Seifert–Van Kampen for n-fold Unions

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The size of a non-disjoint union of sets is computed using an *inclusion–exclusion principle*: $|A \cup B| = |A| + |B| - |A \cap B|$. You can't just add the sizes of the sets, because that would be double-counting the overlap $A \cap B$. You have to subtract the size of the intersection.

The fundamental group of a non-disjoint union of path-connected topological spaces (whose intersection is also path-connected) is computed using the Seifert–Van Kampen theorem: $\pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$. You can't just take the free product, because that would "double-count" the loops in the overlap $U \cap V$. You need to take the amalgamated free product.

How far can we push this analogy? There is a generalized inclusion–exclusion principle for n-fold unions, proved by induction:

$$\left| \bigcup_{i=1}^{n} A_i \right| = \sum_{j=1}^{n} (-1)^{j+1} \left(\sum_{S \in \mathcal{P}(\{A_i\}_{i=1}^{n}); |S|=j} \left| \bigcap_{A_s \in S} A_s \right| \right)$$

Is there a generalized Sifert–van Kampen theorem for n-fold unions?

Let's start with $n := 3$.

For set unions, the $n := 3$ inclusion–exclusion principle is $|A \cup B \cup C| = |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$.

The way we derive that is with a single induction step, by applying the $n := 2$ inclusion exclusion principle to itself:

$$|(A \cup B) \cup C| = \underbrace{|A \cup B|}_{n:=2 \text{ case}} + |C| - |(A \cup B) \cap C|$$

$$= |A| + |B| - |A \cap B| + |C| - |(A \cup B) \cap C|$$

$$= |A| + |B| + |C| - |A \cap B| - \underbrace{|(A \cap C) \cup (B \cap C)|}_{n:=2 \text{ case}}$$

$$= |A| + |B| + |C| - |A \cap B| - (|A \cap C| + |B \cap C| - |(A \cap C) \cap (B \cap C)|)$$

$$= |A| + |B| + |C| - |A \cap B| - |A \cap C| - |B \cap C| + |A \cap B \cap C|$$

To understand if there's an $n := 3$ Siefert–van Kampen principle, I'll need a more detailed understanding of the analogy between "subtracting the intersection" and amalgamation.

The free product $G_1 * G_2$ combines the generators of G_1 and G_2 with no relations.

The fundamental group of joining two topological spaces at a point (or with the intersection being contractible) is the free product of the fundamental groups. If you go around one of the loops of a figure-eight, that doesn't commute with going around the other loop.

The amalgamated free product $G_1 *_H G_2$ over homomorphisms $\varphi_1 : H \to G_1$ and $\varphi_2 : H \to G_2$ combines the generators of G_1 and G_2 and adds relations: for all $h \in H$, we identify $\varphi_1(h) = \varphi_2(h)$.

In the application of amalgamated free products in the SvK theorem, we specifically have $\pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$ with the inclusion homomorphisms. A loop-equivalence-class from $\pi_1(U \cap V)$ should be "the same" whether it appears in $\pi_1(U)$ or $\pi_1(V)$.

$$\begin{tikzcd}
 & \pi_1(U \cap V) \arrow[dl, "\varphi_1"'] \arrow[dr, "\varphi_2"] & \\
\pi_1(V) \arrow[dr] & & \pi_1(U) \arrow[dl] \\
 & \pi_1(U \cup V) &
\end{tikzcd}$$

For three spaces, the diagram becomes:

$$\begin{tikzcd}
 & \pi_1(U \cap V \cap W) \arrow[dl, "i_1"] \arrow[d, "i_2"'] \arrow[dr, "i_3"] & \\
\pi_1(U \cap V) \arrow[d, "j_1"] \arrow[dr, "j_2"] & \pi_1(U \cap W) \arrow[dl, "j_3"] \arrow[dr, "j_5"'] & \pi_1(V \cap W) \arrow[dl, "j_4"'] \arrow[d, "j_6"] \\
\pi_1(U) \arrow[dr, "k_1"] & \pi_1(V) \arrow[d, "k_2"] & \pi_1(W) \arrow[dl, "k_3"] \\
 & \pi_1(U \cup V \cup W) &
\end{tikzcd}$$

SvK tells us that $\pi_1(U \cup V) \simeq \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V)$.
So presumably

$$\pi_1(U \cup V \cup W) = \pi_1((U \cup V) \cup W) \simeq \pi_1(U \cup V) *_{\pi_1((U \cup V) \cap W)} \pi_1(W)$$

$$\simeq \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) *_{\pi_1((U \cup V) \cap W)} \pi_1(W)$$

$$\simeq \pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) *_{\pi_1((U \cap W) \cup (V \cap W))} \pi_1(W)$$

But then we have to unpack what it means to amalgamate over $\pi_1((U \cap W) \cup (V \cap W))$:

$$\pi_1((U \cap W) \cup (V \cap W)) \simeq \pi_1(U \cap W) *_{\pi_1(U \cap V \cap W)} \pi_1(V \cap W)$$

And substitute that back to get:

$$\pi_1(U) *_{\pi_1(U \cap V)} \pi_1(V) *_{\pi_1(U \cap W) *_{\pi_1(U \cap V \cap W)} \pi_1(V \cap W)} \pi_1(W)$$

which is unreadable, but hopefully correct.

It doesn't seem like the order of which amalgamated free products are nested within each other should matter, though: what matters is that we have all the relations in the diagram.

Suppose we have the presentations $\pi_1(U) = \langle S_U | R_U \rangle$, $\pi_1(V) = \langle S_V | R_V \rangle$, and $\pi_1(W) = \langle S_W | R_W \rangle$.

For all $h_{uvw} \in \pi_1(U \cap V \cap W)$, $i_1(h_{uvw}) = i_2(h_{uvw}) = i_3(h_{uvw})$. For all $h_{uv} \in \pi_1(U \cap V)$, $j_1(h_{uv}) = j_2(h_{uv})$. For all $h_{wv} \in \pi_1(U \cap V)$, $j_3(h_{wv}) = j_4(h_{wv})$. For all $h_{uw} \in \pi_1(U \cap V)$, $j_5(h_{uw}) = j_6(h_{uw})$.

Putting that all together, we have

$$\pi_1(U \cup V \cup W) \simeq \langle S_U, S_V, S_W | R_U, R_W, R_W,$$

$$\forall h_{uvw} \in \pi_1(U \cap V \cap W) \; i_1(h_{uvw}) = i_2(h_{uvw}) = i_3(h_{uvw}),$$

$$\forall h_{uv} \in \pi_1(U \cap V) \; j_1(h_{uv}) = j_2(h_{uv})$$

$$\forall h_{wv} \in \pi_1(V \cap W) \; j_3(h_{wv}) = j_4(h_{wv})$$

$$\forall h_{uw} \in \pi_1(U \cap W) j_5(h_{uw}) = j_6(h_{uw})\rangle$$

Gemini 2.5 Pro claims that we don't actually need the triple-intersection relations because they're implied by the pairwise relations by transitivity, so this is actually simpler than the alternating sums of the inclusion–exclusion principle.

We need examples.

Example 1. What about three annuluses, where U overlaps V, and V overlaps W? We can see that the union should just be a thicker annulus (or our proposed generalization should compute the same result formally). We have $\pi_1(U) \simeq \mathbb{Z} \simeq \langle u|\varnothing\rangle$, $\pi_1(V) \simeq \mathbb{Z} \simeq \langle v|\varnothing\rangle$, $\pi_1(W) \simeq \mathbb{Z} \simeq \langle w|\varnothing\rangle$.

Then $\pi_1(U \cap V)$ and $\pi_1(V \cap W)$ are also annuluses. A loop in $\pi_1(U \cap V)$ has to get mapped to the same element in $\pi_1(U)$ and $\pi_1(V)$, and similarly V and W, so we end up with $\langle u, v, w|u = v, v = w\rangle \simeq \mathbb{Z}$.

Example 2. Let's try gluing three punctured toruses together. We know that a punctured torus is homotopy equivalent to a figure eight, because the corners are identified as a point, and then the top and bottom and left and right edges become the loops: $\pi_1(U) \simeq \mathbb{Z} * \mathbb{Z} \simeq \langle a, b|\varnothing\rangle$, $\pi_1(V) \simeq \mathbb{Z} * \mathbb{Z} \simeq \langle c, d|\varnothing\rangle$, $\pi_1(W) \simeq \mathbb{Z} * \mathbb{Z} \simeq \langle e, f|\varnothing\rangle$.

All three of the pairwise intersections are annuluses: $\pi_1(U \cap V) \simeq \mathbb{Z}$, $\pi_1(U \cap W) \simeq \mathbb{Z}$, $\pi_1(V \cap W) \simeq \mathbb{Z}$.

Let's think about the inclusion homomorphisms $j_1 : \pi_1(U \cap V) \to \pi_1(U)$ and $j_2 : \pi_1(U \cap V) \to \pi_1(V)$, for which we must add relations. A loop in $U \cap V$ corresponds to a boundary loop in U and V, and those boundary loops have to be identified. A boundary loop on the punctured torus is $aba^{-1}b^{-1}$.

So our group ends up being $\langle a, b, c, d, e, f|aba^{-1}b^{-1} = cdc^{-1}d^{-1} = efe^{-1}f^{-1}\rangle$?