# CS 216 Homework 2

Zachary DeStefano, 15247592

Due Date: April 25, 2014

## Problem 1

We will prove that $(f * g) * h = f * (g * h)$. First off, let $x = f * g$ and let $y = g * h$.

$$x(t) = (f * g)(t) = \sum_{s=-\infty}^{\infty} f(t-s)g(s)$$

$$y(t) = (g * h)(t) = \sum_{s=-\infty}^{\infty} g(s)h(t-s)$$

This means that

$$(x * h)(t) = \sum_{v=-\infty}^{\infty} x(v)h(t-v) = \sum_{v=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} f(v-s)g(s)h(t-v)$$

Similarly

$$(f * y)(t) = \sum_{v=-\infty}^{\infty} f(t-v)y(v) = \sum_{v=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} f(t-v)g(s)h(v-s)$$

After a change of variables (TODO: More detail), the two sides are equal.

## Problem 2

We will prove that $(f * g) * h \neq f * (g * h)$ if * is correlation. First off, let $x = f * g$ and let $y = g * h$.

$$x(t) = (f * g)(t) = \sum_{s=-\infty}^{\infty} f(t+s)g(s)$$

$$y(t) = (g * h)(t) = \sum_{s=-\infty}^{\infty} g(s)h(t+s)$$

This means that

$$(x * h)(t) = \sum_{v=-\infty}^{\infty} x(v)h(t+v) = \sum_{v=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} f(v+s)g(s)h(t+v)$$

Similarly

$$(f * y)(t) = \sum_{v=-\infty}^{\infty} f(t+v)y(v) = \sum_{v=-\infty}^{\infty} \sum_{s=-\infty}^{\infty} f(t+v)g(s)h(v+s)$$

After a change of variables (TODO: More detail), the two sides are definitely not equal.

## Problem 3

Let $g_1(x) = g_2(y) = N(0, \sigma^2)$. Then

$$g_1(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

$$g_2(y) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{y^2}{2\sigma^2}}$$

This means that

$$g_1(x)g_2(y) = \frac{1}{2\pi\sigma^2} e^{-\frac{x^2+y^2}{2\sigma^2}}$$

This allows us to say that $g(x, y) = g_1(x)g_2(y)$

## Problem 4

For the spatial domain running time, we can think of convolution as for each pixel in the image, we apply the filter to it.
Thus there are $H \cdot W$ iterations of the filter
Each filter will take $M \cdot N$ time.
Thus the time complexity is $O(MNHW)$

If we use FFT, this would be the procedure:
1. Convert the signal to frequency
2. Convert the filter to frequency
3. Do element wise multiplication of the two new vectors.
4. Convert the elements back.

Step 1 will take $MN \cdot log(MN)$ time
Step 2 will take $HW \cdot log(HW)$ time
Step 3 will take $max(HW, MN)$ time. The identity of the max will not matter in the end since the previous step eclipses this one.
Step 4 will take $max(HW, MN)log(max(HW, MN))$ time. Again which one is the max does not matter because of step 1 and 2.

The total running time is thus $O(MN \cdot log(MN) + HW \cdot log(HW))$ time.
Assuming that $HW$ is the max, the running time is $O(HW \cdot log(HW))$.

If the filter is $f(x, y) = f(x)g(y)$ then we could do the following:
1. Compute the horizontal filter
2. Compute the vertical filter

3. Do point wise multiplication

Step 1 will take $HWM$ time and step 2 will be $HWN$ time.
Step 3 will be $HW$ time.
Total running time is thus $O(HW(M+N))$ time.
TODO: Improve the algorithm and its analysis for the case of a filter being that product.

## Problem 5

Our two gaussians are as follows

$$g_1(t) = \frac{1}{\sqrt{2\pi\sigma_1^2}} e^{-\frac{t^2}{2\sigma_1^2}}$$

$$g_2(t) = \frac{1}{\sqrt{2\pi\sigma_2^2}} e^{-\frac{t^2}{2\sigma_2^2}}$$

The convolution formula we will use is the following

$$g_3(t) = \int_{-\infty}^{\infty} g_1(s) g_2(t-s)\, ds$$

Page 6 of this paper has a good explanation

`http://www.tina-vision.net/docs/memos/2003-003.pdf`

Let $F_1$ be the Fourier transform for $g_1$ and $F_2$ be the transform for $g_2$.

$$F_1(t) = \int_{-\infty}^{\infty} g_1(s) e^{-2\pi i s t}\, ds$$

$$F_2(t) = \int_{-\infty}^{\infty} g_2(s) e^{-2\pi i s t}\, ds$$

Expanding we have

$$F_1(t) = \frac{1}{\sqrt{2\pi\sigma_1^2}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2\sigma_1^2}} e^{-2\pi i s t}\, ds$$

$$F_1(t) = \frac{1}{\sqrt{2\pi\sigma_2^2}} \int_{-\infty}^{\infty} e^{-\frac{s^2}{2\sigma_2^2}} e^{-2\pi i s t}\, ds$$