Solving the N-Dimensional Black-Scholes Model with Mimetic Methods

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May 9, 2024

Introduction

- The Black-Scholes Model is a partial differential equation (PDE) developed Fisher Black and Myron Scholes [2] to evaluate the underlying price of European options.
- An option is an agreement where someone can reserve to buy (call) or sell (put) a stock at specific time.
  - European options can only be executed on the maturity date.
- The model is risk-neutral and so it assumes that investors require a return. equivalent to the risk-free rate.
- The Black-Scholes model is expressed as a stochastic PDE, which accounts for the random and unpredictable nature of the underlying asset's price movements.

Introduction

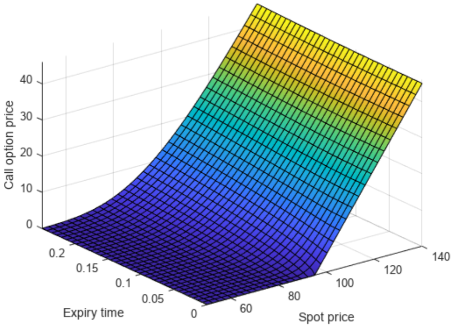


Figure: Example 1D Black-Scholes Model

- Different methods have been used to solve the Black-Scholes model in two-dimensions
  - Finite Differences [5] and Radial Basis Functions (RBF) [4] [3]
- The aim of this work is to compare mimetic methods with those two techniques in two dimensions and to exhibit simple mimetic discrete analogs that can evaluate European options for a general portfolio.

1D Black-Scholes Model

In one dimension, the Black-Scholes model is written as...

1D Black-Scholes

$$\frac{\partial F}{\partial t} = -rS\frac{\partial F}{\partial S} - \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 F}{\partial S^2} + rF \tag{1}$$

$$F(T,S) = \Phi(S) = \max(S-K,0) \tag{2}$$

Because the value of the option is discounted back to the present, the initial condition is the payoff function, $\Phi(S)$, at maturity time (T)

- F is the discounted present value of the call option as a function of time (t) and the underlying asset price (S).
- σ is the volatility of the underlying asset.
- r is the risk free interest rate.
- K is the strike price (option premium).

N-Dimensional Black-Scholes Model

With n underlying assets, the Black-Scholes PDE is expressed as a system of coupled partial differential equations:

N-Dimensional Black-Scholes

$$\frac{\partial F}{\partial t} = -\sum_{i=1}^{n} r_i S_i \frac{\partial F}{\partial S_i} - \frac{1}{2}\sum_{i=1}^{n}\sum_{j=1}^{n} \sigma_i \sigma_j \rho_{ij} S_i S_j \frac{\partial^2 F}{\partial S_i \partial S_j} + rF \tag{3}$$

$$F(T,S) = \Phi(S) = \max\left(\left(\sum_{i=1}^{n} \frac{S_i}{n}\right) - K, 0\right) \tag{4}$$

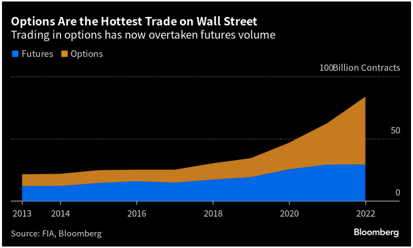
N-Dimensional Black-Scholes Model

- F is the discounted present value (price) of the call option as a function of time (t) and the n underlying asset prices (S_1,S_2,. . .,S_n).
- $\frac{\partial F}{\partial t}$ is the partial derivative of F with respect to time.
- $\frac{\partial F}{\partial S_i}$ is the partial derivative of F with respect to the i-th underlying asset price (S_i).
- $\frac{\partial^2 F}{\partial S_i S_j}$ is the partial second derivative of F with respect to the i-th and j-th underlying asset prices (S_i and S_j).
- σ_i and σ_j are the volatilities of the corresponding underlying assets.
- ρ_{ij} is the correlation between assets i and j.
- r is is the risk-free interest rate applicable to the entire basket of assets.

Impact on Modern Financial Markets

The model has made a significant impact on modern financial markets, particularly in the options trading and risk management sectors.

- For options traders and investors, the PDE has provided them with a quantitative tool to assess option prices, contributing to the growth and efficiency of options markets.
- For risk management sectors, by providing a framework for pricing options, it aids financial institutions in managing their exposure to market volatility.



Problem Statement

- Mimetic differences are a method for numerically approximating spatial operators and when coupled to a time discretization method it produces mimetic schemes [1].
    - They preserve symmetries, conservation laws, and other properties of partial differential equations
- For solving the Black-Scholes model, it is necessary to consider appropriate boundary conditions, as is pointed out by Sundvall and Trang [5].
- Different numerical techniques have been utilized for solving Black-Scholes in two-dimensions, more recently, Radial Basis Functions (RBF) [4] [3].

Proposed Solution

The proposed solutions to the previous section are as follows:

Proposed Solutions

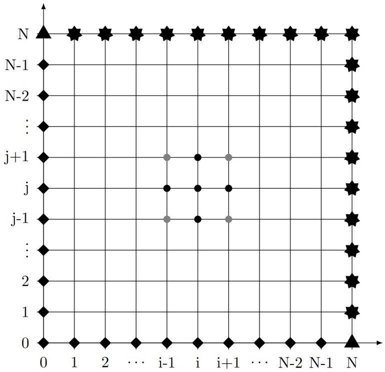
1. Numerically solve the two-dimensions (2D) Black-Scholes model utilizing Mimetic Differences.
2. Replicate the RBF solution of the 2D Black-Scholes model according to Milovanovic's papers, [4] and [3].
3. Compare RBF and Mimetic Differences results.
4. Compare Mimetic Differences model to Real Data
5. Extend the mimetic differences approach for the n-dimensional case in general.

Justification

- The Black-Scholes model forms the basis of modern day options markets.
- Providing a way to solve the model using mimetic operators would have great use by expanding upon the existing implementations of the MOLE library and its application to financial markets.

Background

In [5], Sundvall and Trang explored how different boundary conditions impacted the results of the 2D Black-Scholes model. Due to the nature of options having a minimum value of zero (lower domain) but no maximum price, they divided the boundary conditions into close-field and far-field boundary conditions.



- Diamond: Close-Field
- Star: Far-Field
- Triangle: Either Or (Used as Far-Field in [5])

Background: Far-Field Boundary

For the far-field boundaries, Sundvall and Trang explored Dirichlet, linearity condition, and one-sided differences boundary conditions.

- For example, they derived the Dirichlet boundary condition by extrapolating the 1D case of the limit as the payoff function approaches infinity into two dimensions as:

Far-Field: Dirichlet Boundary Conditions

$$F(t, x_{max}, y) = \frac{x_{\max}+y}{2} - Ke^{-r(T-t)}$$

$$F(t, x, y_{max}) = \frac{x+y_{\max}}{2} - Ke^{-r(T-t)}$$

The paper notes, as a "rule of thumb", to limit the upper domain by setting the maximum value for each asset ($x_{\max}$ and $y_{\max}$ in this case) to be to 4K to 6K times the number of spatial dimensions, which is two.

Background: Close-Field Boundary

For the close-field boundaries, Sundvall and Trang explored Dirichlet one-sided differences, linearity condition, and no boundary conditions.

- For example, for the no boundary condition case, due to $y = 0$ on the x-axis and $x = 0$ on the y-axis, they noted that the x and y axes simplified as:

Close-Field: No Boundary Conditions

$$F_t = -rxF_x - \frac{1}{2}x^2\Sigma^2(1,1)F_{xx} + rF$$

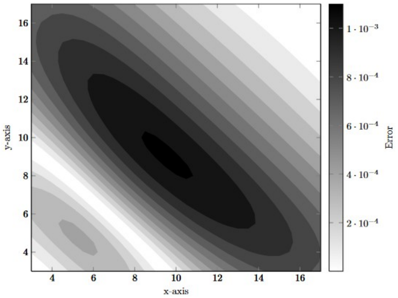
$$F_t = -ryF_y - \frac{1}{2}y^2\Sigma^2(2,2)F_{yy} + rF$$

$$F(t, 0, 0) = 0$$

Background: Results

Although the paper does not produce a figure of the numerical results, the following figure shows the error results for the Dirichlet boundary coundition for the far-field and no boundary counditions for the close-field.

- They specifically focused on the region where $\frac{K}{3} \leq x \leq \frac{K}{3}$ and $\frac{K}{3} \leq y \leq \frac{K}{3}$



2D Black-Scholes Model with Mimetic Methods

The *grad2D* function in the MOLE library returns a matrix that takes the gradient of a vectorized 2D mesh.
The resulting matrix from *grad2D* is:

$$G = \begin{bmatrix} G_x \\ G_y \end{bmatrix}$$

The application of G to a scalar field of nodal (center) points results in a vector field of discrete edge (face) points.

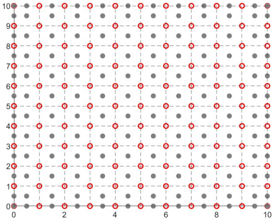


Figure: Center Points (Grey) and Face Points (Red)

2D Black-Scholes Model with Mimetic Methods

In order to have the face points to return back to the center points for further use, the interpolation function, *interpolFacesToCentersG2D*, is implemented.
The resulting matrix from *interpolFacesToCentersG2D* is:

$$I^{FC} = \begin{bmatrix} I_x^{FC} & \\ & I_y^{FC} \end{bmatrix}$$

$I^{FC}G$ results in a gradient operation on each center point interpolated back to that point. Thus,

$$I^{FC}G = \begin{bmatrix} I_x^{FC} & \\ & I_y^{FC} \end{bmatrix} \begin{bmatrix} G_x \\ G_y \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$$

2D Black-Scholes Model with Mimetic Methods

Similar to the *grad2D* function, the *div2D* function in the MOLE library returns a matrix that takes the divergence of a vectorized 2D mesh.
The resulting matrix from *div2D* is:

$$D = \begin{bmatrix} D_x & D_y \end{bmatrix}$$

The application of D to a vector field of discrete edge (face) points results in a scalar field of nodal (center) points. However, since the relevant point lay on the center points, the face points must be interpolated to the center points using the function *interpolCentersToFacesD2D*.
The resulting matrix from *interpolCentersToFacesD2D* is:

$$I^{CF} = \begin{bmatrix} I_x^{CF} & \\ & I_y^{CF} \end{bmatrix}$$

DI^{CF} results in a interpolation of center points to face points, which then allow for a divergence operation that then return the face points to center points. Thus,

$$DI^{CF} = \begin{bmatrix} D_x & D_y \end{bmatrix} \begin{bmatrix} I_x^{CF} & \\ & I_y^{CF} \end{bmatrix} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}$$

Discretization of F_t

The Black-Scholes PDE can be rewritten as

$$F_t = \mathcal{L}F$$

where

$$\mathcal{L} = -rx\frac{\partial}{\partial x} - ry\frac{\partial}{\partial y} - \frac{1}{2}x^2\Sigma^2_{(1,1)}\frac{\partial^2}{\partial x^2} - \frac{1}{2}y^2\Sigma^2_{(2,2)}\frac{\partial^2}{\partial y^2} - xy\Sigma^2_{(1,2)}\frac{\partial^2}{\partial x\partial y} + rI$$

Since $\frac{\partial^2}{\partial x\partial y} = \frac{\partial^2}{\partial y\partial x}$ and $\Sigma_{(1,2)} = \Sigma_{(2,1)}$,

$$\begin{aligned}\mathcal{L} = -r(x\tfrac{\partial}{\partial x} + y\tfrac{\partial}{\partial y}) - \tfrac{1}{2}(x^2\Sigma^2_{(1,1)}\tfrac{\partial^2}{\partial x^2} + xy\Sigma^2_{(1,2)}\tfrac{\partial^2}{\partial x\partial y} + yx\Sigma^2_{(2,1)}\tfrac{\partial^2}{\partial y\partial x} \\ + y^2\Sigma^2_{(2,2)}\tfrac{\partial^2}{\partial x^2} + xy\Sigma^2_{(1,2)}\tfrac{\partial^2}{\partial x\partial y}) + rI\end{aligned}$$

Discretization of F_t

with $x_{\text{diag}} = \text{diag}(x)$, $y_{\text{diag}} = \text{diag}(y)$, and $\mathbf{I} = \begin{bmatrix} I & \\ & I \end{bmatrix}$, further combining and simplifying produces

$$\mathcal{L} = -r \begin{bmatrix} x_{\text{diag}} \\ y_{\text{diag}} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} x \frac{\partial}{\partial x} & y \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} \Sigma^2_{(1,2)} I & \Sigma^2_{(1,2)} I \\ \Sigma^2_{(2,1)} I & \Sigma^2_{(2,2)} I \end{bmatrix} \begin{bmatrix} x \frac{\partial}{\partial x} \\ y \frac{\partial}{\partial y} \end{bmatrix} + r\mathbf{I}$$

$$\mathcal{L} = -r \begin{bmatrix} x_{\text{diag}} \\ y_{\text{diag}} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} - \frac{1}{2} \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix} \begin{bmatrix} x_{\text{diag}} & \\ & y_{\text{diag}} \end{bmatrix} \begin{bmatrix} \Sigma^2_{(1,2)} I & \Sigma^2_{(1,2)} I \\ \Sigma^2_{(2,1)} I & \Sigma^2_{(2,2)} I \end{bmatrix} \begin{bmatrix} x_{\text{diag}} & \\ & y_{\text{diag}} \end{bmatrix} \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix} + r\mathbf{I}$$

Discretization of F_t

substituting $I^{FC}G = \begin{bmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial y} \end{bmatrix}$ and $DI^{CF} = \begin{bmatrix} \frac{\partial}{\partial x} & \frac{\partial}{\partial y} \end{bmatrix}$

$$\mathcal{L} = -r \begin{bmatrix} x_{\text{diag}} \\ y_{\text{diag}} \end{bmatrix} I^{FC}G - \frac{1}{2} DI^{CF} \begin{bmatrix} x_{\text{diag}} & \\ & y_{\text{diag}} \end{bmatrix} \begin{bmatrix} \Sigma^2_{(1,2)} I & \Sigma^2_{(1,2)} I \\ \Sigma^2_{(2,1)} I & \Sigma^2_{(2,2)} I \end{bmatrix} \begin{bmatrix} x_{\text{diag}} & \\ & y_{\text{diag}} \end{bmatrix} I^{FC}G + r\mathbf{I} \tag{5}$$

Discretization of Time

We use the implicit Backward Differentiation Formula of second-order (BDF2), with A the mimetic difference discrete analog of operator $\mathcal{L}$.

$$AF_{n+1} + b = \frac{F_n - F_{n+1}}{\Delta t} - \frac{1}{2}\frac{F_{n+1} - 2F_n + F_{n-1}}{\Delta t}$$

where b is the Dirichlet far-boundary condition. However, we will ignore b as the boundary condition can just be imposed and saved onto the resulting F_{n+1}.

$$\Delta t A F_{n+1} = F_n - F_{n+1} - \tfrac{1}{2}F_{n+1} + F_n - \tfrac{1}{2}F_{n-1}$$

$$\tfrac{3}{2}F_{n+1} + \Delta t A F_{n+1} = 2F_n - \tfrac{1}{2}F_{n-1}$$

Finally, leaving

$$\left(I + \frac{2}{3}\Delta t A\right) F_{n+1} = -\frac{4}{3}F_n + \frac{1}{3}F_{n-1} \tag{6}$$

Parameters

As used in the paper by Sundvall and Trång the project, I will be using the parameters:

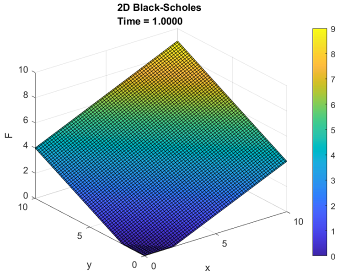
- $r = 0.1$
- $\Sigma = \begin{bmatrix} 0.3 & 0.05 \\ 0.05 & 0.3 \end{bmatrix}$

Being undefined in the paper, I will also be using the parameters:

- $T = 1$
- $K = 1$

Results: Mimetic Differences

Plots of the initial conditions for the entire field and the field of interest $\left(\left[0 \leq x \leq \frac{5K}{3}\right]\right.$ and $\left.\left[0 \leq y \leq \frac{5K}{3}\right]\right)$ are

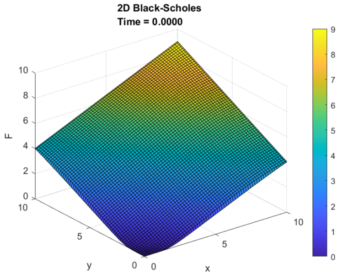


(a) Entire Field (b) Field of Interest

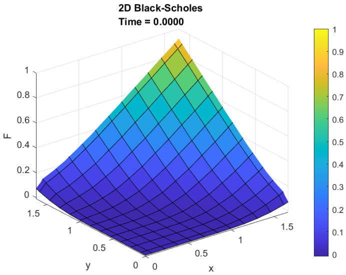
Figure: Initial Conditions: Mimetic Differences

Results: Mimetic Differences

Plots of the results using mimetic differences for the entire field and field of interest are



(a) Entire Field



(b) Field of Interest

Figure: Results: Mimetic Differences

Radial Basis Function - Partition of Unity

Shcherbakov and Larsson use radial basis functions (RBF), specifically the partition of unity (PU) method, to solve the Black-Scholes model in two dimensions for American call options.

- The 2D RBF-PU model is implemented in the MATLAB file $RBFPUMamCallpenalty2D.m$

Modifications to Reflect European Call Options

Since American call options can be exercised at any date before the maturity time, Shcherbakov and Larsson follow the convention of applying a penality parameter, Q ,to the result each timestep in the model.

They also include a parameter, d_i, subtracted from r representing the continuous dividend yield of the ith asset.

- We can just set both of these to zero to have the model represent a European call option with no dividend.

Radial Basis Function - Partition of Unity

Modifications to Compare to 2D Model with Mimetic Differences

The paper and MATLAB file uses a uniform grid for the RBF-PU model so, in order to more easily compare to the Mimetic Differences model, a staggard grid is used instead.

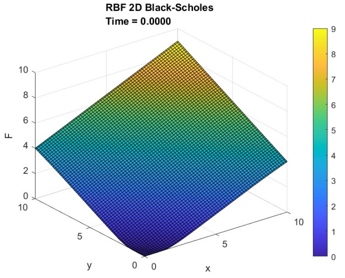
- RBF-PU models are able to work with non-uniform grids so a staggard grid will not impact the results.

The Dirichlet far-boundary conditions in the Mimetic Differences model are also applied to the RBF-PU model at each timestep.

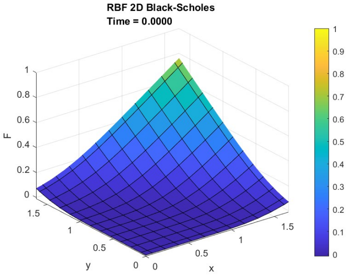
- The close-field boundaries were not able to be implemented which might minimally skew the results.

Results: RBF-PU

Plots of the results for the entire field and the field of interest $\left(\left[0 \leq x \leq \frac{5K}{3}\right] \text{ and } \left[0 \leq y \leq \frac{5K}{3}\right]\right)$ are



(a) Entire Field

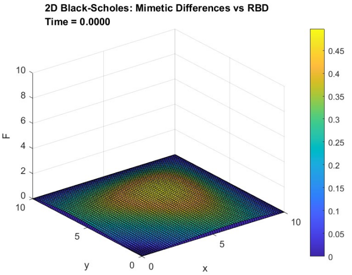


(b) Field of Interest

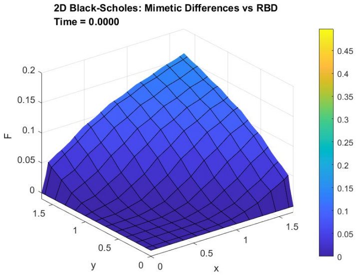
Figure: Results: RBF-PU

Results: Mimetic Differences vs RBF-PU

Plots of the absolute difference between the mimetic differences model and the RBF-PU model for the entire field and field of interest are



(a) Entire Field



(b) Field of Interest

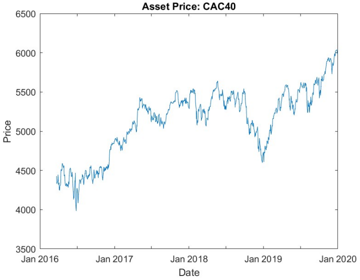
Figure: Results: Mimetic Differences vs RBF-PU

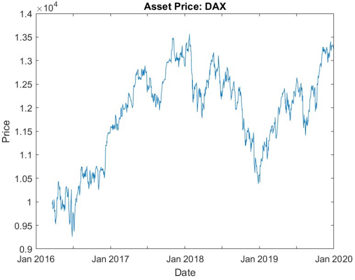
Comparing to Real World Data: Background

Knowing the value of a real call option at each possible price combination of assets is unfeasible.

- By analyzing historical price data, we can estimate the parameters needed to input into our mimetic difference model.

The two assets used to compare to the 2D mimetic difference model are the daily index price data of CAC40 (France) and of the DAX (Germany) from March 2016 to December 2019.





Comparing to Real World Data: Background

To normalize the data, the daily log returns were used instead of nominal prices.
The daily log returns were calculated by

$$r_t = \log\left(\frac{P_t}{P_{t-1}}\right)$$

- where P_t is the price of the index at time, t, with $t-1$ being the trading day beforehand

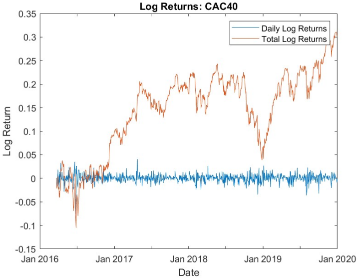
Total log returns were calculated by

$$r_t = \log\left(\frac{P_t}{P_0}\right)$$

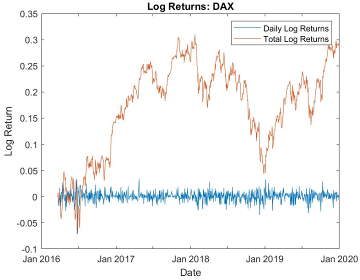
- where P_0 is the initial price of the index.

Comparing to Real World Data: Background

The daily and total log returns for both indexes are displayed below



(a) CAC40



(b) DAX

Figure: Daily and Total Log Returns (March 2016 to December 2019)

Comparing to Real World Data: Assumed Parameters

The following parameters will be used for the

- $r = 0.1$ (assumed risk-free interest rate)
- $T = 3.7728$ (March 2016 to December 2019 in terms of years)
- $K = 1$

Comparing to Real World Data: Volatility and Correlation Coefficients

The correlation volatility matrix is comprised of volatility coefficients (σ) for each asset and a correlation coefficient (ρ) relating both assets.

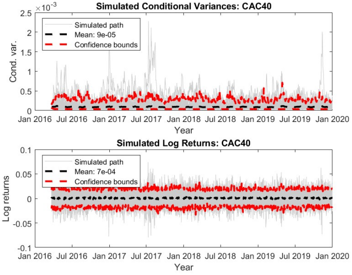
$$\Sigma = \begin{bmatrix} \Sigma_{(1,1)} = \sqrt{\sigma_1 \sigma_1 \rho_{11}} & \Sigma_{(1,2)} = \sqrt{\sigma_1 \sigma_2 \rho_{12}} \\ \Sigma_{(2,1)} = \sqrt{\sigma_2 \sigma_1 \rho_{21}} & \Sigma_{(2,2)} = \sqrt{\sigma_2 \sigma_2 \rho_{22}} \end{bmatrix},$$

$$\Sigma_{(1,2)} = \Sigma_{(2,1)}, \rho_{11} = \rho_{22} = 1, \rho_{12} = \rho_{21}$$

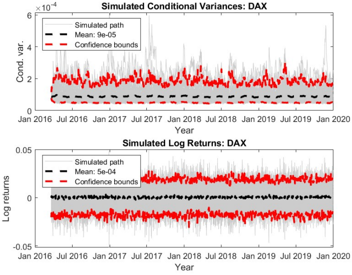
To calculate volatility, the multivariate GARCH method will be used.

Comparing to Real World Data: Volatility and GARCH

To calculate volatility, the multivariate GARCH method is used. The GARCH method uses a ordinary least squares analysis of the daily log returns and runs multiple simulations to produce an approximation for the daily volatility σ^* for each asset.



(a) CAC40



(b) DAX

Comparing to Real World Data: Volatility and GARCH

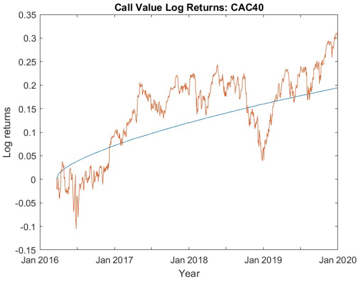
To ease calculation, we will assume that both volatility and correlation will remain constant over time. Thus, we can take the mean of the daily conditional volatility.

Note that after taking the mean daily conditional volatility, we will need to adjust for the length of time the model will be running by multiplying by the square root of the number of trading days in that time period. There are about 252 trading days per year.

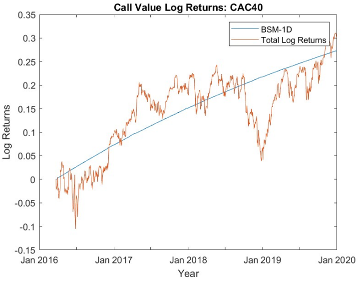
$$\sigma_i = \sigma_i^* \sqrt{252T}$$

Comparing to Real World Data: Volatility and GARCH

After getting the value of the mean daily volatility, we can check its accuracy by plotting a simulated 1D Black-Scholes model using the σ value against the total log returns shown below



(c) CAC40



(d) DAX

Comparing to Real World Data: Correlation and Final Σ

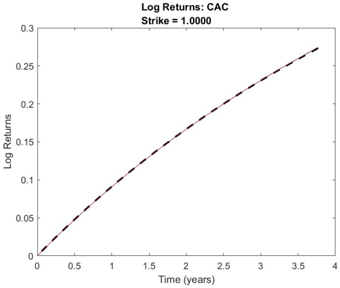
Correlation is just the correlation between DAX and CAC40's log returns. In our case, this ended up being quite close to 1. Combining together, we get

$$\Sigma = \begin{bmatrix} 0.0029 & 0.0027 \\ 0.0027 & 0.0027 \end{bmatrix}$$

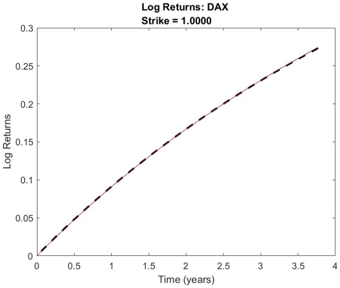
Using the parameters and same initial conditions, we can use the estimated Σ to put into the 2D Black-Scholes model. We can then compare the Mimetic Differences model result to the simulated 1D version of the Black-Scholes model for each asset.

Comparing to Real World Data: Results

I have chosen to compare the points where the initial condition $\Phi(x, y) = 0$ rather than expanding the field as we are only comparing the log returns. The values end up being rather normalized.



(e) CAC40



(f) DAX

Figure: Call Option Value (Black) vs Mimetic Difference Model: CAC40 and DAX

ND Black Scholes Model

The code, $BlackScholesNDMD_{I}mplicit_{O}rder2.m$, is a template to expand the boundary-less Black-Scholes model using mimetic methods.
The only missing parts are the flexible mimetic operators that work for multiple dimensions and the boundary conditions, which can be expanded upon in the future.

Reflection

- The Black-Scholes PDE is quite versatile in the ways it can be approximated.
- In future updates to this project, I would recommend the exploration of different boundary conditions on the PDE, the creation of a better means of comparison to real world data, and the addition of the N-D mimetic operators to the N-D Black-Scholes model.

I am grateful to my committee for their patience with this project, as well as, my friends and family for their love and support throughout graduate school.

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