Graph Theoretic Analysis of Parsimonious Voice Leading Using Pitch Class Set Theory

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Equal Temperament

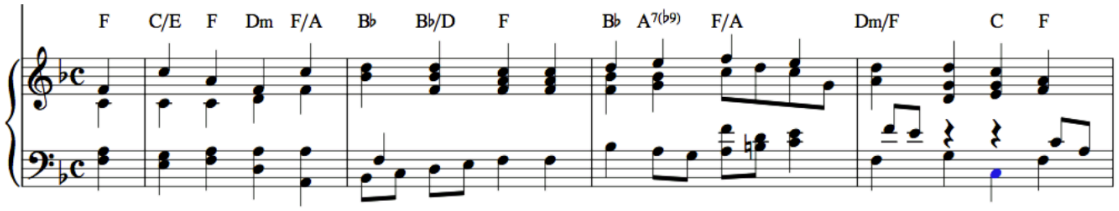
C, C♯, D, E♭, E, F, F♯, G, A♭, A, B♭, B.

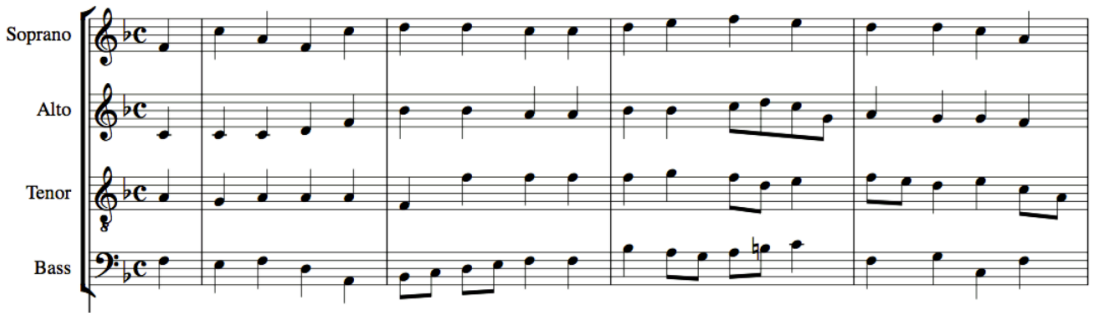
- Almost all Western music is measured using 12 - Tone Equal Temperament (12-TET)
- Music from other parts of the world are typically measured in different temperaments.
- Defines the interval of each octave
  - Number of steps before the scale wraps around on itself
  - Assume octave equivalence

Pitch Class Set Theory

- Class (defines temperament) $C = \{0, 1, 2, 3, ..., 11\}$
- Pitch set $P = \{p_1, p_2, \ldots, p_n\}$
- Chords

Chord	Major	Minor	Augmented	Diminished
Triad	$\{p_i, p_i + 4, p_i + 7\}$	$\{p_i, p_i + 3, p_i + 7\}$	$\{p_i, p_i + 4, p_i + 8\}$	$\{p_i, p_i + 3, p_i + 6\}$
Seventh	$\{p_i, p_i + 4, p_i + 7, p_i + 11\}$	$\{p_i, .., p_i + 10\}$	$\{p_i, .., p_i + 11\}$	$\{p_i, .., p_i + 9\}$

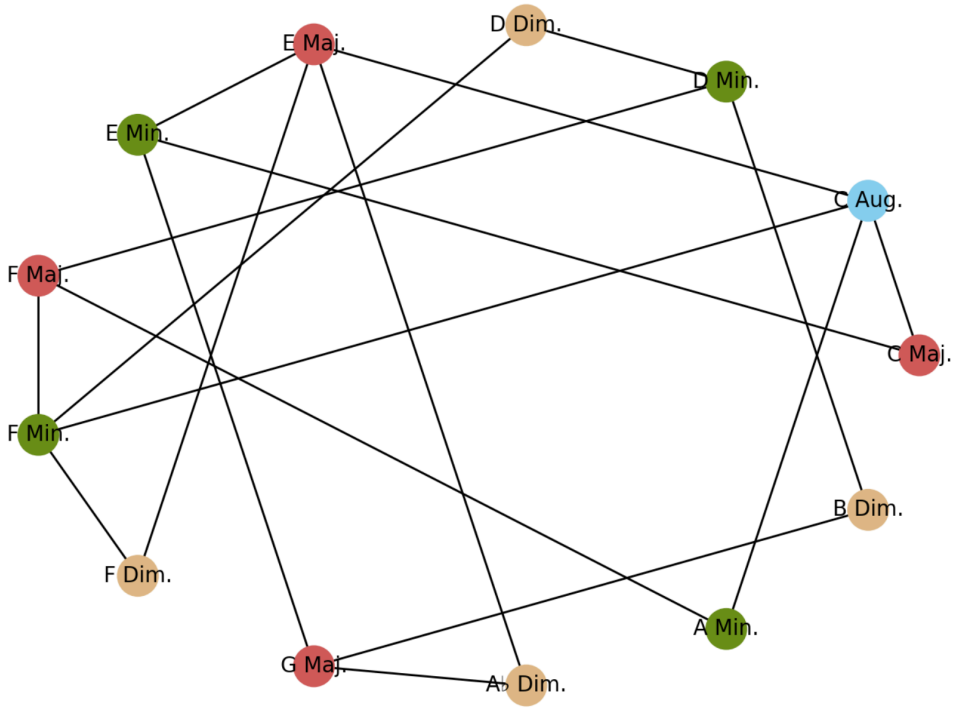




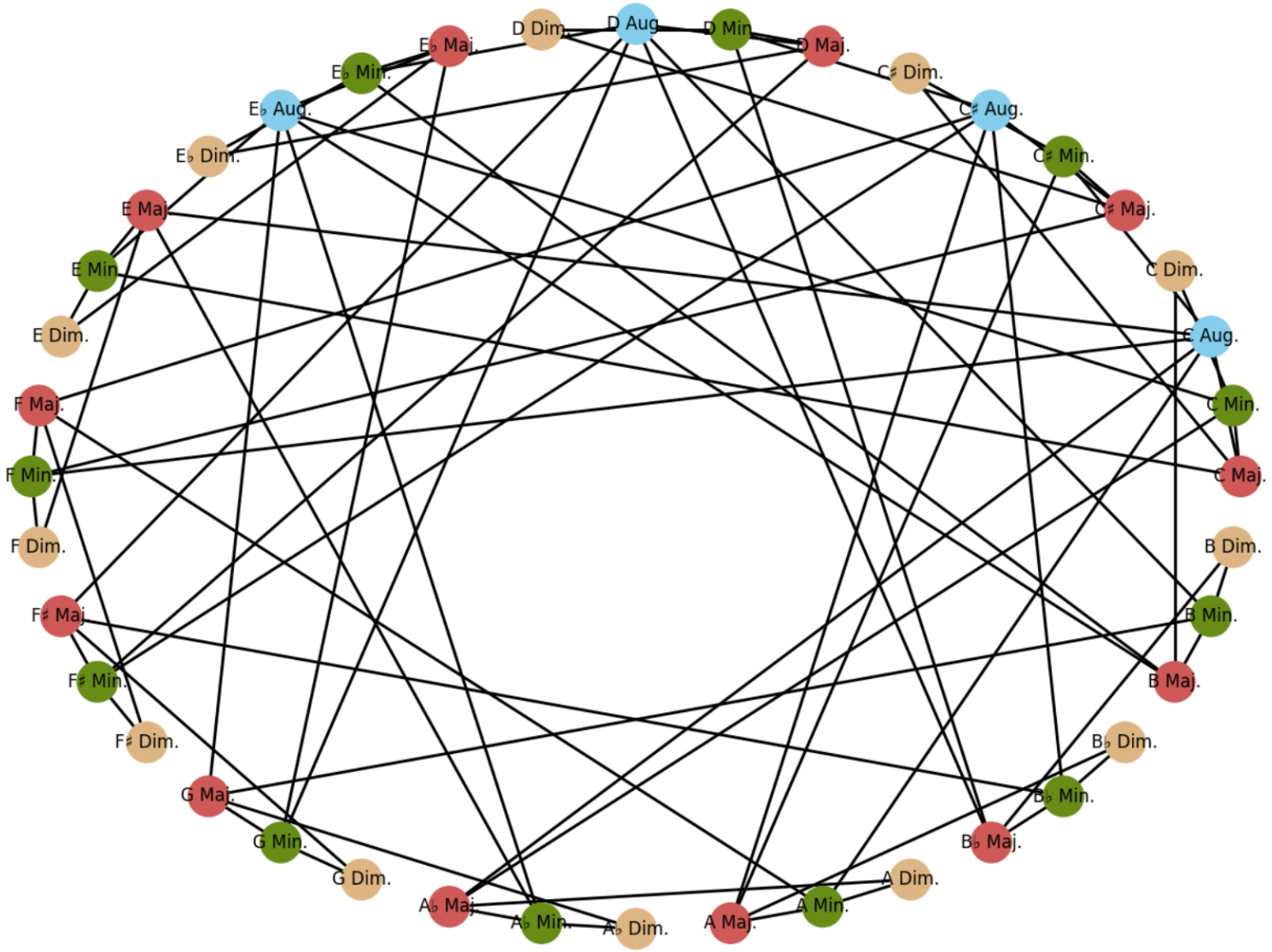
Parsimonious Voice Leading

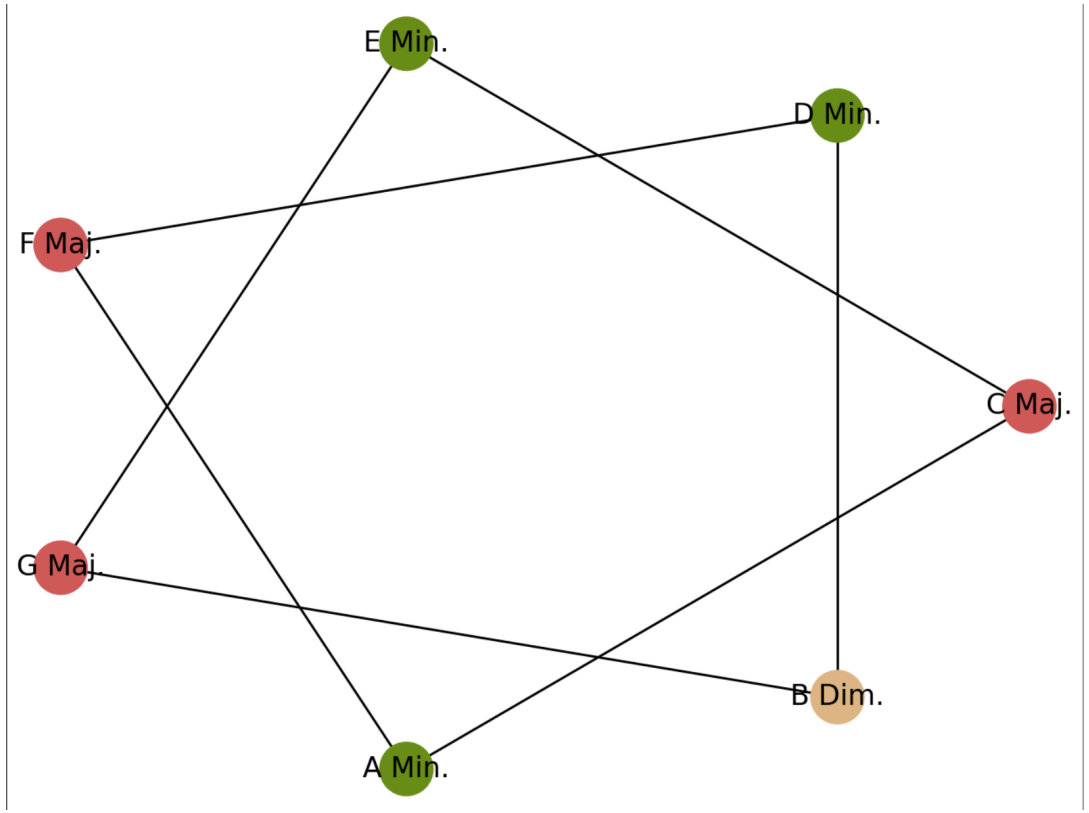
Graph Definitions

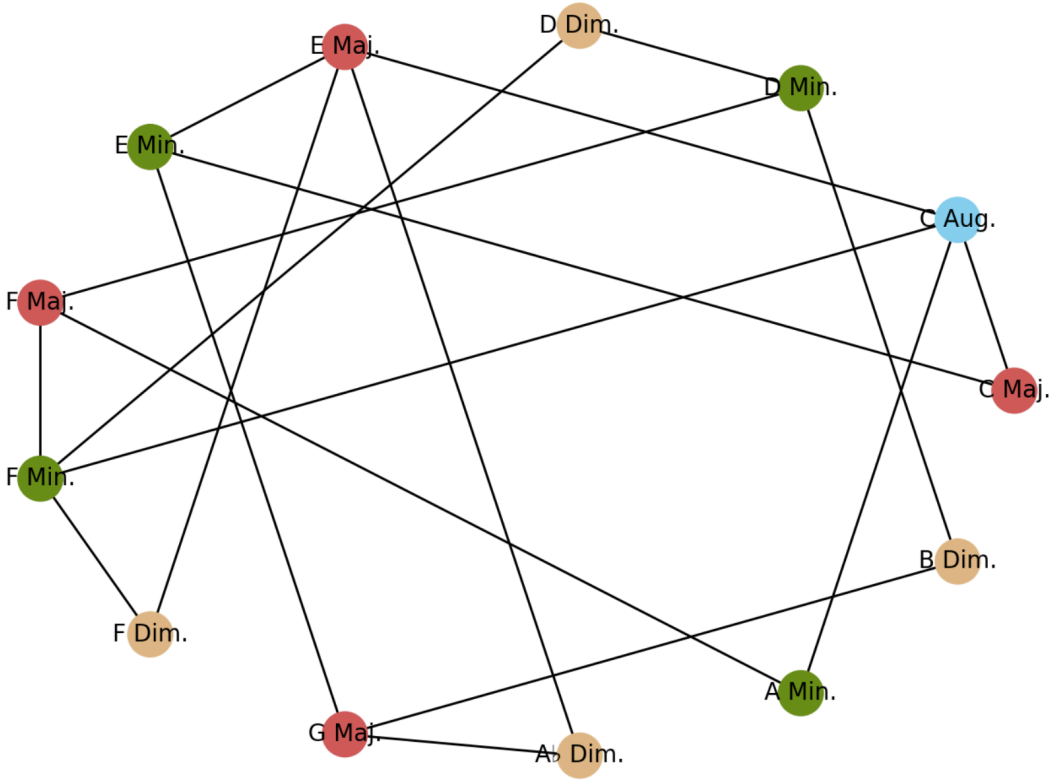
- Major – red
- Minor – green
- Augmented – blue
- Diminished – tan



Graph construction is fairly simple, the major, minor, augmented, and diminished chords form our set of nodes, or vertices V. An edge exists between two vertices if and only if the intersection of $v_i \cap v_j$ contains exactly two elements and if and only if the two elements of the symmetric difference $v_i \Delta v_j$ are neighbors in P, that is they are a semitone apart.


D Dim.
D Aug.
D Min.
D Maj.
E♭ Maj.
E♭ Min.
E♭ Aug.
E♭ Dim.
C♯ Dim.
C♯ Aug.
C♯ Min.
C♯ Maj.
E Maj.
E Min.
E Dim.
C Dim.
C Aug.
C Min.
C Maj.
F Maj.
F Min.
F Dim.
B Dim.
F♯ Maj.
F♯ Min.
F♯ Dim.
B Min.
B Maj.
B♭ Dim.
G Maj.
G Min.
G Dim.
B♭ Min.
B♭ Maj.
A Dim.
A♭ Maj.
A♭ Min.
A♭ Dim.
A Maj.
A Min.


E Min.
D Min.
F Maj.
C Maj.
G Maj.
B Dim.
A Min.


D Dim.
E Maj.
D Min.
E Min.
C Aug.
F Maj.
C Maj.
F Min.
B Dim.
F Dim.
A Min.
G Maj.
A♭ Dim.

Problem Definition

- Expand on previous work in the field
  - Ability to analyze seventh chord (4-note) graphs
  - Ability to analyze N-TET tonality graphs

Analysis Definitions

- Degree Centrality $\frac{deg(V)}{n-1}$
- Closeness Centrality $C(u) = \frac{n-1}{\sum_{v=1}^{n-1} d(u,v)}$
- Betweenness Centrality $C_B(v) = \sum_{s,t \in V} \frac{\sigma(s,t|v)}{\sigma(s,t)}$
- Katz Centrality $x_i = \alpha \sum_j A_{ij} x_j + \beta$
- Load Centrality
- Degree Vitality

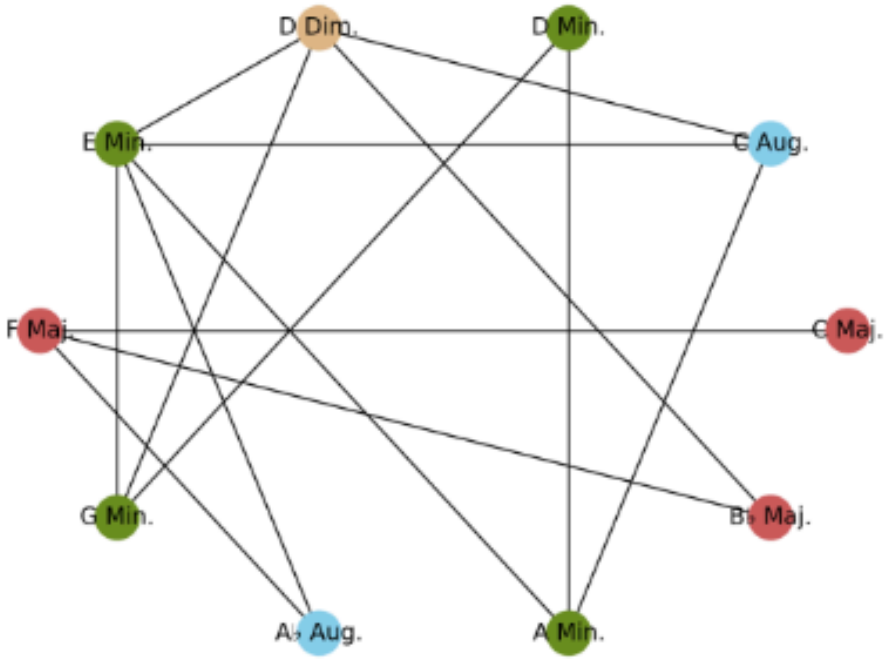


Figure 4: 4-note Chords with Set $\{0, 2, 4, 5, 7, 8, 9, 10, 11\}$

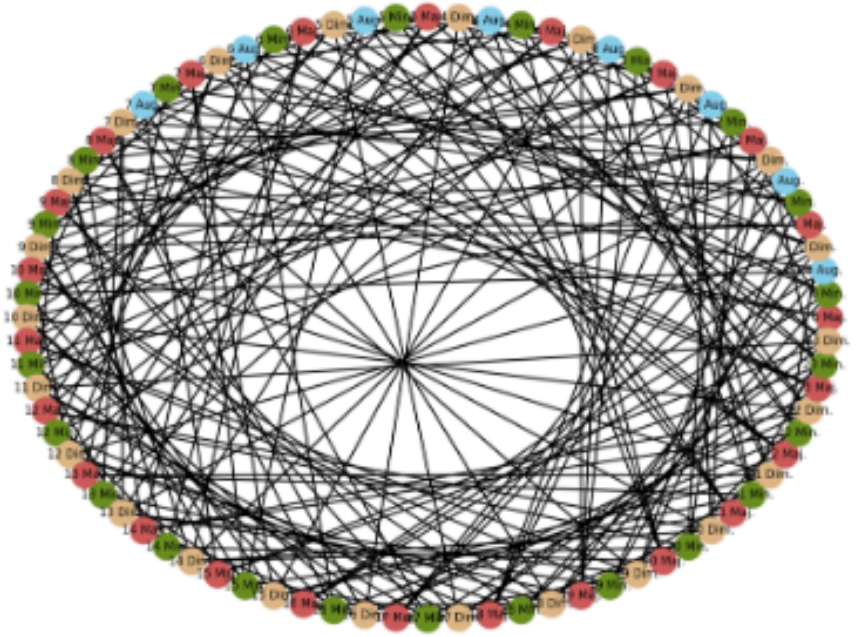


Figure 5: 24-TET Chromatic Scale

Below are the results of comparing triad graphs with seventh graphs using the following pitch set $P = \{0, 2, 4, 5, 7, 8, 9, 10, 11\}$

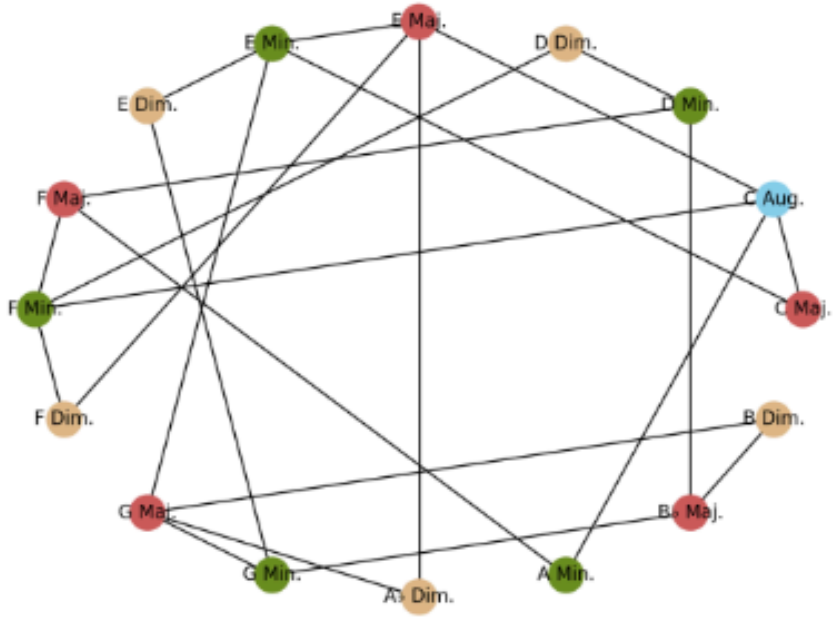


Figure 6: Triads Chords with Set $\{0, 2, 4, 5, 7, 8, 9, 10, 11\}$

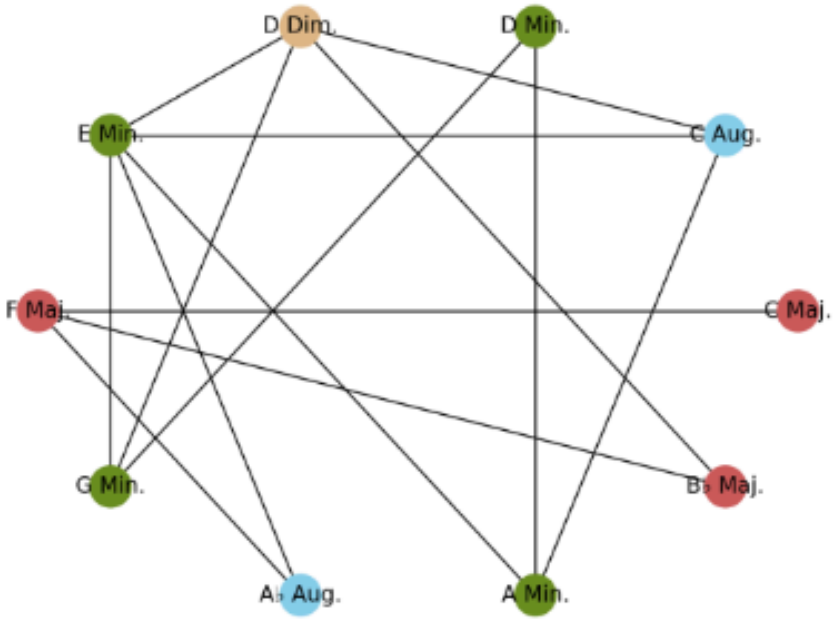


Figure 7: Sevenths Chords with Set $\{0, 2, 4, 5, 7, 8, 9, 10, 11\}$

Results – Triads and Sevenths

Triad	Degree Centrality	Closeness Cent.	Betweenness Cent.	Katz Cent.	Load	Vitality
C Maj.	0.133	0.394	0.047	0.231	0.051	38.0
C Aug.	0.266	0.428	0.193	0.277	0.190	16.0
D Min.	0.2	0.384	0.179	0.248	0.179	1.0
D Dim.	0.133	0.365	0.041	0.228	0.043	41.0
E Maj.	0.266	0.441	0.209	0.277	0.205	14.0
E Min.	0.266	0.428	0.193	0.277	0.190	16.0
E Dim.	0.133	0.365	0.031	0.228	0.033	41.0
F Maj.	0.2	0.384	0.105	0.251	0.104	33.0
F Min.	0.266	0.405	0.165	0.274	0.162	20.0
F Dim.	0.133	0.384	0.044	0.231	0.047	39.0
G Maj.	0.266	0.405	0.165	0.274	0.162	20.0
G Min.	0.2	0.384	0.105	0.251	0.104	33.0
A♭ Dim.	0.133	0.384	0.044	0.231	0.047	39.0
A Min.	0.133	0.365	0.031	0.228	0.033	41.0
B♭ Maj.	0.2	0.384	0.179	0.248	0.179	1.0
B Dim.	0.133	0.365	0.041	0.228	0.043	41.0

Seventh	Degree Centrality	Closeness Cent.	Betweenness Cent.	Katz Cent.	Load	Vitality
C Maj.	0.111	0.321	0.0	0.253	0.0	28.0
C Aug.	0.333	0.5	0.027	0.328	0.027	18.0
D Min.	0.222	0.391	0.013	0.287	0.013	23.0
D Dim.	0.444	0.6	0.226	0.355	0.229	7.0
E Min.	0.555	0.642	0.342	0.384	0.340	2.0
F Maj.	0.333	0.45	0.25	0.305	0.25	∞
G Min.	0.333	0.5	0.120	0.325	0.118	16.0
A♭ Aug.	0.222	0.529	0.203	0.291	0.201	13.0
A Min.	0.333	0.473	0.074	0.322	0.076	18.0
B♭ Maj.	0.222	0.5	0.129	0.288	0.131	16.0

It can be seen that the constraints, even with the ability to change two notes, is a bit more limited when analyzing seventh chords.

Open Questions / Further Research

- Modified voice leading parameters
- Any other area that already benefits from set theory
  - Melodies
  - Rhythms

Questions?