Module 6

Upsampling and Downsampling, Part I

Overview

- Compressor and expander definitions
- The noble identities
- Polyphase representation
- Efficient structures for decimation and interpolation filters
- MATLAB example

Sample Compressor

$$x(n) \rightarrow \boxed{\downarrow M} \rightarrow x_c(n)$$

$$x_c(n) = x(Mn)$$

$$X_c\left(e^{j\omega}\right) = \frac{1}{M} \sum_{r=0}^{M-1} X\left(e^{j\left(\frac{\omega - 2\pi r}{M}\right)}\right)$$

Sample Expander

$$x(n) \rightarrow \boxed{\uparrow L} \rightarrow x_e(n)$$

$$x_e(n) = \begin{cases} x\left(\dfrac{n}{L}\right), & \text{if } n = 0, \pm L, \pm 2L, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$X_e\left(e^{j\omega}\right) = X\left(e^{j\omega L}\right)$$

Noble Identities

- If H(z) is a *rational function* (i.e. a ratio of polynomials in z or z^{-1}) then

$$\rightarrow \boxed{\downarrow M} \rightarrow \boxed{H(z)} \rightarrow \quad \equiv \quad \rightarrow \boxed{H(z^M)} \rightarrow \boxed{\downarrow M} \rightarrow$$

$$\rightarrow \boxed{H(z)} \rightarrow \boxed{\uparrow L} \rightarrow \quad \equiv \quad \rightarrow \boxed{\uparrow L} \rightarrow \boxed{H(z^L)} \rightarrow$$

Polyphase Representation

- Consider the transfer function

$$H(z)=\sum_{n=-\infty}^{\infty} h(n)z^{-n}$$

- Separating the *even* and *odd* samples of the impulse response (M=2)

$$H(z)=\sum_{l=-\infty}^{\infty} h(2l)z^{-2l}+\sum_{l=-\infty}^{\infty} h(2l+1)z^{-(2l+1)}$$

Polyphase Representation

$$
\begin{aligned}
H(z) &= \sum_{l=-\infty}^{\infty} h(2l)(z^2)^{-l} + z^{-1} \sum_{l=-\infty}^{\infty} h(2l+1)(z^2)^{-l} \\
&= E_0(z^2) + z^{-1} E_1(z^2)
\end{aligned}
$$

with

$$
E_0(z) = \sum_{n=-\infty}^{\infty} h(2n) z^{-n} \qquad E_1(z) = \sum_{n=-\infty}^{\infty} h(2n+1) z^{-n}
$$

Polyphase Representation

- In the general case, for any integer M

$$
\begin{aligned}
H(z) &= \sum_{n=-\infty}^{\infty} h(nM) z^{-nM} \\
&+ z^{-1} \sum_{n=-\infty}^{\infty} h(nM+1) z^{-nM} \\
&\vdots \\
&+ z^{-(M-1)} \sum_{n=-\infty}^{\infty} h(nM+M-1) z^{-nM}
\end{aligned}
$$

Polyphase Representation

$$H(z) = \sum_{l=0}^{M-1} z^{-l} E_l(z^M)$$

Type I polyphase

$$E_l(z) = \sum_{n=-\infty}^{\infty} e_l(n) z^{-n}$$

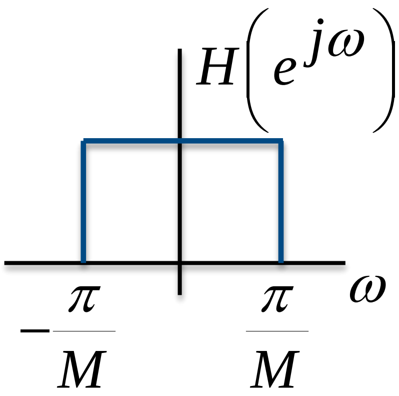
$$e_l(n) = h(Mn+l) \quad 0 \le l \le M-1$$

$$h(n) \rightarrow \boxed{z^l} \rightarrow \boxed{\downarrow M} \rightarrow e_l(n)$$

Block diagram for generating polyphase components from the impulse response

Polyphase Example

- Consider a (zero-phase) lowpass filter with cutoff frequency π/M



$$H\left(e^{j\omega}\right)=\begin{cases}1, & \text{for } |\omega|\le\frac{\pi}{M}\\ 0, & \text{for } \frac{\pi}{M}<|\omega|\le\pi\end{cases}$$

$$E_l\left(e^{j\omega}\right)=\frac{1}{M}\sum_{r=0}^{M-1} e^{j\left(\frac{\omega-2\pi r}{M}\right)l} H\left(e^{j\left(\frac{\omega-2\pi r}{M}\right)}\right)$$

Polyphase Example

- Each $E_l\left(e^{j\omega}\right)$ is *allpass* with phase response $\phi_l(\omega)=\omega l/M$ for $0\le l\le M-1$