Module IA

Review of Discrete-Time Signal Processing

Overview

- Linearity
- Time-Invariance
- Impulse Response/Convolution
- Stability
- Fourier transform
- z-transform
- Sampling Theorem
- Upsampling/Downsampling
- Difference Equations

Linearity

- A discrete-time system is *linear* if superposition holds. Define a system $\mathcal{T}$ such that

$$y(n) = \mathcal{T}[x(n)]$$

- $\mathcal{T}$ is linear if

$$\mathcal{T}[a_1 x_1(n) + a_2 x_2(n)] = a_1 \mathcal{T}[x_1(n)] + a_2 \mathcal{T}[x_2(n)]$$

Time-Invariance

- A discrete-time system is *time-invariant* if the system response does not depend on the time at which the input is applied. If

$$y(n) = \mathcal{T}\left[x(n)\right]$$

then

$$y(n-n_0) = \mathcal{T}\left[x(n-n_0)\right]$$

Example

- The discrete-time system

$$y(n) = [x(n)]^2$$

is *time-invariant* but *non-linear* since

$$\mathcal{T}[x_1(n) + x_2(n)] = [x_1(n)]^2 + [x_2(n)]^2 + 2x_1(n)x_2(n)$$

and

$$\mathcal{T}[x_1(n)] + \mathcal{T}[x_2(n)] = [x_1(n)]^2 + [x_2(n)]^2$$

Impulse Response/Convolution

- A discrete-time system that is both *linear* and *time-invariant* (LTI) can be completely characterized by its response to an impulse input. Let

$$h(n) = \mathcal{T}\left[\delta(n)\right]$$ impulse response

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

convolution sum

Computing the Convolution Sum

- Reflect h(k) about the origin to obtain h(-k)
- Translate h(-k) by n samples to form h(n-k)
- Multiply sample by sample with x(k) for the selected value of n
- Sum the result over all k
- Repeat for all other values of n

Stability

- Every bounded input must produce a bounded output

$$|x(n)| \le B_x \implies |y(n)| \le B_y$$

- A discrete-time LTI system is *stable* in the bounded-input bound-output (BIBO) sense if

$$\sum_{n=-\infty}^{\infty} |h(n)| < \infty$$

Example

- The discrete-time accumulator with impulse response

$$h(n) = \begin{cases} 1, & \text{if } n \geq 0 \\ 0, & \text{if } n < 0 \end{cases}$$

is not BIBO stable since

$$\sum_{n=-\infty}^{\infty} |h(n)| \quad \text{is unbounded}$$

Fourier Transform

- The *Fourier transform* of the discrete sequence x(n) is defined by

$$X\left(e^{j\omega}\right) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

with $0 \leq \omega \leq 2\pi$

Fourier Transform Properties (1 of 2)

$$x(n-n_0) \xleftrightarrow{\mathcal{F}} e^{-j\omega n_0} X\left(e^{j\omega}\right)$$ Time shift

$$e^{j\omega_0 n} x(n) \xleftrightarrow{\mathcal{F}} X\left(e^{j(\omega-\omega_0)}\right)$$ Frequency modulation

$$x(n) * y(n) \xleftrightarrow{\mathcal{F}} X\left(e^{j\omega}\right) Y\left(e^{j\omega}\right)$$ Convolution

Fourier Transform Properties (2 of 2)

$$x(n)w(n) \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}\int_{-\pi}^{\pi} X\left(e^{j\theta}\right) W\left(e^{j(\omega-\theta)}\right) d\theta$$

Windowing

$$\sum_{n=-\infty}^{\infty} |x(n)|^2 = \frac{1}{2\pi}\int_{-\pi}^{\pi} |X\left(e^{j\omega}\right)|^2 d\omega$$

Parseval’s Theorem

Some Fourier Transform Pairs (1 of 2)

$$\delta(n) \stackrel{\mathcal{F}}{\longleftrightarrow} 1$$

$$\delta(n-n_0) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega n_0}$$

$$\mathrm{u(n)} \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{1-e^{-j\omega}} + \sum_{k=-\infty}^{\infty} \pi\delta(\omega + 2\pi k)$$

unit step

$$\frac{\sin \omega_c n}{\pi n} \stackrel{\mathcal{F}}{\longleftrightarrow} X\left(e^{j\omega}\right) = \begin{cases} 1, & \text{for } |\omega| \leq \omega_c \\ 0, & \text{for } \omega_c < |\omega| \leq \pi \end{cases}$$

Some Fourier Transform Pairs (2 of 2)

$$x(n) = \begin{cases} 1, & 0 \le n \le M \\ 0, & otherwise \end{cases} \xleftrightarrow{\mathcal{F}} \frac{\sin(\omega(M+1)/2)}{\sin(\omega/2)} e^{-j\omega M/2}$$

$$e^{j\omega_0 n} \xleftrightarrow{\mathcal{F}} \sum_{k=-\infty}^{\infty} \pi\delta(\omega - \omega_0 + 2\pi k)$$