Superposed Rigid Body Motion (SRBM)

Course from Prof. Chung-Jen Lu
Notes: Chen-Yu Hsin

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1 Invariance under SRBM

The balance laws and constitutive functions holds for all motions, including a SRBM. The balance laws and constitutive response functions are form-invariant under a SRBM.

$$\underset{\sim}{T} = f(\underset{\sim}{E}) \Longrightarrow \underset{\sim}{T}^+ = f(\underset{\sim}{E})^+$$

where $\underset{\sim}{T}$ is Cauchy stress tensor and $\underset{\sim}{E}$ is Green–Lagrange strain tensor. Note that the superscript + indicates rigid body motion.

2 Rigid-Body Motion

From the definition of rigid-body motion,

$$\underset{\sim}{x}^+ = \underset{\sim}{Q}(t)\left(\underset{\sim}{x} - \underset{\sim}{x}_0\right) + \underset{\sim}{d}(t) = \underset{\sim}{Q}(t)\underset{\sim}{x} + \underset{\sim}{c}(t),$$

where

$$\underset{\sim}{Q}\underset{\sim}{Q}^T = \underset{\sim}{Q}^T\underset{\sim}{Q} = \underset{\sim}{1}, \qquad \det \underset{\sim}{Q} = 1. \tag{1}$$

From the derivative of L.H.S. of (1),

$$\dot{\underset{\sim}{Q}}\underset{\sim}{Q}^T + \underset{\sim}{Q}\dot{\underset{\sim}{Q}}^T = 0 \Longrightarrow \dot{\underset{\sim}{Q}}\underset{\sim}{Q}^T = -\left(\dot{\underset{\sim}{Q}}\underset{\sim}{Q}^T\right)^T.$$

Let $\underset{\sim}{\Omega} = \dot{\underset{\sim}{Q}}\underset{\sim}{Q}^T$, clearly $\underset{\sim}{\Omega} = -\underset{\sim}{\Omega}^T$, $\underset{\sim}{\Omega}$ is skew-symmetric tensor. From the definition of dual vector,

$$\underset{\sim}{\Omega}\underset{\sim}{x} = \underset{\sim}{\omega} \times \underset{\sim}{x}.$$

$$\therefore \dot{\underset{\sim}{Q}} = \underset{\sim}{\Omega}\underset{\sim}{Q}$$

$$\underset{\sim}{v}^+ = \dot{\underset{\sim}{x}}^+ = \dot{\underset{\sim}{Q}}\underset{\sim}{x} + \underset{\sim}{Q}\dot{\underset{\sim}{x}} + \dot{\underset{\sim}{c}}(t) = \underset{\sim}{\Omega}\underset{\sim}{Q}\underset{\sim}{x} + \underset{\sim}{Q}\underset{\sim}{v} + \dot{\underset{\sim}{c}}(t) = \underset{\sim}{\omega} \times \left(\underset{\sim}{x}^+ - \underset{\sim}{c}\right) + \underset{\sim}{Q}\underset{\sim}{v} + \dot{\underset{\sim}{c}}(t)$$

$$\Longrightarrow \underset{\sim}{v}^{+} = \underset{\sim}{\Omega}\left(\underset{\sim}{x}^{+} - \underset{\sim}{c}\right) + \underset{\sim}{Q}\underset{\sim}{v} + \dot{\underset{\sim}{c}}(t)$$

$$\underset{\sim}{x}^{+} = \underset{\sim}{Q}(t) + \underset{\sim}{c}(t) \Longrightarrow \underset{\sim}{F}^{+} = \frac{\partial \underset{\sim}{x}^{+}}{\partial \underset{\sim}{X}} = \frac{\partial \underset{\sim}{x}^{+}}{\partial \underset{\sim}{x}}\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{X}} = \underset{\sim}{Q}\underset{\sim}{F}$$

$$J^{+} = \det \underset{\sim}{F}^{+} = \det \underset{\sim}{Q} \det \underset{\sim}{F} = \det \underset{\sim}{F} = J \Longrightarrow dV^{+} = dV \ \text{(isochoric)}.$$

Recall

$$d\underset{\sim}{a} = da\hat{n} = J\underset{\sim}{F}^{-T}d\underset{\sim}{A} = J\underset{\sim}{F}^{-T}dA\hat{N},$$

then

$$d\underset{\sim}{a}^{+} = \hat{n}^{+}da^{+} = d\underset{\sim}{x}_1^{+} \times d\underset{\sim}{x}_2^{+} = J^{+}\left(\underset{\sim}{F}^{+}\right)^{-T}\hat{N}dA = J\left(\underset{\sim}{Q}\underset{\sim}{F}\right)^{-T}\hat{N}dA = J\underset{\sim}{Q}\underset{\sim}{F}^{-T}\underset{\sim}{N}dA = \underset{\sim}{Q}\hat{n}da$$

and

$$\left(da^{+}\right)^2 = d\underset{\sim}{a}^{+} \cdot d\underset{\sim}{a}^{+} = \hat{n}^{+}da^{+} \cdot \hat{n}^{+}da^{+} = \underset{\sim}{Q}\hat{n}da \cdot \underset{\sim}{Q}\hat{n}da = \hat{n} \cdot \underset{\sim}{Q}^{T}\underset{\sim}{Q}\hat{n}\,(da)^2 = (da)^2$$

$$\therefore da^{+} = da, \qquad\qquad \hat{n}^{+} = \underset{\sim}{Q}\hat{n}.$$

2.1 Cauchy-Green deformation tensor

$$\underset{\sim}{C} = \underset{\sim}{F}^{T}\underset{\sim}{F} \Longrightarrow \underset{\sim}{C}^{+} = \left(\underset{\sim}{F}^{+}\right)^{T}\underset{\sim}{F}^{+} = \left(\underset{\sim}{Q}\underset{\sim}{F}\right)^{T}\left(\underset{\sim}{Q}\underset{\sim}{F}\right) = \underset{\sim}{F}^{+}\underbrace{\underset{\sim}{Q}^{T}\underset{\sim}{Q}}_{=\underset{\sim}{1}}\underset{\sim}{F} = \underset{\sim}{F}^{T}\underset{\sim}{F} = \underset{\sim}{C}.$$

2.2 Green–Lagrange strain tensor

$$\underset{\sim}{E} = \frac{1}{2}\left(\underset{\sim}{C} - \underset{\sim}{1}\right) \Longrightarrow \underset{\sim}{E}^{+} = \frac{1}{2}\left(\underset{\sim}{C}^{+} - \underset{\sim}{1}\right) = \frac{1}{2}\left(\underset{\sim}{C} - \underset{\sim}{1}\right) = \underset{\sim}{E}.$$

2.3 Velocity gradient tensor

$$\underset{\sim}{L}^{+} = \frac{\partial \underset{\sim}{v}^{+}}{\partial \underset{\sim}{x}^{+}} = \underset{\sim}{\Omega} + \underset{\sim}{Q}\frac{\partial \underset{\sim}{v}}{\partial \underset{\sim}{x}}\frac{\partial \underset{\sim}{x}}{\partial \underset{\sim}{x}^{+}} \overset{*}{=} \underset{\sim}{\Omega} + \underset{\sim}{Q}\underset{\sim}{L}\underset{\sim}{Q}^{T}$$

$$\underset{\sim}{D}^{+} = \frac{1}{2}\left(\underset{\sim}{L}^{+} + \underset{\sim}{L}^{+T}\right) = \frac{1}{2}\left(\cancel{\underset{\sim}{\Omega}} + \underset{\sim}{Q}\underset{\sim}{L}\underset{\sim}{Q}^{T} + \cancel{\underset{\sim}{\Omega}^{T}} + \underset{\sim}{Q}\underset{\sim}{L}^{T}\underset{\sim}{Q}^{T}\right) = \underset{\sim}{Q}\underset{\sim}{D}\underset{\sim}{Q}^{T}$$

$$\underset{\sim}{W}^{+} = \frac{1}{2}\left(\underset{\sim}{L}^{+} - \underset{\sim}{L}^{+T}\right) = \frac{1}{2}\left(\underset{\sim}{\Omega} + \underset{\sim}{Q}\underset{\sim}{L}\underset{\sim}{Q}^{T} - \underset{\sim}{\Omega}^{T} - \underset{\sim}{Q}\underset{\sim}{L}^{T}\underset{\sim}{Q}^{T}\right) = \underset{\sim}{\Omega} + \underset{\sim}{Q}\underset{\sim}{W}\underset{\sim}{Q}^{T}$$

* $\underset{\sim}{Q}^{-1} = \underset{\sim}{Q}^{T}$, $\underset{\sim}{x} = \underset{\sim}{Q}^{T}\left(\underset{\sim}{x}^{+} - \underset{\sim}{c}(t)\right)$

2.4 Cauchy stress and tensor

We know that $\hat{n}^+ = \underset{\sim}{Q}\hat{n}$, and *assume* the component of stress vector $\underset{\sim}{t}$ in the direction of the outward normal $\hat{n}^+$ remains unchanged under SRBM so that

$$\underset{\sim}{t}^+(\hat{n}^+) \cdot \hat{n}^+ = \underset{\sim}{t}(\hat{n}) \cdot \hat{n}$$

$$\underset{\sim}{t} \cdot \hat{n} = \underset{\sim}{t} \cdot \underset{\sim}{Q}^T \hat{n}^+ = t_i Q_{ij}^T n_j^+ = t_i Q_{ji} n_j^+ = Q_{ji} t_i n_j^+ = \underset{\sim}{Q}\underset{\sim}{t} \cdot \hat{n}^+ = \underset{\sim}{t}^+ \cdot \hat{n}^+$$

$$\Longrightarrow \left(\underset{\sim}{t}^+ - \underset{\sim}{Q}\underset{\sim}{t}\right) \cdot \hat{n}^+ = 0 \Longrightarrow \underset{\sim}{t}^+ = \underset{\sim}{Q}\underset{\sim}{t}$$

From the relation between stress vector and stress tensor, $\underset{\sim}{t} = \underset{\sim}{T}\hat{n}$ so that

$$\underset{\sim}{t}^+ = \underset{\sim}{T}^+ \hat{n}^+ = \underset{\sim}{Q}\underset{\sim}{t} = \underset{\sim}{Q}\underset{\sim}{T}\hat{n} = \underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T \hat{n}^+$$

$$\Longrightarrow \left(\underset{\sim}{T}^+ - \underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T\right) \hat{n}^+ = 0 \Longrightarrow \underset{\sim}{T}^+ = \underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T$$

2.5 Body force

$$\nabla \cdot \underset{\sim}{T} = \rho\left(\dot{\underset{\sim}{v}} - \underset{\sim}{b}\right) \Longrightarrow \nabla^+ \cdot \underset{\sim}{T}^+ = \rho^+\left(\dot{\underset{\sim}{v}}^+ - \underset{\sim}{b}^+\right)$$

From conservation of mass, $\rho_0 = \rho J = \rho^+ J^+ \Longrightarrow \rho = \rho^+$, and

$$\nabla^+ \cdot \underset{\sim}{T}^+ = \frac{\partial T_{ij}^+}{\partial x_j^+}\hat{e}_i = \frac{\partial T_{ij}^+}{\partial x_k}\frac{\partial x_k}{\partial x_j^+}\hat{e}_i = \frac{\partial T_{ij}}{\partial x_k} Q_{jk}\hat{e}_i = Q_{im}\frac{\partial T_{mn}}{\partial x_k}\underbrace{Q_{jn}Q_{jk}}_{=\delta_{nk}}\hat{e}_i = Q_{im}\frac{\partial T_{mk}}{\partial x_k}\hat{e}_i$$

$$= \underset{\sim}{Q}\left(\nabla \cdot \underset{\sim}{T}\right)$$

$$\Longrightarrow \nabla^+ \cdot \underset{\sim}{T}^+ = \underset{\sim}{Q}\left(\nabla \cdot \underset{\sim}{T}\right) = \rho^+\left(\dot{\underset{\sim}{v}}^+ - \underset{\sim}{b}^+\right) \Longrightarrow \underset{\sim}{Q}\rho\left(\dot{\underset{\sim}{v}} - \underset{\sim}{b}\right) = \rho^+\left(\dot{\underset{\sim}{v}}^+ - \underset{\sim}{b}^+\right)$$

$$\Longrightarrow \left(\dot{\underset{\sim}{v}}^+ - \underset{\sim}{b}^+\right) = \underset{\sim}{Q}\left(\dot{\underset{\sim}{v}} - \underset{\sim}{b}\right)$$

3 Derivation of Balance Laws from Balance of Energy

Recall balance of energy

$$\dot{K} + \dot{U} = P + Q$$

where

$$K = \int_V \frac{1}{2}\rho\underset{\sim}{v} \cdot \underset{\sim}{v} dV$$

$$U = \int_V \rho\varepsilon dV$$

$$P = \int_V \rho \underset{\sim}{b} \cdot \underset{\sim}{v} dV + \int_S \underset{\sim}{t} \cdot \underset{\sim}{v} da$$

$$Q = \int_V \rho r dV - \int_S h da, h(\hat{n}) = \underset{\sim}{q} \cdot \hat{n}$$

Assume that $\rho^+ = \rho$, $\varepsilon^+ = \varepsilon$, $r^+ = r$, $\underset{\sim}{q}^+ \cdot \hat{n}^+ = \underset{\sim}{q} \cdot \hat{n} \Longrightarrow \underset{\sim}{q}^+ = \underset{\sim}{Q}\underset{\sim}{q}$

$$\dot{K} + \dot{U} = P + Q \Longrightarrow \rho \dot{\varepsilon} = \underset{\sim}{T} \cdot \underset{\sim}{D} + \rho r - \nabla \cdot \underset{\sim}{q}$$

From

$$\frac{D}{Dt} \int_V \frac{1}{2} \rho \underset{\sim}{v} \cdot \underset{\sim}{v} dV + \frac{D}{Dt} \int_V \rho \varepsilon dV = \int_V \rho \underset{\sim}{b} \cdot \underset{\sim}{v} dV + \int_S \underset{\sim}{t} \cdot \underset{\sim}{v} da + \int_V \rho r dV - \int_S \underset{\sim}{q} \cdot \hat{n} da$$

$$\begin{aligned} \frac{D}{Dt} \int_{V^+} \rho^+ \left(\frac{1}{2} \underset{\sim}{v}^+ \cdot \underset{\sim}{v}^+ + \varepsilon^+ \right) dV^+ = & \int_{V^+} \rho^+ \underset{\sim}{b}^+ \cdot \underset{\sim}{v}^+ dV^+ + \int_{S^+} \underset{\sim}{t}^+ \cdot \underset{\sim}{v}^+ da^+ \\ & + \int_{V^+} \rho^+ r^+ dV^+ - \int_{S^+} \underset{\sim}{q}^+ \cdot \hat{n}^+ da^+ \end{aligned} \tag{$*$}$$

and transport theorem

$$\frac{D}{Dt} \int_V \phi dV = \int_V \left[\frac{D\phi}{Dt} + (\nabla \cdot v) \phi \right] dV \Longrightarrow \frac{D}{Dt} \int_{V^+} \phi dV^+ = \int_{V^+} \left[\frac{D\phi^+}{Dt} + \left(\nabla^+ \cdot v^+ \right) \phi^+ \right] dV^+$$

$$\begin{aligned} \frac{D}{Dt} \int_{V^+} \rho^+ \phi^+ dV^+ &= \int_{V^+} \left[\dot{\rho}^+ \phi^+ + \rho^+ \dot{\phi}^+ + \left(\nabla^+ \cdot \underset{\sim}{v}^+ \right) \rho^+ \phi^+ \right] dV^+ \\ &= \int_{V^+} \left[\left(\dot{\rho}^+ + \rho^+ \nabla^+ \cdot \underset{\sim}{v}^+ \right) \phi^+ + \rho^+ \dot{\phi}^+ \right] dV^+ \end{aligned}$$

$(*)$ becomes

$$\begin{aligned} & \int_{V^+} \left[\left(\dot{\rho}^+ + \rho^+ \nabla^+ \cdot \underset{\sim}{v}^+ \right) \left(\frac{1}{2} \underset{\sim}{v}^+ \cdot \underset{\sim}{v}^+ + \varepsilon^+ \right) + \rho^+ \left(\underset{\sim}{v}^+ \cdot \dot{\underset{\sim}{v}}^+ + \dot{\varepsilon}^+ \right) \right] dV^+ \\ & \qquad = \int_{V^+} \rho^+ \underset{\sim}{b}^+ \cdot \underset{\sim}{v}^+ dV^+ + \int_{S^+} \underset{\sim}{t}^+ \cdot \underset{\sim}{v}^+ da^+ + \int_{V^+} \rho^+ r^+ dV^+ - \int_{S^+} \underset{\sim}{q}^+ \cdot \hat{n}^+ da^+ \end{aligned}$$

where

$$\begin{aligned} \int_{S^+} \underset{\sim}{t}^+ \cdot \underset{\sim}{v}^+ da^+ &= \int_{S^+} t_i^+ v_i^+ da^+ = \int_{S^+} T_{ij}^+ n_j^+ v_i^+ da^+ = \int_{V^+} \left(T_{ij}^+ v_i^+ \right)_{,j} dV^+ \\ &= \int_{V^+} \left(\frac{\partial T_{ij}^+}{\partial x_j^+} v_i^+ + T_{ij}^+ \frac{\partial v_i^+}{\partial x_j^+} \right) dV^+ = \int_{V^+} \left[\underset{\sim}{v}^+ \cdot \left(\nabla^+ \cdot \underset{\sim}{T}^+ \right) + \underset{\sim}{T}^+ \cdot \underset{\sim}{L}^+ \right] dV^+ \end{aligned}$$

$$\therefore \left(\dot{\rho}^+ + \rho^+ \nabla^+ \cdot \underset{\sim}{v}^+\right)\left(\frac{1}{2}\underset{\sim}{v}^+ \cdot \underset{\sim}{v}^+ + \varepsilon^+\right) + \rho^+\left(\underset{\sim}{v}^+ \cdot \dot{\underset{\sim}{v}}^+ + \dot{\varepsilon}^+\right)$$
$$= \rho^+ \underset{\sim}{b}^+ \cdot \underset{\sim}{v}^+ + \underset{\sim}{v}^+ \cdot \left(\nabla^+ \cdot \underset{\sim}{T}^+\right) + \underset{\sim}{T}^+ \cdot \underset{\sim}{L}^+ + \rho^+ r^+ - \nabla^+ \cdot \underset{\sim}{q}^+$$
$$\Longrightarrow \left(\dot{\rho}^+ + \rho^+ \nabla^+ \cdot \underset{\sim}{v}^+\right)\left(\frac{1}{2}\underset{\sim}{v}^+ \cdot \underset{\sim}{v}^+ + \varepsilon^+\right) + \underset{\sim}{v}^+ \cdot \left(\rho^+ \dot{\underset{\sim}{v}}^+ - \rho^+ \underset{\sim}{b}^+ - \nabla^+ \cdot \underset{\sim}{T}^+\right)$$
$$+ \left(\rho^+ \dot{\varepsilon}^+ - \rho^+ r^+ - \underset{\sim}{T}^+ \cdot \underset{\sim}{L}^+ + \nabla^+ \cdot \underset{\sim}{q}^+\right) = 0$$

Recall
$$L_{ij} = \frac{\partial v_i}{\partial x_j} \Longrightarrow L_{ii} = \frac{\partial v_i}{\partial x_i} \Longrightarrow \nabla \cdot \underset{\sim}{v} = \operatorname{tr} \underset{\sim}{L}$$
and
$$\nabla^+ \cdot \underset{\sim}{v}^+ = \operatorname{tr} \underset{\sim}{L}^+ = \operatorname{tr}\left(\underset{\sim}{\Omega} + \underset{\sim}{Q}\underset{\sim}{L}\underset{\sim}{Q}^T\right) = \operatorname{tr} \underset{\sim}{\Omega} + \operatorname{tr}\left(\underset{\sim}{Q}\underset{\sim}{L}\underset{\sim}{Q}^T\right) = \operatorname{tr}\left(\underset{\sim}{Q}\underset{\sim}{L}\underset{\sim}{Q}^T\right)$$
$$= Q_{ij} L_{jk} Q_{ik} = L_{jk} Q_{ik} Q_{ij} = \operatorname{tr}\left(\underset{\sim}{L}\underset{\sim}{Q}^T\underset{\sim}{Q}\right) = \operatorname{tr} \underset{\sim}{L} = \nabla \cdot \underset{\sim}{v}$$
$$\underset{\sim}{T}^+ \cdot \underset{\sim}{L}^+ = \left(\underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T\right) \cdot \left(\underset{\sim}{\Omega} + \underset{\sim}{Q}\underset{\sim}{L}\underset{\sim}{Q}^T\right) = \left(\underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T\right) \cdot \underset{\sim}{\Omega} + \left(\underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T\right) \cdot \left(\underset{\sim}{Q}\underset{\sim}{L}\underset{\sim}{Q}^T\right)$$
$$\overset{\dagger}{=} \left(\underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T\right) \cdot \underset{\sim}{\Omega} + \operatorname{tr}\left(\underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T\underset{\sim}{Q}\underset{\sim}{L}\underset{\sim}{Q}^T\right) = \left(\underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T\right) \cdot \underset{\sim}{\Omega} + \operatorname{tr}\left(\underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{L}\underset{\sim}{Q}^T\right)$$
$$= \left(\underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T\right) \cdot \underset{\sim}{\Omega} + \operatorname{tr}\left(\underset{\sim}{T}\underset{\sim}{L}^T\underset{\sim}{Q}^T\underset{\sim}{Q}\right) = \left(\underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T\right) \cdot \underset{\sim}{\Omega} + \operatorname{tr}\left(\underset{\sim}{T}\underset{\sim}{L}^+\right)$$
$$= \left(\underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T\right) \cdot \underset{\sim}{\Omega} + \underset{\sim}{T} \cdot \underset{\sim}{L}$$
$$\nabla^+ \cdot \underset{\sim}{q}^+ = \frac{\partial q_i^+}{\partial x_i^+} = \frac{\partial Q_{ik} q_k}{\partial x_j}\frac{\partial x_j}{\partial x_i^+} = Q_{ik}\frac{\partial q_k}{\partial x_j} Q_{ji}^T = \delta_{kj}\frac{\partial q_k}{\partial x_j} = \frac{\partial q_j}{\partial x_j} = \nabla \cdot \underset{\sim}{q}$$
then
$$\Longrightarrow \left(\dot{\rho}^+ + \rho^+ \nabla \cdot \underset{\sim}{v}\right)\left(\frac{1}{2}\underset{\sim}{v}^+ \cdot \underset{\sim}{v}^+ + \varepsilon^+\right) + \underset{\sim}{v}^+ \cdot \left(\rho^+ \dot{\underset{\sim}{v}}^+ - \rho^+ \underset{\sim}{b}^+ - \underset{\sim}{Q}\nabla \cdot \underset{\sim}{T}\right)$$
$$+ \left(\rho^+ \dot{\varepsilon}^+ - \rho^+ r^+ - \underset{\sim}{T} \cdot \underset{\sim}{L} - \underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T \cdot \underset{\sim}{\Omega} + \nabla \cdot \underset{\sim}{q}\right) = 0$$

Assume $\rho^+ = \rho$, $\varepsilon^+ = \varepsilon$, $r^+ = r$, and recall $\left(\dot{\underset{\sim}{v}}^+ - \underset{\sim}{b}^+\right) = \underset{\sim}{Q}\left(\dot{\underset{\sim}{v}} - \underset{\sim}{b}\right)$ so that

$$\begin{aligned} &\Longrightarrow \left(\dot{\rho} + \rho \nabla \cdot \underset{\sim}{v}\right)\left(\frac{1}{2}\underset{\sim}{v}^+ \cdot \underset{\sim}{v}^+ + \varepsilon\right) + \underset{\sim}{v}^+ \cdot \underset{\sim}{Q}\left(\rho \dot{\underset{\sim}{v}} - \rho \underset{\sim}{b} - \nabla \cdot \underset{\sim}{T}\right) \\ &+ \left(\rho \dot{\varepsilon} - \rho r - \underset{\sim}{T} \cdot \underset{\sim}{L} - \underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T \cdot \underset{\sim}{\Omega} + \nabla \cdot \underset{\sim}{q}\right) = 0 \end{aligned} \qquad (\bigstar)$$

for all $\underset{\sim}{x}^+ = \underset{\sim}{Q}(t)\underset{\sim}{x} + \underset{\sim}{c}(t)$

† $\underset{\sim}{A} \cdot \underset{\sim}{B} = A_{ij} B_{ij} = \operatorname{tr}\left(\underset{\sim}{A}\underset{\sim}{B}^T\right)$

- No SRBM: $\underset{\sim}{Q} = \underset{\sim}{1}$, $\underset{\sim}{c} = 0$, $\underset{\sim}{\Omega} = \underset{\sim}{0}$, $\underset{\sim}{x}^+ = \underset{\sim}{x}$, and $\underset{\sim}{v}^+ = \underset{\sim}{v}$
  ($\bigstar$) becomes

$$\begin{aligned}\Longrightarrow \left(\dot{\rho} + \rho \nabla \cdot \underset{\sim}{v}\right) \left(\frac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{v} + \varepsilon\right) + \underset{\sim}{v} \cdot \left(\rho \dot{\underset{\sim}{v}} - \rho \underset{\sim}{b} - \nabla \cdot \underset{\sim}{T}\right) \\ + \left(\rho \dot{\varepsilon} - \rho r - \underset{\sim}{T} \cdot \underset{\sim}{L} + \nabla \cdot \underset{\sim}{q}\right) = 0\end{aligned} \tag{2}$$

- Translation: $\underset{\sim}{Q} = \underset{\sim}{1}$, $\dot{\underset{\sim}{c}}(t) = v_0 \hat{e}$, $\dot{\underset{\sim}{Q}} = \underset{\sim}{0}$, $\underset{\sim}{\Omega} = \underset{\sim}{0}$, $\underset{\sim}{x}^+ = \underset{\sim}{x}$, and $\underset{\sim}{v}^+ = \underset{\sim}{v} + v_0 \hat{e}$
  ($\bigstar$) becomes

$$\begin{aligned}\Longrightarrow \left(\dot{\rho} + \rho \nabla \cdot \underset{\sim}{v}\right) \left(\frac{1}{2}\underset{\sim}{v} \cdot \underset{\sim}{v} + \underset{\sim}{v} \cdot (v_0 \hat{e}) + \frac{v_0^2}{2} + \varepsilon\right) \\ + \left(\underset{\sim}{v} + v_0 \hat{e}\right) \cdot \left(\rho \dot{\underset{\sim}{v}} - \rho \underset{\sim}{b} - \nabla \cdot \underset{\sim}{T}\right) \\ + \left(\rho \dot{\varepsilon} - \rho r - \underset{\sim}{T} \cdot \underset{\sim}{L} + \nabla \cdot \underset{\sim}{q}\right) = 0\end{aligned} \tag{3}$$

$$(3) - (2) \Longrightarrow \left(\dot{\rho} + \rho \nabla \cdot \underset{\sim}{v}\right) \left(\underset{\sim}{v} \cdot (v_0 \hat{e}) + \frac{v_0^2}{2}\right) + v_0 \hat{e} \cdot \left(\rho \dot{\underset{\sim}{v}} - \rho \underset{\sim}{b} - \nabla \cdot \underset{\sim}{T}\right) = 0 \forall v_0, \hat{e}$$

$$\Longrightarrow \begin{cases} \dot{\rho} + \rho \nabla \cdot \underset{\sim}{v} = 0 \\ \rho \dot{\underset{\sim}{v}} - \rho \underset{\sim}{b} - \nabla \cdot \underset{\sim}{T} = 0 \end{cases}$$

then ($\bigstar$) becomes

$$\rho \dot{\varepsilon} - \rho r - \underset{\sim}{T} \cdot \underset{\sim}{L} - \underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T \cdot \underset{\sim}{\Omega} + \nabla \cdot \underset{\sim}{q} = 0$$

from $\rho \dot{\varepsilon} - \rho r - \underset{\sim}{T} \cdot \underset{\sim}{L} + \nabla \cdot \underset{\sim}{q} = 0$,

$$\underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T \cdot \underset{\sim}{\Omega} = 0 \Longrightarrow \underset{\sim}{T} = \underset{\sim}{T}^T$$

Summary.

$$\underset{\sim}{x}^+ = \underset{\sim}{Q}(t)\underset{\sim}{x} + \underset{\sim}{c}(t)$$

$$\underset{\sim}{F} = \underset{\sim}{Q}\underset{\sim}{F}, \quad J^+ = J, \quad \hat{n}^+ = \underset{\sim}{Q}\hat{n}, \quad da^+ = da, \quad dV^+ = dV, \quad \rho^+ = \rho, \quad \underset{\sim}{C}^+ = \underset{\sim}{C}, \quad \underset{\sim}{E}^+ = \underset{\sim}{E}$$

$$\underset{\sim}{v}^+ = \underset{\sim}{\Omega}\left(\underset{\sim}{x}^+ - \underset{\sim}{c}\right) + \underset{\sim}{Q}\underset{\sim}{v} + \dot{\underset{\sim}{c}}(t), \qquad \underset{\sim}{\Omega} = \dot{\underset{\sim}{Q}}\underset{\sim}{Q}^T$$

$$\underset{\sim}{L}^+ = \underset{\sim}{\Omega} + \underset{\sim}{Q}\underset{\sim}{L}\underset{\sim}{Q}^T, \quad \underset{\sim}{D}^+ = \underset{\sim}{Q}\underset{\sim}{D}\underset{\sim}{Q}^T, \quad \underset{\sim}{W}^+ = \underset{\sim}{\Omega} + \underset{\sim}{Q}\underset{\sim}{W}\underset{\sim}{Q}^T, \quad \underset{\sim}{t}^+ = \underset{\sim}{Q}\underset{\sim}{t}, \quad \underset{\sim}{T}^+ = \underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T$$

4 Constitutive Equation

4.1 Ideal fluid

An ideal fluid has the following properties:

- It cannot sustain shearing motion.
- The stress vector acting on any surface in the fluid is always directed along the normal to the surface, i.e. $\underset{\sim}{t}$ is parallel to $\underset{\sim}{n}$.

Assume that $\underset{\sim}{T} = \hat{\underset{\sim}{T}}(\rho, \underset{\sim}{v})$. The constitutive equation holds for all motions, including SRBM.

$$\underset{\sim}{T} = \hat{\underset{\sim}{T}}(\rho, \underset{\sim}{v}) \Longrightarrow \underset{\sim}{T}^+ = \hat{\underset{\sim}{T}}(\rho^+, \underset{\sim}{v}^+)$$

A translation: $\underset{\sim}{Q} = \underset{\sim}{1}$, $\underset{\sim}{c}(t) = \underset{\sim}{v}_0 t \Longrightarrow \underset{\sim}{v}^+ = \underset{\sim}{v} + \underset{\sim}{v}_0$

$$\Longrightarrow \underset{\sim}{T} = \underset{\sim}{T}^+ \Longrightarrow \hat{\underset{\sim}{T}}(\rho, \underset{\sim}{v}) = \hat{\underset{\sim}{T}}(\rho^+, \underset{\sim}{v}^+) \Longrightarrow \hat{\underset{\sim}{T}}(\rho, \underset{\sim}{v}) = \hat{\underset{\sim}{T}}(\rho, \underset{\sim}{v} + \underset{\sim}{v}_0) \forall \underset{\sim}{v}_0$$

$$\Longrightarrow \underset{\sim}{T} = \hat{\underset{\sim}{T}}(\rho) \Longrightarrow \underset{\sim}{T}^+ = \hat{\underset{\sim}{T}}(\rho)$$

$$\underset{\sim}{T}^+ = \underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T \Longrightarrow \hat{\underset{\sim}{T}}(\rho) = \underset{\sim}{Q}\hat{\underset{\sim}{T}}(\rho)\underset{\sim}{Q}^T \forall \underset{\sim}{Q}$$

$\hat{\underset{\sim}{T}}(\rho)$ is 2^{nd} order isotropic tensor. Obviously, $\hat{\underset{\sim}{T}}(\rho) = f(\rho)\underset{\sim}{1}$ is a solution. Are there other solutions?

$$\therefore \underset{\sim}{T} = -p(\rho)\underset{\sim}{1}.$$

Equation of motion

$$\rho \left(\underset{\sim}{b} - \dot{\underset{\sim}{v}}\right) = -\nabla \cdot \underset{\sim}{T} = -T_{ij,j}\hat{e}_i = \left(p\delta_{ij}\right)_{,j} \hat{e}_i = p_{,i}\hat{e}_i = \nabla p$$

$$\Longrightarrow \boxed{\rho\dot{\underset{\sim}{v}} = \rho\underset{\sim}{b} - \nabla p}$$

which is famous *Euler's Equation*!

4.2 Viscous fluid

Assume that $\underset{\sim}{T} = \bar{T}(\rho, \underset{\sim}{v}, \underset{\sim}{L}) = \bar{\underset{\sim}{T}}(\rho, \underset{\sim}{v}, \underset{\sim}{D}, \underset{\sim}{W})$, under SRBM $\underset{\sim}{T}^+ = \bar{\underset{\sim}{T}}(\rho^+, \underset{\sim}{v}^+, \underset{\sim}{D}^+, \underset{\sim}{W}^+)$, also $\underset{\sim}{T}^+ = \underset{\sim}{Q}\underset{\sim}{T}\underset{\sim}{Q}^T$

$$\Longrightarrow \underset{\sim}{Q}\bar{\underset{\sim}{T}}(\rho, \underset{\sim}{v}, \underset{\sim}{D}, \underset{\sim}{W})\underset{\sim}{Q}^T = \bar{\underset{\sim}{T}}(\rho^+, \underset{\sim}{v}^+, \underset{\sim}{D}^+, \underset{\sim}{W}^+)$$

and recall

$$\underset{\sim}{D}^+ = \underset{\sim}{Q}\underset{\sim}{D}\underset{\sim}{Q}^T, \qquad \underset{\sim}{W}^+ = \underset{\sim}{\Omega}^+ + \underset{\sim}{Q}\underset{\sim}{W}\underset{\sim}{Q}^T$$

Translation: $\underset{\sim}{v}^+ = \underset{\sim}{v} + \underset{\sim}{v}_0$, $\underset{\sim}{Q} = \underset{\sim}{1}$, $\underset{\sim}{\Omega} = \underset{\sim}{0}$, $\underset{\sim}{D}^+ = \underset{\sim}{D}$, and $\underset{\sim}{W}^+ = \underset{\sim}{W}$

$$\Longrightarrow \bar{\underset{\sim}{T}}(\rho, \underset{\sim}{v}, \underset{\sim}{D}, \underset{\sim}{W}) = \bar{\underset{\sim}{T}}(\rho, \underset{\sim}{v} + \underset{\sim}{v}_0, \underset{\sim}{D}, \underset{\sim}{W}) \forall \underset{\sim}{v}_0$$

$$\Longrightarrow \underset{\sim}{T} = \bar{\underset{\sim}{T}}(\rho, \underset{\sim}{D}, \underset{\sim}{W})$$

Rotation: $\underset{\sim}{x}^+ = \underset{\sim}{Q}(t) + \underset{\sim}{c}(t)$; $\underset{\sim}{c}(t) = 0$, $\underset{\sim}{\Omega} = \underset{\sim}{\Omega}_0 =$ constant. At the instant: $\underset{\sim}{Q} = \underset{\sim}{1}$, $\underset{\sim}{D}^+ = \underset{\sim}{D}$, and $\underset{\sim}{W}^+ = \underset{\sim}{\Omega}_0 + \underset{\sim}{W}$

$$\Longrightarrow \bar{\underset{\sim}{T}}(\rho, \underset{\sim}{D}, \underset{\sim}{W}) = \bar{\underset{\sim}{T}}(\rho, \underset{\sim}{D}, \underset{\sim}{W} + \underset{\sim}{\Omega}_0) \forall \underset{\sim}{\Omega}_0$$

$$\Longrightarrow \underset{\sim}{T} = \bar{\underset{\sim}{T}}(\rho, \underset{\sim}{D})$$

- Newtonian fluid
  $\underset{\sim}{T} = \bar{\underset{\sim}{T}}(\rho, \underset{\sim}{D})$, linear function of $\underset{\sim}{D}$ with the coefficients depending on ρ so that

$$\bar{T}_{ij}(\rho, D_{ij}) = A_{ij}(\rho) + C_{ijkl}(\rho) D_{kl}$$

where $C_{ijkl} = C_{jikl} = C_{jilk}$ because of symmetry: $T_{ij} = T_{ji}$ and $D_{ij} = D_{ji}$

$$\underset{\sim}{Q}\bar{\underset{\sim}{T}}(\rho, \underset{\sim}{D})\underset{\sim}{Q}^T = \bar{\underset{\sim}{T}}(\rho^+, \underset{\sim}{D}^+), \qquad \underset{\sim}{D}^+ = \underset{\sim}{Q}\underset{\sim}{D}\underset{\sim}{Q}^T$$

$$\Longrightarrow \underset{\sim}{Q}\bar{\underset{\sim}{T}}(\rho, \underset{\sim}{D})\underset{\sim}{Q}^T = \bar{\underset{\sim}{T}}(\rho, \underset{\sim}{Q}\underset{\sim}{D}\underset{\sim}{Q}^T) \Longrightarrow Q_{ij}\bar{T}_{jk}(\rho, D_{pq})Q^T_{km} = \bar{T}_{im}(\rho, Q_{pr}D_{rs}Q^T_{sq})$$
$$\Longrightarrow Q_{ij}A_{jk}(\rho)Q^T_{km} + Q_{ij}C_{jkrs}(\rho)D_{rs}Q^T_{km} = A_{im}(\rho) + C_{impq}Q_{pr}D_{rs}Q^T_{sq}$$

choose $\underset{\sim}{D} = \underset{\sim}{0}$,

$$\Longrightarrow Q_{ij}A_{jk}(\rho)Q^T_{km} = A_{im}(\rho) \Longrightarrow \underset{\sim}{A} = -p(\rho)\underset{\sim}{1}$$

and

$$Q_{ij}C_{jkrs}D_{rs}Q^T_{km} = C_{impq}Q_{pr}D_{rs}Q^T_{sq} \Longrightarrow Q_{ij}Q^T_{km}C_{jkrs}D_{rs} = Q_{pr}Q^T_{sq}C_{impq}D_{rs}$$

$$\Longrightarrow Q_{ij}Q^T_{km}C_{jkrs} = Q_{pr}Q^T_{sq}C_{impq} \Longrightarrow Q^T_{ru}Q_{vs}Q_{ij}Q^T_{km}C_{jkrs} = Q^T_{ru}Q_{vs}Q_{pr}Q^T_{sq}C_{impq}$$
$$\Longrightarrow Q^T_{ru}Q_{vs}Q_{ij}Q^T_{km}C_{jkrs} = \delta_{pu}\delta_{vq}C_{impq} \Longrightarrow Q_{ij}Q_{mk}Q_{ur}Q_{vs}C_{jkrs} = C_{imuv}$$

or

$$C_{ijkl} = Q_{ip}Q_{jq}Q_{kr}Q_{ls}C_{pqrs}$$

which is 4^{th} order isotropic tensor

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\delta_{ik}\delta_{jl} + \gamma\delta_{il}\delta_{jk}$$

owing to $C_{ijkl} = C_{jikl} = C_{jilk}$,

$$C_{ijkl} = \lambda\delta_{ij}\delta_{kl} + \mu\left(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk}\right)$$

then

$$T_{ij} = -p\delta_{ij} + \lambda D_{kk}\delta_{ij} + 2\mu D_{ij}$$

Recall $D_{ij} = \left(v_{i,j} + v_{j,i}\right)/2$ so that

$$\begin{aligned}
\nabla \cdot \underset{\sim}{T} = T_{ij,j}\hat{e}_i &= \left(-p\delta_{ij} + \lambda D_{kk}\delta_{ij} + 2\mu D_{ij}\right)_{,j}\hat{e}_i \\
&= \left[-p_{,i} + (\lambda + \mu)\, v_{k,ki} + \mu v_{i,kk}\right]\hat{e}_i
\end{aligned}$$

Equation of motion

$$\rho\left(\underset{\sim}{b} - \dot{\underset{\sim}{v}}\right) = -\nabla \cdot \underset{\sim}{T} = -T_{ij,j}\hat{e}_i$$

$$\Longrightarrow \boxed{\rho\dot{\underset{\sim}{v}} = \rho\underset{\sim}{b} - \nabla p + (\lambda + \mu)\,\nabla\left(\nabla \cdot \underset{\sim}{v}\right) + \mu\nabla^2\underset{\sim}{v}}$$

which is famous *Navier-Stokes Equation*!