Math227A-HW5

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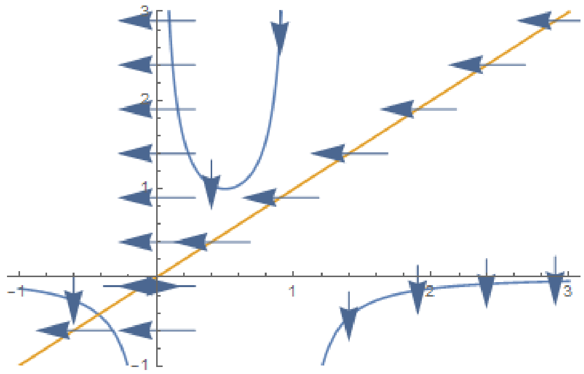

Figure 1: Null Cline analysis when $\beta/r_E = 0.25$

1 Problem 1, 8.1.10 from book

1.1 Part A

r_S is the rate constant for condition of the forest
r_E is rate constant for energy reserve
K_S is unit for forest condition
K_E is unit for energy reserve
P is rate at which energy is used by forest condition

1.2 Part B

For non-dimensionalization, let $S = K_S U$, $E = K_E V$, so that

$$K_S U' = r_S K_S U(1 - \frac{U}{V})$$

$$U' = r_S U(1 - \frac{U}{V})$$

With the other equation

$$K_E V' = r_E K_E V(1 - V) - P\frac{B}{K_S U}$$

Simplifying

$$V' = r_E V(1 - V) - \frac{\beta}{U}$$

where $\beta = PB/(K_S K_E)$

1.3 Part C

Figure 1 shows the nullclines for $\beta/r_E = 0.25$
Solving $U' = 0$ we have that either $U = 0$ or $U = V$
Solving $V' = 0$ we have

$$U = \frac{\beta}{r_E V(1 - V)}$$

$U = 0$ is not a solution so we need to find when $U = V$. We have

$$r_E V^2(1 - V) = \beta$$

For brevity, let $\gamma = \beta/r_E$ then we need to find out for $\gamma > 0$ when

$$V^2(1 - V) = \gamma$$

Let $f(V) = V^2(1 - V)$. We need $f(V) = \gamma$. We only care about $V \geq 0$. If $V = 0$ or $V \geq 1$, then $f(V) \leq 0$ so the interval $0 < V < 1$ matters to us.

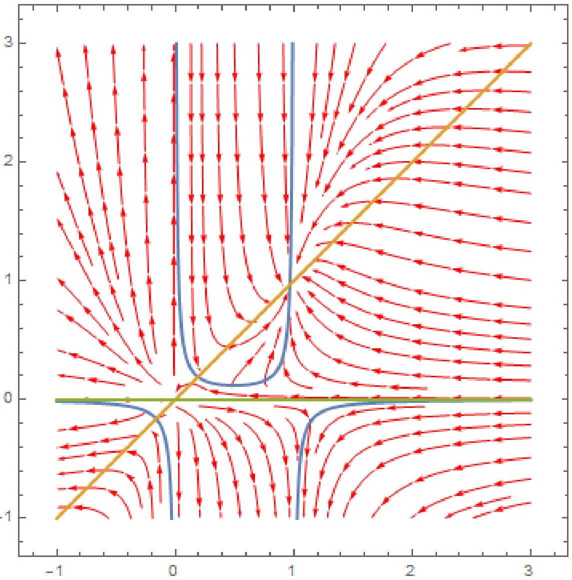

Figure 2: portrait for $\gamma = 0.03$

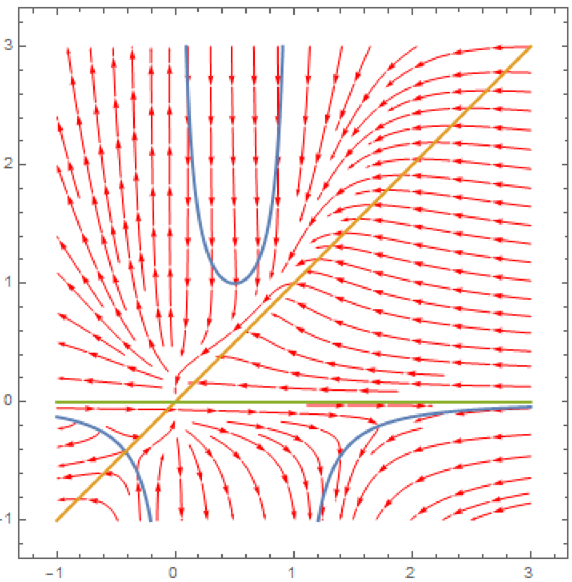

Figure 3: portrait for $\gamma = 0.25$

It holds that $f(V) = V^2 - V^3$ so $f'(V) = 2V - 3V^2$ and thus $f'(V) = 0$ when $V = 2/3$ hence the max value of f in that interval is $(2/3)^2(1/3) = 4/27$.

Thus if $\gamma < 4/27$ then there will be two values were the equation is satsified. Otherwise there will be no values satsifying the equation. This will be saddle node bifurcation.

1.4 Part D

Figures 2 and 3 show phase portraits.

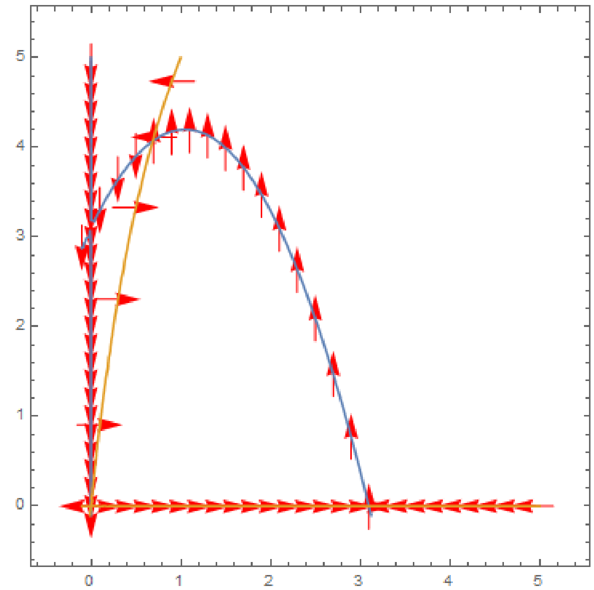

Figure 4: Nullclines for Problem 2

2 Problem 2, Book 8.2.9

2.1 Part A

Figure 4 show the nullclines. From figure 5, it appears that there will is one stable steady state inside the first quadrant that is stable and the other steady states are unstable. This likely indicates saddle node bifurcation.

2.2 Part B

$x' = 0$ means either $x = 0$ or

$$\frac{y}{1+x} = b - x$$
$$y = (b-x)(x+1)$$

y'=0 means $y = 0$ or

$$ay = \frac{x}{x+1}$$
$$y = \frac{x}{a(x+1)}$$

We need $x > 0$ and $y > 0$, so we need to see that

$$(b-x)(x+1) = \frac{x}{a(x+1)}$$

thus

$$a(b-x)(x+1)^2 = x$$

Both sides are continuous functions. The right hand side will start at zero and increase monotonically. The left hand side will start at $a*b$ and decrease monotonically due to the $-x$ term. Thus at some positive x value, both sides will have the same value.

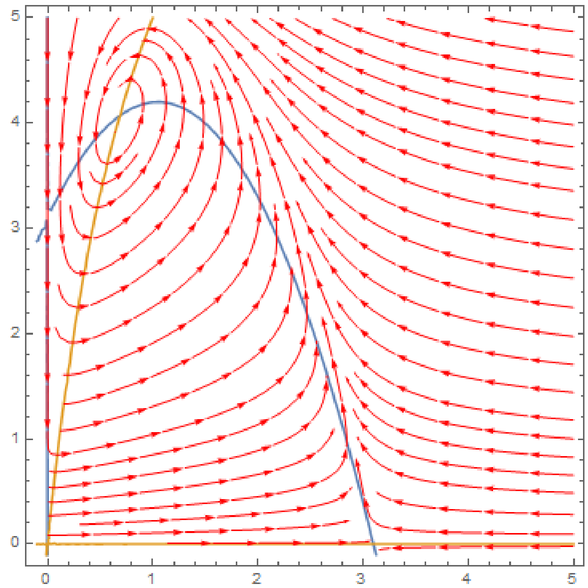

Figure 5: Phase Portrait for Prob 2

2.3 Part C

Let $x' = f_1$ and $y' = f_2$. We need $tr(J) = 0$, thus it needs to hold that

$$\frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} = 0$$

It holds that

$$\frac{\partial f_1}{\partial x} = x(-1 + \frac{y}{(1+x)^2}) + (b - x - \frac{y}{1+x})$$

At the fixed point, using the x' equation, it holds that $y/(1+x) = (b-x)$ hence we can simplify

$$\frac{\partial f_1}{\partial x} = x(-1 + \frac{b-x}{1+x}) + (b - x - (b-x))$$

$$\frac{\partial f_1}{\partial x} = \frac{x(b-x)}{1+x} - x$$

It also holds that

$$\frac{\partial f_2}{\partial y} = \frac{x}{1+x} - 2ay$$

At the fixed point, using the y' equation, it holds that $ay = x/(x+1)$ hence we can simplify

$$\frac{\partial f_2}{\partial y} = \frac{-x}{1+x}$$

Combining the partials it holds that

$$tr(J) = \frac{x(b-x) - x}{1+x} - x$$

$$tr(J) = \frac{x(b-x) - x - x(1+x)}{1+x}$$

Solving for $tr(J) = 0$ we can cancel a factor of $x/(1+x)$ giving us

$$(b-x) - 1 - (1+x) = 0$$

$$-2x + b - 2 = 0$$

$$2x = b - 2$$

We can now say that $b = 2x + 2$ and plug that into the fixed point condition found in Part B to say that

$$a(x+2)(x+1)^2 = x$$

Thus we have

$$a = \frac{x}{(x+2)(x+1)^2}$$

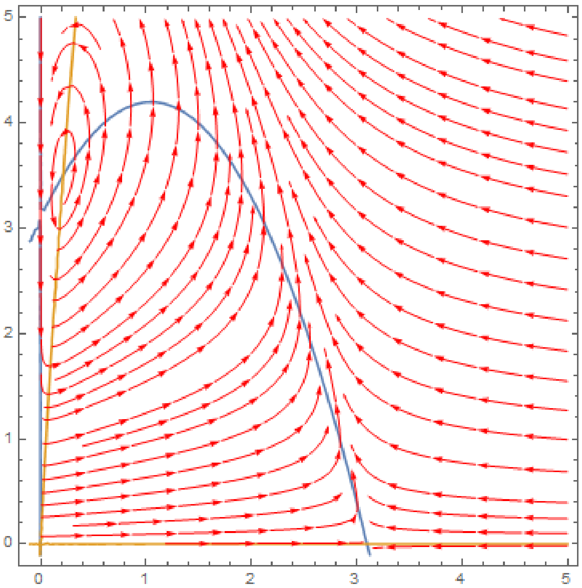

Figure 6: Phase Portrait for $b = 3.1$, $a = 0.05$ so $a < a_c$

Plugging in $x = (b-2)/2$ it holds that

$$x + 2 = \frac{b-2}{2} + 2 = \frac{b-2+4}{2} = \frac{b+2}{2}$$

$$x + 1 = \frac{b-2}{2} + 1 = \frac{b-2+2}{2} = \frac{b}{2}$$

Putting these into a we have

$$a = \frac{4(b-2)}{(b+2)b^2}$$

This is the equation from the book

2.4 Part D

Figure 6, 7, and 8 show the phase portraits before, at, and after the Hopf bifurcation point. Figure 7 seems to show a stable limit cycle, whereas the system in Figure 6 seems to diverge while Figure 8 converges, showing that we have Hopf Bifurcation.

In our case the decay rate seems to become smaller and smaller and then turn to growth when $a > a_c$. This is the behavior of supercritical Hopf Bifurcation.

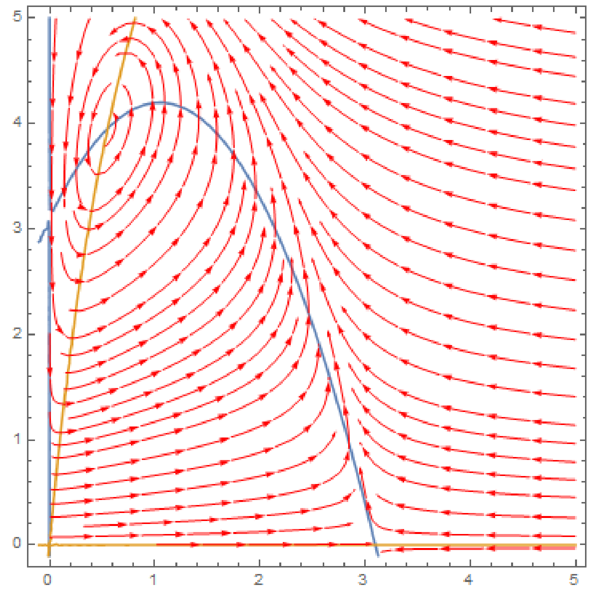

Figure 7: Phase Portrait for $b = 3.1$, $a = 0.089$ so $a = a_c$

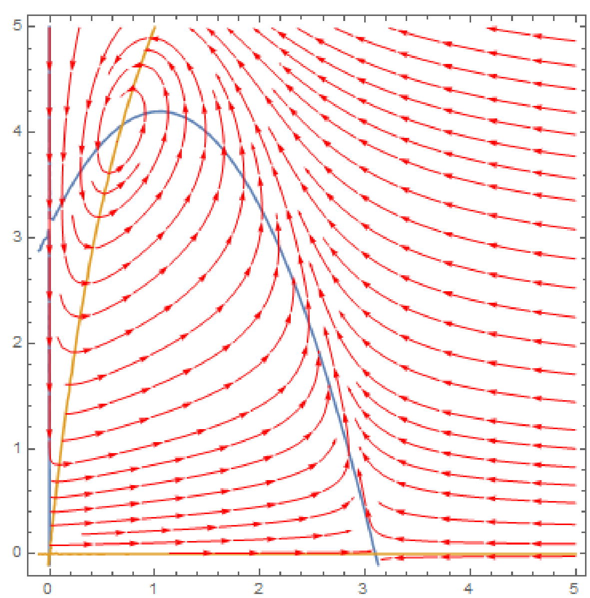

Figure 8: Phase Portrait for $b = 3.1$, $a = 0.1$ so $a > a_c$

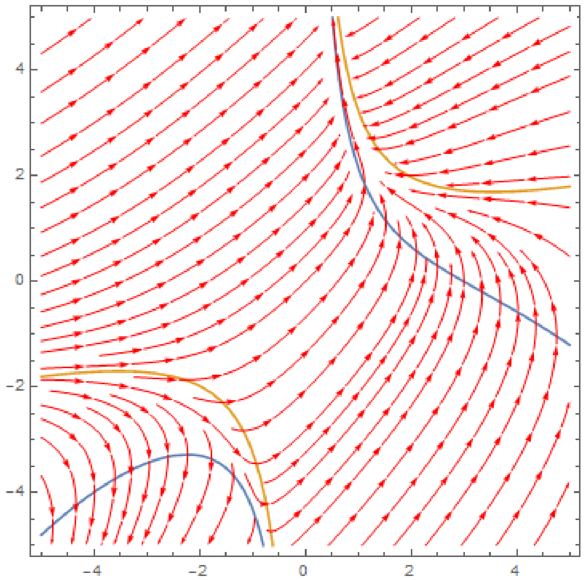

Figure 9: Phase Portrait for $B = 3$, $A = 3$ and $q = 0.08$

3 Problem 3, 8.2.10

To solve for $x' = 0$ we have

$$x' = B - x - \frac{xy}{1 + qx^2} = 0$$

implies that

$$B - x = \frac{xy}{1 + qx^2}$$

To solve $y' = 0$ we have

$$y' = A - \frac{xy}{1 + qx^2} = 0$$

implies that

$$A = \frac{xy}{1 + qx^2}$$

We know $A > 0$ thus $y > 0$. Also $q > 0$ so $1 + qx^2 > 0$. We are thus only left with the following: $B - x = A$ so $x = B - A$

When $B - A = 0$, it holds that $x' = B$ and $y' = A$ thus there is no steady state, as shown in figure 9. According to the stream plots, it appears that there is one stable steady state when $B - A > 0$, as shown in figure 10, and one unstable steady state when $B - A < 0$, as shown in 11.

In the case where $B > A$, there seems to be a trapping region around the steady of $x = B - A$. Figure 12 shows the null clines which further support the proposition.

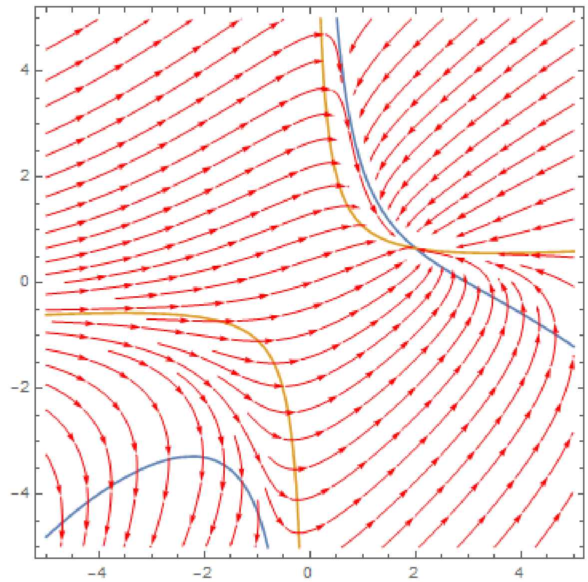

Figure 10: Phase Portrait for $B = 3$, $A = 1$ and $q = 0.08$

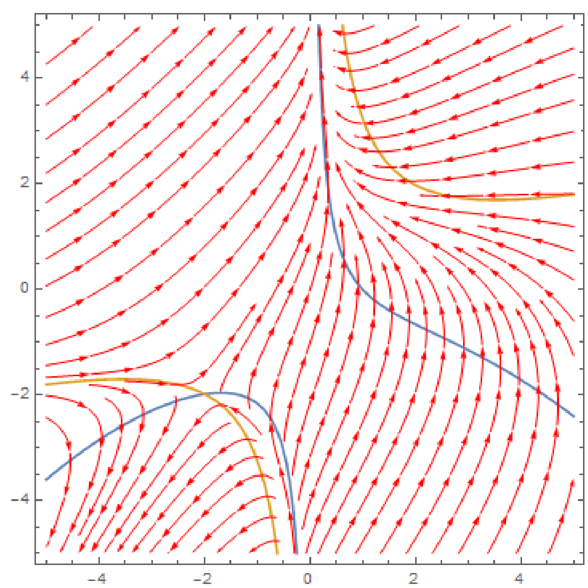

Figure 11: Phase Portrait for $B = 1$, $A = 3$ and $q = 0.08$

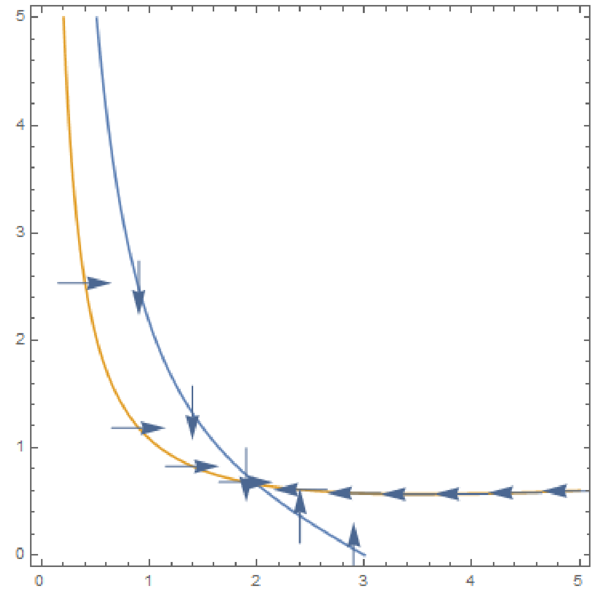

Figure 12: Null Clines for $B = 3$, $A = 1$ and $q = 0.08$

4 Problem 4

Here is the chemical equation with the constants
$3\,\mathrm{M} + 2\,\mathrm{E} \underset{\mathrm{k_2}}{\overset{\mathrm{k_1}}{\rightleftharpoons}} \mathrm{C} \xrightarrow{\mathrm{k_3}} 2\,\mathrm{E} + \mathrm{P}$

4.1 Part A

This will produce the following differential equations

$$E' = -k_1E^2M^3 + k_2C + k_3C$$

$$M' = -k_1E^2M^3 + k_2C$$
$$C' = k_1E^2M^3 - k_2C - k_3C$$
$$P' = k_3C$$

4.2 Part B

The stoichiometry matrix is as follows

.	R_1	R_2	R_3
E	-1	1	1
M	-1	1	0
C	1	-1	-1
P	0	0	1

The reaction speed vector $\omega(y)$ is as follows

$$\begin{array}{c} k_1E^2M^3 \\ k_2C \\ k_3C \end{array}$$

4.3 Part C

Intuitively when I inspect the chemical network, the following equations should be the mass conservation equations

$$M_{tot} = M + C + E + P$$

$$E_{tot} = C + E + P$$
$$C_{tot} = M + C + E + P$$
$$P_{tot} = P$$

4.4 Part D

The rank of the matrix is 2 so that is the number of mass conservation equations.
The first column is a multiple of the second column.
The first and third column are not multiples

The y vectors $(1, 0, 1, 0)$ and $(-1, 1, 0, 1)$ fit the properties of mass conservation vectors, $y \cdot \Gamma = 0$.

They were computed by finding the null space of Γ^T

5 Problem 5, Book 8.3.1

5.1 System that will produce it

We have $x' = ax^2y - bx - x + 1$ and $y' = bx - ax^2y$.
Let A, B, C, D be outside reactants
For x' we need a constant term added to it, so the following reaction will do that

$$\mathrm{A} \xrightarrow{k_1} \mathrm{X}$$

For the $-x$ term we need a reaction that takes from X and does not add back to it, thus we have

$$\mathrm{X} \xrightarrow{k_2} \mathrm{B}$$

To produce ax^2y terms as they stand in both cases, the following reaction would do

$$2\,\mathrm{X} + \mathrm{Y} \xrightarrow{k_3} 3\,\mathrm{X}$$

To produce the bx terms as they stand in both cases, the following reaction does it

$$\mathrm{C} + \mathrm{X} \xrightarrow{k_4} \mathrm{Y} + \mathrm{D}$$

We now have

$$dX/dt = k_1A - k_2X + k_3X^2Y - k_4CX$$

and

$$dY/dt = -k_3X^2Y + k_4CX$$

For non-dimensionalization, let $X = Gx$, $Y = Hy$, $t = I\tau$, so that

$$\frac{Gdx}{Id\tau} = k_1A - k_2Gx + k_3G^2X^2Hy - k_4CGx$$

$$\frac{Hdy}{Id\tau} = -k_3G^2X^2Hy + k_4CGx$$

Dividing G on one equation and H on the other gives us

$$\frac{dx}{Id\tau} = \frac{k_1A}{G} - k_2x + k_3Gx^2Hy - k_4Cx$$

$$\frac{dy}{Id\tau} = -k_3G^2x^2y + \frac{k_4CG}{H}x$$

Multiplying both by I gives us

$$\frac{dx}{d\tau} = \frac{k_1AI}{G} - k_2Ix + k_3GIHx^2y - k_4CIx$$

$$\frac{dy}{d\tau} = -k_3IG^2x^2y + \frac{k_4CGI}{H}x$$

We need $k_3IG^2 = k_3GIH$, so $G = H$
It has now been reduced to

$$\frac{dx}{d\tau} = \frac{k_1AI}{G} - k_2Ix + k_3G^2Ix^2y - k_4CIx$$

$$\frac{dy}{d\tau} = -k_3IG^2x^2y + k_4CIx$$

We need a $-x$ term in x' thus $I = 1/k_2$ reducing the x' term to

$$\frac{dx}{d\tau} = \frac{k_1AI}{G} - x + k_3G^2Ix^2y - k_4CIx$$

Finally we will let $G = k_1AI = \frac{k_1A}{k_2}$ so that

$$\frac{dx}{d\tau} = 1 - x + k_3G^2Ix^2y - k_4CIx$$

Thus with the following parameters

$$G = H = \frac{k_1 A}{k_2}$$

$$I = \frac{1}{k_2}$$

$$a = k_3 G^2 I$$

$$b = k_4 C I$$

We finally obtain the following

$$\frac{dx}{d\tau} = 1 - x + ax^2y - bx$$

$$\frac{dy}{d\tau} = -ax^2y + bx$$

5.2 Part A

Fixed points occur when $y' = 0$ so we have $x = 0$ or $axy = b$

We also need $x' = 0$. That won't happen if $x = 0$. Plugging in $axy = b$ gives us

$$1 - (b+1)x + bx = 0$$

$$x = 1$$

Hence we will just have one fixed point at $(1, b/a)$

Letting $x' = f_1$ and $y' = f_2$ we have

$$\frac{\partial f_1}{\partial x} = -(b+1) + 2axy$$

$$\frac{\partial f_2}{\partial x} = b - 2axy$$

$$\frac{\partial f_1}{\partial y} = ax^2$$

$$\frac{\partial f_2}{\partial y} = -ax^2$$

Plugging in the fixed point

$$\frac{\partial f_1}{\partial x} = b - 1$$

$$\frac{\partial f_2}{\partial x} = -b$$

$$\frac{\partial f_1}{\partial y} = a$$

$$\frac{\partial f_2}{\partial y} = -a$$

hence we have J =

$$\begin{pmatrix} b-1 & a \\ -b & -a \end{pmatrix}$$

thus we have $tr(J) = b - a - 1$ and $det(J) = a$
Since $det(J) > 0$ it holds that the steady state is either a node or spiral.
If $tr(J)^2 > 4 \cdot det(J)$ so $(b-a-1)^2 > 4a$ we have a node.
If $tr(J)^2 < 4 \cdot det(J)$ so $(b-a-1)^2 < 4a$ we have a spiral.

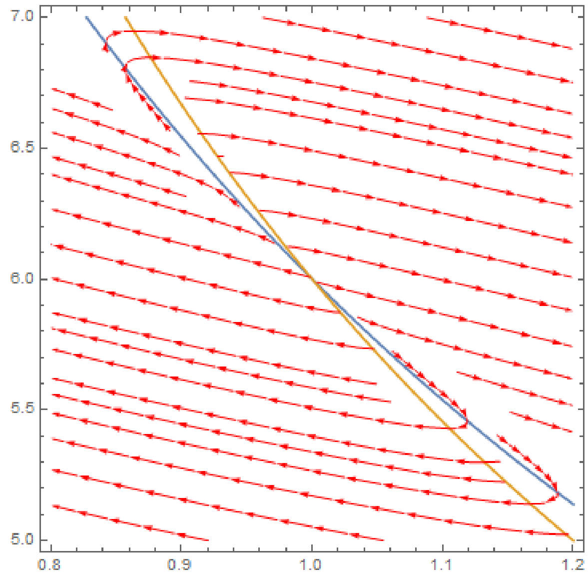

Figure 13: Phase Portrait for $a = 1$ and $b = 6$

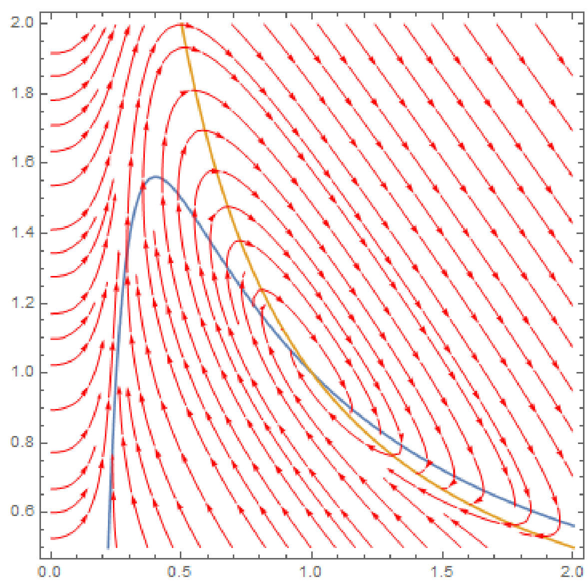

Figure 14: Phase Portrait for $a = 4$ and $b = 4$

5.3 Part B

Figure 13 shows a node example, so there is no trapping region. Figure 14 shows a spiral example, so there is a trapping region.

5.4 Part C

Hopf Bifurcation occurs when $tr(J) = 0$ so $b_c = a + 1$