CSE 417
Introduction to Algorithms

NP-Completeness

(Chapter 8)

What can we feasibly compute?

Focus so far has been to give good algorithms for specific problems (and general techniques that help do this).

Now shifting focus to problems where we think this is impossible. Sadly, there are many…

Some History

1930/40's

Gödel, Church, Turing, ...: What is (is not) computable

1960/70's

What is (is not) *feasibly* computable

Goal – a (largely) technology-independent theory of time required by algorithms

Key modeling assumptions/approximations

- Asymptotic (Big-O), worst case is revealing
- Polynomial, exponential time – qualitatively different

Polynomial Time

The class P

(defined later)

Definition: P = the set of (decision) problems solvable by computers in *polynomial time*, i.e., $T(n) = O(n^k)$ for some fixed k (indp of input). K depends on problem

These problems are sometimes called *tractable* problems.

Examples: sorting, shortest path, MST, connectivity, RNA folding & other dyn. prog., flows & matching – i.e.: most of this qtr

(exceptions: Change-Making/Stamps, Knapsack, TSP)

Why "Polynomial"?

- n^{2000} is *not* a nice time bound
- differences among n, 2n and n^2 are *not* negligible.

But, simple theoretical tools don't easily capture such differences, while exponential vs polynomial is a qualitative difference potentially more amenable to theoretical analysis.

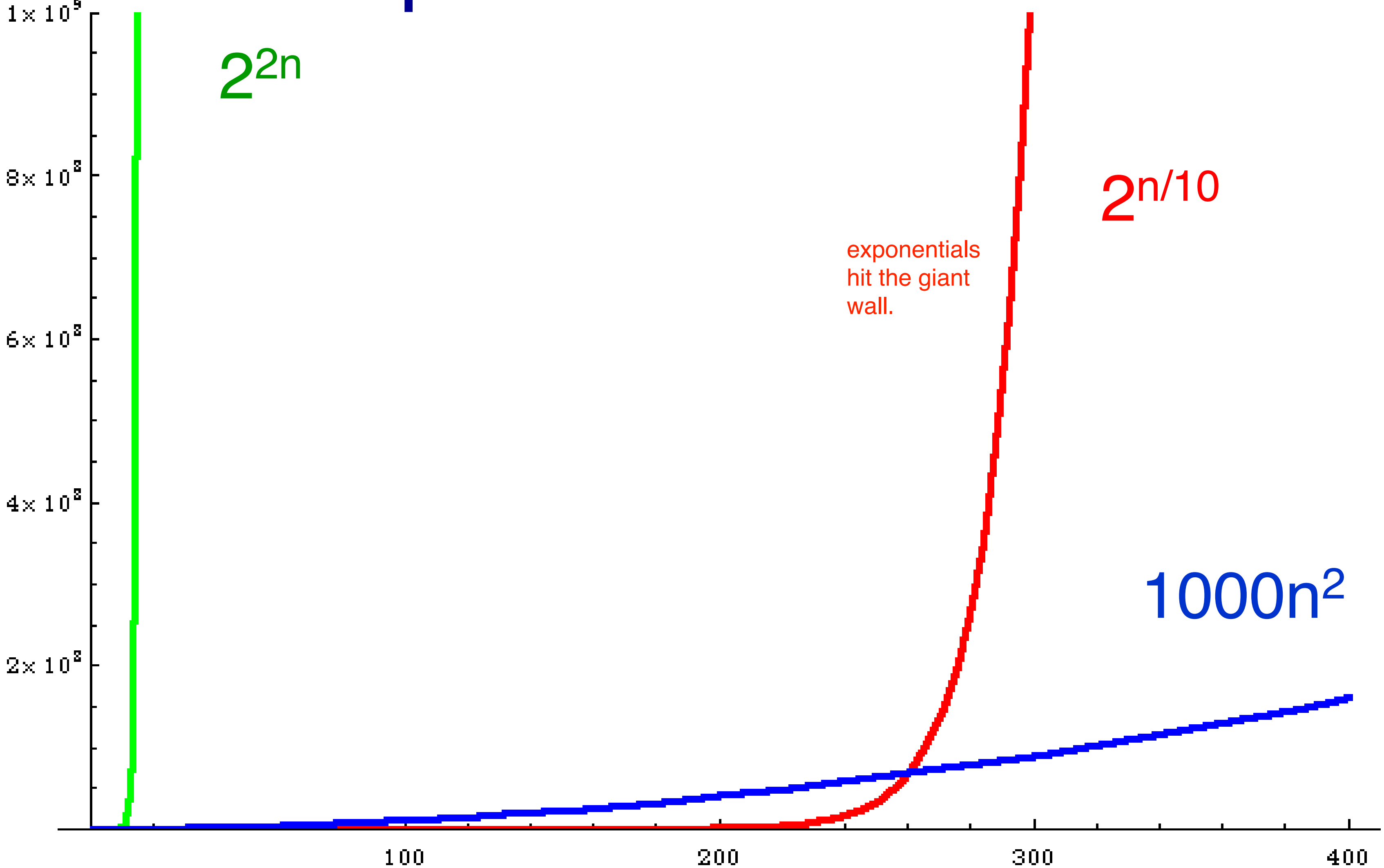
"Problem is in P" a starting point for more detailed analysis

"Problem is not in P" may suggest that you need to shift to a more tractable variant / lower your expectations

approximation of solution instead?

Reminder

Polynomial vs Exponential Growth



Reminder

Another view of Poly vs Exp

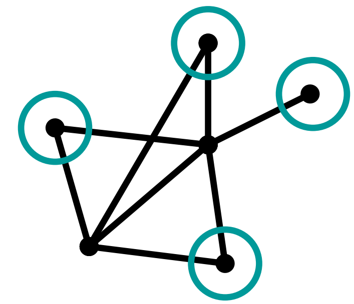
Next year's computer will be 2x faster. If I can solve problem of size n_0 today, how large a problem can I solve in the same time next year?

Complexity	Increase	E.g. $T=10^{12}$	
$O(n)$	$n_0 \rightarrow 2n_0$	10^{12}	2×10^{12}
$O(n^2)$	$n_0 \rightarrow \sqrt{2}\, n_0$	10^6	1.4×10^6
$O(n^3)$	$n_0 \rightarrow \sqrt[3]{2}\, n_0$	10^4	1.25×10^4
$2^{n/10}$	$n_0 \rightarrow n_0+10$	400	410
2^n	$n_0 \rightarrow n_0+1$	40	41

Two Problems

How hard are they? We don't fully know…

The Independent Set Problem



Given: a graph G=(V,E) and an integer k

Question: is there $U \subseteq V$ with $|U| \geq k$ s.t. no pair of vertices in U is joined by an edge?

What's it good for?

E.g., if nodes = web pages, and edges join "similar" pages, then pages forming an independent set are likely to represent distinctly different topics

E.g., if nodes = courses, and edge = a student is co-enrolled, then an independent set is a set of courses whose finals could be scheduled simultaneously

How hard is it? Don't fully know. Exponential time is easily possible (try all 2^n subsets). But no poly time solution is known

A seemingly completely different problem from the independent set problem.

Boolean Satisfiability

Boolean variables $x_1, \ldots, x_n$

taking values in $\{0,1\}$. 0=false, 1=true

Literals

x_i or $\neg x_i$ for $i = 1, \ldots, n$

Clause

a logical OR of one or more literals

e.g. $(x_1 \vee \neg x_3 \vee x_7 \vee x_{12})$

CNF formula ("conjunctive normal form")

a logical AND of a bunch of clauses

Boolean Satisfiability

CNF formula example

$(x_1 \vee \neg x_3 \vee x_7) \wedge (\neg x_1 \vee \neg x_4 \vee x_5 \vee \neg x_7)$

If there is some assignment of 0's and 1's to the variables that makes it true then we say the formula is *satisfiable*

the one above is, the following isn't

$x_1 \wedge \overbrace{(\underbrace{\neg x_1}_{\text{literal}} \vee x_2)}^{\text{clause}} \wedge (\neg x_2 \vee x_3) \wedge \neg x_3$

Satisfiability: Given a CNF formula F, is it satisfiable?

3SAT = satisifiable problem with each clause having AT MOST 3 literals

Naive algorithm; try all 2^n states of x1, x2, …., xn (each is either a zero/false or a one/true)

Satisfiable?

$$(x \vee y \vee z) \wedge (\neg x \vee y \vee \neg z) \wedge$$
$$(x \vee \neg y \vee z) \wedge (\neg x \vee \neg y \vee z) \wedge$$
$$(\neg x \vee \neg y \vee \neg z) \wedge (x \vee y \vee z) \wedge$$
$$(x \vee \neg y \vee z) \wedge (x \vee y \vee \neg z)$$

$$(x \vee y \vee z) \wedge (\neg x \vee y \vee \neg z) \wedge$$
$$(x \vee \neg y \vee \neg z) \wedge (\neg x \vee \neg y \vee z) \wedge$$
$$(\neg x \vee \neg y \vee \neg z) \wedge (\neg x \vee y \vee z) \wedge$$
$$(x \vee \neg y \vee z) \wedge (x \vee y \vee \neg z)$$

Satisfiability

What's it good for?

- Theorem provers
- Circuit validation
- Analysis of program logic
- Etc.

How hard is it?

- Don't know fully
- Exponential time is easily possible (try all 2^n assignments)
- But no poly time solution is known

Reduction, I

Reduce a problem I do not know how to do to a problem that I do know how to do.

Reductions: a useful tool

Definition: To "reduce A to B" means to solve A, given a subroutine solving B.

Example: reduce MEDIAN to SORT

Solution: sort, then select $(n/2)^{nd}$

Example: reduce SORT to FIND_MAX

Solution: FIND_MAX, remove it, repeat

Example: reduce MEDIAN to FIND_MAX

Solution: transitivity: compose solutions above.

can compose reductions!

Reductions & Time

Definition: To reduce A to B means to solve A, given a subroutine solving B.

If setting up call, etc., is fast, then a fast algorithm for B implies (nearly as) fast an algorithm for A

poly-time,
for our uses

Contrapositive: If every algorithm for A is slow, then no algorithm for B can be fast.

$$\boxed{\text{“complexity of A”} \leq \textit{“complexity of B”} + \textit{“complexity of reduction”}}$$

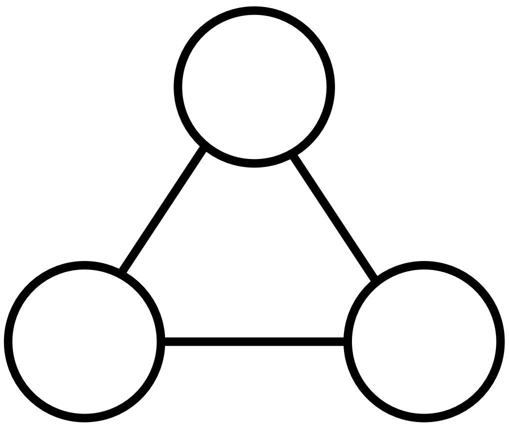
If every algorithm for A is slow, then no algorithm for B is fast assuming we can efficiently convert A to B

CAREFUL: we cannot have a slow reduction algorithm as it is part of the complexity analysis of our algorithm A

SAT and Independent Set

They are superficially different problems, but are intimately related at a deep level

3SAT $\leq_p$ IndpSet



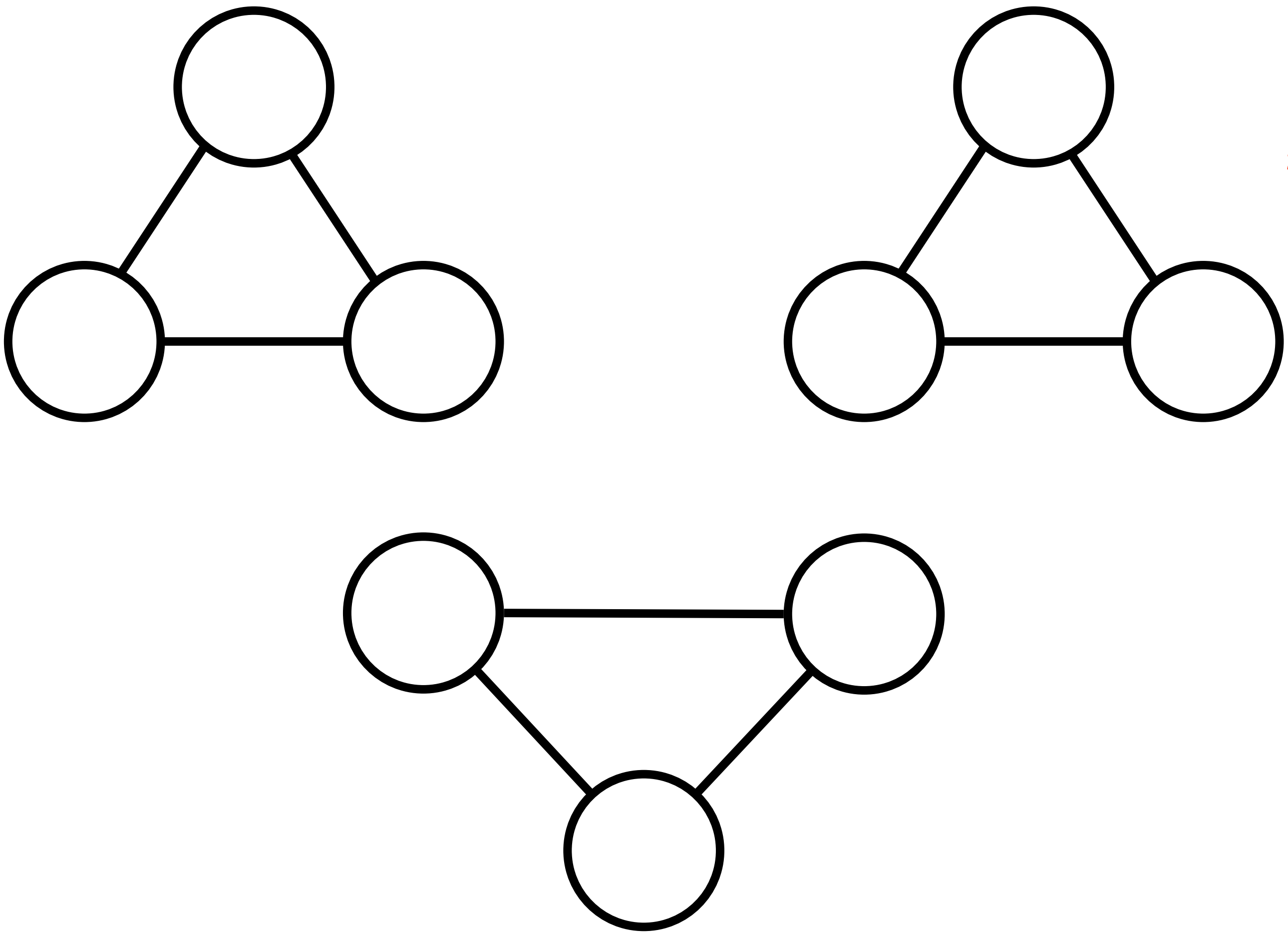
what indp sets?

how large?

how many?

3 possible independent sets of size 1.

$3SAT \leq_p$ IndpSet



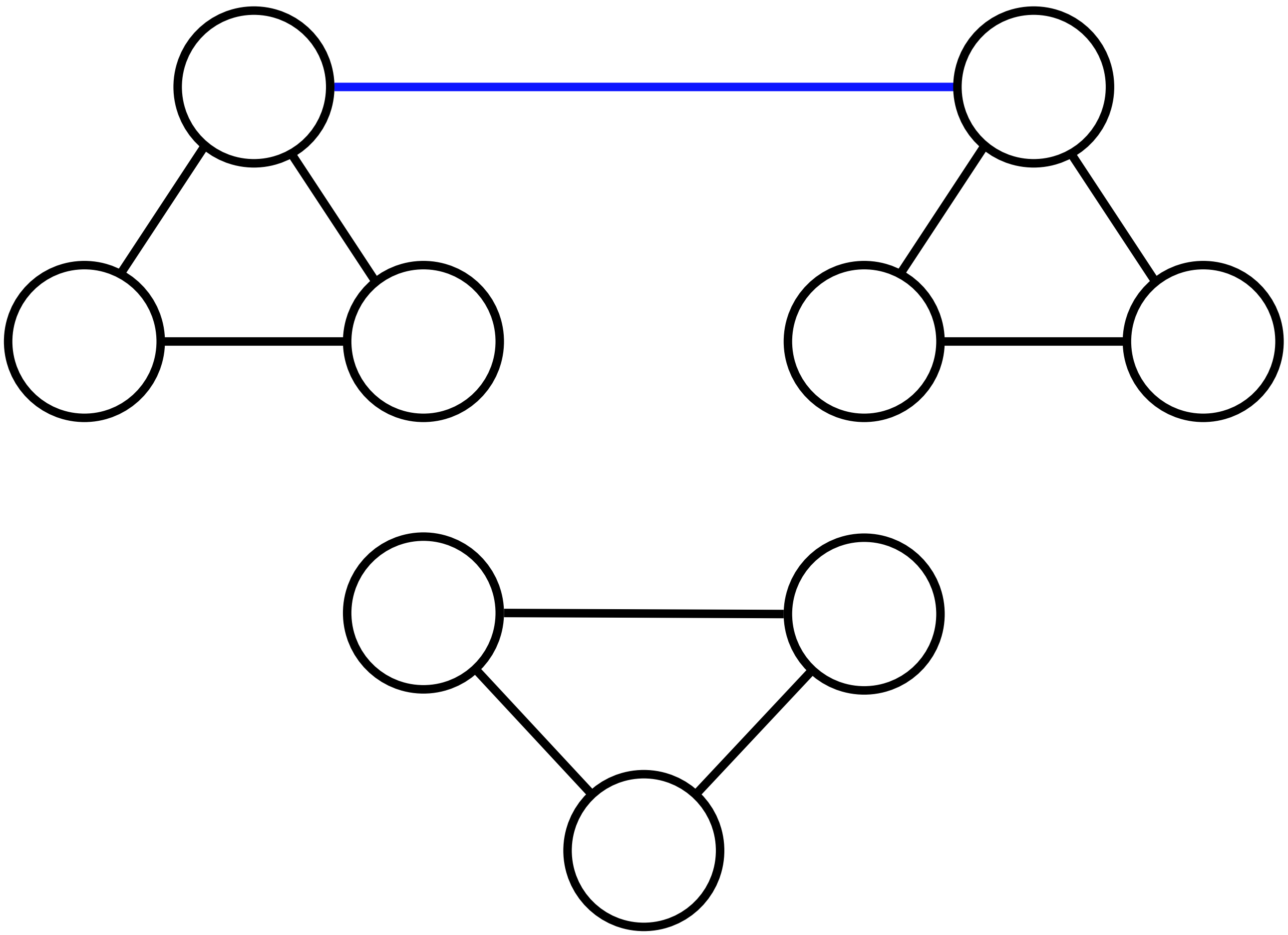
what indp sets?
how large?
how many?

27 independent sets of size 3

3SAT $\leq_p$ IndpSet

what indp sets?
how large?
how many?

24 independent sets of size 4

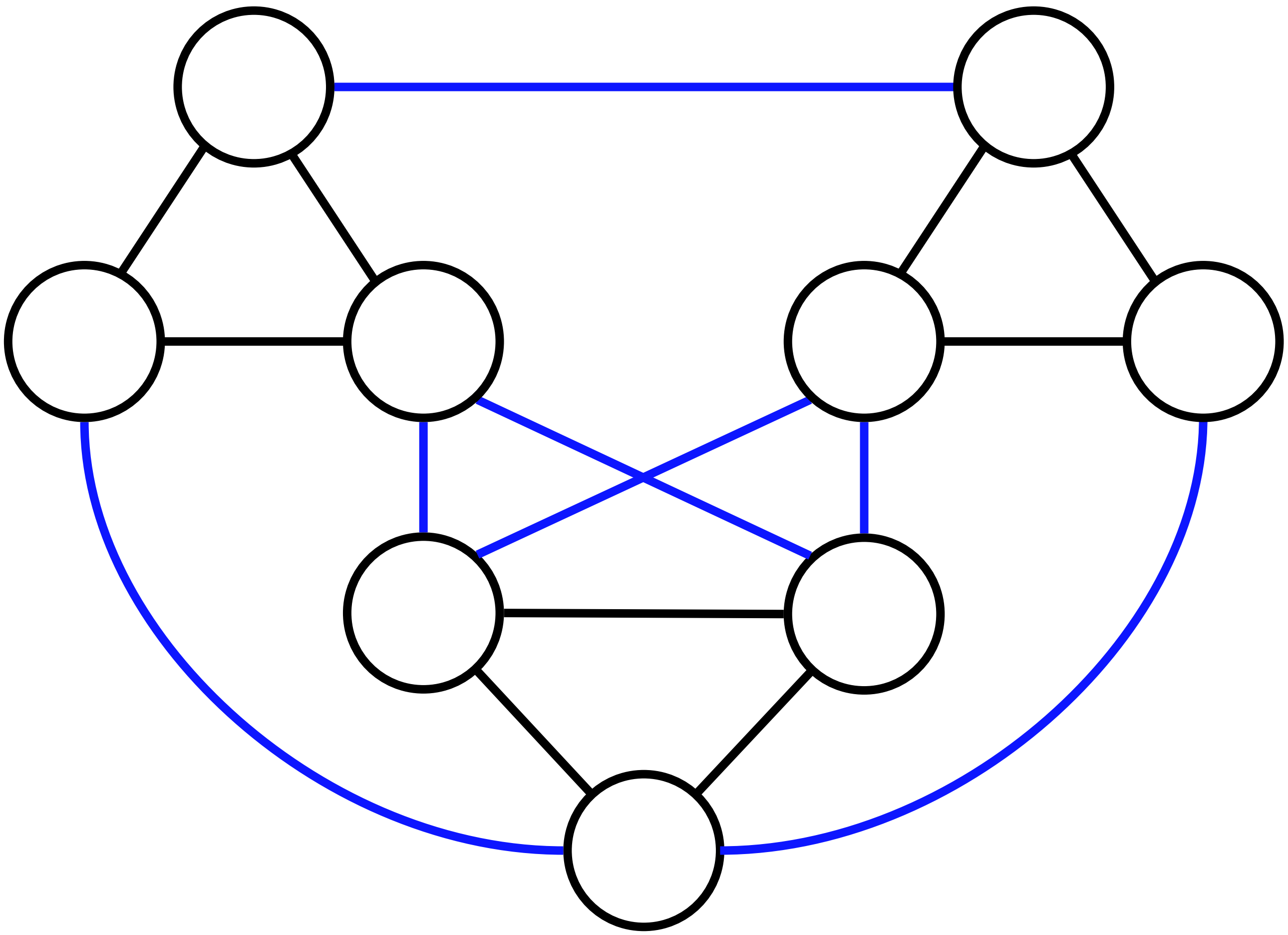


3SAT $\leq_p$ IndpSet

what indp sets?
how large?
how many?

? independent sets of size 3

k=3



reduction; reducing the satisfiability problem to the independent set problem.

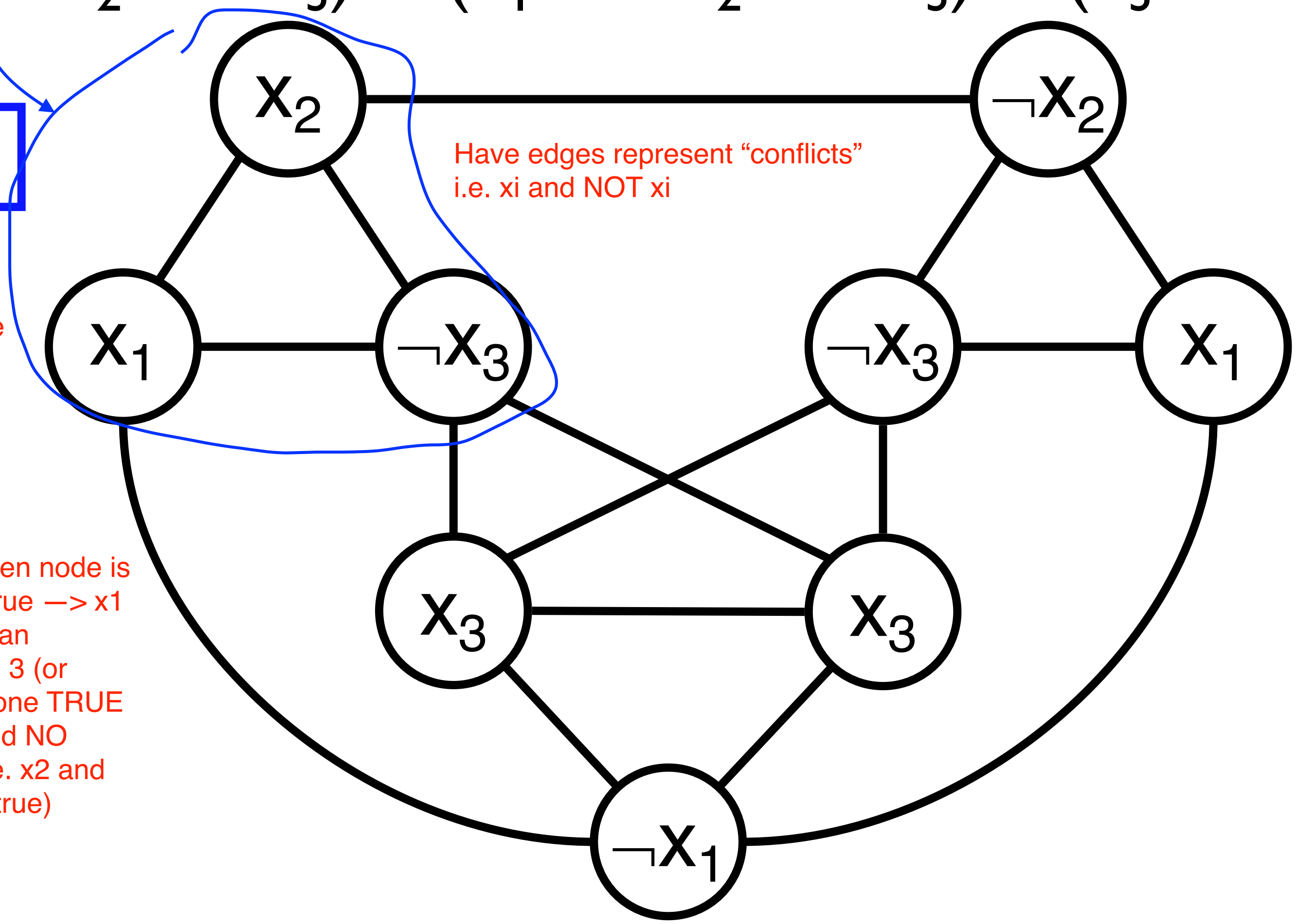
$3SAT \leq_p$ IndpSet

clause1

$$(x_1 \lor x_2 \lor \neg x_3) \land (x_1 \lor \neg x_2 \lor \neg x_3) \land (x_3 \lor \neg x_1 \lor x_3)$$

$\boxed{k=3}$

Have edges between every pair of vertices within the same clause



Have edges represent "conflicts" i.e. xi and NOT xi

Take a vertex if that given node is TRUE (not NOT x1 is true —> x1 is FALSE). If I can find an independent set of size 3 (or more) there is at least one TRUE node in each clause and NO conflicting variables (i.e. x2 and NOT x2 are both NOT true)

$3SAT \leq_p IndpSet$

f (

3-SAT Instance:

- Variables: $x_1, x_2, \ldots$
- Literals: $y_{i,j}$, $1 \leq i \leq q$, $1 \leq j \leq 3$
- Clauses: $c_i = y_{i1} \vee y_{i2} \vee y_{i3}$, $1 \leq i \leq q$
- Formula: $c = c_1 \wedge c_2 \wedge \ldots \wedge c_q$

) =

IndpSet Instance:

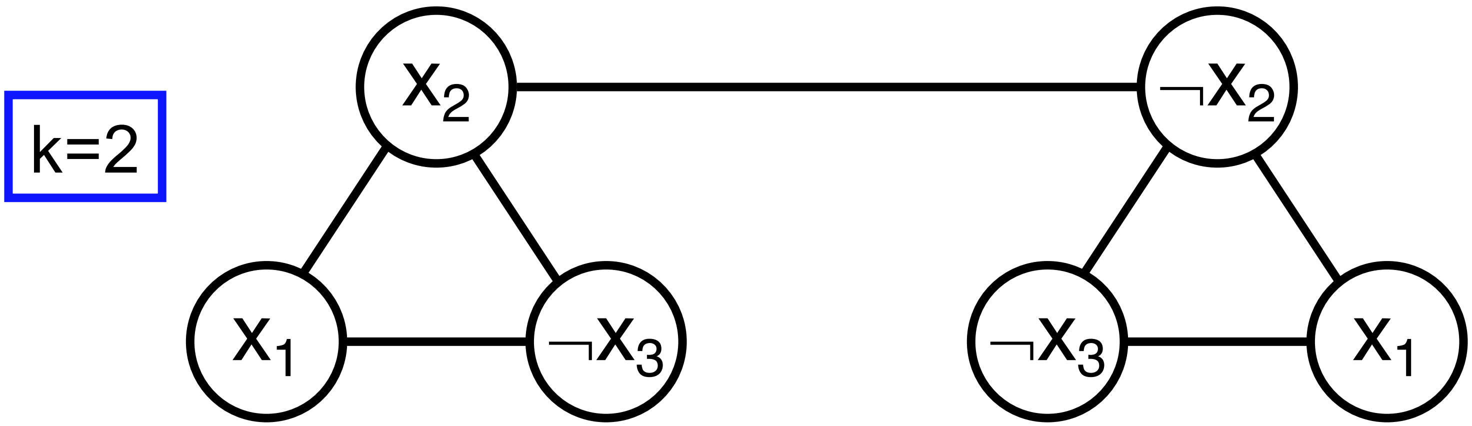
- $k = q$
- $G = (V, E)$
- $V = \{ [i,j] \mid 1 \leq i \leq q, 1 \leq j \leq 3 \}$
- $E = \{ ([i,j], [k,l]) \mid i = k \text{ or } y_{ij} = \neg y_{kl} \}$

Does NOT require a satisfying assignment to the given SAT problem to form this graph.

connect two nodes if they are in the same clause or if they are negations of the same literal

3SAT $\leq_p$ IndpSet

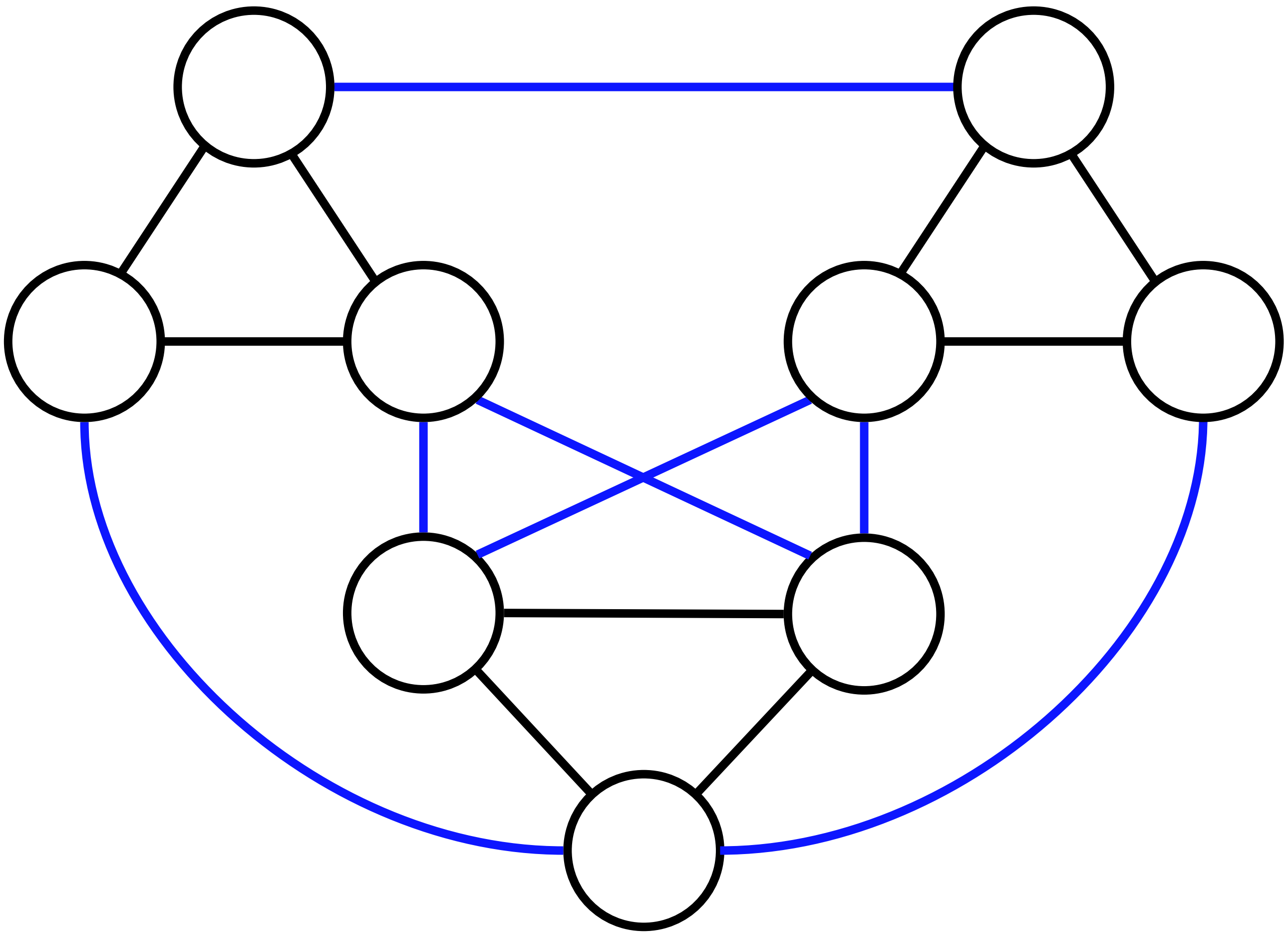
$$(x_1 \vee x_2 \vee \neg x_3) \wedge (x_1 \vee \neg x_2 \vee \neg x_3)$$



3SAT $\leq_p$ IndpSet

polynomial less than. i.e. there is a polynomial reduction problem between the 2.

k=3



Correctness of "3SAT $\leq_p$ IndpSet"

Summary of reduction function f: Given formula, make graph G with one group per clause, one node per literal. Connect each to all nodes in same group; connect all complementary literal pairs (x, ¬x). Output graph G plus integer k = number of clauses. Note: *f does not know whether formula is satisfiable or not; does not know if G has k-IndpSet; does not try to find satisfying assignment or set.*

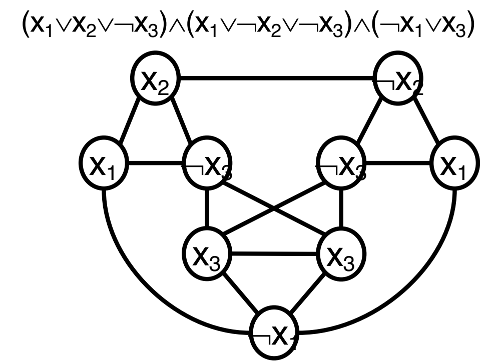
Correctness:

- Show f poly time computable: A key point is that graph size is polynomial in formula size; mapping basically straightforward.
- Show c in 3-SAT iff f(c)=(G,k) in IndpSet:

(⇒) Given an assignment satisfying c, pick one true literal per clause. Add corresponding node of each triangle to set. Show it is an IndpSet: 1 per triangle never conflicts w/ another in same triangle; only true literals (but perhaps not all true literals) picked, so not both ends of any (x, ¬x) edge.

(⇐) Given a k-Independent Set in G, selected labels define a valid (perhaps partial) truth assignment since no (x, ¬x) pair picked. It satisfies c since there is one selected node in each clause triangle (else some other clause triangle has > 1 selected node, hence not an independent set.)

Utility of "3SAT $\leq_p$ IndpSet"



Suppose we had a fast algorithm for IndpSet, then we could get a fast algorithm for 3SAT:

Given 3-CNF formula *w*, build Independent Set instance $y = f(w)$ as above, run the fast IS alg on *y*; say "YES, *w* is satisfiable" iff IS alg says "YES, *y* has a Independent Set of the given size"

On the other hand, *suppose* no fast alg is possible for 3SAT, then we know none is possible for Independent Set either.

“3SAT $\leq_p$ IndpSet” Retrospective

Previous slides: two suppositions

Somewhat clumsy to have to state things that way.

Alternative: abstract out the key elements, give it a name (“polynomial time mapping reduction”), then properties like the above always hold.

Reduction, II

Polynomial time “mapping” reduction

Polynomial-Time Reductions

Definition: Let A and B be two decision problems.

A is *polynomially (mapping) reducible* to B ($A \leq_p B$) if there exists a polynomial-time algorithm f that converts each instance x of problem A to an instance $f(x)$ of B such that:

Decision Problems: just determining IF a solution exists, not necessarily finding it.

x is a YES instance of A iff $f(x)$ is a YES instance of B

x is a solution to problem A

f(x) is a solution to problem B

$$x \in A \iff f(x) \in B$$

The notation "$A \leq_p B$" is meant to suggest "A is easier than B", or more precisely, "A is not more than polynomially harder than B"

Polynomial-Time Reductions (cont.)

Defn: $A \leq_p B$ "A is polynomial-time reducible to B," iff there is a polynomial-time computable function f such that: $x \in A \iff f(x) \in B$

Why the notation?

polynomial

"complexity of A" ≤ "complexity of B" + "complexity of f"

Theorem: If B is in P and I can convert B to A via a polynomial time algorithm, then A is in P
If there is a fast algorithm for B and a fast algorithm to make B into A, we can solve A fast.

(1) $A \leq_p B$ and $B \in P \Rightarrow A \in P$

(2) $A \leq_p B$ and $A \notin P \Rightarrow B \notin P$

If There are no poly algorithms for A, but I could convert B to A in polynomial time, there are no polynomial algorithms for B as well

(3) $A \leq_p B$ and $B \leq_p C \Rightarrow A \leq_p C$ (transitivity)

Another Example Reduction

SAT to Subset Sum (Knapsack)

Subset-Sum, AKA Knapsack

sum to EXACTLY C

KNAP = { (w_1, w_2, …, w_n, C) | a subset of the w_i sums to C }

at least binary

w_i's and C encoded in radix r ≥ 2. (Decimal used in following example.)

I can convert KNAP to 3SAT

Theorem: 3-SAT $\leq_p$ KNAP

Pf: given formula with p variables & q clauses, build KNAP instance with 2(p+q) w_i's, each with (p+q) decimal digits. See examples below.

3-SAT $\leq_p$ KNAP

Formula: (x z)

		Variables	Clauses	
		x	(x)	
Literals	w_1 (x)	1	1	eleven
	w_2 ($\neg$x)	1	0	ten
Slack	w_7 (s_{11})		1	one
	w_8 (s_{12})		1	one
C		1	3	

What/How Many Satisfying Assignments?

What/How Many KNAP solutions?

Can I find the subset that gives the given capacity 13? Yes, eleven, one, one.

3-SAT $\leq_p$ KNAP

Formula: $(x \quad\quad z) \wedge (\neg x \quad\quad z)$

		Variables	Clauses	
		x	$(x \quad)$	$(\neg x \quad)$
Literals	w_1 (x)	1	1	0
	w_2 $(\neg x)$	1	0	1
Slack	w_7 (s_{11})		1	0
	w_8 (s_{12})		1	0
	w_9 (s_{21})			1
	w_{10} (s_{22})			1
	C	1	3	3

What/How Many Satisfying Assignments?

What/How Many KNAP solutions?

3-SAT $\leq_P$ KNAP

Formula: (x V y V z)

		Variables			Clauses
		x	y	z	$(x \vee y \vee z)$
Literals	w_1 (x)	1	0	0	1
	w_2 ($\neg$x)	1	0	0	0
	w_3 (y)		1	0	1
	w_4 ($\neg$y)		1	0	0
	w_5 (z)			1	1
	w_6 ($\neg$z)			1	0
Slack	w_7 (s_{11})				1
	w_8 (s_{12})				1
	C	1	1	1	3

Add two slack variables for each clause. i.e. only have to choose at least one literal to be true to satisfy a clause

literals in clause - 1 slack variables per clause

What/How Many Satisfying Assignments?

What/How Many KNAP solutions?

Choose specific rows such that sum of each column matches the Capacity row. i.e. must have the sum of column x which includes w(x) and w(NOT x) can only be one; cannot choose both.

We need the column of each clause to sum to at least one (i.e. at least one true literal in each clause) to get to three.

3-SAT $\leq_p$ KNAP

Formula: $(x \lor y \lor z) \land (\neg x \lor y \lor \neg z) \land (\neg x \lor \neg y \lor z)$

		Variables			Clauses		
		x	y	z	$(x \lor y \lor z)$	$(\neg x \lor y \lor \neg z)$	$(\neg x \lor \neg y \lor z)$
Literals	w_1 (x)	1	0	0	1	0	0
	w_2 ($\neg x$)	1	0	0	0	1	1
	w_3 (y)		1	0	1	1	0
	w_4 ($\neg y$)		1	0	0	0	1
	w_5 (z)			1	1	0	1
	w_6 ($\neg z$)			1	0	1	0
Slack	w_7 (s_{11})				1	0	0
	w_8 (s_{12})				1	0	0
	w_9 (s_{21})					1	0
	w_{10} (s_{22})					1	0
	w_{11} (s_{31})						1
	w_{12} (s_{32})						1
	C	1	1	1	3	3	3

w1 = 100100

w2 = 100011

…

What/How Many Satisfying Assignments/KNAP solutions?

$3\text{-SAT} \le_p \text{KNAP}$

f (

3-SAT Instance:

- Variables: $x_1, x_2, \ldots, x_p$
- Literals: $y_{i,j}$, $1 \le i \le q$, $1 \le j \le 3$
- Clauses: $c_i = y_{i1} \vee y_{i2} \vee y_{i3}$, $1 \le i \le q$
- Formula: $c = c_1 \wedge c_2 \wedge \ldots \wedge c_q$

) =

KNAP Instance: See past example.

- 2(p+q) wi's, each with (p+q) decimal digits, mostly 0
- For the 2p "literal" weights, a single 1 in H.O. (high order) p digits marks which variable; 1's in L.O. q digits mark each claus containing that literal. lower order
- Two "slacks" per clause; single 1 marks the clause.
- Knapsack Capacity C = 11..133..3 (p 1's, q 3's)

Correctness

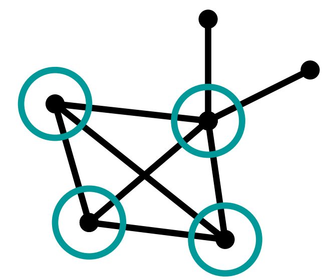
Poly time for reduction is routine; details omitted. Note that it does *not* look at satisfying assignment(s), if any, nor at subset sums (but the problem instance it builds captures one via the other...)

If formula is satisfiable, select the literal weights corresponding to the true literals in a satisfying assignment. If that assignment satisfies k literals in a clause, also select (3 - k) of the "slack" weights for that clause. Total = C.

Conversely, suppose KNAP instance has a solution. Columns are decoupled since ≤ 5 one's per column, so no "carries" in sum (recall – weights are decimal). Since H.O. p digits of C are 1, exactly one of each pair of literal weights included in the subset, so it defines a valid assignment. Since L.O. q digits of C are 3, but at most 2 "slack" weights contribute to each, at least one of the selected literal weights must be 1 in that clause, hence the assignment satisfies the formula.

Decision vs Search Problems

The Clique Problem



Given: a graph G=(V,E) and an integer k

Question: is there a subset U of V with |U| ≥ k such that every pair of vertices in U is joined by an edge.

complete subgraph. Ramsey Theory.

E.g., if nodes are web pages, and edges join "similar" pages, then pages forming a clique are likely to be about the same topic

Decision Problems

Computational complexity usually analyzed using decision problems

Answer is just 1 or 0 (yes or no).

Why?

Much simpler to deal with

Deciding whether G has a k-clique, is certainly no harder than finding a k-clique in G, so a lower bound on deciding is also a lower bound on finding

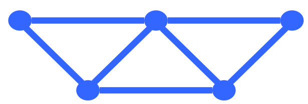
Less important, but if you have a good decider, you can often use it to get a good finder. (Ex.: does G still have a k-clique after I remove this vertex?)

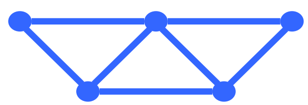
Some Convenient Technicalities

“Problem” – the general case

Ex: The Clique Problem: Given a graph G and an integer k, does G contain a k-clique?

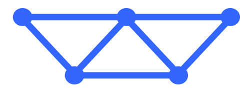
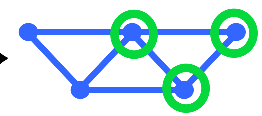
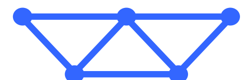
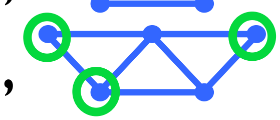
“Problem Instance” – the specific cases

Ex: Does  contain a 4-clique? (no)

Ex: Does  contain a 3-clique? (yes)

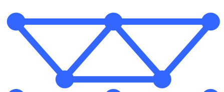
Some Convenient Technicalities

Three kinds of problem:

- Search: *Find* a k-clique in G (3, ) → 
- Decision: *Is there* a k-clique in G (3, ) → yes
- Verification: Is *this* a k-clique in G (3, ) → no

Problems as Sets of "Yes" Instances

Ex: CLIQUE = { (G,k) | G contains a k-clique }

E.g., ( , 4) $\notin$ CLIQUE

E.g., ( , 3) $\in$ CLIQUE

But we'll sometimes be a little sloppy and use CLIQUE to mean the associated search problem

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Beyond P

Algebraic Satisfiability

Given positive integers a, b, c

Question 1: does there exist a positive integer x such that $ax = c$?

Question 2: does there exist a positive integer x such that $ax^2 + bx = c$?

Question 3: do there exist positive integers x and y such that $ax^2 + by = c$?

SAT and 3SAT

Satisfiability: A Boolean formula in conjunctive normal form (CNF) is satisfiable if there exists an assignment of 0's and 1's to its variables such that the value of the expression is 1.

Example:

$$S=(x \vee y \vee \neg z) \wedge (\neg x \vee y \vee z) \wedge (\neg x \vee \neg y \vee \neg z)$$

Example above is satisfiable. (E.g., set x=1, y=1 and z=0.)

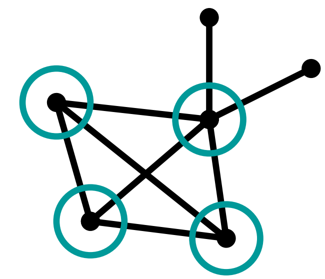
SAT = the set of satisfiable CNF formulas

3SAT = … having at most 3 literals per clause

More Problems

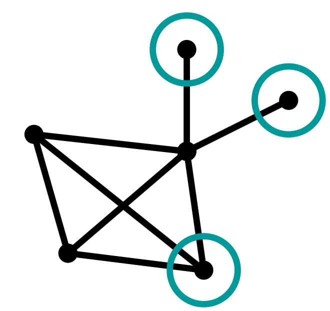
Clique:

Pairs ⟨G,k⟩, where G=(V,E) is a graph and k is an integer k, for which there is a subset U of V with |U| ≥ k such that *every* pair of vertices in U is joined by an edge.



Independent-Set:

Pairs ⟨G,k⟩, where G=(V,E) is a graph and k is an integer, for which there is a subset U of V with |U| ≥ k such that *no* pair of vertices in U is joined by an edge.



More Problems

Euler Tour:

Graphs G=(V,E) for which there is a cycle traversing each edge once.

Hamilton Tour:

Graphs G=(V,E) for which there is a simple cycle of length |V|, i.e., traversing each vertex once.

TSP:

Pairs ⟨G,k⟩, where G=(V,E,w) is a a weighted graph and k is an integer, such that there is a Hamilton tour of G with total weight $\leq$ k.

More Problems

Short Path:

4-tuples $\langle G, s, t, k\rangle$, where $G=(V,E)$ is a digraph with vertices s, t, and an integer k, for which there is a path from s to t of length $\leq k$

Long Path:

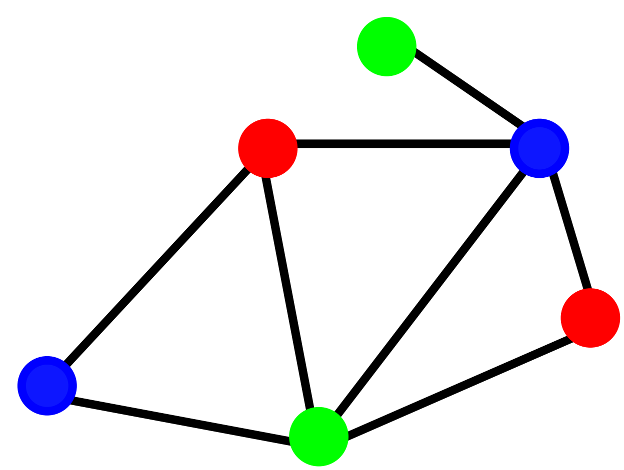
4-tuples $\langle G, s, t, k\rangle$, where $G=(V,E)$ is a digraph with vertices s, t, and an integer k, for which there is an acyclic path from s to t of length $\geq k$

More Problems

3-Coloring:

Graphs G=(V,E) for which there is an assignment of at most 3 colors to the vertices in G such that no two adjacent vertices have the same color.

Example:



Beyond P?

There are many natural, practical problems for which we don't know any polynomial-time algorithms:

e.g. CLIQUE:

Given an undirected graph G and an integer k, does G contain a k-clique?

e.g. quadratic Diophantine equations:

Given a, b, c $\in$ N, $\exists$ x, y $\in$ N s.t. $ax^2 + by = c$?

e.g., most of others just mentioned (excl: shortpath, Euler)

Lack of imagination or intrinsic barrier?

NP

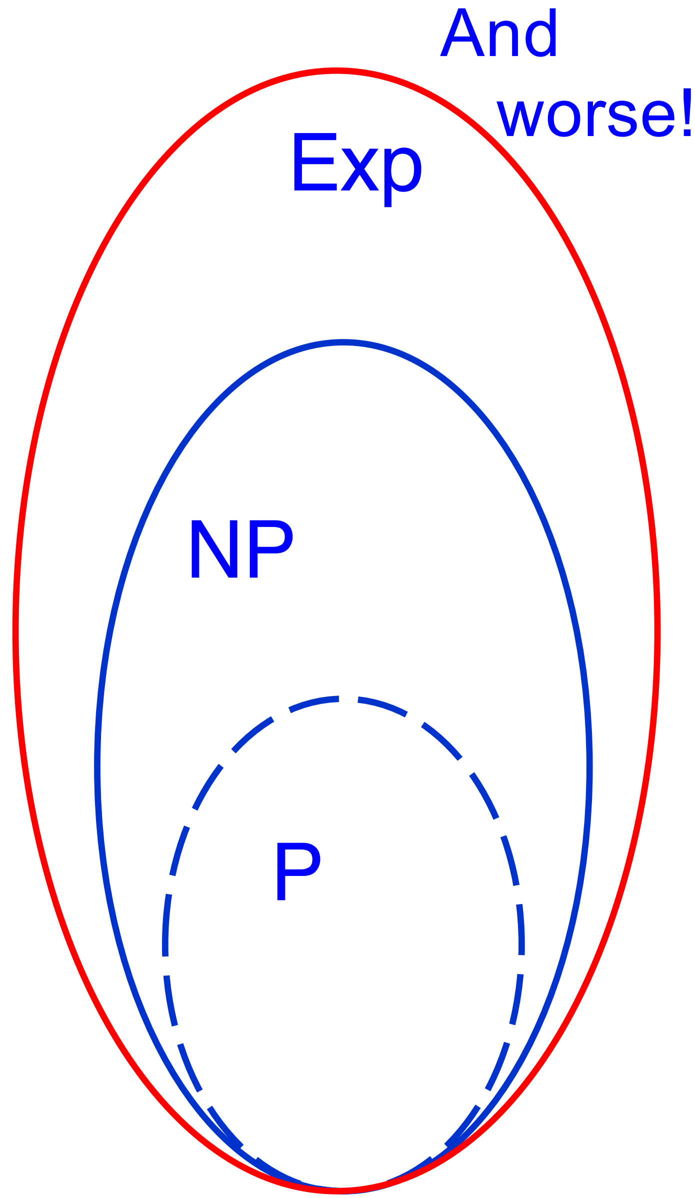
Roadmap

Not every problem is easy (in P)

Exponential time is bad

Worse things happen, too

There is a very commonly-seen class of problems, called *NP*, that *appear* to require exponential time (but unproven)



Review: Some Problems

Quadratic Diophantine Equations
Clique
Independent Set
Euler Tour
Hamilton Tour
TSP
3-Coloring
Partition
Satisfiability
Short Paths
Long Paths

All of the form: Given input X, is there a Y with property Z? Furthermore, if I had a purported Y, I could quickly test whether it had property Z

Common property of these problems: Discrete Exponential Search Loosely–find a needle in a haystack

"Answer" to a decision problem is literally just yes/no, but there's always a somewhat more elaborate "solution" (aka "hint" or "certificate"; what the search version would report) that *transparently*‡ justifies each "yes" instance (and only those) – but it's *buried in an exponentially large search space of potential solutions.*

The decision problem just tell us whether or not a given structure satisfying a given property exists. The "solution" here is that actual structure. Given this "hint/ certificate", the solution structure we are looking for, we could easily find if it satisfies the property we are interested in

‡*Transparently* = verifiable in polynomial time

Defining NP: The Idea

NP consists of all decision problems where

You can verify the YES answers efficiently (in polynomial time) given a short (polynomial-size) hint

one among exponentially many; "know it when you see it"

And

No hint can fool your polynomial time verifier into saying YES for a NO instance

NOTE: everything in P is also in NP

Defining NP: formally

x is only in L if it is a YES answer to decision problem.

A decision problem L is in *NP* iff there is a polynomial time procedure $v(-,-)$, (the "verifier") and an integer k such that

for every $x \in L$ there is a "hint" h with $|h| \leq |x|^k$ such that $v(x,h)$ = YES and

For every YES element; every solution to the problem L

cannot fool our verifier into thinking a nonsolution is a solution

for every $x \notin L$ there is *no* hint h with $|h| \leq |x|^k$ such that $v(x,h)$ = YES

For every NO element / nonsolution

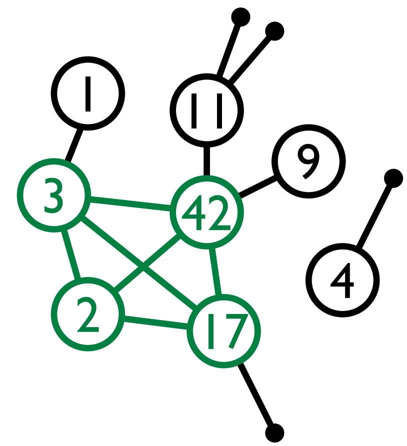
("Hints," sometimes called "certificates," or "witnesses", are just strings. Think of them as exactly what the search version would output.)

Note 1: a problem is "in NP" if it can be *posed* as an exponential search problem, even if there may be other ways to *solve* it.

NOTE: everything in P is also in NP; so in NP does not imply hard to solve.

Note 2: his definition is not quickly actionable without a way to find *h*.

Example: Clique



"Is there a *k*-clique in this graph?"

any subset of *k* vertices *might* be a clique

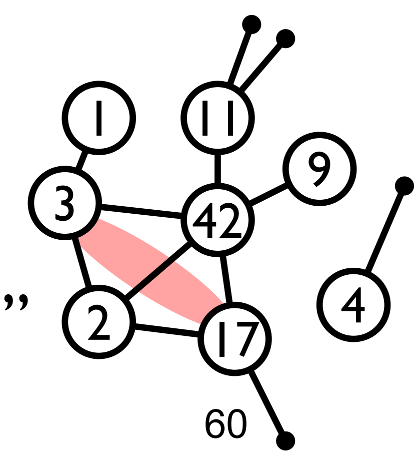
there are *many* such subsets, but I only need to find one

if I knew where it was, I could describe it succinctly, e.g. "look at vertices 2, 3, 17, 42, ...",

I'd know one if I saw one: "yes, there are edges between 2 & 3, 2 & 17,... so it's a *k*-clique"

this can be *quickly checked*

And if there is *no* *k*-clique, I wouldn't be fooled by a statement like "look at vertices 2, 3, 17, 42, ..."



More Formally: CLIQUE is in NP

procedure $v(x,h)$

if

x is a well-formed representation of a graph $G = (V, E)$ and an integer k,

and

h is a well-formed representation of a k-vertex subset U of V,

and

U is a clique in G,

then output "YES"

else output "I'm unconvinced"

Important note: this answer does NOT mean $x \notin$ CLIQUE; just means *this h* isn't a *k*-clique (but some other might be)

Is it correct?

For every $x = (G,k)$ such that G contains a k-clique, there is a hint h that will cause $v(x,h)$ to say YES, namely h = a list of the vertices in such a k-clique

and

No hint can fool v into saying yes if either x isn't well-formed (the uninteresting case) or if $x = (G,k)$ but G does not have any cliques of size k (the interesting case)

And $|h| < |x|$ and $v(x,h)$ takes time $\sim (|x|+|h|)^2$

Example: SAT

"Is there a satisfying assignment for this Boolean formula?"

- any assignment might work
- there are lots of them
- I only need one
- if I had one I could describe it succinctly, e.g., "x_1=T, x_2=F, ..., x_n=T"
- I'd know one if I saw one: "yes, plugging that in, I see formula = T..." and this can be quickly checked
- And if the formula is unsatisfiable, I wouldn't be fooled by , "x_1=T, x_2=F, ..., x_n=F"

More Formally: SAT $\in$ NP

Hint: the satisfying assignment A

Verifier: v(C, A) = syntax(C, A) && satisfies(C, A)

- Syntax: True iff C is a well-formed CNF formula & A is a truth-assignment to its variables
- Satisfies: plug A into C; check that it evaluates to True

Correctness:

- If C is satisfiable, it has some satisfying assignment A, and we'll recognize it
- If C is unsatisfiable, it doesn't, and we won't be fooled

Analysis: $|A| < |C|$, and time for v(C,A) ~ linear in $|C|+|A|$

IndpSet is in NP

procedure v(x,h)

if

x is a well-formed representation of a graph $G = (V, E)$ and an integer k,

and

h is a well-formed representation of a k-vertex subset U of V,

and

U is an Indp Set in G,

then output "YES"

else output "I'm unconvinced"

Important note: this answer does NOT mean $x \notin$ IndpSet; just means *this* h isn't a k-IndpSet (but some other might be)

Is it correct?

For every $x = (G,k)$ such that G contains a k-IndpSet, there is a hint h that will cause $v(x,h)$ to say YES, namely h = a list of the vertices in such a set

and

No hint can fool v into saying yes if either x isn't well-formed (the uninteresting case) or if $x = (G,k)$ but G does not have any Indp Set of size k (the interesting case)

And $|h| < |x|$ and $v(x,h)$ takes time $\sim (|x|+|h|)^2$

Example: Quad Diophantine Eqns

"Is there an integer solution to this equation?"

any pair of integers *x* & *y* might be a solution

there are lots of potential pairs

I only need to find one such pair

if I knew a solution, I could easily describe it, e.g. "try *x*=42 and *y* = 321" [A slight subtlety here: some algebra will show that if there's any int solution, there's one involving ints with only polynomially many digits...]

I'd know one if I saw one: "yes, plugging in 42 for *x* & 321 for *y* I see ..."

And wouldn't be fooled by (42,321) if there's no solution

Short Path

"Is there a short path (< *k*) from *s* to *t* in this graph?"

Any path might work

There are lots of them

I only need one

If I knew one I could describe it succinctly, e.g., "go from *s* to node 2, then node 42, then ... "

I'd know one if I saw one: "yes, I see there's an edge from *s* to 2 and from 2 to 42... and the total length is < *k*"

And if there isn't a short path, I wouldn't be fooled by, e.g., "go from *s* to node 2, then node 42, then ... "

Long Path

"Is there a long (acyclic) path (> *k*) from *s* to *t* in this graph?"

Any path might work

There are lots of them

I only need one

If I knew one I could describe it succinctly, e.g., "go from *s* to node 2, then node 42, then ... "

I'd know one if I saw one: "yes, I see there's an edge from *s* to 2 and from 2 to 42..., no dups, & total length is > *k*"

And if there isn't a long path, I wouldn't be fooled by, e.g., "go from *s* to node 2, then node 42, then ... "

Keys to showing that a problem is in NP

What's the output? (must be YES/NO)

What's the input? Which are YES?

For every given YES input, is there a hint that would help, i.e. allow verification in polynomial time? Is it polynomial length?

OK if some inputs need no hint

For any given NO input, is there a hint that would trick you?

Two Final Points About "Hints"

1. Hints/verifiers aren't unique. The "… there is a …" framework often suggests their form, but many possibilities

   "is there a clique" could be verified from its vertices, or its edges, or all but 3 of each, or all non-vertices, or… Details of the hint string, the verifier and its time bound all shift, but same bottom line.

2. In NP doesn't prove its hard

   "Short Path" or "Small Spanning Tree" or "Large Flow" can be formulated as "…there is a…," but, due to very special structure of these problems, we can quickly find the solution even without a hint. The mystery is whether that's possible for the other problems, too.

Contrast: problems *not* in NP (probably)

Rather than "there is a…" maybe it's
"*no*…" or "*for all*…" or "*the smallest/largest…*"

E.g.

UNSAT: "*no* assignment satisfies formula," or
"*for all* assignments, formula is false"

Or

NOCLIQUE: "*every* subset of k vertices is not a k-clique"

MAXCLIQUE: "the largest clique has size k"

Unlikely that a single, short hint is sufficiently informative to allow poly time verification of properties like these (but this is also an important open problem).

Another Contrast: *Mostly* Long Paths

"Are the *majority* of paths from *s* to *t* long (>*k*)?"

Any path might work

Yes! → There are lots of t…

I only need on…

If I knew or… r
succinct… o node
2, the…
I'd … one: "yes, I
see a… to 2 and from
2 to 42… al length > k"

And if there… sn't a long path, I wouldn't be fooled …

This problem is not believed to be in NP; probably *harder*

No, this is a *collective* property of the set of all paths in the graph, and no one path overrules the rest

Problems in P can also be verified in polynomial-time

Short Path: Given a graph G with edge lengths, is there a path from s to t of length $\leq$ k?

- Verify: Given a purported path from s to t, is it a path, is its length $\leq$ k?

Small Spanning Tree: Given a weighted undirected graph G, is there a spanning tree of weight $\leq$ k?

- Verify: Given a purported spanning tree, is it a spanning tree, is its weight $\leq$ k?

(But the hints aren't really needed in these cases…)

NP-completeness

NP-Completeness

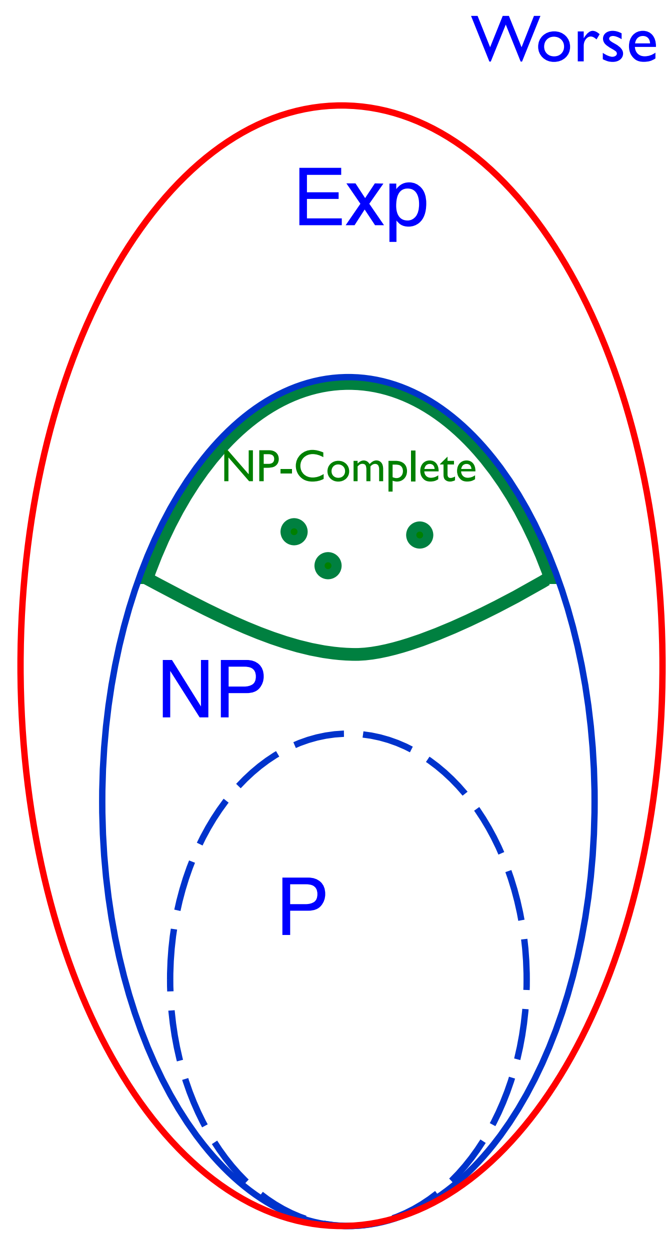
Definition: Problem B is *NP-complete* if:

(1) B belongs to NP, and

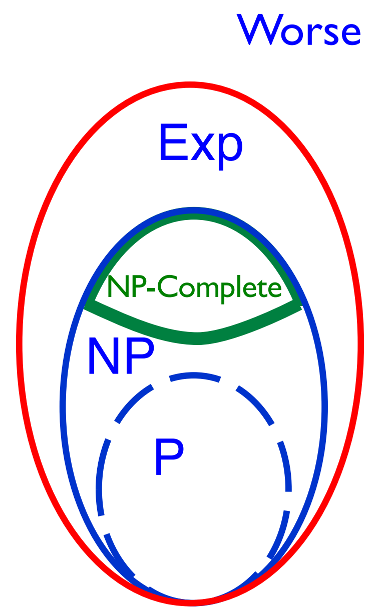
(2) every problem in NP is polynomially reducible to B.

Intuitively, these are the "hardest problems" in NP

They are also all deeply related–solving any solves them all!



NP-completeness (cont.)



Thousands of important problems have been shown to be NP-complete.

The general belief is that there is no efficient algorithm for any NP-complete problem, but no proof of that belief is known.

Examples: SAT, clique, vertex cover, IndpSet, Ham tour, TSP, bin packing… Basically, everything we've seen that's in NP but not known to be in P

Alt way to prove NP-completeness

After showing that the first problem is NP-complete, we can just show that each NP problem is complete iff we can reduce another NP-complete problem to it.

Lemma: Problem B is NP-complete iff:

(1) B belongs to NP, and

just need one!

(2') Some NP-complete problem A is polynomial-time reducible to B.

i.e. show that 3SAT <= B

That is, to show NP-completeness of a new problem B in NP, it suffices to show that SAT or any other NP-complete problem is polynomial-time reducible to B.

Ex: IndpSet is NP-complete

3-SAT is NP-complete (S. Cook; see below)

we can reduce 3SAT to IndpSet in poly time

$\left.\begin{array}{l}\text{3-SAT} \le_p \text{IndpSet} \\ \text{IndpSet is in NP}\end{array}\right\}$ we showed these earlier

Therefore IndpSet is also NP-complete

So, poly-time algorithm for IndpSet would give poly-time algs for *everything* in NP

Ditto for KNAP, 3COLOR, …

Cook's Theorem

SAT is NP-Complete

Cook’s Theorem

Theorem: Every problem in NP is reducible to SAT

Proof Sketch: SAT assignment = hint; formula = verifier.

hint = input that specifies possible solution to problem

Generic “NP” problem: is there a poly size “hint,” verifiable in poly time

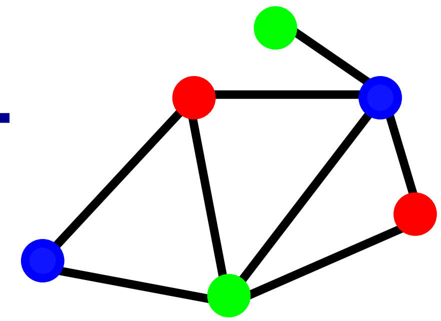
Encode “hint” using Boolean variables. SAT mimics “is there a hint” via “is there an assignment”. The “verifier” runs on a digital computer, and digital computers just do Boolean logic. “SAT” can mimic that, too, hence can verify that the assignment *actually* encodes a hint the verifier would accept.

“SAT”: is there an assignment (the hint) satisfying the formula (the verifier)

Pf uses *generic* NP problems, but a few specific examples will give the flavor

SAT is NP-complete, so I should be able to reduce any NP problem to SAT in polynomial time

3-Coloring $\leq_p$ SAT



Given G = (V, E)

$\forall$ i in V, variables r_i, g_i, b_i encode color of i ← hint

Color the vertex EXACTLY one color

$$\bigwedge_{i \in V} [(r_i \vee g_i \vee b_i) \wedge$$
$$(\neg r_i \vee \neg g_i) \wedge (\neg g_i \vee \neg b_i) \wedge (\neg b_i \vee \neg r_i)] \wedge$$
$$\bigwedge_{(i,j) \in E} [(\neg r_i \vee \neg r_j) \wedge (\neg g_i \vee \neg g_j) \wedge (\neg b_i \vee \neg b_j)]$$

← verifier

adjacent vertices not both red

adj nodes ⇔ diff colors
no node gets 2
every node gets a color

Equivalently:
$(\neg(r_i \wedge g_i)) \wedge (\neg(g_i \wedge b_i)) \wedge (\neg(b_i \wedge r_i)) \wedge$
$\bigwedge_{(i,j) \in E} [(r_i \Rightarrow \neg r_j) \wedge (g_i \Rightarrow \neg g_j) \wedge (b_i \Rightarrow \neg b_j)]$

Independent Set $\leq_p$ SAT

Given G = (V, E) and k

$\forall$ i in V, variable x_i encodes inclusion of i in IS ← hint

$$\underbrace{\bigwedge_{(i,j)\in E}(\neg x_i \vee \neg x_j)}_{\substack{\text{every edge has one end} \\ \text{or other not in IS} \\ \text{(no edge connects 2 in IS)}}} \wedge \underbrace{\text{“number of True } x_i \text{ is} \geq k\text{”}}_{\substack{\text{possible in 3 CNF, but technically} \\ \text{messy, so details omitted;} \\ \text{basically, count 1's}}} \leftarrow \text{verifier}$$

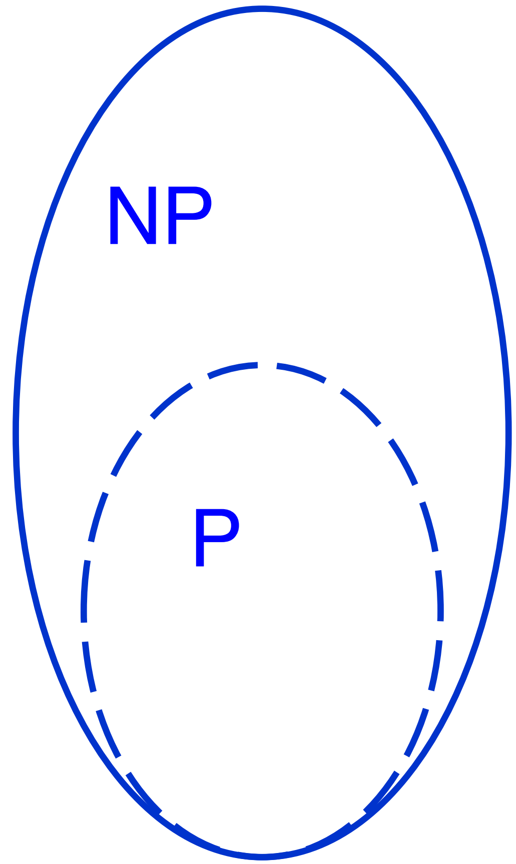
Relating P to NP

Complexity Classes

NP = Polynomial-time
verifiable

P = Polynomial-time
solvable

P ⊆ NP: "verifier" is
just the P-time alg;
ignore "hint"



Solving NP problems without hints

The most obvious algorithm for most of these problems is brute force:

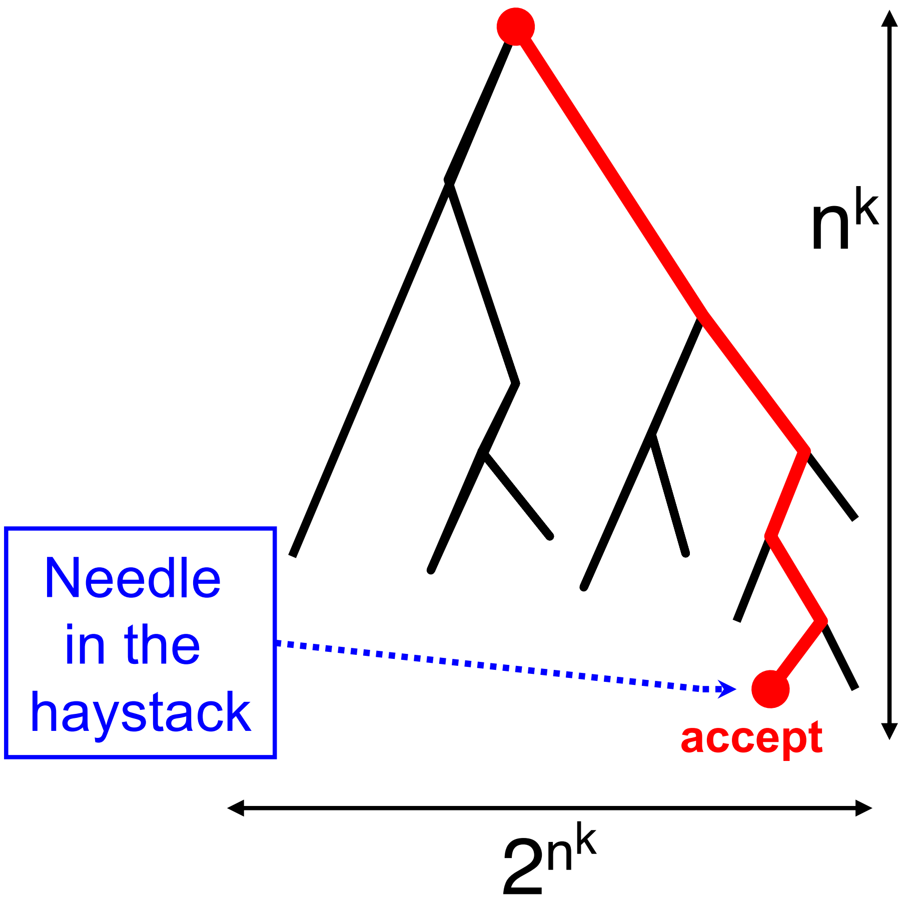
- try all possible hints; check each one to see if it works.
- Exponential time:
  - 2^n truth assignments for n variables
  - n! possible TSP tours of n vertices
  - $\binom{n}{k}$ possible k element subsets of n vertices, perhaps k = log n or n/3
  - etc.

…and to date, every alg, even much less-obvious ones, are slow, too

P vs NP vs Exponential Time

Theorem: Every problem in NP can be solved (deterministically) in exponential time

Proof: “hints” are only n^k long; try all 2^{n^k} possibilities, say, by backtracking. If any succeed, answer YES; if all fail, answer NO.



P and NP

Every problem in P is in NP

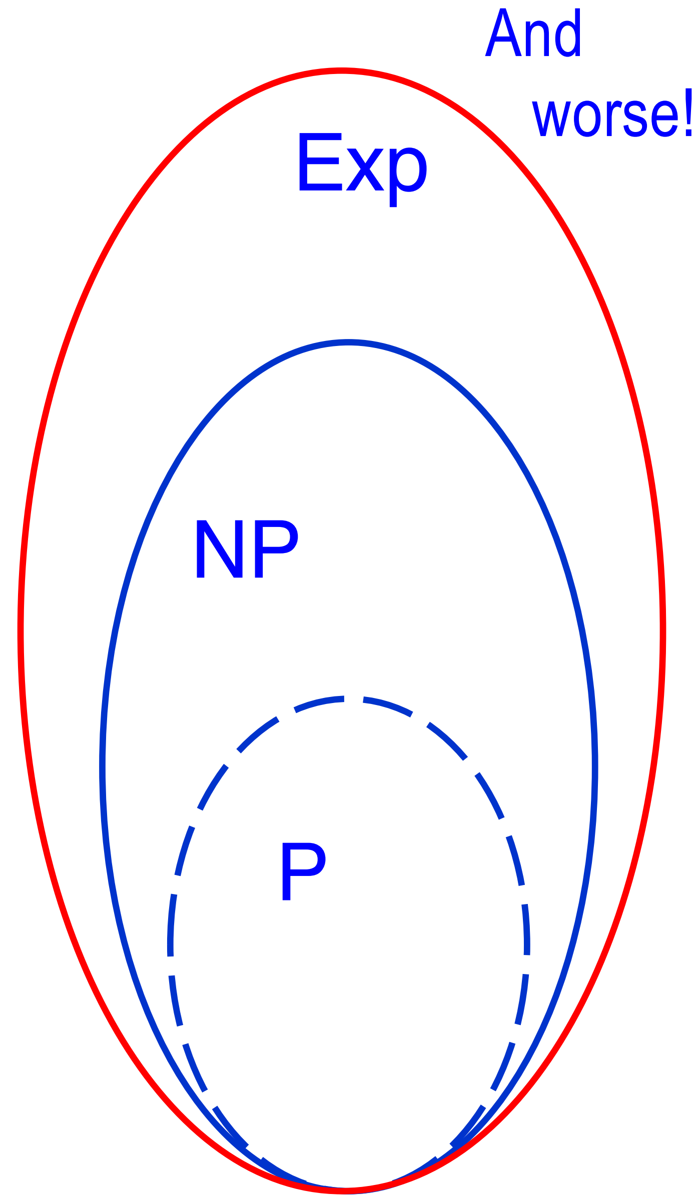
one doesn't even need a hint for problems in P so just ignore any hint you are given

Every problem in NP is in exponential time

I.e., $P \subseteq NP \subseteq Exp$

We know $P \neq Exp$, so either $P \neq NP$, or $NP \neq Exp$ (most likely both)

E.g., see CSE 431



Does P = NP?

This is the big open question!

To show that P = NP, we have to show that every problem that belongs to NP can be solved by a polynomial time deterministic algorithm.

Would be very cool, but no one has shown this yet.

(And it seems unlikely to be true.)

(Also seems daunting: there are infinitely many problems in NP; do we have to pick them off one at a time…?)

Does P = NP?

This is the big open question!

To show that P = NP, we have to show that every problem that belongs to NP can be solved by a polynomial time deterministic algorithm.

Would be very cool, but no one has shown this yet.

(And it seems unlikely to be true.)

More History – As of 1970

Many of the above problems had been studied for decades

All had real, practical applications

None had poly time algorithms; exponential was best known

But, it turns out they all have a very deep similarity under the skin

Some Problem Pairs

Euler Tour	Hamilton Tour
2-SAT	3-SAT
2-Coloring	3-Coloring
Min Cut	Max Cut
Shortest Path	Longest Path

Superficially different; similar computationally

Similar pairs; seemingly different computationally

Polynomial Time Reduction, III

Two definitions of “A $\leq_p$ B”

Book uses general definition: “could solve A in poly time, if I had a poly time *subroutine* for B.” Cook

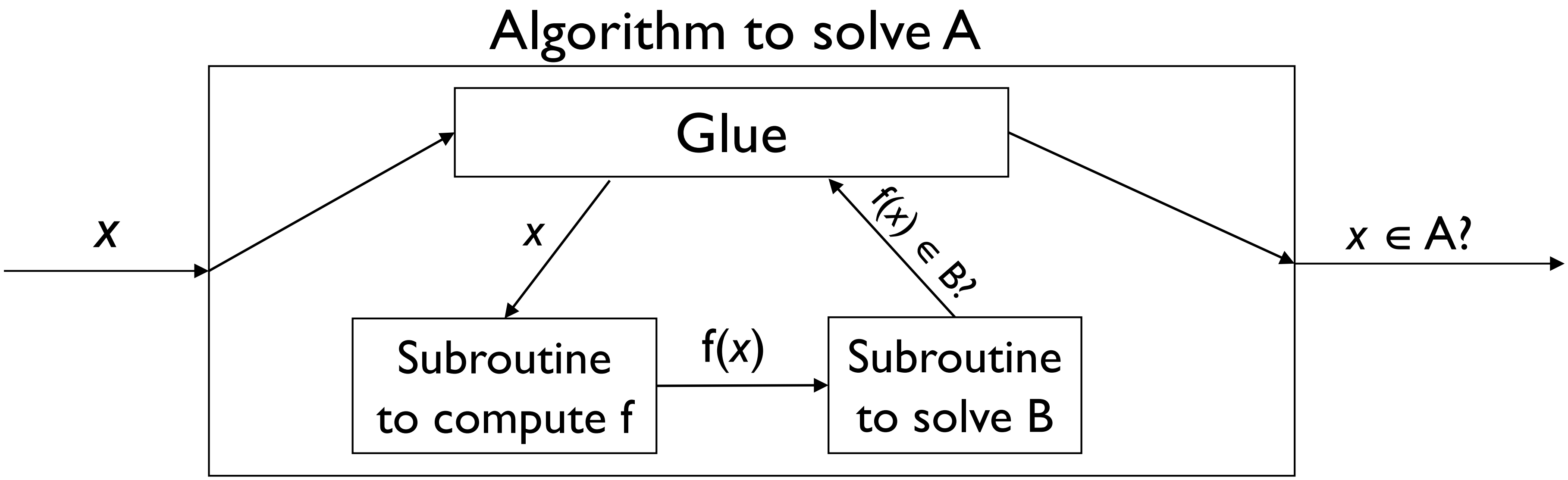
Examples on previous slides are special case:

- call the subroutine *once*, report *its* answer. Karp

This special case is used in ~98% of all reductions

Largely irrelevant for this course, but if you seem to need 1st defn, e.g. on HW, fine, but there’s perhaps a simpler way…

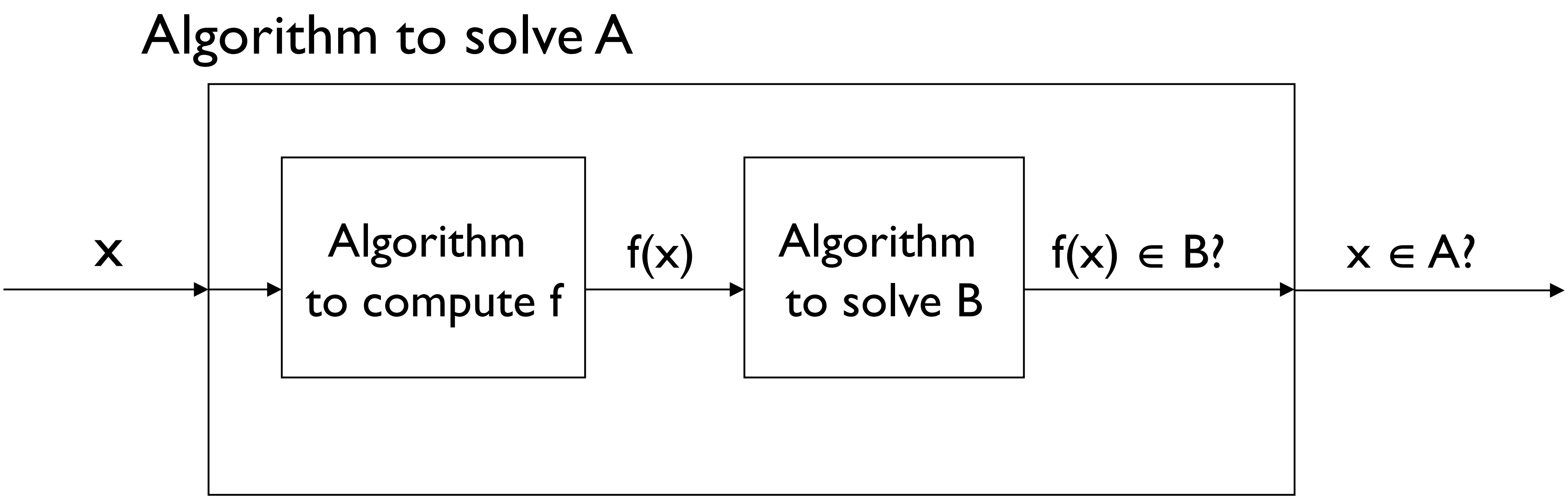
Using an Algorithm for *B* to Solve *A*



"If A $\leq_P$ B, and we can solve B in polynomial time, then we can solve A in polynomial time also."

Key issue: Can we (quickly) turn an A-instance x into one (or more) B-instance(s) f(x) so that answer(s) to "f(x) $\in$ B" help us decide x $\in$ A"?

Using an Algorithm for *B* to Solve *A*



"If A $\leq_P$ B, and we can solve B in polynomial time, then we can solve A in polynomial time also."

Ex: suppose f takes $O(n^3)$ and algorithm for B takes $O(n^2)$.
How long does the above algorithm for A take?

P vs NP

Theory

- P = NP ?
- Open Problem!
- I bet against it

Practice

- Many interesting, useful, natural, well-studied problems known to be NP-complete
- With rare exceptions, no one routinely finds exact solutions to large, arbitrary instances

P vs NP: Summary so far

P = "poly time solvable"

NP = "poly time verifiable" *(nondeterministic poly time solvable)*

Defined only for *decision* problems, but fundamentally about *search*: can cast *many* problems as searching for a poly size, poly time verifiable "solution" in a 2^{poly} size "search space."

Examples:

- is there a big clique? Space = all big subsets of vertices; solution = one subset; verify = check all edges
- is there a satisfying assignment? Space = all assignments; solution = one asgt; verify = eval formula

Sometimes we can do that quickly (is there a small spanning tree?); P = NP would mean we could *always* do it quickly.

NP: Yet to come

NP-Completeness: the “hardest” problems in NP.

Surprisingly, most known problems in NP are equivalent, in a strong sense, despite great superficial differences.

Reductions: key to showing those facts.

More Reductions

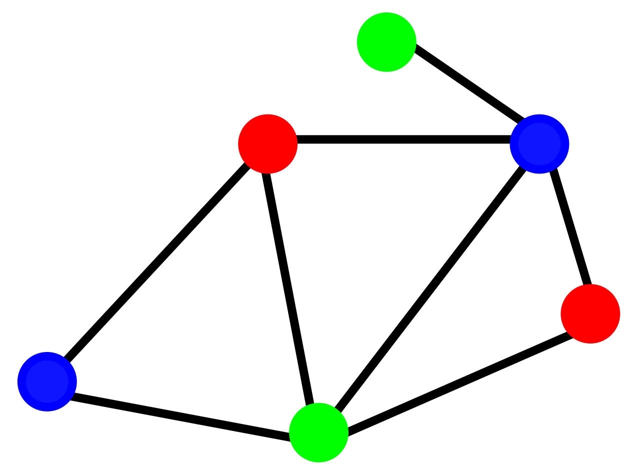
SAT to Coloring

NP-complete problem: 3-Coloring

Input: An undirected graph G=(V,E).

Output: True iff there is an assignment of at most 3 colors to the vertices in G such that no two adjacent vertices have the same color.

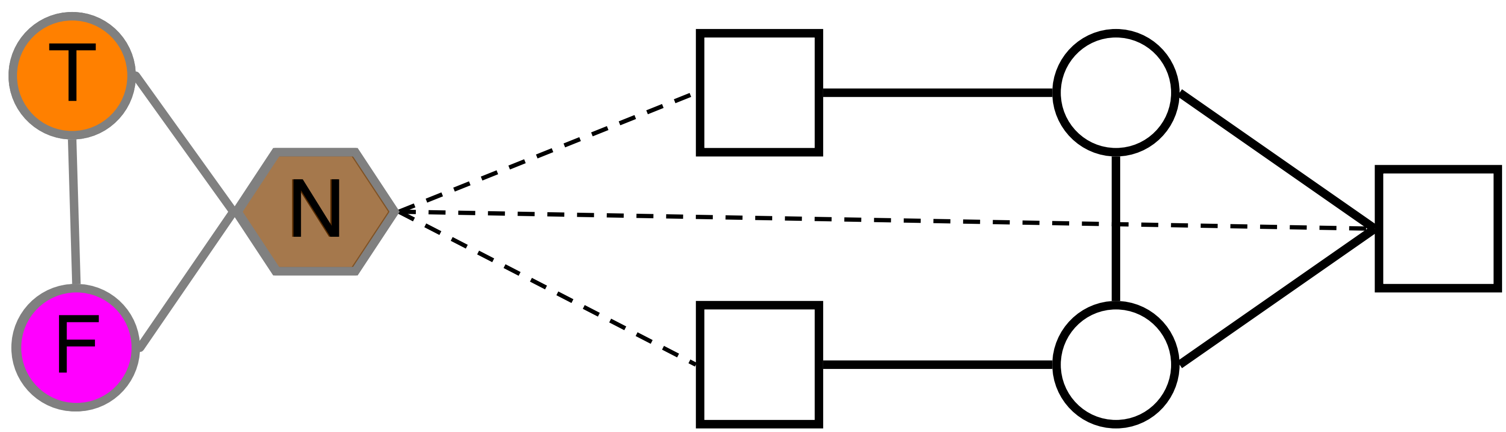
Example:



In NP? Exercise

A 3-Coloring Gadget:

In what ways can this be 3-colored?



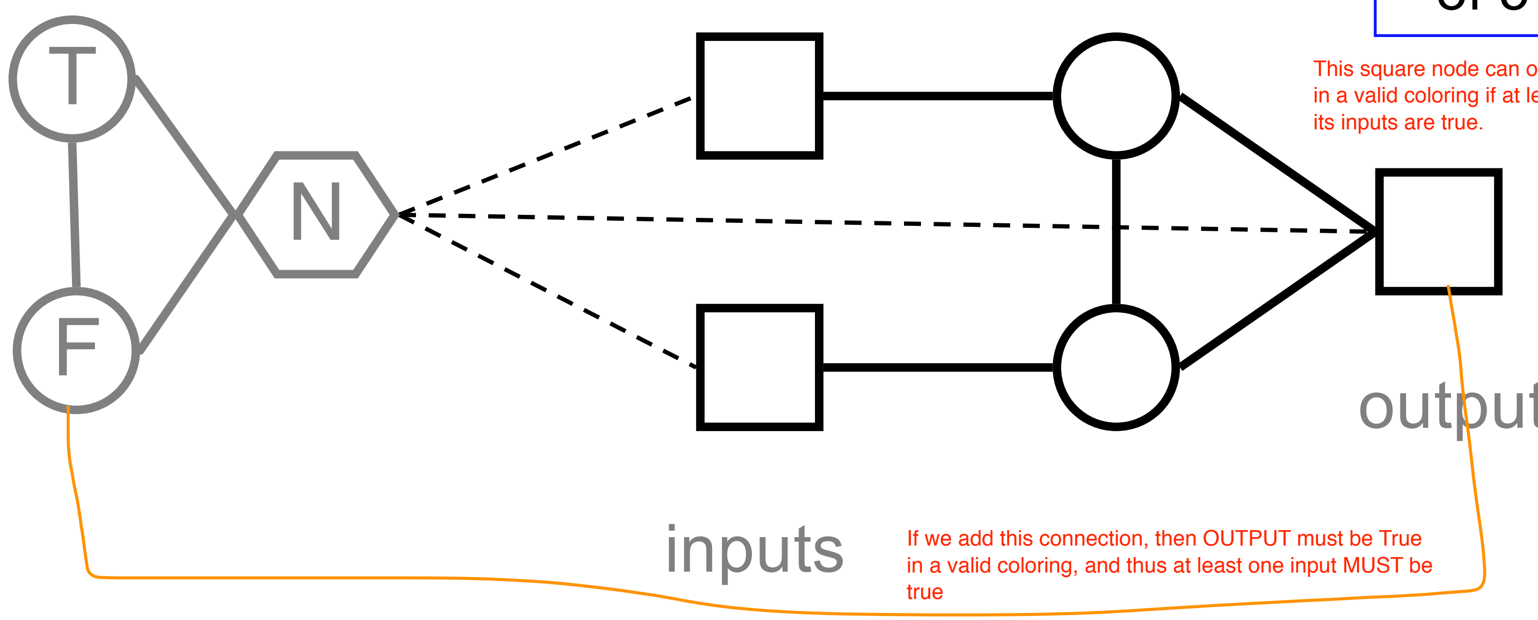
A 3-Coloring Gadget: “Sort of an OR gate“

This square node can only be true in a valid coloring if at least one of its inputs are true.

if output is T, some input must be T

if some input is T, output may be T

Exercise: find all colorings of 5 nodes



NB: this is *not* the same gadget as used in KT 8.7

$3SAT \leq_p 3Color$

f (**3-SAT Instance:**

- Variables: $x_1, x_2, \ldots$
- Literals: $y_{i,j}$, $1 \leq i \leq q$, $1 \leq j \leq 3$
- Clauses: $c_i = y_{i1} \vee y_{i2} \vee y_{i3}$, $1 \leq l \leq q$
- Formula: $c = c_1 \wedge c_2 \wedge \ldots \wedge c_q$

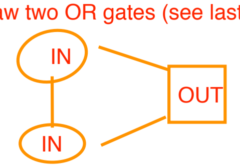
) =

3Color Instance:

- G = (V, E)
- 6 q + 2 n + 3 vertices
- 13 q + 3 n + 3 edges
- (See Example for details)

Construction

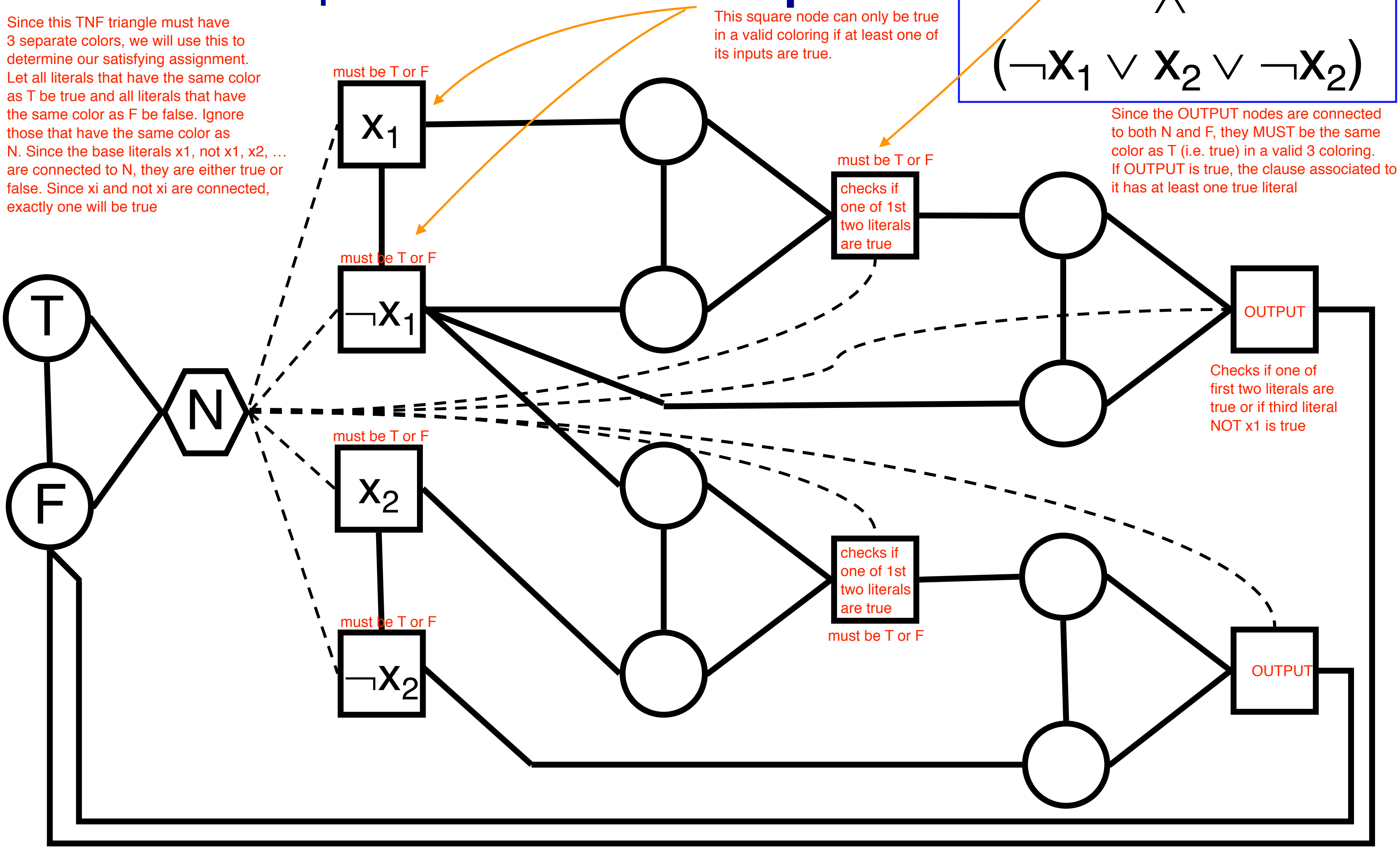
For each variable x1, x2, …, xn make a node for x1, NOT x1, …, xn, NOT xn
Draw an edge between xi and NOT xi, between xi and N, and between NOT xi and N.
For each clause draw two OR gates (see last slide)



Connect each OUT gate to both N and F.
For each clause, connect the IN nodes of one OR gate to the first two literals in the clause. For the other OR gate, have connect one IN node to the output of the first OR gate and the other IN to the third literal.

3SAT $\leq_p$ 3Color Example

$$(x_1 \vee \neg x_1 \vee \neg x_1)$$
$$\wedge$$
$$(\neg x_1 \vee x_2 \vee \neg x_2)$$



6 q + 2 n + 3 vertices

13 q + 3 n + 3 edges

Correctness of "3SAT $\leq_p$ 3Coloring"

<u>Summary of reduction function f</u>:
Given formula, make G with T-F-N triangle, 1 pair of literal nodes per variable, 2 "or" gadgets per clause, connected as in example.
Note: *again, f does not know or construct satisfying assignment or coloring.*

<u>Correctness</u>:

- Show f poly time computable: A key point is that graph size is polynomial in formula size; graph looks messy, but pattern is basically straightforward.
- Show c in 3-SAT iff f(c) is 3-colorable:

($\Rightarrow$) Given an assignment satisfying c, color literals T/F as per assignment; can color "or" gadgets so output nodes are T since each clause is satisfied.
($\Leftarrow$) Given a 3-coloring of f(c), name colors T-N-F as in example. All square nodes are T or F (since all adjacent to N). Each variable pair (x_i, $\neg x_i$) must have complementary labels since they're adjacent. Define assignment based on colors of x_i's. Clause "output" nodes must be colored T since they're adjacent to both N & F. By fact noted earlier, output can be T only if at least one input is T, hence it is a satisfying assignment.

Coping with NP-hardness

Coping with NP-Hardness

- Is your real problem a special subcase?
  - E.g. 3-SAT is NP-complete, but 2-SAT is not; ditto 3- vs 2-coloring
  - E.g. only need planar-/interval-/degree 3 graphs, trees,…?
- Guaranteed approximation good enough?
  - E.g. Euclidean TSP within 1.5 * Opt in poly time
- Fast enough in practice (esp. if n is small),
  - E.g. clever exhaustive search like dynamic programming, backtrack, branch & bound, pruning
- Heuristics – usually a good approx and/or fast

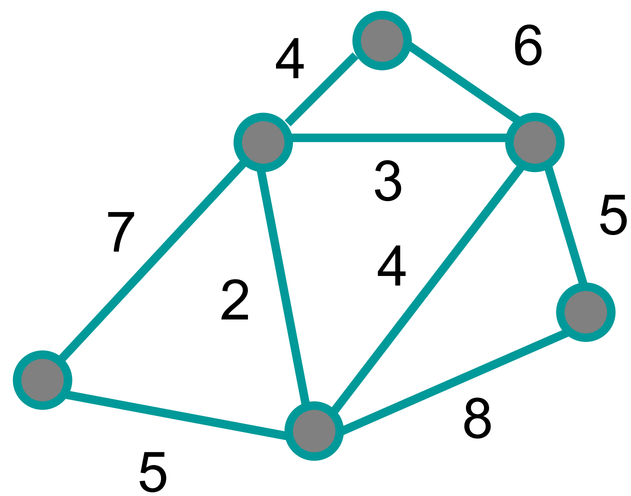
NP-complete problem: TSP

Input: An undirected graph G=(V,E) with integer edge weights, and an integer b.

Output: YES iff there is a simple cycle in G passing through all vertices (once), with total cost ≤ b.

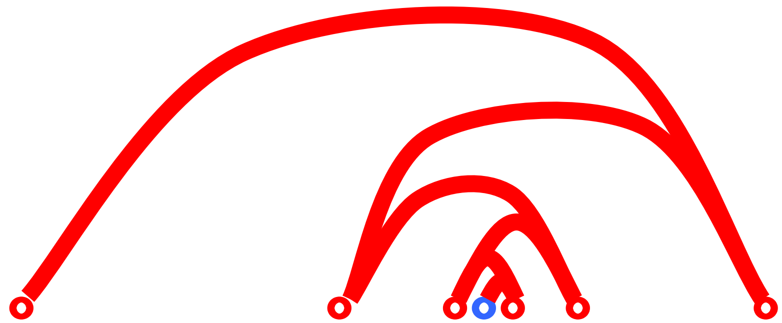
Example:

b = 34



TSP - Nearest Neighbor Heuristic

Recall NN Heuristic–go to nearest unvisited vertex



Fact: NN tour can be about (log n) x opt, i.e.

$$\lim_{n \to \infty} \frac{NN}{OPT} \to \infty$$

(above example is not that bad)

2x Approximation to EuclideanTSP

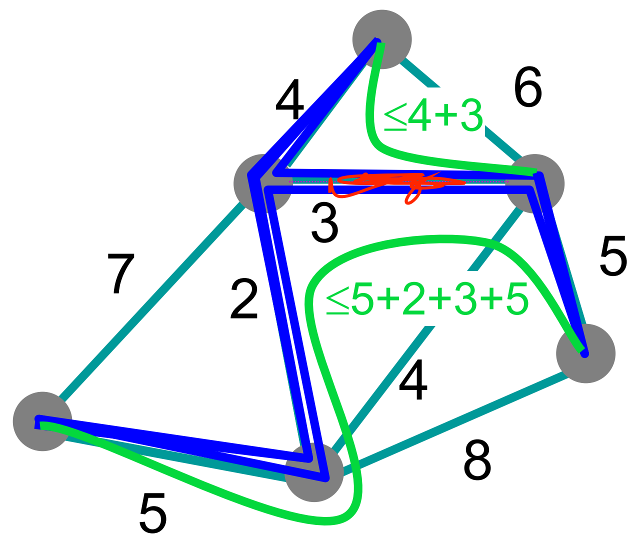
n points in space, Euclidean distance, all possible edges; example omits edges for clarity

A TSP tour visits all vertices, so contains a spanning tree, so cost of min spanning tree < TSP cost.

Find MST

Find “DFS” Tour

Shortcut



TSP ≤ shortcut < DFST = 2 * MST < 2 * TSP

P / NP Summary

P

Many important problems are in P: solvable in deterministic polynomial time

- Details are the fodder of algorithms courses. We've seen a few examples here, plus many other examples in other courses

Few problems *not* in P are routinely solved;

- For those that are, practice is usually restricted to small instances, or we're forced to settle for approximate, suboptimal, or heuristic "solutions"

A major goal of complexity theory is to delineate the boundaries of what we can feasibly solve

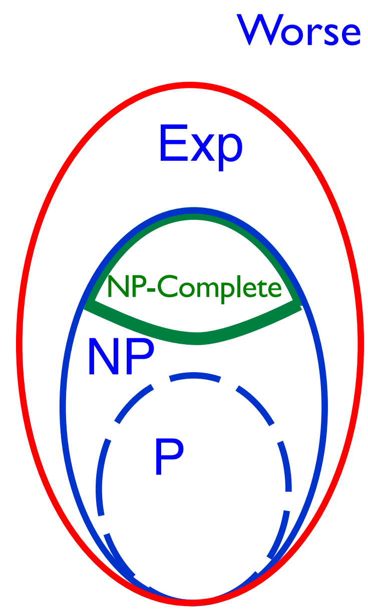
NP

The tip-of-the-iceberg in terms of problems conjectured not to be in P, but a very important tip, because

a) they're very commonly encountered, probably because

b) they arise naturally from basic "search" and "optimization" questions.

Definition: poly time verifiable;
"guess and check", "is there a…" – are also useful views



NP-completeness

Defn & Properties of $\leq_p$

A is NP-complete: in NP & everything in NP reducible to A

“the hardest problems in NP”

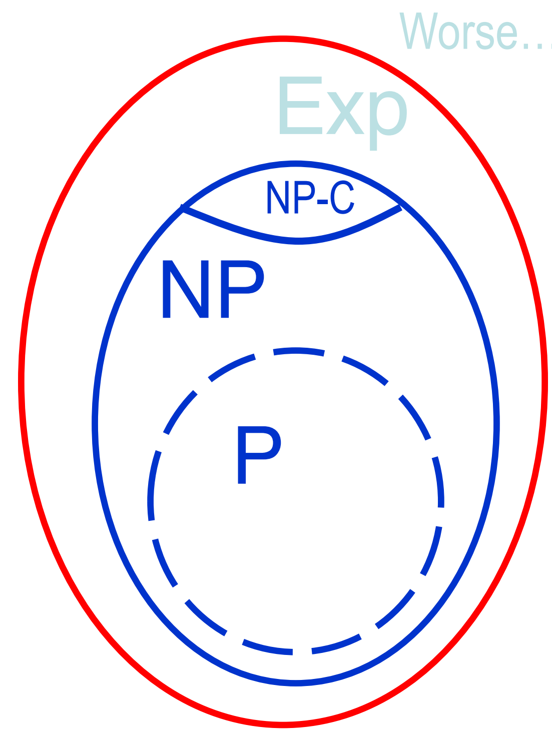
“All alike under the skin”

Most known natural problems in NP are complete

#1: 3CNF-SAT

Many others: Clique, IndpSet, 3Color, KNAP, HamPath, TSP, …

Summary



Big-O – good

P – good

Exp – bad

Exp, but hints help? NP

NP-hard, NP-complete – bad (I bet)

To show NP-complete – reductions

NP-complete = hopeless? – no, but you need to lower your expectations: heuristics, approximations and/or small instances.

Common Errors in NP-completeness Proofs

Backwards reductions

Bipartiteness $\leq_p$ SAT is true, but not so useful.
(XYZ $\leq_p$ SAT shows XYZ in NP, doesn't show it's hard.)

Sloooow Reductions

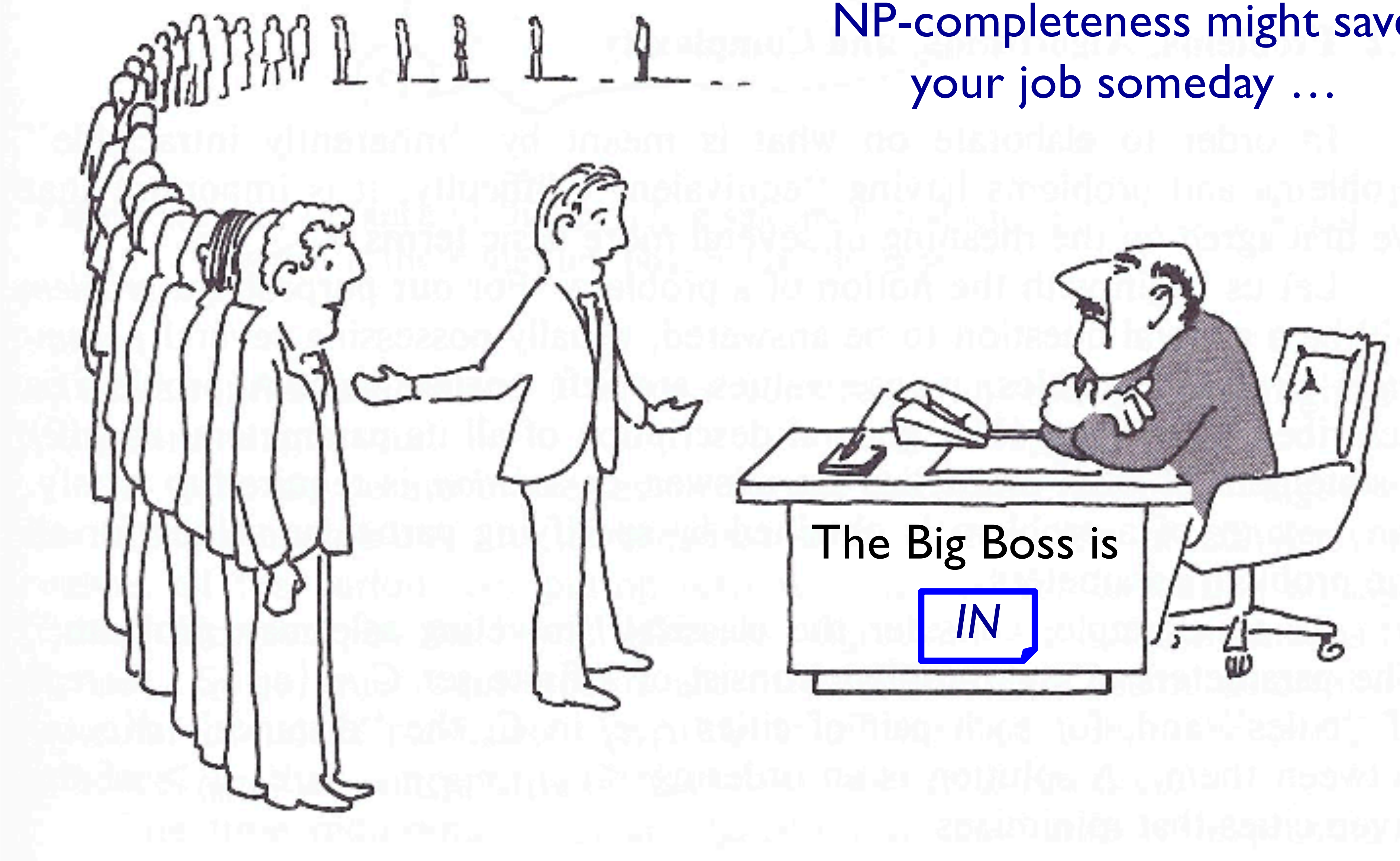
"Find a satisfying assignment, then output…"

Half Reductions

E.g., delete dashed edges in 3Color reduction. It's still true that "c satisfiable $\Rightarrow$ G is 3 colorable", but 3-colorings don't necessarily give satisfying- (or valid) assignments.

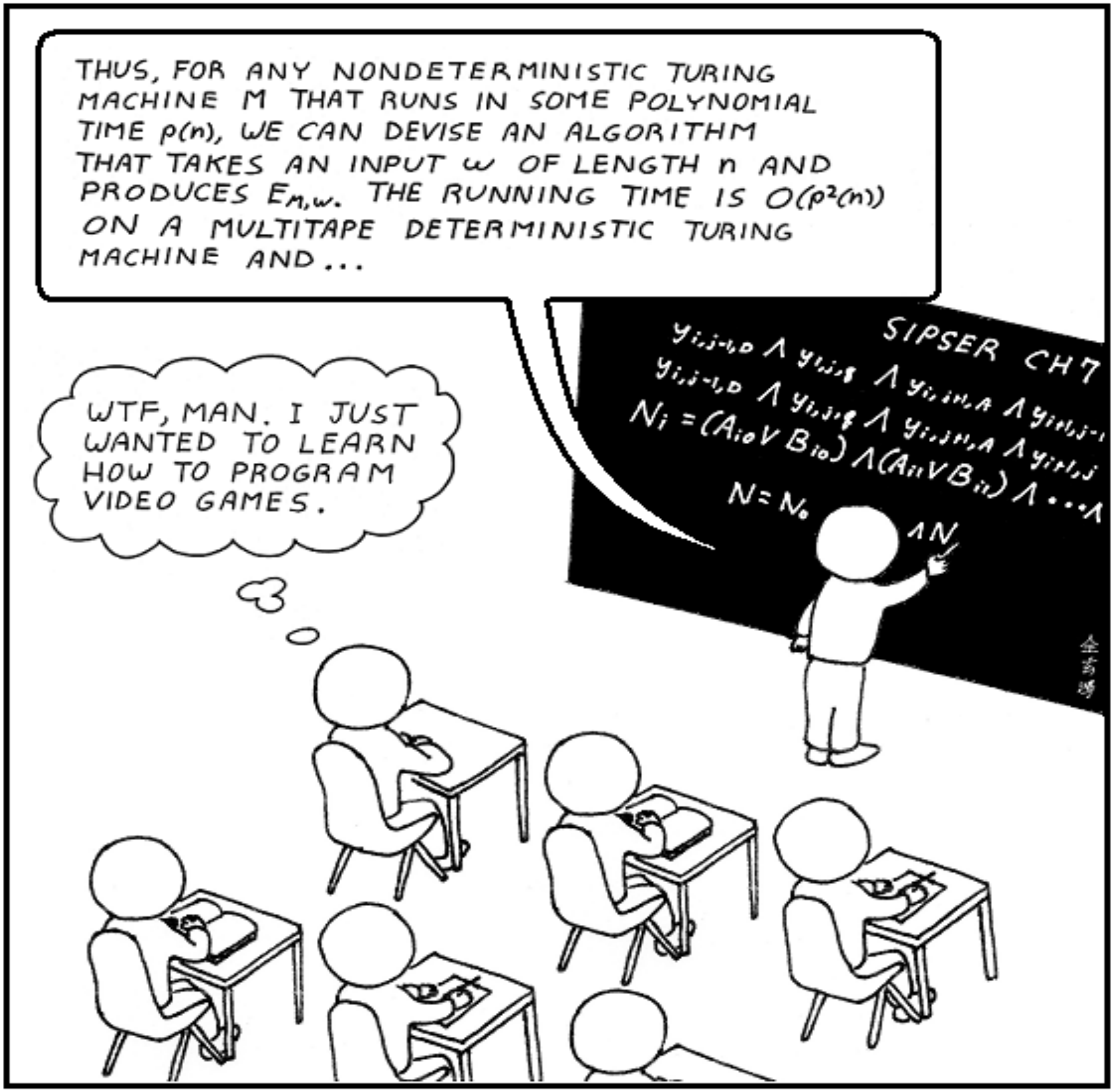
E.g., add or delete slacks in KNAP: similar troubles

NP-completeness might save your job someday …



"I can't find an efficient algorithm, but neither can all these famous people."

[Garey & Johnson, 1979]


THUS, FOR ANY NONDETERMINISTIC TURING MACHINE M THAT RUNS IN SOME POLYNOMIAL TIME $p(n)$, WE CAN DEVISE AN ALGORITHM THAT TAKES AN INPUT w OF LENGTH n AND PRODUCES $E_{M,w}$. THE RUNNING TIME IS $O(p^2(n))$ ON A MULTITAPE DETERMINISTIC TURING MACHINE AND...
WTF, MAN. I JUST WANTED TO LEARN HOW TO PROGRAM VIDEO GAMES.
SIPSER CH7