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1 Original Implementation in GEODYN

1.1 In MSIS.f90

DRHODZ is computed in the subroutine that serves as a wrapper for the density models. In the case of MSIS this is the `MSIS.f90` subroutine.

Original Calculation in MSIS.f90:

```
        !  Calculate drho/dz.
           IF(IDRV .NE. 0) THEN
        !
        !   D      O            D(1) - HE NUMBER DENSITY(CM-3)
        !                       D(2) - O NUMBER DENSITY(CM-3)
        !                       D(3) - N2 NUMBER DENSITY(CM-3)
        !                       D(4) - O2 NUMBER DENSITY(CM-3)
        !                       D(5) - AR NUMBER DENSITY(CM-3)
        !                       D(6) - TOTAL MASS DENSITY(GM/CM3)
        !                       D(7) - H NUMBER DENSITY(CM-3)
        !                       D(8) - N NUMBER DENSITY(CM-3)
        !   T                   T(1) - EXOSPHERIC TEMPERATURE
        !                       T(2) - TEMPERATURE AT ALT
        !
             TERM1 = -1.66D-24*(16.D0*DEN(1) + 256.D0*DEN(2) + 784.D0*DEN(3) &
           &          + DEN(7) + 196.D0*DEN(8) )
             TERM2 = GSURF/(TEMP(2)*RGAS)/(1.D0 + ZL/RE)**2
             TERM3 = ((RE+ZL)/(RE+ALTKM))**2
             DRHODZ = TERM1*TERM2*TERM3
        !    DRHODZ IS NOW IN G/CC/KM.  THIS IS DIMENSIONALLY EQUAL TO KG/M4.

           ENDIF

```

In more readable math, this is:

$$\frac{d\rho}{dz} = -m_p\Bigg((m_{He}^2 n_{He}) + (m_O^2 n_O) + (m_{N2}^2 n_{N2}) + (m_H^2 n_H) + (m_N^2 n_N)\Bigg) * \frac{\left(\frac{g_{surf}}{T_{alt}R_{gas}}\right)}{(1+Z_L/R_E)^2}\left(\frac{R_E+Z_L}{R_E+Z_{alt}}\right) \tag{1}$$

Where:

- m_p .... mass of a proton ($= 1.66E^{-24}$ grams)
- m_{He} .... atomic mass of helium ($= 4$ amu)
- m_O .... atomic mass of oxygen ($= 16$ amu)
- m_{N2} .... atomic mass of molecular nitrogen ($= 28$ amu)
- m_H .... atomic mass of hydrogen ($= 1$ amu)

- m_N .... atomic mass of atomic nitrogen (= 14 amu)
- g_{surf} .... gravity at Earth's surface
- T_{alt} .... temperature at the given altitude
- R_{gas} .... gas constant (= 831.4)
- Z_L .... lower boundary for diffusive equilibrium in the MSIS model ($= 120km$)
- R_E .... radius of earth
- Z_{alt} .... altitude of the satellite

1.2 How is DRHODZ used in a POD sense?

In general, the drag equation used by GEODYN is as follows:

$$\bar{A}_D = -\frac{1}{2}C_d\frac{A_{sat}}{m_{sat}}\rho\, v_r\bar{v}_r \tag{2}$$

Where:

- $\bar{A}_D$ acceleration due to atmospheric drag force
- C_d is the satellite drag coefficient
- A_{sat} is the cross-sectional area of the satellite
- m_{sat} is the mass of the satellite
- ρ is the-density of the atmosphere at the satellite position
- $\bar{v}_r$ is the velocity vector of the satellite relative to the atmosphere, and
- v_r is the magnitude of the velocity vector, vr

In GEODYN, the drag is incorporated into the orbit determination run through the D_r matrix in the Variational Equations (not included here). The D_r matrix includes the partial derivatives of the drag acceleration with respect to the Cartesian orbital elements:

$$D_r = -\frac{1}{2}C_d\frac{A_{sat}}{m_{sat}}\left[\rho v_r\frac{\delta\bar{v}_r}{\delta\bar{x}_t} + \rho\frac{\delta v_r}{\delta\bar{x}_t}\bar{v}_r + \frac{\delta\rho}{\delta\bar{x}_t}\bar{v}_r v_r\right] \tag{3}$$

Where $\bar{x}_t$ is $(x, y, z, \dot{x}, \dot{y}, \dot{z})$, (so it spans $\bar{r}$ and $\bar{\dot{r}}$).

$\frac{\delta\rho}{\delta\bar{x}_t}$ is the matrix containing the partial derivatives of the atmospheric density with respect to the Cartesian coordinates and can also be written in terms of the positional vector as follows:

$$\frac{\delta\rho}{\delta\bar{r}} = \frac{\delta\rho}{\delta\phi}\frac{\delta\phi}{\delta\bar{r}} + \frac{\delta\rho}{\delta\lambda}\frac{\delta\lambda}{\delta\bar{r}} + \frac{\delta\rho}{\delta h}\frac{\delta h}{\delta\bar{r}} \tag{4}$$

where h is spheroid height of the satellite (perpendicular height above the reference spheroid), ϕ is the sub-satellite latitude, λ is the sub satellite longitude, and $\bar{r}$ is the true of date position vector of the satellite.

Because variations in atmospheric density are primarily due to changes in altitude, GEODYN assumes the longitude and latitudinal changes in density are zero:

$$\frac{\delta\rho}{\delta\phi} = \frac{\delta\rho}{\delta\lambda} = 0$$

Consequently Equation 5 becomes:

$$\frac{\delta\rho}{\delta\bar{r}} = \frac{\delta\rho}{\delta h}\frac{\delta h}{\delta\bar{r}} \tag{5}$$

$\frac{\delta h}{\delta\bar{r}}$ is shown in Section 5.1 of the GEODYN Vol. 1 Documentation.

$\frac{\delta\rho}{\delta h}$ can be written in terms of the cartesian height coordinate z such that

$$\frac{\delta\rho}{\delta h} = \frac{\delta\rho}{\delta z}$$

In Summary: $\frac{\delta\rho}{\delta z}$ is included in the variational equations as a prominent term in the D_r (drag) matrix.

2 Derivation of DRHODZ

2.1 Background information on scale heights

Specific relationships amongst scale heights for individual species and the total gas can be derived assuming the ideal gas law, molecular and thermal diffusion, and hydrostatic equilibrium.

A pressure scale height can be derived starting with the hydrostatic equation, $(\frac{dP}{dz} + \rho g = 0)$, and applying the ideal gas law $(\rho = \frac{P\bar{m}}{kT})$,

$$\frac{dP}{dz} + \frac{P\bar{m}g}{kT} = 0$$

Where $H_P = \frac{kT}{\bar{m}g}$, such that, $H_P\frac{dP}{dz} + P = 0$. Using this, we can define a pressure scale height in general to be,

$$H_P = \frac{-P}{\partial P/\partial z} \tag{6}$$

Similarly for density, temperature, and mean molecular mass we can write:

$$H_\rho = -\frac{\rho}{\partial\rho/\partial z} \tag{7}$$

$$H_T = \frac{T}{\partial T/\partial z} \tag{8}$$

$$H_{\bar{m}} = -\frac{\bar{m}}{\partial\bar{m}/\partial z} \tag{9}$$

We can define a relation between the scale heights using Equations 7 and 9, to differentiate the ideal gas law with respect to height.

$$\begin{aligned}
P &= \frac{\rho}{\bar{m}}kT \\
\frac{\partial P}{\partial z} &= \frac{\partial}{\partial z}\left(\frac{\rho}{\bar{m}}kT\right) \\
\frac{\partial P}{\partial z} &= \frac{\partial\rho}{\partial z}\left(\frac{kT}{\bar{m}}\right) + \frac{\partial T}{\partial z}\left(\frac{k\rho}{\bar{m}}\right) - \frac{1}{\bar{m}^2}\frac{\partial\bar{m}}{\partial z}\left(k\rho T\right)
\end{aligned}$$

Simplifying, we get:

$$-\frac{1}{P}\frac{\partial P}{\partial z} = -\frac{1}{\rho}\frac{\partial\rho}{\partial z} - \frac{1}{T}\frac{\partial T}{\partial z} + \frac{1}{\bar{m}}\frac{\partial\bar{m}}{\partial z}$$

$$\frac{1}{H_P} = \frac{1}{H_\rho} - \frac{1}{H_T} - \frac{1}{H_{\bar{m}}} \tag{10}$$

2.2 Deriving DRHODZ in a molecularly diffuse atmosphere

Assuming an atmosphere in diffusive equilibrium (which is the case above 120km) the following scale height relation can written for an individual species.

$$\frac{1}{H_{\rho_s}} = \frac{m_s g}{kT} + \frac{1}{H_T} \tag{11}$$

Plugging in Equations 7 and 8, we can rewrite Equation 11.

$$
\begin{aligned}
\frac{1}{H_{\rho_s}} &= \frac{m_s g}{kT} + \frac{1}{H_T} \\
\frac{-\delta\rho_s/\delta z}{\rho_s} &= \frac{m_s g}{kT} + \frac{\delta T/\delta z}{T} \\
\frac{\delta\rho_s}{\delta z} &= -\rho_s\left(\frac{m_s g}{kT} + \frac{\delta T/\delta z}{T}\right) \\
\frac{\delta\rho_s}{\delta z} &= -\rho_s\left(\frac{m_s g}{kT} + \frac{\delta T/\delta z}{T}\right)
\end{aligned}
$$

We note here that the total gas mass density can be written as $\rho = m_p(\sum_s m_s n_s)$. With this in mind, the above equations can be written for the total gas density as follows:

$$\frac{\delta\rho}{\delta z} = -m_p(\sum_s m_s n_s)\left(\frac{m_s g}{kT} + \frac{\delta T/\delta z}{T}\right) \tag{12}$$

Finally, we note that above 300-400 km (depending on solar cycle), the temperature becomes isothermal with respect to height, meaning that $\delta T/\delta z = 0$. We can remove the temperature scale height term and simplify.

$$\frac{\delta\rho}{\delta z} = -m_p(\sum_s m_s^2\, n_s)\frac{g}{kT} \tag{13}$$

Equation 13 is nearly how the DRHODZ variable appears in GEODYN in the MSIS.f90 subroutine. **In its final implementation, there is included an additional normalization factor to account for gravity (discussed in Section 3.2)**

We expand Equation 13 to account for the constituents returned by MSIS (versions 00 and 2.0 only, v86 does not include AnomO): (He, O, N_2, O_2, Ar, H, N, $Anomalous\,O$).

$$
\begin{aligned}
\frac{\delta\rho}{\delta z} = -m_p\bigg(&(m_{He}^2 n_{He}) + (m_O^2 n_O) + (m_{N2}^2 n_{N2}) + (m_H^2 n_H) + \\
&(m_N^2 n_N) + (m_{Ar}^2 n_{Ar}) + (m_{O_2}^2 n_{O_2}) + (m_{Anom\_O}^2 n_{Anom\_O})\bigg)\frac{g}{kT}
\end{aligned} \tag{14}
$$

3 Modifications from Original Implementation

A summary of modification discussions:

1. For completeness sake, we should include all constituents that are outputted by the MSIS models. This means including O_2 and Ar in all versions of MSIS and additionally $Anom\_O$ for the later versions of MSIS.
2. Since anomalous oxygen is computed in MSIS using a different temperature than the other constituents, we should calculate its contribution to DRHODZ separately using its own scale height given the anomalous temperature. Discussed in Section 3.1
3. $GSURF$ and ZL should be made consistent such that the normalization accounts for the gravitational acceleration at $ZL = 120km$ as opposed to the acceleration at Earth's surface. Discussed in Section 3.2

3.1 Including Anomalous Oxygen

The anomalous oxygen scale height should be computed separately from the other constituents such that the anomalous temperature is used in its calculation.

$$\begin{aligned}
\frac{\delta\rho_{O_{anom}}}{\delta z} &= -m_p(m_{O_{anom}} n_{O_{anom}})\left(\frac{m_{O_{anom}} g}{kT_{anom}} + \frac{\delta T_{anom}/\delta z}{T_{anom}}\right) \\
\frac{\delta\rho_{O_{anom}}}{\delta z} &= -m_p(m^2_{O_{anom}} n_{O_{anom}})\frac{g}{kT_{anom}}
\end{aligned} \tag{15}$$

finish this

3.2 Normalization term

finish this