Math227A-HW4

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1 Problem 1, Strogatz 5.2.1

1.1 Part A

The following matrix is A

$$\begin{pmatrix} 4 & -1 \\ 2 & 1 \end{pmatrix}$$

This means that the characteristic polynomial is as follows

$$(4-\lambda)(1-\lambda)+2=0$$

which can be transformed as follows

$$4-5\lambda+\lambda^2+2=0$$

$$\lambda^2-5\lambda+6=0$$

Factoring this gives us

$$(\lambda-2)(\lambda-3)=0$$

So the eigenvalues are as follows

$$\lambda_1=2\,,\ \lambda_2=3$$

To get the first eigenvector v_1 we solve

$$4x - y = 2x$$

$$2x + y = 2y$$

Adding the equations yields

$$6x = 2(x + y)$$

which means $4x = 2y$ or $y = 2x$ so $v_1 = (1, 2)$

To get the second eigenvector v_2 we solve

$$4x - y = 3x$$

$$2x + y = 3y$$

Adding the equations yields

$$6x = 3(x + y)$$

$$3x = 3y$$

thus $v_2 = (1, 1)$

1.2 Part B

The general solution is thus as follows

$$\left(\begin{array}{c} x(t) \\ y(t) \end{array} \right) = c_1 e^{2t} \left(\begin{array}{c} 1 \\ 2 \end{array} \right) + c_2 e^{3t} \left(\begin{array}{c} 1 \\ 1 \end{array} \right)$$

which can also be expressed as

$$x(t) = c_1 e^{2t} + c_2 e^{3t}$$

$$y(t) = 2c_1 e^{2t} + c_2 e^{3t}$$

1.3 Part C

The origin is an unstable steady state since the positive eigenvalues means the system will diverge to infinity.

1.4 Part D

Assuming $x(0) = 3$ and $y(0) = 4$ means

$$3 = c_1 + c_2$$

$$4 = 2c_1 + c_2$$

Subtracting the first equation from the second one yields

$$1 = c_1$$

This means

$$c_2 = 2$$

Thus the ending solution is as follows

$$x(t) = e^{2t} + 2e^{3t}$$

$$y(t) = 2e^{2t} + 2e^{3t}$$

2 Problem 2

2.1 Part A

Here it holds that A is equal to

$$\left(\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right)$$

The characteristic equation would thus be

$$(1-\lambda)^2 + 1 = 0$$

Expanding this becomes

$$\lambda^2 - 2\lambda + 2 = 0$$

The solutions to this would be as follows

$$\lambda = \frac{2 \pm \sqrt{4 - 4(2)}}{2}$$

$$\lambda = \frac{2 \pm 2i}{2}$$

Thus $\lambda_1 = 1 + i$ and $\lambda_2 = 1 - i$ are the two eigenvalues.

To get an eigenvector we need to solve

$$\left(\begin{array}{cc} 1 & -1 \\ 1 & 1 \end{array} \right)\left(\begin{array}{c} x \\ y \end{array} \right) = (1+i)\left(\begin{array}{c} x \\ y \end{array} \right)$$

and this gives us the two equations

$$x - y = (1+i)x$$

$$x + y = (1+i)y$$

Adding the two equations we obtain

$$2x = x + ix + y + iy$$

Rearranging we have

$$x - ix = y + iy$$

$$x(1-i) = y(1+i)$$

Multiplying both sides by $(1+i)$ for conjugation we have

$$2x = y(2i)$$

$$x = iy$$

Thus we have our first eigenvector $v_1 = (i, 1)$

To get the other eigenvector we have to solve

$$(\begin{matrix} 1 & -1 \\ 1 & 1 \end{matrix})(\begin{matrix} x \\ y \end{matrix}) = (1-i)(\begin{matrix} x \\ y \end{matrix})$$

giving us the equations

$$x - y = (1-i)x$$

$$x + y = (1-i)y$$

Adding the two equations yields

$$2x = x - ix + y - iy$$

Rearranging gives us

$$x(1+i) = y(1-i)$$

Multiplying by $(1-i)$ for conjugation gives us

$$2x = -2iy$$

$$x = -iy$$

Thus our second eigenvector $v_2 = (-i, 1)$

2.2 Part B

Plugging in λ_1 and λ_2 we have

$$\mathbf{x}(t) = c_1 e^{(1+i)t}\mathbf{v}_1 + c_2 e^{(1-i)t}\mathbf{v}_2$$

$$\mathbf{x}(t) = c_1 e^t e^{it}\mathbf{v}_1 + c_2 e^t e^{-it}\mathbf{v}_2$$

Rewriting in terms of sines and cosines we have

$$\mathbf{x}(t) = e^t[c_1(cos(t) + i \cdot sin(t))\mathbf{v}_1 + c_2(cos(t) - i \cdot sin(t))\mathbf{v}_2]$$

$$\mathbf{x}(t) = e^t cos(t)(c_1\mathbf{v}_1 + c_2\mathbf{v}_2) + i \cdot e^t sin(t)(c_1\mathbf{v}_1 - c_2\mathbf{v}_2)$$

Solving for the second component tell us

$$y(t) = e^t cos(t)(c_1 + c_2) + i \cdot e^t sin(t)(c_1 - c_2)$$

Solving for the first component we have

$$x(t) = e^t cos(t)(c_1 i - c_2 i) + i \cdot e^t sin(t)(c_1 i + c_2 i)$$

Factoring out the i parts we have

$$x(t) = -e^t sin(t)(c_1 + c_2) + ie^t cos(t)(c_1 - c_2)$$

Let $A = c_1 + c_2$ and $B = c_1 - c_2$ then
$c_1 = (1/2)(A + B)$ and $c_2 = (1/2)(A - B)$ so for any A, B
there will be a corresponding c_1, c_2 so A, B can be treated as arbitrary constants

In terms of A and B a solution is

$$x_0(t) = -Ae^t sin(t) + iBe^t cos(t)$$

$$y_0(t) = Ae^t cos(t) + iBe^t sin(t)$$

Let $B_1 = -B$ then the following are also solutions

$$x_1(t) = -Ae^t sin(t) - iBe^t cos(t)$$

$$y_1(t) = Ae^t cos(t) - iBe^t sin(t)$$

We will use $x(t) = (1/2)(x_0(t)+x_1(t))$ and $y(t) = (1/2)(y_0(t)+y_1(t))$, which also counts as a solution giving us

$$\mathbf{x}_1(t) = (c_1 + c_2)e^t\left(\begin{array}{c} -sin(t) \\ cos(t) \end{array} \right)$$

For the other solution, it holds that $x(t) = (1/2)(x_0(t) - x_1(t))(-i)$ and $y(t) = (1/2)(y_0(t) - y_1(t))(-i)$, which also counts as a solution giving us

$$\mathbf{x}_2(t) = (c_1 - c_2)e^t\left(\begin{array}{c} cos(t) \\ sin(t) \end{array} \right)$$

Our final form is thus as follows

$$\mathbf{x}(t) = Ae^t\left(\begin{array}{c} -sin(t) \\ cos(t) \end{array} \right) + Be^t\left(\begin{array}{c} cos(t) \\ sin(t) \end{array} \right)$$

for arbitrary constants A, B

This is the form predicted by the theorem.
For our case, $a = 1$, $b = 1$, $u = (0, 1)$, $v = (1, 0)$

3 Problem 3

3.1 Part A

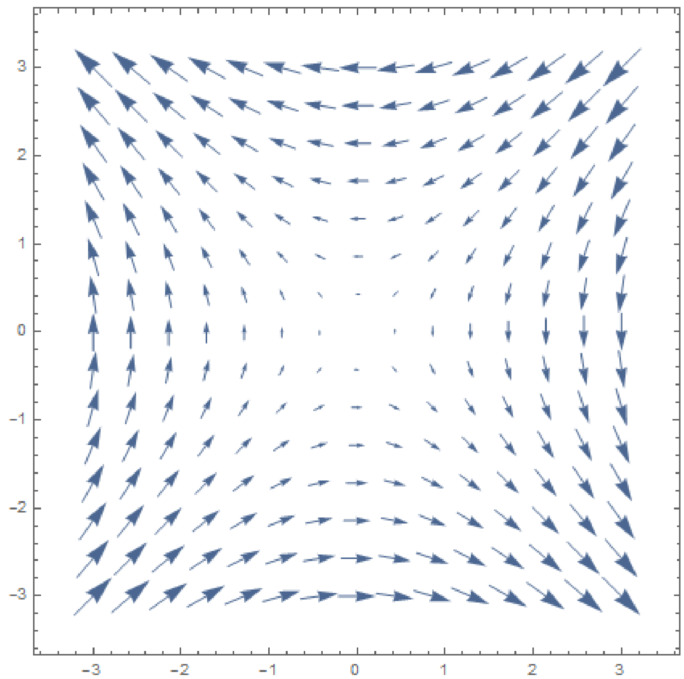

Figure 1: Vector Field Sketch for Problem 3, Part A

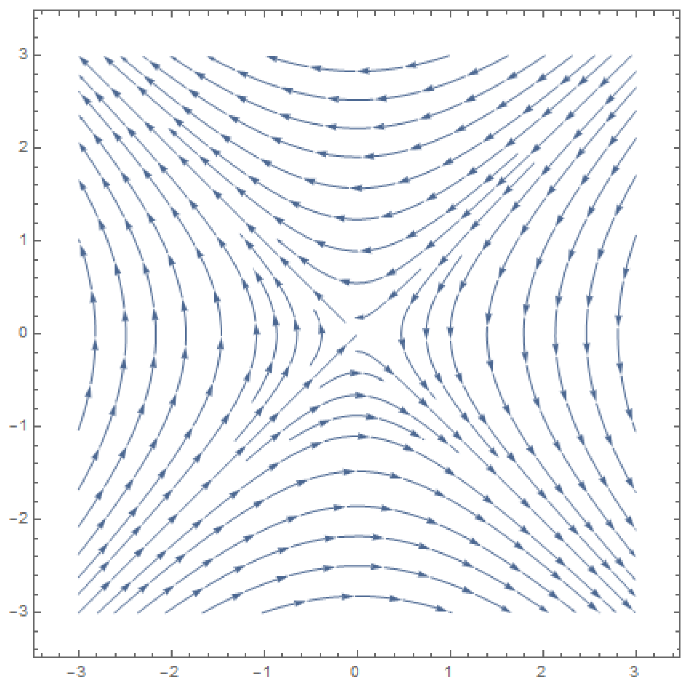

Figure 2: Stream Plot of Problem 3, Part A Vector Field

3.2 Part B

We are given $x'(t) = -y(t)$ and $y'(t) = -x(t)$
thus it holds that $x'(t)x(t) - y(t)y'(t) = -x(t)y(t) + x(t)y(t) = 0$

When we integrate the leftmost side and rightmost side we have

$$\int x(t)x'(t)\,dt - \int y(t)y'(t)\,dt = K$$

for some constant K
By the rule of integration by substitution $x'(t)dt = dx$ and $y'(t)dt = dy$ thus we have

$$\int x\,dx - \int y\,dy = K$$

for some constant K
Doing the integration we now have

$$x^2 - y^2 = C$$

for some constant C

3.3 Part D

It holds by derivative rules that

$$u'(t) = x'(t) + y'(t)$$

Plugging in the equations we have

$$u'(t) = -y - x = -(x + y)$$

This means that $u' = -u$

It also holds that

$$v'(t) = x'(t) - y'(t)$$

Plugging in equations gives us

$$v'(t) = -y + x = x - y$$

Thus we have $v' = v$

Since $u' = -u$ the solution is of the form $u(t) = A_1 e^{-t}$ for some constant A_1
Since $v' = v$ the solution has the form $v(t) = A_2 e^t$ for some constant A_2

It will then hold that $u(0) = A_1 = u_0$ and $v(0) = A_2 = v_0$

Thus the solution will be as follows

$$u(t) = u_0 e^{-t}$$

$$v(t) = v_0 e^t$$

3.4 Part F

It holds that

$$x(t) = \frac{u(t) + v(t)}{2}$$

$$y(t) = \frac{u(t) - v(t)}{2}$$

which means that

$$x(t) = \frac{u_0 e^{-t} + v_0 e^t}{2}$$

$$y(t) = \frac{u_0 e^{-t} - v_0 e^t}{2}$$

We need to express u_0 and v_0 in terms of $x_0 = x(0)$ and $y_0 = y(0)$
By definition then $u_0 = x_0 + y_0$ and $v_0 = x_0 - y_0$ thus it holds that

$$x(t) = \frac{(x_0 + y_0)e^{-t} + (x_0 - y_0)e^t}{2}$$

$$y(t) = \frac{(x_0 + y_0)e^{-t} - (x_0 - y_0)e^t}{2}$$

Thus our ending solution is as follows

$$x(t) = \frac{x_0(e^{-t} + e^t) + y_0(e^{-t} - e^t)}{2}$$

$$y(t) = \frac{x_0(e^{-t} - e^t) + y_0(e^{-t} + e^t)}{2}$$

4 Problem 4

4.1 Part A

The following shows the command done and its results:

```
Amat1 = [3 -4;1 -1];
[P1,D1] = eig(Amat1)

%{
Results:
P1 =

    0.8944    0.8944
    0.4472    0.4472


D1 =

     1     0
     0     1
%}
```

I simplified the eigenvector as shown:

```
P1A = P1./(0.2*sqrt(5))
%{
Result:
P1A =

    2.0000    2.0000
    1.0000    1.0000
%}
```

Thus we have

$$w = \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Since the eigenvalue $\lambda = 1$ the general form is as follows

$$x(t) = 2c_1e^t$$

$$y(t) = c_1e^t$$

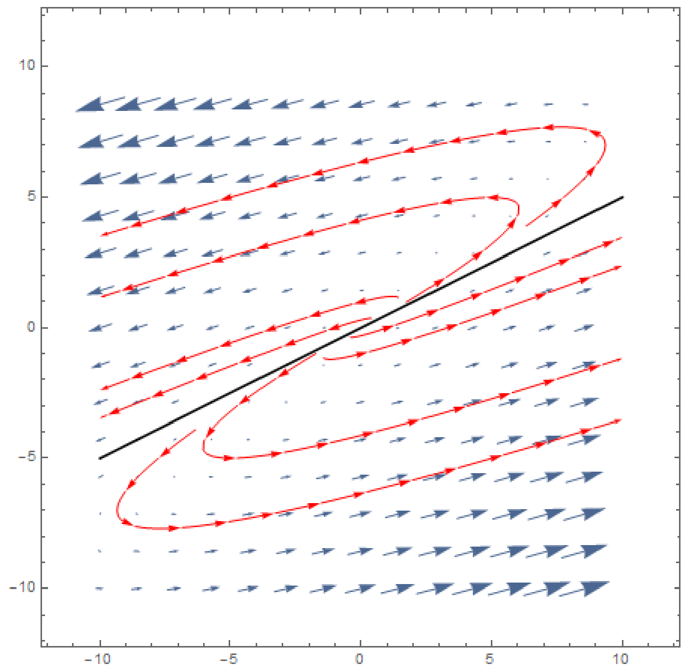

Figure 3: Phase Diagram for 4A. Trajectories are in red, eigenvector in black, and vector field in blue

4.2 Part B

The following shows the command done and its results

```
Amat2 = [5  2;-17  -5];
[P2,D2] = eig(Amat2)

%{
Results:
P2 =

   0.2782 + 0.1669i    0.2782 - 0.1669i
  -0.9459             -0.9459



D2 =

        0 + 3.0000i         0
        0                   0 - 3.0000i
%}
```

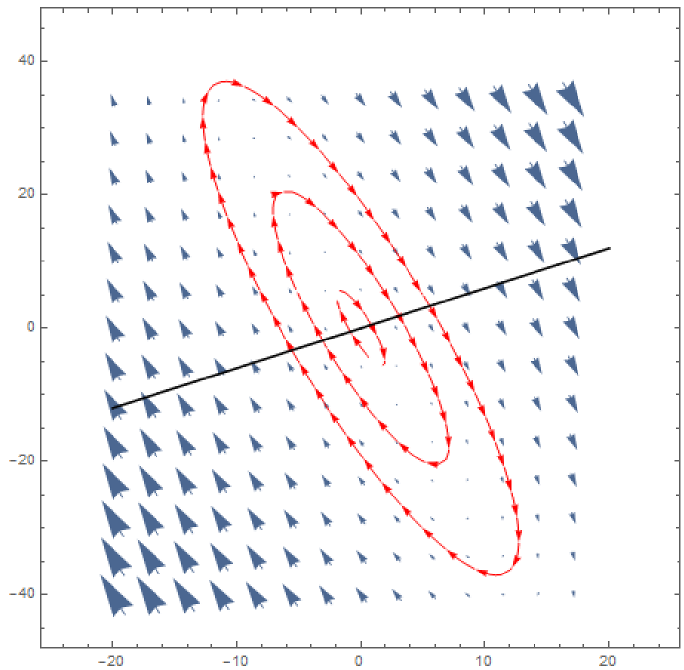

Figure 4: Phase Diagram for 4B. Trajectories are in red and vector field in blue. Real part of eignvector is in black

If we let $\lambda = a + ib$ then in our case $a = 0$,$b = 3$
If the eigenvector is $w = u + iv$ then $u = (0.2782, -0.9459)$ and $v = (0.1669, 0)$
According to the theorem then, the following are solutions

$$sin(3t)u + cos(3t)v$$

$$cos(3t)u - sin(3t)v$$

Thus the general form will be as follows

$$x(t) = c_1[0.2782sin(3t) + 0.1669cos(3t)] + c_2[0.2782cos(3t) - 0.1669sin(3t)]$$

$$y(t) = c_1[-0.9459sin(3t)] + c_2[-0.9459cos(3t)] = -0.9459[c_1sin(3t) + c_2cos(3t)]$$

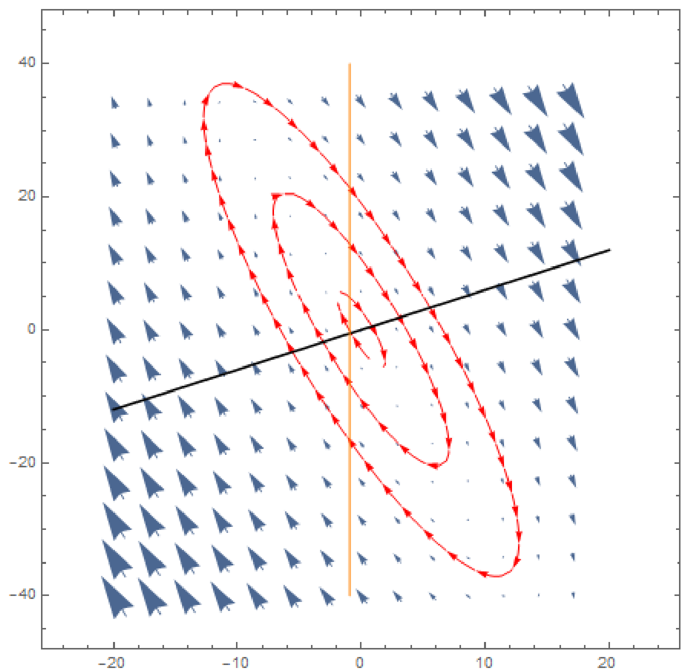

Figure 5: Phase Diagram for 4B with both eigenvector parts. Trajectories are in red, real part of eigenvector in black, imaginary part of eigenvector in orange, and vector field in blue

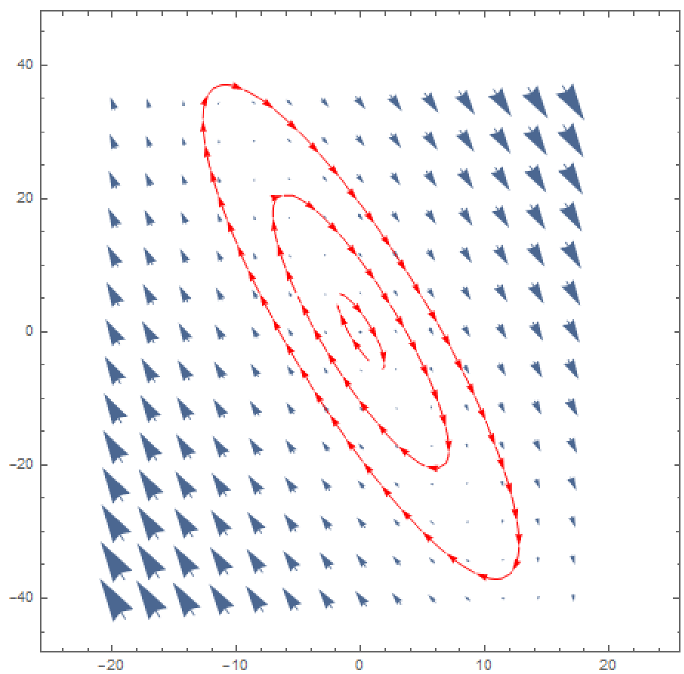

Figure 6: Phase Diagram for 4B without the eigenvectors. Trajactories are in red and vector field in blue

4.3 Part C

The following shows the command line, its results, and then simplifications of the eigenvectors

```
%Part C matrix
Amat3 = [4 -3;8 -6];
[P3,D3] = eig(Amat3)

%{
Results:
P3 =

    0.6000    0.4472
    0.8000    0.8944


D3 =

    0.0000         0
         0   -2.0000
%}

P3A = P3./(0.2*sqrt(5))
%{
Results:

1.3416    1.0000
1.7889    2.0000
%}

P3B = P3.*5
%{
Result:
P3B =

    3.0000    2.2361
    4.0000    4.4721
%}
```

We can now say that $\lambda_1 = 0$ and $\lambda_2 = -2$ and
$w_1 = (3, 4)$ and $w_2 = (1, 2)$
thus the solution in general form is as follows

$$x(t) = 3c_1 + c_2e^{-2t}$$

$$y(t) = 4c_1 + 2c_2e^{-2t}$$

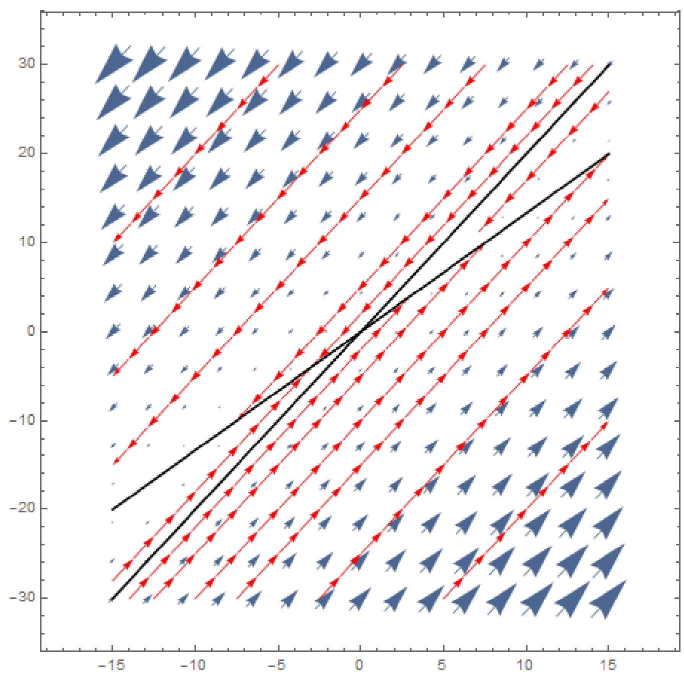

Figure 7: Phase Diagram for 4C. Trajactories are in red, eigenvectors in black, and vector field in blue

5 Problem 5

5.1 Part A

Since $\lambda = 4$ is the largest eigenvalue, the system will be unstable and diverge to infinity

5.2 Part B

There is a complex eigenvalue $\lambda = -1 + 3i$ with a negative real root and the other eigenvalues are negative. Because of the negatives, it will converge. Due to the imaginary part of the one of the eigenvalues, there will be oscillatory behavior as it converges toward its fixed point.

5.3 Part C

Since $\lambda = 2 + i$ is a complex eigenvalue with a positive real root, the system will diverge to infinity and oscillate as it does that.

5.4 Part D

Same behavior as B: there is a complex eigenvalue $\lambda = -3 + 4i$ with a negative real root and the other eigenvalues are negative. Because of the negatives, it will converge. Due to the imaginary part of the one of the eigenvalues, there will be oscillatory behavior as it converges toward its fixed point.

6 Problem 6

Solving the system, we first have

$$y_3'(t) = \lambda y_3$$

Solving this one yields

$$y_3 = c_3 e^{\lambda t}$$

for some constant c_3

Solving for y_2 we first have

$$y_2'(t) = \lambda y_2 + y_3$$

Plugging in above result yields

$$y_2'(t) = \lambda y_2 + c_3 e^{\lambda t}$$

We now have a linear ODE with $p(t) = -\lambda$ and $q(t) = c_3 e^{\lambda t}$
The integrating factor is thus $E(t) = e^{-\lambda t}$ and our solution will be

$$y_2 = e^{\lambda t} \int e^{-\lambda t} e^{\lambda t} c_3 dt = e^{\lambda t} \int c_3 dt = (c_3 t + c_2) e^{\lambda t}$$

hence finally

$$y_2 = c_2 e^{\lambda t} + c_3 t e^{\lambda t}$$

Solving for y_1 we have

$$y_1'(t) = \lambda y_1 + y_2$$

Plugging in above result we have

$$y_1'(t) = \lambda y_1 + c_2 e^{\lambda t} + c_3 t e^{\lambda t}$$

We have a linear ODE with $p(t) = -\lambda$ and $q(t) = c_2 e^{\lambda t} + c_3 t e^{\lambda t}$
Integrating factor $E(t) = e^{-\lambda t}$ and our solution will thus be

$$y_1 = e^{\lambda t} \int e^{-\lambda t} (c_2 e^{\lambda t} + c_3 t e^{\lambda t})\, dt = e^{\lambda t} \int c_2 + c_3 t\, dt = e^{\lambda t}(c_1 + c_2 t + c_3 t^2/2)$$

Putting y_1, y_2, y_3 together from above, it holds as required that

$$\begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} = e^{\lambda t} \begin{pmatrix} 1 & t & t^2/2 \\ 0 & 1 & t \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ c_3 \end{pmatrix}$$