Fourth-order Isotropic Tensor Derivation

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1 Introduction

Recall

$$Q_{ip}Q_{jq}Q_{kr}Q_{ls}\cdots T_{pqrs\cdots} = T_{ijkl\cdots} \qquad \forall Q_{mn}, \tag{$*$}$$

and

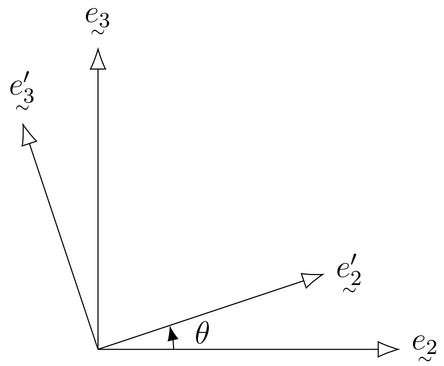
$$\underset{\sim}{Q} = \underset{\sim}{Q}(\underset{\sim}{n},\theta), \qquad \underset{\sim}{Q} = \underset{\sim}{Q}(\underset{\sim}{n},0) = \underset{\sim}{1},$$

then differentiate with respect to θ, $(*)$ becomes

$$\left(Q'_{ip}Q_{jq}Q_{kr}Q_{ls}\cdots + Q_{ip}Q'_{jq}Q_{kr}Q_{ls} + \cdots\right)T_{pqrs\cdots} = 0 \qquad \forall Q_{mn}.$$

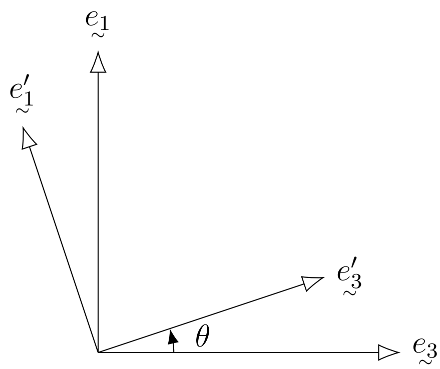
Let $\theta = 0$, $\underset{\sim}{\Omega} = \underset{\sim}{Q}'\Big|_{\theta=0}$, then

$$\left(\Omega_{ip}\delta_{jq}\delta_{kr}\delta_{ls}\cdots + \delta_{ip}\Omega_{jq}\delta_{kr}\delta_{ls}\cdots\right)T_{pqrs\cdots} = 0 \qquad \forall Q_{mn}.$$



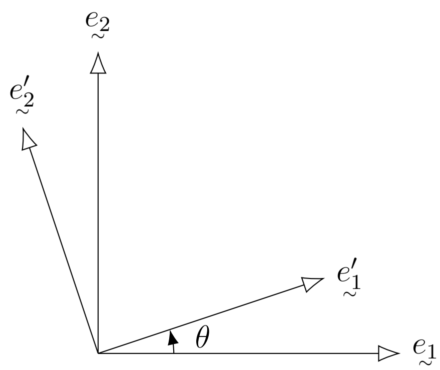
$$\left[\underset{\sim}{Q}^{(1)}\right] = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{bmatrix} \xrightarrow{\frac{d}{d\theta}} \begin{bmatrix} 0 & 0 & 0 \\ 0 & -\sin\theta & \cos\theta \\ 0 & -\cos\theta & -\sin\theta \end{bmatrix}.$$

$$\Longrightarrow \left[\underset{\sim}{\Omega}^{(1)}\right] = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{bmatrix} \Longrightarrow \Omega_{ij}^{(1)} = \epsilon_{1ij}.$$



$$\left[\underset{\sim}{Q}^{(2)}\right] = \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{bmatrix} \xrightarrow{\frac{d}{d\theta}} \begin{bmatrix} -\sin\theta & 0 & -\cos\theta \\ 0 & 0 & 0 \\ \cos\theta & 0 & -\sin\theta \end{bmatrix}.$$

$$\Longrightarrow \left[\underset{\sim}{\Omega}^{(2)}\right] = \begin{bmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix} \Longrightarrow \Omega_{ij}^{(2)} = \epsilon_{2ij}.$$



$$\left[\underset{\sim}{Q}^{(2)}\right] = \begin{bmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{bmatrix} \xrightarrow{\frac{d}{d\theta}} \begin{bmatrix} -\sin\theta & \cos\theta & 0 \\ -\cos\theta & -\sin\theta & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

$$\Longrightarrow \left[\underset{\sim}{\Omega}^{(3)}\right] = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \Longrightarrow \Omega_{ij}^{(3)} = \epsilon_{3ij}.$$

$$\therefore \Omega_{ij}^{(r)} = \epsilon_{rij}.$$

In particular, for rotation about a-th coordinate axis, $\Omega_{ij}^{(a)} = \epsilon_{aij}$.

$$\therefore \left(\epsilon_{aip}\delta_{jq}\delta_{kr}\delta_{ls}\cdots + \delta_{ip}\epsilon_{ajq}\delta_{kr}\delta_{ls}\cdots + \delta_{ip}\delta_{jq}\epsilon_{akr}\delta_{ls}\cdots\right) T_{pqrs\cdots} = 0 \qquad \forall a.$$

2 Rank 1 (vector)

$$Q_{ip}v_p = v_i \overset{\frac{d}{d\theta}}{\Longrightarrow} Q'_{ip}v_p = 0 \overset{\theta=0}{\Longrightarrow} \Omega_{ip}v_p = 0 \overset{a\text{-axis}}{\Longrightarrow} \epsilon_{aip}v_p = 0 \qquad \forall a, i.$$

$$\begin{aligned}
&\Longrightarrow \epsilon_{ai1}v_1 + \epsilon_{ai2}v_2 + \epsilon_{ai3}v_3 = 0,\\
&\underset{i=2}{\overset{a=1}{\Longrightarrow}} \epsilon_{12p}v_p = 0 \Longrightarrow v_3 = 0.\\
&\underset{i=3}{\overset{a=2}{\Longrightarrow}} \epsilon_{23p}v_p = 0 \Longrightarrow v_1 = 0.\\
&\underset{i=1}{\overset{a=3}{\Longrightarrow}} \epsilon_{31p}v_p = 0 \Longrightarrow v_2 = 0.
\end{aligned}$$

3 Rank 2 (matrix)

$$\begin{aligned}
Q_{ip}Q_{jq}T_{pq} = T_{ij} &\overset{\frac{d}{d\theta}}{\Longrightarrow} Q'_{ip}Q_{jq}T_{pq} + Q_{ip}Q'_{jq}T_{pq} = 0\\
&\overset{\theta=0}{\Longrightarrow} \Omega_{ip}\delta_{jq}T_{pq} + \delta_{ip}\Omega_{jq}T_{pq} = 0\\
&\overset{a\text{-axis}}{\Longrightarrow} \epsilon_{aip}\delta_{jq}T_{pq} + \delta_{ip}\epsilon_{ajq}T_{pq} = 0\\
&\Longrightarrow \epsilon_{aip}T_{pj} + \epsilon_{ajq}T_{iq} = 0, \qquad a, i, j \text{ are free indices}\\
&\underset{\text{by } \epsilon_{aik}}{\overset{\text{multiply}}{\Longrightarrow}} \epsilon_{aik}\epsilon_{aip}T_{pj} + \epsilon_{aik}\epsilon_{ajq}T_{iq} = 0\\
&\Longrightarrow \left(\delta_{ii}\delta_{kp} - \delta_{ip}\delta_{ki}\right)T_{pj} + \left(\delta_{ij}\delta_{kq} - \delta_{iq}\delta_{kj}\right)T_{iq} = 0\\
&\Longrightarrow 2\delta_{kp}T_{pj} + \delta_{ij}\delta_{kq}T_{iq} - \delta_{iq}\delta_{kj}T_{iq} = 0\\
&\left.\begin{aligned}&\Longrightarrow 2T_{kj} + T_{jk} - \delta_{kj}T_{ii} = 0\\ &\overset{k\leftrightarrow j}{\Longrightarrow} 2T_{jk} + T_{kj} - \delta_{jk}T_{ii} = 0\end{aligned}\right\} \underset{\text{system of eqn.}}{\overset{\text{solve}}{\Longrightarrow}} T_{jk} = \frac{T_{ii}}{3}\delta_{jk}.
\end{aligned}$$

If a 2^{nd} order $\underset{\sim}{T}$ is isotropic, then $\underset{\sim}{T} = \alpha\underset{\sim}{1}$.

4 Rank 3

$$Q_{ip}Q_{jq}Q_{kr}T_{pqr} = T_{ijk}$$

contraction on i, j

$$\begin{aligned}
&\Longrightarrow Q_{ip}Q_{iq}Q_{kr}T_{pqr} = T_{iik}\\
&\Longrightarrow \delta_{pq}Q_{kr}T_{pqr} = T_{iik}\\
&\Longrightarrow Q_{kr}T_{ppr} = T_{iik}\\
&\Longrightarrow T_{iik} \text{ is a rank 1 isotropic tensor}\\
&\Longrightarrow T_{iik} = 0
\end{aligned}$$

contraction on i, k, T_{iji} is a rank 1 isotropic tensor

$$\Longrightarrow T_{iji} = 0$$

contraction on j, k, T_{ijj} is a rank 1 isotropic tensor

$$\Longrightarrow T_{ijj} = 0$$

Contraction on any two indices leads to a zero 1^{st} order tensor

$$\left(\epsilon_{aip}\delta_{jq}\delta_{kr} + \delta_{ip}\epsilon_{ajq}\delta_{kr} + \delta_{ip}\delta_{jq}\epsilon_{akr}\right) T_{pqr} = 0$$

$\Longrightarrow \left(\epsilon_{aip}T_{pjk} + \epsilon_{ajq}T_{iqk} + \epsilon_{akr}T_{ijr}\right) = 0,$ $\qquad a, i, j, k$ are free indices

$\Downarrow$ contraction on (a, i), (a, j), (a, k) in turn and introduce a free index c

$\xRightarrow{\times\epsilon_{abc}} \epsilon_{abc}\epsilon_{aip}T_{pjk} + \epsilon_{abc}\epsilon_{ajq}T_{iqk} + \epsilon_{abc}\epsilon_{akr}T_{ijr} = 0$

$$\Longrightarrow \left(\delta_{bi}\delta_{cp} - \delta_{bp}\delta_{ci}\right) T_{pjk} + \left(\delta_{bj}\delta_{cq} - \delta_{bq}\delta_{cj}\right) T_{iqk} + \left(\delta_{bk}\delta_{cr} - \delta_{br}\delta_{ck}\right) T_{ijr} = 0 \qquad (\star)$$

$$(\star) \xRightarrow{b=i} \left(3\delta_{cp} - \delta_{cp}\right) T_{pjk} + \left(\delta_{ij}\delta_{cq} - \delta_{iq}\delta_{cj}\right) T_{iqk} + \left(\delta_{ik}\delta_{cr} - \delta_{ir}\delta_{ck}\right) T_{ijr} = 0$$

$$\begin{aligned}
&\Longrightarrow 2T_{cjk} + T_{jck} - \cancel{\delta_{cj}T_{iik}} + T_{kjc} - \cancel{\delta_{ck}T_{iji}} = 0\\
&[\text{Recall } T_{iik} = T_{iji} = T_{ijj} = 0]\\
&\Longrightarrow \boxed{2T_{cjk} + T_{jck} + T_{kjc} = 0}
\end{aligned}$$

$$(\star) \xRightarrow{b=j} \left(\delta_{ji}\delta_{cp} - \delta_{jp}\delta_{ci}\right) T_{pjk} + \left(3\delta_{cq} - \delta_{cq}\right) T_{iqk} + \left(\delta_{jk}\delta_{cr} - \delta_{jr}\delta_{ck}\right) T_{ijr} = 0$$

$$\begin{aligned}
&\Longrightarrow T_{cik} - \cancel{\delta_{ci}T_{jjk}} + 2T_{ick} + T_{ikc} - \cancel{\delta_{ck}T_{ijj}} = 0\\
&\Longrightarrow 2T_{cjk} + T_{jck} + T_{kjc} = 0\\
&\xRightarrow[c\to j]{i\to c} \boxed{2T_{cjk} + T_{jck} + T_{ckj} = 0}
\end{aligned}$$

$$(\star) \xRightarrow{b=k} \left(\delta_{ki}\delta_{cp} - \delta_{kp}\delta_{ci}\right) T_{pjk} + \left(\delta_{kj}\delta_{cq} - \delta_{kq}\delta_{jc}\right) T_{iqk} + (3\delta_{cr} - \delta_{cr})\, T_{ijr} = 0$$

$$\Longrightarrow T_{cji} - \cancel{\delta_{ci}T_{kjk}} + T_{icj} - \cancel{\delta_{jc}T_{ikk}} + 2T_{ijc} = 0$$

$$\Longrightarrow 2T_{ijc} + T_{cji} + T_{icj} = 0$$

$$\underset{c\to k}{\overset{i\to c}{\Longrightarrow}} \boxed{2T_{cjk} + T_{kjc} + T_{ckj} = 0}$$

$$\begin{cases} 2T_{cjk} + T_{jck} + T_{kjc} = 0 & (1) \\ 2T_{cjk} + T_{jck} + T_{ckj} = 0 & (2) \\ 2T_{cjk} + T_{kjc} + T_{ckj} = 0 & (3) \end{cases}$$

$$\Longrightarrow \begin{cases} T_{kjc} = T_{ckj} \\ T_{jck} = T_{kjc} \Longrightarrow T_{jck} = T_{kjc} = T_{ckj} \\ T_{jck} = T_{ckj} \end{cases}$$

$$(1) \xRightarrow{T_{kjc}=T_{jck}} 2T_{cjk} + 2T_{jck} = 0 \text{ or } 2T_{cjk} + 2T_{kjc} = 0$$

$$(2) \xRightarrow{T_{jck}=T_{ckj}} 2T_{cjk} + 2T_{ckj} = 0 \text{ or } 2T_{cjk} + 2T_{jck} = 0$$

Summary.

$$-T_{ijk} = T_{jik} = T_{kji} = T_{ikj}$$

$$\begin{cases} T_{\alpha\alpha\beta} = -T_{\alpha\alpha\beta} = 0 \\ T_{\alpha\beta\alpha} = -T_{\alpha\beta\alpha} = 0 \quad \text{No summation on Greek letters.} \\ T_{\beta\alpha\alpha} = -T_{\beta\alpha\alpha} = 0 \end{cases}$$

If a 3rd order tensor is isotropic, then $\underset{\sim}{T} = \mu\underset{\sim}{\epsilon}$.

5 Rank 4

$$\begin{aligned} &Q_{ip}Q_{jq}Q_{kr}Q_{ls}T_{pqrs} = T_{ijkl} \\ &\Downarrow \text{ contraction on } i, j \\ &Q_{ip}Q_{iq}Q_{kr}Q_{ls}T_{pqrs} = T_{iikl} \\ \Longrightarrow\ &\delta_{pq}Q_{kr}Q_{ls}T_{pqrs} = T_{iikl} \\ \Longrightarrow\ &Q_{kr}Q_{ls}T_{pprs} = T_{iikl} \\ \Longrightarrow\ &T_{iikl} \text{ is a rank 2 isotropic tensor} \\ \Longrightarrow\ &T_{iikl} = \alpha_{12}\delta_{kl} \end{aligned}$$

Contraction on any two indices leads to a rank 2 isotropic tensor, which is a scalar multiple of the identity tensor. ($C_2^4 = 6$)

$$T_{pprs} = \alpha_{12}\delta_{rs}, \qquad T_{pqps} = \alpha_{13}\delta_{qs}, \qquad T_{pqrp} = \alpha_{14}\delta_{qr},$$

$$T_{pqqs} = \alpha_{23}\delta_{ps}, \qquad T_{pqrq} = \alpha_{24}\delta_{pr}, \qquad T_{pqrr} = \alpha_{34}\delta_{pq}.$$

$$\left(\epsilon_{aip}\delta_{jq}\delta_{kr}\delta_{ls} + \delta_{ip}\epsilon_{ajp}\delta_{kr}\delta_{ls} + \delta_{ip}\delta_{jq}\epsilon_{akr}\delta_{ls} + \delta_{ip}\delta_{jq}\delta_{kr}\epsilon_{als}\right) T_{pqrs} = 0$$

$$\Longrightarrow \epsilon_{aip}T_{pjkl} + \epsilon_{ajq}T_{iqkl} + \epsilon_{akr}T_{ijrl} + \epsilon_{als}T_{ijks} = 0, \qquad a, i, j, k, l \text{ are free indices}$$

$\Downarrow$ contraction on (a, i), (a, j), (a, k), (a, l) in turn and introduce a free index c

$$\xRightarrow{\times\epsilon_{abc}} \epsilon_{abc}\epsilon_{aip}T_{pjkl} + \epsilon_{abc}\epsilon_{ajq}T_{iqkl} + \epsilon_{abc}\epsilon_{akr}T_{ijrl} + \epsilon_{abc}\epsilon_{als}T_{ijks} = 0$$

$$\Longrightarrow \left(\delta_{bi}\delta_{cp} - \delta_{bp}\delta_{ci}\right) T_{pjkl} + \left(\delta_{bj}\delta_{cq} - \delta_{bq}\delta_{cj}\right) T_{iqkl} + \left(\delta_{bk}\delta_{cr} - \delta_{br}\delta_{ck}\right) T_{ijrl} + \left(\delta_{bl}\delta_{cs} - \delta_{bs}\delta_{cl}\right) T_{ijks} = 0$$

$$\xRightarrow{b=i} \left(\delta_{ii}\delta_{cp} - \delta_{ip}\delta_{ci}\right) T_{pjkl} + \left(\delta_{ij}\delta_{cq} - \delta_{iq}\delta_{cj}\right) T_{iqkl} + \left(\delta_{ik}\delta_{cr} - \delta_{ir}\delta_{ck}\right) T_{ijrl} + \left(\delta_{il}\delta_{cs} - \delta_{is}\delta_{cl}\right) T_{ijks} = 0$$

$$\Longrightarrow 2T_{cjkl} + T_{jckl} - \underbrace{T_{qqkl}}_{=\alpha_{12}\delta_{kl}} \delta_{cj} + T_{kjcl} - \underbrace{T_{rjrl}}_{=\alpha_{13}\delta_{jl}} \delta_{ck} + T_{ljkc} - \underbrace{T_{sjks}}_{=\alpha_{14}\delta_{jk}} \delta_{cl}$$

$$\Longrightarrow \boxed{2T_{cjkl} + T_{jckl} + T_{kjcl} + T_{ljkc} = \alpha_{12}\delta_{cj}\delta_{kl} + \alpha_{13}\delta_{jl}\delta_{ck} + \alpha_{14}\delta_{jk}\delta_{cl}}$$

$$\xRightarrow{k=j} 2T_{cjjl} + T_{jcjl} + T_{jjcl} + T_{ljjc} = \alpha_{12}\delta_{cj}\delta_{jl} + \alpha_{13}\delta_{jl}\delta_{cj} + \alpha_{14}\delta_{jj}\delta_{cl}$$

$$\Longrightarrow 2\alpha_{23}\delta_{cl} + \cancel{\alpha_{13}\delta_{cl}} + \cancel{\alpha_{12}\delta_{cl}} + \alpha_{23}\delta_{lc} = \cancel{\alpha_{12}\delta_{cl}} + \cancel{\alpha_{13}\delta_{cl}} + 3\alpha_{14}\delta_{cl}$$

$$\Longrightarrow \alpha_{23} = \alpha_{14}$$

$$\xRightarrow{l=j} \alpha_{24} = \alpha_{13}$$

$$\xRightarrow{l=k} \alpha_{34} = \alpha_{12}$$

$$\xRightarrow{b=j} \left(\delta_{ji}\delta_{cp} - \delta_{jp}\delta_{ci}\right) T_{pjkl} + \left(\delta_{jj}\delta_{cq} - \delta_{jq}\delta_{cj}\right) T_{iqkl} + \left(\delta_{jk}\delta_{cr} - \delta_{jr}\delta_{ck}\right) T_{ijrl} + \left(\delta_{jl}\delta_{cs} - \delta_{js}\delta_{cl}\right) T_{ijks} = 0$$

$$\Longrightarrow T_{cikl} - \underbrace{T_{ppkl}}_{=\alpha_{12}\delta_{kl}} \delta_{ci} + 2T_{ickl} + T_{ikcl} - \underbrace{T_{ijjl}}_{=\alpha_{23}\delta_{il}} \delta_{ck} + T_{ilkc} - \underbrace{T_{isks}}_{=\alpha_{24}\delta_{ik}} \delta_{cl} = 0$$

$$\Longrightarrow T_{cikl} + 2T_{ickl} + T_{ikcl} + T_{ilkc} = \alpha_{12}\delta_{kl}\delta_{ci} + \alpha_{23}\delta_{il}\delta_{ck} + \alpha_{24}\delta_{ik}\delta_{cl}$$

$$\xRightarrow[c\to j]{i\to c} \boxed{T_{jckl} + 2T_{cjkl} + T_{ckjl} + T_{clkj} = \alpha_{12}\delta_{kl}\delta_{cj} + \alpha_{14}\delta_{cl}\delta_{jk} + \alpha_{13}\delta_{ck}\delta_{jl}}$$

$$\xRightarrow{b=k} \cdots \xRightarrow[c\to k]{i\to c} \boxed{T_{kjcl} + T_{ckjl} + 2T_{cjkl} + T_{cjlk} = \alpha_{13}\delta_{jl}\delta_{kc} + \alpha_{14}\delta_{cl}\delta_{kj} + \alpha_{12}\delta_{cj}\delta_{kl}}$$

$$\xRightarrow{b=l} \cdots \xRightarrow[c\to l]{i\to c} \boxed{T_{ljkc} + T_{clkj} + T_{cjlk} + 2T_{cjkl} = \alpha_{14}\delta_{jk}\delta_{lc} + \alpha_{13}\delta_{ck}\delta_{jl} + \alpha_{12}\delta_{cj}\delta_{kl}}$$

Let $S=\alpha_{12}\delta_{cj}\delta_{kl}+\alpha_{13}\delta_{ck}\delta_{jl}+\alpha_{14}\delta_{cl}\delta_{jk}$

$$\begin{cases} 2T_{cjkl}+T_{jckl}+T_{kjcl}+T_{ljkc}=S & (4)\\ T_{jckl}+2T_{cjkl}+T_{ckjl}+T_{clkj}=S & (5)\\ T_{kjcl}+T_{ckjl}+2T_{cjkl}+T_{cjlk}=S & (6)\\ T_{ljkc}+T_{clkj}+T_{cjlk}+2T_{cjkl}=S & (7)\end{cases}$$

$$\begin{cases} \big[(4)+(5)\big]-\big[(6)+(7)\big]\Longrightarrow T_{jckl}=T_{cjlk}\\ \big[(4)+(6)\big]-\big[(5)+(7)\big]\Longrightarrow T_{kjcl}=T_{clkj}\text{ , Change 1}^{\text{st}}\text{ index of every term to } c\\ \big[(4)+(7)\big]-\big[(5)+(6)\big]\Longrightarrow T_{ljkc}=T_{ckjl}\end{cases}$$

$$(4)\Longrightarrow 2T_{cjkl}+T_{cjlk}+T_{clkj}+T_{ckjl}=\alpha_{12}\delta_{cj}\delta_{kl}+\alpha_{13}\delta_{ck}\delta_{jl}+\alpha_{14}\delta_{cl}\delta_{jk} \qquad (*)$$

then $$\begin{tikzcd} & j \arrow[dr, bend left] & \\ l \arrow[ur, bend left] & & k \arrow[ll, bend left] \end{tikzcd}$$

$$\Longrightarrow 2T_{cklj}+T_{ckjl}+T_{cjlk}+T_{clkj}=\alpha_{12}\delta_{ck}\delta_{lj}+\alpha_{13}\delta_{cl}\delta_{kj}+\alpha_{14}\delta_{cj}\delta_{kl} \qquad (**)$$

then $$\begin{tikzcd} & j \arrow[dr, bend left] & \\ l \arrow[ur, bend left] & & k \arrow[ll, bend left] \end{tikzcd}$$

$$\Longrightarrow 2T_{cljk}+T_{clkj}+T_{ckjl}+T_{cjlk}=\alpha_{12}\delta_{cl}\delta_{jk}+\alpha_{13}\delta_{cj}\delta_{lk}+\alpha_{14}\delta_{ck}\delta_{lj} \qquad (***)$$

$$\begin{aligned} &(*)+(**)+(***)\\ &\Longrightarrow 2\boxed{\left(T_{cjkl}+T_{cklj}+T_{cljk}\right)}+3\boxed{\left(T_{clkj}+T_{ckjl}+T_{cjlk}\right)} \qquad (\bigstar)\\ &=\left(\alpha_{12}+\alpha_{13}+\alpha_{14}\right)\left(\delta_{cl}\delta_{jk}+\delta_{cj}\delta_{lk}+\delta_{ck}\delta_{lj}\right)\\ &\xRightarrow{l\leftrightarrow k} 2\boxed{\left(T_{cjlk}+T_{clkj}+T_{ckjl}\right)}+3\boxed{\left(T_{cklj}+T_{cljk}+T_{cjkl}\right)} \qquad (\bigstar\bigstar)\\ &=\left(\alpha_{12}+\alpha_{13}+\alpha_{14}\right)\left(\delta_{ck}\delta_{jl}+\delta_{cj}\delta_{kl}+\delta_{cl}\delta_{kj}\right)\end{aligned}$$

$$\begin{aligned}(\bigstar)-(\bigstar\bigstar)\Longrightarrow \left(T_{cjkl}+T_{cklj}+T_{cljk}\right)&=\left(T_{clkj}+T_{ckjl}+T_{ckjl}\right)\\ &=\underbrace{\boxed{\frac{1}{5}\left(\alpha_{12}+\alpha_{13}+\alpha_{14}\right)}}_{=K}\left(\delta_{cl}\delta_{jk}+\delta_{cj}\delta_{lk}+\delta_{ck}\delta_{lj}\right)\end{aligned}$$

$$(4)\Longrightarrow 2T_{cjkl}+K\left(\delta_{cl}\delta_{jk}+\delta_{cj}\delta_{lk}+\delta_{ck}\delta_{lj}\right)=\alpha_{12}\delta_{cj}\delta_{kl}+\alpha_{13}\delta_{ck}\delta_{jl}+\alpha_{14}\delta_{cl}\delta_{jk}$$

$$\xRightarrow{c\rightarrow i} T_{ijkl}=\lambda\delta_{ij}\delta_{kl}+\mu\delta_{ik}\delta_{jl}+\gamma\delta_{il}\delta_{jk}$$