(*Math 227A Homework 2*)

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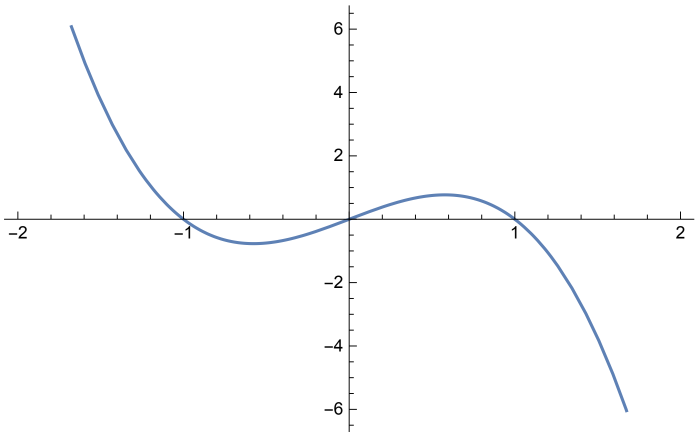
Due October 7 2016*)

In[191]:= (*Part 1A*)

In[192]:=
```
Plot[2 y - 2 y^3, {y, -2, 2}]
```

Out[192]=



In[193]:=
```
(*Steady States at y=-1, y=0, y=1
   y=-1 is stable
    y=0 is unstable
     y=1 is stable*)

(*Evident from graph as well as fact that f'(y)=2-6y^2 so that
    f'(1)=f'(-1)=-4 < 0 so those are stable
      and f'(0)=2 > 0 so that one is unstable*)
```

In[194]:=
```
(*I will choose the following initial values for the sample solutions:
   Initial Value 1 is y(0) = 1.5
   Initial Value 2 is y(0) = 0.5
       Initial Value 3 is y(0) = -0.5
     Initial Value 4 is y(0) = -1.5 *)
```

In[195]:=
```
prob1Ainit1 = NDSolveValue[{y'[t] == 2 y[t] - 2 y[t]^3, y[0] == 1.5}, y, {t, 0, 20}];
```

In[196]:=
```
prob1Ainit2 = NDSolveValue[{y'[t] == 2 y[t] - 2 y[t]^3, y[0] == .5}, y, {t, 0, 20}];
```

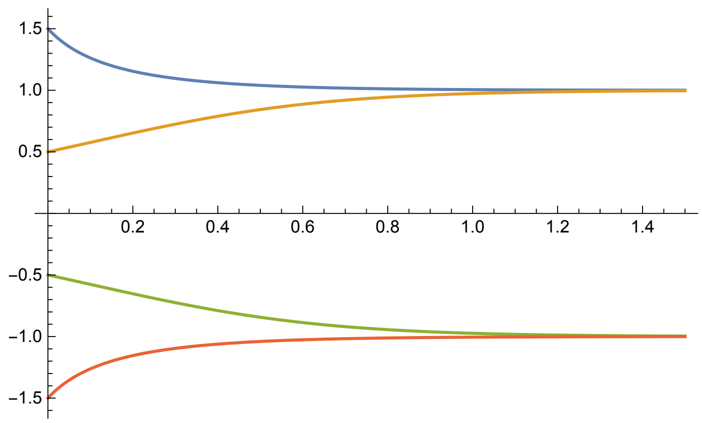
In[197]:=
```
prob1Ainit3 = NDSolveValue[{y'[t] == 2 y[t] - 2 y[t]^3, y[0] == -.5}, y, {t, 0, 20}];
```

In[198]:=
```
prob1Ainit4 = NDSolveValue[{y'[t] == 2 y[t] - 2 y[t]^3, y[0] == -1.5}, y, {t, 0, 20}];
```

In[199]:=
```
(*Here is the graph of sample solutions*)
```

In[200]:=
```
Plot[{prob1Ainit1[t], prob1Ainit2[t], prob1Ainit3[t], prob1Ainit4[t]}, {t, 0, 1.5}]
```

Out[200]=



In[201]:=

(*Problem 1B*)

```
(*Steady state if e^(-y)=0 or sin(y)=0
    First condition will never happen
    sin(y)=0 if y=n*Pi for any integer n*)
(*It holds that f'(y)=e^(-y) (cos(y)-sin(y)) *)
(*Here is verification*)
D[Exp[-y] * Sin[y], y]
```

Out[201]= $e^{-y}\,\mathrm{Cos}[y] - e^{-y}\,\mathrm{Sin}[y]$

In[202]:=
```
(*Steady states occur at n*Pi
   e^(-y)>0 everywhere so we care about the sign of the other term
   If y=
 2*n*Pi for integer n then cos(y)=1 so f'(y)>0 making those steady states unstable
     If y=(2*n+1)*Pi for integer n then cos(y)=
    -1 so f'(y)<0 making those steady states stable *)
```

In[203]:=
```
(*Here are the initial values that I chose:
   y(0) = Pi/4
   y(0) = 5Pi/4
    y(0) = -Pi/4
     y(0) = -5Pi/4*)
```

In[204]:=
```
prob1Binit1 = NDSolveValue[{y'[t] == Exp[-y[t]] * Sin[y[t]], y[0] == Pi / 4}, y, {t, 0, 40}];
```

In[205]:=
```
prob1Binit2 =
  NDSolveValue[{y'[t] == Exp[-y[t]] * Sin[y[t]], y[0] == 5 * Pi / 4}, y, {t, 0, 40}];
```

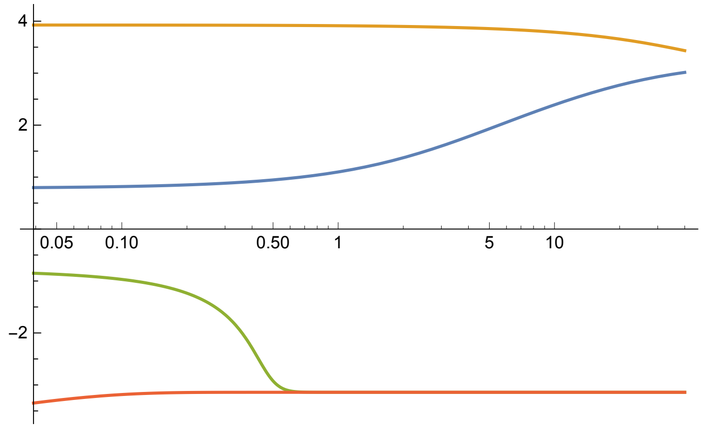
In[206]:=
```
prob1Binit3 = NDSolveValue[{y'[t] == Exp[-y[t]] * Sin[y[t]], y[0] == -Pi / 4}, y, {t, 0, 40}];
```

In[207]:=
```
prob1Binit4 =
  NDSolveValue[{y'[t] == Exp[-y[t]] * Sin[y[t]], y[0] == -5 * Pi / 4}, y, {t, 0, 40}];
```

In[208]:=
```
(*Here is the log linear graph of sample solutions*)
```

In[209]:=
```
LogLinearPlot[{prob1Binit1[t], prob1Binit2[t], prob1Binit3[t], prob1Binit4[t]}, {t, 0, 40}]
```

Out[209]=



In[210]:=
```
(*Problem 2*)
(*Need a function with zeros at 0,1,2,3. For simplicity I will try a polynomial*)
(*One possibility is the form f(y)=Ky(y-1)(y-2)(y-3) *)
(*Here I am computing f'(y) to get info about steady states*)
Expand[D[KK * y (y - 1) (y - 2) (y - 3), y]]
```

Out[210]= $-6\,KK + 22\,KK\,y - 18\,KK\,y^2 + 4\,KK\,y^3$

In[211]:=
```
fPrime[y_] := -6 KK + 22 KK y - 18 KK y^2 + 4 KK y^3
```

In[212]:=
```
(*Here I am computing f'(y) at the steady state points*)
```

In[213]:=
```
{fPrime[0], fPrime[1], fPrime[2], fPrime[3]}
```

Out[213]= $\{-6\,KK, 2\,KK, -2\,KK, 6\,KK\}$

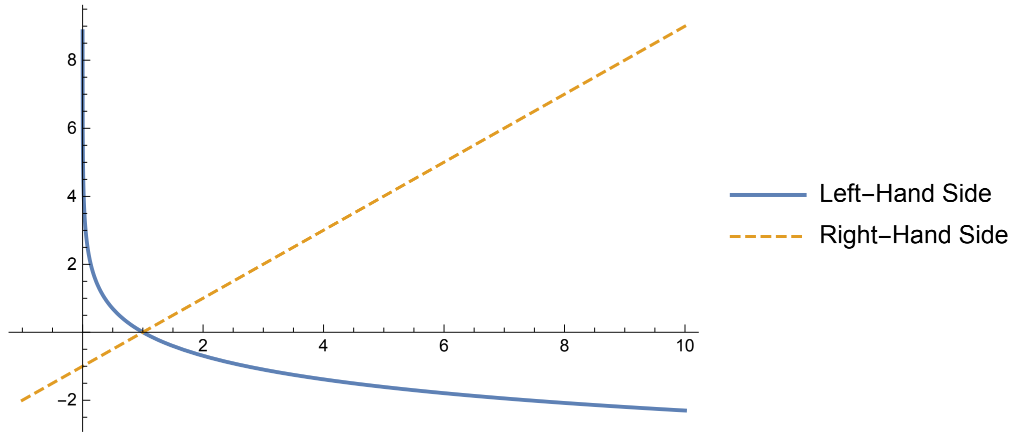
In[214]:=
```
(*As long as K>0 the properties are satified,
thus every function f(y)=Ky(y-1)(y-2)(y-3) where K>0
   satisfies our requirements*)
```

In[215]:=
```
(*Problem 3A*)
```

In[216]:=
```
(*Here are four graphs showing both functions*)
```

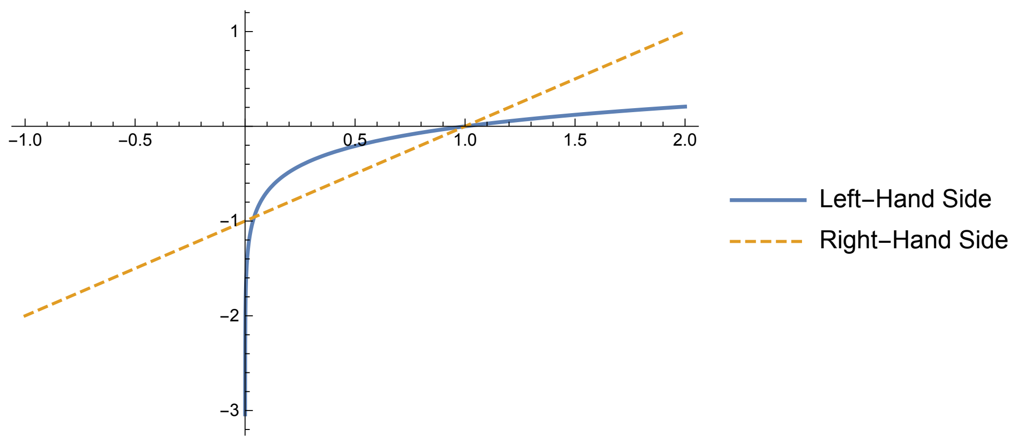
In[217]:=
```
Plot[{-1 * Log[y], y - 1}, {y, -1, 10}, PlotLegends → {"Left-Hand Side", "Right-Hand Side"},
 PlotStyle → {Thick, Dashed}] (* r=-1 *)
```

Out[217]=



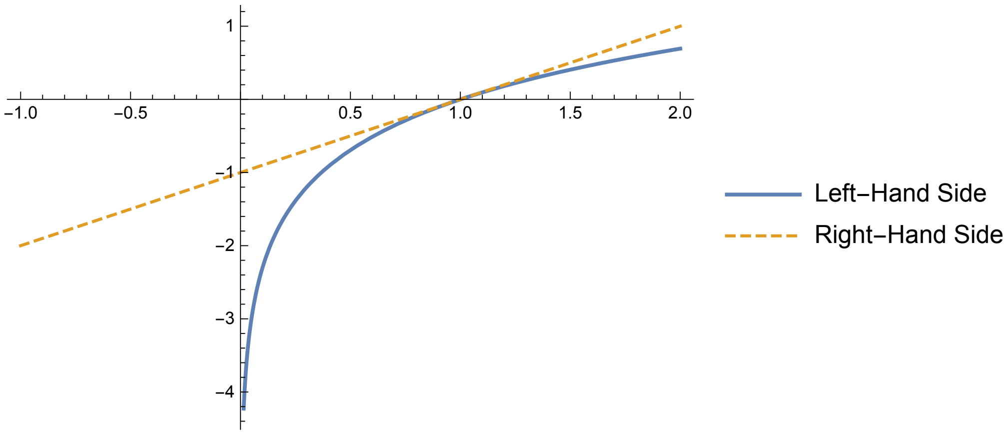
In[218]:=
```
Plot[{0.3 * Log[y], y - 1}, {y, -1, 2}, PlotLegends → {"Left-Hand Side", "Right-Hand Side"},
 PlotStyle → {Thick, Dashed}] (* r=0.3 *)
```

Out[218]=



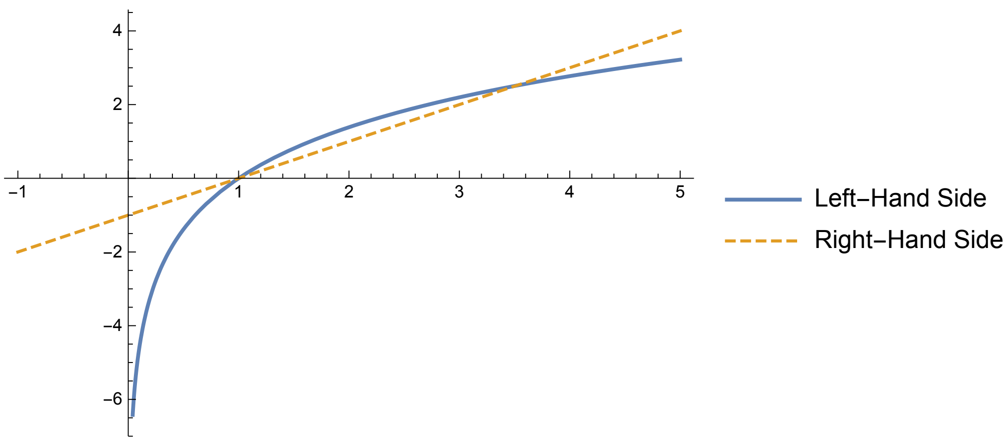
In[219]:=
```
Plot[{Log[y], y - 1}, {y, -1, 2}, PlotLegends → {"Left-Hand Side", "Right-Hand Side"},
 PlotStyle → {Thick, Dashed}] (* r=1 *)
```

Out[219]=



In[220]:=
```
Plot[{2 * Log[y], y - 1}, {y, -1, 5}, PlotLegends → {"Left-Hand Side", "Right-Hand Side"},
 PlotStyle → {Thick, Dashed}] (* r=2 *)
```

Out[220]=



In[221]:=
```
(* At the points where the two graphs intersect,
it holds that y'=0 so the intersection points
  will be the steady states and that holds for any value of r *)
(*It appears for large r, there are two steady states.
   At r=1 there is one steady state.
    Between 0 and 1 there appear to be two steady states.
    For r<0 there appears to be one steady state.
    This is Saddle Node Bifurcation*)
```

In[222]:=
```
(*If y=1, the ln(y)=0 so the value of r is irrelevant*)
```

```
(*Problem 3B*)
(*Equation is now the following
  u'=-r ln(u+1) + u
*)
(*Problem 3C*)
(* If we assume that ln(u+1) =
 u - (1/2)u^2 then when expanding the part b equation using that assumption we have
    u' = -r (u - (1/2)u^2 ) + u
     u' = -r*u + (1/2)r*u^2 + u  (1)
      When you expand the part c equation you end up with
      u' = (r/2)u( -2 + 2/r + u)
      u' = u( -r + 1 + r*u/2)
       u' = -r*u + u + r*u^2/2   (2)
        Equation (1) matches equation (2) above so the given equation for u' is valid*)
```

In[224]:=
```
(*Since (-r+1)*(2/r) is a constant just like mu,
our system will have very similar behavior around r=1.
   At r=1 and mu=
   0 the only difference is a constant factor. They both have a single steady state.
      For r>1 the zeros are at 0 and a positive number, just like for mu<0.
  For r<1 the zeros are at 0 and a negative number, just like for mu>0.
  Both also have a parabolic shape pointing upward, so the first zero is
 a stable steady state and the second zero is an unstable steady state*)
```

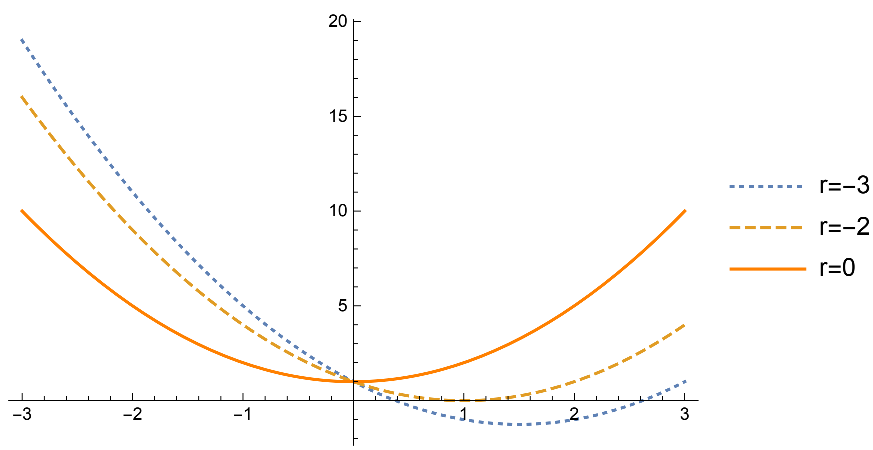
In[225]:=
```
(*Problem 4*)
```

In[226]:=
```
(*For large positive and negative values, y' will be positive.
       The quadratic equation 1+r*y+y^2 has a discriminant of r^2-4
      Meaning that if |r|<2 there are no real roots so y' is positive everywhere.
       If r=
   2 or r=-2 then there is one unstable steady state at y=-1 and y=1 respectively
        If |r|>2 there are two steady states at
 (-r-sqrt(r^2-4))/2 and(-r+sqrt(r^2-4))/2
            The first one is stable and the second one is unstable respectively
 This is an example of Saddle Node Bifurcation*)
```

In[227]:=
```
yPrimeFunc[r_] := 1 + r * y + y^2;
```

In[228]:=
```
(*This shows the phase diagram for
  r=-3 (two steady states),
r=-2 (one steady state), and
  r=2 (no steady states)*)
Plot[{yPrimeFunc[-3], yPrimeFunc[-2], yPrimeFunc[0]}, {y, -3, 3},
 PlotLegends → {"r=-3", "r=-2", "r=0"}, PlotStyle → {Dotted, Dashed, Orange}]
```

Out[228]=

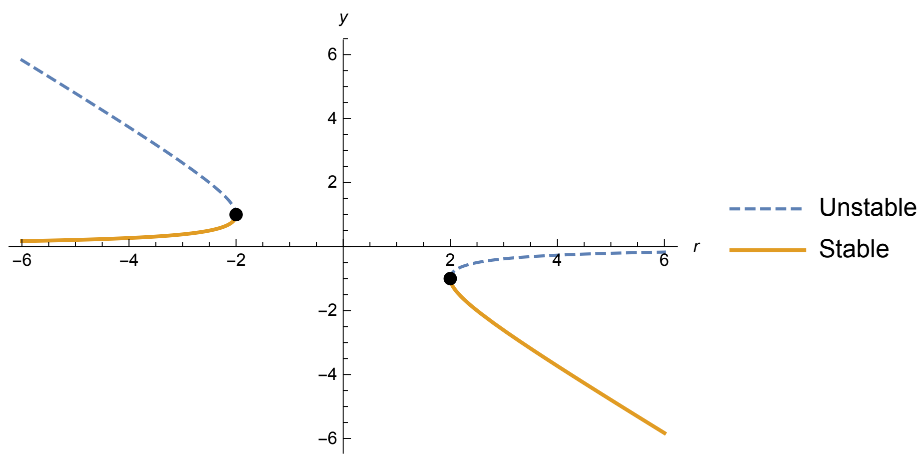


In[229]:=
```
(*It holds that f'(y)=r+2y
   For a given r value, the two steady state y values are described by
  y1(r) = (-r+sqrt(r^2-4))/2
   y2(r) = (-r-sqrt(r^2-4))/2

     For y1 it holds that r+2y = sqrt(r^2-4) ≥ 0 so that branch is unstable
       For y2 it holds that r+2y = -sqrt(r^2-4) ≤ 0 so that branch is stable

        The bifurcation points occur when r+2y=
     0 thus (r,y) is equal to (-2,1) and (2,-1)

      The plot below shows the stable and unstable
      branches as well as the bifurcation points as black dots*)
```

In[230]:=
```
prob4Plot = Plot[{(-r + Sqrt[r^2 - 4]) / 2, (-r - Sqrt[r^2 - 4]) / 2}, {r, -6, 6},
   PlotStyle → {Dashed, Thick}, AxesLabel → {r, y}, PlotLegends → {"Unstable", "Stable"}];
```

In[231]:=
```
Show[prob4Plot, Graphics[{PointSize[0.02], Point[{-2, 1}]}],
 Graphics[{PointSize[0.02], Point[{2, -1}]}]]
```

Out[231]=



In[232]:=
```
(*Problem 5*)
```

In[233]:=
```
prob5Func[r_] := y * (r - Exp[y])
```

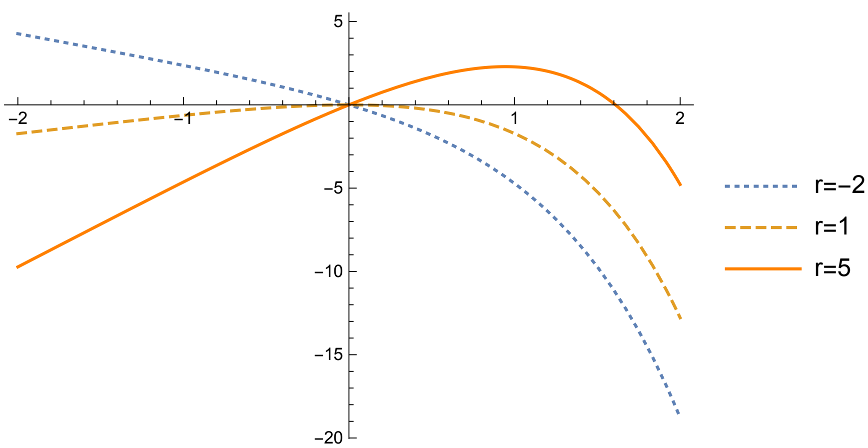
In[234]:=
```
(*For large positive values of y it holds that r won't matter as -
  y*e^(y) will be a large negative number
   For large negative values of y it holds that e^(y)
   is near zero thus the graph will be similar to r*y
   So it r<0 and y negative then y' will be positive,
but if r>0 and y negative then y' will be negative *)
```

In[235]:=
```
(*The steady states will occur when y=0 or r=e^y
     If r≤0 then there will only be one steady state at y=0 and it will be stable
      If r>0 then there will be two steady states at y=
    0 and y=ln(r). The lower one is unstable and the higher value is stable.
        In the case of r=1 there is one steady state at y=1 and it is unstable.*)
```

In[236]:=
```
(*This is Transcritical Bifurcation*)
```

In[237]:=
```
Plot[{prob5Func[-2], prob5Func[1], prob5Func[5]}, {y, -2, 2},
 PlotLegends → {"r=-2", "r=1", "r=5"}, PlotStyle → {Dotted, Dashed, Orange}]
```

Out[237]=

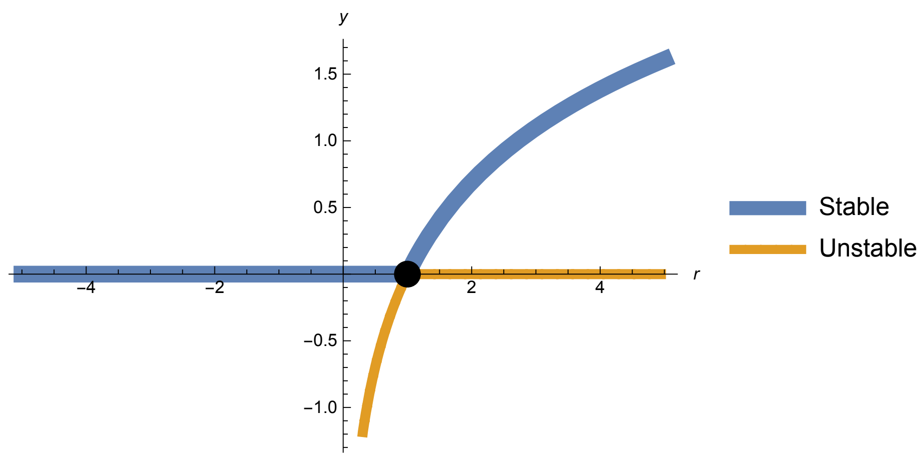


In[238]:=
```
(*It holds that f'(y)=y(-e^y)+(r-e^y)= r - e^y - y*e^y

      For the y=0 bifurcation branch it holds that f'(y)=r-1
         so f'(y)<0 for r<1 (stable part) and f'(y)≥0 for r≥1 (unstable part)

        For the r=e^y branch it tholds that f'(y)=-y*e^y so since e^y>0 for all values,
it holds that f'(y)<0 for y>0 (stable part) and f'(y)≥0 for y<0 (unstable part)

  The Bifurcation Point occurs when
  (r,y) is equal to (1,0) and is given a black dot below*)
```

In[239]:=
```
prob5Plot =
  Plot[{Piecewise[{{0, r < 1}, {Log[r], r ≥ 1}}], Piecewise[{{Log[r], r < 1}, {0, r ≥ 1}}]},
   {r, -5, 5}, AxesLabel → {r, y}, PlotLegends → {"Stable", "Unstable"},
   PlotStyle → {Thickness[0.025], {Thickness[0.015], Dashed}}];
```

In[240]:=
```
Show[prob5Plot, Graphics[{PointSize[0.04], Point[{1, 0}]}]]
```

Out[240]=



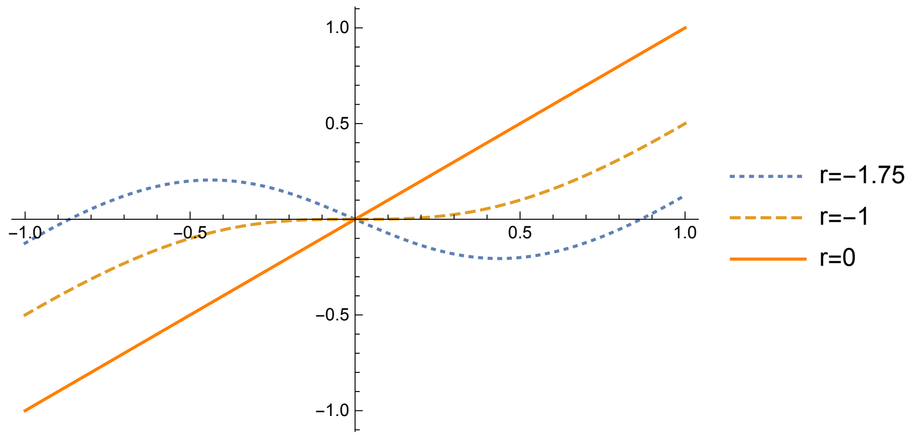
In[241]:=
```
(*Problem 6*)
```

In[242]:=
```
(*It holds that our equation is y(1 + r/(1+y^2) )=0 so either y=0 or 1 + r/(1+y^2)=0.
     In the later case we can manipulate it to be
     y^2=-(r+1) so it must hold that r<-1 for a root to exist.
        For r<-1 we have three roots:
        y =
     -sqrt(-r-1) 0 sqrt(-r-1) and they are unstable stable and unstable respectively
        For r≥-1 we have an unstable root at y=0

       This is Pitch Fork Bifurcation*)
```

In[243]:=
```
prob6Func[r_] := y + r * y / (1 + y^2)
```

In[244]:=
```
Plot[{prob6Func[-1.75], prob6Func[-1], prob6Func[0]}, {y, -1, 1},
 PlotLegends → {"r=-1.75", "r=-1", "r=0"}, PlotStyle → {Dotted, Dashed, Orange}]
```

Out[244]=



In[245]:=

In[246]:=
```
D[y + r * y / (1 + y^2), y]
```

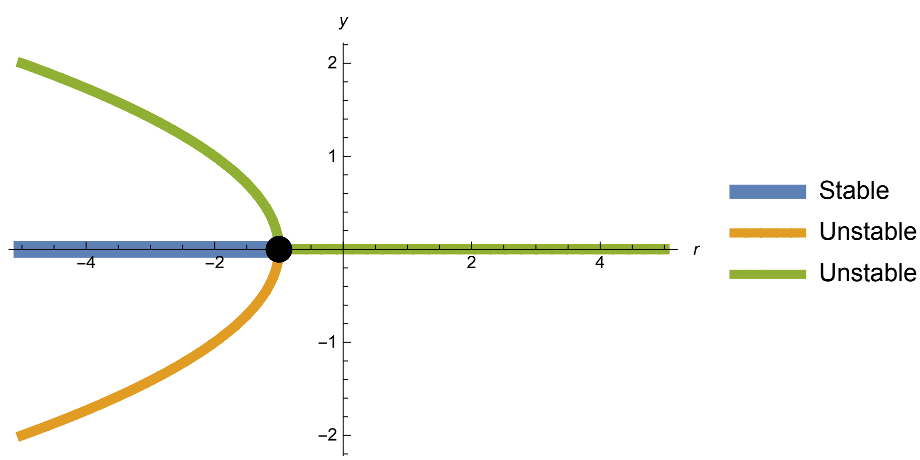
Out[246]= $1 - \frac{2\, r\, y^2}{\left(1 + y^2\right)^2} + \frac{r}{1 + y^2}$

In[247]:=
```
(*It thus holds that f'(y) = 1-r( (y^2-1) / (1+y^2)^2 ) *)
```

In[248]:=
```
(* The branches of the graph are y=0 and y^2=-(r+1)

    If y=
  0 then f'(y)=1+r so if r<-1 the branch is stable but if r≥-1 the branch is unstable

       If y^2=-(r+1) then f'(y) = 1 - r( (-r - 2)/r^2 ) = 1 + (r+2)/r = 2(r+1)/r,
thus since r≤-1 it holds that f'(y)≥0 hence the branches are unstable

  f'(y)=0 in our branches only when r=-1 thus
   the Bifurcation Point is (r,y)=(-1,0)
   which is given a black dot below*)
```

In[249]:=
```
prob6Plot = Plot[{Piecewise[{{0, r < -1}, {Sqrt[-(r + 1)], r ≥ -1}}],
    Piecewise[{{-Sqrt[-(r + 1)], r < -1}, {0, r ≥ -1}}],
    Piecewise[{{Sqrt[-(r + 1)], r < -1}}]}, {r, -5, 5}, AxesLabel → {r, y},
  PlotLegends → {"Stable", "Unstable", "Unstable"}, PlotStyle →
    {Thickness[0.025], {Thickness[0.015], Dashed}, {Thickness[0.015], Dashed}}];
```

In[250]:=
```
Show[prob6Plot, Graphics[{PointSize[0.04], Point[{-1, 0}]}]]
```

Out[250]=


In[251]:=
```
(*Extra Problem*)
```

In[252]:=
```
(*Show that y' = y^(1/3) and y(0)=0 has more than one solution*)
```

In[253]:=
```
(*After doing separation of variables,
we have y^(-1/3) dy = dt. After integrating we have
   (3/2) y^(2/3) = t + C
     If we assume y(0)=0 then it holds that C=0.

      After rearranging our solution is
      y^2 = (2t/3)^3

      The positive and negative square root of the right hand side are both solutions,
hence there is more than one solution*)

(*Additionally,
the function y(t)=0 satisfies both properties and is thus an additional solution*)
```