Uncertainty in AI: Probabilistic Reasoning

CSE 415: Introduction to Artificial Intelligence
University of Washington
Winter 2019

© S. Tanimoto and University of Washington, 2019

Outline

- Motivation
- Definitions and Laws of Probability
- Generalizing Modus Ponens
- Bayes' Rule
- Odds
- The Monty Hall Problem

Motivation

In the real world:

- data is often noisy,
- many processes cannot be modeled deterministically in a reliable and practical way;
  - parts may be insufficiently understood, other
  - parts too complex for efficient computation.
- logical reasoning does not match the available information,
- adversarial agents may not always obey rationality assumptions.

OR the two agents may have different ways of analyzing the system

Motivation (continued)

Therefore, we need a means to

- represent the certainty/uncertainty of information,
- compute certainty/uncertainty of related information.

The mathematics of probability is our tool of choice.

Definitions

A *random variable* X is a symbol that represents a class of events that may occur any number of times, and which take on values in a given set D called a *domain*.

Example: Let C be a coin-toss random variable with domain D={H, T}.

A *probability distribution* P for a random variable is a function that assigns to each domain element d_i a value p_i in the range 0 to 1. If D is finite then P is often given as a table.

d_i	p_i
H	0.5
T	0.5

Definitions

A *joint distribution* over a set of random variables $X_1, X_2, \ldots, X_n$ is a function that assigns to each n-tuple of domain elements $(d_{1,i_1}, d_{2,i_2}, \ldots, d_{n,i_n})$ a value $p_{i_1,i_2,\ldots,i_n}$ in the range 0 to 1. If all the domains D_j are finite then P is often given as a table.

Example with n=2:

$d_{1,i_1} \in D_1$	$d_{2,i_2} \in D_2$	$P(d_{1,i_1}, d_{2,i_2})$
rain	no crash	1/4
rain	crash	1/8
clear	no crash	5/16
clear	crash	1/16
snow	no crash	1/8
snow	crash	1/8

$P(X_1=d_{1,i_1}, X_2=d_{2,i_2})$

P(X1 =d1, X2 = d2) = P(X1 = d1)P(X2 = d2) iff X1 and X2 are independent

Definitions

The *conditional probability* of $P(X=d_i \mid Y=d_j)$ is the probability of $X=d_i$, given $Y=d_j$.

Example:
Let D={0, 1, 2, 3, 4, 5, 6, 7, 8, 9}.

Let Odd be the set of outcomes X=1, X=3, ..., X=9.
Let Even " " X=0, X=2, ..., X=8.
Let Prime " " X=2, X=3, X=5, X=7.
P(Odd) = 5/10 = 0.5
P(Prime) = 4/10 = 0.4
P(Odd | Prime) = |odd and prime | / | prime | = 3/4 = 0.75.
P(Prime | Odd) = |odd and prime | / | odd | = 3/5 = 0.6.

Two Laws of Probability

The sum rule: $\Sigma_{1<i<m} P(X = d_i) = 1$
Adding the probabilities of all the possible outcomes for a random variable must give a total of 1.0.

The product rule: $P(X=x, Y=y) = P(X=x)\, P(Y=y \mid X=x)$.
The joint probability of two random variables is equal to the product of the marginal of one times the conditional of the other given the one.

If P implies Q
AND
P is truem
THEREFORE
Q is true

Logic vs Probability: Generalizing Modus Ponens

Modus Ponens:

$P \rightarrow Q$	If it's raining then I do my homework.
P	It's raining.
Q	I do my homework.

Bayes' Rule: (general idea)

If P then sometimes Q	If it's raining then I might do my homework.
P	It's raining.
Maybe Q	I might do my homework.

(Bayes' rule lets us calculate the probability of Q, taking P into account.)

Bayes' Rule

E: Some evidence exists, i.e., a particular condition is true
H: some hypothesis is true.

P(E|H) = probability of E given H.
P(E|~H) = probability of E given not H.
P(H) = probability of H, independent of E.

$$P(H|E) = \frac{P(E|H)\,P(H)}{P(E)}$$

P(E) = P(E|H) P(H) + P(E|~H)(1 - P(H))

Applying Bayes' Rule

E: The patient's white blood cell count exceeds 110% of average.

H: The patient is infected with tetanus.

P(E|H) = 0.8 class-conditional probability
P(E|~H) = 0.3 "
P(H) = 0.01 prior probability

posterior probability:

$$P(H|E) = \frac{P(E|H)\,P(H)}{P(E)} = \frac{(0.8)\ (0.01)}{(0.8)\ (0.01) + (0.3)(0.99)} = \frac{0.008}{0.305} = 0.0262$$

P(E) = P(E|H) P(H) + P(E|~H)(1 - P(H))

Odds

Odds are 10 to 1 it will rain tomorrow.

$$P(rain) = \frac{10}{10+1} = \frac{10}{11}$$

Suppose P(A) = 1/4
Then O(A) = (1/4) / (3/4) = 1/3

in general: $$O(A) = \frac{P(A)}{P(\sim A)} = \frac{P(A)}{1 - P(A)}$$

Un
Uncertainty

Bayes' Rule reformulated...

$$P(H|E) = \frac{P(E|H)\,P(H)}{P(E)}$$

$$P(\sim H|E) = \frac{P(E|\sim H)\,P(\sim H)}{P(E)}$$

$$O(H|E) = \frac{P(E|H)}{P(E|\sim H)}\,O(H)$$

Un
Uncertainty

Odds-Likelihood Form of Bayes' Rule

E: The patient's white blood cell count exceeds 110% of average.

H: The patient is infected with tetanus.

O(H) = 0.01/0.99

O(H|E) = λ O(H) lambda is called the *sufficiency* factor.

O(H|~E) = λ' O(H) lambda prime is called the *necessity* factor.

$\lambda = P(E|H)/P(E|\sim H) = 0.8/0.3 \approx 2.67$

$\lambda' = P(\sim E|H)/P(\sim E|\sim H) = 0.2/0.7 \approx 0.286$

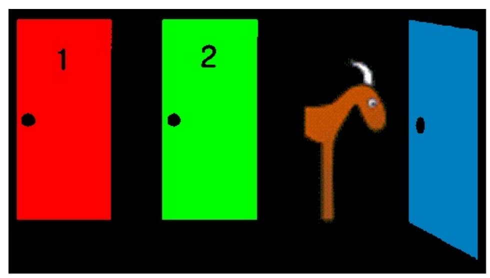
Un
Uncertainty

Now For A Brain Teaser

The following puzzle is an effective stimulant for thinking about probabilities, and how judgments about probabilities should take into consideration the results of observations.

Un
Uncertainty

The Monty Hall Problem



Un
Uncertainty

The Monty Hall Problem

There are three doors: a red door, green door, and blue door.
Behind one is a car, and behind the other two are goats.
You get to keep whatever is behind the door you choose.

You choose a door (say, red).

The host opens one of the other doors (say, green), which reveals a goat.

The host says, "Would you like to select the OTHER door?"

Should you switch?