Integers II

CSE 351 Spring 2019

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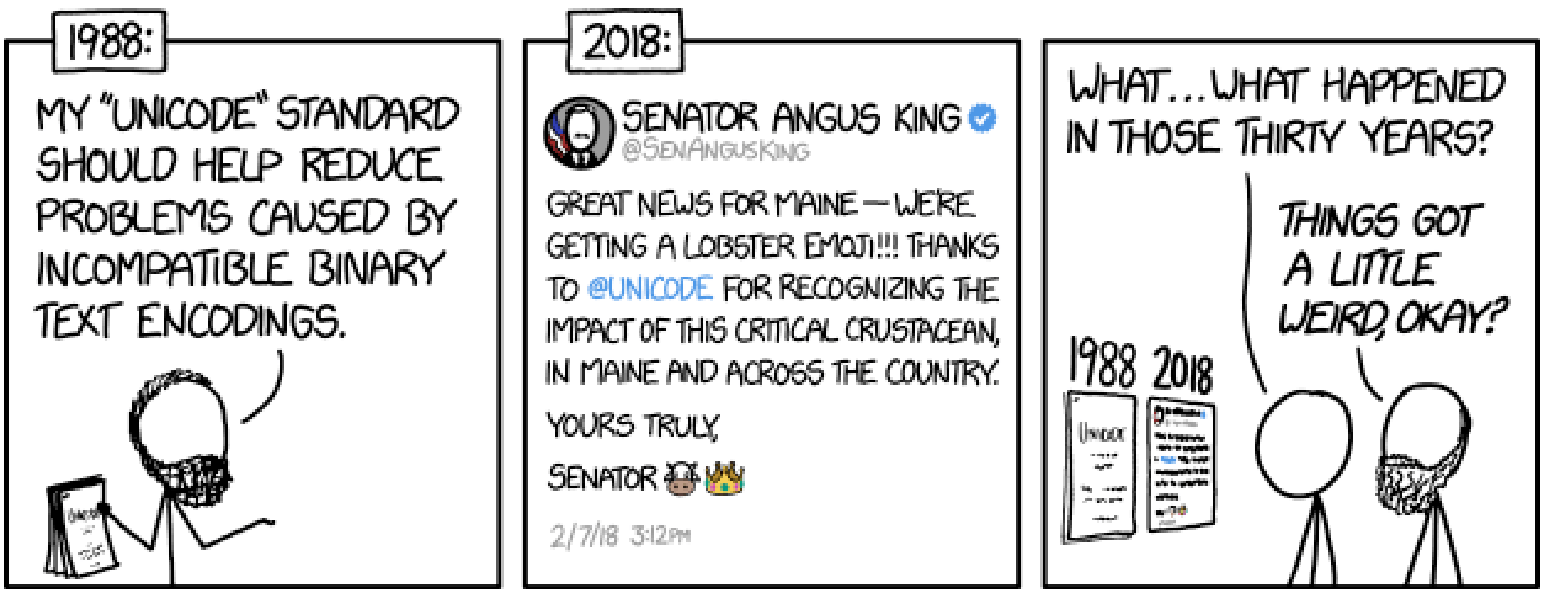


http://xkcd.com/1953/

Administrivia

- Homework 1 due TONIGHT (4/10)
  - Reminder: autograded, 20 tries, no late submissions
- Lab 1a due Monday (4/15)
  - Submit `pointer.c` and `lab1Areflect.txt` to Canvas
- Lab 1b released soon, due Monday 4/22
  - Bit puzzles on number representation
  - Have much of what you need after today, will need floating point, coming soon
  - Section tomorrow will be useful!
  - Bonus slides at the end of today's lecture have relevant examples

Extra Credit

- All labs starting with Lab 1b have extra credit portions
  - These are meant to be fun extensions to the labs
- Extra credit points *don't* affect your lab grades
  - From the course policies: "they will be accumulated over the course and will be used to bump up borderline grades at the end of the quarter."
  - Make sure you finish the rest of the lab before attempting any extra credit

Integers

- **Binary representation of integers**
  - **Unsigned and signed**
  - **Casting in C**
- Consequences of finite width representations
  - Overflow, sign extension
- Shifting and arithmetic operations

Two's Complement Arithmetic

- The same addition procedure works for both unsigned and two's complement integers
  - **Simplifies hardware:** only one algorithm for addition
  - **Algorithm:** simple addition, discard the highest carry bit
    - Called modular addition: result is sum *modulo* 2^w

Can implement subtraction in a similar way; see slide 15

- **4-bit Examples:**

4 +3	0100 +0011	-4 +3	1100 +0011	4 -3	0100 +1101
=7	0111	=-1	1111	=1	1 $\boxed{0001}$

Why Does Two's Complement Work?

- For all representable positive integers x, we want:

additive inverse

$$\begin{array}{r} \textit{bit representation of } \;\; x \\ + \textit{ bit representation of } -x \\ \hline 0 \end{array}$$

(ignoring the carry-out bit)

  - What are the 8-bit negative encodings for the following?

$$\begin{array}{r} \text{00000001} \\ 11111111 \\ + \;\; \text{????????} \\ \hline \cancel{1}\;\text{00000000} \end{array} \qquad \begin{array}{r} \text{00000010} \\ 11111110 \\ + \;\; \text{????????} \\ \hline \cancel{1}\;\text{00000000} \end{array} \qquad \begin{array}{r} 1\;1\;11\;111 \\ \text{11000011} \\ 00111101 \\ + \;\; \text{????????} \\ \hline \cancel{1}\;\text{00000000} \end{array}$$

Why Does Two's Complement Work?

- For all representable positive integers x, we want:

$$\begin{array}{r} \textit{bit representation of } x \\ +\ \textit{bit representation of } {-x} \\ \hline 0 \end{array}$$ (ignoring the carry-out bit)

0b1…1

$x + (\sim x) =$

$x + (\sim x) = -1$

$x + (\sim x + 1) = 0$

$\boxed{-x = \sim x + 1}$

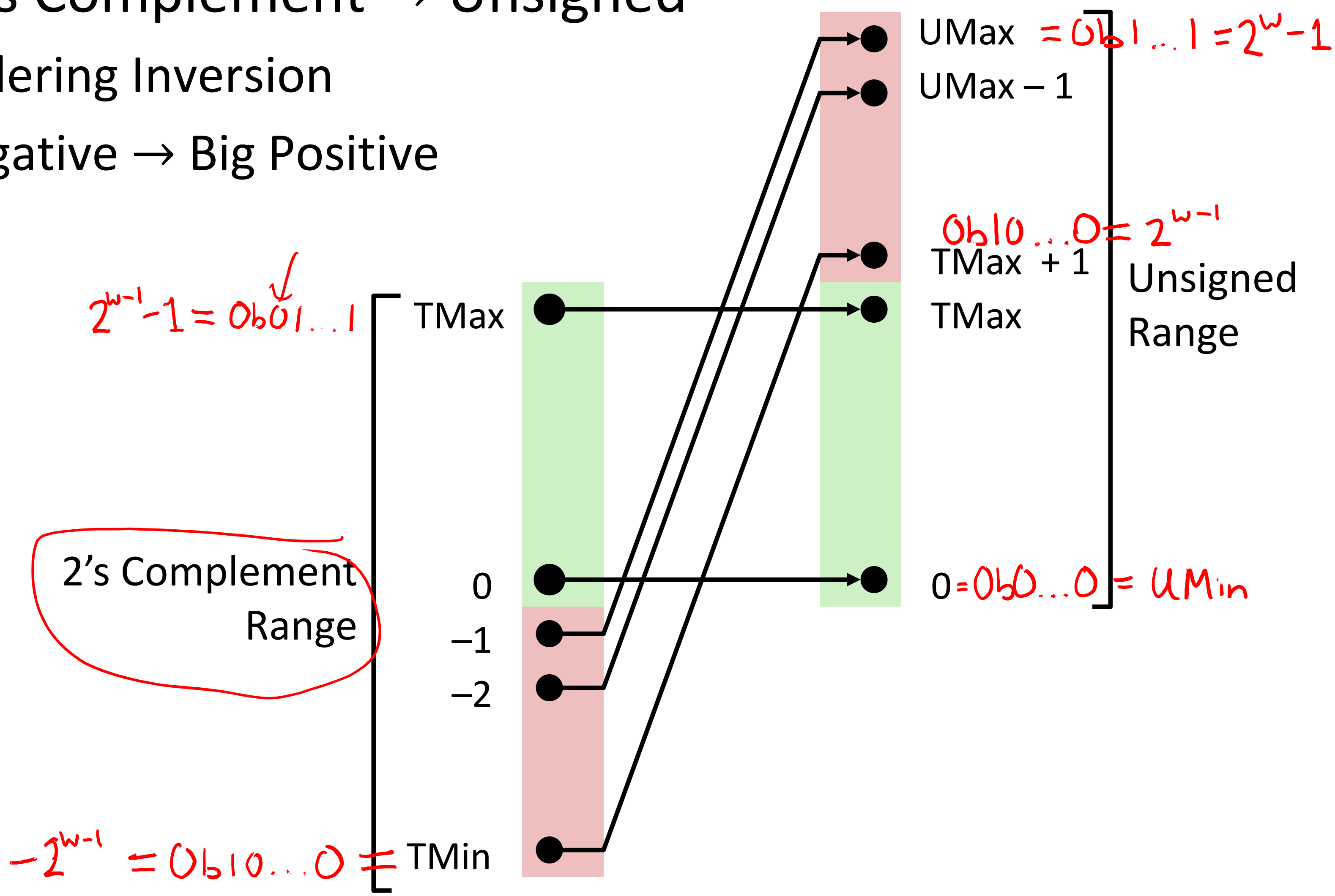
- What are the 8-bit negative encodings for the following?

```
   00000001       00000010       11000011
 + 11111111     + 11111110     + 00111101
  100000000      100000000      100000000
```

These are the bitwise complement plus 1!
`-x == ~x + 1`

Signed/Unsigned Conversion Visualized

- Two's Complement → Unsigned
  - Ordering Inversion
  - Negative → Big Positive



Values To Remember

- Unsigned Values
  - UMin = `0b00…0` = 0
  - UMax = `0b11…1` = $2^w - 1$
- Two's Complement Values
  - TMin = `0b10…0` = -2^{w-1}
  - TMax = `0b01…1` = $2^{w-1} - 1$
  - -1 = `0b11…1`
- **Example:** Values for $w = 64$

	Decimal	Hex
UMax	18,446,744,073,709,551,615	FF FF FF FF FF FF FF FF
TMax	9,223,372,036,854,775,807	7F FF FF FF FF FF FF FF
TMin	-9,223,372,036,854,775,808	80 00 00 00 00 00 00 00
-1	-1	FF FF FF FF FF FF FF FF
0	0	00 00 00 00 00 00 00 00

In C: Signed vs. Unsigned

- Casting
  - Bits are unchanged, just interpreted differently!
    - **int** tx, ty;
    - **unsigned int** ux, uy;
  - *Explicit* casting
    - tx = (**int**) ux;
    - uy = (**unsigned int**) ty;
  - *Implicit* casting can occur during assignments or function calls cast to target variable/parameter type
    - tx = ux;
    - uy = ty; (also implicitly occurs with printf format specifiers)

Casting Surprises

!!!

- Integer literals (constants)
  - By default, integer constants are considered *signed* integers
    - Hex constants already have an explicit binary representation
  - Use "U" (or "u") suffix to explicitly force *unsigned*
    - **Examples:** 0U, 4294967259u

- Expression Evaluation
  - When you mix unsigned and signed in a single expression, then **signed values are implicitly cast to unsigned** (unsigned "dominates")
  - Including comparison operators <, >, ==, <=, >=

Casting Surprises

!!!

- 32-bit examples:
  - TMin = -2,147,483,648, TMax = 2,147,483,647

Left Constant	Order	Right Constant	Interpretation
0 0000 0000 0000 0000 0000 0000 0000 0000	==	0U 0000 0000 0000 0000 0000 0000 0000 0000	unsigned
-1 1111 1111 1111 1111 1111 1111 1111 1111	<	0 0000 0000 0000 0000 0000 0000 0000 0000	signed
-1 1111 1111 1111 1111 1111 1111 1111 1111	>	0U 0000 0000 0000 0000 0000 0000 0000 0000	unsigned
2147483647 0111 1111 1111 1111 1111 1111 1111 1111	>	-2147483648 1000 0000 0000 0000 0000 0000 0000 0000	signed
2147483647U 0111 1111 1111 1111 1111 1111 1111 1111	<	-2147483648 1000 0000 0000 0000 0000 0000 0000 0000	unsigned
-1 1111 1111 1111 1111 1111 1111 1111 1111	>	-2 1111 1111 1111 1111 1111 1111 1111 1110	signed
(unsigned) -1 1111 1111 1111 1111 1111 1111 1111 1111	>	-2 1111 1111 1111 1111 1111 1111 1111 1110	unsigned
2147483647 0111 1111 1111 1111 1111 1111 1111 1111	<	2147483648U 1000 0000 0000 0000 0000 0000 0000 0000	unsigned
2147483647 0111 1111 1111 1111 1111 1111 1111 1111	>	(int) 2147483648U 1000 0000 0000 0000 0000 0000 0000 0000	signed

Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C
- **Consequences of finite width representations**
  - **Overflow, sign extension**
- Shifting and arithmetic operations

Arithmetic Overflow

Umin

Bits	Unsigned	Signed
0000	0	0
0001	1	1
0010	2	2
0011	3	3
0100	4	4
0101	5	5
0110	6	6
0111	7	7
1000	8	-8
1001	9	-7
1010	10	-6
1011	11	-5
1100	12	-4
1101	13	-3
1110	14	-2
1111	15	-1

Umax

Tmax

Tmin

- When a calculation produces a result that can't be represented in the current encoding scheme
  - Integer range limited by fixed width
  - Can occur in both the positive and negative directions e.g. larger than Umax or smaller than Umin; cannot be represented by the current number of bits
- C and Java ignore overflow exceptions
  - You end up with a bad value in your program and no warning/indication… oops!

Assigned arithmetic

Overflow: Unsigned

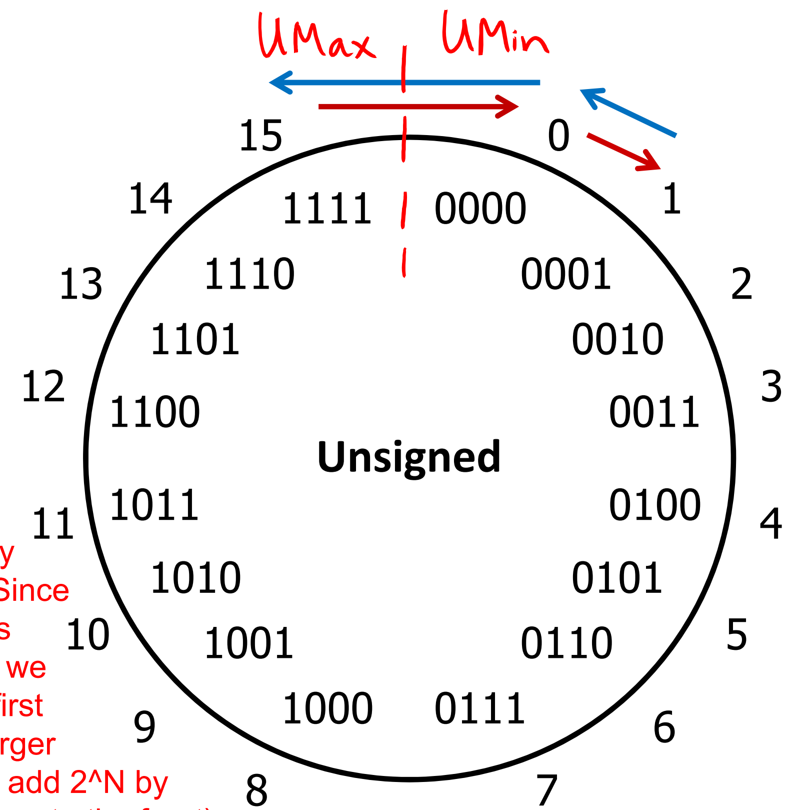
- **Addition:** drop carry bit (-2^N)

```
  15        1111
+  2      + 0010
----      ------
  17       10001
   1
```

- **Subtraction:** borrow ($+2^N$)

```
   1       10001
-  2     -  0010
----      ------
  -1        1111
  15
```

subtract and carry just like normal. Since unsigned integers are nonnegative, we always treat the first number like its larger (if necessary, we add 2^N by adding the grey one to the front)



$\pm 2^N$ because of modular arithmetic

$2^4 = 16$

i.e. addition and subtraction are modulo 2^w where w is the number of bits

Overflow: Two's Complement

sum is going above Tmax

- **Addition:** (+) + (+) = (−) result?

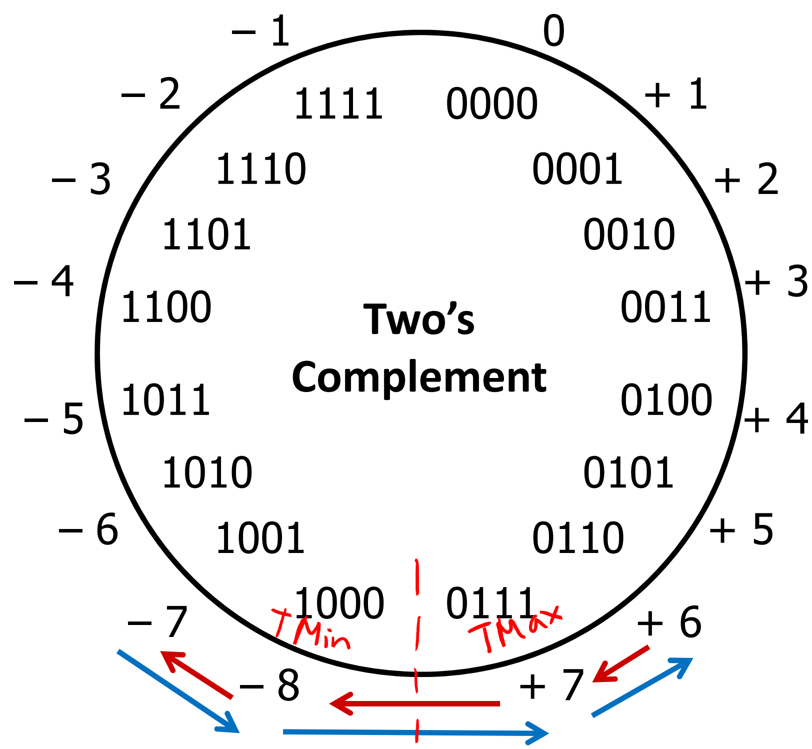
```
   6       0110
+  3    +  0011
-----   -------
   9       1001
  -7
```

sum is going below Tmin

- **Subtraction:** (−) + (−) = (+)?

```
  -7       1001
-  3    -  0011
-----   -------
 -10       0110
   6
```

Same addition/subtraction as before; add/subtract and carry.
Again, everything is modulo 2^N



For signed: overflow if operands have same sign and result's sign is different

Sign Extension

- What happens if you convert a *signed* integral data type to a larger one?
  - *e.g.* `char` → `short` → `int` → `long`
    1 byte, 2 bytes, 4 bytes, 8 bytes
- **4-bit → 8-bit Example:**
  - Positive Case
    - ✓ Add 0's?
  - Negative Case?

4-bit:	`0010`	=	+2
8-bit:	`00000010`	=	+2

Peer Instruction Question

- Which of the following 8-bit numbers has the same *signed* value as the 4-bit number **0b1100**?

  $-8 \; 4 \; 2 \; 1$

  $-8+4 = -4$

  $-x = 0011 + 1 = 0100 = 4 \Rightarrow x = -4$

  - Underlined digit = MSB
  - Vote at http://pollev.com/rea

~~A.~~ 0b 0000 1100 — positive!

~~B.~~ 0b 1000 1100 — much too negative: $-2^7+2^3+2^2 = -116$

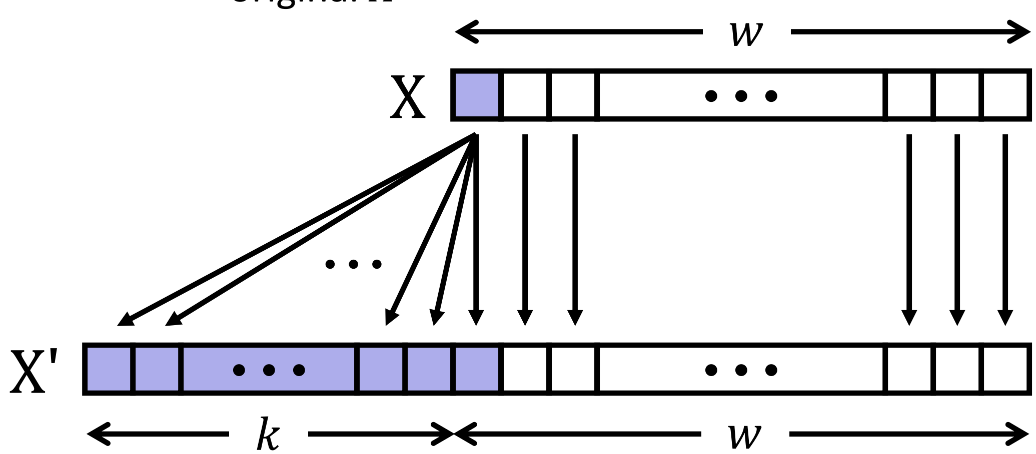
$\boxed{\text{C. 0b 1111 1100}}$ — correct! $-y = \text{0b } 0000\ 0011 + 1 = 4, \; y = -4$

~~D.~~ 0b 1100 1100 — $-2^7+2^6+2^3+2^2 = -52$

E. We're lost…

Sign Extension

- **Task:** Given a w-bit signed integer X, convert it to w+k-bit signed integer X' *with the same value*
- **Rule: Add k copies of sign bit**
  - Let x_i be the i-th digit of X in binary
  - $\mathrm{X}' = \underbrace{x_{w-1}, \dots, x_{w-1}}_{k \text{ copies of MSB}}, \underbrace{x_{w-1}, x_{w-2}, \dots, x_1, x_0}_{\text{original } \mathrm{X}}$



Sign Extension Example

- Convert from smaller to larger integral data types
- C automatically performs sign extension
  - Java too

```
short int x =   12345;
int      ix = (int) x;
short int y =  -12345;
int      iy = (int) y;
```

0b 0011

Var	Decimal	Hex	Binary
x	12345	30 39	00110000 00111001
ix	12345	00 00 30 39	00000000 00000000 00110000 00111001
y	-12345	CF C7	11001111 11000111
iy	-12345	FF FF CF C7	11111111 11111111 11001111 11000111

0b 1100

Integers

- Binary representation of integers
  - Unsigned and signed
  - Casting in C
- Consequences of finite width representations
  - Overflow, sign extension
- **Shifting and arithmetic operations**

Shift Operations

- Left shift (`x<<n`) bit vector `x` by `n` positions
  - Throw away (drop) extra bits on left
  - Fill with `0`s on right
- Right shift (`x>>n`) bit-vector `x` by `n` positions
  - Throw away (drop) extra bits on right
  - Logical shift (for unsigned values)
    - Fill with `0`s on left
  - Arithmetic shift (for signed values)
    - Replicate most significant bit on left
    - Maintains sign of `x`

Shift Operations

8-bit example:

- Left shift (x<<n)
  - Fill with 0s on right
- Right shift (x>>n)
  - Logical shift (for unsigned values)
    - Fill with 0s on left
  - Arithmetic shift (for signed values)
    - Replicate most significant bit on left

	x	0010 0010
	x<<3	0001 0000
logical:	x>>2	0000 1000
arithmetic:	x>>2	0000 1000

	x	1010 0010
	x<<3	0001 0000
logical:	x>>2	0010 1000
arithmetic:	x>>2	1110 1000

- Notes:
  - Shifts by n<0 or n≥w (w is bit width of x) are ***undefined*** behavior not guaranteed
  - **In C:** behavior of >> is determined by compiler
    - In gcc / C lang, depends on data type of x (signed/unsigned) arithmetic/logical
  - **In Java:** logical shift is >>> and arithmetic shift is >>

Shifting Arithmetic?

- What are the following computing?
  - x>>n
    - 0b 0100 >> 1 = 0b 0010
    - 0b 0100 >> 2 = 0b 0001
    - Divide by 2^n
  - x<<n
    - 0b 0001 << 1 = 0b 0010
    - 0b 0001 << 2 = 0b 0100
    - Multiply by 2^n
- Shifting is faster than general multiply and divide operations

Left Shifting Arithmetic 8-bit Example

- No difference in left shift operation for unsigned and signed numbers (just manipulates bits)
  - Difference comes during interpretation: x*2^n?

			Signed	Unsigned
x = 25;	00011001	=	25	25
L1=x<<2;	~~00~~01100100	=	100	100
L2=x<<3;	~~000~~11001000	=	-56	200
L3=x<<4;	~~0001~~10010000	=	-112	144

200
-256 → 2^8
= -56

signed overflow

400
-256 → 2^8
= 144

unsigned overflow

Right Shifting 8-bit Examples

- **Reminder:** C operator >> does *logical* shift on unsigned values and *arithmetic* shift on signed values
  - Logical Shift: x/2ⁿ?

`xu = 240u;` `11110000` = 240 $/8 = 30$

`R1u=xu>>3;` `00011110`~~`000`~~ = 30 $/4 = 7.5$

`R2u=xu>>5;` `00000111`~~`10000`~~ = 7

rounding (down)

Right Shifting Arithmetic 8-bit Examples

- **Reminder:** C operator >> does *logical* shift on unsigned values and *arithmetic* shift on signed values
  - Arithmetic Shift: $x/2^n$?

xs = -16;   11110000 = -16

R1s=xu>>3; 11111110~~000~~ = -2 /4 = -0.5

R2s=xu>>5; 11111111~~10000~~ = -1

rounding (down)

Question

UMin = 0, UMax = 255

8-bits, so TMin = -128, TMax = 127

For the following expressions, find a value of **`signed char`** `x`, if there exists one, that makes the expression TRUE. Compare with your neighbor(s)!

- Assume we are using 8-bit arithmetic:

Expression	Example:	All solutions:
`x == (unsigned char) x` (unsigned)	x = 0	works for all x
`x >= 128U` (unsigned; 0b1 000 0000)	x = -1	any x < 0
`x != (x>>2)<<2`	x = 3	any x where lowest two bits are not 0b00
`x == -x` • Hint: there are two solutions	x = 0	① x = 0b0...0 = 0 ② x = 0b10...0 = -128
`(x < 128U) && (x > 0x3F)`	x = 64	any x where upper two bits are exactly 0b01

Summary

- Sign and unsigned variables in C
  - Bit pattern remains the same, just *interpreted* differently
  - Strange things can happen with our arithmetic when we convert/cast between sign and unsigned numbers
    - Type of variables affects behavior of operators (shifting, comparison)
- We can only represent so many numbers in *w* bits
  - When we exceed the limits, *arithmetic overflow* occurs
  - *Sign extension* tries to preserve value when expanding
- Shifting is a useful bitwise operator
  - Right shifting can be arithmetic (sign) or logical (0)
  - Can be used in multiplication with constant or bit masking

BONUS SLIDES

Some examples of using shift operators in combination with bitmasks, which you may find helpful for Lab 1.

- Extract the 2nd most significant byte of an `int`
- Extract the sign bit of a signed `int`
- Conditionals as Boolean expressions

Using Shifts and Masks

- Extract the 2nd most significant *byte* of an `int`:
  - First shift, then mask: `(x>>16) & 0xFF`

`x`	`00000001 00000010 00000011 00000100`
`x>>16`	`00000000 00000000 00000001 00000010`
`0xFF`	`00000000 00000000 00000000 11111111`
`(x>>16) & 0xFF`	`00000000 00000000 00000000 00000010`

  - Or first mask, then shift: `(x & 0xFF0000)>>16`

`x`	`00000001 00000010 00000011 00000100`
`0xFF0000`	`00000000 11111111 00000000 00000000`
`x & 0xFF0000`	`00000000 00000010 00000000 00000000`
`(x&0xFF0000)>>16`	`00000000 00000000 00000000 00000010`

Using Shifts and Masks

- Extract the *sign bit* of a signed `int`:
  - First shift, then mask: `(x>>31) & 0x1`
    - Assuming arithmetic shift here, but this works in either case
    - Need mask to clear 1s possibly shifted in

`x`	`00000001 00000010 00000011 00000100`
`x>>31`	`00000000 00000000 00000000 00000000`
`0x1`	`00000000 00000000 00000000 00000001`
`(x>>31) & 0x1`	`00000000 00000000 00000000 00000000`

`x`	`10000001 00000010 00000011 00000100`
`x>>31`	`11111111 11111111 11111111 11111111`
`0x1`	`00000000 00000000 00000000 00000001`
`(x>>31) & 0x1`	`00000000 00000000 00000000 00000001`

Using Shifts and Masks

- Conditionals as Boolean expressions
  - For `int x`, what does `(x<<31)>>31` do?

`x=!!123`	`00000000 00000000 00000000 00000001`
`x<<31`	`10000000 00000000 00000000 00000000`
`(x<<31)>>31`	`11111111 11111111 11111111 11111111`
`!x`	`00000000 00000000 00000000 00000000`
`!x<<31`	`00000000 00000000 00000000 00000000`
`(!x<<31)>>31`	`00000000 00000000 00000000 00000000`

  - Can use in place of conditional:
    - In C: `if(x) {a=y;} else {a=z;}` equivalent to `a=x?y:z;`
    - `a=(((x<<31)>>31)&y) | (((!x<<31)>>31)&z);`