Surface Area for Final Project

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The following page has a good explanation of solid angle as well as the surface area of the patch: http://mathworld.wolfram.com/SolidAngle.html

From that page we know that

$$\Omega = \int \int sin(\phi) \, d\theta \, d\phi$$

We want to find the surface area of a patch $[\phi_1, \phi_2] \times [\theta_1, \theta_2]$. This ends up being

$$\Omega = \int_{\phi_1}^{\phi_2} \int_{\theta_1}^{\theta_2} sin(\phi) \, d\theta \, d\phi$$

$$\Omega = (\theta_2 - \theta_1) \int_{\phi_1}^{\phi_2} sin(\phi) \, d\phi$$

$$\Omega = (\theta_1 - \theta_2)(cos\phi_2 - cos\phi_1)$$

This page lists more information: http://luki.webzdarma.cz/eng_12_en.htm

Our current patches are in the form $[x_1, y_1] \times [x_2, y_2]$ which are squares in the unit disk. In order to accomplish what we want, we will use the following unit hemisphere parameterization:

$$r(x, y) = (x, y, \sqrt{1 - x^2 - y^2})$$

The partials are as follows

$$r_x = (1, 0, \frac{-x}{\sqrt{1 - x^2 - y^2}})$$

$$r_y = (0, 1, \frac{-y}{\sqrt{1 - x^2 - y^2}})$$

The cross product ends up being

$$r_x \times r_y = \frac{x}{z} i + \frac{y}{z} j + k$$

The magnitude ends up being

$$|r_x \times r_y| = \sqrt{(\frac{x}{z})^2 + (\frac{y}{z})^2 + 1}$$

$$|r_x \times r_y| = \sqrt{\frac{x^2 + y^2 + z^2}{z^2}}$$

Now we know that $x^2 + y^2 + z^2 = 1$ and that $z \geq 0$ since this is the unit hemisphere, thus

$$|r_x \times r_y| = \frac{1}{z}$$

We know need to find

$$\Omega = \int_{y_1}^{y_2} \int_{x_1}^{x_2} \frac{1}{z} \, dx \, dy$$

The exact expression is quite large and cumbersome to compute, so we will approximate it. Let z' be the average of the z values in the 4 corners of the patch.We will thus have

$$\Omega \approx (x_2 - x_1)(y_2 - y_1) \frac{1}{z'}$$