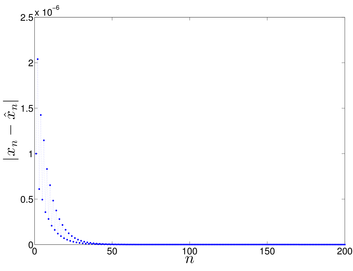
Homework problems in this course are broken into three types and marked by the following symbols:

● — Mandatory problems that should be accessible to all students.

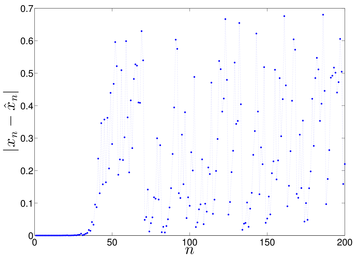
■ — Optional problems that are somewhat harder. We strongly suggest that you work through these problems; completing them will give you greater insight into the topics at hand. Please discuss them and their results on the forum.

◆— Challenging problems that are intended for experts. These are optional; not all students will be able to complete them. Again, feel free to discuss them on the forum.

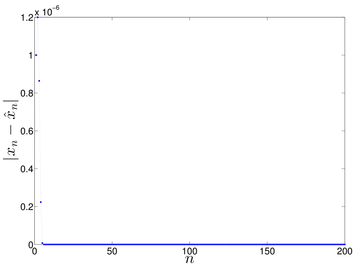
1. Use your logistic map program to explore the idea of *sensitive dependence on initial conditions*. Generate two trajectories of the logistic map $\{x_n\}$ and $\{\hat{x}_n\}$. For each of these experiments the initial condition for $\{x_n\}$ should be $x_0 = 0.2$ and for $\{\hat{x}_n\}$ you should use $x_0 = 0.200001$. For now, fix $r = 2$, generate these two trajectories and then plot $|x_n - \hat{x}_n|$ vs. n. Now repeat this for $r = 3.4$, and $r = 3.72$. Compare your results to the three plots below and then answer the questions that follow.



(a)



(b)



(c)

Figure 1: Plots of $|x_n - \hat{x}_n|$ vs. n for the logistc map with three different r values.

(a) ● Which of the plots in Figure **??** corresponds to $r = 2$?
   (a)
   (b)
   (c)
   None of them

(b) ● Which of the plots in Figure **??** corresponds to $r = 3.4$?
   (a)
   (b)
   (c)
   None of them

(c) ● Which of the plots in Figure **??** corresponds to $r = 3.72$?
   (a)
   (b)
   (c)
   None of them

(d) ■ What is the overall difference between these plots, if any? Do any of them seem to exhibit *sensitive dependence on initial conditions*? What does this mean about the underlying dynamics of each r value?

(e) ● For $r = 2$, what is the difference between the 500th iterates, *i.e.*, what is $|x_{500} - \hat{x}_{500}|$?
   0
   0.25
   1
   None of the above

(f) ● For $r = 3.4$, what is the difference between the 500th iterates, *i.e.*, what is $|x_{500} - \hat{x}_{500}|$?
   0
   0.25
   0.5
   None of the above

(g) ● For $r = 3.72$, what is the *average* absolute difference for the first 5,000 iterates? (*i.e.*, $\frac{1}{5{,}000}\sum_j^{5{,}000} |x_j - \hat{x}_j|$) *Depending on your implementation your exact answer may be slightly different; choose the answer that is closest to the one that you computed.*
   0
   0.2436
   0.3532
   0.5
   None of the above

(h) ● For $r = 3.72$, what is the *average* absolute difference for the first 500.000 iterates? (*i.e.*, $\frac{1}{500{,}000}\sum_j^{500{,}000} |x_j - \hat{x}_j|$) *Again, depending on your implementation, your exact answer may be slightly different; choose the answer that is closest to the one you computed.*
   0
   0.3532
   0.2441
   0.2436
   None of the above

2. Point your browser to:

   `http://tuvalu.santafe.edu/~jgarland/LogisticTools.html`

(a) ● Generate a 50-point trajectory of the logistic map starting at $x_0 = 0.2$ using parameter value $r = 3.68725$. What kind of dynamics does this *appear* to be?
   Fixed Point
   Periodic (9-cycle)
   Chaotic
   None of the above

(b) ● What if you click "Start Animation" and watch for a while. Does your conclusion change? Wait for a while until something changes. What kind of dynamics is this?
   Fixed Point
   Periodic (9-cycle)
   Chaotic
   None of the above

(c) ■ What is the take away here?

3. ■ Set $r = 3.828$, plot 50 iterates, and click "Start Animation". Raise r slowly to 3.8285, remembering to click "Restart Simulation" after each change. For $r \in (3.828, 3.8285)$ the dynamics are very deceiving—be patient! Describe & explain what you see. If you don't see anything interesting, you aren't being patient enough.

4. ◆In Quiz 1.5 you estimated the r value for which the dynamics bifurcated from a fixed point to a 2-cycle. Try to analytically find this number: *i.e.*, start with $x_n = x_{n+2}$ and solve for the first r that makes this valid. *Warning: This problem involves a significant amount of algebra and factoring.*