Homework 1

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1. Consider the sequence $x(n) = \alpha^{|n|}$ with α complex. Find its z-transform $X(z)$ and associated region of convergence. Under what conditions does the Fourier Transform exist?

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty} \alpha^{|n|}z^{-n} = X(z)_1 + X(z)_2$$

$$X(z)_1 = \sum_{n=0}^{\infty} \alpha^{|n|}z^{-n} = \sum_{n=0}^{\infty} (\alpha z^{-1})^n = \frac{1}{1-\alpha z^{-1}} \; for \; |\alpha z^{-1}| < 1 \; and \; |z| > |\alpha|$$

$$X(z)_2 = \sum_{n=-\infty}^{-1} \alpha^{|n|}Z^{-n} = \sum_{m=1}^{\infty} \alpha^{|m|}Z^{m} = \sum_{m=1}^{\infty} (\alpha z)^m = -1 + \sum_{m=0}^{\infty} (\alpha z)^m = -1 + \frac{1}{1-\alpha z}$$

$$= \frac{\alpha z}{1-\alpha z} = \quad for \; |z| < \frac{1}{|\alpha|}$$

$$\boldsymbol{X(z) = \frac{1}{1-\alpha z^{-1}} + \frac{\alpha z}{1-\alpha z} \; for \begin{cases} |z| > |\alpha|, \\ |z| < \dfrac{1}{|\alpha|} \end{cases}}$$

$X(e^{j\omega})$ exists as long as **$|\alpha|<1$**

2. Consider the non-linear discrete-time system $y(n) = [x(n)]^{\beta}$ with β a positive integer. Compute analytically the Fourier transform of the output for arbitrary input and β=2. Evaluate $y(n)$ directly for $x(n) = \cos(\omega_0 n)$ and $\beta = 2,3$. What conclusions can you make regarding the bandwidth properties of the non-linear system $y(n) = [x(n)]^{\beta}$ for arbitrary β?

$$Y(e^{j\omega}) = \sum_{n=-\infty}^{\infty} [x(n)]^{|\beta|}e^{-j\omega n}$$

For an arbitrary signal and β=2 the system looks like the windowing property $F\{x(n)w(n)\} = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})W(e^{j(\omega-\theta)})\,d\theta$ but instead w(n) also equals x(n).

$$\boldsymbol{Y(e^{j\omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(e^{j\theta})X(e^{j(\omega-\theta)})\,d\theta}$$

<u>For β=2</u>

$$y(n) = [\cos(\omega_0 n)\,]^2 = \frac{1}{2} + \frac{\cos(2\omega_0 n)}{2} = \frac{1}{2} + \frac{1}{4}e^{2j\omega_0 n} + \frac{1}{4}e^{-2j\omega_0 n}$$

$$Y(z) = \sum_{n=-\infty}^{\infty} \frac{1}{2}e^{-j\omega n} + \frac{1}{4}\sum_{n=-\infty}^{\infty} e^{2j\omega_0 n}e^{-j\omega n} + \frac{1}{4}\sum_{n=-\infty}^{\infty} e^{-2j\omega_0 n}e^{-j\omega n}$$

$$Y(z) = \sum_{n=-\infty}^{\infty} \frac{1}{2}e^{-j\omega n} + \frac{1}{4}\sum_{n=-\infty}^{\infty} e^{-j(\omega-2\omega_0)n} + \frac{1}{4}\sum_{n=-\infty}^{\infty} e^{-j(\omega+2\omega_0)n}$$

$\boldsymbol{Y(z) = \frac{1}{2}\delta(\omega) + \frac{\pi}{2}\delta(\omega - 2\omega_0) + \frac{\pi}{2}\delta(\omega + 2\omega_0)}$ **where Y(z) repeats with period 2π**

<u>For β=3</u>

$$y(n) = [\cos(\omega_0 n)\,]^3 = \left[\frac{1}{2} + \frac{\cos(2\omega_0 n)}{2}\right] cos(\omega_0 n) = \left[\frac{1}{2} + \frac{1}{4}e^{2j\omega_0 n} + \frac{1}{4}e^{-2j\omega_0 n}\right]\left[\frac{e^{j\omega_0 n}}{2} + \frac{e^{-j\omega_0 n}}{2}\right]$$

$$= \frac{1}{4}e^{j\omega_0 n} + \frac{1}{4}e^{-j\omega_0 n} + \frac{1}{8}e^{j3\omega_0 n} + \frac{1}{8}e^{j\omega_0 n} + \frac{1}{8}e^{-j\omega_0 n} + \frac{1}{8}e^{-j3\omega_0 n}$$

$$= \left(\frac{1}{4} + \frac{1}{8}\right)e^{j\omega_0 n} + \left(\frac{1}{4} + \frac{1}{8}\right)e^{-j\omega_0 n} + \frac{e^{j3\omega_0 n}}{8} + \frac{e^{-j3\omega_0 n}}{8} = \frac{3}{8}e^{j\omega_0 n} + \frac{3}{8}e^{-j\omega_0 n} + \frac{e^{j3\omega_0 n}}{8} + \frac{e^{-j3\omega_0 n}}{8}$$

$$Y(z) = \frac{3}{8}\sum_{n=-\infty}^{\infty} e^{j\omega_0 n}e^{-j\omega n} + \frac{3}{8}\sum_{n=-\infty}^{\infty} e^{j\omega_0 n}e^{-j\omega n} + \frac{1}{8}\sum_{n=-\infty}^{\infty} e^{j3\omega_0 n}e^{-j\omega n} + \frac{1}{8}\sum_{n=-\infty}^{\infty} e^{-j3\omega_0 n}e^{-j\omega n}$$

$$Y(z) = \frac{3}{8}\sum_{n=-\infty}^{\infty} e^{-j(\omega-\omega_0)n} + \frac{3}{8}\sum_{n=-\infty}^{\infty} e^{-j(\omega+\omega_0)n} + \frac{1}{8}\sum_{n=-\infty}^{\infty} e^{-j(\omega-3\omega_0)n} + \frac{1}{8}\sum_{n=-\infty}^{\infty} e^{-j(\omega+3\omega_0)n}$$

$$\boldsymbol{Y(z) = \frac{3\pi}{4}\delta(\omega - \omega_0) + \frac{3\pi}{4}\delta(\omega + \omega_0) + \frac{\pi}{4}\delta(\omega + 3\omega_0) + \frac{\pi}{4}\delta(\omega - 3\omega_0)}$$

where Y(z) repeats with period 2π

We can see that the bandwidth increases as β increases. Eventually aliasing will occur at higher values of β due to the periodic nature of the DTFT.

3. Consider the system block diagram and input as shown in the following sketch. Compute $Y(e^{j\omega})$ for the following cases: $\omega_c = \frac{\pi}{2}$ and M=2,3. Comment on the effects of the M-sample compressor for each value of M. Is the information in $X(e^{j\omega})$ preserved in both cases?

<u>**Case M = 2:**</u>

$$Y(e^{j\pi}) = \frac{1}{2}\sum_{r=0}^{1} X(e^{j\left(\frac{\pi-2\pi r}{2}\right)} = \frac{1}{2}\left[X\left(e^{j\left(\frac{\pi-0}{2}\right)}\right) + X\left(e^{j\left(\frac{\pi-2\pi}{2}\right)}\right)\right] = \frac{1}{2}\left[X\left(e^{j\left(\frac{\pi}{2}\right)}\right) + X\left(e^{j\left(\frac{-\pi}{2}\right)}\right)\right] =$$

$$\mathbf{\frac{1}{2}[0+0] = 0}$$

$X(e^{j\omega})$ is preserved when down sampling by a factor of 2 because the signal was sampled exactly at the Nyquist frequency (Nyquist = pi).

<u>**Case M = 3:**</u>

$$Y(e^{j\pi}) = \frac{1}{3}\sum_{r=0}^{2} X(e^{j\left(\frac{\pi-2\pi r}{3}\right)} = \frac{1}{3}\left[X\left(e^{j\left(\frac{\pi-0}{3}\right)}\right) + X\left(e^{j\left(\frac{\pi-2\pi}{3}\right)}\right) + X\left(e^{j\left(\frac{\pi-4\pi}{3}\right)}\right)\right]$$
$$= \frac{1}{3}\left[X\left(e^{j\left(\frac{\pi}{3}\right)}\right) + X\left(e^{j\left(\frac{-\pi}{3}\right)}\right) + X(e^{j(-\pi)})\right] = \frac{1}{3}\left[X\left(e^{j\left(\frac{\pi}{3}\right)}\right) + X\left(e^{j\left(\frac{-\pi}{3}\right)}\right) + 0\right]$$
$$= \boldsymbol{\frac{1}{3}\left[X\left(e^{j\left(\frac{\pi}{3}\right)}\right) + X\left(e^{j\left(\frac{-\pi}{3}\right)}\right)\right]}$$

$X(e^{j\omega})$ is NOT preserved when down sampling by a factor of 3. We know this because $\frac{\pi}{3}$ becomes pi and since there is spectral content at $\frac{\pi}{2}$ the original signal will not be preserved. Another way to verify the answer is that the signal was sampled at exactly the Nyquist frequency. If this is the case then the most we can down sample is by a factor of 2.