^{40} \in O(c)$

$14n, 25n^{0.5}, n^{0.01}, \sqrt{n} \in O(n)$ $\boxed{\sqrt{n} \in \Theta(25n^{0.5})}$

$\log\log n \in O(\log\log n)$

$5n\log n, n\log_5 n \in O(n\log n)$

$n^2\log n, (n\log n)^2 \in O(n^2)$ $\boxed{(n\log n)^2 \in \Theta(n^2\log n)}$

$n^3 \in O(n^3)$

$7n^{5/2} \in O(n^c)$

$2^{2^n}, 5^{n^2}, 4^n \in O(c^n)$

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