

# Seifert–Van Kampen for $n$ -fold Unions

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The size of a non-disjoint union of sets is computed using an *inclusion–exclusion principle*:  $|A \cup B| = |A| + |B| - |A \cap B|$ . You can’t just add the sizes of the sets, because that would be double-counting the overlap  $A \cap B$ . You have to subtract the size of the intersection.

The fundamental group of a non-disjoint union of path-connected topological spaces (whose intersection is also path-connected) is computed using the Seifert–Van Kampen theorem:  $\pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$ . You can’t just take the free product, because that would “double-count” the loops in the overlap  $U \cap V$ . You need to take the amalgamated free product.

How far can we push this analogy? There is a generalized inclusion–exclusion principle for  $n$ -fold unions, proved by induction:

$$\left| \bigcup_{i=1}^n A_i \right| = \sum_{j=1}^n (-1)^{j+1} \left( \sum_{S \in \mathcal{P}(\{A_i\}_{i=1}^n); |S|=j} \left| \bigcap_{A_s \in S} A_s \right| \right)$$

Is there a generalized Sifert–van Kampen theorem for  $n$ -fold unions?

Let’s start with  $n := 3$ .

For set unions, the  $n := 3$  inclusion–exclusion principle is  $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$ .

The way we derive that is with a single induction step, by applying the  $n := 2$  inclusion exclusion principle to itself:

$$\begin{aligned} |(A \cup B) \cup C| &= \underbrace{|A \cup B|}_{n:=2 \text{ case}} + |C| - |(A \cup B) \cap C| \\ &= |A| + |B| - |A \cap B| + |C| - |(A \cup B) \cap C| \\ &= |A| + |B| + |C| - |A \cap B| - \underbrace{|(A \cap C) \cup (B \cap C)|}_{n:=2 \text{ case}} \\ &= |A| + |B| + |C| - |A \cap B| - (|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)|) \\ &= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C| \end{aligned}$$

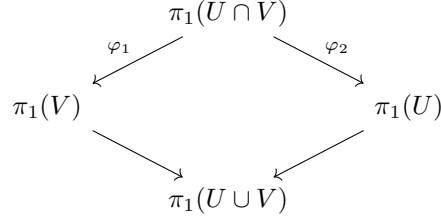
To understand if there’s an  $n := 3$  Seifert–van Kampen principle, I’ll need a more detailed understanding of the analogy between “subtracting the intersection” and amalgamation.

The free product  $G_1 * G_2$  combines the generators of  $G_1$  and  $G_2$  with no relations.

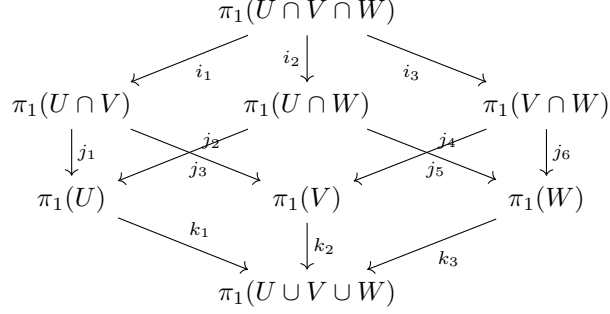
The fundamental group of joining two topological spaces at a point (or with the intersection being contractible) is the free product of the fundamental groups. If you go around one of the loops of a figure-eight, that doesn’t commute with going around the other loop.

The amalgamated free product  $G_1 *_H G_2$  over homomorphisms  $\varphi_1 : H \rightarrow G_1$  and  $\varphi_2 : H \rightarrow G_2$  combines the generators of  $G_1$  and  $G_2$  and adds relations: for all  $h \in H$ , we identify  $\varphi_1(h) = \varphi_2(h)$ .

In the application of amalgamated free products in the SvK theorem, we specifically have  $\pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$  with the inclusion homomorphisms. A loop-equivalence-class from  $\pi_1(U \cap V)$  should be “the same” whether it appears in  $\pi_1(U)$  or  $\pi_1(V)$ .



For three spaces, the diagram becomes:



SvK tells us that  $\pi_1(U \cup V) \simeq \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$ .

So presumably

$$\pi_1(U \cup V \cup W) = \pi_1((U \cup V) \cup W) \simeq \pi_1(U \cup V) *_{\pi_1((U \cup V) \cap W)} \pi_1(W)$$

$$\simeq \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) *_{\pi_1((U \cup V) \cap W)} \pi_1(W)$$

$$\simeq \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) *_{\pi_1((U \cap W) \cup (V \cap W))} \pi_1(W)$$

But then we have to unpack what it means to amalgamate over  $\pi_1((U \cap W) \cup (V \cap W))$ :

$$\pi_1((U \cap W) \cup (V \cap W)) \simeq \pi_1(U \cap W) *_{\pi_1(U \cap V \cap W)} \pi_1(V \cap W)$$

And substitute that back to get:

$$\pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) *_{\pi_1(U \cap W) *_{\pi_1(U \cap V \cap W)} \pi_1(V \cap W)} \pi_1(W)$$

which is unreadable, but hopefully correct.

It doesn't seem like the order of which amalgamated free products are nested within each other should matter, though: what matters is that we have all the relations in the diagram.

Suppose we have the presentations  $\pi_1(U) = \langle S_U | R_U \rangle$ ,  $\pi_1(V) = \langle S_V | R_V \rangle$ , and  $\pi_1(W) = \langle S_W | R_W \rangle$ .

For all  $h_{uvw} \in \pi_1(U \cap V \cap W)$ ,  $i_1(h_{uvw}) = i_2(h_{uvw}) = i_3(h_{uvw})$ . For all  $h_{uv} \in \pi_1(U \cap V)$ ,  $j_1(h_{uv}) = j_2(h_{uv})$ . For all  $h_{wv} \in \pi_1(U \cap V)$ ,  $j_3(h_{wv}) = j_4(h_{wv})$ . For all  $h_{uw} \in \pi_1(U \cap V)$ ,  $j_5(h_{uw}) = j_6(h_{uw})$ .

Putting that all together, we have

$$\pi_1(U \cup V \cup W) \simeq \langle S_U, S_V, S_W | R_U, R_W, R_W, \$$

$$\forall h_{uvw} \in \pi_1(U \cap V \cap W) \ i_1(h_{uvw}) = i_2(h_{uvw}) = i_3(h_{uvw}),$$

$$\forall h_{uv} \in \pi_1(U \cap V) \ j_1(h_{uv}) = j_2(h_{uv})$$

$$\forall h_{wv} \in \pi_1(V \cap W) \ j_3(h_{wv}) = j_4(h_{wv})$$

$$\forall h_{uw} \in \pi_1(U \cap W) j_5(h_{uw}) = j_6(h_{uw})$$

Gemini 2.5 Pro claims that we don't actually need the triple-intersection relations because they're implied by the pairwise relations by transitivity, so this is actually simpler than the alternating sums of the inclusion-exclusion principle.

We need examples.

**Example 1.** What about three annuluses, where  $U$  overlaps  $V$ , and  $V$  overlaps  $W$ ? We can see that the union should just be a thicker annulus (or our proposed generalization should compute the same result formally). We have  $\pi_1(U) \simeq \mathbb{Z} \simeq \langle u | \emptyset \rangle$ ,  $\pi_1(V) \simeq \mathbb{Z} \simeq \langle v | \emptyset \rangle$ ,  $\pi_1(W) \simeq \mathbb{Z} \simeq \langle w | \emptyset \rangle$ .

Then  $\pi_1(U \cap V)$  and  $\pi_1(V \cap W)$  are also annuluses. A loop in  $\pi_1(U \cap V)$  has to get mapped to the same element in  $\pi_1(U)$  and  $\pi_1(V)$ , and similarly  $V$  and  $W$ , so we end up with  $\langle u, v, w | u = v, v = w \rangle \simeq \mathbb{Z}$ .

**Example 2.** Let's try gluing three punctured toruses together. We know that a punctured torus is homotopy equivalent to a figure eight, because the corners are identified as a point, and then the top and bottom and left and right edges become the loops:  $\pi_1(U) \simeq \mathbb{Z} * \mathbb{Z} \simeq \langle a, b | \emptyset \rangle$ ,  $\pi_1(V) \simeq \mathbb{Z} * \mathbb{Z} \simeq \langle c, d | \emptyset \rangle$ ,  $\pi_1(W) \simeq \mathbb{Z} * \mathbb{Z} \simeq \langle e, f | \emptyset \rangle$ .

All three of the pairwise intersections are annuluses:  $\pi_1(U \cap V) \simeq \mathbb{Z}$ ,  $\pi_1(U \cap W) \simeq \mathbb{Z}$ ,  $\pi_1(V \cap W) \simeq \mathbb{Z}$ .

Let's think about the inclusion homomorphisms  $j_1 : \pi_1(U \cap V) \rightarrow \pi_1(U)$  and  $j_2 : \pi_1(U \cap V) \rightarrow \pi_1(V)$ , for which we must add relations. A loop in  $U \cap V$  corresponds to a boundary loop in  $U$  and  $V$ , and those boundary loops have to be identified. A boundary loop on the punctured torus is  $aba^{-1}b^{-1}$ .

So our group ends up being  $\langle a, b, c, d, e, f | aba^{-1}b^{-1} = cdc^{-1}d^{-1} = efe^{-1}f^{-1} \rangle$ ?