Math227A-HW1

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1 Problem 1

1.1 Part A

This is a linear ODE

$$p(t) = 1$$

$$q(t) = te^{-t} + 1$$

The integrating factor $E(t)$ is as follows: $E(t) = e^t$ This means the solution looks like the following:

$$y = \frac{1}{e^t}(\int e^t(te^{-t} + 1)dt + C)$$

Simplifying and solving the integral we obtain

$$y = \frac{1}{e^t}(\int (t + e^t)dt + C)$$

$$y = \frac{1}{e^t}(\frac{t^2}{2} + e^t + C)$$

Finally we obtain

$$y = \frac{t^2}{2e^t} + C$$

1.2 Part B

We start with

$$2y' - y = e^{t/3}$$

Dividing by 2 on both sides

$$y' - \frac{y}{2} = \frac{e^{t/3}}{2}$$

We now have a linear ODE

$$p(t) = -1/2$$

$$q(t) = \frac{e^{t/3}}{2}$$

The integrating factor is thus as follows

$$E(t) = e^{-t/2}$$

Thus the solution will be as follows

$$y = \frac{e^{t/2}}{2}(\int e^{-t/2}e^{t/3}dt + C)$$

$$y = \frac{e^{t/2}}{2}(\int e^{t/6}dt + C)$$

Integrating we obtain

$$y = 3e^{t/2}(e^{t/6} + C)$$

Finally we have

$$y = 3\sqrt[3]{e^{2t}} + C\sqrt{e^t}$$

2 Problem 2

2.1 Part A

Using separation of variables we obtain

$$\frac{dy}{y} = 5dx$$

Integrating both sides gives us the following

$$ln(y) = 5x + C$$

The final solution is thus

$$y = Ae^{5x}$$

for some positive constant A

2.2 Part B

Our equation is as follows

$$\frac{dy}{dt} = (t+1)^2 y$$

Separating the variables we obtain

$$\frac{dy}{y} = (t+1)^2 dt$$

Integrating both sides we obtain

$$ln(y) = \frac{1}{3}(t+1)^3 + C$$

Solving for y we obtain

$$y = A\sqrt[3]{e^{(t+1)^3}}$$

for some positive constant A

2.3 Part C

Rearranging this equation we obtain

$$y' = -2ty^2$$

Separating variables we obtain

$$\frac{dy}{y^2} = -2tdt$$

Integrating both sides we obtain

$$\frac{-1}{y} = -t^2 + C$$

Our final solution is thus as follows

$$y = \frac{1}{t^2 + C}$$

for some constant C

2.4 Part D

Rearranging and separating variables we obtain

$$\frac{dy}{\sqrt{1-y^2}} = tdt$$

Integrating both sides we obtain

$$arcsin(y) = \frac{t^2}{2} + C$$

The final solution is thus

$$y = sin(\frac{1}{2}t^2 + C)$$

for some constant C.

3 Problem 3

3.1 Part A

Rearranging we have

$$\frac{dy}{y} = -2dx$$

Integrating we have

$$ln(y) = -2x + C$$

Thus the general solution is as follows

$$y = Ae^{-2x}$$

for some positive constant A
Since y(0)=2 we can say that

$$2 = Ae^{-2*0}$$

which simplifies to $A = 2$
Thus our unique solution is as follows

$$y = 2e^{-2x}$$

3.2 Part B

Rearranging we end up with

$$\frac{dy}{y^2 - 1} = \frac{dt}{t^2 - 1}$$

After doing partial fractions we have

$$\frac{1}{2}(\frac{1}{y-1} - \frac{1}{y+1})dy = \frac{1}{2}(\frac{1}{t-1} - \frac{1}{t+1})dt$$

After multiplying both sides by 2 and integrating both sides we have

$$ln(1-y) - ln(1+y) = ln(1-t) - ln(1+t) + C$$

Simplifying we obtain

$$ln(\frac{1-y}{1+y}) = C + ln(\frac{1-t}{1+t})$$

Exponentiating both sides we obtain

$$(1-y)(1+t) = A(1-t)(1+y)$$

To solve for A, I plug in the initial condition of $y = 1$, $t = 2$
The equation then becomes $A = 0$
It must thus hold that either $1 - y = 0$ or $1 + t = 0$
Since we need a function in terms of t it must then be true that $y = 1$
Thus the final equation is the following

$$y(t) = 1$$

This equation satisfies the differential equation and initial condition

4 Problem 4

4.1 Part A

Dividing by t^2 gives us

$$y' + \frac{y}{t} = \frac{1}{t^2}$$

Thus I will use the linear ODE method with
$p(t) = 1/t$ and $q(t) = 1/(t^2)$
The integrating factor is thus as follows

$$E(t) = exp(ln(t)) = t$$

Thus our solution is

$$y = \frac{1}{t}\int \frac{1}{t}$$

$$y = \frac{ln(t)}{t}$$

4.2 Part B

Rearranging we have

$$y' + \frac{y}{t+1} = \frac{ln(t)}{t+1}$$

I will use the linear ODE method so

$$p(t) = \frac{1}{t+1}$$

$$q(t) = \frac{ln(t)}{t+1}$$

The integrating factor is thus as follows

$$E(t) = exp(ln(t+1))$$

$$E(t) = t+1$$

Our solution would thus be

$$y(t) = \frac{1}{t+1}(\int (t+1)\frac{ln(t)}{t+1}dt + C)$$

$$y(t) = \frac{1}{t+1}(\int ln(t)dt + C)$$

$$y(t) = \frac{t \cdot ln(t) - t + C}{t+1}$$

4.3 Part C

Rearranging this becomes

$$ty' + y'y = y$$

Dividing both sides by y we obtain

$$y'(1 + t/y) = 1$$

$$y' = \frac{1}{1 + (y/t)^{-1}}$$

This is a homogeneous ODE. Let

$$v(t) = y/t$$

Writing the ODE in terms of v we obtain

$$v't + v = \frac{1}{1 + 1/v}$$

This can be written as

$$v't + v = \frac{v}{v + 1}$$

Bringing the v over we obtain

$$v't = \frac{v - v(v + 1)}{v + 1}$$

$$v' = -\frac{v^2}{v + 1}\frac{1}{t}$$

This is a separable equation so we obtain

$$\frac{(v + 1)dv}{v^2} = -\frac{dt}{t}$$

$$(\frac{1}{v} + \frac{1}{v^2})dv = -\frac{dt}{t}$$

Integrating both sides gives us

$$ln(v) - \frac{1}{v} = -ln(t) + C$$

Expanding v we obtain the following

$$ln(y) - ln(t) - \frac{t}{y} = -ln(t) + C$$

The $ln(t)$ term cancels. Multiplying y on both sides now yields

$$yln(y) - t = Cy$$

The final implicit form is as follows

$$y(ln(y) - C) - t = 0$$

5 Problem 5

Here is a sketch of the possible justification of the separation of variables formula
We assume the equation has the following form

$$\frac{dy}{dx} = f(x)g(y)$$

Assume that x and y are functions of a parameter t
I am then assuming the following

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt}$$

Let $h(y) = 1/g(y)$ and assume prime denotes dt then we can say that

$$y'(t)h(y(t)) = x'(t)f(x(t))$$

Let H and F be antiderivatives of h and f respectively
Integrating over the same variable t on both sides and applying
the rule from integration by substitution it holds that

$$H(y) = F(x)$$

This is the same result as what the separation of variables method produces