

We are assuming that energy is not flowing into the system.
Let E be energy.
By conservation of energy, in a given spatial region, the following holds

$$E_{gained} = E_{in} - E_{out}$$

Let J_i be energy flow in direction i
We are assuming that

$$E_{gained} = \frac{\partial u}{\partial t}$$

I am also assuming that

$$E_{in} - E_{out} = -(\Delta E)$$

since we are looking at the reverse change in energy.

For simplification, the total change in energy is the change in energy flow from all directions summed up, hence

$$\Delta E = \frac{\partial}{\partial x} J_x + \frac{\partial}{\partial y} J_y = \nabla \cdot \vec{J}$$

Combining change in energy with what we know about u , we have the following equation

$$\frac{\partial u}{\partial t} = -(\frac{\partial}{\partial x} J_x + \frac{\partial}{\partial y} J_y) = \nabla \cdot \vec{J}$$

By Fick's law of diffusion

$$J_x = -\alpha \frac{\partial u}{\partial x}$$

$$J_y = -\alpha \frac{\partial u}{\partial y}$$

Plugging in the Fick's law equation into our energy equation gives us

$$\frac{\partial u}{\partial t} = \alpha (\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2})$$

This is the heat equation