Module IB

Review of Discrete-Time Signal Processing

Overview

- Linearity
- Time-Invariance
- Impulse Response/Convolution
- Stability
- Fourier transform
- z-transform
- Sampling Theorem
- Upsampling/Downsampling
- Difference Equations

z-transform

- The *z-transform* of the discrete sequence x(n) is defined by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- The region over which X(z) is finite is termed the *region of convergence* (ROC)

Example (1 of 2)

- Let

$$x(n) = \begin{cases} a^n, & \text{if } n \geq 0 \\ 0, & \text{if } n < 0 \end{cases}$$

- The Fourier transform of x(n) is

$$X\left(e^{j\omega}\right) = \sum_{n=0}^{\infty} a^n e^{-j\omega n}$$

$$= \frac{1}{1 - ae^{-j\omega}} \quad |a| < 1$$

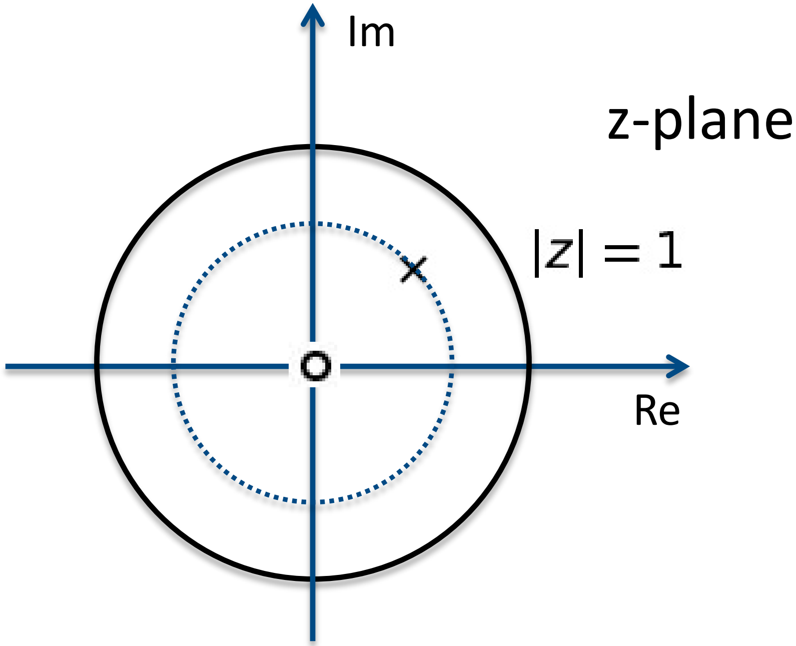
Example (2 of 2)

- The z-transform of x(n) is

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n}$$

$$= \frac{1}{1 - az^{-1}} \quad |z| > |a|$$

Pole/zero diagram for X(z)



Pole at z=a and a zero at z=0

Sampling Theorem (1 of 2)

- A *bandlimited* continuous-time signal $x_c(t)$ is completely represented by its samples $x(n) = x_c(nT)$ where $f_s = \frac{1}{T} \geq 2f_c$ and

$$X_c(f) = 0 \text{ for } |f| > f_c$$

- The Fourier transform of $x(n)$ is

$$X\left(e^{j\omega}\right) = \frac{1}{T}\sum_{k=-\infty}^{\infty} X_c\left(j\frac{\omega}{T} - j\frac{2\pi k}{T}\right)$$

Sampling Theorem (2 of 2)

- $x_c(t)$ is reconstructed from its samples according to

$$x_c(t) = \sum_{n=-\infty}^{\infty} x(n) \frac{\sin \pi(t-nT)}{\pi(t-nT)}$$

Upsampling (1 of 2)

- Block diagram for L-sample expander

$$x(n) \longrightarrow \boxed{\uparrow L} \longrightarrow x_e(n)$$

$$x_e(n) = \begin{cases} x\left(\frac{n}{L}\right), & \text{if } n = 0, \pm L, \pm 2L, \ldots \\ 0, & \text{otherwise} \end{cases}$$

$$X_e\left(e^{j\omega}\right) = X\left(e^{j\omega L}\right)$$

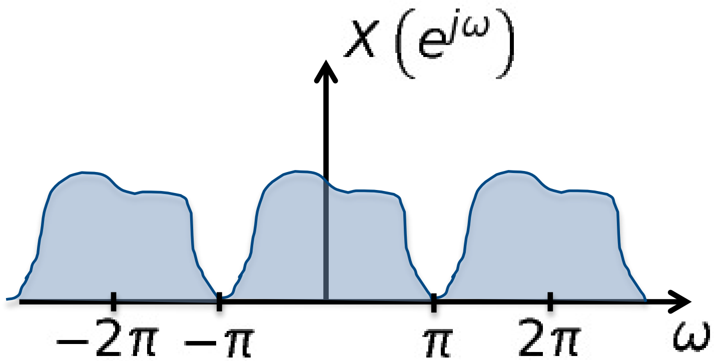
Upsampling (2 of 2)

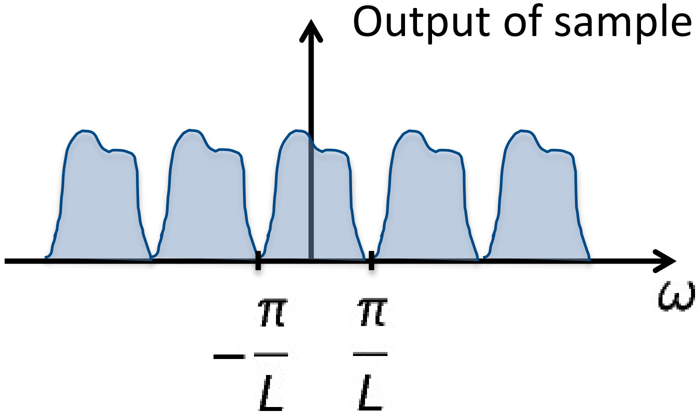
- Block diagram for upsampling

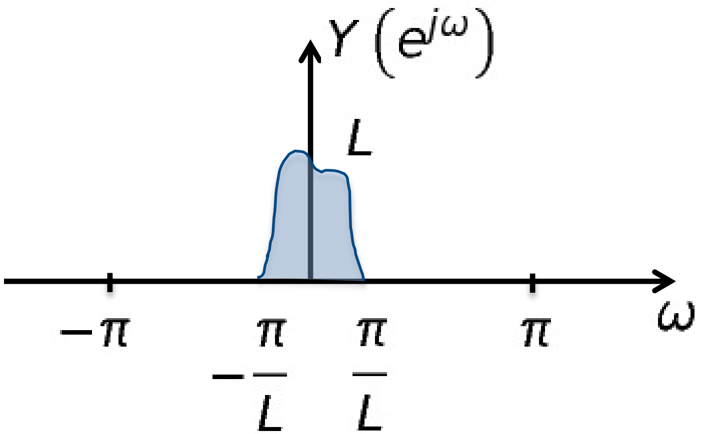
$$x(n) \longrightarrow \boxed{\uparrow L} \longrightarrow \boxed{H\left(e^{j\omega}\right)} \longrightarrow y(n)$$

$$H\left(e^{j\omega}\right) = \begin{cases} L, & \text{for } |\omega| \le \frac{\pi}{L} \\ 0, & \text{for } \frac{\pi}{L} < |\omega| \le \pi \end{cases}$$

Upsampling Example







Downsampling (1 of 2)

- Block diagram for M-sample compressor

$$x(n) \longrightarrow \boxed{\downarrow M} \longrightarrow x_c(n)$$

$$x_c(n) = x(Mn)$$

$$X_c\left(e^{j\omega}\right) = \frac{1}{M}\sum_{r=0}^{M-1} X\left(e^{j\left(\frac{\omega-2\pi r}{M}\right)}\right)$$

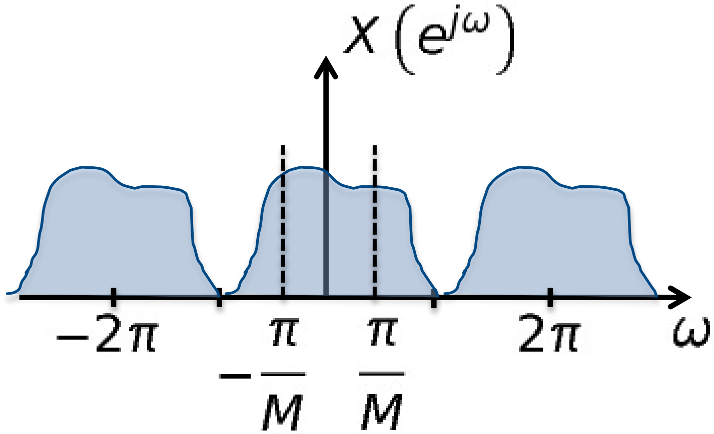
Downsampling (2 of 2)

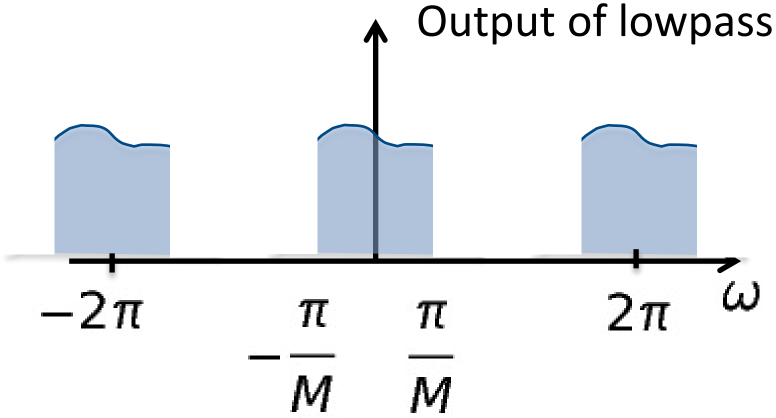
- Block diagram for downsampling

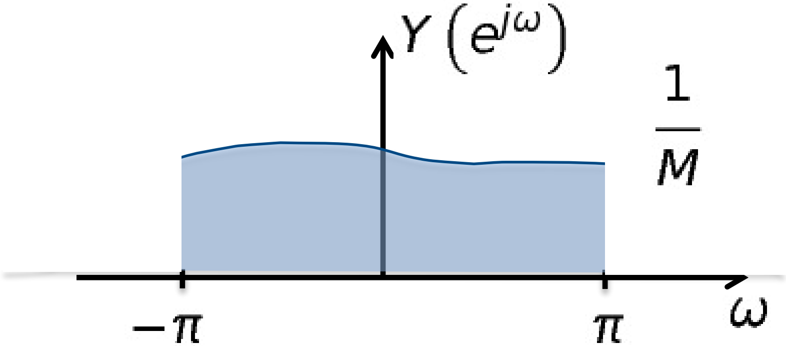
$$x(n) \longrightarrow \boxed{H\left(e^{j\omega}\right)} \longrightarrow \boxed{\downarrow M} \longrightarrow y(n)$$

$$H\left(e^{j\omega}\right) = \begin{cases} 1, & \text{for } |\omega| \le \frac{\pi}{M} \\ 0, & \text{for } \frac{\pi}{M} < |\omega| \le \pi \end{cases}$$

Downsampling Example







Difference Equations (1 of 2)

- Difference equation representation of a discrete-time system

$$y(n) = \underbrace{\sum_{k=1}^{N} a_k y(n-k)}_{\text{recursive terms}} + \underbrace{\sum_{k=0}^{M} b_k x(n-k)}_{\text{non-recursive terms}}$$

Difference Equations (2 of 2)

- Transfer function representation of system described by the difference equation

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 - \sum_{k=1}^{N} a_k z^{-k}}$$

Example

- The discrete-time accumulator can be represented in difference equation form as

$$y(n) = y(n-1) + x(n) \qquad y(0) = 0$$

The transfer function is

$$H(z) = \frac{1}{1 - z^{-1}}, \text{ for } |z| > 1$$