CS 216 Homework 2

Zachary DeStefano, 15247592

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Problem 1

We will prove that $(f * g) * h = f * (g * h)$. First off, let $x = f * g$ and let $y = g * h$.

$$x(t) = (f * g)(t) = \sum_{s=-\infty}^{\infty} f(t-s)g(s)$$

$$y(t) = (g * h)(t) = \sum_{s=-\infty}^{\infty} g(s)h(t-s)$$

This means that

$$(x * h)(t) = \sum_{v=-\infty}^{\infty} x(v)h(t-v) = \sum_{v=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} f(v-s)g(s)h(t-v)$$

Similarly

$$(f * y)(t) = \sum_{v=-\infty}^{\infty} f(t-v)y(v) = \sum_{v=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} f(t-v)g(s)h(v-s)$$

In the second equation, do a change of variables $v = t - v' + s$ which does not change the final result since we are going from negative infinity to infinity. For the second equation, we end up with

$$(f * y)(t) = \sum_{v'=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} f(v'-s)g(s)h(t-v')$$

As can be observed, the first and second equation are now equal. This proves that $(x * h)(t) = (f * y)(t)$ and since this applies for any three functions, we have proven that convolution is associative.

Problem 2

If we correlate a function f with an impulse function, we get $f(-t)$,
so the function gets flipped around the y-axis.
Here is the proof:
Take the impulse function g, so $g(0) = 1$ and $g(x) = 0$ for all other x
Let * be the correlation operation in this case.

$$(f * g)(t) = \sum_{s=-\infty}^{\infty} f(s)g(s+t)$$

In our case, the inner term is always 0 except for when $s + t = 0$ so $s = -t$,
thus $(f * g)(t) = f(-t)$, proving our assertion.

Now let $h(t) = g(t)$. We will prove that $(f * g) * h \neq f * (g * h)$.

For the right hand side, by what was shown above,
$g * h = g$ since the impulse function is symmetric around the y-axis.
This means that $(f * (g * h))(t) = (f * g)(t) = f(-t)$.

For the left hand side, by what was shown above,
$(f * g)(t) = f(-t)$ thus $((f * g) * h)(t) = (f(-t)) * h(t)$
This will flip the function back to its original position, thus the left hand side is equal to $f(t)$

For any non-symmetric function f, such as $f(t) = 3t + 5$, it holds that $f(t) \neq f(-t)$
thus the left hand side is not equal to the right hand side.

Since we have example functions whose correlations are not associative, there is no way correlation is associative.

Problem 3

Let $g_1(x) = g_2(y) = N(0, \sigma^2)$. Then

$$g_1(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

$$g_2(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}$$

This means that

$$g_1(x)g_2(y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

This is equal to $g(x, y)$, thus allowing us to say that $g(x, y) = g_1(x)g_2(y)$

Problem 4

For the spatial domain running time, we can think of convolution as the following:
- For each pixel in the image, we apply the filter to it.
Thus there are $H \cdot W$ iterations of the filter
Each filter will take $M \cdot N$ time.
Thus the time complexity is $O(MNHW)$

If we use FFT, this would be the procedure:
1. Convert the signal to frequency
2. Convert the filter to frequency
3. Do element wise multiplication of the two new vectors.
4. Convert the elements back.

Step 1 will take $MN \cdot log(MN)$ time
Step 2 will take $HW \cdot log(HW)$ time
Step 3 will take $max(HW, MN)$ time. The identity of the max will not matter in the end since the previous step eclipses this one.
Step 4 will take $max(HW, MN)log(max(HW, MN))$ time. Again which one is the max does not matter because of step 1 and 2.

The total running time is thus $O(MN \cdot log(MN) + HW \cdot log(HW))$ in the case where we use FFT. Assuming that HW is the max, the running time is $O(HW \cdot log(HW))$.

If the filter is $f(x, y) = f_1(x)f_2(y)$, then assuming that I is the image, it holds that $I * f = (I * f_1) * f_2$
Here is the proof:

$$(f * I)(x, y) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f(i, j)I(x - i, y - j)$$

By separability,

$$(f * I)(x, y) = \sum_{i=-\infty}^{\infty} \sum_{j=-\infty}^{\infty} f_1(i)f_2(j)I(x - i, y - j)$$

$$(f * I)(x, y) = \sum_{i=-\infty}^{\infty} f_1(i)(\sum_{j=-\infty}^{\infty} f_2(j)I(x - i, y - j))$$

The inner summation is just convolution, thus we have

$$(f * I)(x, y) = \sum_{i=-\infty}^{\infty} f_1(i)(I * f_2)(i)$$

The summation shown here is convolution again, thus

$$(f * I)(x, y) = (I * f_2) * f_1$$

By associativity and commutativity of convolution, $f * I = (I * f_1) * f_2$, proving our assertion.

By what was just shown, we can do this procedure:
1. Convolve the image horizontally.
2. Convolve the result vertically.
Step 1 will take $O(HWM)$ time and step 2 will take $O(HWN)$ time,
thus the total running time is $O(HW \cdot max(M, N))$.
Assuming M is the max, the running time is $O(HWM)$.
Thus, by doing the horizontal filter then vertical filter, we can do even better than if we convert to frequency space.

Problem 5

Our two gaussians are as follows

$$g_1(t) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{t^2}{2\sigma_1^2}}$$

$$g_2(t) = \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{t^2}{2\sigma_2^2}}$$

The convolution formula we will use is the following

$$g_3(t) = \int_{-\infty}^{\infty} g_1(s) g_2(t-s)\, ds$$

Let F_1 be the Fourier transform for g_1 and F_2 be the transform for g_2.

$$F_1(t) = \int_{-\infty}^{\infty} g_1(s) e^{-2\pi ist}\, ds$$

$$F_2(t) = \int_{-\infty}^{\infty} g_2(s) e^{-2\pi ist}\, ds$$

Expanding we have

$$F_1(t) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2\sigma_1^2}} e^{-2\pi ist}\, ds$$

$$F_2(t) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2\sigma_2^2}} e^{-2\pi ist}\, ds$$

Using Euler's Formula we can now say that

$$F_1(t) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2\sigma_1^2}} (cos(2\pi st) + isin(2\pi st))\, ds$$

$$F_2(t) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2\sigma_2^2}} (cos(2\pi st) + isin(2\pi st))\, ds$$

Since the sine is an odd function, the integral of it will go to zero, thus we can simplify the above equations

$$F_1(t) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2\sigma_1^2}} cos(2\pi st)\, ds$$

$$F_2(t) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2\sigma_2^2}} cos(2\pi st)\, ds$$

This Wolfram Alpha web page explains the result of that integration.

`http://mathworld.wolfram.com/FourierTransformGaussian.html`

We can now use that fact to assert that

$$F_1(t) = \frac{1}{\sqrt{2\pi\sigma_1^2}}\sqrt{2\pi\sigma_1^2}\, e^{-2\pi^2 t^2 \sigma_1^2}$$

$$F_2(t) = \frac{1}{\sqrt{2\pi\sigma_2^2}}\sqrt{2\pi\sigma_2^2}\, e^{-2\pi^2 t^2 \sigma_2^2}$$

Simplifying we have the following

$$F_1(t) = e^{-2\pi^2 t^2 \sigma_1^2}$$

$$F_2(t) = e^{-2\pi^2 t^2 \sigma_2^2}$$

Let $F = F_1 \cdot F_2$ since we want to multiply the frequency functions, then we have

$$F(t) = e^{-2\pi^2 t^2 (\sigma_1^2 + \sigma_2^2)}$$

Let $\sigma = -2\pi^2(\sigma_1^2 + \sigma_2^2)$, then

$$F(t) = e^{\sigma t^2}$$

We want the inverse transform of F to get the convolution. Thus we can say that

$$g_3(x) = \int_{-\infty}^{\infty} e^{2\pi i x t} F(t) dt$$

$$g_3(x) = \int_{-\infty}^{\infty} e^{2\pi i x t} e^{\sigma t^2} dt$$

Expanding using Euler's formula and using the fact that sine is an odd function, we can say that

$$g_3(x) = \int_{-\infty}^{\infty} cos(2\pi x t) e^{\sigma t^2} dt$$

This is the same integral as was in the Wolfram Alpha web page, thus we can assert that

$$g_3(x) = \sqrt{-\frac{\pi}{\sigma}}\, e^{\frac{\pi^2 x^2}{\sigma}}$$

Expanding σ and then simplifying, we have

$$g_3(x) = \frac{1}{\sqrt{2\pi(\sigma_1^2 + \sigma_2^2)}}\, e^{-\frac{x^2}{2(\sigma_1^2 + \sigma_2^2)}}$$

If we let $\sigma_3 = \sqrt{\sigma_1^2 + \sigma_2^2}$ then we have

$$g_3(x) = \frac{1}{\sqrt{2\pi\sigma_3^2}}\, e^{-\frac{x^2}{2\sigma_3^2}}$$

This is clearly a Gaussian with variance σ_3 as defined above, thus g_3 is a Gaussian as desired.

Problem 6

Here are the initial plots

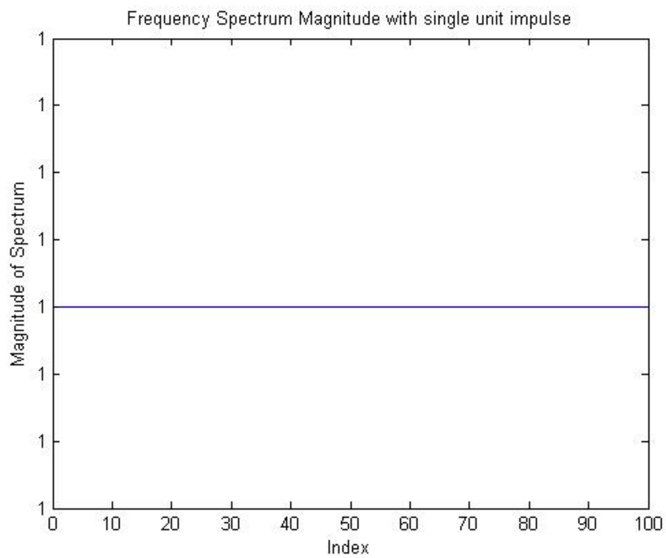

Figure 1: Frequency Spectrum Magnitude Plot for single impulse

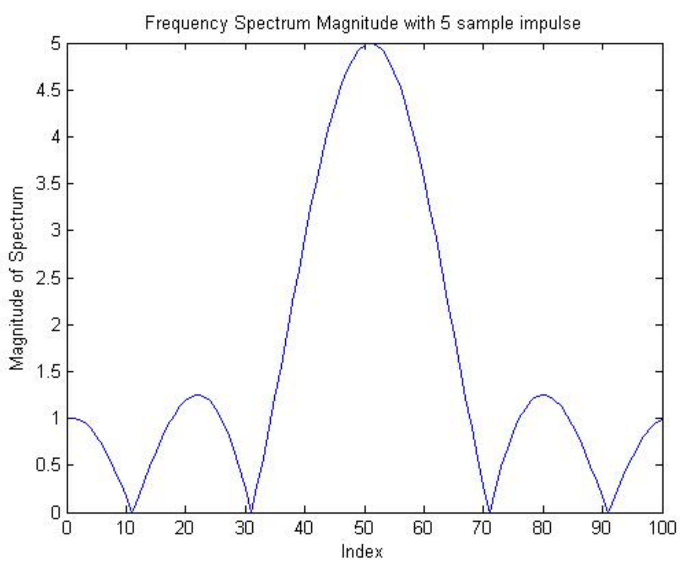

Figure 2: Frequency Spectrum Magnitude Plot for 5 sample impulse

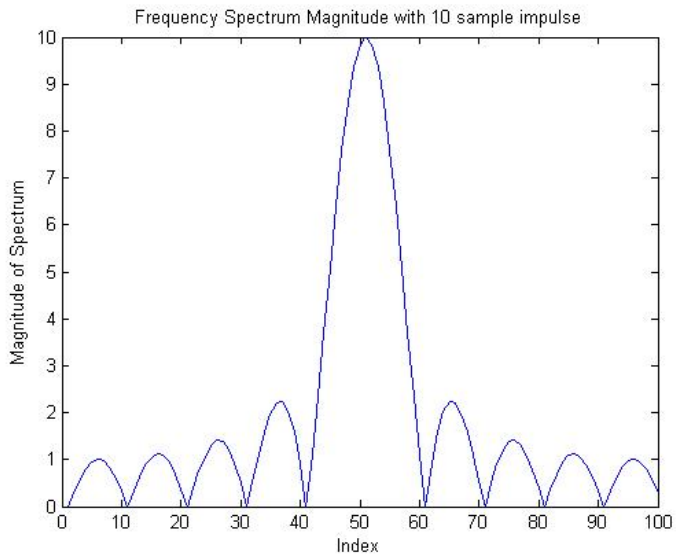

Figure 3: Frequency Spectrum Magnitude Plot for 10 sample impulse

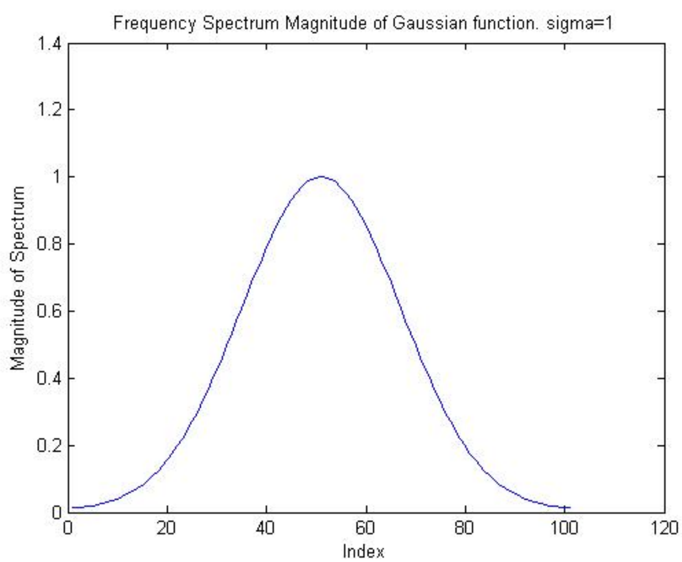

Figure 4: Frequency Spectrum Magnitude Plot for Gaussian Function with $\sigma = 1$

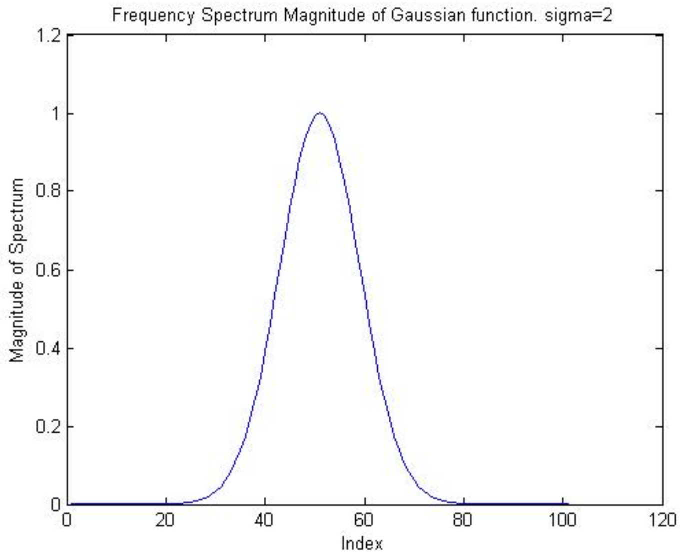

Figure 5: Frequency Spectrum Magnitude Plot for Gaussian Function with $\sigma = 2$

Here is the magnitude, phase experiment results

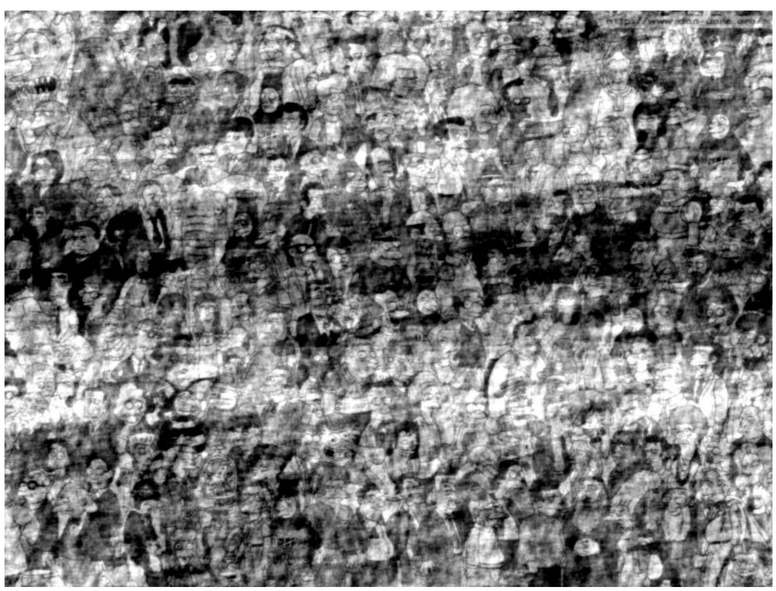

Figure 6: Magnitude of Zebra picture with Phase of Simpsons picture

Problem 7

Pictures for the Zebra image

Figure 7: Magnitude of the Gradient of the zebra picture where $\sigma = 3$

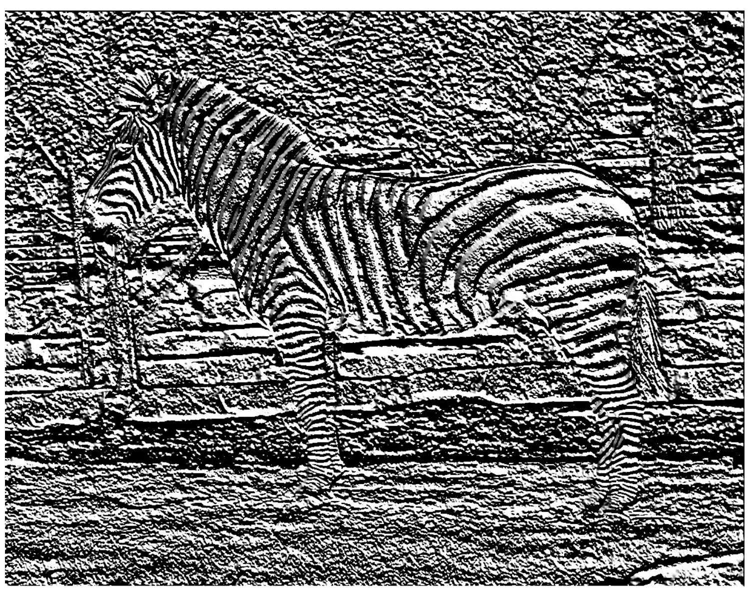

Figure 8: Orientation of the Gradient of the zebra picture where $\sigma = 3$

With new sigma

Figure 9: Magnitude of the Gradient of the zebra picture where $\sigma = 2$

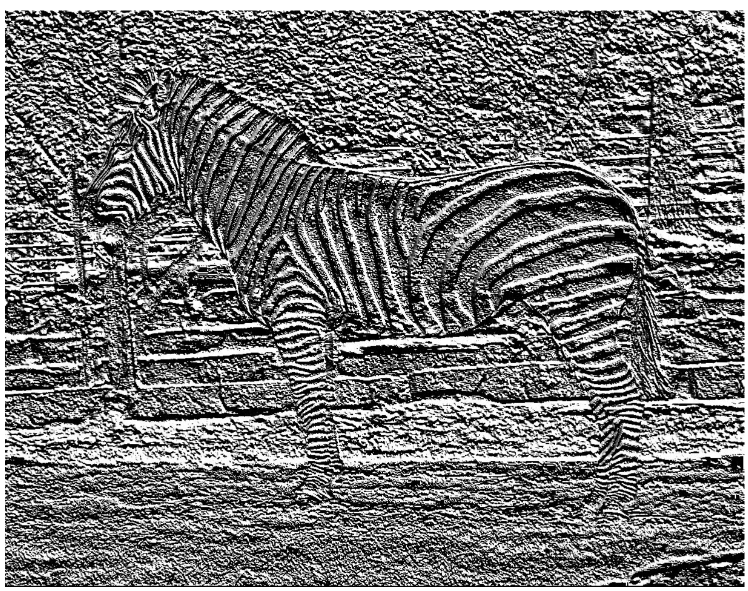

Figure 10: Orientation of the Gradient of the zebra picture where $\sigma = 2$