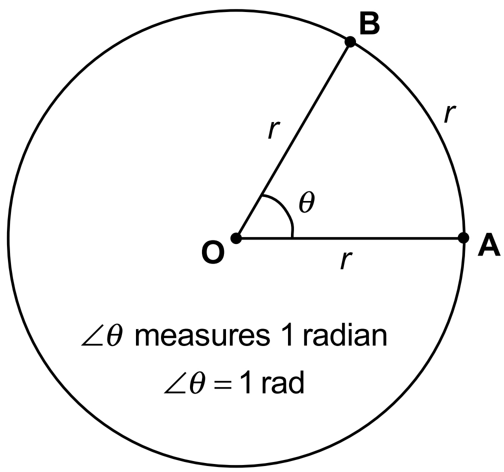
Pre-Calculus 12

UNIT 4 - NOTES

4.1 - Angles and Angle Measure

One **RADIAN** is the measure of the angle formed by rotating the radius of a circle through an arc equal in length to the radius.



$\angle\theta$ measures 1 radian

$\angle\theta = 1$ rad

One full rotation is $360°$ or 2π radians.

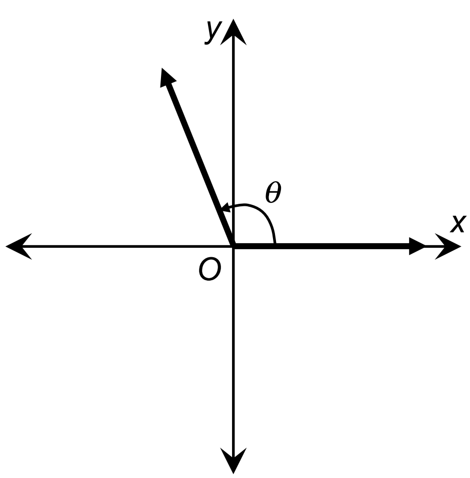
One half rotation is $180°$ or π radians.

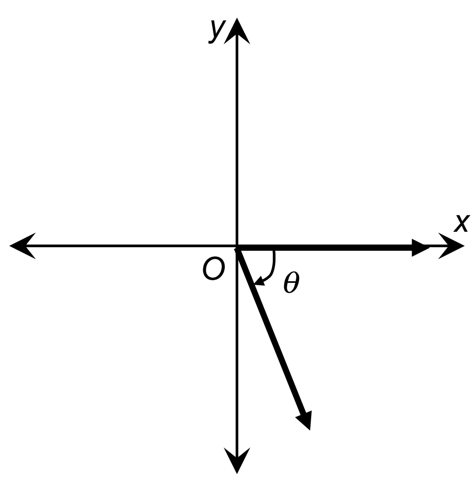
One quarter rotation is $90°$ or $\frac{\pi}{2}$ radians.

One eighth rotation is $45°$ or $\frac{\pi}{4}$ radians.

* any angle measure given without a degree symbol $(°)$ is assumed to be in radians.

Angles in Standard Position





By convention, angles measured in a **Counter-Clock-Wise** direction are **Positive**.

Those measured in a **Clock-Wise** direction are **Negative**.

Converting Between Degrees & Radians $\left(180^\circ = \pi \text{rad}\right)$

We can convert between degrees & radians by writing and solving proportions.

<u>Example 1</u>: Convert degrees to radians and radians to degrees.

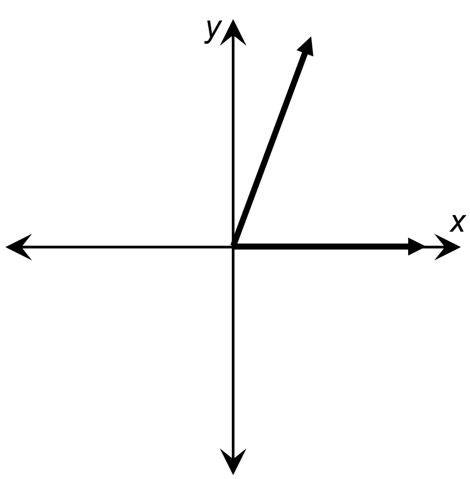
a) 30°

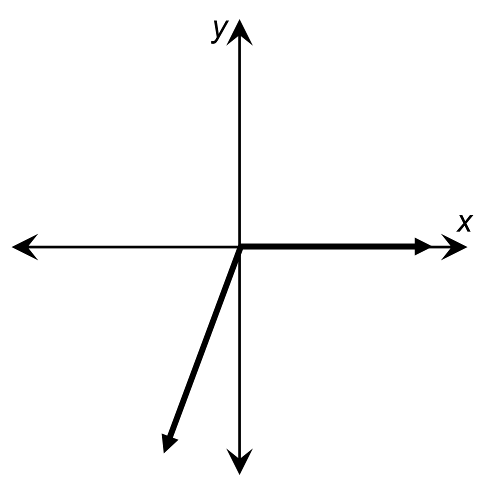
b) -120°

c) $\frac{5\pi}{4}$rad

d) 2.57rad

Coterminal Angles - angles in standard position with the same terminal arms.





Coterminal Angles in General Form

By adding or subtracting multiples of one full rotation (degrees or radians), we can write an infinite number of angles that are coterminal with any given angle.

Angles coterminal with any angle θ can be described using the expressions:

Degrees	**Radians**	
$\theta \pm (360°)n$	$\theta \pm (2\pi)n$	where $n \in \mathbb{N}$

Example 2: Identify angles coterminal with $110°$ that satisfy the domain $-720° \le \theta \le 720°$

n	**1**	**2**	**3**	**4**
$110° + (360°)n$				
$110° - (360°)n$				

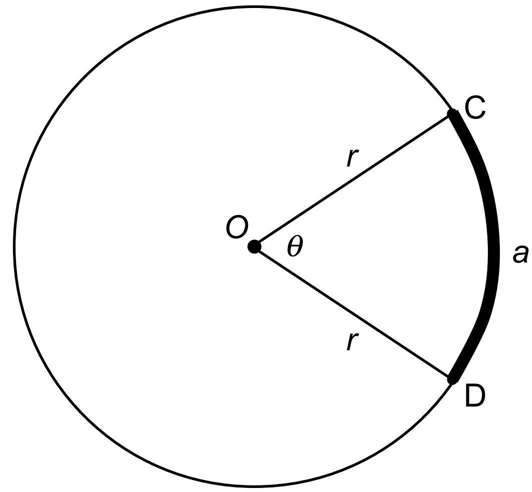
Example 3: Identify angles coterminal with $\frac{8\pi}{3}$ that satisfy the domain $-4\pi \le \theta \le 4\pi$

n	**1**	**2**	**3**	**4**
$\frac{8\pi}{3} + (2\pi)n$				
$\frac{8\pi}{3} - (2\pi)n$				

Arc Length of a Circle

An arc is formed by the endpoints of 2 radii on a circle

The length of an arc in a circle is proportional to the central angle formed by the two radii.



In Radians:

$$a = \frac{\theta}{2\pi}(2\pi r)$$

$$\boxed{a = \theta r}$$

as long as ***a*** and ***r*** are measured in the same units.

<u>Example 4</u>: If a represents the length of an arc in a circle with radius r, subtended by central angle θ, determine the missing quantities.

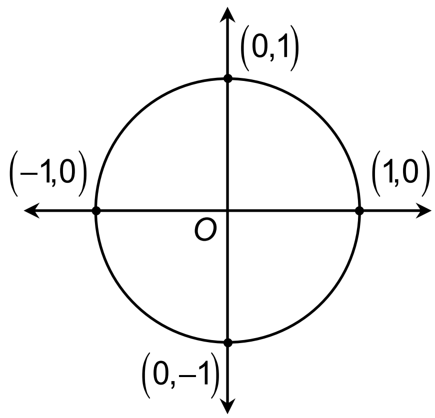
a) $r = 8.7\text{cm}$, $\theta = 75°$, $a =$ _____ cm

b) $r =$ _____ cm, $\theta = 1.8$, $a = 4.7\text{cm}$

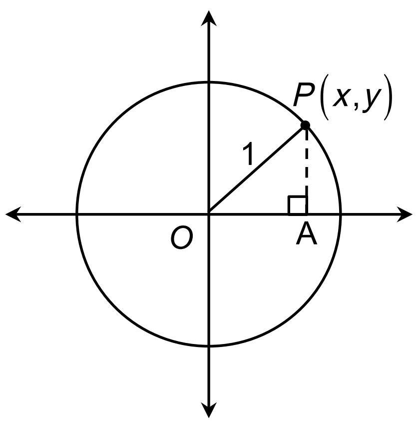
c) $r = 5$ m, $\theta =$ _____, $a = 13$ m

4.2 - The Unit Circle

The Unit Circle is a circle of radius 1 unit with center at the origin of the Cartesian plane.



Using the Pythagorean Theorem, we can write the equation for the Unit Circle:



$OP = 1$

$OA = x$

$PA = y$

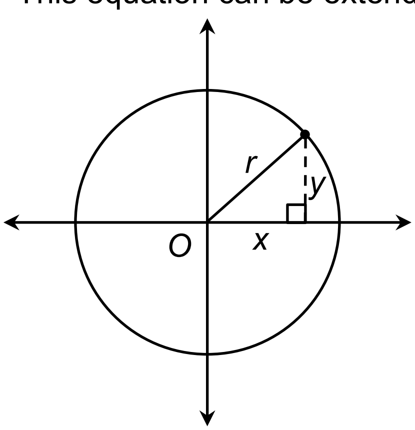
$\therefore$

$(x)^2 + (y)^2 = (1)^2$

or

$x^2 + y^2 = 1$

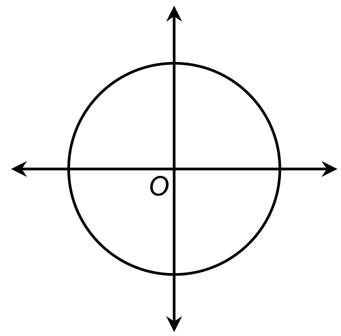
This equation can be extended to any circle, as long as it is centered at the origin:



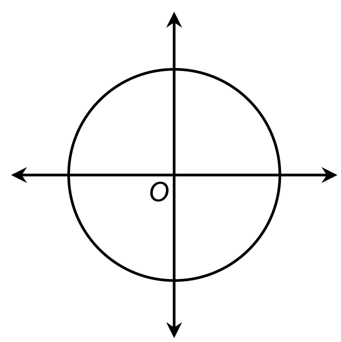
$x^2 + y^2 = r^2$

Example 1: Determine the coordinates for all points on the unit circle that satisfy the given conditions. Draw a diagram for each case.

a) The x-coordinate is $\frac{2}{3}$.



b) The y-coordinate is $-\frac{1}{\sqrt{2}}$ and the point is in Q-3.

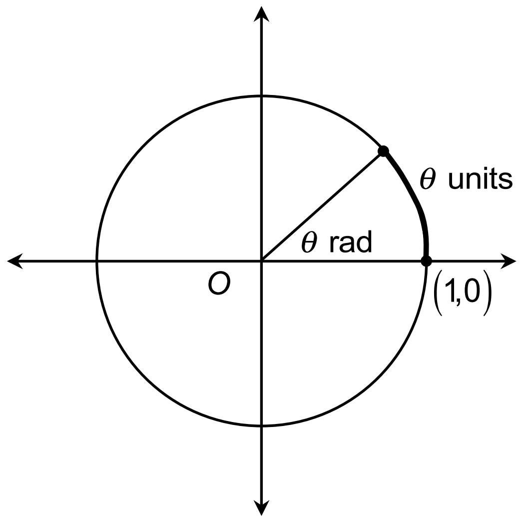


Arc Length and Angle Measures

The formula $a = \theta r$ (where *a* is the arc length, θ is the central angle in radians, and *r* is the radius), applies to any circle, as long as *a* and *r* are measured in the same units.

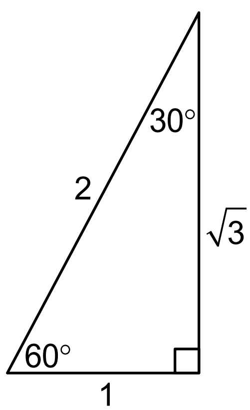
In the Unit Circle, the formula becomes $a = \theta(1)$ or $a = \theta$.

This means that a central angle and its subtended arc on the unit circle have the same numerical value.

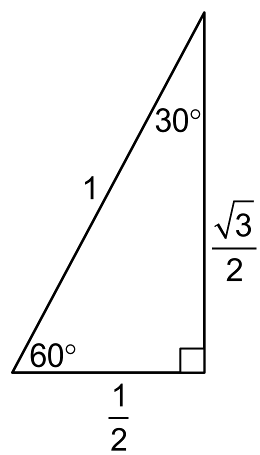


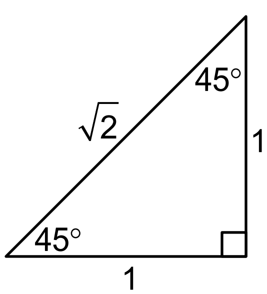
Special Triangles

Recall the $30° - 60° - 90°$ and $45° - 45° - 90°$ triangles.

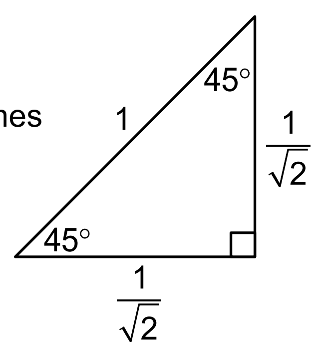


on the Unit Circle, becomes

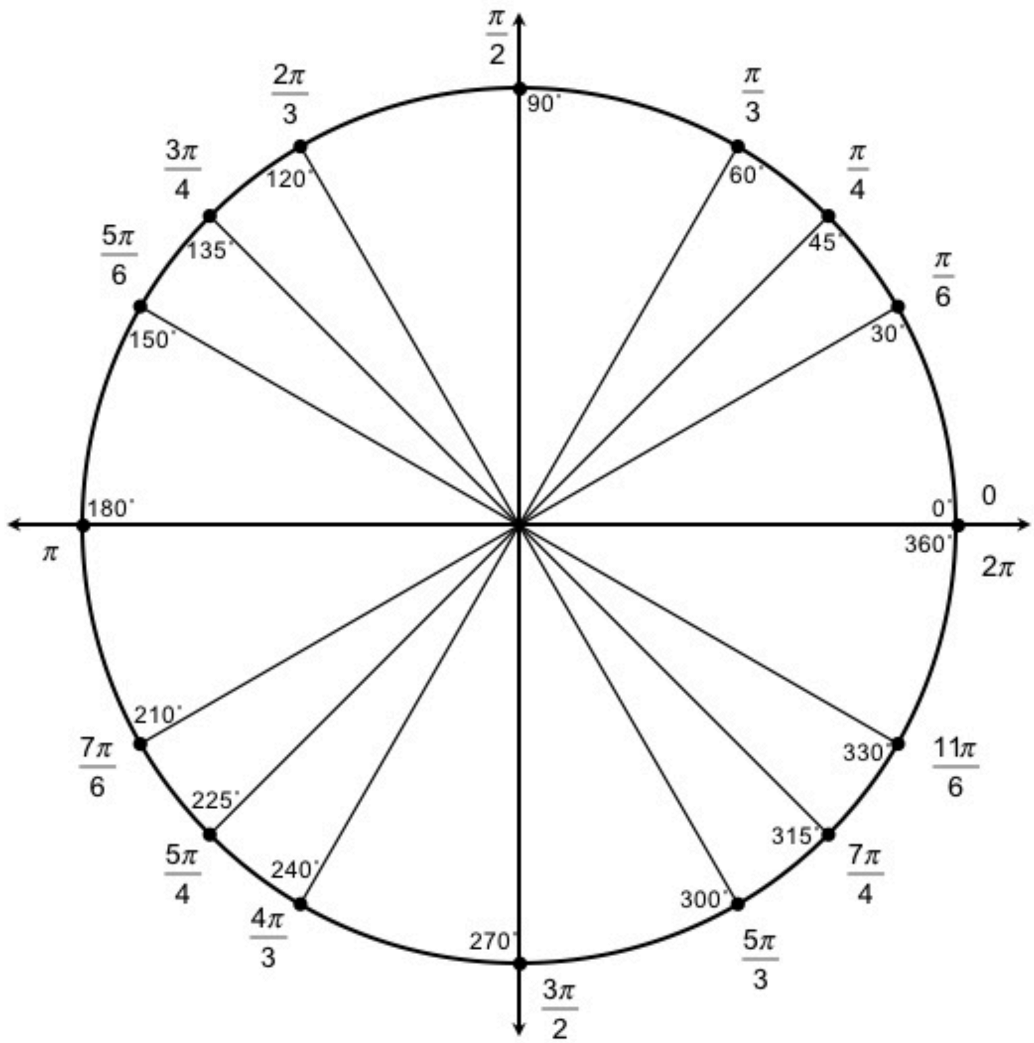




on the Unit Circle, becomes



Common Angle Measures on the Unit Circle (Degrees and Radians)

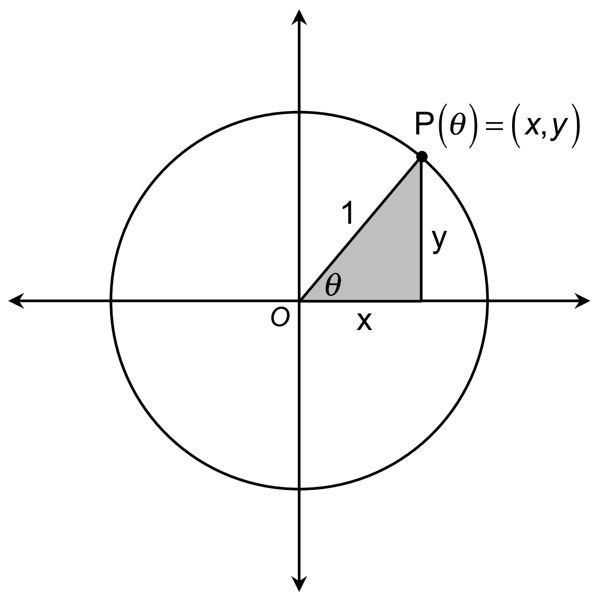


Label the coordinates for each point (hint – use the ‘special’ triangles, if necessary)

4.3 – Trigonometric Ratios (part 1)

Primary Trigonometric Ratios

If $P(\theta) = (x, y)$ is the point on the terminal arm of angle θ that intersects the unit circle, then



$$\sin\theta = \frac{y}{1} = y$$

$$\cos\theta = \frac{x}{1} = x$$

$$\tan\theta = \frac{y}{x}$$

Reciprocal Trigonometric Ratios

Three other trigonometric ratios are defined - the reciprocals of sine, cosine and tangent:

COSECANT (csc) is the reciprocal of the SINE ratio.

For $P(\theta) = (x, y)$ on the unit circle, $\csc\theta = \frac{1}{y}$.

SECANT (sec) is the reciprocal of the COSINE ratio.

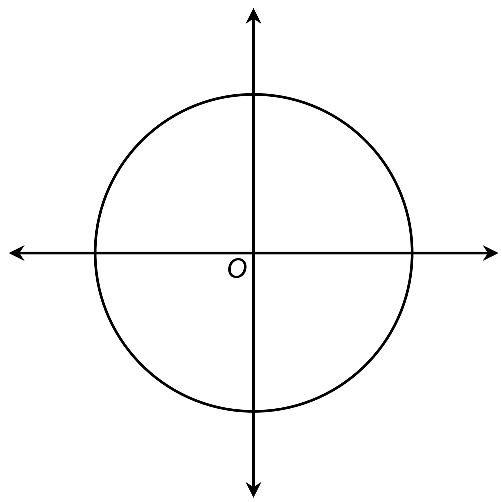
For $P(\theta) = (x, y)$ on the unit circle, $\sec\theta = \frac{1}{x}$.

COTANGENT (cot) is the reciprocal of the TANGENT ratio.

For $P(\theta) = (x, y)$ on the unit circle, $\cot\theta = \frac{x}{y}$.

(if $\tan\theta = 0$, $\cot\theta$ is undefined)

Example 1: The point $A\left(-\frac{3}{5},-\frac{4}{5}\right)$ lies at the intersection of the unit circle and the terminal arm of angle θ in standard position. Draw a diagram to model the situation, **AND** determine the values of the six trigonometric ratios for angle θ.



$\sin\theta = \boxed{}$ $\csc\theta = \boxed{}$

$\cos\theta = \boxed{}$ $\sec\theta = \boxed{}$

$\tan\theta = \boxed{}$ $\cot\theta = \boxed{}$

Example 2: Determine the exact value for each trigonometric ratio.

a) $\sin 135°$

b) $\sec\frac{4\pi}{3}$

4.3 – Trigonometric Ratios (part 2)

EXACT values for trig ratios can be determined using special triangles (30º - 60º - 90º or 45º - 45º - 90º), and multiples of $\theta = 0, \frac{\pi}{2}, \frac{\pi}{3}, \frac{\pi}{4}, \frac{\pi}{6}$ or $\theta = 0°, 30°, 45°, 60°, \& \ 90°$ for points $\mathrm{P}(\theta)$ on the unit circle.

Example 1: Determine the exact value of each trig ratio. Use diagrams to illustrate your solution.

a) $\cos 150°$

b) $\sec \frac{7\pi}{4}$

APPROXIMATE values for trig ratios can be determined using a scientific calculator. Most calculators can determine values for trig ratios in either degrees or radians, but you will need to set the mode to the correct angle measure.

For example:

$\sin 45° = 0.7071067812...$ (in degree mode)

$\sin \frac{\pi}{4} = 0.7071067812...$ (in radian mode)

Most calculators can compute trig ratios for negative angles, **however**, you should use your knowledge of reference angles and the signs of trig ratios to check that the calculator's answer is reasonable.

Finding Angles

Given the value of a trig ratio, you can determine the measure of an angle using an inverse trig function on your calculator.

For example:

$\sin 30° = 0.5 \quad \Rightarrow \quad \sin^{-1} 0.5 = 30°$

Note that $\sin^{-1} 30°$ is an abbreviation for "the inverse of sine" **NOT** to be confused with $\left(\sin 30°\right)^{-1}$ which represents "the reciprocal of $\sin 30°$" or $\frac{1}{\sin 30°}$ or $\csc 30°$.

Using an inverse trig function on a calculator will only return one solution, when there are often two angles with the same trig function value in any full rotation. It is generally safest to use the reference angle, θ_R, in the appropriate quadrant(s) containing the terminal arm of the angle.

Example 2: Determine the measures of all angles that satisfy the given conditions.

a) $\sin\theta = 0.879$ in the domain $0 \le \theta < 2\pi$ (to the nearest tenth)

b) $\cos\theta = -0.366$ in the domain $0° \le \theta < 360°$ (to the nearest tenth)

c) $\tan\theta = \sqrt{3}$ in the domain $-180^\circ \le \theta < 180^\circ$ (exact solution)

d) $\sec\theta = \dfrac{2}{\sqrt{3}}$ in the domain $-2\pi \le \theta < 2\pi$ (exact solution)

4.4 – Introduction to Trigonometric Equations

A trigonometric equation involves a trigonometric ratio. We can utilize many of the algebraic strategies and techniques used to solve linear and quadratic equations, to solve trigonometric equations.

Example 1: Solve each equation in the specified domain. Determine exact solutions where possible. Otherwise give approximate angle measures to the nearest hundredth of a degree or radian.

a) $2\cos\theta + 1 = 0,\ 0^\circ \le \theta < 360^\circ$

b) $5\sin x + 2 = 1 + 3\sin x,\ 0 \le x < 2\pi$

c) $\tan^2\theta - 5\tan\theta + 4 = 0,\ [0,\ 2\pi)$

note: $\tan^2\theta \Leftarrow \quad \Rightarrow (\tan\theta)^2$ and $[0,\ 2\pi) \Leftarrow \quad \Rightarrow 0 \le \theta < 2\pi$

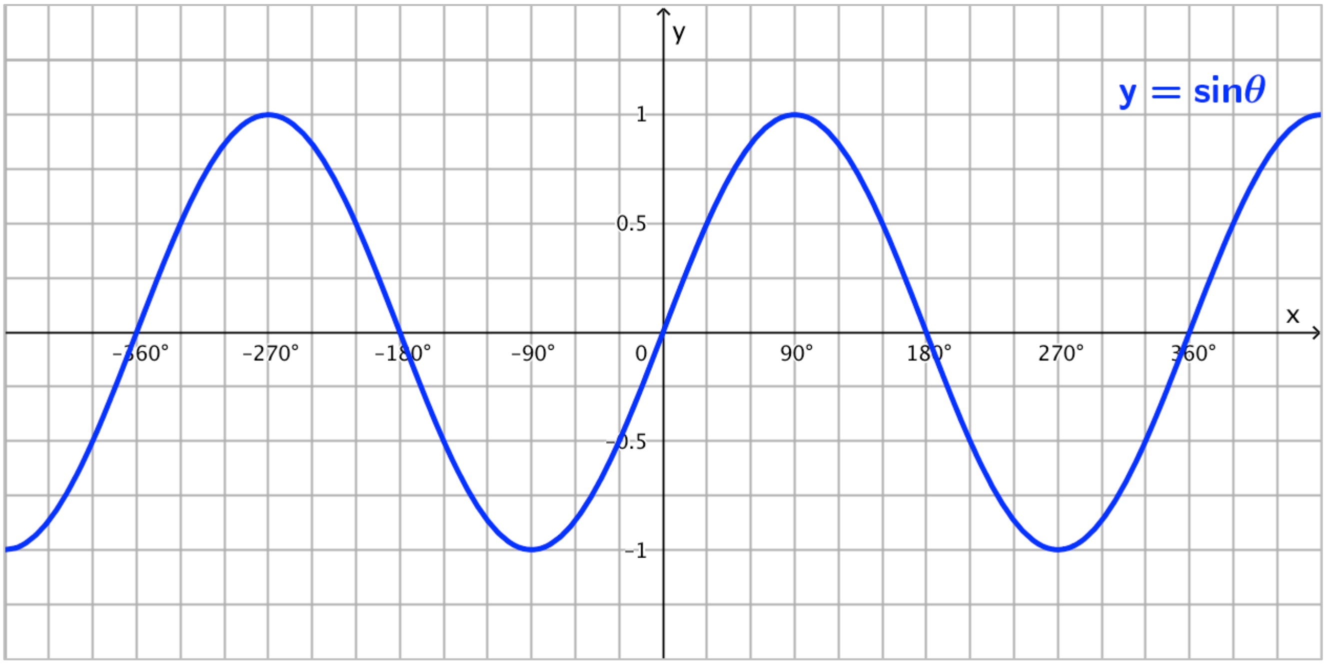
The domain of trigonometric equations is typically defined for one positive rotation. If the domain is not specified, or is defined for all Real numbers, there will be infinite solutions to the equation (think coterminal angles). The **General Solution** to a trigonometric equation represents all possible solutions to the equation.

Example 2: Determine the general solution to $\cos^2 x - 1 = 0$ where the domain is real numbers measured in degrees.

5.1 – Graphing Sine and Cosine Functions

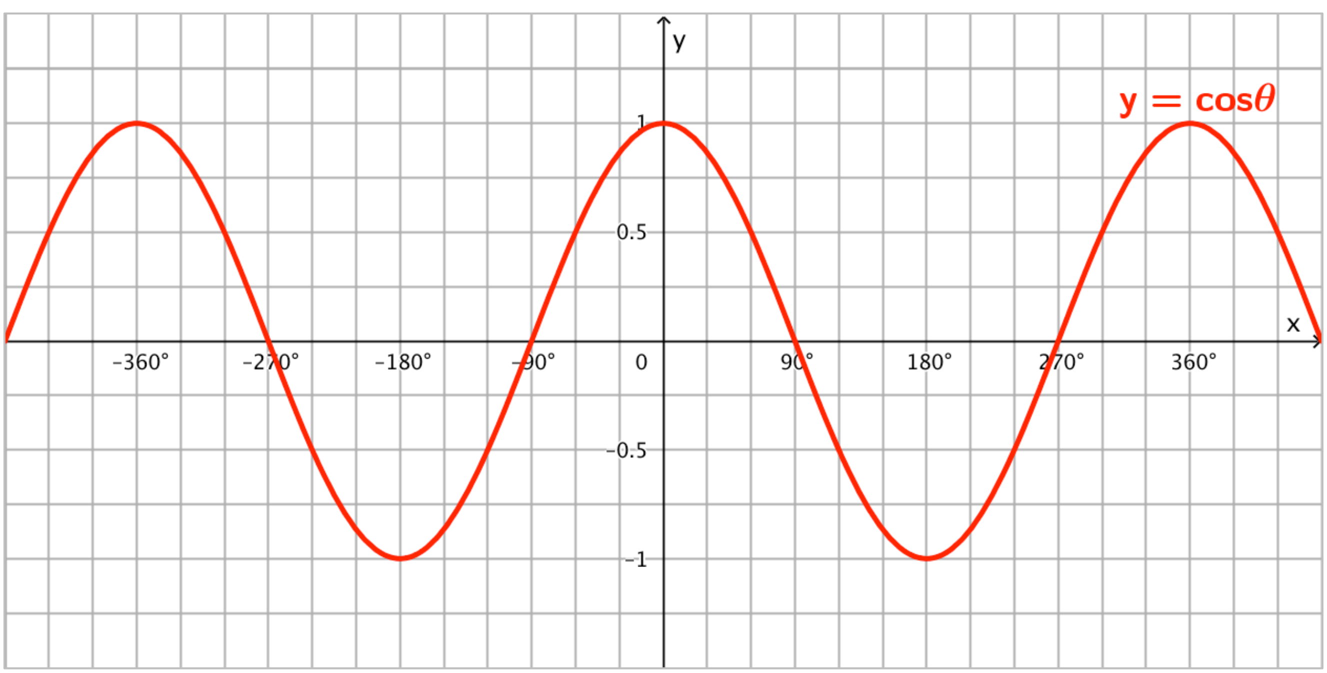
Sine and Cosine are **PERIODIC FUNCTIONS**. The values of these functions repeat over a specified **PERIOD**.

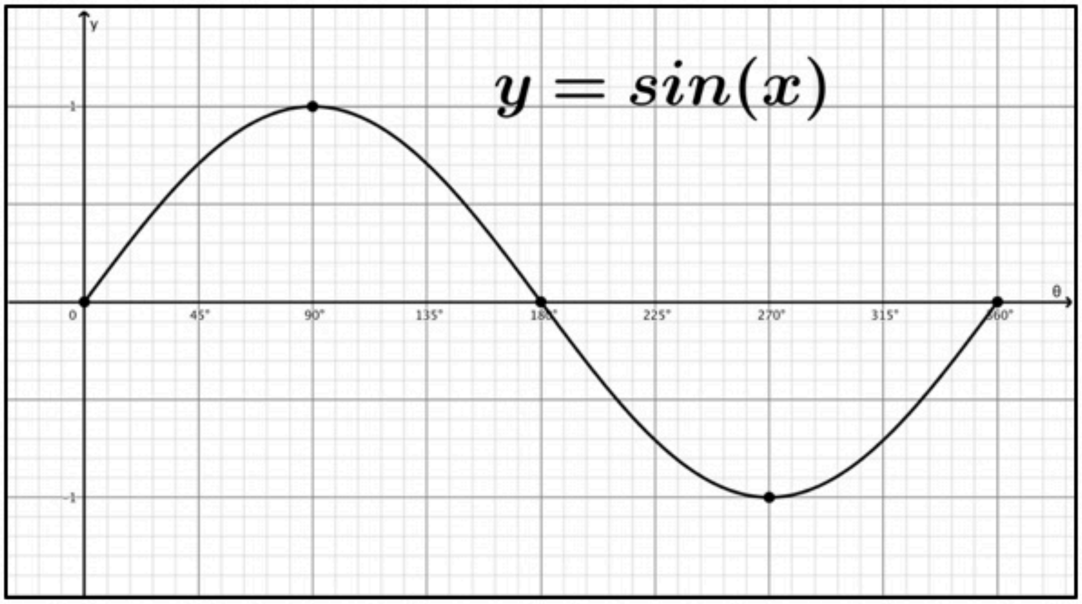
A sine graph is the graph of the function $y = \sin\theta$. The sine graph is described as a **SINUSOIDAL CURVE**.



Period - the length of the interval of the domain over which a graph repeats itself.

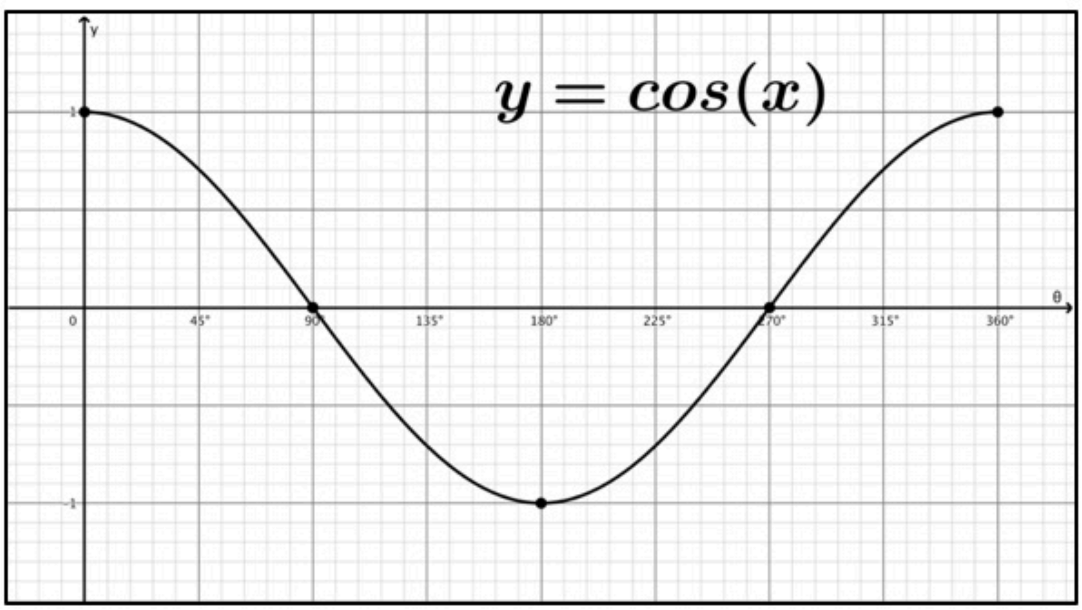
Amplitude - the maximum vertical distance the graph of a sinusoidal function varies above and below the central axis of the curve.





KEY POINTS

θ	*y*
0˚	0
90˚	1
180˚	0
270˚	-1
360˚	0

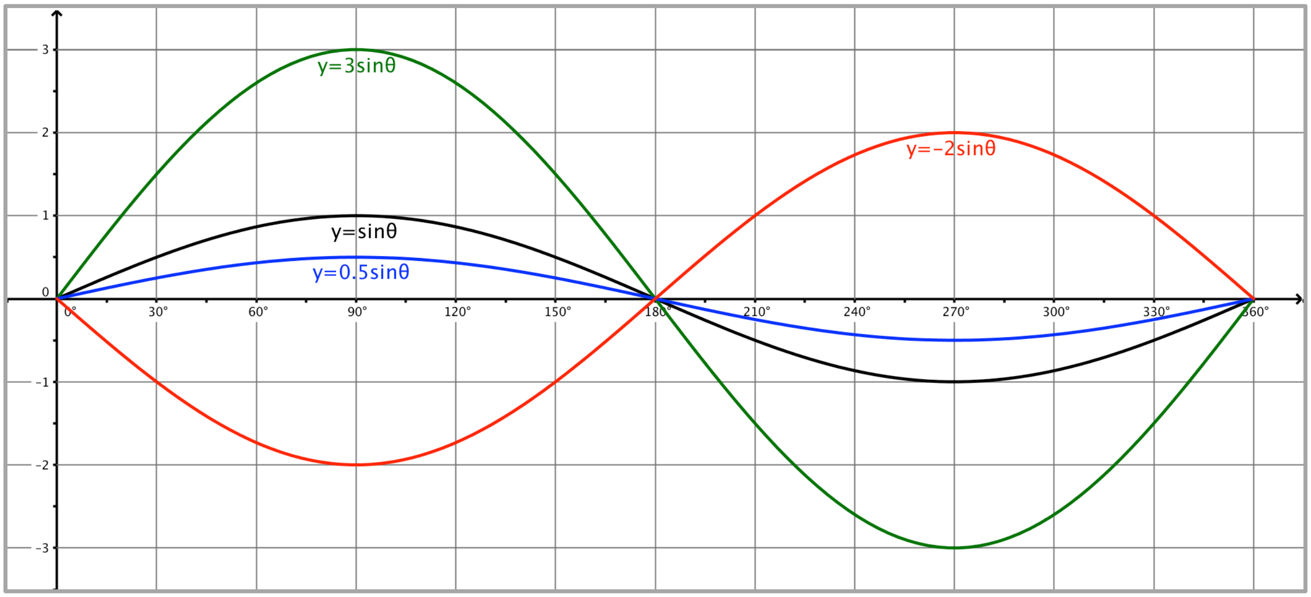


KEY POINTS

θ	*y*
0˚	1
90˚	0
180˚	-1
270˚	0
360˚	1

AMPLITUDE of Sine or Cosine Functions

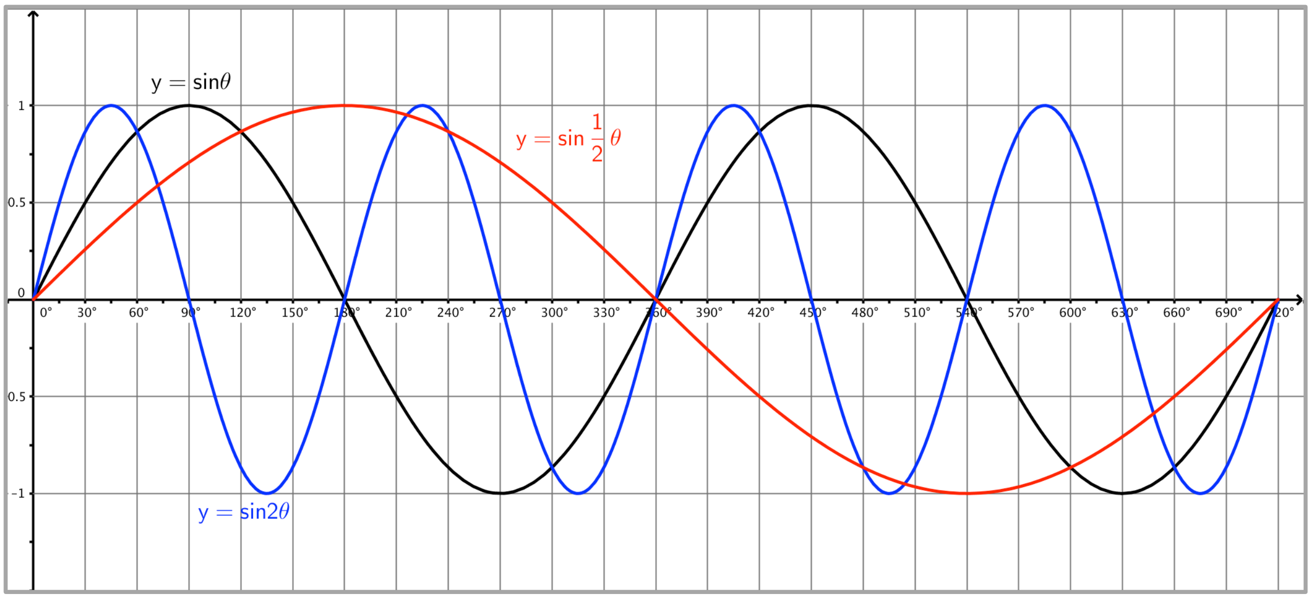
Any function in the form $y = af(x)$ is related to $y = f(x)$ by a vertical stretch by a factor $|a|$ about the *x*-axis. If $a < 0$, the function is also reflected in the *x*-axis.



Changing the value of *a* affects the **amplitude** of a sinusoidal function. For the functions $y = a\sin\theta$ or $y = a\cos\theta$, the amplitude is $|a|$ .or $\text{Amplitude} = \dfrac{\text{max. value} - \text{min. value}}{2}$.

PERIOD of Sine or Cosine Functions

Any function in the form $y = f(bx)$ is related to $y = f(x)$ by a horizontal stretch by a factor of $\frac{1}{|b|}$ about the *y*-axis. If $b < 0$, the function is also reflected in the *y*-axis.



Changing the value of *b* affects the **period** of a sinusoidal function. For the functions $y = \sin b\theta$ or $y = \cos b\theta$, the period can be found using the formula:

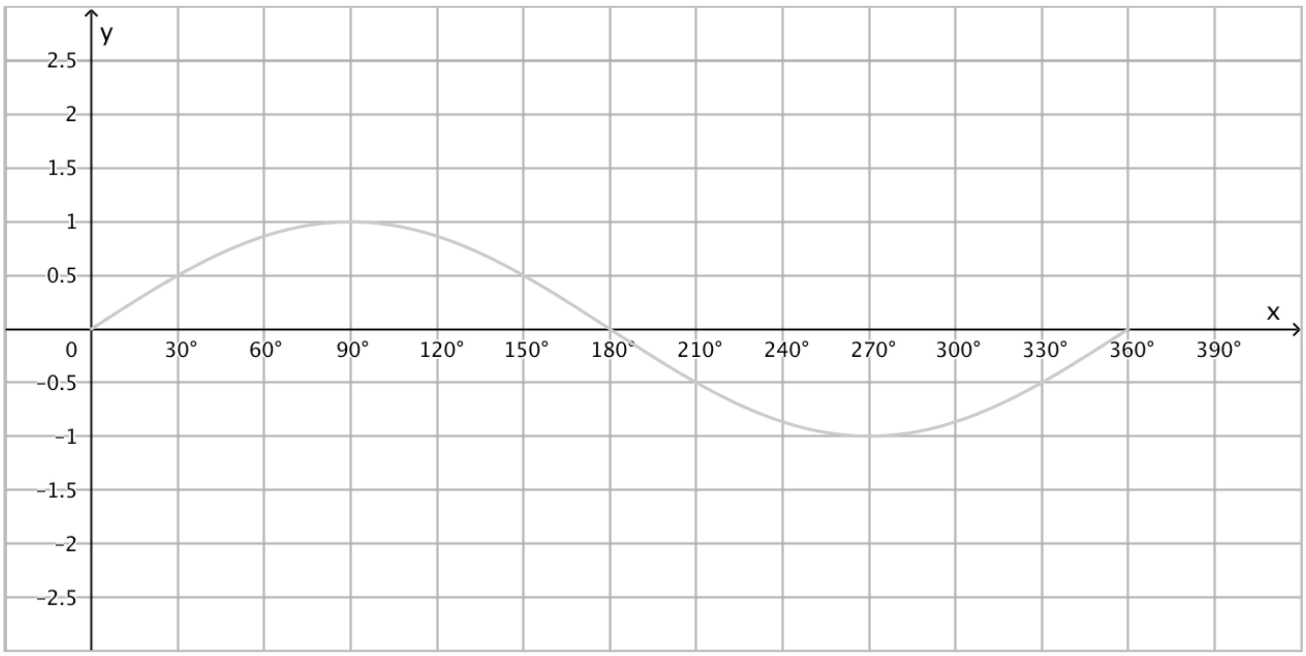
$$\text{Period} = \frac{360^\circ}{|b|} \qquad \text{or} \qquad \text{Period} = \frac{2\pi}{|b|}$$

Example 1:

a) graph $y = 2\sin 3x$, showing at least two cycles.

b) Determine:

- the amplitude
- the period
- the maximum and minimum value
- the x-intercepts and y-intercept
- the domain and range



5.2 – Transformations of Sinusoidal Functions

Like we've seen with other types of functions, $f(x) = \sin x$ and $f(x) = \sin x + d$ are related by a vertical translation of *d*-units. With a sinusoidal function, we call this a:

VERTICAL DISPLACEMENT

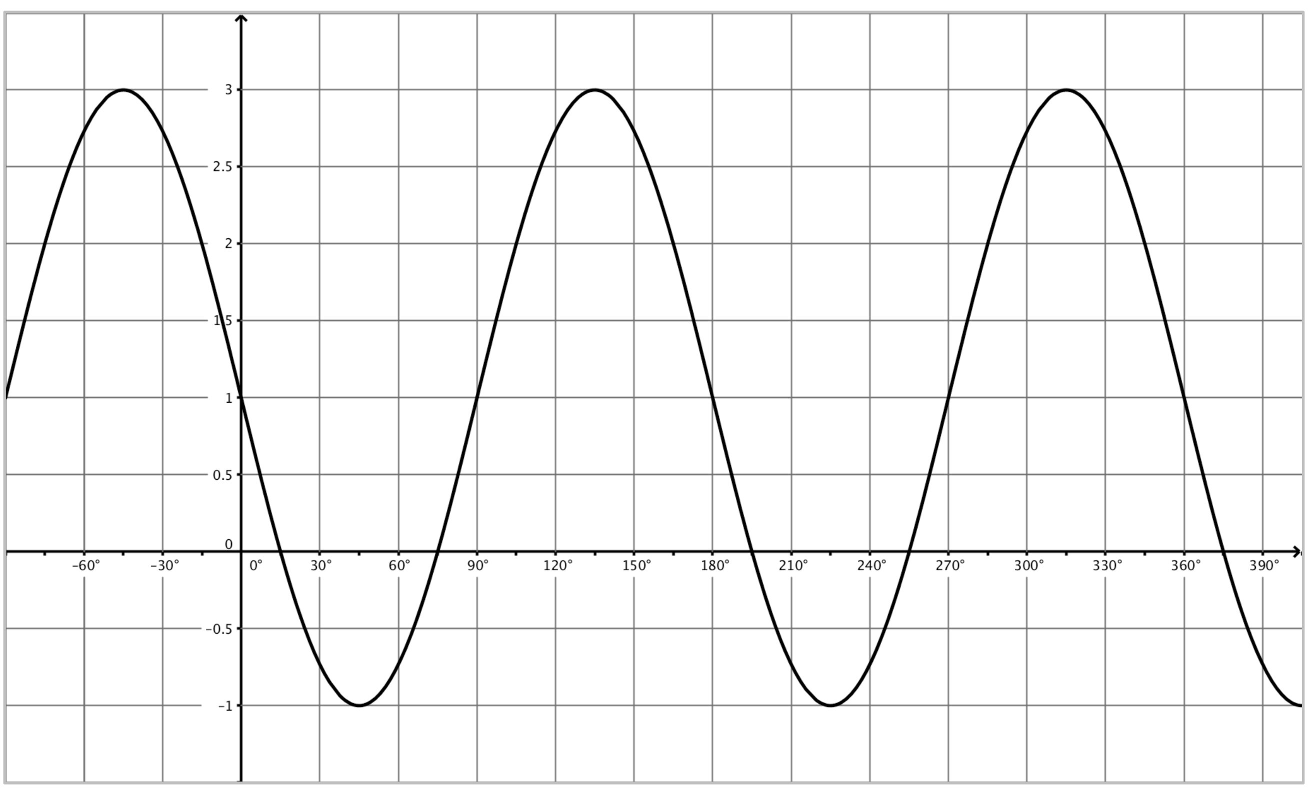
d > 0 shifts UP *d*-units / *d* < 0 shifts DOWN *d*-units

As well, $f(x) = \cos x$ and $f(x) = \cos(x - c)$ are related by a horizontal translation of *c* degrees or radians. With a sinusoidal function, we call this a:

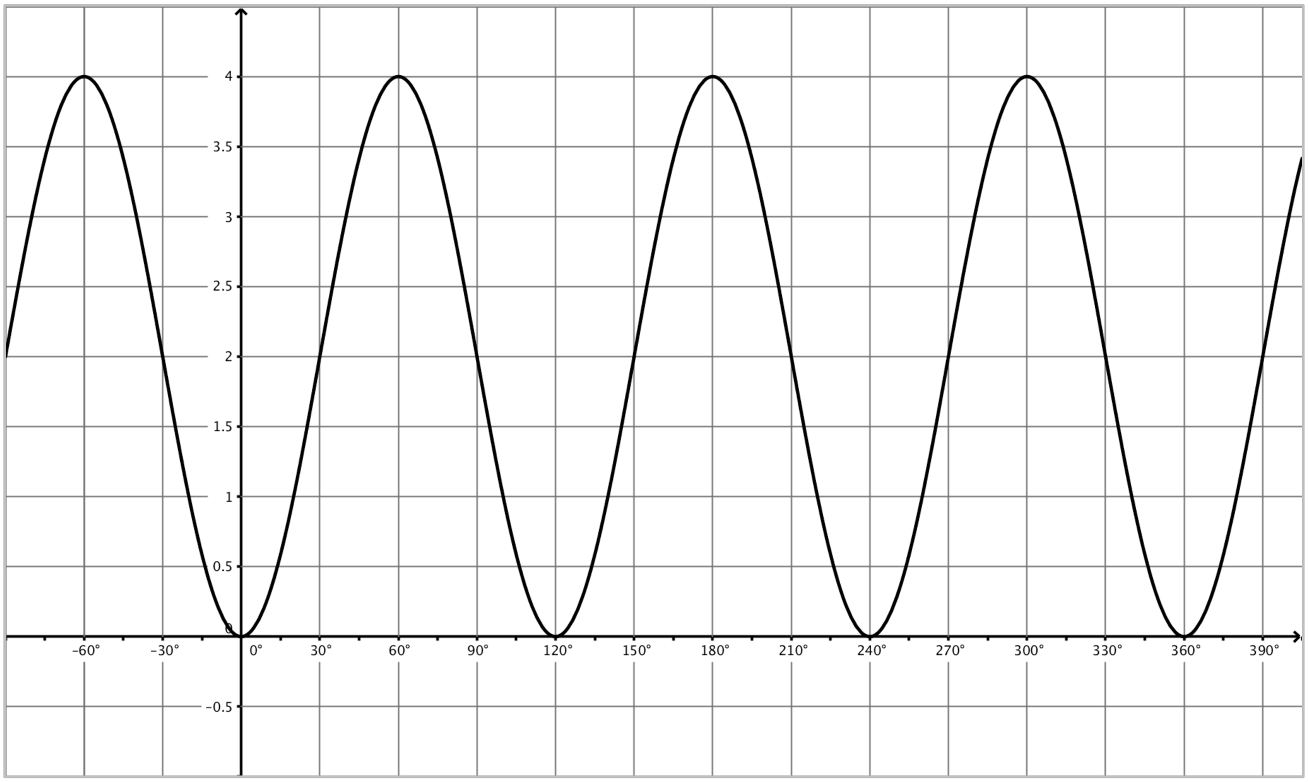
PHASE SHIFT

(*x* - *c*) shifts RIGHT *c*-units / (*x* + *c*) shifts LEFT *c*-units

We can determine the equation of a sinusoidal function in the form $y = a\sin b(x - c) + d$ or $y = a\cos b(x - c) + d$, given its properties or its graph.



Example 1: The graph shows the function $y = f(x)$.



a) Write the equation of the function in the form: $y = a\sin b(x-c)+d,\ a>0$

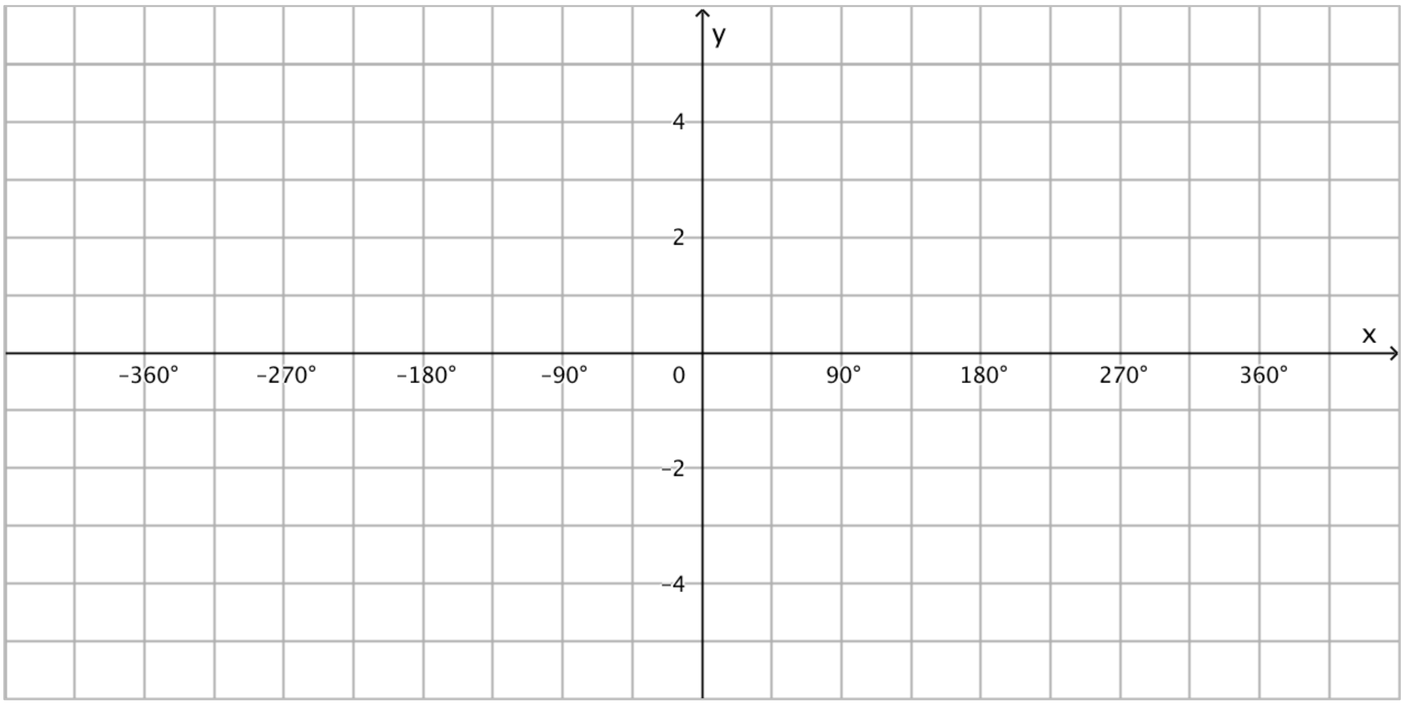
b) Write the equation of the function in the form: $y = a\cos b(x-c)+d,\ a>0$

<u>Example 2</u>: Sketch the graph of the function $y = 3\sin\left(2x - \frac{2\pi}{3}\right) + 2$

5.3 – The Tangent Function

The graph of $y = \tan\theta$ is unlike the graphs of $y = \sin\theta$ or $y = \cos\theta$.

θ	-360°	-315°	-270°	-225°	-180°	-135°	-90°	-45°	0°	45°	90°	135°	180°	225°	270°	315°	360°
$\tan\theta$																	



- as you can see from the graph above, $y = \tan\theta$ is not a continuous graph.

- there are breaks in the graph at regular intervals, where $y = \tan\theta$ is undefined.
  Vertical Asymptotes at $\theta = \ldots, -270°, -90°,\ 90°,\ 270, \ldots$

- the period of $y = \tan\theta$ is $180°$.

- the graph of $y = \tan\theta$ has no amplitude, since it has no maximum or minimum value.

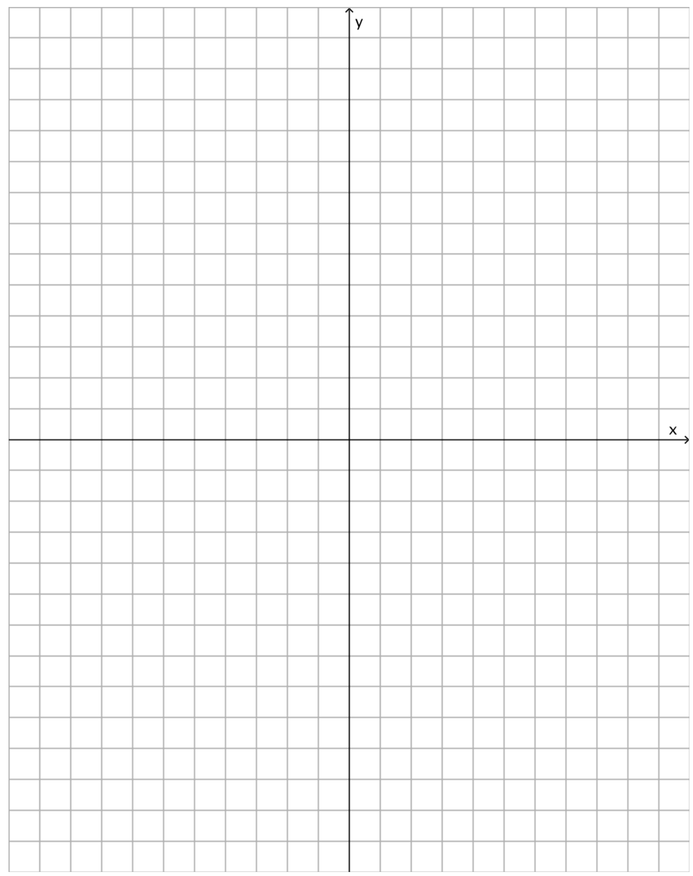
- the range of $y = \tan\theta$ is $\{y \mid y \in \mathbb{R}\}$.

- the domain of $y = \tan\theta$ is $\{\theta \mid \theta \neq 90° \pm 180°n,\ \theta \in \mathbb{R},\ n \in \mathbb{N}\}$.

Example 1: A small plane is flying at a constant altitude of 6000 m directly towards an observer on the ground.

a) sketch a diagram to model the situation and determine the relation between the horizontal distance, in meters, from the observer to the plane and the angle, in degrees, formed from the vertical to the plane.

b) Sketch a graph of the function.



c) Where are the asymptotes located in this graph. What do they represent?

d) What happens when the angle is equal to 0° ?

5.4 – Equations and Graphs of Trigonometric Functions

We can use the graphs of trigonometric functions to solve trigonometric equations:

Example 1: Solve $2\cos^2\theta - 1 = 0$ for the interval $0° \le \theta < 360°$

Option 1 - graph $y = 2\cos^2\theta - 1$ and find the zeros/roots (x-intercepts).

Option 2 - graph $y = \cos^2 x$ & $y = \frac{1}{2}$ and find the points of intersection.

Example 2: Determine the general solutions to $16 = 6\cos\left(\frac{\pi}{6}\theta\right) + 14$. Round answers to the nearest hundredth.

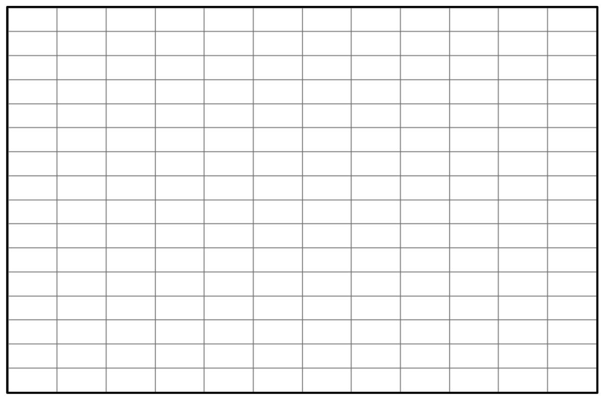
Option 1 - graph $y = 6\cos\left(\frac{\pi}{6}\theta\right) - 2$ and find the zeros/roots (*x*-intercepts).

Option 2 - graph $y = \cos\left(\frac{\pi}{6}\theta\right)$ & $y = \frac{1}{3}$ and find the points of intersection.

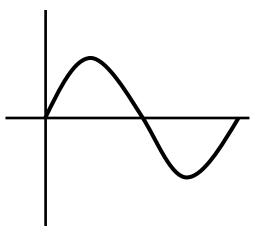
Example 3: Windsor, Ontario is located a latitude 42˚ N. The table below shows the number of hours of daylight in Windsor on the 21st day of each month through one full year.

Hours of Daylight by Day of the Year for Windsor, Ontario											
21	52	80	111	141	172	202	233	264	294	325	355
9.62	10.87	12.20	13.64	14.79	15.28	14.81	13.64	12.22	10.82	9.59	9.08

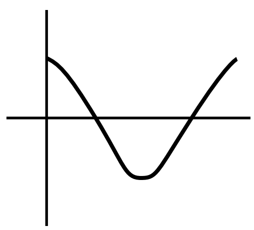
1. Create a scatter plot:



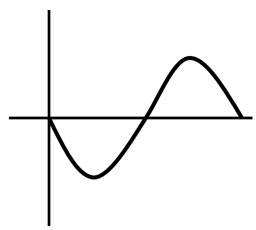
2. Choose a function:



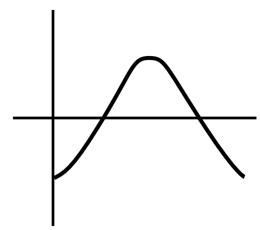
$f(x) = \sin\theta$



$f(x) = \cos\theta$



$f(x) = -\sin\theta$



$f(x) = -\cos\theta$

3. Find:

a=

b=

c=

d=

4. Write equation:

y =

6.1 – Reciprocal, Quotient and Pythagorean Identities (part 1)

Trigonometric Identity - a trigonometric equation that is true (equal on both sides) for all permissible values of the variable.

RECIPROCAL IDENTITIES

$\csc\theta = \frac{1}{\sin\theta}$ $\qquad$ $\sec\theta = \frac{1}{\cos\theta}$ $\qquad$ $\cot\theta = \frac{1}{\tan\theta}$

QUOTIENT IDENTITIES

$\tan\theta = \frac{\sin\theta}{\cos\theta}$ $\qquad$ $\cot\theta = \frac{\cos\theta}{\sin\theta}$

Trigonometric Identities can be verified graphically, numerically and algebraically.

Example 1: Determine whether $\sin\theta = \cos\theta\tan\theta$ is a trigonometric identity

Option 1 - graph $y = \sin x$ and $y = \cos x \tan x$ on the same grid $\left[0°, 360°\right)$

Option 2 - evaluate both $\sin\theta$ and $\cos\theta\tan\theta$ for $60°$ and $\frac{\pi}{4}\,rad$

Option 3 - verify algebraically

Example 2: Determine whether $\sec\theta = \dfrac{\tan\theta}{\sin\theta}$ is a trigonometric identity

Option 1 - graph $y = \sec x$ and $y = \dfrac{\tan x}{\sin x}$ on the same grid $\left[0°, 360°\right)$

Option 2 - evaluate both $\sec\theta$ and $\dfrac{\tan\theta}{\sin\theta}$ for $60°$ and $\dfrac{\pi}{4}rad$

Option 3 - verify algebraically

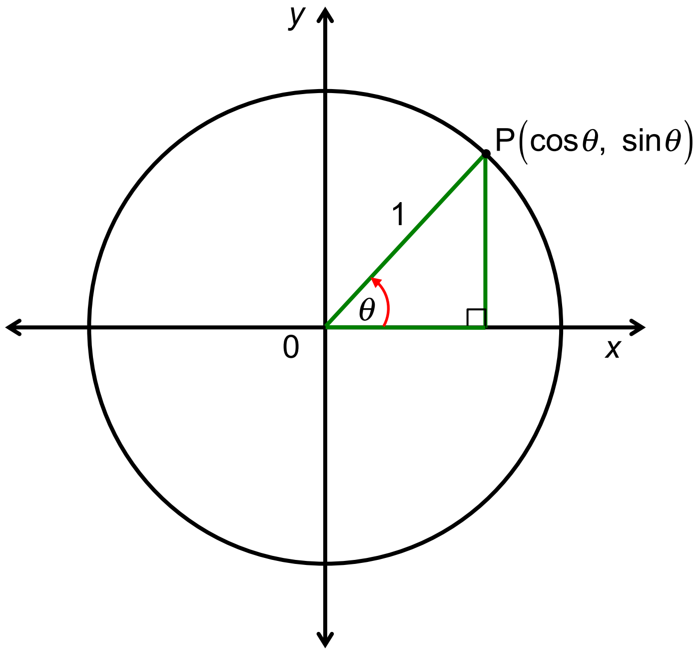
Example 3: Given the expression $\dfrac{\cot x}{\csc x \cos x}$,

a) Determine the non-permissible values, in degrees, of the variable.

b) Use identities to simplify the expression.

6.1 – Reciprocal, Quotient and Pythagorean Identities (part 2)

Recall that point P on the terminal arm of an angle θ in standard position on the Unit Circle has coordinates $(\cos\theta,\ \sin\theta)$.



If we apply the Pythagorean Theorem to this triangle, we get:

PYTHAGOREAN IDENTITIES

$\sin^2\theta + \cos^2\theta = 1$ $\quad$ $1 + \cot^2\theta = \csc^2\theta$ $\quad$ $1 + \tan^2\theta = \sec^2\theta$

<u>Example 1</u>: Determine whether $(\tan\theta - 1)^2 = \sec^2\theta - 2\tan\theta$ is a trigonometric identity

Option 1 - graph $y = (\tan x - 1)^2$ and $y = \sec^2 x - 2\tan x$ on the same grid $[0°, 360°)$

Option 2 - evaluate both $(\tan\theta - 1)^2$ and $\sec^2 x - 2\tan x$ for $60°$ and $\frac{\pi}{4}\,rad$

Option 3 - verify algebraically

6.2 – Sum, Difference and Double Angle Identities

SUM IDENTITIES

$$\sin(A+B) = \sin A\cos B + \cos A\sin B$$

$$\cos(A+B) = \cos A\cos B - \sin A\sin B$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A\tan B}$$

DIFFERENCE IDENTITIES

$$\sin(A-B) = \sin A\cos B - \cos A\sin B$$

$$\cos(A-B) = \cos A\cos B + \sin A\sin B$$

$$\tan(A-B) = \frac{\tan A - \tan B}{1 + \tan A\tan B}$$

DOUBLE ANGLE IDENTITIES

$$\sin 2\theta = 2\sin\theta\cos\theta$$

$$\cos 2\theta = \cos^2\theta - \sin^2\theta$$

$$\tan 2\theta = \frac{2\tan\theta}{1-\tan^2\theta}$$

Example 1: Write each expression as a single trigonometric function.

a) $\sin(48°)\cos(17°) - \cos(48°)\sin(17°)$

b) $\cos^2\frac{\pi}{3} - \sin^2\frac{\pi}{3}$

Example 2: Consider the expression $\frac{1-\cos 2x}{\sin 2x}$

a) Determine the non-permissible values for the expression.

b) Simplify the expression into one of the three primary trigonometric ratios.

Example 3: Determine the exact value of each expression.

a) $\tan 105^\circ$

b) $\sin\frac{\pi}{12}$

6.3 – Proving Identities

To prove a trigonometric identity algebraically, separately simplify both sides of the identity into identical expressions.

General Strategies:

- use known identities to make substitutions.
- re-write the expression using sine and cosine only.
- factor to simplify expressions.
- if quadratics are present, the Pythagorean Identity or one of its alternate forms can often be used.
- multiply the numerator and denominator by the conjugate of an expression.
- it is usually better to make a complicated expression simpler, than making a simple expression more complicated.

Examples: Prove each identity algebraically.

a) $1 - \sin^2 x = \sin x \cos x \cot x$

b) $\dfrac{\sin 2x}{\cos 2x + 1} = \tan x$

c) $\dfrac{1 - \cos x}{\sin x} = \dfrac{\sin x}{1 + \cos x}$

d) $\cot x - \csc x = \dfrac{\cos 2x - \cos x}{\sin 2x + \sin x}$

6.4 – Solving Trigonometric Equations Using Identities

• when solving trigonometric equations, it may be possible to make substitutions using the trigonometric identities covered in this chapter.

• this often involves expressing the equation in terms of ONE trigonometric function.

Example 1: Solve algebraically over the domain $0° \le x < 360°$

a) $\cos 2x + 1 - \cos x = 0$

b) $1 - \cos^2 x = 3\sin x - 2$

<u>Example 2</u>: Determine the general solution. Provide an exact solution, in radians.

a) $\cos^2 x = \cot x \sin x$

b) $2\sin x = 7 - 3\csc x$