Minimum Spanning Trees

Data Structures and Algorithms

Announcements

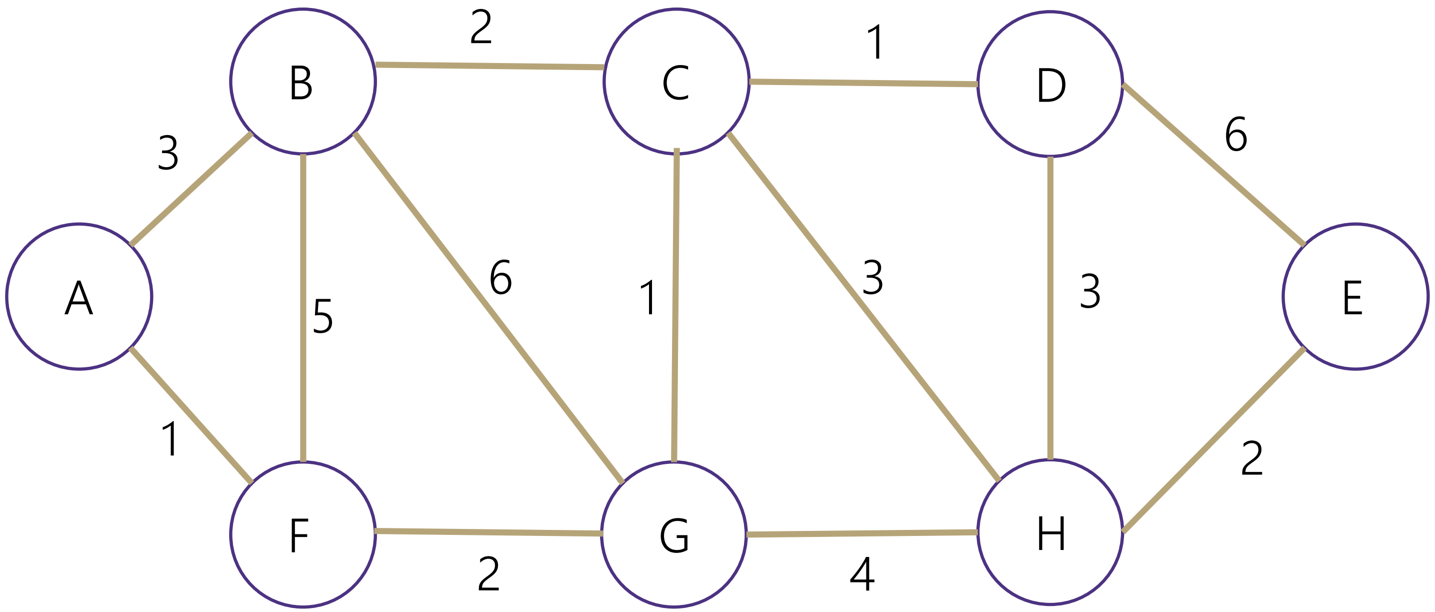
- Project 3 Due Tonight
- Project 4 Assigned Today
  - Same partners as project 3
  - We will re-run project 3 grading on project 4, just like the checkpoint from project 1 (this is why you are keeping your partners)
  - If you are curious about the missing part2 of this project, look at last quarter's website (change 18su to 18sp in the web address)

Goal for today: Learn the algorithm you will be implementing in project 4.

Review: Minimum Spanning Trees

Spanning Tree – A **subtree** of a graph that **spans** (includes) all of the vertices

- connected
- acyclic



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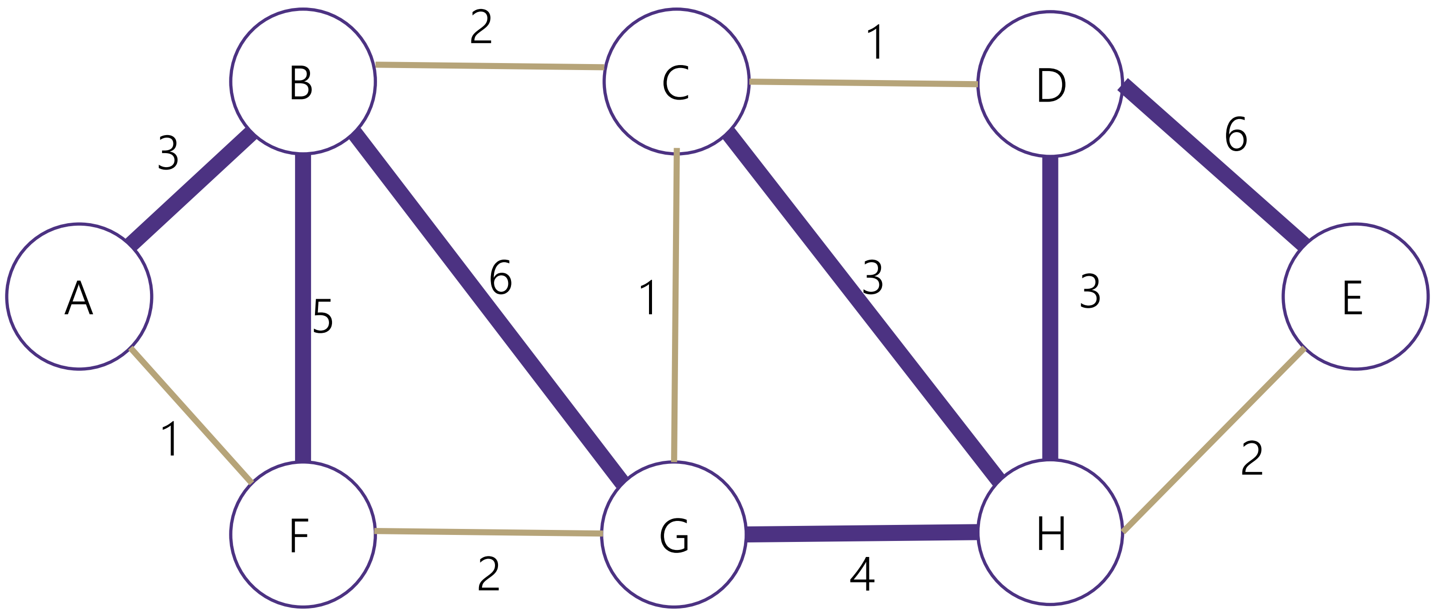

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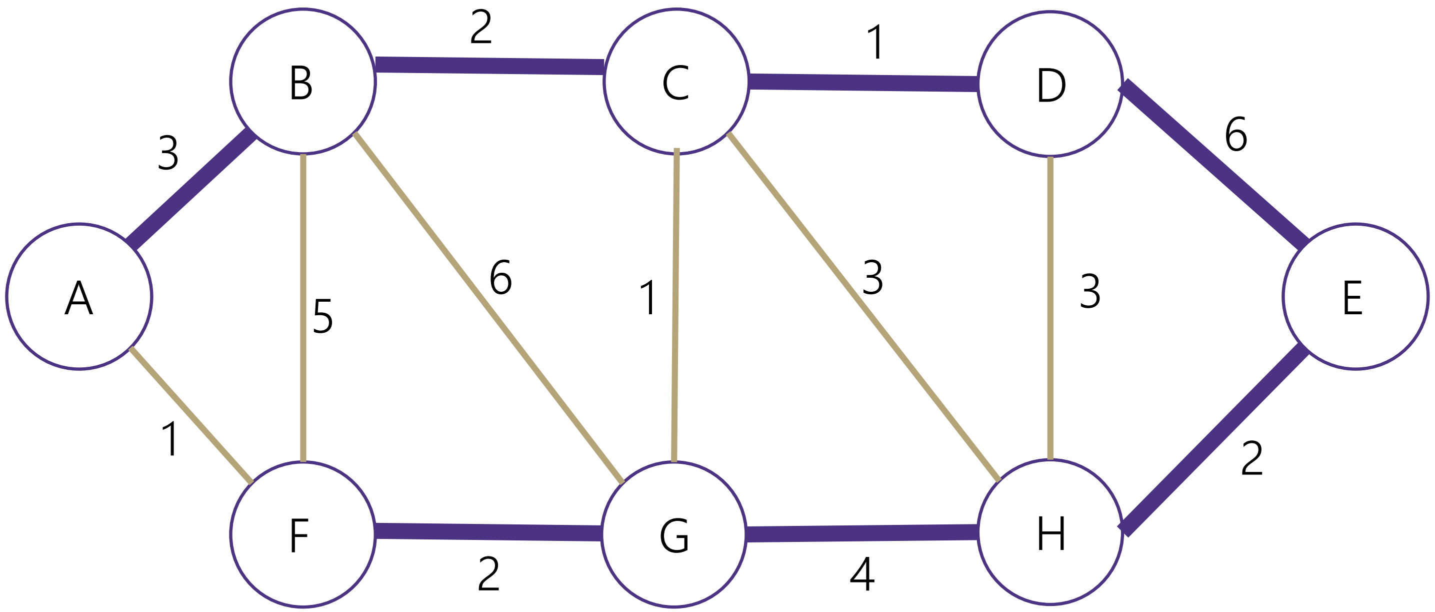

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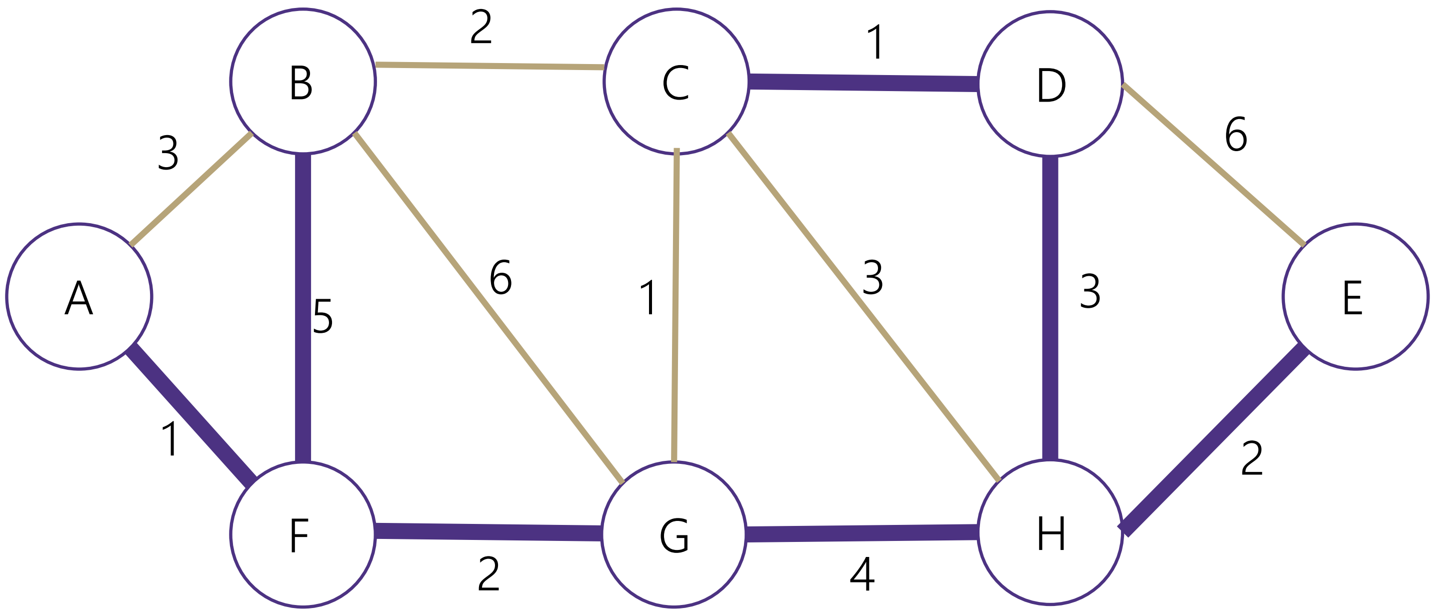
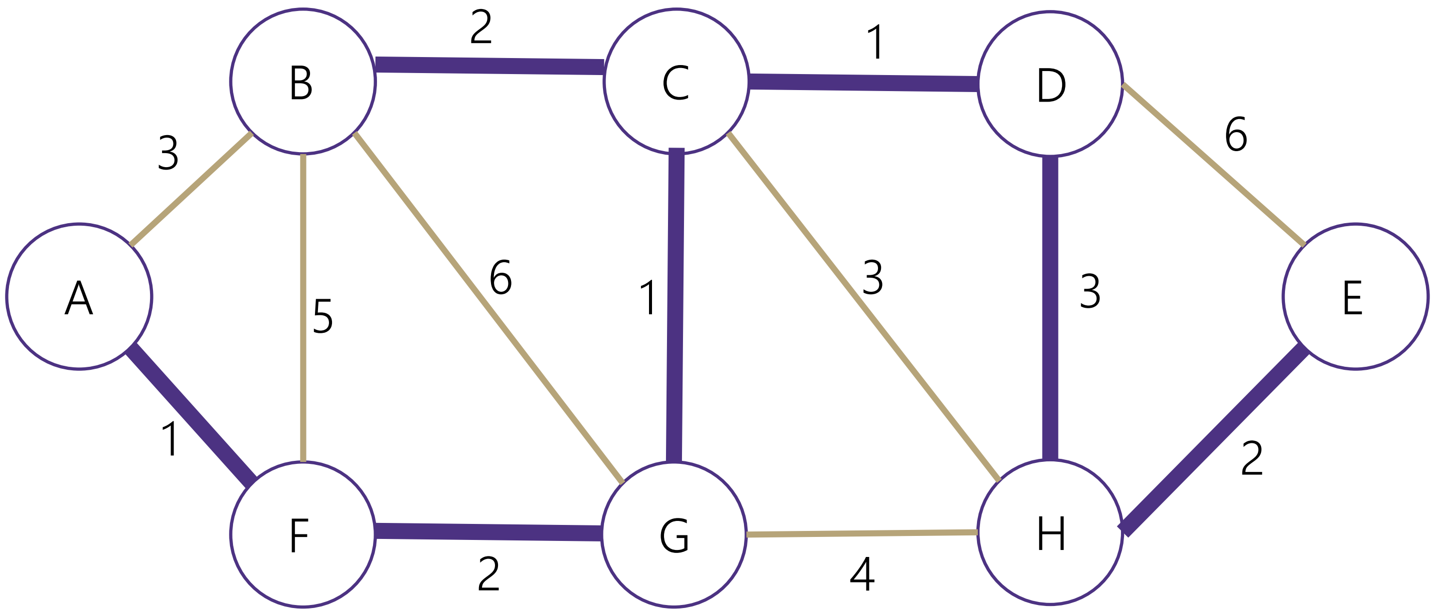


Review: Minimum Spanning Trees

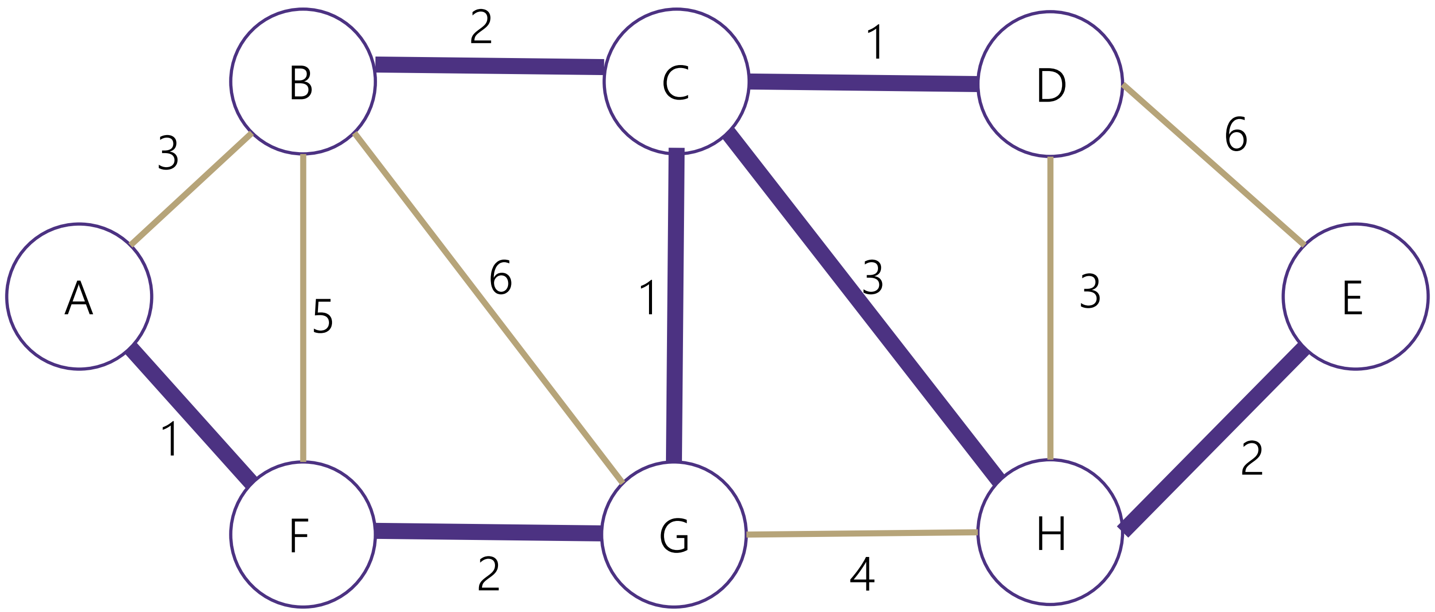
Minimum Spanning Tree – The **lowest weight subtree** of a graph that **spans** (includes) all of the vertices.



Review: Minimum Spanning Trees

Minimum Spanning Tree – The **lowest weight subtree** of a graph that **spans** (includes) all of the vertices.

- A graph can have more than one



How Do We Find One?

Discuss with your neighbors – how could we try to find the minimum spanning tree?

- Modify Dijkstra's Alg - only find a tree

- Topo Sort?

Greedy Algorithms

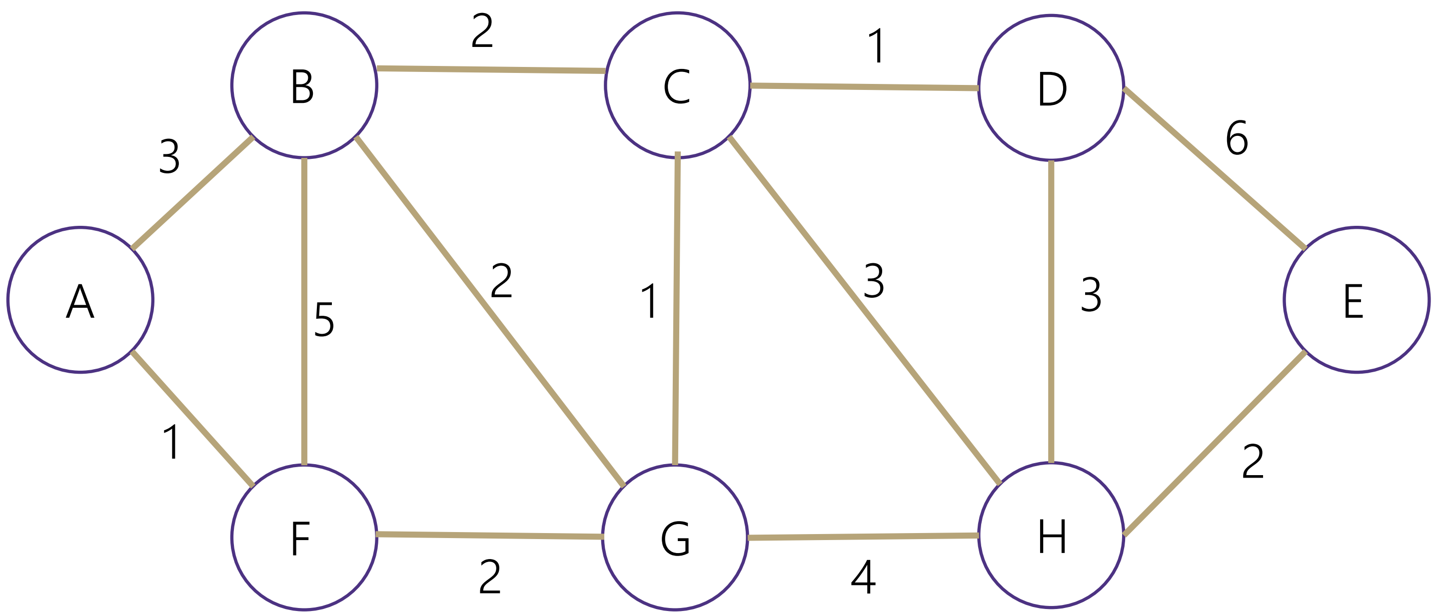
Strategy: Take the best we can get right now, ignoring long-term optimality.

- Usually fast to implement
- Does not always get the "best" result
  - But often is "good enough"

Does a greedy approach work for MST?

A Greedy Approach to MST

Strategy: Pick the smallest edge until we're done.



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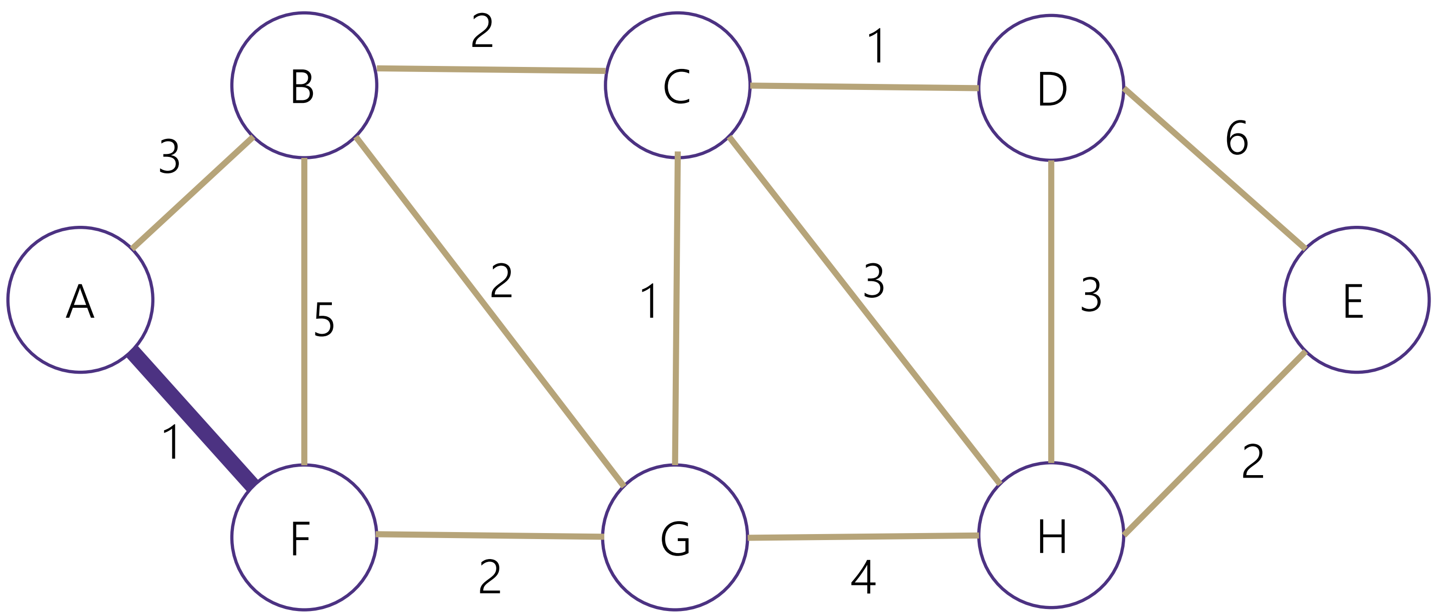

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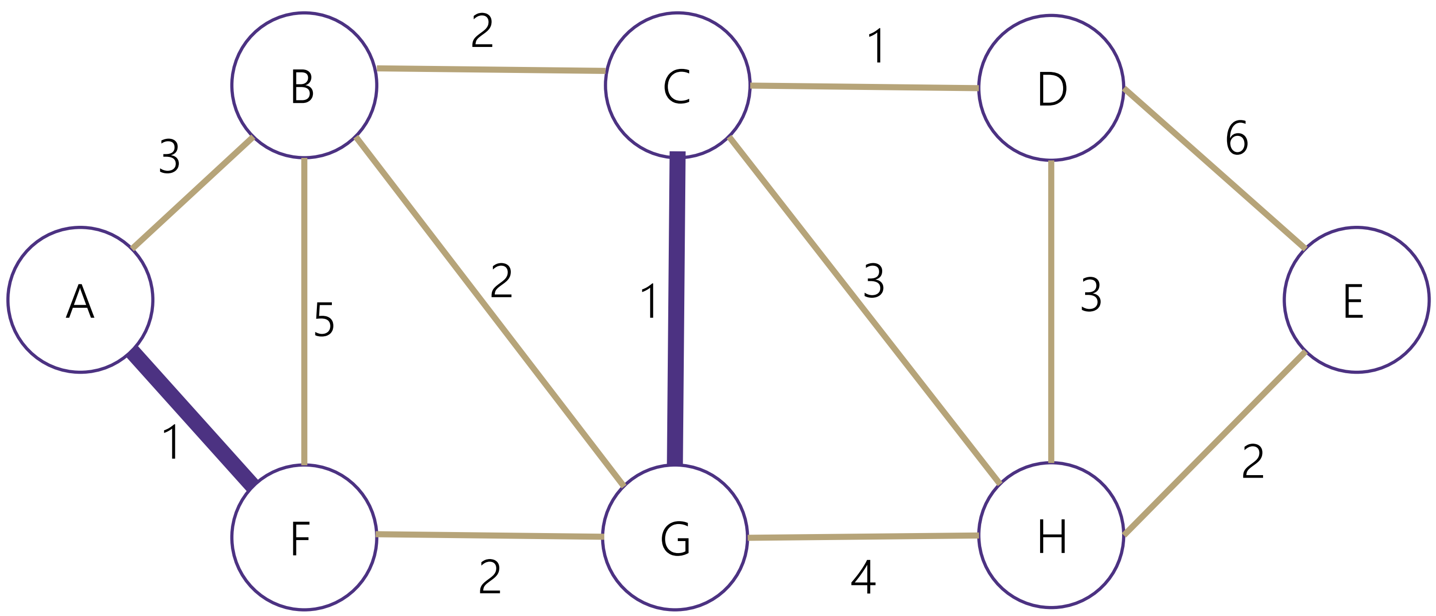

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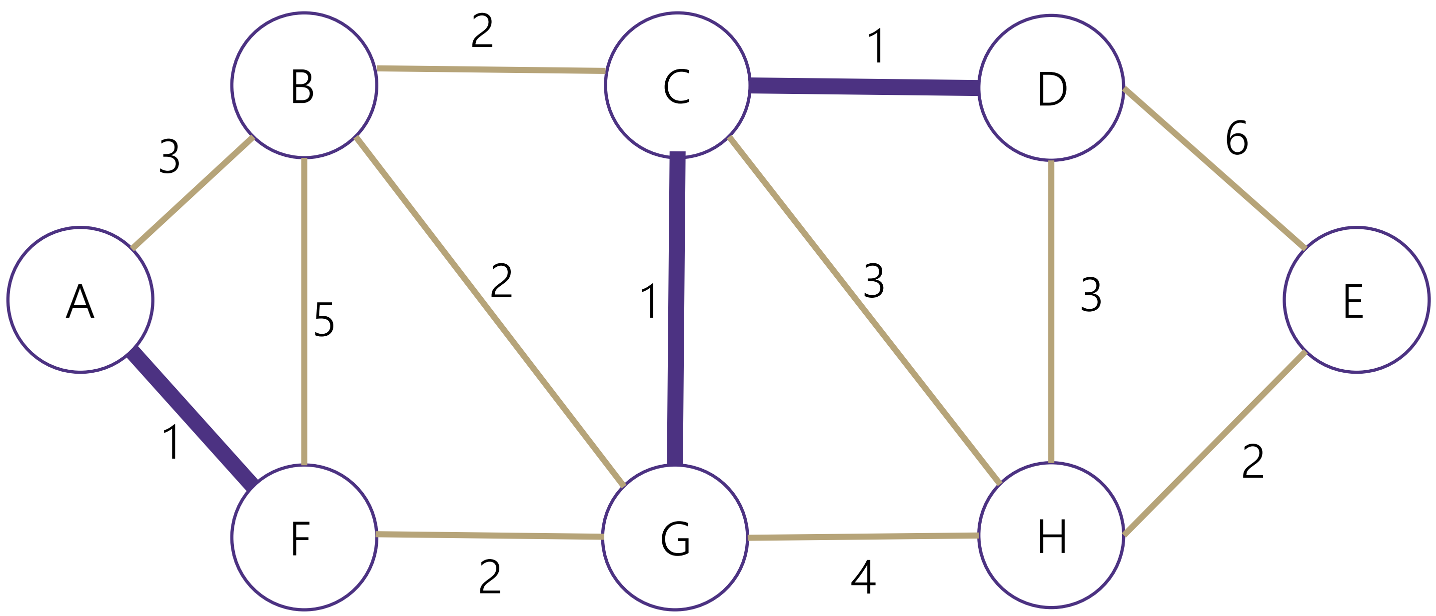

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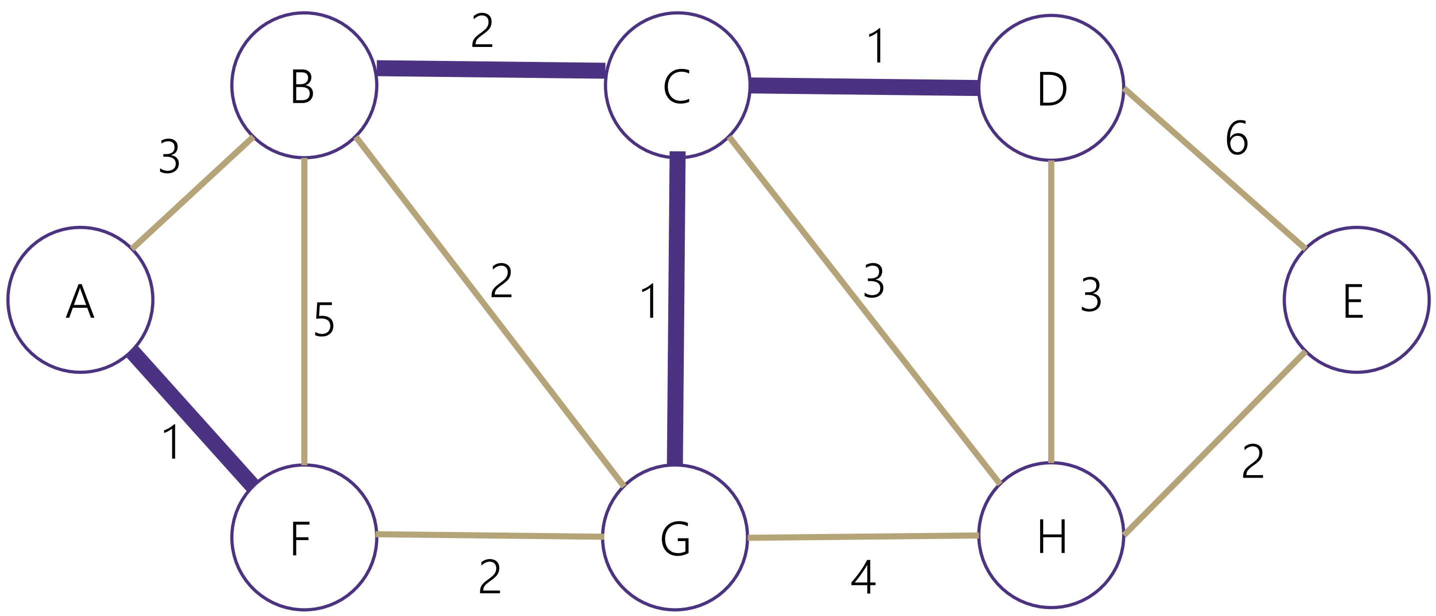

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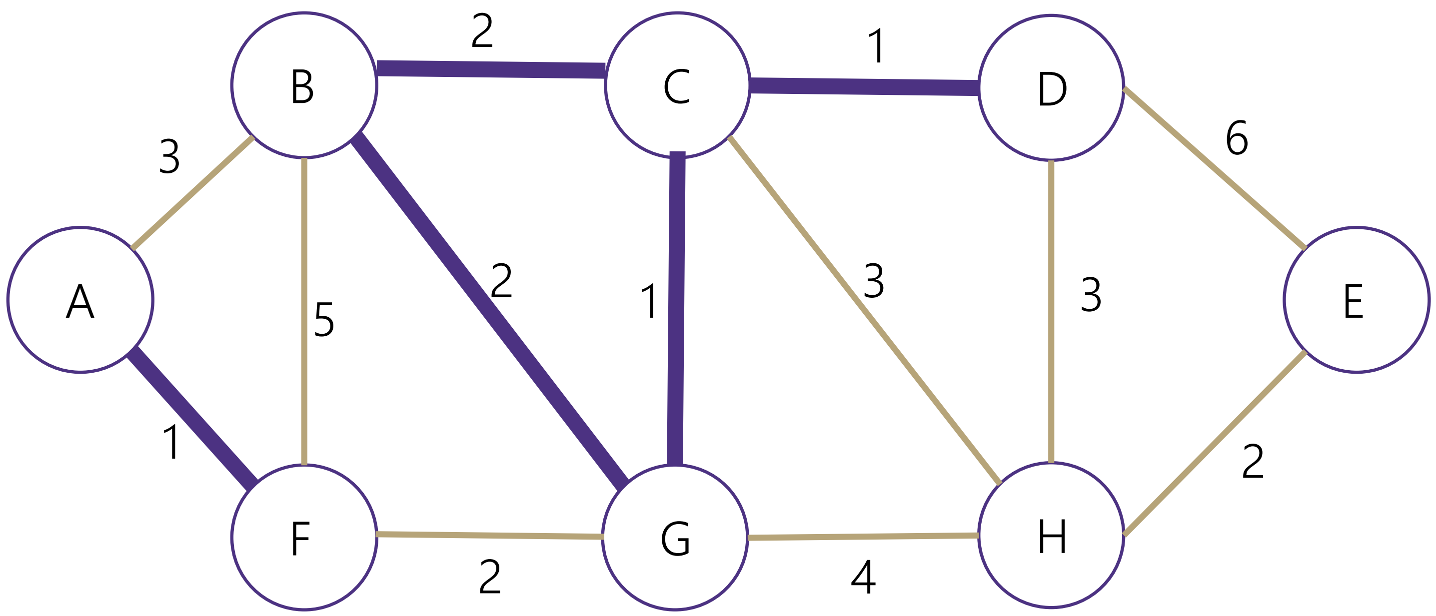

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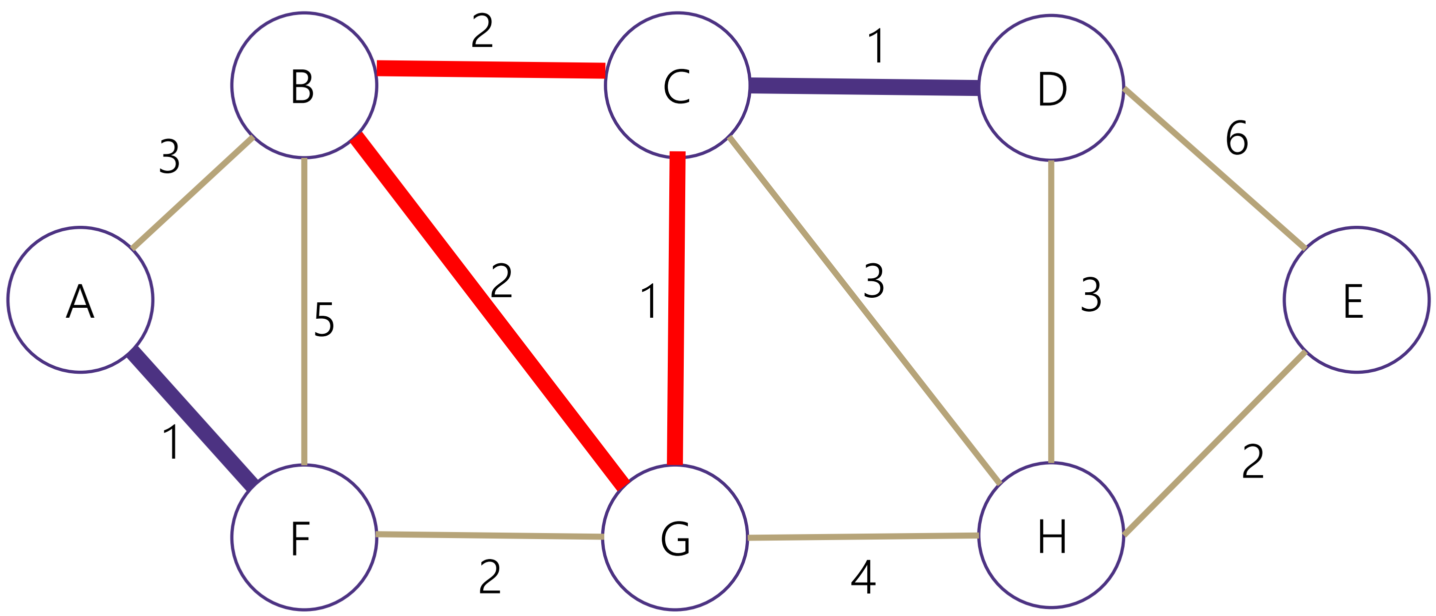
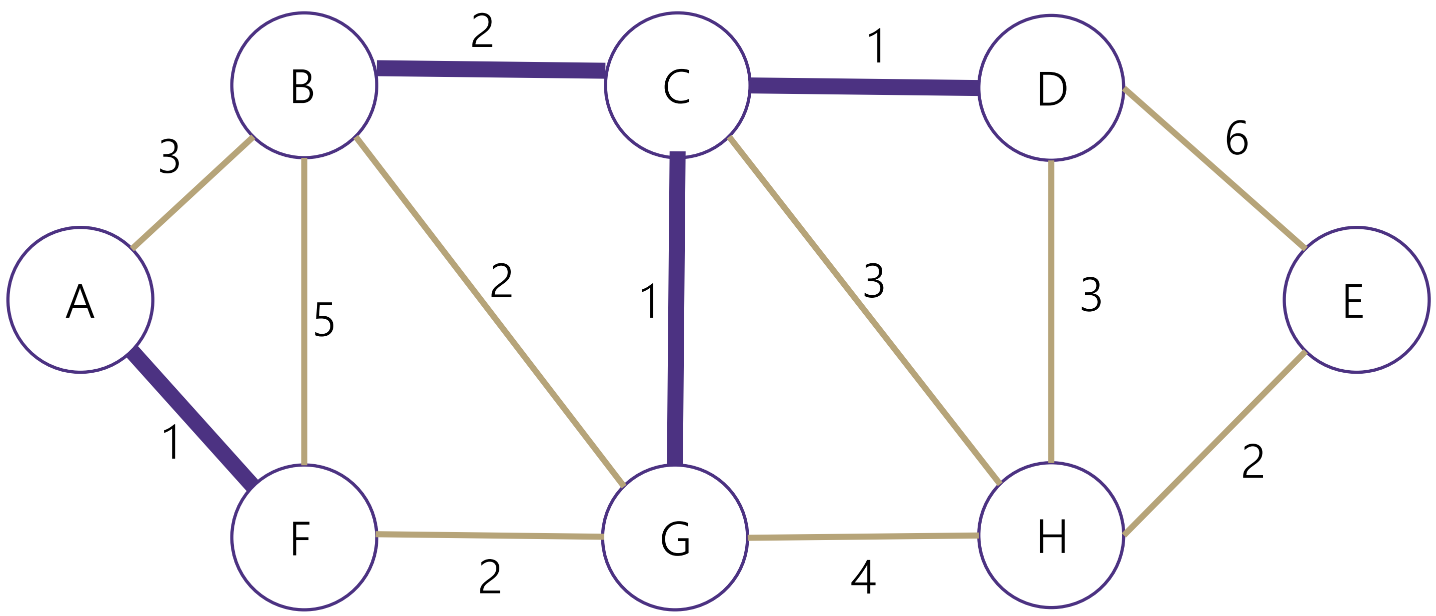


A Greedy Approach to MST

Strategy: Pick the smallest edge *that doesn't create a cycle* until we're done.



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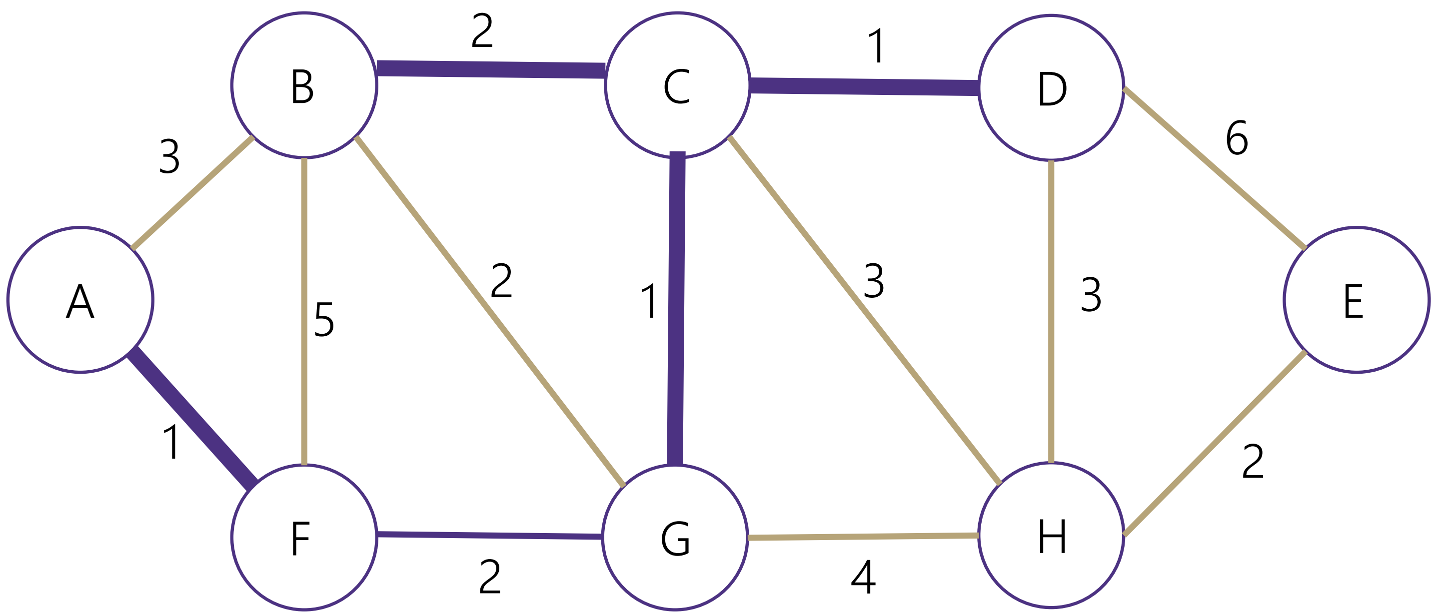

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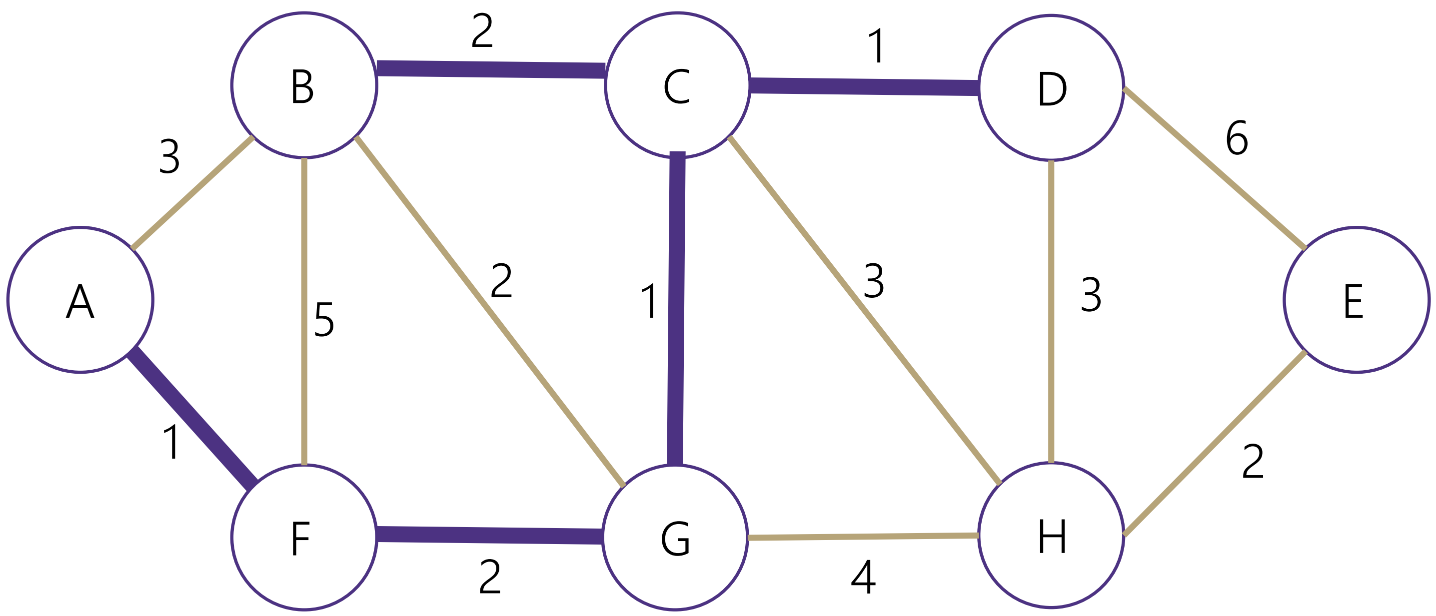

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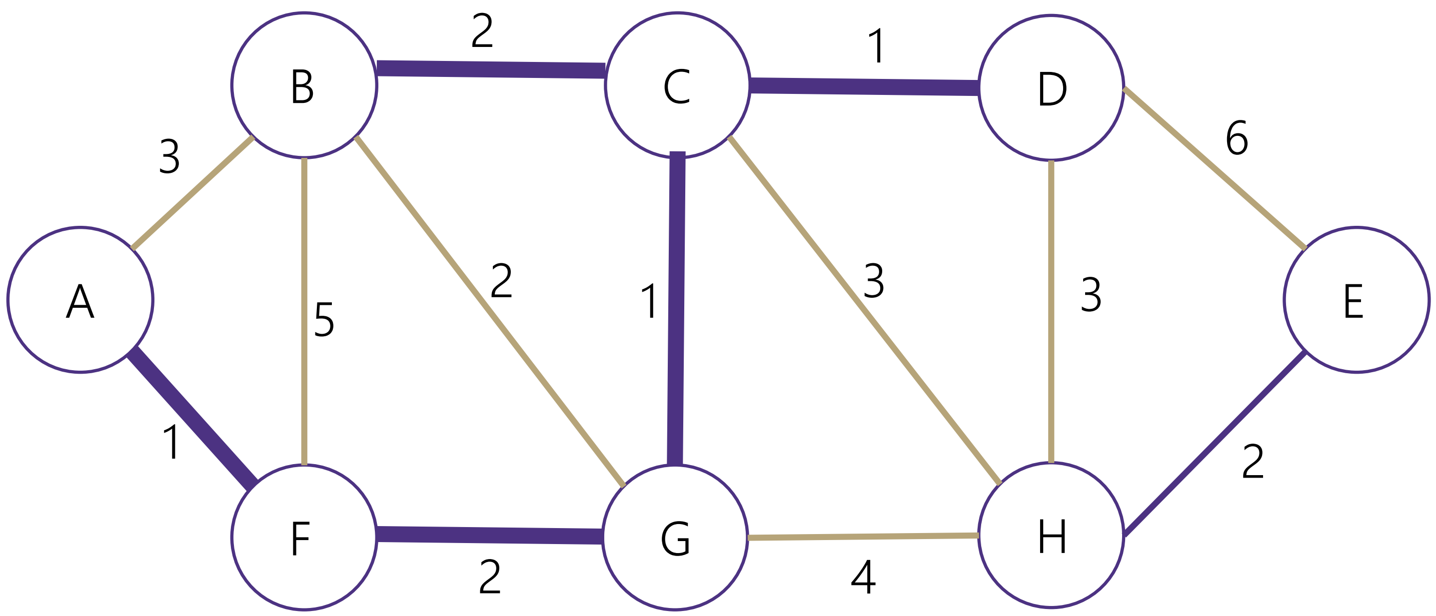

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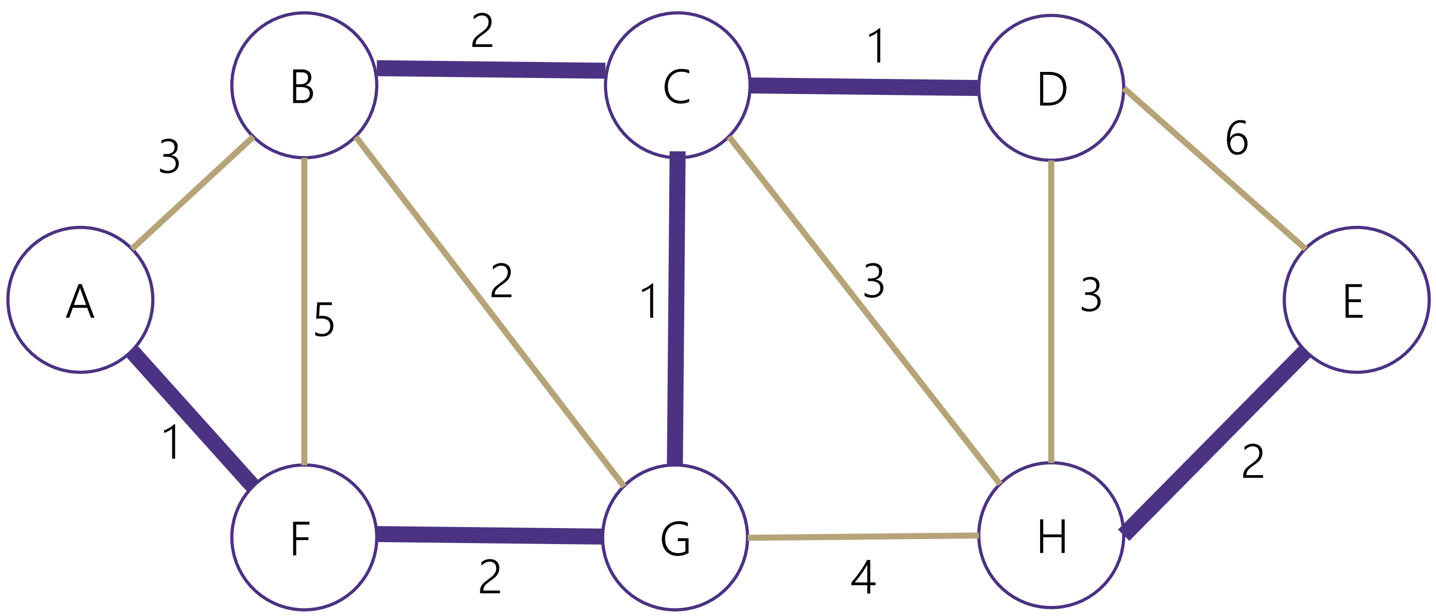

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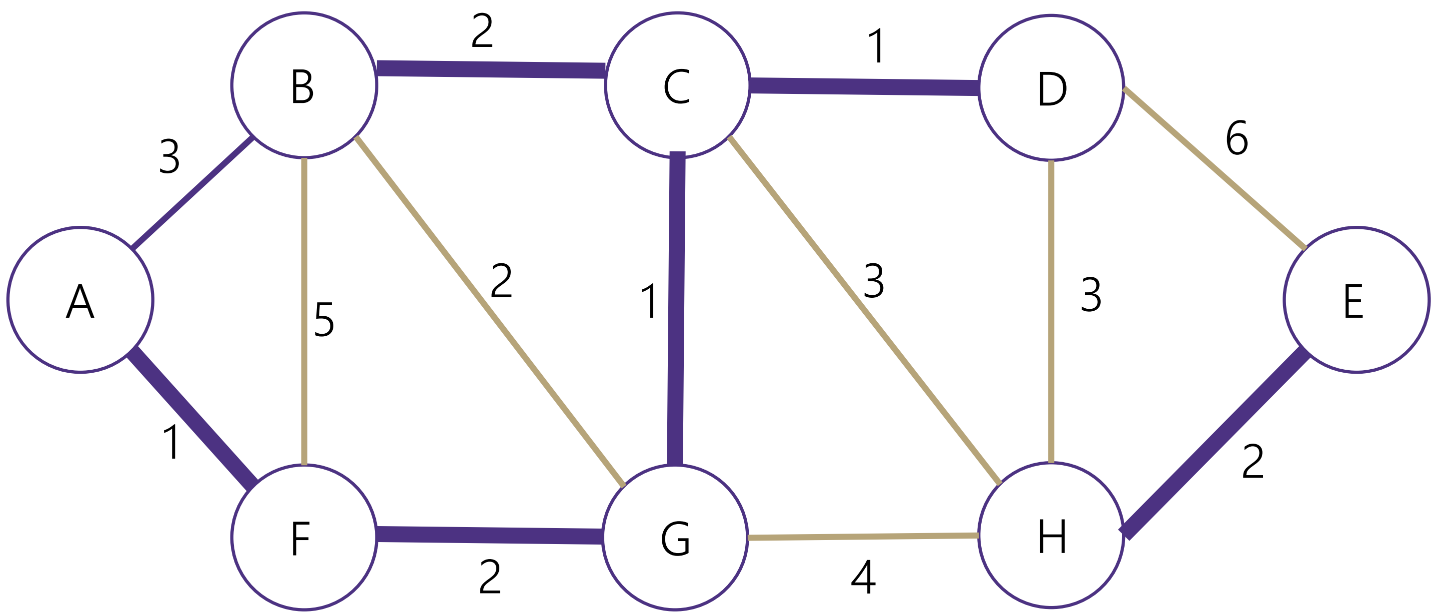

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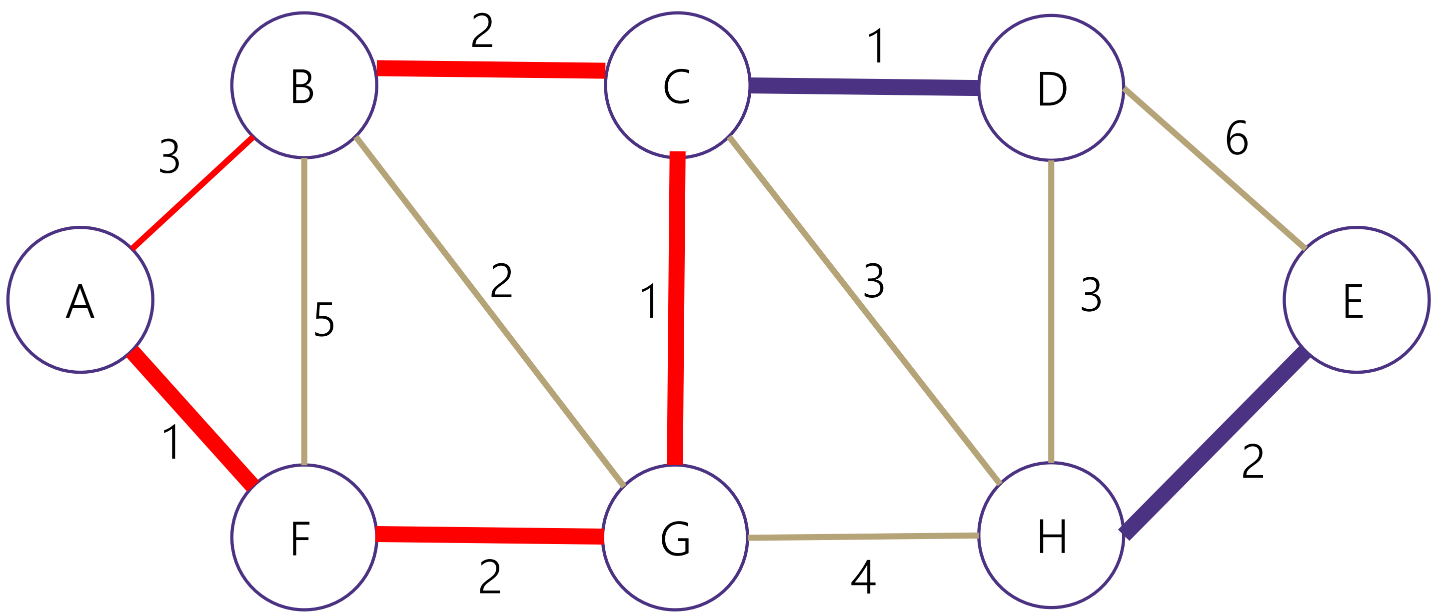

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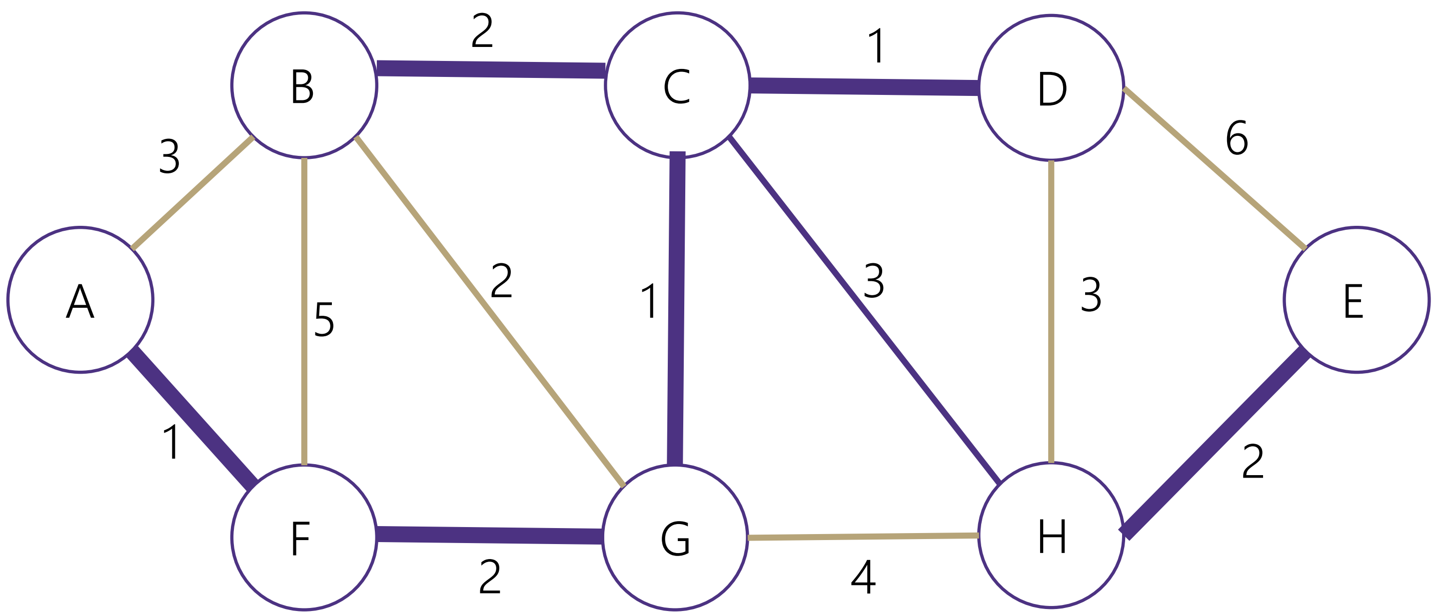

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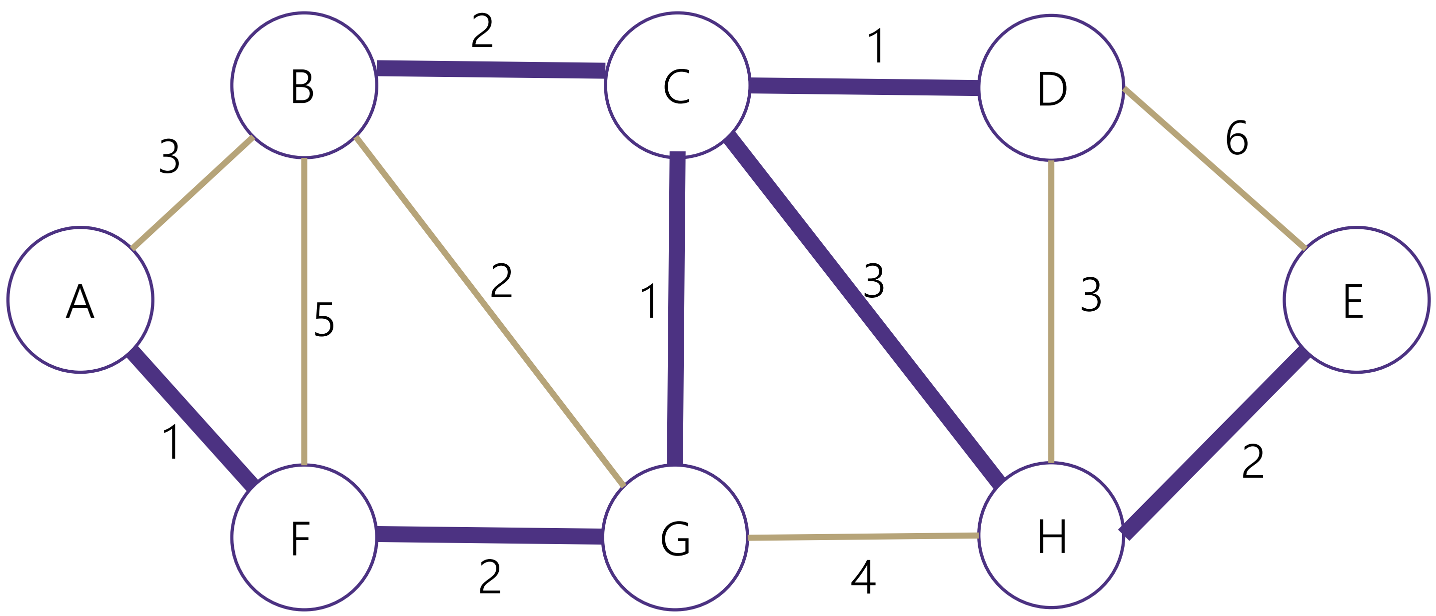
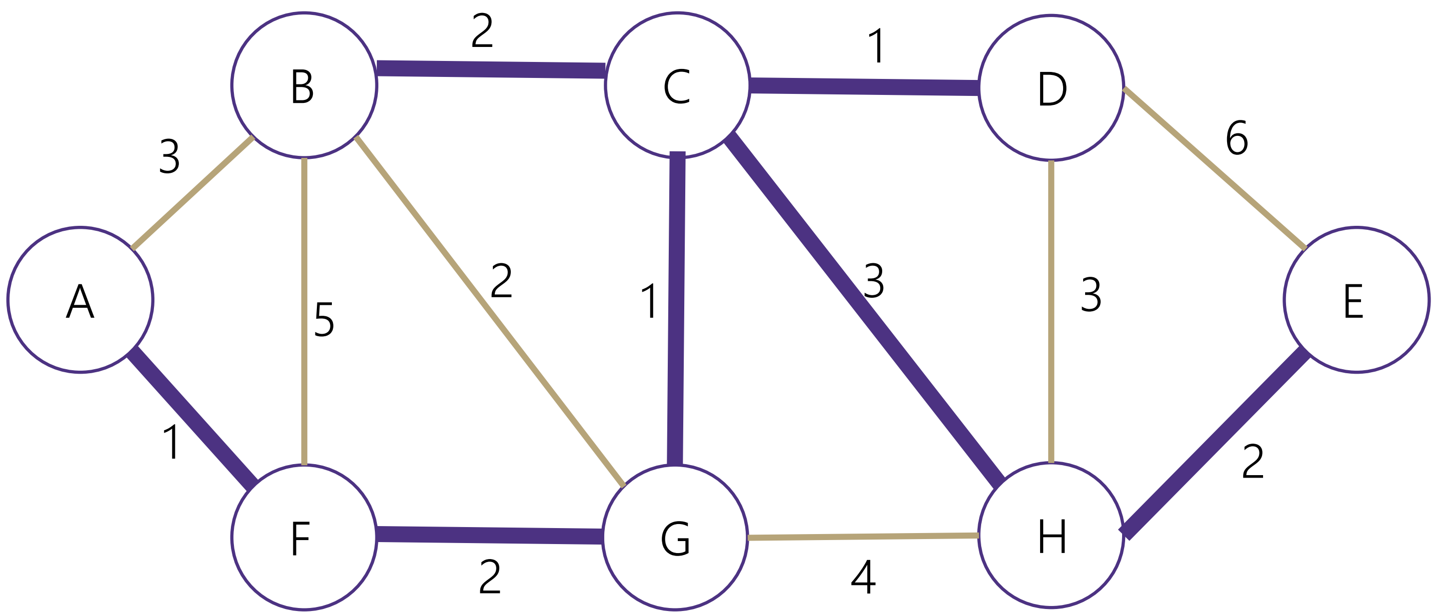


A Greedy Approach to MST

Strategy: Pick the smallest edge *that doesn't create a cycle* until *we have* ***n – 1*** *edges.*



Does this always work?

Proof Sketch: (you don't need to remember this – just remember greedy algorithms don't *always* find the optimum solution, but this one does).

At every step we have a forest (never add edges that make a cycle).

At the end, we have a spanning tree (an acyclic graph with n-1 edges can only be a tree with |V| = n).

Suppose we found T, and T* is a minimum spanning tree. If we repeatedly swap in the smallest edge we **didn't pick** from T*, we will eventually transform our tree into T*. No swap will ever increase the weight of our tree, since we picked edges in order from smallest to largest.

So T is at least as small as T*.

To really prove this, use induction! (See CSE 417/421)

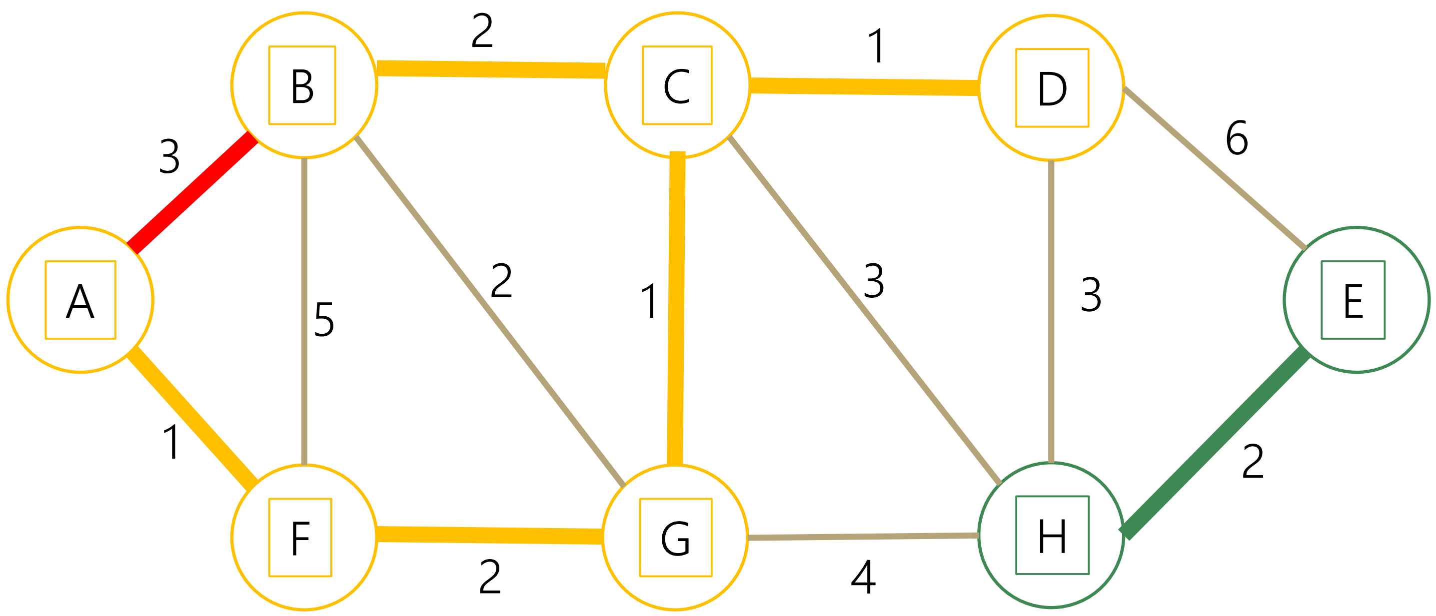
Kruskal's Algorithm

```
Kruskal(G = (V, E)):
    queue = priorityQueue(E)      O(|E|) – Floyd's Build-Heap
    mst = empty list   O(1)
    while (size(mst) < |V| - 1):   At most |E| iterations
        e = queue.deleteMin()    O(log |E|)
        if adding e would not create a cycle: ???  O(|V|+|E|) – DFS from section
            mst.add(e)  O(1)
    return mst
```

$O(|E|^2)$ Can we do better?

A Criteria for Cycle Checking

Observation: An edge will create a cycle if and only if **both endpoints are in the same connected component**.



Strategy: Build a data structure that can quickly answer **sameCC(A, B)**.

Properties of sameCC(A, B)

Recall: A is in the same connected component as B if and only if there is a path from A to B

- sameCC(A, A) = True
  - There is always a (trivial) path from a vertex to itself

REFLEXIVITY

- sameCC(A, B) = sameCC(B, A)
  - Reversing a path from A to B makes a path from B to A

SYMMETRY

- If sameCC(A,B) and sameCC(B, C), then sameCC(A, C)
  - Can join a path from A to B to a path from B to C, yielding a path from A to C

TRANSITIVITY

In mathematics, we call anything with these properties and **equivalence relation**.

Equivalence Relations

Equivalence Relation: A **binary relation** (boolean valued function with two arguments of the same type) that is **reflexive**, **symmetric**, and **transitive**.

Namesake: Equals (==)

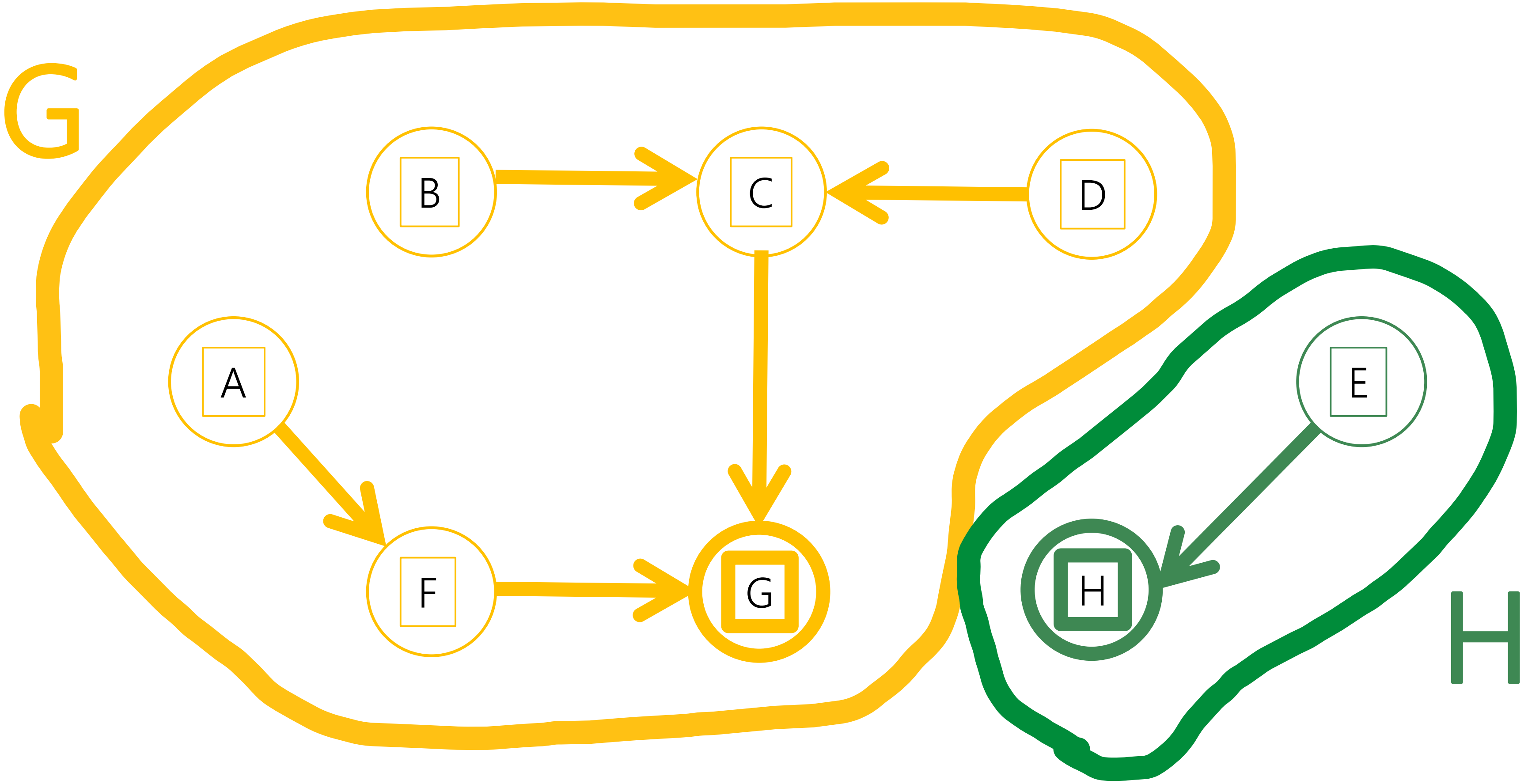
- A == A (reflexive)
- A == B ⇔ B == A (symmetric)
- A == B and B == C ➔ A == C (transitive)

The collection of all objects that are equivalent under an equivalence relation is called an **equivalence class.**

Connected components are equivalence classes under “sameCC” (i.e. pathExists(A,B))

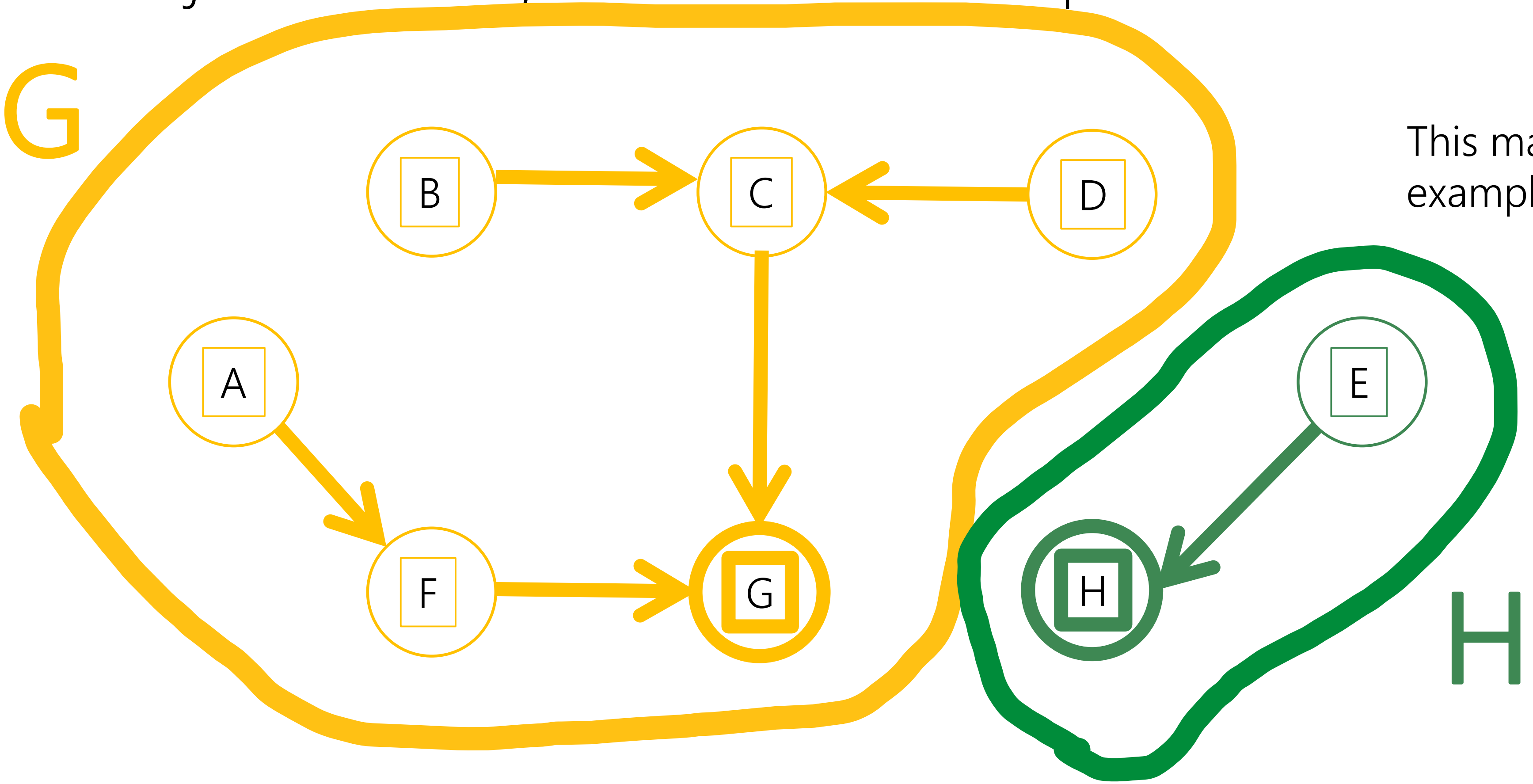
A Datastructure for Equivalence Classes

Main Idea: Link together elements in an equivalence class, pointing towards a **representative element**.



A Datastructure for Equivalence Classes

Notice: Equivalence classes are **disjoint** – they don't share elements. They also **cover** the entire set of objects – each object is contained in an equivalence class.



This makes them an example of **disjoint sets**.

ADT: Disjoint Sets

Requirements:

- Keeps track of which set each element is in
- Dynamic: can combine sets (union)
- Online: can find the set an element is in on-the-fly (and then continue modifying)

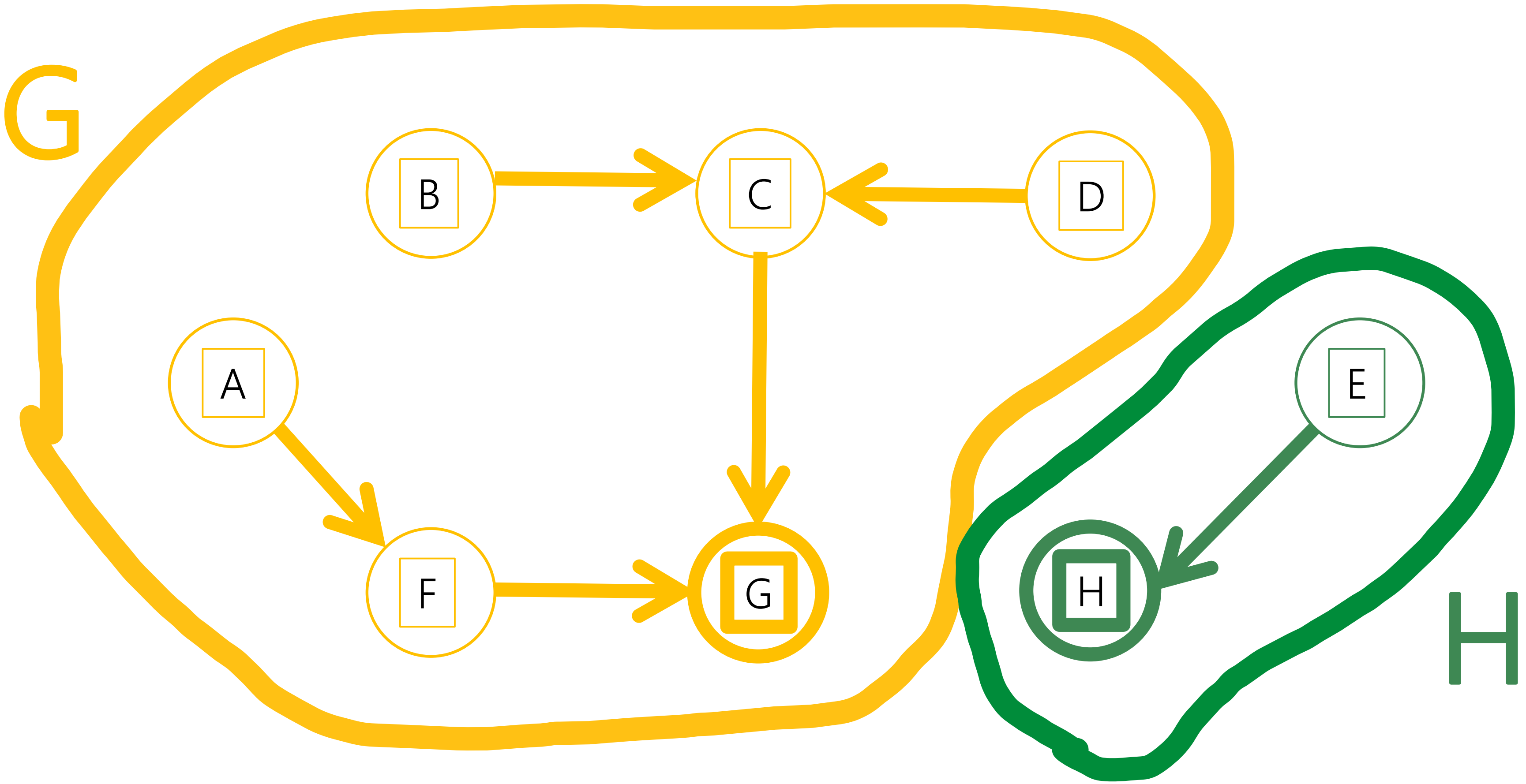
ADT: Disjoint Sets

- union(A, B) – Joins together the sets which A and B belong to
- find(A) - finds a **representative element** for the set that A is in

- [constructor – all elements start in their own separate disjoint set]

Find

Find: Return the **representative element** of an element's set. Example: find(D)



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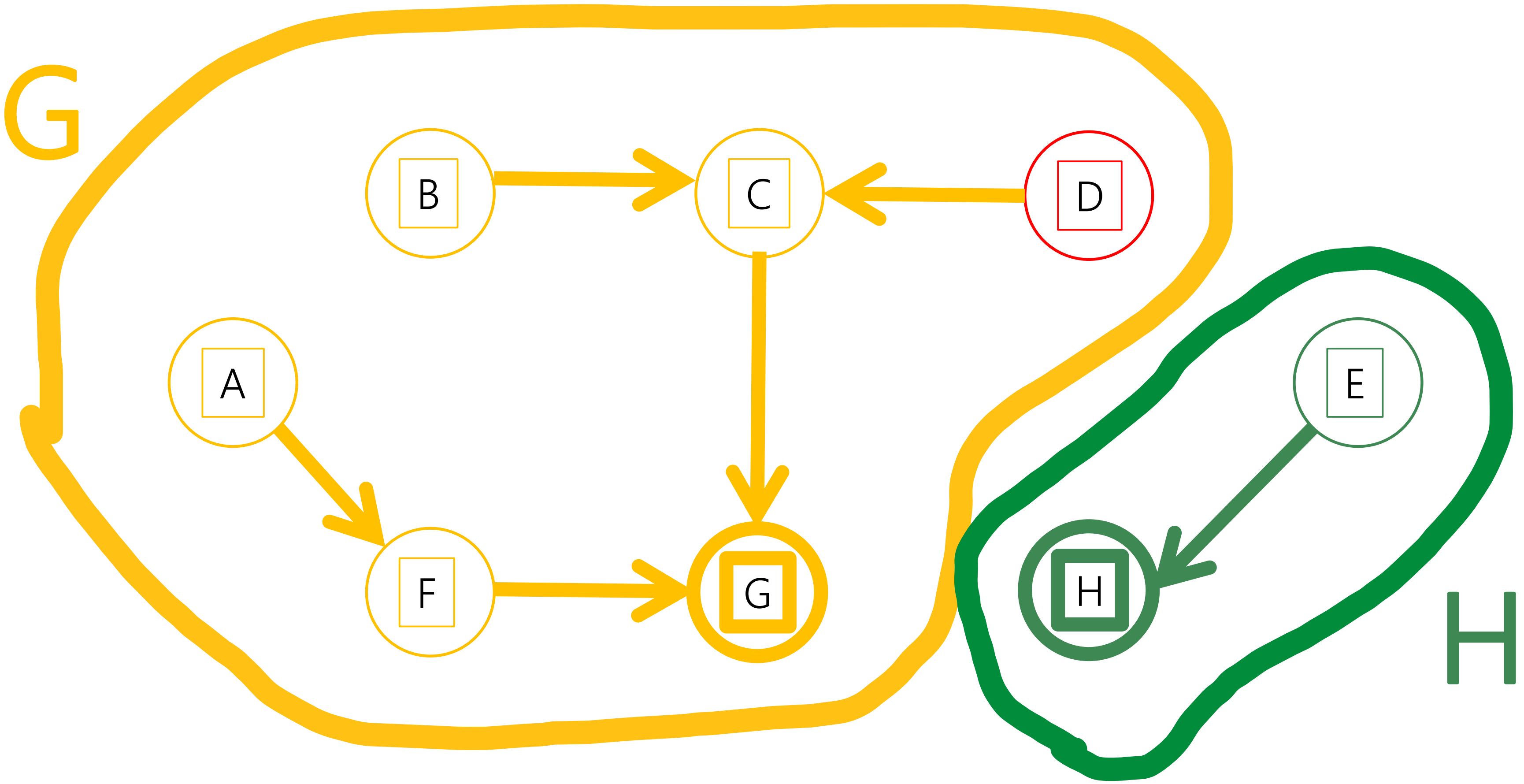

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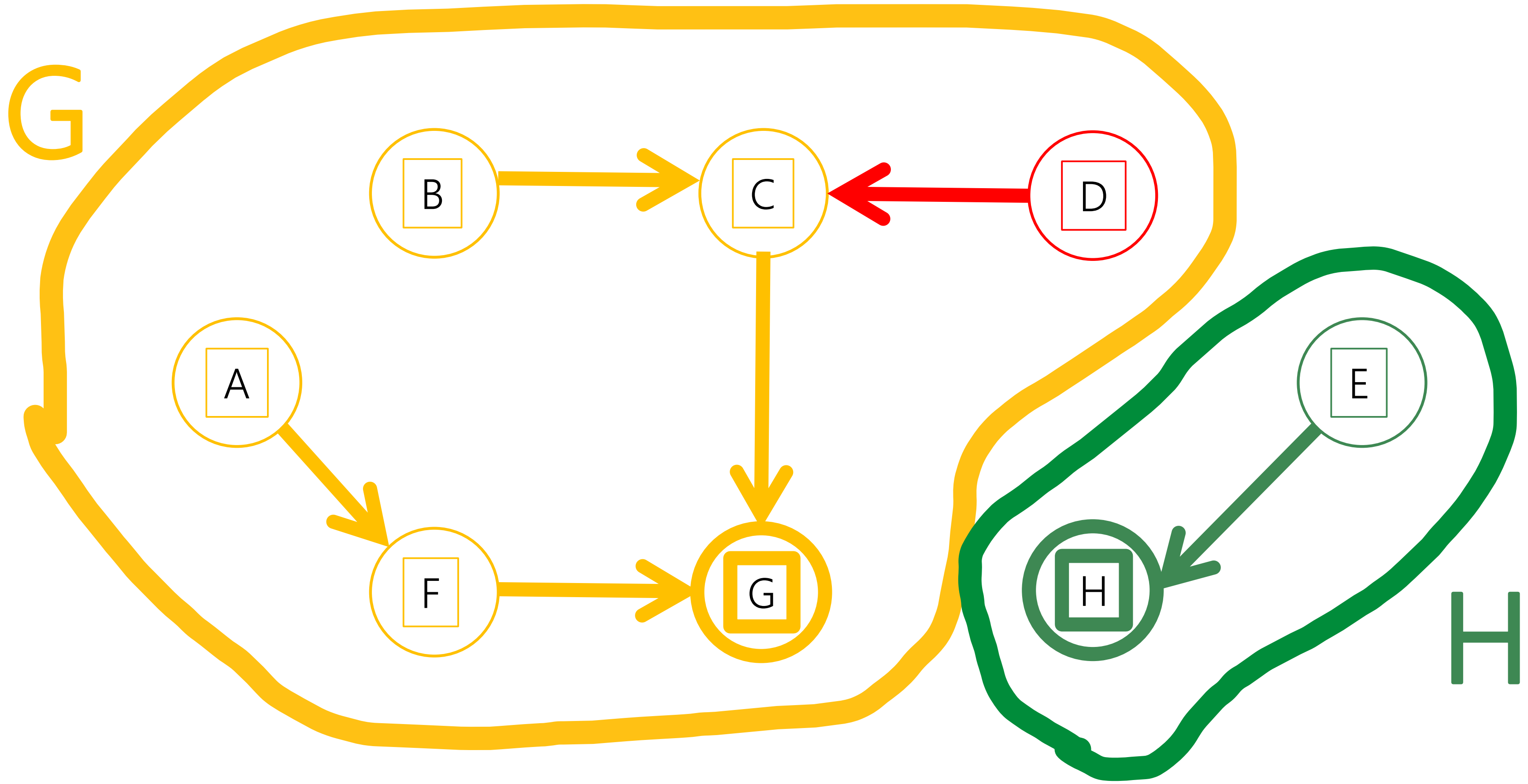

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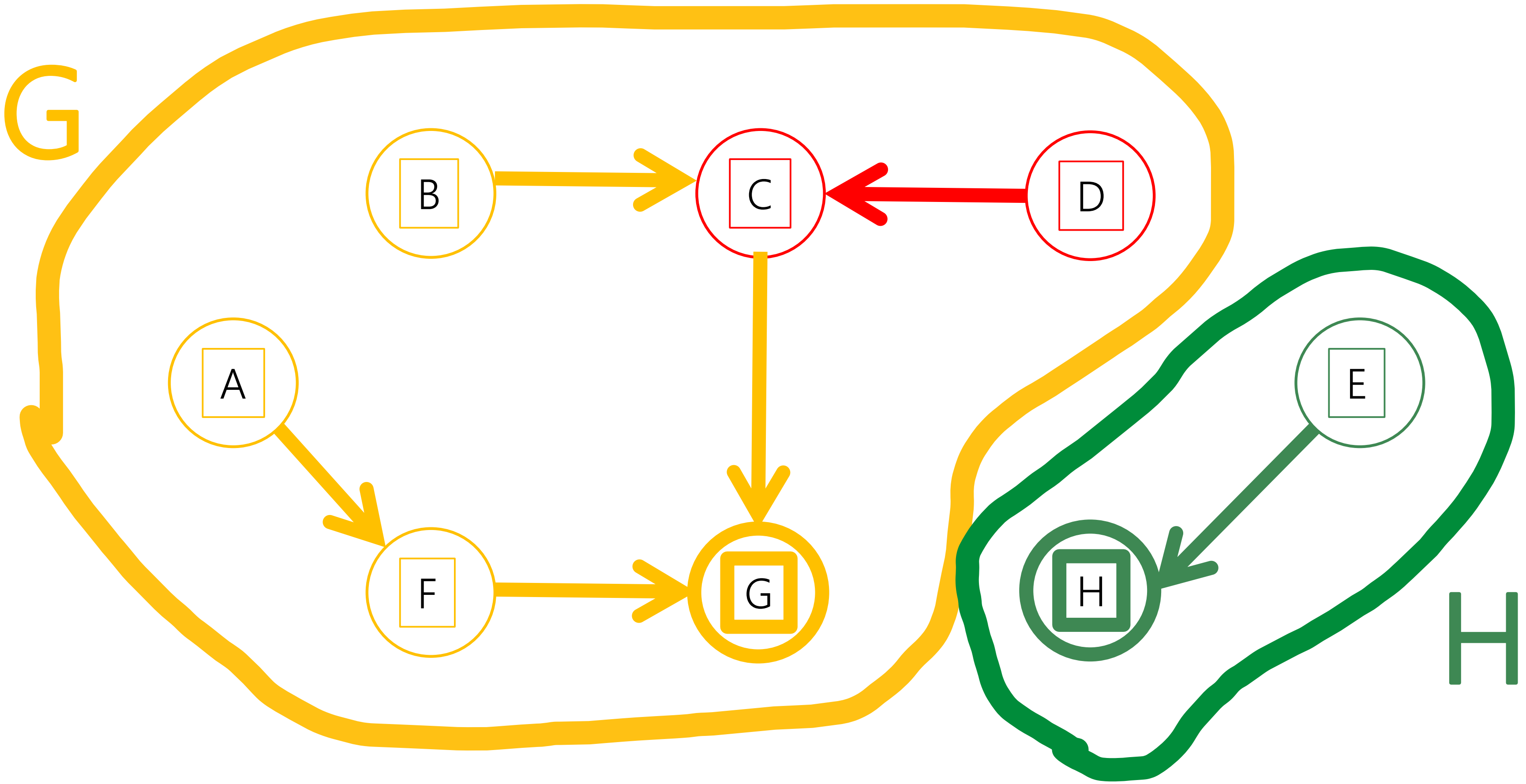

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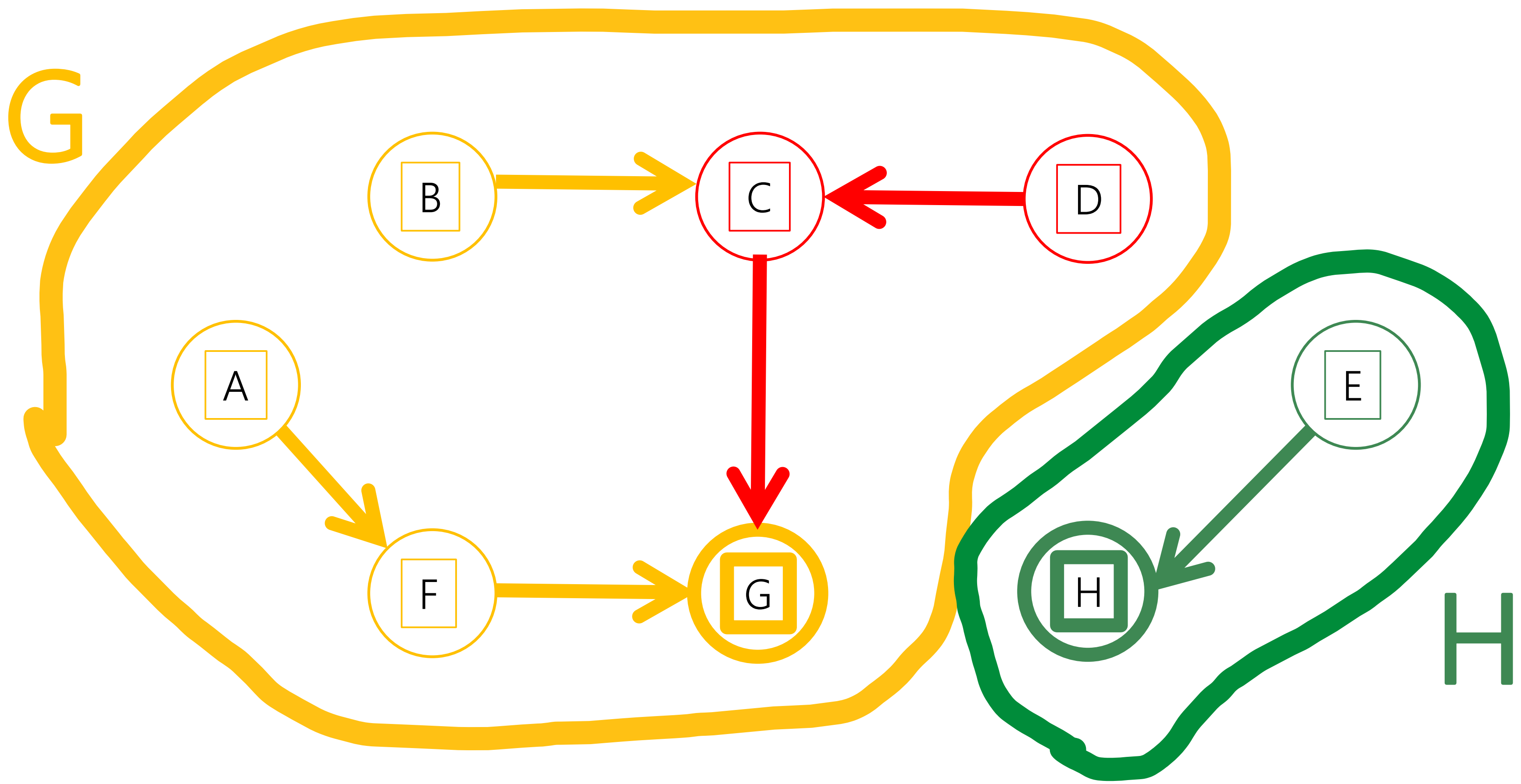
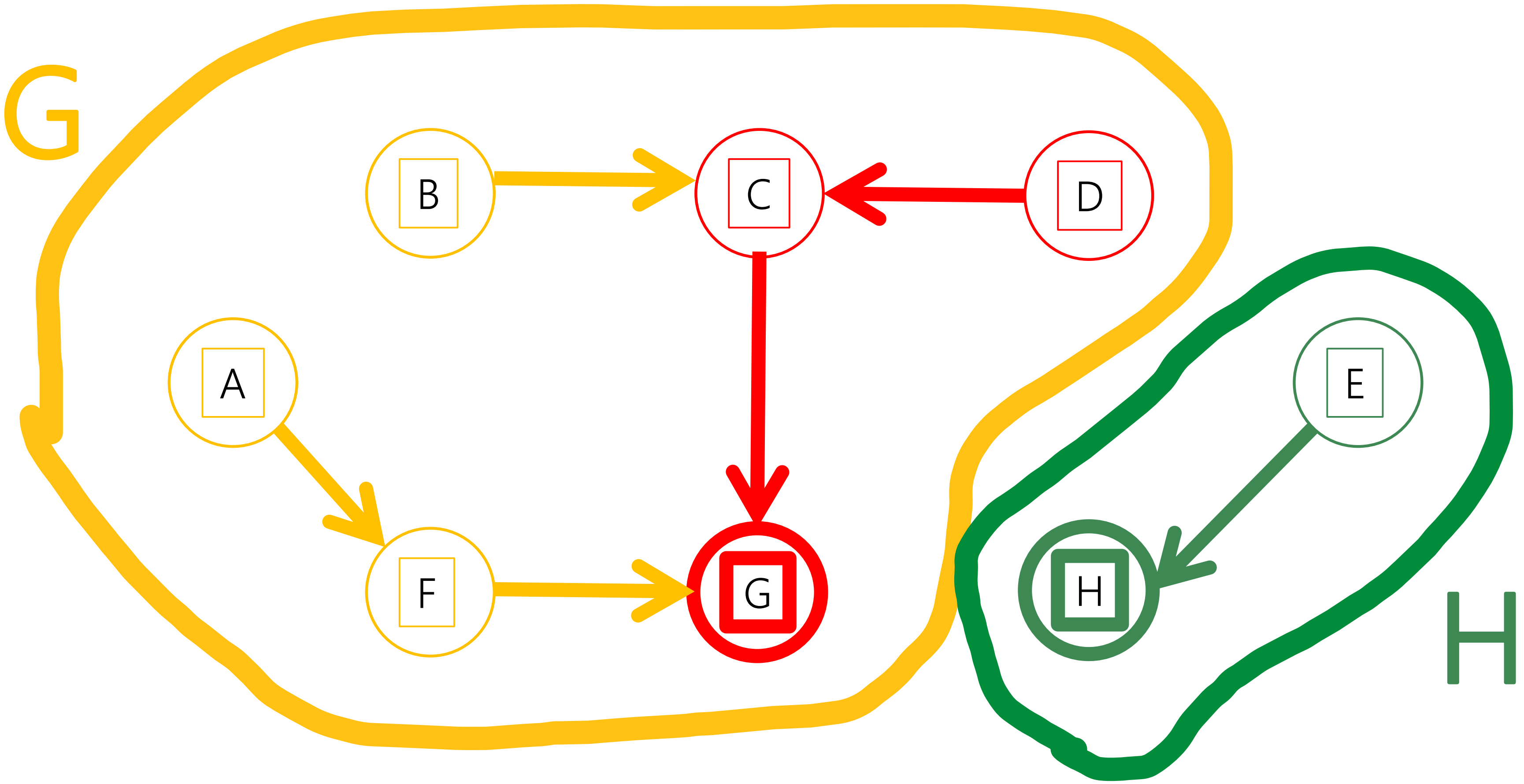


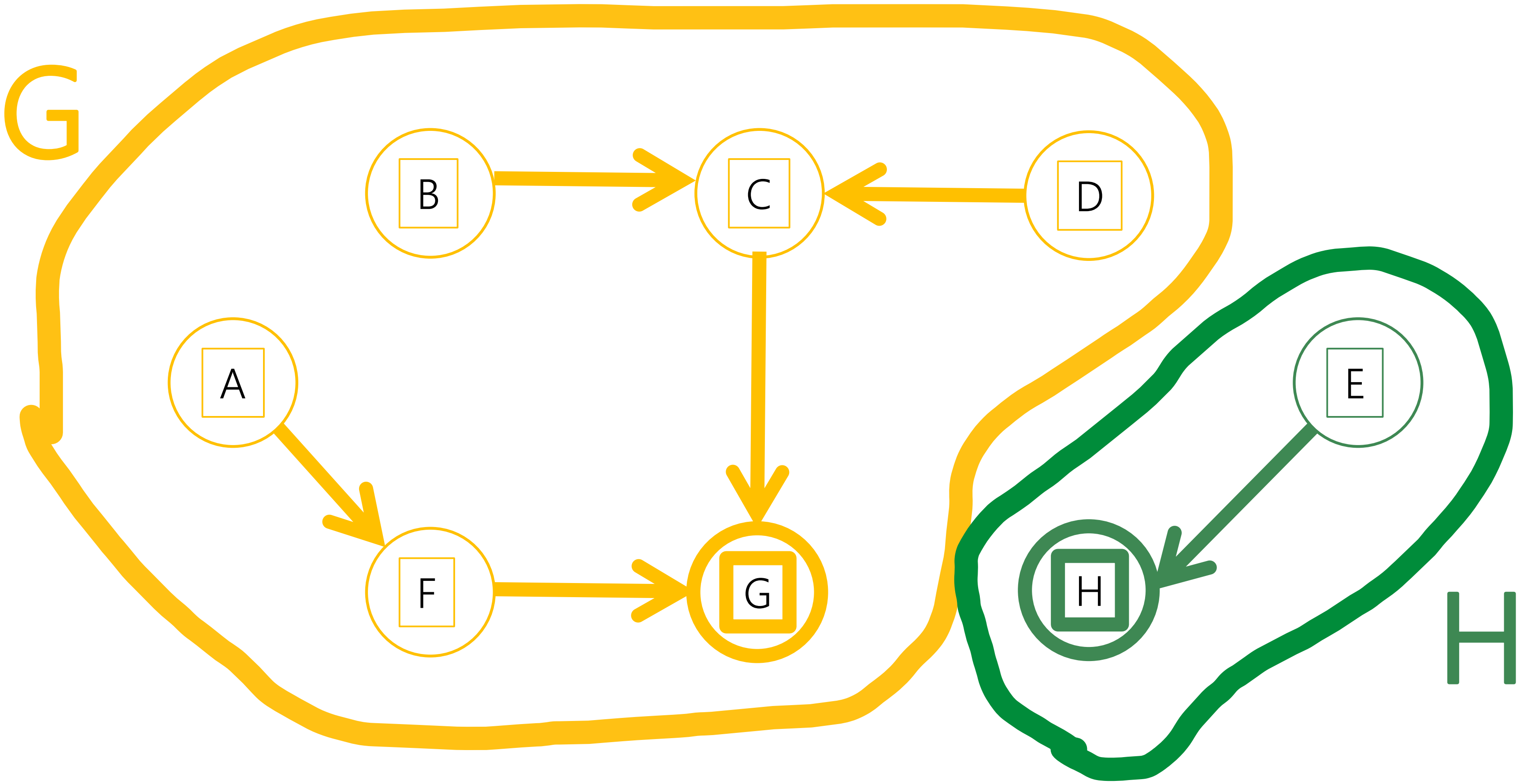
Find

Find: Return the **representative element** of an element's set. Example: find(D) = G



Union

Union: Combine two disjoint sets. Example: Union(D, E)



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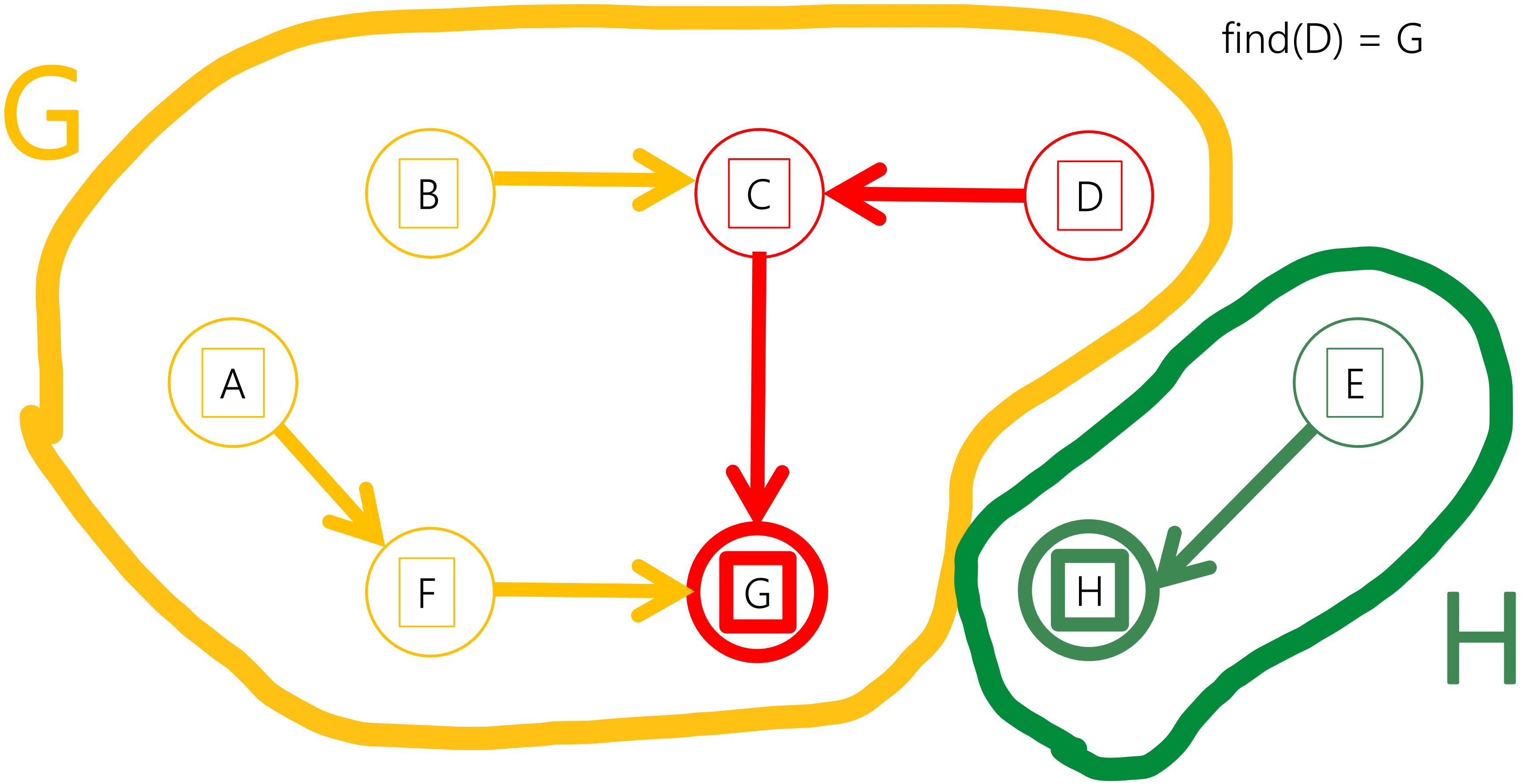

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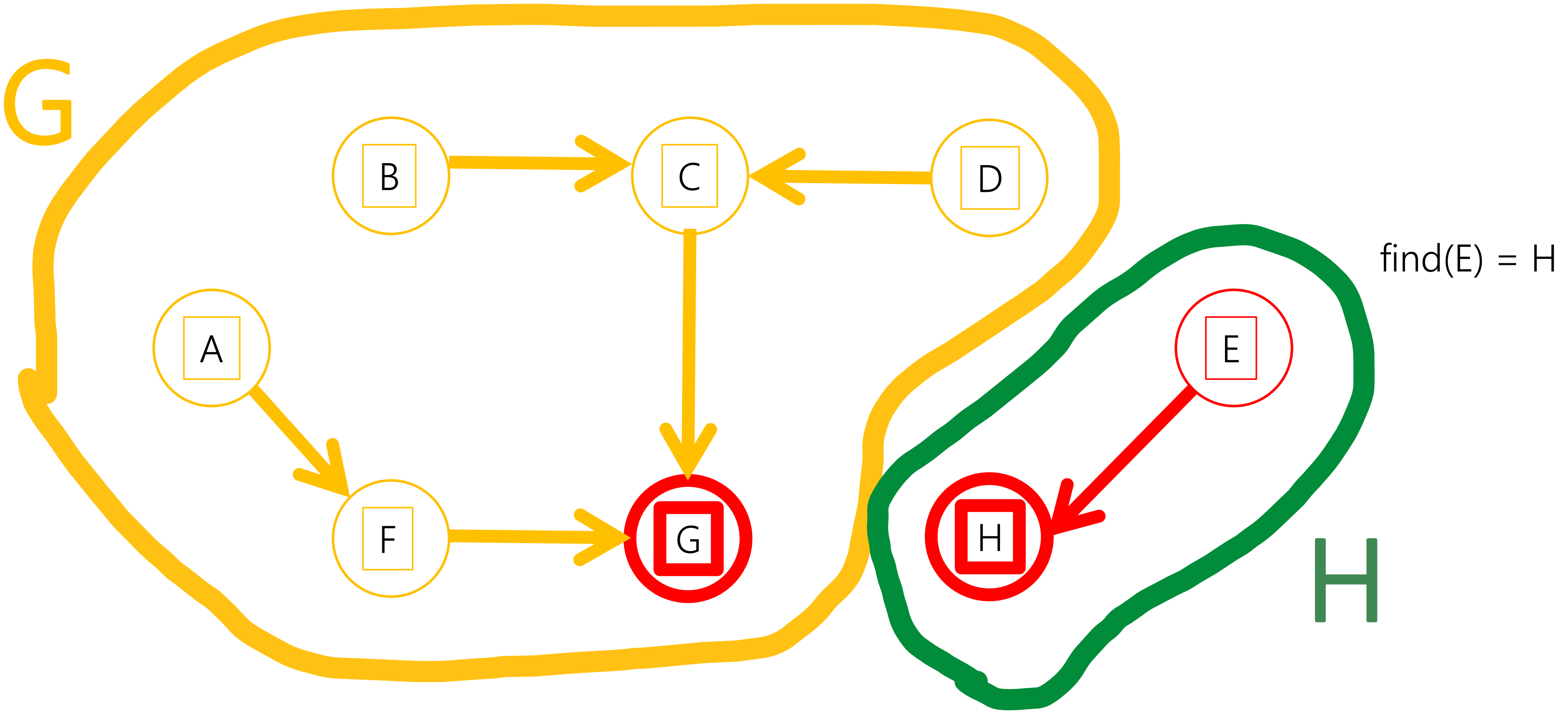

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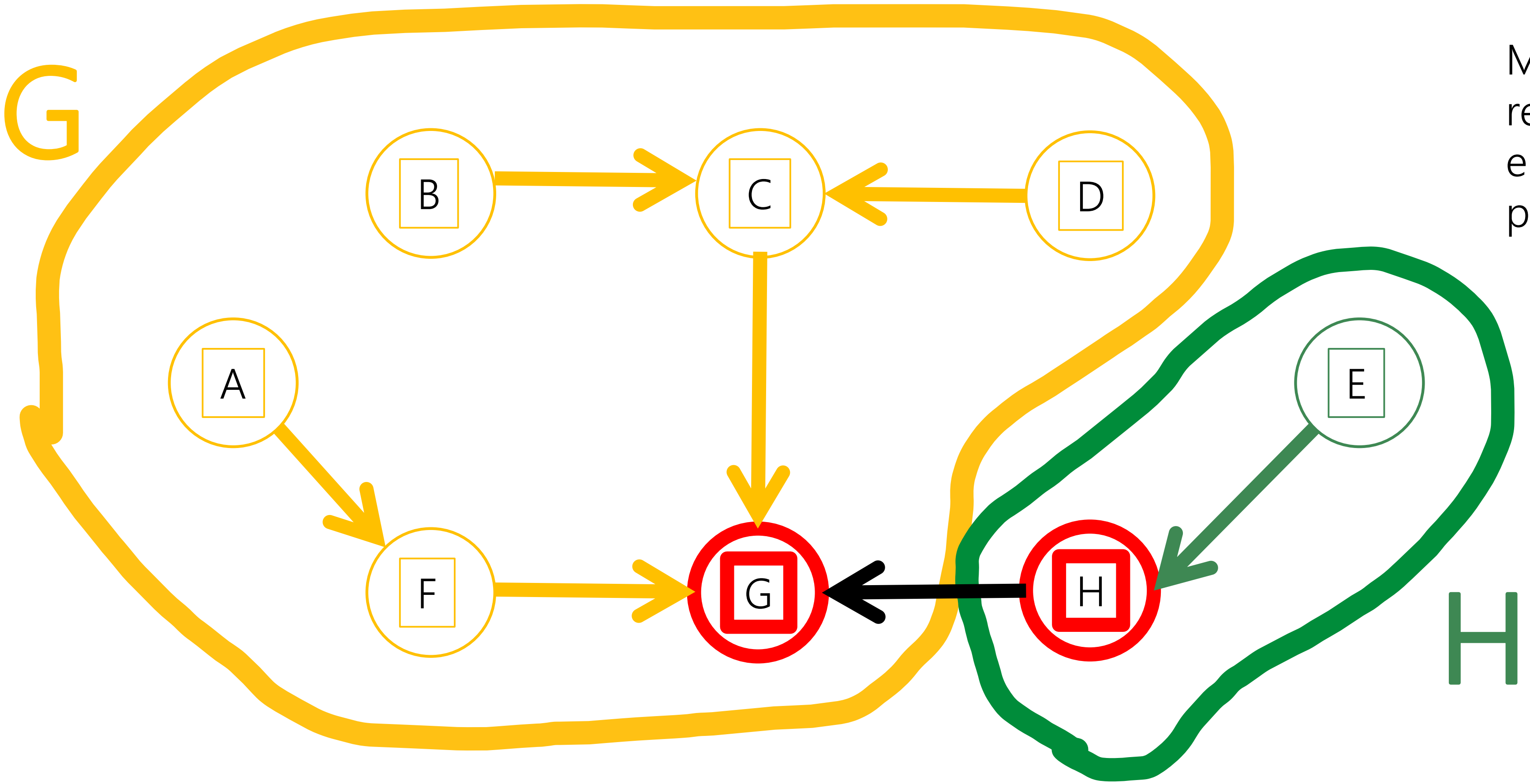

Make one of the representative elements the parent of the other

Union

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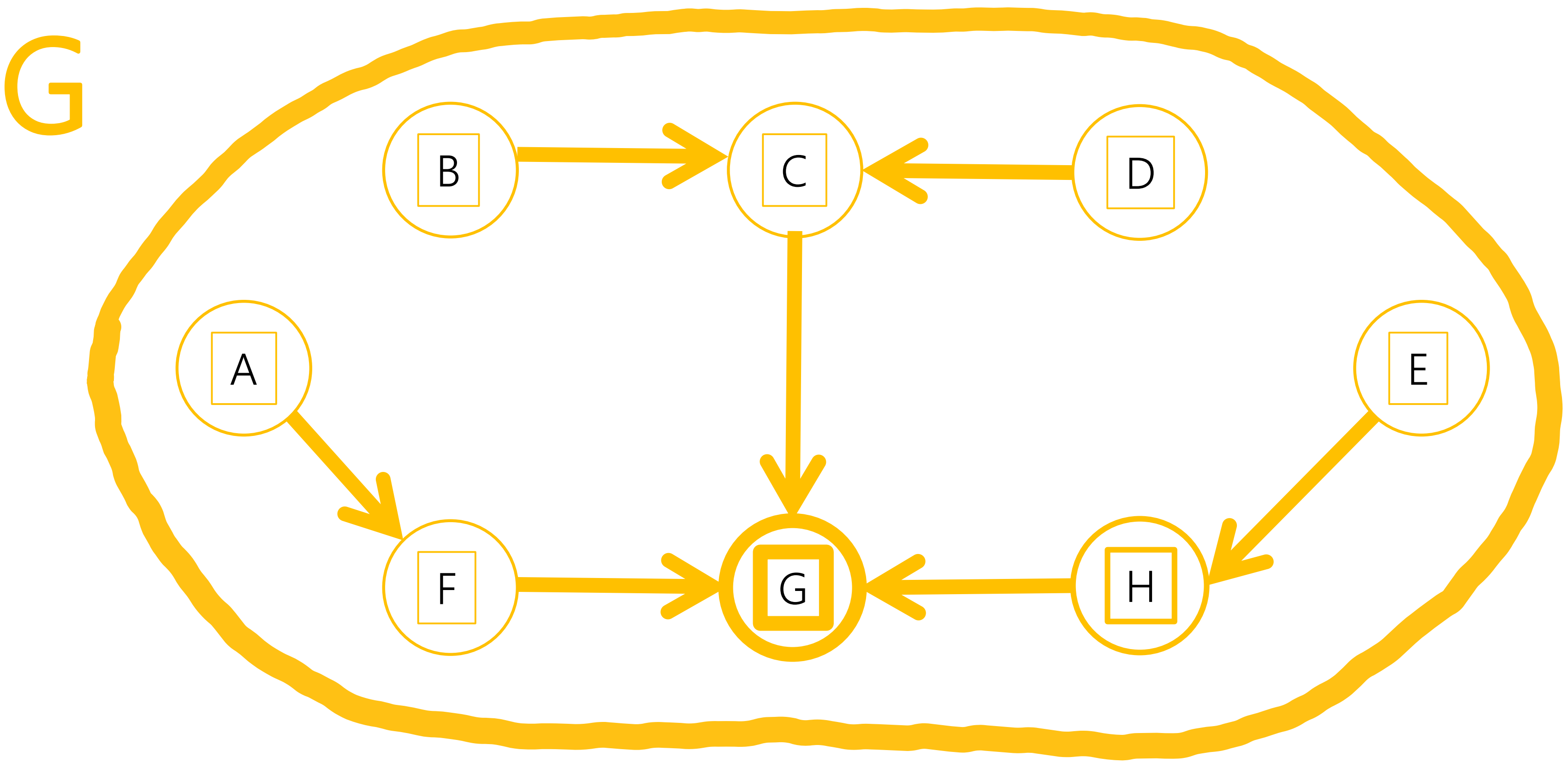
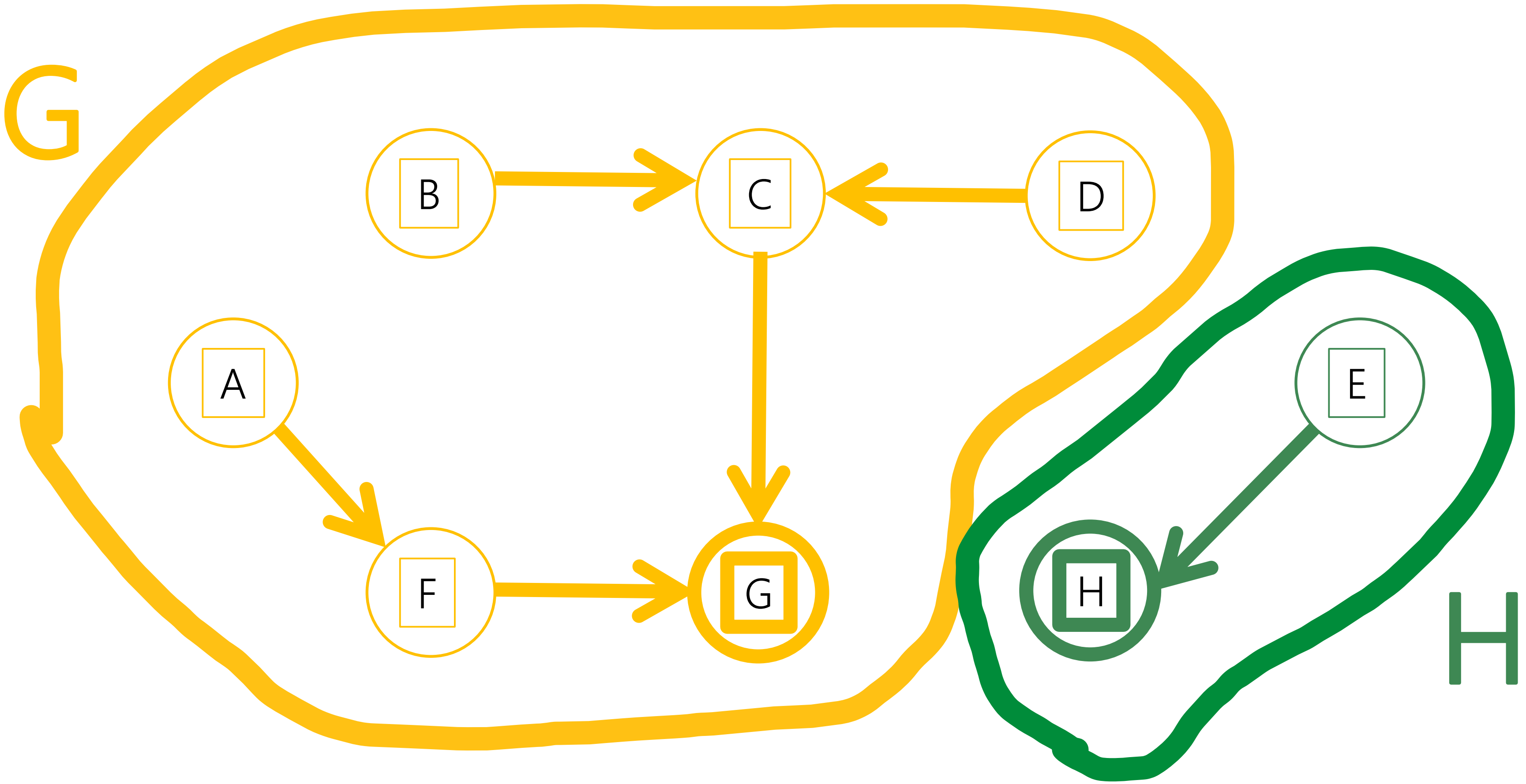


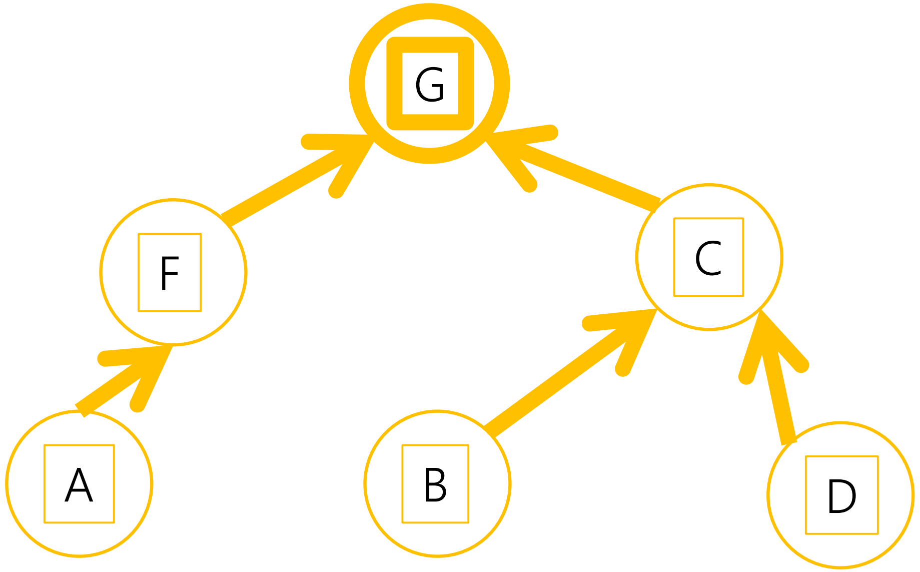
Representation

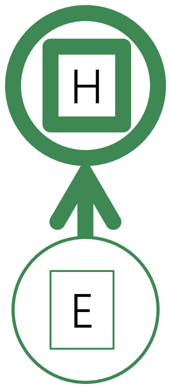
Observe: This is a **forest**. How can we represent these **trees**?



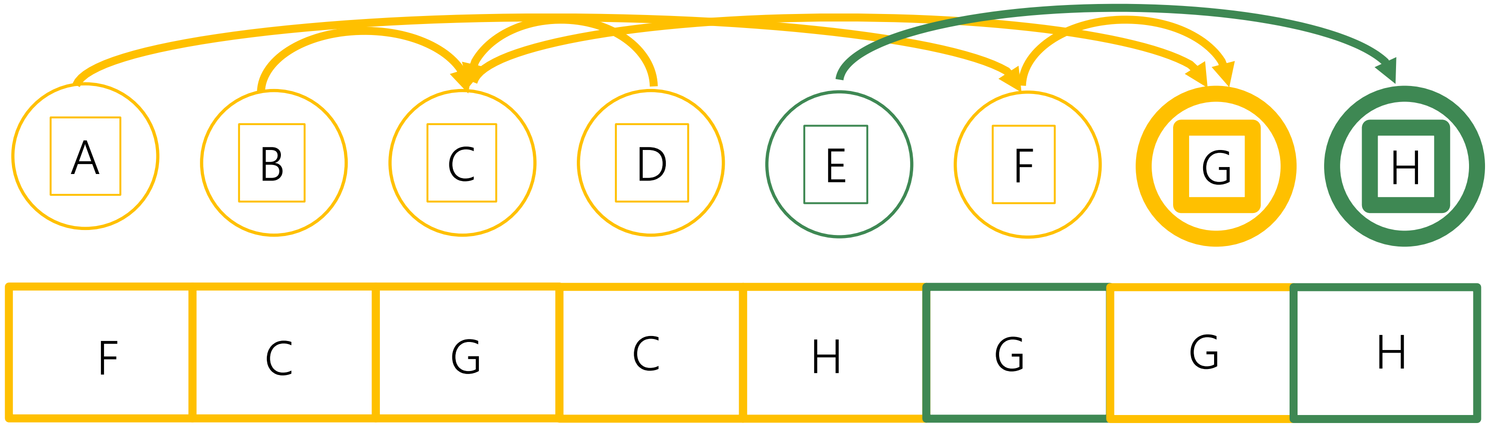
Disjoint Set Trees (aka Union-Find Trees)

Observe: Each element has at most 1 parent (the links point up towards the root).





Only 1 piece of data is needed for each element, so we can use an **array**.



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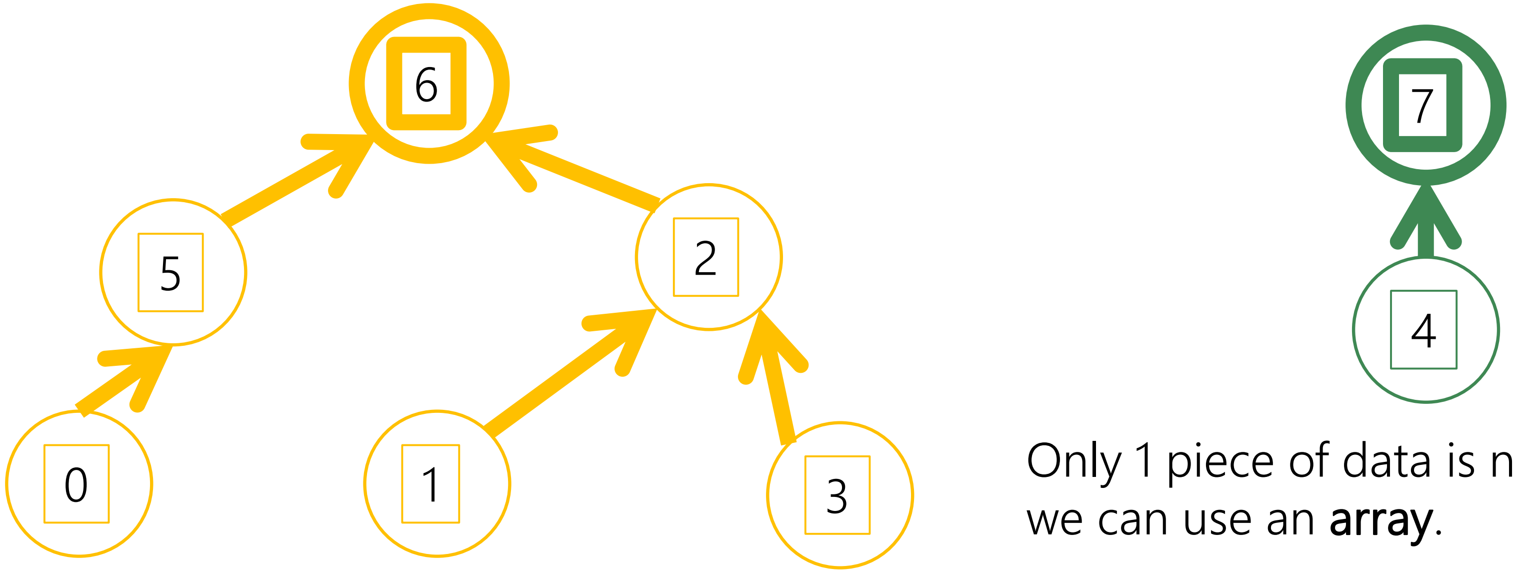

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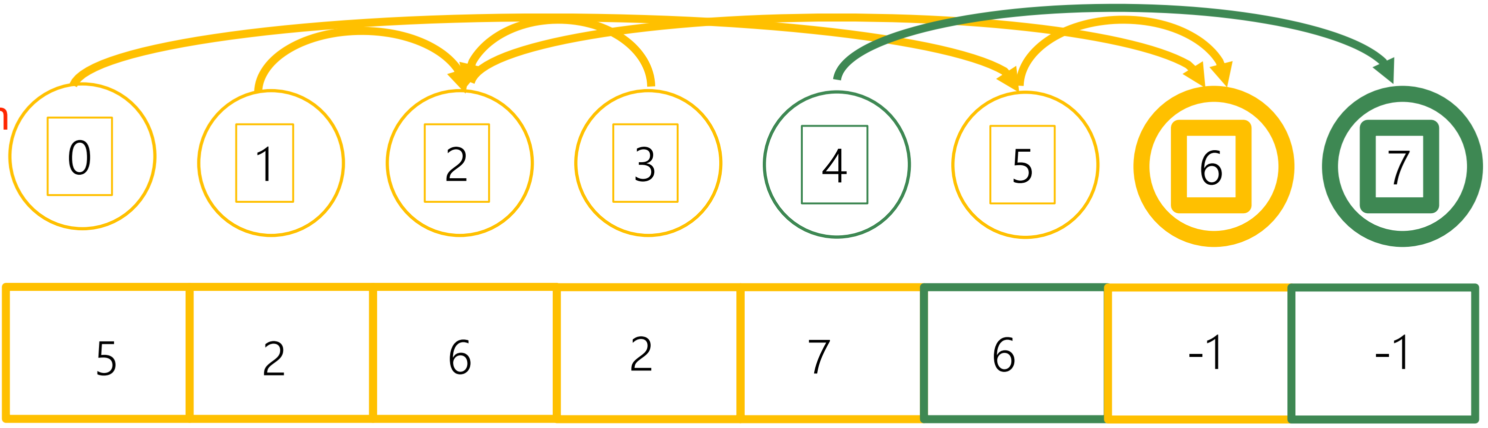
each element is assigned to an index, and at each index the value is
a) index of parent
b) negative value (could store size of disjoint set)



-1 is used as a **sentinel value** representing a root.

Disjoint Set (Simple Version)

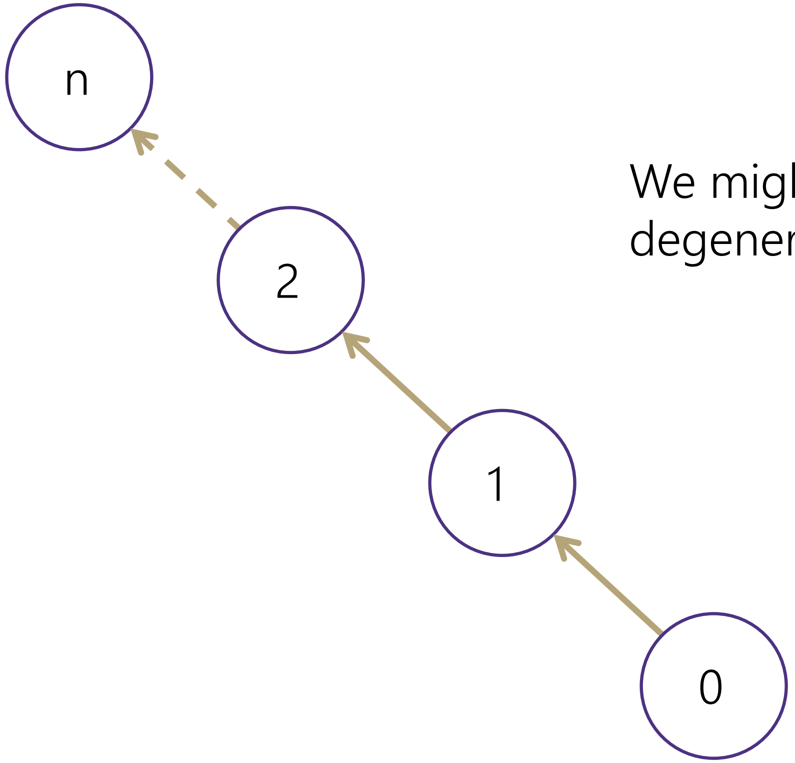
```
constructor:
    s = [-1, -1, -1, …, -1]

find(a):
    if (s[a] < 0): return a
    return find(s[a])

union(rootA, rootB): ← assumes you already ran "find", so these are representative elements
   s[rootA] = rootB
```

Is it fast?

Run union(0,1), union(0,2), … union(0, n):



We might form degenerate trees.

Remember balanced trees? Can we try and make this more balanced?

Union by Size

Strategy: Point the smaller tree at the larger to avoid deep chains.

```
union(rootA, rootB):
   if size(rootB) > size(rootA):
       s[rootA] = rootB
       updateSize(rootB)
   else:
      s[rootB] = rootA
      updateSize(rootA)
```

Problem: How to keep track of size?

Solution: Use the sentinel values! Instead of -1, store the **negative of the size.** -1 will still initializes!

Union by Size (in one array)

Strategy: Point the smaller tree at the larger to avoid deep chains.

```
union(rootA, rootB):
   if s[rootB] < s[rootA]: // Note the flipped sign, since we are using the negative of the size!!!
       s[rootB] = s[rootB] + s[rootA]
       s[rootA] = rootB
   else:
      s[rootA] = s[rootA] + s[rootB]
      s[rootB] = rootA
```

Problem: How to keep track of size?

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Analysis of Union by Size

How deep can the trees get?

If the depth of a node increases after a union, it must have been in a smaller subtree.

Therefore, the size of its subtree has at least doubled.

We can double the size of a subtree at most log n times before everything is in one set.

Therefore the depth of any node can only increase at most log n times.

This means that the maximum depth of a union-by-size tree is O(log n)!

Corollary: A sequence of M operations on a disjoint sets collection with N elements takes at most O(M log N) time.

Union by Height (in one array)

Strategy: Point the shallower tree at the larger to avoid deep chains.

```
union(rootA, rootB):
   if s[rootB] < s[rootA]: // Note the flipped sign, since we are using the negative of the height!!!
      s[rootA] = rootB
   else:
      if ( s[rootA] == s[rootB] ): // Total height only increases when both trees are equally deep!
         s[rootA]--  // Subtracting increases the height
       s[rootB] = rootA
```

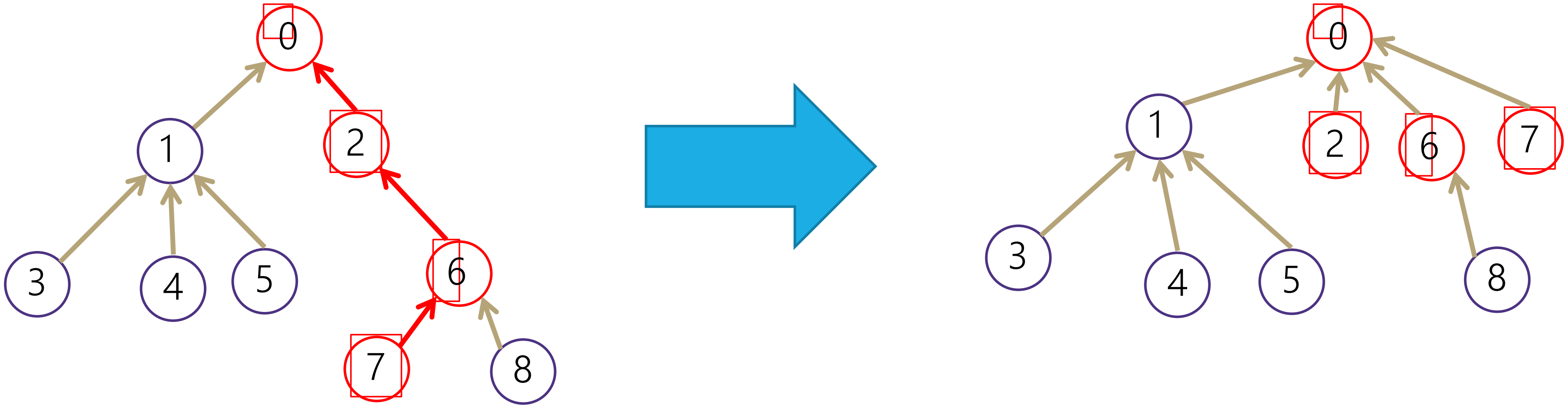
Note that we are actually storing **-(height + 1)** so that height 0 trees are still negative (still start at -1)

More Optimization!

It's not hard to hit the worst case, but there's not much more left to do!

We haven't changed **find** yet – what could we do here?

Idea: Whenever we run find, "flatten" the tree for the path we explore (i.e. set the parent of all intermediate nodes to the root:



Find with Path Compression

```
find(a):
    if s[a] < 0:
        return a
    else
        return s[a] = find( s[a] )
```

Runtime for M operations on a size N data structure: $\Theta(M\ \alpha(M, N))$

The $\alpha(M, N)$ function is **very very** slow growing (effectively <= 5), but this is not quite linear. See chapter 8.6 in the book. It is an instance of an **iterated logarithm** (log*).

Bringing it back to MSTs: Kruskal's Alg.

```
Kruskal(G = (V, E)):
    queue = priorityQueue(E)
    ds = new DisjointSets( |V| )
    mst = empty list
    while (size(mst) < |V| - 1):
        e = (u,v) = queue.deleteMin()
        repU = ds.find(u)
        repV = ds.find(v)
        if repU != repV:
            mst.add(e)
            ds.union(repU, repV)
    return mst
```

At most 3|E| union-find operations, so these lines contribute at most $\theta\left(|E|\alpha(|E|,|V|)\right) \leq \theta(|E|\log(|E|))$ to the running time.

Therefore the O(|E| log(|E|)) time of the heap operations dominates!

Since $|E| = |V|^2$, and $\log(|V|^2) = 2\log(|V|)$, we can write it as $O(|E|\log(|V|))$.

In practice we don't usually need to iterate over all of the edges, so it's even faster.

Another Approach to MSTs: Prim's Alg.

Strategy – Grow an MST from a starting node, just like Dijkstra's algorithm.

```
Dijkstra(Graph G, Vertex source)
  initialize distances to ∞, source.dist to 0
      mark all vertices unprocessed
      initialize MPQ as a Min Priority Queue
      add source at priority 0
      while(MPQ is not empty){
          u = MPQ.getMin()
          foreach(edge (u,v) leaving u){
              if(u.dist+w(u,v) < v.dist){
                  if(v.dist == ∞ )
                      MPQ.insert(v, u.dist+w(u,v))
                  else
                      MPQ.decreasePriority(v, u.dist+w(u,v))
                  v.dist = u.dist+w(u,v)
                  v.predecessor = u
              }
          }
          mark u as processed
      }
```

```
Prim(Graph G, Vertex source)
  initialize distances to ∞, source.dist to 0
      mark all vertices unprocessed
      initialize MPQ as a Min Priority Queue
      add source at priority 0
      while(MPQ is not empty){
          u = MPQ.getMin()
          foreach(edge (u,v) leaving u){
              if(w(u,v) < v.dist){
                  if(v.dist == ∞ )
                      MPQ.insert(v, w(u,v))
                  else
                      MPQ.decreasePriority(v, w(u,v))
                  v.dist = w(u,v)
                  mst.add(u,v)
              }
          }
          mark u as processed
      }
```